THE NONLOCAL NEUMANN PROBLEM

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Preface

As part of this thesis, a preprint was written which appears as [8] in the bibliography. The content of this work can be found in Chapters 2 to 7.

Acknowledgements

First and foremost, I want to express my sincere gratitude to my supervisor Leonhard Frerick for accepting me as his PhD student and for his guidance. The constant support he has provided is immeasurable and without him I would have been lost many times. Next, I would like to thank Ekkehard Sachs for his continuous interest in my work and the suggestions he has given me. Furthermore, I want to thank Ekkehard Sachs for accepting to be a reviewer of this thesis. In the same manner, I want to give many thanks to Max Gunzburger. I am very honored that he accepted to be a reviewer as his scientific contributions greatly influenced this thesis. And also I would like to thank Christian Vollmann for the many discussions we had and Martin Schmidt as he is the reason that optimization is part of this thesis.

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Abstract

Instead of presuming only local interaction, we assume nonlocal interactions. By doing so, mass at a point in space does not only interact with an arbitrarily small neighborhood surrounding it, but it can also interact with mass somewhere far, far away. Thus, mass jumping from one point to another is also a possibility we can consider in our models. So, if we consider a region in space, this region interacts in a local model at most with its closure. While in a nonlocal model this region may interact with the whole space. Therefore, in the formulation of nonlocal boundary value problems the enforcement of boundary conditions on the topological boundary may not suffice. Furthermore, choosing the complement as nonlocal boundary may work for Dirichlet boundary conditions, but in the case of Neumann boundary conditions this may lead to an overfitted model.

In this thesis, we introduce a nonlocal boundary and study the well-posedness of a nonlocal Neumann problem. We present sufficient assumptions which guarantee the existence of a weak solution. As in a local model our weak formulation is derived from an integration by parts formula. However, we also study a different weak formulation where the nonlocal boundary conditions are incorporated into the nonlocal diffusion-convection operator.

After studying the well-posedness of our nonlocal Neumann problem, we consider some applications of this problem. For example, we take a look at a system of coupled Neumann problems and analyze the difference between a local coupled Neumann problems and a nonlocal one. Furthermore, we let our Neumann problem be the state equation of an optimal control problem which we then study. We also add a time component to our Neumann problem and analyze this nonlocal parabolic evolution equation.

As mentioned before, in a local model mass at a point in space only interacts with an arbitrarily small neighborhood surrounding it. We analyze what happens if we consider a family of nonlocal models where the interaction shrinks so that, in limit, mass at a point in space only interacts with an arbitrarily small neighborhood surrounding it.

Contents

Chapter 1

Introduction

Our aim is to formulate and study the weak formulation of nonlocal equations in the form of

$$\begin{cases} \mathcal{L}_\gamma u(x) = \displaystyle\int_{\mathbb{R}^d} u(x)\gamma(x,y) - u(y)\gamma(y,x)\,\mathrm{d}y & = f(x) \quad \text{for } x \in \Omega, \\ \alpha(y)u(y) + (1-\alpha(y))\mathcal{N}_\gamma u(y) & = g(y) \quad \text{for } y \in \widehat{\Gamma}(\Omega,\gamma), \end{cases} \tag{P}$$

where

$$\begin{aligned} \Omega \subset \mathbb{R}^d & \qquad \text{is a nonempty and open set,} \\ \gamma\colon \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty) & \qquad \text{is a measurable function,} \\ \widehat{\Gamma}(\Omega,\gamma) := \{y \in \mathbb{R}^d \setminus \Omega\colon \int_\Omega \gamma(x,y) + \gamma(y,x)\,\mathrm{d}x > 0\} & \qquad \text{is the nonlocal boundary of } \Omega, \\ \alpha\colon \widehat{\Gamma}(\Omega,\gamma) \to [0,1] & \qquad \text{is measurable} \end{aligned}$$

and where we set

$$\mathcal{N}_\gamma u(y) := \int_\Omega u(y)\gamma(y,x) - u(x)\gamma(x,y)\,\mathrm{d}x \quad \text{for } y \in \widehat{\Gamma}(\Omega,\gamma).$$

We recall that a set $A \subset \mathbb{R}^d$ is said to be (Lebesgue) measurable if a Lebesgue measure can be assigned to it which we denote by $\lambda(A)$. With this characterization functions are measurable if the pre-image of all measurable sets are measurable. We call γ in problem (P) *kernel (function)* and define the space of all kernels by

$$\mathcal{K} := \{\gamma\colon \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty) \text{ measurable}\}.$$

Furthermore, we define for each $\gamma \in \mathcal{K}$ the function $\gamma^\top \in \mathcal{K}$ by $\gamma^\top(x,y) := \gamma(y,x)$ for $x,y \in \mathbb{R}^d$ and we call $\gamma \in \mathcal{K}$ *symmetric* if we have $\gamma = \gamma^\top$. We call $\gamma \in \mathcal{K}$ *regional* (in Ω) if $\lambda(\widehat{\Gamma}(\Omega,\gamma)) = 0$ holds. For example, $\gamma \in \mathcal{K}$ is regional if γ vanishes identically in the complement of $\Omega \times \Omega$, i.e., if we have

$$\gamma = 0 \text{ a.e. on } (\mathbb{R}^d \times \mathbb{R}^d) \setminus \Omega \times \Omega.$$

The measurable function α indicates which boundary conditions are enforced on $\widehat{\Gamma}(\Omega,\gamma)$. For example, we have pure Neumann boundary conditions on the set $\{y \in \widehat{\Gamma}(\Omega,\gamma)\colon \alpha(y) = 0\}$. And if

$\{y \in \widehat{\Gamma}(\Omega,\gamma)\colon \alpha(y) = 0\}$ is up to a null set equal to $\widehat{\Gamma}(\Omega,\gamma)$, then we obtain the nonlocal Neumann problem

$$\begin{cases} \mathcal{L}_\gamma u(x) &= \displaystyle\int_{\mathbb{R}^d} u(x)\gamma(x,y) - u(y)\gamma(y,x)\,\mathrm{d}y &= f(x) \quad \text{for } x \in \Omega, \\ \mathcal{N}_\gamma u(y) &= \displaystyle\int_{\Omega} u(y)\gamma(y,x) - u(x)\gamma(x,y)\,\mathrm{d}x &= g(y) \quad \text{for } y \in \widehat{\Gamma}(\Omega,\gamma). \end{cases} \tag{NP}$$

Pure Dirichlet boundary conditions are enforced on $\{y \in \widehat{\Gamma}(\Omega,\gamma)\colon \alpha(y) = 1\}$ and the nonlocal Dirichlet problem is therefore defined by

$$\begin{cases} \mathcal{L}_\gamma u(x) &= f(x), \quad \text{for } x \in \Omega, \\ u(y) &= g(y), \quad \text{for } y \in \widehat{\Gamma}(\Omega,\gamma). \end{cases} \tag{DP}$$

In the case that $\lambda(\widehat{\Gamma}(\Omega,\gamma)) = 0$ holds, problem (P) is considered to be a pure Neumann problem which we then call regional problem.

For simplicity, we omit γ if possible, and write $\mathcal{L}$ instead of $\mathcal{L}_\gamma$, and $\mathcal{N}$ instead of $\mathcal{N}_\gamma$. Moreover, we set

$$\widehat{\Gamma} := \widehat{\Gamma}(\Omega,\gamma) = \{y \in \mathbb{R}^d \setminus \Omega\colon \int_\Omega \gamma(x,y) + \gamma(y,x)\,\mathrm{d}x > 0\}.$$

As in the local case, the boundary conditions are required for the well-posedness of problem (P). However, in the local case boundary conditions are enforced on the topological boundary $\partial\Omega$, i.e., the intersection of the closure of Ω with the closure of its complement. Due to the nonlocal nature of problem (P), boundary conditions on $\partial\Omega$ are in general not sufficient to guarantee well-posedness. As a consequence of this, we define the nonlocal boundary $\widehat{\Gamma}$. Note that in the pure Dirichlet case or if $\lambda(\mathbb{R}^d \setminus \widehat{\Gamma}) = 0$ holds, boundary conditions can be enforced on the whole complement of Ω.

To the best of our knowledge, Gunzburger and Lehoucq [15] are the first to formulate and consider nonlocal boundary value problems in the form of (P). Although, we consider a different Neumann operator the approaches to derive a weak formulation coincide. Furthermore, in [15] the nonlocal boundary does not depend on γ.

The nonlocal Dirichlet problem is further studied by Felsinger et al. [21] and by Du et al. [28], for instance. Note that in the pure Dirichlet case or if $\widehat{\Gamma}$ is up to a null set equal to $\mathbb{R}^d \setminus \Omega$, the boundary conditions can be enforced on the whole complement of Ω. However, for symmetric kernels, Du et al. [28] enforced the nonlocal Dirichlet boundary condition on

$$\Omega_I = \left\{ y \in \mathbb{R}^d \setminus \Omega \text{ such that } \gamma(x,y) \neq 0 \text{ for some } x \in \Omega \right\}.$$

And they also consider a different nonlocal Neumann problem

$$\begin{cases} \mathcal{L}_\gamma u(x) &= \displaystyle\int_{\mathbb{R}^d} u(x)\gamma(x,y) - u(y)\gamma(y,x)\,\mathrm{d}y &= f(x) \quad \text{for } x \in \Omega, \\ \mathcal{N}^G_\gamma u(y) &= \displaystyle\int_{\Omega\cup\Omega_I} u(y)\gamma(y,x) - u(x)\gamma(x,y)\,\mathrm{d}x &= g(y) \quad \text{for } y \in \Omega_I, \end{cases} \tag{NP2}$$

where $\mathcal{N}_\gamma^G$ was first introduced by Gunzburger and Lehoucq [15]. Then, for a symmetric kernel, we obtain $\widehat{\Gamma} \subset \Omega_I$ and

$$\mathcal{N}_\gamma^G u(y) - \mathcal{N}_\gamma u(y) = \int_{\Omega_I} u(y)\gamma(y,x) - u(x)\gamma(x,y)\,\mathrm{d}x \quad \text{for } y \in \widehat{\Gamma}.$$

Roughly speaking, $\mathcal{N}_\gamma$ only controls the mass leaving Ω. While, $\mathcal{N}_\gamma^G$ takes into account the mass leaving Ω and the mass movement within Ω_I.

There have been several other approaches for nonlocal Neumann problem in the literature. For instance, Barles et al. [9] formulate a Neumann–type boundary value problem by imposing a reflection condition on $\mathbb{R}^d \setminus \Omega$. Cortazar et al. [4] and You et al. [40] derive a Neumann-type boundary value problem from a decomposition of the operator $\mathcal{L}$. A connection between the decomposition approach and the Neumann problem (NP) is shown in Chapter 7. And a comparison between the nonlocal Neumann operator $\mathcal{N}$ and $\mathcal{N}_\gamma^G$ is made in Chapter 2.

Our approach follows Dipierro et al. [32] who consider problem (NP) for

$$\gamma_s(y,x) := \frac{c_{d,s}}{\|y-x\|^{d+2s}}\chi_{\mathbb{R}^d\setminus\{0\}}(y-x) \quad \text{for } y,x \in \mathbb{R}^d$$

where $c_{d,s}$ is a normalization constant and $s \in (0,1)$. Foghem [11] and Foghem and Kaßmann [13] further generalize this approach by considering all $\gamma \in \mathcal{K}$ such that the convolution kernel $\nu\colon \mathbb{R}^d \setminus \{0\} \to (0,\infty)$, defined by $\gamma(y,x) := \nu(y-x)\chi_{\mathbb{R}^d\setminus\{0\}}(y-x)$ for $y,x \in \mathbb{R}^d$, is even and Lévy integrable, i.e., ν satisfies

$$\nu(x) = \nu(-x) \quad \text{for } x \in \mathbb{R}^d \setminus \{0\} \text{ and } \int_{\mathbb{R}^d\setminus\{0\}} \min\{1, \|x\|^2\}\nu(x)\,\mathrm{d}x < \infty.$$

In this thesis, we broaden this approach to kernels in $\mathcal{K}$.

In order to define a weak solution for (NP), we require a weak formulation and a test function space. To be more precise, we want to have a test function space $F \subset \{u\colon \mathbb{R}^d \to \mathbb{R} \text{ measurable}\}$ and a bounded bilinearform $\mathfrak{F}\colon F \times F \to \mathbb{R}$ with

$$\int_\Omega (\mathcal{L}u(x))v(x)\,\mathrm{d}x + \int_{\widehat{\Gamma}} (\mathcal{N}u(y))v(y)\,\mathrm{d}y = \mathfrak{F}(u,v) \text{ for all } v \in F$$

and all sufficiently regular functions $u \in F$. Then, our aim is to study under which assumptions a solution to this weak formulation exists, i.e., we want to present assumptions which guarantee that there is at least one function $u \in F$ solving

$$\int_\Omega f(x)v(x)\,\mathrm{d}x + \int_{\widehat{\Gamma}} g(y)v(y)\,\mathrm{d}y = \mathfrak{F}(u,v) \text{ for all } v \in F.$$

We recall that for a normed space $(X, \|\cdot\|_X)$ the *(continuous) dual space* X^* is defined by

$$X^* := \{p\colon X \to \mathbb{R} \text{ linear with } \sup_{x\in X, \|x\|_X \leqslant 1} |p(x)| < \infty\}.$$

If $(F, \mathfrak{F})$ is a Hilbert space and $\Phi\colon F \to \mathbb{R}$ defined by

$$\Phi(v) := \int_\Omega f(y)v(x)\,\mathrm{d}x + \int_{\widehat{\Gamma}} g(y)v(y)\,\mathrm{d}y \tag{1.1}$$

is a bounded linear form, i.e., $\Phi \in F^*$, then the Riesz representation Theorem yields the existence of a unique weak solution. Note that we in fact obtain that for every $\Phi \in F^*$ there is a unique $u \in F$ solving

$$\Phi(v) = \mathfrak{F}(u, v) \text{ for all } v \in F.$$

A unique existence result which holds under weaker assumptions is the Lax-Milgram Theorem (see [6, 6.2.1 Lax-Milgram Theorem]). If $(F, \langle\cdot,\cdot\rangle_F)$ is a Hilbert space such that there are constants $c, C > 0$ with

$$|\mathfrak{F}(u, v)| \leqslant C\|u\|_F\|v\|_F \quad \text{and} \quad c\|u\|_F^2 \leqslant \mathfrak{F}(u, u) \quad \text{for all } u, v \in F, \tag{1.2}$$

where $\|u\|_F^2 := \langle u, u\rangle_F$ for $u \in F$, then we obtain by the Lax-Milgram Theorem that for every $\Phi \in F^*$ there is a unique $u \in F$ solving

$$\Phi(v) = \mathfrak{F}(u, v) \text{ for all } v \in F.$$

To sum it up, we want to have an inner product on F such that (1.2) is satisfied and that $\Phi\colon F \to \mathbb{R}$ defined by (1.1) is an element F^*.

This thesis is structured in the following way.

First, we present in Chapter 2, an interpretation of problem (NP) and our nonlocal Neumann operator $\mathcal{N}$. This interpretation leads to Theorem 2.1, the nonlocal integration by parts formula. This nonlocal integration by parts formula provides us a test space $\mathrm{V}(\Omega;\gamma)$ and a bilinearform $\mathfrak{B}$ which is symmetric if $\gamma \in \mathcal{K}$ is symmetric.

Then, we define in Chapter 3 a bilinearform $\langle\cdot,\cdot\rangle_{\mathrm{V}(\Omega;\gamma)}$ on our test space $\mathrm{V}(\Omega;\gamma)$ such that for a symmetric $\gamma \in \mathcal{K}$ we obtain

$$\frac{1}{2}\langle u, v\rangle_{\mathrm{V}(\Omega;\gamma)} \leqslant \langle u, v\rangle_{\mathrm{L}^2(\Omega)} + \mathfrak{B}(u, v) \leqslant \langle u, v\rangle_{\mathrm{V}(\Omega;\gamma)} \quad \text{for all } u, v \in \mathrm{V}(\Omega;\gamma).$$

Then, we show that $(\mathrm{V}(\Omega;\gamma), \langle\cdot,\cdot\rangle_{\mathrm{V}(\Omega;\gamma)})$ is Hilbert space (see Corollary 3.2).

The weak solution of problem (NP) is defined in Chapter 4 and we show an existence result for problem (NP) (see Theorem 4.5) where we assume, among other things, that $\gamma \in \mathcal{K}$ is symmetric, that the nonlocal counterpart of the Poincaré inequality holds, i.e., there is a $C > 0$ with

$$\left\| v - \frac{1}{\lambda(\Omega)}\int_\Omega v(x)\,\mathrm{d}x \right\|_{\mathrm{L}^2(\Omega)} \leqslant C\mathfrak{B}(u, u) \quad \text{for all } u, v \in \mathrm{V}(\Omega;\gamma),$$

that a continuous linear functional on $\mathrm{V}(\Omega;\gamma)$ is defined by

$$v \mapsto \int_\Gamma g(y)v(y)\,\mathrm{d}y,$$

and that a compatibility condition is satisfied, i.e., we have

$$\int_\Omega f(y)\,\mathrm{d}x + \int_\Gamma g(y)\,\mathrm{d}y = 0.$$

Note that we in fact assume an equivalent formulation of the nonlocal Poincaré inequality in Theorem 4.5. This is shown in Chapter 5 as this chapter is dedicated to the nonlocal Poincaré inequality.

As mentioned before, we require $\Phi\colon \mathrm{V}(\Omega;\gamma) \to \mathbb{R}$ defined by (1.1) to be an element of the dual space of the Hilbert space $\mathrm{V}(\Omega;\gamma)$ because $f \in \mathrm{L}^2(\Omega)$ implies

$$\int_\Omega f(x)v(x)\,\mathrm{d}x \leqslant \|f\|_{\mathrm{L}^2(\Omega)}\|v\|_{\mathrm{L}^2(\Omega)} \leqslant \|f\|_{\mathrm{L}^2(\Omega)}\|v\|_{\mathrm{V}(\Omega;\gamma)} \quad \text{for all } v \in \mathrm{V}(\Omega;\gamma) \tag{1.3}$$

it remains to verify if a continuous linear functional on $\mathrm{V}(\Omega;\gamma)$ is defined by

$$v \mapsto \int_\Gamma g(y)v(y)\,\mathrm{d}y.$$

In order to do so, we study the restriction of functions in $\mathrm{V}(\Omega;\gamma)$ on Γ in Chapter 6. In other words, we analyze the behaviour of our test functions on the nonlocal boundary and we investigate which functions defined on the nonlocal boundary Γ can be extended into a function in $\mathrm{V}(\Omega;\gamma)$, i.e., we study a nonlocal Trace space.

Chapter 7 introduces another weak formulation (see Definition 7.3) where we incorporate the boundary conditions into the kernel. To be more precise, with boundary conditions we modify the kernel into a regional kernel.

In Chapter 8, we consider coupled nonlocal Neumann problems. Due to the nonlocal nature, these coupled problems are quite different compared to the local coupled Neumann problems.

And in Chapter 9, we consider an optimal control problem where problem (NP) is the state equation.

Finally, Chapter 11 deals with the convergence of nonlocal Neumann problem to local Neumann problems which is achieved by considering a family of kernels with shrinking support.

Chapter 2

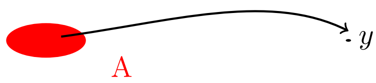
Interpretation

In [32], a probabilistic interpretation of problem (NP) is presented. We, however, provide a more physical interpretation of problem (NP) by following along the lines of [29]. Until further notice, let $\Omega \subset \mathbb{R}^d$ be a bounded, nonempty, and open set, $\gamma \in \mathcal{K}$ and $u\colon \mathbb{R}^d \to \mathbb{R}$ be measurable with

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (|u(x)| + |u(y)|)\gamma(x, y) \, \mathrm{d}y \, \mathrm{d}x < \infty.$$

Then, the function u is considered to be the mass density and the flux is determined by γ in the following manner. While mass which travels from an open subset $A \subset \mathbb{R}^d$ to a point $y \in \mathbb{R}^d$ is given by

$$u(y) \int_A \gamma(y, x) \, \mathrm{d}x,$$



Mass traveling from A to y

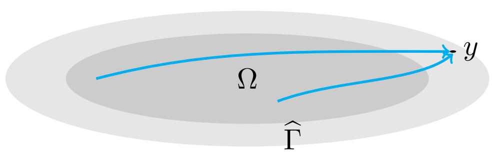
mass traveling from this point y to the set A is given by

$$\int_A u(x)\gamma(x, y) \, \mathrm{d}x.$$

Mass traveling from y to A

Hence, the nonlocal boundary $\widehat{\Gamma}$ consists of all the points in the complement of $\Omega \subset \mathbb{R}^d$ interacting with Ω. Thus, we have that

$$\mathcal{N}u(y) = \int_\Omega u(y)\gamma(y, x) - u(x)\gamma(x, y) \, \mathrm{d}x$$

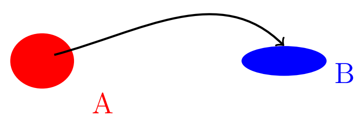


Mass leaving Ω to $y \in \widehat{\Gamma}$

is the (total) mass leaving the open set Ω to the point y belonging to the interaction set $\widehat{\Gamma}$. So, by prescribing $\mathcal{N}u$ on $\widehat{\Gamma}$, we control the mass leaving Ω.

The (total) flux from the open subset $A \subset \mathbb{R}^d$ to the open subset $B \subset \mathbb{R}^d$ is the mass traveling from A to B subtracted with the mass traveling from B to A, i.e.,

$$\begin{aligned}&\int_A \int_B u(y)\gamma(y,x) - u(x)\gamma(x,y)\,\mathrm{d}y\,\mathrm{d}x\\ =&\int_B \int_A u(y)\gamma(y,x) - u(x)\gamma(x,y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$



Flux from A to B

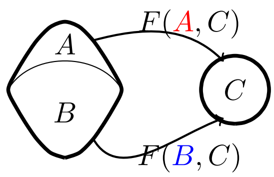
Therefore, we define the flux operator from the open set $A \subset \mathbb{R}^d$ to the open set $B \subset \mathbb{R}^d$ as

$$F(A,B) := F_\gamma(A,B;u) = \int_A \int_B u(y)\gamma(y,x) - u(x)\gamma(x,y)\,\mathrm{d}y\,\mathrm{d}x.$$

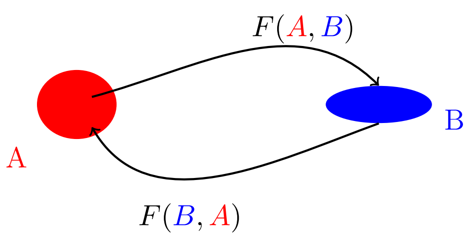
Note that the flux operator F can be extended to fluxes between two measurable sets.

By the linearity of the integral and Fubini's Theorem, the action-reaction principle is satisfied (see [29]), i.e., for all open sets $A, B, C \subset \mathbb{R}^d$, we have

$$F(A \cup B, C) = F(A,C) + F(B,C) - F(A \cap B, C) \quad \text{and} \quad F(A,B) = -F(B,A). \tag{ARP}$$



Flux from A and B to C



Flux from A to B and from B to A

Notice, that if γ is symmetric, then for all open subsets $A, B \subset \mathbb{R}^d$, we obtain

$$\begin{aligned}\int_A u(x) \int_B \gamma(x,y)\,\mathrm{d}y\,\mathrm{d}x &= \int_A u(x) \int_B \gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x\\ \text{and} \quad F(A,B) &= \int_A \int_B (u(y) - u(x))\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

We want to emphasize that $\mathcal{L}u \in \mathrm{L}^1(\Omega)$ holds and that the mass leaving Ω is given by $F(\Omega, \mathbb{R}^d \setminus \Omega)$. So by the action-reaction principle (ARP) and the definition of $\widehat{\Gamma}$ and $\mathcal{N}$ we further have

$$\begin{aligned}\int_\Omega -\mathcal{L}u(x)\,\mathrm{d}x &= -\int_\Omega \int_{\mathbb{R}^d} u(x)\gamma(x,y) - u(y)\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x\\ &= F(\Omega, \mathbb{R}^d)\\ &= F(\Omega, \mathbb{R}^d \setminus \Omega) + F(\Omega,\Omega)\\ &= -F(\mathbb{R}^d \setminus \Omega, \Omega)\\ &= \int_{\mathbb{R}^d \setminus \Omega} \int_\Omega u(y)\gamma(y,x) - u(x)\gamma(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\widehat{\Gamma}} \mathcal{N}u(y)\,\mathrm{d}y.\end{aligned}$$

The equation

$$\int_{\Omega} -\mathcal{L}u(x)\,\mathrm{d}x = \int_{\widehat{\Gamma}} \mathcal{N}u(y)\,\mathrm{d}y$$

is the nonlocal counterpart of the well-known Gauss Theorem.

Now, we want to further motivate that $-\mathcal{L}$ is the nonlocal divergence operator by following the argumentation made for the local case (see Purcell and Morin [27, pp. 78-80]).

Similar to the local case, we define the nonlocal divergence at a point x_0 of the flux F by

$$(\operatorname{ndiv} F)(x_0) := \lim_{r\to 0} \frac{1}{\lambda(\mathrm{B}_r(x_0))} F(\mathrm{B}_r(x_0), \mathbb{R}^d \setminus \mathrm{B}_r(x_0)).$$

Since for $r > 0$ and almost all $x_0 \in \Omega$ it is valid that $F(\mathrm{B}_r(x_0), \mathrm{B}_r(x_0)) = 0$, we obtain

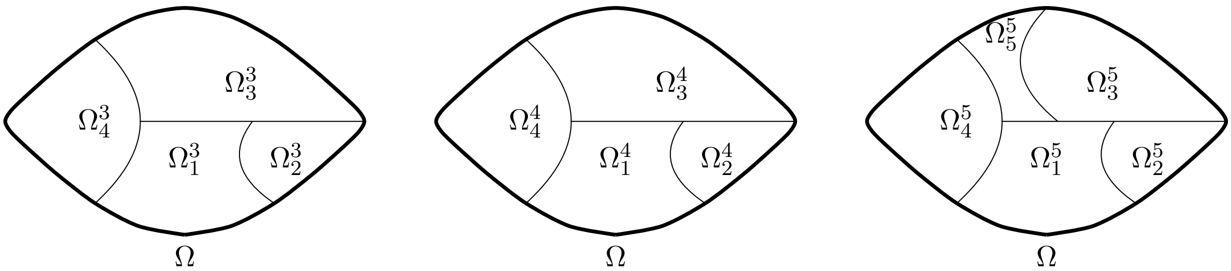
$$\begin{aligned}
(\operatorname{ndiv} F)(x_0) &= \lim_{r\to 0} \frac{1}{\lambda(\mathrm{B}_r(x_0))} F(\mathrm{B}_r(x_0), \mathbb{R}^d \setminus \mathrm{B}_r(x_0)) + F(\mathrm{B}_r(x_0), \mathrm{B}_r(x_0)) \\
&= \lim_{r\to 0} \frac{1}{\lambda(\mathrm{B}_r(x_0))} F(\mathrm{B}_r(x_0), \mathbb{R}^d) \\
&= \lim_{r\to 0} \frac{1}{\lambda(\mathrm{B}_r(x_0))} \int_{\mathrm{B}_r(x_0)} \int_{\mathbb{R}^d} u(y)\gamma(y,x) - u(x)\gamma(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \lim_{r\to 0} \frac{1}{\lambda(\mathrm{B}_r(x_0))} \int_{\mathrm{B}_r(x_0)} -\mathcal{L}u(x)\,\mathrm{d}x.
\end{aligned}$$

By Theorem 7.10 in Rudin [37], the Lebesgue differentiation Theorem, we conclude that

$$\operatorname{ndiv} F = -\mathcal{L}u$$

holds almost everywhere in Ω. Note that Theorem 7.10 in Rudin [37] is also applicable if we consider a sequence of measurable sets nicely shrinking to x_0, instead of shrinking balls centered at x_0.

Similiar to the local case (see Purcell and Morin [27, pp. 78-80]), we now divide Ω into countable disjoint and nonempty subsets of Ω. To be more precise, for each $n \in \mathbb{N}$ we assume that there is a disjoint family of sets $(\Omega_i^n)_{i\in\mathbb{N}, i\leqslant k_n}$ where $(k_n)_{n\in\mathbb{N}}$ is a monotonically increasing sequence in $\mathbb{N}$ with $\bigcup_{i=1}^{k_n} \Omega_i^n = \Omega$ and that for each $x \in \Omega$, there is an $i \in \mathbb{N}$ such that Ω_i^n nicely shrinks to x as $n \to \infty$.



Example for $n = 3, 4, 5$

For every fixed $n \in \mathbb{N}$ the action-reaction principle (ARP) yields

$$F(\Omega, \mathbb{R}^d \setminus \Omega) = \sum_{i=1}^{k_n} F(\Omega_i^n, \mathbb{R}^d \setminus \Omega_i^n).$$

In general, we obtain that the flux leaving Ω is equal to the combined flux leaving every element of a partition of Ω.

For all $n \in \mathbb{N}$, we therefore have

$$\begin{aligned}
&\int_{\widehat{\Gamma}} \mathcal{N}u(y)\,\mathrm{d}y \\
=&F(\Omega, \mathbb{R}^d \setminus \Omega) = \sum_{i=1}^{k_n} \left(\int_{\Omega_i^n} \frac{1}{\lambda(\Omega_i^n)} F(\Omega_i^n, \mathbb{R}^d \setminus \Omega_i^n)\,\mathrm{d}x \right) \\
=&\int_{\Omega} \sum_{i=1}^{k_n} \left(\frac{1}{\lambda(\Omega_i^n)} F(\Omega_i^n, \mathbb{R}^d \setminus \Omega_i^n) \chi_{\Omega_i^n}(x) \right) \mathrm{d}x.
\end{aligned}$$

And by our assumptions and Theorem 7.10 in [37], we see that for all $x \in \Omega$, there is an $i \in \mathbb{N}$ with

$$\lim_{n \to \infty} \left| \frac{1}{\lambda(\Omega_i^n)} F(\Omega_i^n, \mathbb{R}^d \setminus \Omega_i^n) - (\operatorname{ndiv} F)(x) \right| = 0.$$

Hence, we conclude

$$\int_{\widehat{\Gamma}} \mathcal{N}u(y)\,\mathrm{d}y = \int_{\Omega} (\operatorname{ndiv} F)(x)\,\mathrm{d}x.$$

All in all, we have now shown the nonlocal counterpart of the Gauss Theorem. However, as in the local case, we extend it into the nonlocal counterpart of the integration by parts formula.

Theorem 2.1 (Nonlocal integration by parts formula)**.**
Let $\Omega \subset \mathbb{R}^d$ be an open and bounded subset and $\gamma \in \mathcal{K}$. For a given $k \in \mathcal{K}$ with

$$k > 0 \quad \text{a.e. on } \{(y,x) \in \mathbb{R}^d \times \Omega \colon |\gamma(x,y) - \gamma(y,x)| > 0\} \quad \text{and} \quad \left\| \int_{\mathbb{R}^d} k(y,\cdot)\,\mathrm{d}y \right\|_{\mathrm{L}^\infty(\Omega)} < \infty,$$

set

$$\begin{aligned}
\widehat{\gamma}(y,x) &:= \max \left\{ \gamma(y,x), \frac{(\gamma(x,y) - \gamma(y,x))^2}{k(y,x)} \chi_{\{k(y,x)>0\}} \right\} \quad \text{for} \quad y, x \in \mathbb{R}^d, \\
\mathcal{V}(\Omega; \widehat{\gamma}) &:= \left\{ v \colon \mathbb{R}^d \to \mathbb{R} \text{ measurable} : \|v\|_{\mathcal{V}(\Omega;\widehat{\gamma})} < \infty \right\}, \text{ where} \\
\|v\|^2_{\mathcal{V}(\Omega;\widehat{\gamma})} &:= \int_{\Omega} v^2(x) + \int_{\mathbb{R}^d} (v(x) - v(y))^2 \widehat{\gamma}(y,x)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

If $u \in \mathcal{V}(\Omega; \widehat{\gamma})$ satisfies

$$\int_{\Omega} \left(\int_{\mathbb{R}^d} |u(x)\gamma(x,y) - u(y)\gamma(y,x)|\,\mathrm{d}y \right)^2 \mathrm{d}x < \infty,$$

then for all $v \in \mathcal{V}(\Omega; \widehat{\gamma})$, we have

$$\begin{aligned}
&\int_{\Omega} \mathcal{L}u(x)\,v(x)\,\mathrm{d}x + \int_{\widehat{\Gamma}} \mathcal{N}u(y)\,v(y)\,\mathrm{d}y \\
=&\frac{1}{2} \int_{\Omega} \int_{\Omega} (u(x)\gamma(x,y) - u(y)\gamma(y,x))(v(x) - v(y))\,\mathrm{d}y\,\mathrm{d}x \qquad (2.1) \\
&+ \int_{\Omega} \int_{\widehat{\Gamma}} (u(x)\gamma(x,y) - u(y)\gamma(y,x))(v(x) - v(y))\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

In particular, we obtain

$$\int_\Omega -\mathcal{L}u(x)\,\mathrm{d}x = \int_{\widehat{\Gamma}} \mathcal{N}u(y)\,\mathrm{d}y \tag{2.2}$$

and for $\widehat{\mathfrak{B}}_\gamma \colon \mathcal{V}(\Omega;\widehat{\gamma}) \times \mathcal{V}(\Omega;\widehat{\gamma}) \to \mathbb{R}$ *defined by*

$$\begin{aligned}\widehat{\mathfrak{B}}_\gamma(u,v) :=& \frac{1}{2}\int_\Omega\int_\Omega (u(x)\gamma(x,y) - u(y)\gamma(y,x))(v(x)-v(y))\,\mathrm{d}y\,\mathrm{d}x \\ &+ \int_\Omega\int_{\widehat{\Gamma}} (u(x)\gamma(x,y) - u(y)\gamma(y,x))(v(x)-v(y))\,\mathrm{d}y\,\mathrm{d}x,\end{aligned}$$

there is a $C>0$ *with* $\widehat{\mathfrak{B}}_\gamma(u,v) \leqslant C\|u\|_{\mathcal{V}(\Omega;\widehat{\gamma})}\|v\|_{\mathcal{V}(\Omega;\widehat{\gamma})}$.

Proof. Our desired result is obtained via Fubini's Theorem. Let $u,v\in\mathcal{V}(\Omega;\widehat{\gamma})$ be given and assume

$$\int_\Omega \left(\int_{\mathbb{R}^d} |u(x)\gamma(x,y) - u(y)\gamma(y,x)|\,\mathrm{d}y\right)^2 \mathrm{d}x < \infty.$$

Note that by our assumptions, we have $\mathcal{L}u\in\mathrm{L}^2(\Omega)$ and by applying Hölder's inequality, we get for the first term in (2.1) that $\int_\Omega \mathcal{L}u(x)\,v(x)\,\mathrm{d}x \leqslant \|\mathcal{L}u\|_{\mathrm{L}^2(\Omega)}\|v\|_{\mathrm{L}^2(\Omega)} < \infty$. Fubini's Theorem and the triangle inequality yield

$$\begin{aligned}\int_{\widehat{\Gamma}} \mathcal{N}u(y)\,v(y)\,\mathrm{d}y &\leqslant \int_{\widehat{\Gamma}}\int_\Omega |(u(y)\gamma(y,x) - u(x)\gamma(x,y))v(y)|\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\widehat{\Gamma}}\int_\Omega |(u(y)\gamma(y,x) - u(x)\gamma(x,y))(v(x)-v(x)-v(y))|\,\mathrm{d}x\,\mathrm{d}y \\ &\leqslant \int_{\mathbb{R}^d}\int_\Omega |(u(y)\gamma(y,x) - u(x)\gamma(x,y))v(x)|\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^d}\int_\Omega |(u(y)\gamma(y,x) - u(x)\gamma(x,y))(v(x)-v(y))|\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Therefore, it remains to show that there is a $C>0$ with

$$|\widehat{\mathfrak{B}}_\gamma(u,v)| \leqslant \int_\Omega\int_{\mathbb{R}^d} |(u(x)\gamma(x,y) - u(y)\gamma(y,x))(v(x)-v(y))|\,\mathrm{d}y\,\mathrm{d}x \leqslant C\|u\|_{\mathcal{V}(\Omega;\widehat{\gamma})}\|v\|_{\mathcal{V}(\Omega;\widehat{\gamma})}.$$

By using the triangle inequality again, we obtain

$$\begin{aligned}&\int_\Omega\int_{\mathbb{R}^d} |(u(x)\gamma(x,y) - u(y)\gamma(y,x))(v(x)-v(y))|\,\mathrm{d}y\,\mathrm{d}x \\ =& \int_\Omega\int_{\mathbb{R}^d} |(u(x)(\gamma(x,y) - \gamma(y,x)) + (u(x)-u(y))\gamma(y,x))(v(x)-v(y))|\,\mathrm{d}y\,\mathrm{d}x \\ \leqslant& \int_\Omega\int_{\mathbb{R}^d} |u(x)(\gamma(x,y) - \gamma(y,x))(v(x)-v(y))|\,\mathrm{d}y\,\mathrm{d}x \\ &+ \int_\Omega\int_{\mathbb{R}^d} |(u(x)-u(y))\gamma(y,x)(v(x)-v(y))|\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

While for the second term Hölder's inequality yields

$$\int_\Omega\int_{\mathbb{R}^d} |(u(x)-u(y))\gamma(y,x)(v(x)-v(y))|\,\mathrm{d}y\,\mathrm{d}x \leqslant \|u\|_{\mathcal{V}(\Omega;\widehat{\gamma})}\|v\|_{\mathcal{V}(\Omega;\widehat{\gamma})} < \infty,$$

we estimate the first term by

$$
\begin{aligned}
&\int_\Omega \int_{\mathbb{R}^d} |u(x)(\gamma(x,y)-\gamma(y,x))(v(x)-v(y))| \,\mathrm{d}y\,\mathrm{d}x \\
=&\int_\Omega \int_{\mathbb{R}^d} |u(x)\frac{(\gamma(x,y)-\gamma(y,x))}{\sqrt{k(y,x)}}\sqrt{k(y,x)}(v(x)-v(y))| \,\mathrm{d}y\,\mathrm{d}x \\
\leqslant& \left\| \int_{\mathbb{R}^d} k(y,\cdot)\,\mathrm{d}y \right\|_{\mathrm{L}^\infty(\Omega)}^{\frac{1}{2}} \|u\|_{\mathcal{V}(\Omega;\widehat{\gamma})}\|v\|_{\mathcal{V}(\Omega;\widehat{\gamma})} \\
<&\infty.
\end{aligned}
$$

Therefore, all integrals are, in the Lebesgue-sense, well defined and finite. And the linearity of the integral yields

$$
\begin{aligned}
&\int_\Omega \int_{\mathbb{R}^d} (u(x)\gamma(x,y)-u(y)\gamma(y,x))v(x)\,\mathrm{d}y\,\mathrm{d}x \\
=&\int_\Omega \int_\Omega (u(x)\gamma(x,y)-u(y)\gamma(y,x))v(x)\,\mathrm{d}y\,\mathrm{d}x \\
&+\int_\Omega \int_{\mathbb{R}^d\setminus\Omega} (u(x)\gamma(x,y)-u(y)\gamma(y,x))v(x)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}
$$

Note that $\gamma(x,y)=\gamma(y,x)=0$ holds for a.e. $(x,y)\in\Omega\times(\mathbb{R}^d\setminus(\Omega\cup\widehat{\Gamma}))$ and thereby,

$$
\begin{aligned}
&\int_\Omega \int_{\mathbb{R}^d\setminus\Omega} (u(x)\gamma(x,y)-u(y)\gamma(y,x))v(x)\,\mathrm{d}y\,\mathrm{d}x \\
=&\int_\Omega \int_{\widehat{\Gamma}} (u(x)\gamma(x,y)-u(y)\gamma(y,x))v(x)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}
$$

By Fubini's Theorem, we conclude

$$
\begin{aligned}
&\int_\Omega \int_\Omega (u(x)\gamma(x,y)-u(y)\gamma(y,x))v(x)\,\mathrm{d}y\,\mathrm{d}x \\
=&\frac{1}{2}\int_\Omega \int_\Omega (u(x)\gamma(x,y)-u(y)\gamma(y,x))v(x)\,\mathrm{d}y\,\mathrm{d}x \\
&-\frac{1}{2}\int_\Omega \int_\Omega (u(y)\gamma(y,x)-u(x)\gamma(x,y))v(x)\,\mathrm{d}y\,\mathrm{d}x \\
=&\frac{1}{2}\int_\Omega \int_\Omega (u(x)\gamma(x,y)-u(y)\gamma(y,x))(v(x)-v(y))\,\mathrm{d}y\,\mathrm{d}x
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\Omega \int_{\mathbb{R}^d\setminus\Omega} (u(x)\gamma(x,y)-u(y)\gamma(y,x))v(x)\,\mathrm{d}y\,\mathrm{d}x \\
=&\int_\Omega \int_{\widehat{\Gamma}} (u(x)\gamma(x,y)-u(y)\gamma(y,x))(v(x)-v(y))\,\mathrm{d}y\,\mathrm{d}x \\
&-\int_{\widehat{\Gamma}} \int_\Omega (u(y)\gamma(y,x)-u(x)\gamma(x,y))\mathrm{d}x\, v(y)\,\mathrm{d}y.
\end{aligned}
$$

Due to $\chi_{\mathbb{R}^d}\in\mathcal{V}(\Omega;\widehat{\gamma})$, the special case (2.2) follows. □

Recall that a bounded domain $\Omega \subset \mathbb{R}^d$ is called Lipschitz if for each point $y \in \partial\Omega$ there is a $r > 0$ such that $\partial\Omega \cap \mathrm{B}_r(y)$ is the graph of a Lipschitz function. However, we want to remark that if Ω is not bounded, we require rather complicated conditions on Ω and $\partial\Omega$ (see [2, 4.9]). As a consequence of Brenner and Scott [31, (5.1.5) Proposition], we see that for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ the integration by parts formula on Sobolev spaces holds, i.e.,

$$\int_\Omega -\operatorname{div}(A\nabla u)(x)\, v(x)\, \mathrm{d}x + \int_{\partial\Omega} \nabla_\nu^A u(y)\, v(y)\, \mathrm{d}y = \int_\Omega \langle A(x)\nabla u(x), \nabla v(x)\rangle\, \mathrm{d}x$$

for all $u \in \mathrm{H}^2(\Omega)$ and $v \in \mathrm{H}^1(\Omega)$ with the local Neumann operator defined by

$$\nabla_\nu^A u(y) = \langle A(y)\nabla u(y), \nu(y)\rangle, \quad y \in \partial\Omega,$$

where $\langle\cdot\,,\cdot\rangle$ is the dot product, ν denotes the outward unit normal to $\partial\Omega$, and $A \in (\mathrm{C}^1(\overline{\Omega}))^{d\times d}$.

For a symmetric kernel $\gamma \in \mathcal{K}$ and a bounded open set $\Omega \subset \mathbb{R}^d$, Theorem 2.1 yields

$$\begin{aligned}
&\int_\Omega (\mathcal{L}u(x))\, v(x)\, \mathrm{d}x + \int_{\widehat{\Gamma}} (\mathcal{N}u(y))\, v(y)\, \mathrm{d}y \\
=&\frac{1}{2}\int_\Omega\int_\Omega (u(x)-u(y))(v(x)-v(y))\,\gamma(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&+\int_\Omega\int_{\widehat{\Gamma}} (u(x)-u(y))(v(x)-v(y)\,\gamma(x,y)\,\mathrm{d}y\,\mathrm{d}x
\end{aligned} \tag{2.3}$$

for all $u, v \in \mathcal{V}(\Omega;\gamma)$ with

$$\int_\Omega \left(\int_{\mathbb{R}^d} |u(x)) - u(y)|\gamma(y,x)\,\mathrm{d}y\right)^2 \mathrm{d}x < \infty.$$

And the nonlocal Neumann operator can be written as

$$\mathcal{N}u(y) = \langle \gamma(y,\cdot)\mathcal{G}u(y,\cdot), \mu(y,\cdot)\rangle_{\mathrm{L}^2(\mathbb{R}^d)} \qquad \text{for } y \in \widehat{\Gamma}$$

where $\mathcal{G}\colon \{v\colon \mathbb{R}^d \to \mathbb{R} \text{ measurable}\} \to \{v\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \text{ measurable}\}$ is defined by

$$\mathcal{G}u(y,x) = (u(y)-u(x))\frac{y-x}{\|y-x\|} \qquad \text{for } (y,x) \in \mathbb{R}^d \times \mathbb{R}^d.$$

and where we set $\mu\colon \mathbb{R}^d \times \mathbb{R}^d \setminus \Omega \to \mathbb{R}^d$, $\mu(y,x) = \frac{y-x}{\|y-x\|}\chi_\Omega(x)$. Furthermore, by (2.3) we have

$$\begin{aligned}
&\int_\Omega (\mathcal{L}u(x))\, v(x)\, \mathrm{d}x + \int_{\widehat{\Gamma}} (\mathcal{N}u(y))\, v(y)\, \mathrm{d}y \\
=&\int_\Omega \frac{1}{2}\langle \gamma(\cdot,x)\mathcal{G}u(\cdot,x), \mathcal{G}v(\cdot,x)\rangle_{\mathrm{L}^2(\Omega\times\Omega)}\,\mathrm{d}x \\
&+\int_\Omega \langle \gamma(\cdot,x)\mathcal{G}u(\cdot,x), \mathcal{G}v(\cdot,x)\rangle_{\mathrm{L}^2((\mathbb{R}^d\setminus\Omega)\times\Omega)}\,\mathrm{d}x
\end{aligned}$$

for all $u, v \in \mathcal{V}(\Omega;\gamma)$ with

$$\int_\Omega \left(\int_{\mathbb{R}^d} |u(x)) - u(y)|\gamma(y,x)\,\mathrm{d}y\right)^2 \mathrm{d}x < \infty.$$

Recall the Neumann Problem introduced and studied by Du et al. [28], i.e.,

$$\begin{cases} \mathcal{L}u(x) &= \displaystyle\int_{\mathbb{R}^d} u(x)\gamma(x,y) - u(y)\gamma(y,x)\,\mathrm{d}y &= f(x) \quad \text{for } x \in \Omega, \\ \mathcal{N}^G u(y) &= \displaystyle\int_{\Omega\cup\Omega_I} u(y)\gamma(y,x) - u(x)\gamma(x,y)\,\mathrm{d}x &= g(y) \quad \text{for } y \in \Omega_I, \end{cases} \tag{NP2}$$

where $\gamma \in \mathcal{K}$ is symmetric and the interaction set is defined by

$$\Omega_I := \left\{ y \in \mathbb{R}^d \setminus \Omega \text{ such that } \gamma(x,y) \neq 0 \text{ for some } x \in \Omega \right\}.$$

Assume $\gamma \in \mathcal{K}$ is symmetric, then we have $\widehat{\Gamma} \subset \Omega_I$ and

$$\mathcal{N}^G u(y) - \mathcal{N}u(y) = \int_{\Omega_I} u(y)\gamma(y,x) - u(x)\gamma(x,y)\,\mathrm{d}x \quad \text{for } y \in \widehat{\Gamma}.$$

Furthermore, by Du et al. [29, Corollary 4.2] and Fubini's Theorem we have

$$\begin{aligned} &\int_\Omega (\mathcal{L}u(x))\, v(x)\,\mathrm{d}x + \int_{\Omega_I} (\mathcal{N}^G u(y))\, v(y)\,\mathrm{d}y \\ =&\frac{1}{2}\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} (u(x)-u(y))(v(x)-v(y))\,\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\ =&\frac{1}{2}\int_\Omega\int_\Omega (u(x)-u(y))(v(x)-v(y))\,\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\ &+\int_\Omega\int_{\Omega_I} (u(x)-u(y))(v(x)-v(y))\,\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\ &+\frac{1}{2}\int_{\Omega_I}\int_{\Omega_I} (u(x)-u(y))(v(x)-v(y))\,\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x, \end{aligned}$$

for all $u, v \in \mathcal{V}(\Omega\cup\Omega_I; \gamma\chi_{(\Omega\cup\Omega_I)\times(\Omega\cup\Omega_I)})$ with

$$\int_{\Omega\cup\Omega_I} \left(\int_{\Omega\cup\Omega_I} |u(x)) - u(y)|\gamma(y,x)\,\mathrm{d}y \right)^2 \mathrm{d}x < \infty.$$

Then, $\mathcal{V}(\Omega\cup\Omega_I; \gamma\chi_{(\Omega\cup\Omega_I)\times(\Omega\cup\Omega_I)}) \subset \mathcal{V}(\Omega; \gamma)$. And by definition of $\widehat{\Gamma}$ we have

$$\int_{\widehat{\Gamma}}\int_\Omega (u(x)-u(y))(v(x)-v(y))\,\gamma(y,x)\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^d\setminus\Omega}\int_\Omega (u(x)-u(y))(v(x)-v(y))\,\gamma(y,x)\,\mathrm{d}x\,\mathrm{d}y.$$

Hence, we obtain

$$\int_{\Omega_I} (\mathcal{N}^G u(y))v(y)\,\mathrm{d}y - \int_\Gamma (\mathcal{N}u(y))v(y)\,\mathrm{d}y = \frac{1}{2}\int_{\Omega_I}\int_{\Omega_I} (u(x)-u(y))(v(x)-v(y))\,\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x$$

for all $u, v \in \mathcal{V}(\Omega\cup\Omega_I; \gamma\chi_{(\Omega\cup\Omega_I)\times(\Omega\cup\Omega_I)})$ with

$$\int_\Omega \left(\int_{\mathbb{R}^d} |u(x)) - u(y)|\gamma(y,x)\,\mathrm{d}y \right)^2 \mathrm{d}x < \infty.$$

Therefore, a function $u \in \mathcal{V}(\Omega \cup \Omega_I; \gamma\chi_{(\Omega\cup\Omega_I)\times(\Omega\cup\Omega_I)})$ solves

$$\begin{aligned}
&\frac{1}{2}\int_\Omega\int_\Omega (u(x)-u(y))(v(x)-v(y))\,\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
&+\int_\Omega\int_{\widehat{\Gamma}} (u(x)-u(y))(v(x)-v(y))\,\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
=&\frac{1}{2}\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} (u(x)-u(y))(v(x)-v(y))\,\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x
\end{aligned}$$

for all $v \in \mathcal{V}(\Omega \cup \Omega_I; \gamma\chi_{(\Omega\cup\Omega_I)\times(\Omega\cup\Omega_I)})$ if and only if we have

$$\int_{\Omega_I}\int_{\Omega_I} (u(x)-u(y))^2\,\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x = 0.$$

As mentioned before, another method for modeling a nonlocal Neumann problem is to decompose $\mathcal{L}$. To be more precise, because for $x \in \Omega$ we have

$$\begin{aligned}
\mathcal{L}u(x) &= \int_{\mathbb{R}^d} u(x)\gamma(x,y) - u(y)\gamma(y,x)\,\mathrm{d}y \\
&= \int_\Omega u(x)\gamma(x,y) - u(y)\gamma(y,x)\,\mathrm{d}y + \int_{\mathbb{R}^d\setminus\Omega} u(x)\gamma(x,y) - u(y)\gamma(y,x)\,\mathrm{d}y,
\end{aligned}$$

the aim is now to control

$$\int_{\mathbb{R}^d\setminus\Omega} u(x)\gamma(x,y) - u(y)\gamma(y,x)\,\mathrm{d}y$$

with a nonlocal Neumann condition. Both Cortazar et al. [4] and Glusa et al. [14] followed this approach with different nonlocal Neumann conditions. In Chapter 7, we see that such an approach is also applicable for our Neumann problem (NP).

Chapter 3

A Test Function Space

In accordance with the nonlocal integration by parts formula (see Theorem 2.1), we define the function space which was first introduced in [21]

$$\mathcal{V}(\Omega;\gamma) := \Big\{ v\colon \mathbb{R}^d \to \mathbb{R} \text{ measurable} : \|v\|_{\mathcal{V}(\Omega;\gamma)} < \infty \Big\}$$
$$\text{where} \quad \|v\|^2_{\mathcal{V}(\Omega;\gamma)} := \int_\Omega v^2(x) + \int_{\mathbb{R}^d} (v(x)-v(y))^2 \gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x$$

for a nonempty, open set $\Omega \subset \mathbb{R}^d$ and a kernel $\gamma \in \mathcal{K}$. We further set

$$\Gamma := \Gamma(\Omega,\gamma) := \{y \in \mathbb{R}^d \setminus \Omega\colon \int_\Omega \gamma(y,x)\mathrm{d}x > 0\}. \tag{3.1}$$

Then, we have $\widehat{\Gamma}(\Omega,\gamma) = \Gamma(\Omega,\gamma) \cup \Gamma(\Omega,\gamma^\top)$ and especially for a symmetric $\gamma \in \mathcal{K}$, we obtain

$$\widehat{\Gamma}(\Omega,\gamma) = \Gamma(\Omega,\gamma) = \Gamma(\Omega,\gamma^\top).$$

For $\gamma \in \mathcal{K}$, we define the symmetric bilinear form $\mathfrak{B}\colon \mathcal{V}(\Omega;\gamma) \times \mathcal{V}(\Omega;\gamma) \to \mathbb{R}$ by

$$\begin{aligned}\mathfrak{B}(u,v) := \mathfrak{B}_\gamma(u,v) :=& \frac{1}{2}\int_\Omega\int_\Omega (u(x)-u(y))(v(x)-v(y))\,\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\ &+ \int_\Omega\int_\Gamma (u(x)-u(y))(v(x)-v(y))\,\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x,\end{aligned}$$

and the mapping $\langle\cdot,\cdot\rangle_{\mathcal{V}(\Omega;\gamma)}\colon \mathcal{V}(\Omega;\gamma)\times\mathcal{V}(\Omega;\gamma) \to \mathbb{R}$ by

$$\langle u,v\rangle_{\mathcal{V}(\Omega;\gamma)} := \int_\Omega u(x)v(x) + \int_{\Omega\cup\Gamma} (u(x)-u(y))(v(x)-v(y))\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x.$$

We observe that $\langle\cdot,\cdot\rangle_{\mathcal{V}(\Omega;\gamma)}$ is a semi-inner-product and, therefore, $\langle\cdot,\cdot\rangle_{\mathcal{V}(\Omega;\gamma)}$ induces a seminorm $\|\cdot\|_{\mathcal{V}(\Omega;\gamma)}$ on $\mathcal{V}(\Omega;\gamma)$. Furthermore, $\mathfrak{B}$ is bounded by definition and for all functions $u,v \in \mathcal{V}(\Omega;\gamma)$, we have

$$\frac{1}{2}\langle u,u\rangle_{\mathcal{V}(\Omega;\gamma)} \leqslant \int_\Omega u^2(x)\,\mathrm{d}x + \mathfrak{B}(u,u) \leqslant \langle u,u\rangle_{\mathcal{V}(\Omega;\gamma)} \tag{3.2}$$

and

$$\int_\Omega \int_\Gamma (u(x) - u(y))(v(x) - v(y))\,\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_\Omega \int_{\mathbb{R}^d \setminus \Omega} (u(x) - u(y))(v(x) - v(y))\,\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x.$$

By following the arguments used in [32, Proposition 3.1], [21, Lemma 2.3], [39, Therorem 3.1] and [11, Theorem 3.46], we obtain:

Theorem 3.1.
Let $\Omega \subset \mathbb{R}^d$ be a nonempty and open set and $\gamma \in \mathcal{K}$. Then $\mathcal{V}(\Omega;\gamma)$ is a complete seminormed vector space with respect to $\|\cdot\|_{\mathcal{V}(\Omega;\gamma)}$ and $\langle\cdot,\cdot\rangle_{\mathcal{V}(\Omega;\gamma)}$ is a semi-inner-product on $\mathcal{V}(\Omega;\gamma)$.

Proof.

As mentioned before, $(\mathcal{V}(\Omega;\gamma), \|\cdot\|_{\mathcal{V}(\Omega;\gamma)})$ is a seminormed vector space because $\langle\cdot,\cdot\rangle_{\mathcal{V}(\Omega;\gamma)}$ is bilinear and positive semi-definite, i.e., $\langle\cdot,\cdot\rangle_{\mathcal{V}(\Omega;\gamma)}$ is a semi-inner-product on $\mathcal{V}(\Omega;\gamma)$.

In order to show that $(\mathcal{V}(\Omega;\gamma), \|\cdot\|_{\mathcal{V}(\Omega;\gamma)})$ is complete, let $(v_n)_{n\in\mathbb{N}}$ be a Cauchy sequence with respect to the seminorm $\|\cdot\|_{\mathcal{V}(\Omega;\gamma)}$. Then it remains to show that $(v_n)_{n\in\mathbb{N}}$ converges to a function $v \in \mathcal{V}(\Omega;\gamma)$ with respect to $\|\cdot\|_{\mathcal{V}(\Omega;\gamma)}$. Since $(v_n)_{n\in\mathbb{N}}$ is Cauchy, we have

$$\begin{aligned} &\lim_{k,\ell\to\infty} \int_\Omega (v_k(x) - v_\ell(x))^2\,\mathrm{d}x = 0,\\ \text{and}\quad &\lim_{k,\ell\to\infty} \int_\Omega \int_{\mathbb{R}^d} (v_k(x) - v_k(y) - (v_\ell(x) - v_\ell(y)))^2 \gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x = 0. \end{aligned}$$

Hence, due to the completeness of $\mathrm{L}^2(\Omega)$ and $\mathrm{L}^2(\Omega \times \mathbb{R}^d)$, there is a $v \in \mathrm{L}^2(\Omega)$ and $w \in \mathrm{L}^2(\mathbb{R}^d \times \Omega)$ with

$$\begin{aligned} &\lim_{k\to\infty} \int_\Omega (v_k(x) - v(x))^2\,\mathrm{d}x = 0,\\ \text{and}\quad &\lim_{k\to\infty} \int_\Omega \int_{\mathbb{R}^d} ((v_k(x) - v_k(y))\sqrt{\gamma(y,x)} - w(y,x))^2\,\mathrm{d}y\,\mathrm{d}x = 0. \end{aligned}$$

We now choose a subsequence $(v_{n_\ell})_{\ell\in\mathbb{N}}$, such that we have

$$\begin{aligned} &\lim_{\ell\to\infty} v_{n_\ell}(x) = v(x) && \text{for a.e. } x \in \Omega,\\ \text{and}\quad &\lim_{\ell\to\infty} (v_{n_\ell}(x) - v_{n_\ell}(y))\sqrt{\gamma(y,x)} = w(y,x) && \text{for a.e. } (y,x) \in \mathbb{R}^d \times \Omega. \end{aligned}$$

Without loss of generality, we now assume that $\Omega \neq \mathbb{R}^d$. In view of (3.1), we set $v(y) = 0$ for $y \in \mathbb{R}^d \setminus (\Omega \cup \Gamma)$. For a.e. $y \in \Gamma$ we choose a measurable subset

$$\Omega_y \subset \{x \in \Omega \text{ such that } \gamma(y,x) > 0\} \subset \mathbb{R}^d \quad \text{with} \quad 0 < \lambda(\Omega_y) < \infty.$$

The existence of such subsets is given by the definition of Γ and the σ-finiteness of the Lebesgue measure on $\mathbb{R}^d$. We set

$$v(y) = \frac{1}{\lambda(\Omega_y)} \int_{\Omega_y} v(x) - \frac{w(y,x)}{\sqrt{\gamma(y,x)}}\,\mathrm{d}x \text{ for a.e. } y \in \Gamma.$$

Because for any $n \in \mathbb{N}$, a.e. $y \in \Gamma$ and a.e. $x \in \Omega_y$, we have

$$|v_n(x) - \frac{(v_n(x) - v_n(y))\sqrt{\gamma(y,x)}}{\sqrt{\gamma(y,x)}}| = |v_n(y)| \leqslant \sup_{n \in \mathbb{N}} |v_n(y)| < \infty.$$

Thus, by Lebesgue's Dominated Convergence Theorem, we see that, for a.e. $y \in \Gamma$,

$$v(y) = \frac{1}{\lambda(\Omega_y)} \int_{\Omega_y} v(x) - \frac{w(y,x)}{\sqrt{\gamma(y,x)}} \, \mathrm{d}x = \lim_{\ell \to \infty} \frac{1}{\lambda(\Omega_y)} \int_{\Omega_y} v_{n_\ell}(y) \, \mathrm{d}x = \lim_{\ell \to \infty} v_{n_\ell}(y).$$

Recalling that $y \in \mathbb{R}^d \setminus (\Omega \cup \Gamma)$ implies $\int_\Omega \gamma(y,x) \, \mathrm{d}x = 0$ and, therefore, $\gamma(y,x) = 0$ holds for a.e. $x \in \Omega$. We conclude that for a.e. $(y,x) \in \mathbb{R}^d \times \Omega$, we have

$$w(y,x) = \lim_{\ell \to \infty} (v_{n_\ell}(x) - v_{n_\ell}(y))\sqrt{\gamma(y,x)} = (v(x) - v(y))\sqrt{\gamma(y,x)}.$$

Now we have found a function $v \in \mathcal{V}(\Omega;\gamma)$ for which $\lim_{\ell \to \infty} \|v_{n_\ell} - v\|_{\mathcal{V}(\Omega;\gamma)} = 0$ holds and because $(v_n)_{n \in \mathbb{N}}$ is Cauchy with respect to $\|\cdot\|_{\mathrm{V}(\Omega;\gamma)}$, we obtain $\lim_{n \to \infty} \|v_n - v\|_{\mathcal{V}(\Omega;\gamma)} = 0$ as well. Therefore, $(\mathcal{V}(\Omega;\gamma), \|\cdot\|_{\mathcal{V}(\Omega;\gamma)})$ is complete. $\square$

In the same manner as Lebesgue spaces are treated, we now identify the elements of $\mathcal{V}(\Omega;\gamma)$ with their respective equivalent class. For this reason, we define

$$\mathrm{N} := \{v \in \mathcal{V}(\Omega;\gamma) \colon v = 0 \text{ a.e. on } \Omega \cup \Gamma\}.$$

By definition of $\|\cdot\|_{\mathcal{V}(\Omega;\gamma)}$, we have $u \in \ker(\|\cdot\|_{\mathcal{V}(\Omega;\gamma)})$ if and only if $u = 0$ a.e. in Ω and

$$\int_\Omega \int_\Gamma u^2(y)\gamma(y,x) \, \mathrm{d}y \, \mathrm{d}x = 0,$$

so that $\mathrm{N} = \ker(\|\cdot\|_{\mathcal{V}(\Omega;\gamma)})$. Now, by defining

$$[u] := u + \mathrm{N} := \{v \in \mathcal{V}(\Omega;\gamma) \colon v = u \text{ a.e. on } \Omega \cup \Gamma\} \quad \text{for } u \in \mathcal{V}(\Omega;\gamma)$$

the corresponding quotient space is given by

$$\mathrm{V}(\Omega;\gamma) := \mathcal{V}(\Omega;\gamma)/\mathrm{N} = \{[u] \colon u \in \mathcal{V}(\Omega;\gamma)\}.$$

For any $u, v \in \mathcal{V}(\Omega;\gamma)$, we then have $\langle u_1, v_1 \rangle_{\mathcal{V}(\Omega;\gamma)} = \langle u_2, v_2 \rangle_{\mathcal{V}(\Omega;\gamma)}$ and $\mathfrak{B}(u_1, v_1) = \mathfrak{B}(u_2, v_2)$ for all $u_1, u_2 \in [u]$ and $v_1, v_2 \in [v]$. This implies that both mappings $\langle \cdot, \cdot \rangle_{\mathrm{V}(\Omega;\gamma)} \colon \mathrm{V}(\Omega;\gamma) \times \mathrm{V}(\Omega;\gamma) \to \mathbb{R}$ and $\mathfrak{B} \colon \mathrm{V}(\Omega;\gamma) \times \mathrm{V}(\Omega;\gamma) \to \mathbb{R}$ defined by

$$\langle [u], [v] \rangle_{\mathrm{V}(\Omega;\gamma)} := \langle u, v \rangle_{\mathcal{V}(\Omega;\gamma)} \quad \text{and} \quad \mathfrak{B}([u], [v]) := \mathfrak{B}(u, v) \text{ for } u \in [u], v \in [v]$$

are well-defined. And for $v \in \mathcal{V}(\Omega;\gamma)$, we have $\langle [v], [v] \rangle_{\mathrm{V}(\Omega;\gamma)} = 0$ if and only if $[v] = \mathrm{N}$.

Although the elements of $\mathrm{V}(\Omega;\gamma)$ are in fact equivalent classes and not functions, we consider the elements of $\mathrm{V}(\Omega;\gamma)$ to be functions which are considered identical if they are representatives of the same equivalent class. To be more precise, in our view, the elements of $\mathrm{V}(\Omega;\gamma)$ are functions which are defined a.e. on $\Omega \cup \Gamma$.

Thanks to Theorem 3.1, we therefore conclude:

Corollary 3.2.
Let $\Omega \subset \mathbb{R}^d$ be nonempty and open and $\gamma \in \mathcal{K}$. Then, $\mathrm{V}(\Omega;\gamma)$ is a separable Hilbert space with the inner product $\langle \cdot,\cdot \rangle_{\mathrm{V}(\Omega;\gamma)}$. In particular, for every $D \in (\mathrm{V}(\Omega;\gamma))^*$, there exists a unique $u \in \mathrm{V}(\Omega;\gamma)$ such that for all $v \in \mathrm{V}(\Omega;\gamma)$, we have

$$D(v) = \int_\Omega u(x)v(x)\,\mathrm{d}x + \mathfrak{B}(u,v).$$

Proof.

Because of Theorem 3.1, the definition of $\mathrm{V}(\Omega;\gamma)$, eq. (3.2), and the Riesz representation theorem, it solely remains to show that $\mathrm{V}(\Omega;\gamma)$ is separable. The mapping

$$\mathcal{I}\colon \mathrm{V}(\Omega;\gamma) \to \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Omega \times \mathbb{R}^d),\, u \mapsto \Big(u, (u(x)-u(y))\sqrt{\gamma(y,x)}\Big)$$

is isometric due the definition of the norm in $\mathrm{V}(\Omega;\gamma)$. Due to the completeness of $\mathrm{V}(\Omega;\gamma)$, we obtain that $\mathcal{I}(\mathrm{V}(\Omega;\gamma))$ is a closed subspace of $\mathrm{L}^2(\Omega)\times\mathrm{L}^2(\Omega\times\mathbb{R}^d)$. This product is separable which implies the separability of $\mathrm{V}(\Omega;\gamma)$. □

Remark 3.3.
We note that $\gamma \in \mathcal{K}$ is not required to be symmetric in this subsection. In fact, also for a nonsymmetric kernel $\gamma \in \mathcal{K}$, we have that $\mathcal{V}(\Omega;\gamma)$, $\langle \cdot,\cdot \rangle_{\mathcal{V}(\Omega;\gamma)}$ and $\mathfrak{B}$ are all well-defined and $\mathcal{V}(\Omega;\gamma)$ is a complete seminormed vector space with respect to $\|\cdot\|_{\mathcal{V}(\Omega;\gamma)}$. And therefore, $\mathrm{V}(\Omega;\gamma)$ is a separable Hilbert space also in this case.

Chapter 4

A Nonlocal Neumann Problem

Let $u \in \mathrm{V}(\Omega;\gamma)$ be a solution of problem (NP) which satisfies

$$\int_\Omega \left(\int_{\mathbb{R}^d} |u(x)-u(y)|\gamma(y,x)\,\mathrm{d}y \right)^2 \mathrm{d}x < \infty.$$

Then, by the nonlocal integration by parts formula (see Theorem 2.1), we get, for all $v \in \mathrm{V}(\Omega;\gamma)$,

$$\begin{aligned}\int_\Omega f(x)v(x)\,\mathrm{d}x &= \int_\Omega \mathcal{L}u(x)v(x)\,\mathrm{d}x \\ &= \widehat{\mathfrak{B}}(u,v) - \int_\Gamma \mathcal{N}u(y)\,v(y)\,\mathrm{d}y = \widehat{\mathfrak{B}}(u,v) - \int_\Gamma g(y)\,v(y)\,\mathrm{d}y.\end{aligned}$$

However, for a symetric kernel $\gamma \in \mathcal{K}$, we obtain $\widehat{\mathfrak{B}} = \mathfrak{B}$. Hence, we define our weak solution accordingly.

Definition 4.1.
Let $\Omega \subset \mathbb{R}^d$ be nonempty and open and $\gamma \in \mathcal{K}$ be symmetric. For given measurable functions $f\colon \Omega \to \mathbb{R}$ and $g\colon \Gamma \to \mathbb{R}$, we say that the function $u \in \mathrm{V}(\Omega;\gamma)$ is a weak solution *of the Neumann problem* (NP) *if*

$$\int_\Omega f(x)v(x)\,\mathrm{d}x + \int_\Gamma g(y)\,v(y)\,\mathrm{d}y = \mathfrak{B}(u,v) \quad \textit{holds for all } v \in \mathrm{V}(\Omega;\gamma).$$

We call the Neumann problem homogeneous *if $g = 0$.*

First, we highlight that the symmetry assumption on $\gamma \in \mathcal{K}$ can be relaxed.

Remark 4.2.
Consider the nonlocal problem

$$\begin{cases} \mathcal{L}_\eta u(x) = \displaystyle\int_{\mathbb{R}^d} (u(x)-u(y))\eta(y,x)\,\mathrm{d}y &= f(x) \quad \textit{for } x \in \Omega, \\ \mathcal{N}_\eta u(y) = \displaystyle\int_\Omega (u(y)-u(x))\eta(y,x)\,\mathrm{d}x &= g(y) \quad \textit{for } y \in \Gamma(\Omega,\eta), \end{cases} \tag{*}$$

where the measurable function $\eta\colon \mathbb{R}^d \times \Omega \to [0,\infty)$ *is only assumed to be symmetric a.e. on* $\Omega \times \Omega$. *By setting*

$$\gamma\colon \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty),\ \gamma(y,x) = \begin{cases} \eta(y,x) & \textit{for } (y,x) \in \Omega \times \Omega, \\ \eta(y,x) & \textit{for } (y,x) \in \mathbb{R}^d \setminus \Omega \times \Omega, \\ \eta(x,y) & \textit{for } (y,x) \in \Omega \times \mathbb{R}^d \setminus \Omega, \end{cases}$$

we obtain a well-defined kernel $\gamma \in \mathcal{K}$ *which is symmetric a.e. on* $\mathbb{R}^d \times \mathbb{R}^d$ *with* $\Gamma(\Omega,\gamma) = \Gamma(\Omega,\eta)$ *and*

$$\mathfrak{B}_\eta(u,v) = \mathfrak{B}_\gamma(u,v) \ \ \textit{for all } u,v \in \mathrm{V}(\Omega;\gamma) = \mathrm{V}(\Omega;\eta).$$

Moreover, $u\colon \mathbb{R}^d \to \mathbb{R}$ *is a strong solution of problem* (*) *if and only if it is a strong solution of problem* (NP). *Therefore, we define the weak solution of problem* (*) *accordingly: A function* $u \in \mathrm{V}(\Omega;\eta)$ *is a weak solution of the problem* (*) *if*

$$\int_\Omega f(x)v(x)\,\mathrm{d}x - \int_{\Gamma(\Omega,\eta)} g(y)\,v(y)\,\mathrm{d}y = \mathfrak{B}_\eta(u,v) \quad \textit{holds for all } v \in \mathcal{V}(\Omega;\eta).$$

In the next remark, we study under which assumptions the weak and strong solutions of the homogeneous Neumann problem are equivalent.

Remark 4.3.
Let $f \in \mathrm{L}^2(\Omega)$ *and let* $u \in \mathrm{V}(\Omega;\gamma)$ *be a weak solution of the homogeneous Neumann problem* (NP) *satisfying*

$$\int_\Omega \left(\int_{\mathbb{R}^d} |u(x) - u(y)| \gamma(y,x)\,\mathrm{d}y \right)^2 \mathrm{d}x < \infty.$$

By the definition of the weak solution (Definition 4.1) and the nonlocal integration by parts formula (Theorem 2.1), we find

$$\int_\Omega \mathcal{L}u(x)v(x)\,\mathrm{d}x + \int_\Gamma \mathcal{N}u(y)v(y)\,\mathrm{d}y = \int_\Omega f(x)v(x)\,\mathrm{d}x \textit{ for all } v \in \mathrm{V}(\Omega;\gamma).$$

If every infinitely differentiable function with compact support is an element of $\mathrm{V}(\Omega;\gamma)$ *(i.e.,* $\mathrm{C}_0^\infty(\mathbb{R}^d) \subset \mathrm{V}(\Omega;\gamma)$*), then also*

$$\int_\Omega \mathcal{L}u(x)v(x)\,\mathrm{d}x + \int_\Gamma \mathcal{N}u(y)v(y)\,\mathrm{d}y = \int_\Omega f(x)v(x)\,\mathrm{d}x \textit{ for all } v \in \mathrm{C}_0^\infty(\mathbb{R}^d),$$

so that by [16, Corollary 4.24.], we have $\mathcal{L}u = f$ *a.e. on* Ω *and* $\mathcal{N}u = 0$ *a.e. on* Γ.

If for a.e. $x \in \Omega \cup \Gamma$ *there is an* $R > 0$ *such that for all* $0 < r < R$ *we have* $\chi_{\mathrm{B}_r(x)} \in \mathrm{V}(\Omega;\gamma)$*, then by [37, Theorem 7.7], we get*

$$\mathcal{L}u(x_0) = \lim_{r\to 0} \int_\Omega \mathcal{L}u(x) \frac{\chi_{\mathrm{B}_r(x_0)}(x)}{\lambda(\mathrm{B}_r(0))}\,\mathrm{d}x = \lim_{r\to 0} \int_\Omega f(x) \frac{\chi_{\mathrm{B}_r(x_0)}(x)}{\lambda(\mathrm{B}_r(0))}\,\mathrm{d}x = f(x_0)$$

for a.e. $x_0 \in \Omega$ *and*

$$\mathcal{N}u(y_0) = \lim_{r\to 0} \int_\Gamma \mathcal{N}u(y)\chi_{\mathrm{B}_r(y_0)}(y)\,\mathrm{d}y = 0$$

for a.e. $y_0 \in \Gamma$.

Before we present an existence result for the weak solution of problem (NP), we show that Corollary 3.2 ensures the existence of a weak solution for the following regularized variant of (NP) with homogeneous Neumann constraints.

Remark 4.4.
Let us consider the problem

$$\begin{cases} \mathcal{L}u(x) + \kappa(x)u(x) & = f(x) \quad \textit{for } x \in \Omega, \\ \mathcal{N}u(y) & = 0 \qquad \textit{for } y \in \Gamma, \end{cases}$$

for $\Omega \subset \mathbb{R}^d$ *bounded and open,* $f \in \mathrm{L}^2(\Omega)$*,* $\kappa \colon \Omega \to [\alpha, \beta]$ *measurable with* $0 < \alpha < \beta < \infty$*, and a symmetric function* $\gamma \in \mathcal{K}$*. Because*

$$\min\left\{\frac{1}{2}, \alpha\right\} \|u\|^2_{\mathrm{V}(\Omega;\gamma)} \leqslant \mathfrak{B}(u,u) + \int_\Omega \kappa(x)u^2(x)\,\mathrm{d}x \leqslant \max\{1,\beta\}\|u\|^2_{\mathrm{V}(\Omega;\gamma)}$$

holds for all $u \in \mathrm{V}(\Omega;\gamma)$*, the existence of a unique weak solution of this problem, i.e., the function* $u \in \mathrm{V}(\Omega;\gamma)$ *solving*

$$\int_\Omega f(x)v(x)\,\mathrm{d}x = \mathfrak{B}(u,v) + \int_\Omega \kappa(x)u(x)v(x)\,\mathrm{d}x \quad \textit{for all } v \in \mathrm{V}(\Omega;\gamma),$$

is given by Corollary 3.2. The case $\kappa = 0$*, however, requires special treatment since neither the existence nor the uniqueness is guaranteed. Note that all constant functions belong to* $\mathrm{V}(\Omega;\gamma)$ *and that we have* $\mathfrak{B}(u,v) = \mathfrak{B}(u+c,v)$ *for all* $u,v \in \mathrm{V}(\Omega;\gamma)$ *and constants* $c \in \mathbb{R}$*. Therefore, the weak solution cannot be unique in* $\mathrm{V}(\Omega;\gamma)$*. And the existence of a weak solution to our Neumann problem already implies*

$$\int_\Omega f(x)\,\mathrm{d}x + \int_\Gamma g(y)\,\mathrm{d}y = 0.$$

In other words, a compatibility condition is necessary for the existence.

Using a similar approach as is used for the local Neumann problem in Brenner and Scott [31, Section 2.5.]), we now present an existence and uniqueness result for the weak solution of the homogeneous Neumann problem (NP) (i.e., $\kappa = 0$ in view of Remark 4.4).

Theorem 4.5.
Let $\Omega \subset \mathbb{R}^d$ *be bounded, nonempty and open,* $f \in \mathrm{L}^2(\Omega)$*, and* $\gamma \in \mathcal{K}$ *be symmetric. Then, the homogeneous Neumann problem* (NP)*, i.e.,*

$$\begin{cases} \mathcal{L}u & = f \quad \textit{on } \Omega, \\ \mathcal{N}u & = 0 \quad \textit{on } \Gamma, \end{cases}$$

has a weak solution if

$$\int_\Omega f(x)\,\mathrm{d}x = 0 \qquad \text{(Compatibility condition)}$$

and if there is a constant $C > 0$ *such that for all* $v \in \mathrm{V}(\Omega;\gamma)$*, we have*

$$\int_\Omega \int_\Omega (v(x)-v(y))^2\,\mathrm{d}y\,\mathrm{d}x \leqslant C \int_\Omega \int_{\mathbb{R}^d} (v(x)-v(y))^2 \gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x. \qquad \text{(Poincaré inequality)}$$

Furthermore, the weak solution is unique up to an additive constant.

Proof.

First, we remark that for all $u \in \mathrm{V}(\Omega;\gamma)$, we have

$$2\left(\lambda(\Omega)\int_\Omega u^2(x)\,\mathrm{d}x - \left(\int_\Omega u(x)\,\mathrm{d}x\right)^2\right) = \int_\Omega\int_\Omega (u(x)-u(y))^2\,\mathrm{d}y\,\mathrm{d}x$$
$$\leqslant C\int_\Omega\int_{\mathbb{R}^d}(u(x)-u(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x$$

and we introduce the space

$$\widehat{V}(\Omega;\gamma) := \left\{v \in \mathrm{V}(\Omega;\gamma) \colon \int_\Omega v(x)\,\mathrm{d}x = 0\right\}.$$

Let us define the average

$$u_\Omega := \frac{1}{\lambda(\Omega)}\int_\Omega u(x)\,\mathrm{d}x$$

for $u \in \mathrm{V}(\Omega;\gamma)$. Since $u - u_\Omega$ is for all $u \in \mathrm{V}(\Omega;\gamma)$ an element of $\widehat{V}(\Omega;\gamma)$, we have

$$\int_\Omega (u(x)-u_\Omega)^2\,\mathrm{d}x \leqslant \frac{C}{2\lambda(\Omega)}\int_\Omega\int_{\mathbb{R}^d}(u(x)-u(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \leqslant \frac{C}{\lambda(\Omega)}\,\mathfrak{B}(u,u).$$

For all $u \in \widehat{V}(\Omega;\gamma)$ the Poincaré inequality yields

$$\min\left\{\frac{1}{4},\frac{\lambda(\Omega)}{2\,C}\right\}\|u\|^2_{\mathrm{V}(\Omega;\gamma)} \leqslant \frac{\lambda(\Omega)}{2\,C}\int_\Omega u^2(x)\,\mathrm{d}x + \frac{1}{4}\int_\Omega\int_{\mathbb{R}^d}(u(x)-u(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \tag{4.1}$$
$$\leqslant \mathfrak{B}(u,u) \tag{4.2}$$
$$\leqslant \left(\int_\Omega u^2(x)\,\mathrm{d}x + \int_\Omega\int_{\mathbb{R}^d}(u(x)-u(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x\right) \tag{4.3}$$
$$= \|u\|^2_{\mathrm{V}(\Omega;\gamma)}. \tag{4.4}$$

We obtain that $\mathfrak{B}(w,w) = 0$ for $w \in \widehat{V}(\Omega;\gamma)$ implies $\|w\|_{\mathrm{V}(\Omega;\gamma)} = 0$, i.e., $w = 0$ a.e. on $\Omega \cup \Gamma$. Thereby, $\mathfrak{B}$ is an inner product on the space $\widehat{V}(\Omega;\gamma)$. Because Ω is bounded and because of (4.1), we see that $\widehat{V}(\Omega;\gamma)$ is a Hilbert space with respect to the inner product $\mathfrak{B}$. Let us consider the linear functional

$$\Lambda(v) = \int_\Omega f(x)v(x)\,\mathrm{d}x \quad \text{on } \widehat{V}(\Omega;\gamma),$$

which is bounded by the Cauchy–Schwarz inequality Λ. Therefore, by the Riesz representation Theorem, there is a unique $u \in \widehat{V}(\Omega;\gamma)$ such that

$$\Lambda(v) = \mathfrak{B}(u,v) \quad \text{for all } v \in \widehat{V}(\Omega;\gamma).$$

Because $\int_\Omega f(x)\,\mathrm{d}x = 0$ holds, we get

$$\int_\Omega f(x)v(x)\,\mathrm{d}x = \int_\Omega f(x)(v(x)-v_\Omega)\,\mathrm{d}x$$
$$= \mathfrak{B}(u, v - v_\Omega) = \mathfrak{B}(u,v) \text{ for all } v \in \mathrm{V}(\Omega;\gamma).$$

It remains to show that weak solutions in $\mathrm{V}(\Omega;\gamma)$ are unique up to a constant. For that reason let both $u, v \in \mathrm{V}(\Omega;\gamma)$ be weak solutions for our nonlocal Neumann problem. This means $u - u_\Omega$ and $v - v_\Omega$ are weak solutions as well and because the weak solution is unique in $\widehat{V}(\Omega;\gamma)$, we obtain $u = v - v_\Omega + u_\Omega$. □

Now, we study the nonhomogeneous nonlocal Neumann problem. For the existence of a weak solution, the following linear operator

$$\mathrm{V}(\Omega;\gamma) \to \mathbb{R}, \quad v \mapsto \int_\Gamma g(y)v(y)\,\mathrm{d}y$$

must exist and be well-defined.

Definition 4.6.
Let $\Omega \subset \mathbb{R}^d$ be bounded, nonempty and open and let $\gamma \in \mathcal{K}$.

(i) We say that a measurable function $g\colon \Gamma \to \mathbb{R}$ satisfies the functional condition (on $\mathrm{V}(\Omega;\gamma)$) *if*

$$v \mapsto \int_\Gamma g(y)v(y)\,\mathrm{d}y,$$

is a linear functional on $\mathrm{V}(\Omega;\gamma)$. If the functional is continuous, we say that $g\colon \Gamma \to \mathbb{R}$ satisfies the continuous functional condition.

(ii) We say that a measurable function $g\colon \mathbb{R}^d \to \mathbb{R}$ satisfies the (continuous) functional condition *if $g|_\Gamma$ satisfies the (continuous) functional condition.*

Imposing the continuous functional condition on the Neumann data is sufficient for the existence of a weak solution to the nonhomogeneous Neumann problem as shown in the next theorem.

Theorem 4.7.
Let $\Omega \subset \mathbb{R}^d$ be bounded, nonempty and open, $f \in \mathrm{L}^2(\Omega)$, $\gamma \in \mathcal{K}$ be symmetric, and $g\colon \Gamma \to \mathbb{R}$ satisfy the continuous functional condition. Then, the nonhomogeneous Neumann problem (NP)*, i.e.,*

$$\begin{cases} \mathcal{L}u &= f \quad \text{on } \Omega, \\ \mathcal{N}u &= g \quad \text{on } \Gamma, \end{cases}$$

has a weak solution if

$$\int_\Omega f(x)\,\mathrm{d}x + \int_\Gamma g(y)\,\mathrm{d}y = 0 \qquad \text{(Compatibility condition)}$$

and if there is a constant $C > 0$ such that for all $v \in \mathrm{V}(\Omega;\gamma)$, we have

$$\int_\Omega \int_\Omega (v(x) - v(y))^2\,\mathrm{d}y\,\mathrm{d}x \leqslant C \int_\Omega \int_{\mathbb{R}^d} (v(x) - v(y))^2 \gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x. \qquad \text{(Poincaré inequality)}$$

Furthermore, the weak solution is unique up to an additive constant.

Proof.

As in the proof of Theorem 4.5, we see that

$$\widehat{V}(\Omega;\gamma) := \left\{ v \in \mathrm{V}(\Omega;\gamma) \colon \int_\Omega v(x)\,\mathrm{d}x = 0 \right\}$$

is a Hilbert space with respect to the inner product $\mathfrak{B}$. Next, we consider the linear functional

$$\Lambda(v) = \int_\Omega f(x)v(x)\,\mathrm{d}x + \int_\Gamma g(y)v(y)\,\mathrm{d}y \quad \text{for } v \in \widehat{V}(\Omega;\gamma).$$

This functional is bounded, so that, by the Riesz representation Theorem, there is a unique function $u \in \widehat{V}(\Omega;\gamma)$ satisfying

$$\Lambda(v) = \mathfrak{B}(u,v) \quad \text{for all } v \in \widehat{V}(\Omega;\gamma).$$

Similarly to the proof of Theorem 4.5, we find that the compatibility condition implies

$$\Lambda(v) = \mathfrak{B}(u,v) \quad \text{for all } v \in \mathrm{V}(\Omega;\gamma).$$

Finally, for two weak solutions $u_1, u_2 \in \mathrm{V}(\Omega;\gamma)$ we get that the mean–centered versions are equal, i.e.,

$$u_1 - \frac{1}{\lambda(\Omega)} \int_\Omega u_1(x)\,\mathrm{d}x = u_2 - \frac{1}{\lambda(\Omega)} \int_\Omega u_2(x)\,\mathrm{d}x.$$

Thus, weak solutions are unique up to an additive constant. □

In chapter 6 we present some sufficient assumptions for the continuous functional condition.

An approach to study the local nonhomogeneous Neumann problem is to transform it into an equivalent homogeneous problem by using the linear dependence of the weak solution on the right hand side.

Remark 4.8.
Let the assumptions of Theorem 4.7 be satisfied and $u \in \mathrm{V}(\Omega;\gamma)$ be a weak solution of the non-homogeneous Neumann problem (NP). *Furthermore, let $\widetilde{g} \in \mathrm{V}(\Omega;\gamma)$ be the weak solution of the Neumann problem*

$$\begin{cases} \mathcal{L}u + \kappa u &= 0 \quad \textit{on } \Omega, \\ \mathcal{N}u &= g \quad \textit{on } \Gamma, \end{cases}$$

for a given $\kappa > 0$. More precisely, let $\widetilde{g}$ satisfy

$$\mathfrak{B}(\widetilde{g}, v) + \kappa \int_\Omega \widetilde{g}(x)v(x)\,\mathrm{d}x = \int_\Gamma g(y)v(y)\,\mathrm{d}y \qquad \textit{for all } v \in \mathrm{V}(\Omega;\gamma).$$

The existence of $\widetilde{g}$ is given by the Lax–Milgram Theorem. Then, we obtain that $\widetilde{u} = u - \widetilde{g}$ is a weak solution of

$$\begin{cases} \mathcal{L}u &= f + \kappa\widetilde{g} \quad \textit{on } \Omega, \\ \mathcal{N}u &= 0 \qquad\quad \textit{on } \Gamma. \end{cases}$$

We now show that depending on the nonlocal Neumann boundary condition, there is an explicit way to transform our nonhomogeneous Neumann problem into an equivalent homogeneous problem:

Theorem 4.9.
Let $\Omega \subset \mathbb{R}^d$ be a bounded, nonempty and open set, $\gamma \in \mathcal{K}$ be symmetric, and $g\colon \Gamma \to \mathbb{R}$ such that for a.e. $y \in \Gamma$ with $\int_\Omega \gamma(y,x)\,\mathrm{d}x = \infty$, we have $g(y) = 0$ and such that both integrals

$$\int_\Gamma \frac{g^2(y)}{\int_\Omega \gamma(z,y)\,\mathrm{d}z}\,\mathrm{d}y \quad \textit{and} \quad \int_\Omega \Big(\int_\Gamma \frac{|g(y)|\gamma(y,x)}{\int_\Omega \gamma(z,y)\,\mathrm{d}z}\,\mathrm{d}y \Big)^2 \,\mathrm{d}x$$

are finite.

(i) Then, the continuous functional condition is satisfied by g and we have $g \in \mathrm{L}^1(\Gamma)$.

(ii) Let $\widetilde{g}$ be the zero extension of $\frac{g(\cdot)}{\int_\Omega \gamma(\cdot,z)\,\mathrm{d}z}$ outside Γ, i.e.,

$$\widetilde{g}\colon \mathbb{R}^d \to \mathbb{R},\ \widetilde{g}(y) = \begin{cases} \frac{g(\cdot)}{\int_\Omega \gamma(\cdot,z)\,\mathrm{d}z}, & y \in \Gamma, \\ 0, & y \in \mathbb{R}^d \setminus \Gamma. \end{cases}$$

Then, $\widetilde{g} \in \mathrm{V}(\Omega;\gamma)$ is a strong solution of the nonhomogeneous Neumann problem

$$\begin{cases} \mathcal{L}u(x) &= -\int_\Gamma \frac{g(y)\gamma(y,x)}{\int_\Omega \gamma(y,z)\,\mathrm{d}z}\,\mathrm{d}y \quad \textit{for } x \in \Omega, \\ \mathcal{N}u(y) &= g(y) \quad \textit{for } y \in \Gamma. \end{cases}$$

(iii) A function $u \in \mathrm{V}(\Omega;\gamma)$ is a weak solution of the nonhomogeneous Neumann problem (NP)*, i.e.,*

$$\begin{cases} \mathcal{L}u(x) &= f(x) \quad \textit{for } x \in \Omega, \\ \mathcal{N}u(y) &= g(y) \quad \textit{for } y \in \Gamma, \end{cases}$$

if and only if $u(\cdot) + \frac{g(\cdot)}{\int_\Omega \gamma(z,\cdot)\,\mathrm{d}z}\chi_\Gamma(\cdot) \in \mathrm{V}(\Omega;\gamma)$ is a weak solution of the following homogeneous Neumann problem

$$\begin{cases} \mathcal{L}\widetilde{u}(x) &= f(x) + \int_\Gamma \frac{g(y)\gamma(x,y)}{\int_\Omega \gamma(z,y)\,\mathrm{d}z}\,\mathrm{d}y \quad \textit{for } x \in \Omega, \\ \mathcal{N}\widetilde{u}(y) &= 0 \quad \textit{for } y \in \Gamma. \end{cases} \tag{4.5}$$

Proof.

For all $v \in \mathrm{V}(\Omega;\gamma)$, we find

$$
\begin{aligned}
&\left| \int_\Gamma g(y)v(y)\,\mathrm{d}y \right| \\
&\leqslant \int_\Gamma |\,g(y)v(y)|\,\mathrm{d}y \\
&= \int_\Omega \int_\Gamma \frac{|g(y)|\gamma(y,x)}{\int_\Omega \gamma(y,z)\,\mathrm{d}z} |v(y)|\,\mathrm{d}y\ \mathrm{d}x \\
&\leqslant \int_\Omega \int_\Gamma \frac{|g(y)|\gamma(y,x)}{\int_\Omega \gamma(y,z)\,\mathrm{d}z}\,\mathrm{d}y\,|v(x)|\,\mathrm{d}x + \int_\Omega \int_\Gamma \frac{|g(y)|}{\int_\Omega \gamma(y,z)\,\mathrm{d}z}|v(x)-v(y)|\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
&\leqslant \|v\|_{\mathrm{V}(\Omega;\gamma)} \max\left\{ \int_\Omega \Big(\int_\Gamma \frac{|g(y)|\gamma(y,x)}{\int_\Omega \gamma(y,z)\,\mathrm{d}z}\,\mathrm{d}y \Big)^2 \mathrm{d}x,\ \int_\Gamma \frac{g^2(y)}{\int_\Omega \gamma(y,z)\,\mathrm{d}z}\,\mathrm{d}y \right\},
\end{aligned}
$$

where the two terms in the max–function are finite due to our assumptions. Thus, g satisfies the continuous functional condition. And by choosing a constant v, we see that $g \in \mathrm{L}^1(\Gamma)$. Because of our assumptions, we further have

$$
\|\widetilde{g}\|_{\mathrm{V}(\Omega;\gamma)} = \int_\Omega \int_{\mathbb{R}^d} \widetilde{g}^2(y)\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x = \int_\Gamma \frac{g^2(y)}{\int_\Omega \gamma(z,y)\,\mathrm{d}z}\,\mathrm{d}y < \infty,
$$

so that $\widetilde{g} \in \mathrm{V}(\Omega;\gamma)$. The rest follows by evaluating both $\mathcal{L}\widetilde{g}$ and $\mathcal{N}\widetilde{g}$, and by the linear dependence of the weak solution on the right hand side. □

Remark 4.10.
In the case that $\gamma \in \mathcal{K}$ is nonsymmetric, the variational formulation of problem (NP) *can also be obtained by Theorem 2.1, the nonlocal integration by parts formula. However, to the best of our knowledge, there is, in general, no "natural" test function space in the nonsymmetric case. Furthermore, for all $\gamma \in \mathcal{K}$, we get*

$$
\begin{aligned}
&\mathcal{N}u(y_1) = -\int_\Omega u(x)\gamma(x,y_1)\,\mathrm{d}x && \text{for } y_1 \in \widehat{\Gamma}(\Omega,\gamma) \setminus \Gamma(\Omega,\gamma) \\
\text{and}\quad &\mathcal{N}u(y_2) = u(y_2)\int_\Omega \gamma(y_2,x)\,\mathrm{d}x && \text{for } y_2 \in \widehat{\Gamma}(\Omega,\gamma) \setminus \Gamma(\Omega,\gamma^\top).
\end{aligned}
$$

So, on $\widehat{\Gamma}(\Omega,\gamma) \setminus \Gamma(\Omega,\gamma)$, our Neumann condition reduces to a weighted volume constraint and on $\widehat{\Gamma}(\Omega,\gamma) \setminus \Gamma(\Omega,\gamma^\top)$, our Neumann condition reduces to a Dirichlet condition. For our nonlocal operator $\mathcal{L}$, we further have by Fubini's Theorem

$$
\int_\Omega \int_{\widehat{\Gamma}(\Omega,\gamma)\setminus\Gamma(\Omega,\gamma)} \gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x = \int_{\widehat{\Gamma}(\Omega,\gamma)\setminus\Gamma(\Omega,\gamma)} \int_\Omega \gamma(y,x)\,\mathrm{d}x\,\mathrm{d}y = 0
$$

and therefore, for a.e. $x \in \Omega$,

$$
\begin{aligned}
\mathcal{L}u(x) = &\int_\Omega u(x)\gamma(x,y) - u(y)\gamma(y,x)\,\mathrm{d}y \\
&+ \int_{\widehat{\Gamma}(\Omega,\gamma)\setminus\Gamma(\Omega,\gamma)} u(x)\gamma(x,y)\,\mathrm{d}y \\
&+ \int_{\Gamma(\Omega,\gamma)} u(x)\gamma(x,y) - u(y)\gamma(y,x)\,\mathrm{d}y.
\end{aligned}
$$

In the following theorem, we discuss some sufficient assumptions on a not necessarily symmetric kernel such that our nonlocal Neumann problem has a weak solution.

Theorem 4.11.
Let $\Omega \subset \mathbb{R}^d$ be bounded, nonempty and open, $f \in \mathrm{L}^2(\Omega)$, $\alpha \in \mathrm{L}^\infty(\Omega)$, and $\gamma \in \mathcal{K}$ with

$$\Big\| \frac{1}{2} \int_\Omega |\gamma(\cdot, y) - \gamma(y, \cdot)| \,\mathrm{d}y \Big\|_{\mathrm{L}^\infty(\Omega)} + \Big\| \int_{\widehat{\Gamma}} |\gamma(\cdot, y) - \gamma(y, \cdot)| \,\mathrm{d}y \Big\|_{\mathrm{L}^\infty(\Omega)} < \infty.$$

We assume that there is a constant $C > 0$ with $\gamma(x, y) \leqslant C\gamma(y, x)$ for a.e. $(y, x) \in \widehat{\Gamma} \times \Omega$ and such that for a.e. $x \in \Omega$, we have

$$0 < c < \alpha(x) + \frac{1}{2} \int_\Omega \gamma(x, y) - \gamma(y, x) \,\mathrm{d}y - \frac{C+1}{2} \int_{\widehat{\Gamma}} |\gamma(x, y) - \gamma(y, x)| \,\mathrm{d}y.$$

Then, the Neumann problem

$$\begin{cases} \mathcal{L}u(x) + \alpha(x)u(x) & = f(x) \quad \text{for } x \in \Omega, \\ \mathcal{N}u(y) & = 0 \qquad \text{for } y \in \widehat{\Gamma}, \end{cases}$$

has a weak solution, namely there is a function $u \in \mathrm{V}(\Omega; \gamma)$ solving

$$\widetilde{\mathfrak{B}}(u, v) = \int_\Omega f(x)v(x) \,\mathrm{d}x \qquad \text{for all } v \in \mathrm{V}(\Omega; \gamma)$$

where for $u, v \in \mathrm{V}(\Omega; \gamma)$, we set

$$\begin{aligned} \widetilde{\mathfrak{B}}(u, v) :=& \widehat{B}(u, v) + \int_\Omega u(x)v(x)\alpha(x) \,\mathrm{d}x \\ =& \frac{1}{2} \int_\Omega \int_\Omega (u(x)\gamma(x, y) - u(y)\gamma(y, x))(v(x) - v(y)) \,\mathrm{d}y \,\mathrm{d}x \\ &+ \int_\Omega \int_{\widehat{\Gamma}} (u(x)\gamma(x, y) - u(y)\gamma(y, x))(v(x) - v(y)) \,\mathrm{d}y \,\mathrm{d}x \\ &+ \int_\Omega u(x)v(x)\alpha(x) \,\mathrm{d}x. \end{aligned}$$

Proof.

By Theorem 2.1, we see that the definition of the weak solution is justified. By Hölder's inequality there is a constant $K > 0$ such that for all $u, v \in \mathrm{V}(\Omega; \gamma)$ we have

$$\begin{aligned} |\widetilde{\mathfrak{B}}(u, v)| \leqslant& \int_\Omega \int_{\mathbb{R}^d} |u(x)\gamma(x, y) - u(y)\gamma(y, x)||v(x) - v(y)| \,\mathrm{d}y \,\mathrm{d}x \\ &+ \int_\Omega |u(x)||v(x)|\alpha(x) \,\mathrm{d}x \\ \leqslant& \int_\Omega \int_{\mathbb{R}^d} |u(x) - u(y)||v(x) - v(y)|\gamma(y, x) \,\mathrm{d}y \,\mathrm{d}x \\ &+ \int_\Omega \int_{\mathbb{R}^d} |u(x)||\gamma(x, y) - \gamma(y, x)||v(x) - v(y)| \,\mathrm{d}y \,\mathrm{d}x \\ &+ \int_\Omega |u(x)||v(x)|\alpha(x) \,\mathrm{d}x \\ \leqslant& K\|u\|_{\mathrm{V}(\Omega;\gamma)} \|v\|_{\mathrm{V}(\Omega;\gamma)}. \end{aligned}$$

Therefore $\widetilde{\mathfrak{B}}$ is bounded on $\mathrm{V}(\Omega;\gamma) \times \mathrm{V}(\Omega;\gamma)$ with

$$\begin{aligned}\widetilde{\mathfrak{B}}(u,v) =& \mathfrak{B}(u,v) \\ &+ \frac{1}{2}\int_\Omega\int_\Omega u(x)(\gamma(x,y)-\gamma(y,x))(u(x)-u(y))\,\mathrm{d}y\,\mathrm{d}x \\ &+ \int_\Omega\int_\Gamma u(x)(\gamma(x,y)-\gamma(y,x))(u(x)-u(y))\,\mathrm{d}y\,\mathrm{d}x \\ &+ \int_\Omega u(x)v(x)\alpha(x)\,\mathrm{d}x.\end{aligned}$$

And because of

$$\begin{aligned}&\int_\Omega\int_\Omega u(x)(\gamma(x,y)-\gamma(y,x))(v(x)-v(y))\,\mathrm{d}y\,\mathrm{d}x \\ &\qquad = \int_\Omega\int_\Omega u(y)(\gamma(x,y)-\gamma(y,x))(v(x)-v(y))\,\mathrm{d}y\,\mathrm{d}x,\end{aligned}$$

we obtain, for $u, v \in \mathrm{V}(\Omega;\gamma)$,

$$\begin{aligned}&\int_\Omega\int_\Omega u(x)(\gamma(x,y)-\gamma(y,x))(u(x)-u(y))\,\mathrm{d}y\,\mathrm{d}x \\ &\quad = \frac{1}{2}\int_\Omega\int_\Omega (u(x)+u(y))(\gamma(x,y)-\gamma(y,x))(u(x)-u(y))\,\mathrm{d}y\,\mathrm{d}x \\ &\quad = \int_\Omega u^2(x)\int_\Omega(\gamma(x,y)-\gamma(y,x))\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Moreover we estimate

$$\begin{aligned}&\left|\int_\Omega\int_{\widehat{\Gamma}} u(x)(\gamma(x,y)-\gamma(y,x))(u(x)-u(y))\sqrt{\frac{C+1}{C+1}}\,\mathrm{d}y\,\mathrm{d}x\right| \\ \leqslant& \frac{C+1}{2}\int_\Omega u^2(x)\int_{\widehat{\Gamma}}|\gamma(x,y)-\gamma(y,x)|\,\mathrm{d}y\,\mathrm{d}x \\ &+ \frac{1}{2}\int_\Omega\int_{\widehat{\Gamma}}(u(x)-u(y))^2\frac{|\gamma(x,y)-\gamma(y,x)|}{C+1}\,\mathrm{d}y\,\mathrm{d}x \\ \leqslant& \frac{C+1}{2}\int_\Omega u^2(x)\int_{\widehat{\Gamma}}|\gamma(x,y)-\gamma(y,x)|\,\mathrm{d}y\,\mathrm{d}x \\ &+ \frac{1}{2}\int_\Omega\int_{\widehat{\Gamma}}(u(x)-u(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x\end{aligned}$$

for $u \in \mathrm{V}(\Omega;\gamma)$. In other words $\widetilde{\mathfrak{B}}$ is coercive on $\mathrm{V}(\Omega;\gamma)$ and by the Lax–Milgram Theorem there exists a unique weak solution. □

Chapter 5

A Nonlocal Poincaré Inequality

Similar to the local Neumann problem, the Lax–Milgram Theorem provides us an existence and uniqueness result for the weak solution of problem (NP). Note that because in Theorem 4.5 the bilinear form is symmetric, we could directly apply the Riesz representation Theorem.

The Poincaré inequality (see [6, Chapter 5.8.1.]) holds if there is a constant $C > 0$ such that for all $u \in \mathrm{H}^1(\Omega)$ we have

$$\begin{aligned} 2\lambda(\Omega)\int_\Omega u^2(x)\,\mathrm{d}x - 2\left(\int_\Omega u(y)\,\mathrm{d}y\right)^2 =& \int_\Omega\int_\Omega (u(x)-u(y))^2\,\mathrm{d}y\,\mathrm{d}x \\ \leqslant& C\int_\Omega \|\nabla u(x)\|^2\,\mathrm{d}x. \end{aligned}$$

This inequality is most commonly used in order to show that the corresponding bilinear form is coercive. In the literature, the inequalities - most commonly called nonlocal Poincaré type inequality - are either in the shape of (see for example [5])

$$\int_\Omega\int_\Omega (u(x)-u(y))^2\,\mathrm{d}y\,\mathrm{d}x \leqslant C\int_\Omega\int_\Omega (u(x)-u(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x$$

or (see for example [28])

$$\int_\Omega\int_\Omega (u(x)-u(y))^2\,\mathrm{d}y\,\mathrm{d}x \leqslant C\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} (u(x)-u(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x$$

where $\gamma \in \mathcal{K}$ and

$$\Omega_I := \{y \in \mathbb{R}^d \setminus \Omega \text{ such that there is a } x \in \Omega \text{ with } \gamma(x,y) > 0\}.$$

We, however, call

$$\int_\Omega\int_\Omega (u(x)-u(y))^2\,\mathrm{d}y\,\mathrm{d}x \leqslant C\int_\Omega\int_{\mathbb{R}^d} (u(x)-u(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x$$

nonlocal Poincaré inequality. We easily see that

$$\begin{aligned} \int_\Omega\int_\Omega (u(x)-u(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \leqslant& \int_\Omega\int_{\mathbb{R}^d} (u(x)-u(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\ \leqslant& \int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I} (u(x)-u(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \end{aligned}$$

holds.

Definition 5.1.
Let $\Omega \subset \mathbb{R}^d$ be a nonempty, open, and bounded set, and $\gamma \in \mathcal{K}$. We say that the nonlocal Poincaré inequality holds (on $\mathrm{V}(\Omega;\gamma)$) *if there is a constant $C > 0$ such that for all $u \in \mathrm{V}(\Omega;\gamma)$, we have*

$$\int_\Omega \int_\Omega (u(x) - u(y))^2 \, \mathrm{d}y \, \mathrm{d}x \leqslant C \int_\Omega \int_{\mathbb{R}^d} (u(x) - u(y))^2 \gamma(y,x) \, \mathrm{d}y \, \mathrm{d}x.$$

Every constant $C > 0$ for which the nonlocal Poincaré inequality is satisfied is called Poincaré constant.

Recalling that for every $\eta \in \mathcal{K}$, we set

$$\begin{aligned}\mathfrak{B}_\eta(u,v) :=& \frac{1}{2} \int_\Omega \int_\Omega (u(x) - u(y))(v(x) - v(y)) \, \eta(y,x) \, \mathrm{d}y \, \mathrm{d}x \\ &+ \int_\Omega \int_\Gamma (u(x) - u(y))(v(x) - v(y)) \, \eta(y,x) \, \mathrm{d}y \, \mathrm{d}x\end{aligned}$$

for $u, v \in \mathrm{V}(\Omega;\eta)$, we obtain

Lemma 5.2.
Let $\Omega \subset \mathbb{R}^d$ be a nonempty, open, and bounded set and $\gamma \in \mathcal{K}$. Then, the following statements are equivalent.

(i) The nonlocal Poincaré inequality holds.

(ii) There is a constant $C > 0$ such that for all $u \in \mathrm{V}(\Omega;\gamma)$, we have $\mathfrak{B}_{\chi_{\Omega\times\Omega}}(u,u) \leqslant C \mathfrak{B}_\gamma(u,u)$.

(iii) There is a constant $C > 0$ such that for all $u \in \mathrm{V}(\Omega;\gamma)$, we have $\|u - u_\Omega\|^2_{\mathrm{L}^2(\Omega)} \leqslant C \mathfrak{B}_\gamma(u,u)$ where $u_\Omega = \frac{1}{\lambda(\Omega)} \int_\Omega u(x) \, \mathrm{d}x$.

Proof.

Let $u \in \mathrm{V}(\Omega;\gamma)$. Then, the binomial theorem and linearity of the integral yields

$$\begin{aligned}2\lambda(\Omega) \, \|u - u_\Omega\|^2_{\mathrm{L}^2(\Omega)} =& 2\lambda(\Omega) \int_\Omega (u^2(x) - 2u_\Omega u(x) + (u_\Omega)^2) \, \mathrm{d}x \\ =& 2\lambda(\Omega) \int_\Omega u^2(x) \, \mathrm{d}x - 4 \left(\int_\Omega u(y) \, \mathrm{d}y \right)^2 + 2\lambda(\Omega)^2 (u_\Omega)^2 \\ =& 2\lambda(\Omega) \int_\Omega u^2(x) \, \mathrm{d}x - 2 \left(\int_\Omega u(y) \, \mathrm{d}y \right)^2 \\ =& \int_\Omega \int_\Omega (u^2(x) - 2u(x)u(y) + u^2(y)) \, \mathrm{d}y \, \mathrm{d}x \\ =& \int_\Omega \int_\Omega (u(x) - u(y))^2 \, \mathrm{d}y \, \mathrm{d}x \\ =& 2\mathfrak{B}_{\chi_{\Omega\times\Omega}}(u,u).\end{aligned}$$

And because of

$$\frac{1}{2}\int_\Omega\int_{\mathbb{R}^d}(u(x)-u(y))^2\gamma(x,y)\,\mathrm{d}y\,\mathrm{d}x \leqslant \mathfrak{B}_\gamma(u,u) \leqslant \int_\Omega\int_{\mathbb{R}^d}(u(x)-u(y))^2\gamma(x,y)\,\mathrm{d}y\,\mathrm{d}x,$$

we get the equivalences. □

Remark 5.3.
Let $\Omega \subset \mathbb{R}^d$ be a nonempty, open, and bounded set. Then, we have for all measurable functions $u\colon \mathbb{R}^d \to \mathbb{R}$ with $\mathfrak{B}_{\chi_{\Omega\times\Omega}}(u,u) < \infty$ that $u|_\Omega \in \mathrm{L}^2(\Omega)$ holds (see proof of Proposition 2.1 in [12]). Furthermore, this implies that the nonlocal Poincaré inequality holds on $\mathrm{V}(\Omega;\gamma)$ if and only if there is a constant $C > 0$ such that for every measurable function $v\colon \mathbb{R}^d \to \mathbb{R}$ with $v|_\Omega \in \mathrm{L}^2(\Omega)$, we have

$$\int_\Omega\int_\Omega(v(x)-v(y))^2\,\mathrm{d}y\,\mathrm{d}x \leqslant C\int_\Omega\int_{\mathbb{R}^d}(v(x)-v(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x.$$

Now, we give some sufficient assumptions such that the nonlocal Poincaré inequality holds:

Theorem 5.4.
Let $\Omega \subset \mathbb{R}^d$ be a nonempty, open, and bounded set and $\gamma \in \mathcal{K}$ with

$$0 < \int_{\mathbb{R}^d}\operatorname*{ess\,inf}_{x\in\Omega}\gamma(y,x)\,\mathrm{d}y,$$

Then, the nonlocal Poincaré inequality holds.

Proof.

Choose a constant $C > 0$ such that there is a bounded measurable set

$$A \subset \{y \in \mathbb{R}^d\colon \operatorname*{ess\,inf}_{x\in\Omega}\gamma(y,x) < C\} \text{ with } 0 < c := \int_A \operatorname*{ess\,inf}_{x\in\Omega}\gamma(y,x)\,\mathrm{d}y.$$

Then, for every $u \in \mathrm{V}(\Omega;\gamma)$ Jensen's inequality yields

$$\begin{aligned}
&\int_\Omega\int_\Omega(u(x)-u(y))^2\,\mathrm{d}y\,\mathrm{d}x\\
=&\frac{1}{c}\int_\Omega\int_\Omega\int_A(u(x)-u(t)+u(t)-u(y))^2\operatorname*{ess\,inf}_{s\in\Omega}(\gamma(t,s))\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x\\
\leqslant&\frac{2\lambda(\Omega)}{c}\int_\Omega\int_{\mathbb{R}^d}(u(x)-u(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

□

Due to Theorem 6.7 in [5], we see that the assumptions in Theorem 5.4 are only sufficient conditions for the nonlocal Poincaré inequality. For this reason, we now relax the assumptions on γ by using the following Lemma.

Lemma 5.5.
Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $\varepsilon > 0$. Then, there is a $C > 0$ such that for all $u \in \mathrm{L}^2(\Omega)$, we have

$$\int_\Omega\int_\Omega(u(x)-u(y))^2\,\mathrm{d}y\,\mathrm{d}x \leqslant C\int_\Omega\int_\Omega(u(x)-u(y))^2\chi_{\{\|x-y\|<\varepsilon\}}\,\mathrm{d}y\,\mathrm{d}x.$$

Proof.

Due to the Heine–Borel Theorem, we know that $\overline{\Omega}$ is compact. Therefore, there is a $M \in \mathbb{N}$ and a sequence $(a_i)_{i\in\mathbb{N}}$ in Ω with

$$\overline{\Omega} \subset \bigcup_{i\in\mathbb{M}, i\leqslant N} \mathrm{B}_{\frac{\varepsilon}{4}}(a_i).$$

Recall, that a chain of sets is a finite series of sets $X_1, X_2 \ldots, X_k$ such that $X_j \cap X_{j+1} \neq \emptyset$ for $j = 1, 2, \ldots, k-1$. Then, by Theorem 1.6 in [23, Chapter IV], we know that Ω is connected if and only if for any open covering, or finite closed covering, C of Ω, any two elements $\Omega_1, \Omega_k \in C$ are the first and last elements of a chain $\Omega_1, \Omega_2, \ldots, \Omega_k \in C$. Therefore, there is a $N \in \mathbb{N}$ and a sequence $(\omega_i)_{i\in\mathbb{N}}$ in Ω such that

$$\overline{\Omega} \subset \bigcup_{i\in\mathbb{N}, i\leqslant N} \mathrm{B}_{\frac{\varepsilon}{4}}(\omega_i) \quad \text{and} \quad \|\omega_{j+1} - \omega_j\| \leqslant \frac{\varepsilon}{2} \text{ for } j = 1, \ldots, N-1.$$

Let $x, y \in \Omega$. Then, we assume , without loss of generality, that $x \in \mathrm{B}_{\frac{\varepsilon}{4}}(\omega_1)$. Let $j \in \{1, \ldots, N\}$ satisfy $y \in \mathrm{B}_{\frac{\varepsilon}{4}}(\omega_j)$ and let $y_\ell \in \mathrm{B}_{\frac{\varepsilon}{4}}(0)$ for $\ell = 1, \ldots, N$. Then,

$$\begin{aligned} &\|x - \omega_1 + y_1\| \leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leqslant \varepsilon, \quad \|\omega_j - y + y_j\| \leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leqslant \varepsilon \\ \text{and} \quad &\|\omega_i + y_i - w_{i+1} - y_{i+1}\| \leqslant \|\omega_i - \omega_{i+1}\| + \|y_i\| + \|y_{i+1}\| \leqslant \varepsilon \end{aligned}$$

for $i = 1, \ldots, \ell - 1$. In other words, for every $x, y \in \Omega$, there is a sequence $(z_i)_{i\in\mathbb{N}}$ in Ω such that

$$\begin{aligned} &\|x - z_1 + y_1\| \leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leqslant \varepsilon, \quad \|z_N - y + y_\ell\| \leqslant \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \leqslant \varepsilon \\ \text{and} \quad &\|z_i + y_i - z_{i+1} - y_{i+1}\| \leqslant \|z_i - z_{i+1}\| + \|y_i\| + \|y_{i+1}\| \leqslant \varepsilon \end{aligned}$$

for $i = 1, \ldots, N-1$. Setting $D_i(x, y) = \left(\mathrm{B}_{\frac{\varepsilon}{4}}(z_i)\right) \cap \Omega$ for $i = 1, \ldots, N$, we conclude that $(x_1, \ldots, x_N) \in \bigtimes_{i=1}^{N} D_i(x, y)$ implies $\|x - x_1\|$, $\|x_N - y\|, \|x_j - x_{j+1}\| \leqslant \varepsilon$ for all $j = 1, \ldots, N-1$ and therefore,

$$\begin{aligned} &\bigtimes_{i=1}^{N} D_i(x, y) \\ \subset &\{(v_1, \ldots, v_N) \in \Omega^N \colon \|x - v_1\|, \|v_N - y\|, \|v_j - v_{j+1}\| \leqslant \varepsilon \text{ for } j = 1, \ldots, N-1\}. \end{aligned}$$

Because $\overline{\Omega}$ is compact and Ω is open, there exists a constant $c > 0$ for which $c \leqslant \inf_{x\in\Omega} \lambda(\mathrm{B}_{\frac{\varepsilon}{4}}(x) \cap \Omega)$ holds. Hence, we estimate

$$\begin{aligned} &\int_\Omega \int_\Omega (u(x) - u(y))^2 \,\mathrm{d}y \,\mathrm{d}x \\ \leqslant &\frac{1}{c^N} \int_\Omega \int_\Omega \int_{D_1(x,y)} \int_{D_2(x,y)} \cdots \int_{D_N(x,y)} (u(x) - u(y))^2 \,\mathrm{d}x_N \ldots \mathrm{d}x_2 \,\mathrm{d}x_1 \,\mathrm{d}y \,\mathrm{d}x \\ \leqslant &\frac{1}{c^N} \underbrace{\int_\Omega \int_{\mathrm{B}_\varepsilon(x)} \int_{\mathrm{B}_\varepsilon(x_1)} \cdots \int_{\mathrm{B}_\varepsilon(x_N)} \int_\Omega (u(x) - u(y))^2 \prod_{i=1}^{N} \chi_\Omega(x_i) \,\mathrm{d}y \,\mathrm{d}x_N \ldots \mathrm{d}x_2 \,\mathrm{d}x_1 \,\mathrm{d}x}_{=:J}. \end{aligned}$$

For $j = 1, 2, \ldots, N-1$, set

$$J_0 := \int_\Omega \int_{\mathrm{B}_\varepsilon(x)} \cdots \int_{\mathrm{B}_\varepsilon(x_N)} \int_\Omega (u(x) - u(x_1))^2 \prod_{i=1}^{N} \chi_\Omega(x_i) \,\mathrm{d}y \,\mathrm{d}x_N \ldots \mathrm{d}x_1 \,\mathrm{d}x,$$

$$J_j := \int_\Omega \int_{\mathrm{B}_\varepsilon(x)} \cdots \int_{\mathrm{B}_\varepsilon(x_N)} \int_\Omega (u(x_j) - u(x_{j+1}))^2 \prod_{i=1}^{N} \chi_\Omega(x_i) \,\mathrm{d}y \,\mathrm{d}x_N \ldots \mathrm{d}x_1 \,\mathrm{d}x,$$

$$\text{and } J_N := \int_\Omega \int_{\mathrm{B}_\varepsilon(x)} \cdots \int_{\mathrm{B}_\varepsilon(x_N)} \int_\Omega (u(x_N) - u(y))^2 \prod_{i=1}^{N} \chi_\Omega(x_i) \,\mathrm{d}y \,\mathrm{d}x_N \ldots \mathrm{d}x_1 \,\mathrm{d}x.$$

Then, for all $j = 0, 1, 2, \ldots, N$, we estimate

$$J_j \leqslant (\lambda(\Omega))^N \int_\Omega \int_{\|x-y\| \leqslant \varepsilon} (u(x) - u(y))^2 \chi_\Omega(y) \,\mathrm{d}y \,\mathrm{d}x.$$

And by iteratively applying Jensen's inequality we get

$$\begin{aligned}
&(u(x) - u(y))^2 \\
=& \left(u(x) - u(x_1) + \left(\sum_{j=1}^{N-1} (u(x_j) - u(x_{j+1})) \right) + u(x_N) - u(y) \right)^2 \\
\leqslant& (N+1) \left((u(x) - u(x_1))^2 + \left(\sum_{j=1}^{N-1} (u(x_j) - u(x_{j+1}))^2 \right) + (u(x_N) - u(y))^2 \right)
\end{aligned}$$

and, therefore, conclude

$$\begin{aligned}
J \leqslant& (N+1) \left(J_0 + \left(\sum_{j=1}^{N-1} J_j \right) + J_N \right) \\
\leqslant& (N+1)^2 \, (\lambda(\Omega))^N \int_\Omega \int_{\|x-y\| \leqslant \varepsilon} (u(x) - u(y))^2 \chi_\Omega(y) \,\mathrm{d}y \,\mathrm{d}x.
\end{aligned}$$

□

Theorem 5.6.
Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $\gamma \in \mathcal{K}$, and $\varepsilon > 0$. Furthermore, let $0 < C_1 \leqslant \gamma(y, x)$ hold for a.e. $x, y \in \Omega$ with $\|x - y\| \leqslant \varepsilon$. Then, the nonlocal Poincaré inequality holds.

Proof.

Follows as a consequence of Lemma 5.2 and Lemma 5.5. □

Theorem 5.7.
Let $\Omega \subset \mathbb{R}^d$ be a bounded, nonempty domain, $\varepsilon > 0$, and $\gamma \in \mathcal{K}$ with

$$0 < \operatorname*{ess\,inf}_{x,y \in \Omega, \|x-y\| < \varepsilon} \int_{\mathbb{R}^d} \min\{\gamma(z, x), \gamma(z, y)\} \,\mathrm{d}z.$$

Then, the nonlocal Poincaré inequality holds.

Proof.

Choose a constant $C > 0$ such that there is a bounded measurable set

$$A \subset \{z \in \mathbb{R}^d \colon \min\{\gamma(z,x), \gamma(z,y)\} < C \text{ for a.e. } x, y \in \Omega \text{ with } \|x-y\| < \varepsilon\}$$
$$\text{with } 0 < c := \operatorname*{ess\,inf}_{x,y\in\Omega, \|x-y\|<\varepsilon} \int_A \min\{\gamma(z,x), \gamma(z,y)\} \,\mathrm{d}z.$$

By Lemma 5.5, there is a $C_1 > 0$ with

$$\int_\Omega \int_\Omega (u(x)-u(y))^2 \,\mathrm{d}y \,\mathrm{d}x \leqslant C_1 \int_\Omega \int_\Omega (u(x)-u(y))^2 \chi_{\{\|x-y\|<\varepsilon\}} \,\mathrm{d}y \,\mathrm{d}x \text{ for } u \in \mathrm{V}(\Omega;\gamma).$$

Hence, for $u \in \mathrm{V}(\Omega;\gamma)$, Jensen's inequality yields

$$\begin{aligned}
&C_1 \int_\Omega \int_\Omega (u(x)-u(y))^2 \chi_{\{\|x-y\|<\varepsilon\}} \,\mathrm{d}y \,\mathrm{d}x \\
\leqslant &\frac{C_1}{c} \int_\Omega \int_\Omega \int_A \min\{\gamma(z,x), \gamma(z,y)\} (u(x)-u(z)+u(z)-u(y))^2 \,\mathrm{d}z \,\mathrm{d}y \,\mathrm{d}x \\
\leqslant &\frac{2C_1\lambda(\Omega)}{c} \int_\Omega \int_{\mathbb{R}^d} (u(x)-u(z))^2 \gamma(z,x) \,\mathrm{d}z \,\mathrm{d}x.
\end{aligned}$$

□

Corollary 5.8.
Let $\Omega \subset \mathbb{R}^d$ be a bounded, nonempty domain and let $\gamma \in \mathcal{K}$ satisfy

$$0 < \gamma_0 < \gamma(y,x) \quad \textit{for all} \quad 0 < r_0 < \|x-y\| < r_1 < \infty.$$

Then, the nonlocal Poincaré inequality holds.

Proof.

For $x \in \Omega$, we define $A_x = \{y \in \mathbb{R}^d \colon r_0 < \|x-y\| < r_1\}$ and for $y, z \in \mathbb{R}^d$, we set $\widetilde{\gamma}(y,z) = \chi_{A_z}(y)\gamma_0$. Then, we choose $\varepsilon > 0$ such that

$$0 < \operatorname*{ess\,inf}_{x,y\in\Omega, \|x-y\|<\varepsilon} \lambda(A_x \cap A_y) = \operatorname*{ess\,inf}_{x,y\in\Omega, \|x-y\|<\varepsilon} \int_{\mathbb{R}^d} \min\{\widetilde{\gamma}(z,y), \widetilde{\gamma}(z,x)\} \,\mathrm{d}z,$$

holds and by Theorem 5.7, we get

$$\begin{aligned}
\int_\Omega \int_\Omega (u(x)-u(y))^2 \,\mathrm{d}y \,\mathrm{d}x &\leqslant C \int_\Omega \int_{A_x} (u(x)-u(y))^2 \widetilde{\gamma}(y,x) \,\mathrm{d}y \,\mathrm{d}x \\
&\leqslant C \int_\Omega \int_{\mathbb{R}^d} (u(x)-u(y))^2 \gamma(y,x) \,\mathrm{d}y \,\mathrm{d}x \text{ for } u \in \mathrm{V}(\Omega;\gamma).
\end{aligned}$$

□

We define

$$\mathrm{V}_0(\Omega;\gamma) := \{u \in \mathrm{V}(\Omega;\gamma) \text{ such that } u = 0 \text{ on } \mathbb{R}^d \setminus \Omega\}.$$

Then, we have for all $u \in \mathrm{V}_0(\Omega;\gamma)$ that

$$\begin{aligned}&\int_\Omega \int_{\mathbb{R}^d} (u(x)-u(y))^2 \gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\ =&\int_\Omega u^2(x) \left(\int_\Gamma \gamma(y,x)\,\mathrm{d}y\right)\mathrm{d}x + \int_\Omega\int_\Omega (u(x)-u(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\ <&\infty.\end{aligned} \tag{5.1}$$

We say the nonlocal Friedrich's inequality holds on $\mathrm{V}_0(\Omega;\gamma)$ if there is a $C > 0$ such that

$$\int_\Omega u^2(x)\,\mathrm{d}x \leqslant C \int_\Omega\int_{\mathbb{R}^d} (u(x)-u(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \quad \text{holds for all } u \in \mathrm{V}_0(\Omega;\gamma).$$

By (5.1), we see that the Friedrich's inequality holds on $\mathrm{V}_0(\Omega;\gamma)$ if we have

$$\operatorname*{ess\,inf}_{x\in\Omega} \int_\Gamma \gamma(y,x)\,\mathrm{d}y > 0.$$

However, this assumption can be relaxed.

Theorem 5.9.
Let $\Omega \subset \mathbb{R}^d$ be a nonempty, open, and bounded set and $\gamma \in \mathcal{K}$ such that, there is a measurable $\widetilde{\Omega} \subset \Omega$ with

$$\operatorname*{ess\,inf}_{x\in\widetilde{\Omega}} \int_\Gamma \gamma(y,x)\,\mathrm{d}y > 0 \quad \textit{and} \quad \operatorname*{ess\,inf}_{(y,x)\in\widetilde{\Omega}\times\Omega} \gamma(y,x) > 0.$$

Then, the nonlocal Friedrich's inequality holds. Hence, $\mathrm{V}_0(\Omega;\gamma)$ is a Hilbert space with respect to

$$\langle u,v\rangle_0 = \int_\Omega\int_{\mathbb{R}^d} (u(x)-u(y))(v(x)-v(y))\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \quad \textit{for } u,v \in \mathrm{V}_0(\Omega;\gamma).$$

Proof.

For the first part, let $u \in \mathrm{V}_0(\Omega;\gamma)$. Then, we have

$$\begin{aligned}\operatorname*{ess\,inf}_{x\in\widetilde{\Omega}} \int_\Gamma \gamma(y,x)\,\mathrm{d}y \int_{\widetilde{\Omega}} u^2(x)\,\mathrm{d}x &\leqslant \int_{\widetilde{\Omega}} u^2(x)\int_\Gamma \gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\ &\leqslant \int_\Omega\int_\Gamma (u(x)-u(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\ &\leqslant \int_\Omega\int_{\mathbb{R}^d} (u(x)-u(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\lambda(\widetilde{\Omega})\int_\Omega u^2(x)\,\mathrm{d}x &= \int_\Omega\int_{\widetilde{\Omega}} (u(x)-u(y)+u(y))^2\,\mathrm{d}y\,\mathrm{d}x \\ &\leqslant \int_\Omega\int_{\widetilde{\Omega}} 2(u(x)-u(y))^2 + 2u^2(y)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Therefore, the Friedrich's inequality is satisfied. And if the Friedrich's inequality is satisfied, then there is a $\alpha > 0$ and $\beta < \infty$ such that for all $u \in \mathrm{V}_0(\Omega;\gamma)$, we have

$$\alpha\|u\|_{\mathrm{V}(\Omega;\gamma)} \leqslant \langle u,u\rangle_0 \leqslant \beta\|u\|_{\mathrm{V}(\Omega;\gamma)}.$$

This means that $\langle\cdot,\cdot\rangle_0$ defines an inner product on $\mathrm{V}_0(\Omega;\gamma)$. □

Corollary 5.10.
Let $\Omega \subset \mathbb{R}^d$ be nonempty, open, and bounded and $s \in (0,1)$. Then, the assumption for the Friedrich's inequality in Theorem 5.9 is satisfied if we consider

$$\gamma_s(x,y) := \frac{1}{\|x-y\|^{d+2s}} \chi_{\mathbb{R}^d \setminus \{0\}}(y-x) \quad \textit{for } x, y \in \mathbb{R}^d.$$

Proof.

For the proof we refer to Lemma A.1. in [24]. □

Chapter 6

A Nonlocal Trace Theorem

In this section, we study the nonlocal trace space (for the local trace space, see Leoni [19, Chapter 18.]). In other words, we study the "restriction" of the elements of $\mathrm{V}(\Omega;\gamma)$ on Γ. Like the elements of the Sobolev space, the elements of $\mathrm{V}(\Omega;\gamma)$ are in fact equivalence classes, so pointwise evaluations of these elements are in general impractical. However, unlike the local case, the elements of $\mathrm{V}(\Omega;\gamma)$ are defined a.e. on the nonlocal boundary Γ.

Most recently, nonlocal trace spaces for the fractional Laplacian kernel were introduced by Bersetche and Borthagaray [7] as well as Dyda and Kassmann [22]. Furthermore, Tian and Du [30] studied a nonlocal trace space for regional kernels by using density arguments. As in [7], [36] and [22], we will prove the existence of a weighted Lebesgue space $\mathrm{L}^2(\Gamma;w)$, for which

$$\mathrm{Tr}\colon \mathrm{V}(\Omega;\gamma) \to \mathrm{L}^2(\Gamma;w), \quad v \mapsto v|_\Gamma$$

is a continuous linear operator, i.e., there is a constant $C > 0$ with

$$\|\mathrm{Tr}(u)\|_{\mathrm{L}^2(\Gamma;w)} \leqslant C\|u\|_{\mathrm{V}(\Omega;\gamma)} \quad \text{for all } u \in \mathrm{V}(\Omega;\gamma).$$

However, while the connection between the weighted Lebesgue space which we study and the one introduced in [36] is clear, the connection between the nonlocal trace space of [7] and [22] and the one we study remains an open question.

Also, we will find that the measurable weight function $w\colon \Gamma \to [0,\infty]$ only depends on γ and Ω. Because we have $\mathrm{Tr}(u) = 0$ for all $u \in \mathrm{V}_0(\Omega;\gamma)$, we see that Tr is injective if and only if $\mathrm{V}_0(\Omega;\gamma) = \{0\}$. As in [17], we present a characterization of the trace space for some example kernels.

Note that by using Fubini's Theorem, we see

$$\int_\Omega\int_\Gamma \gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x = \int_\Gamma\int_\Omega \gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x.$$

Therefore, Γ is a null set if and only if $\{x \in \Omega\colon \int_\Gamma \gamma(y,x)\,\mathrm{d}y > 0\}$ is a null set. Furthermore, if

$$\Omega = \{x \in \Omega\colon \int_\Gamma \gamma(y,x)\,\mathrm{d}y = \infty\}$$

holds, then we get $\mathrm{V}_0(\Omega;\gamma) = \{0\}$ by (5.1).

Theorem 6.1.
Let $\Omega \subset \mathbb{R}^d$ be a bounded, nonempty, open, and $\gamma \in \mathcal{K}$. For a.e. $x \in \Omega$, we assume

$$\int_\Gamma \gamma(z,x)\,\mathrm{d}z < \infty.$$

Furthermore, let $c \in [0,\infty)$ satisfy $\operatorname*{ess\,inf}_{x\in\Omega} \int_\Gamma \gamma(y,x)\,\mathrm{d}y + c > 0$ and define the function $w\colon \Gamma \to (0,\infty]$ by

$$w(y) = \int_\Omega \frac{\gamma(y,x)}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c}\,\mathrm{d}x.$$

Then, $\{y \in \Gamma\colon\ w(y) = \infty\}$ is a null set and

$$\mathrm{Tr}\colon \mathrm{V}(\Omega;\gamma) \to \mathrm{L}^2(\Gamma;w),\ \mathrm{Tr}(v) = v|_\Gamma$$

is a continuous linear operator.

Proof.

By definition, we have $w(y) > 0$ and because $\int_\Gamma w(y)\,\mathrm{d}y \leqslant \lambda(\Omega)$ holds, w is finite almost everywhere. For $u \in \mathrm{V}(\Omega;\gamma)$, Jensen's inequality and Fubini's Theorem yield

$$\begin{aligned}
\|\mathrm{Tr}(u)\|^2_{\mathrm{L}^2(\Gamma;w)} &= \int_\Gamma u^2(y)w(y)\,\mathrm{d}y \\
&= \int_\Gamma \int_\Omega (u(x) - u(y) - u(x))^2 \frac{\gamma(y,x)}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c}\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant 2\int_\Omega u^2(x)\,\mathrm{d}x + 2\int_\Omega \int_{\mathbb{R}^d} (u(x) - u(y))^2 \frac{\gamma(y,x)}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c}\,\mathrm{d}y\,\mathrm{d}x \\
&\leqslant 2\max\left\{ \frac{1}{\operatorname*{ess\,inf}_{x\in\Omega} \int_\Gamma \gamma(y,x)\,\mathrm{d}y + c}, 1 \right\} \|u\|^2_{\mathrm{V}(\Omega;\gamma)}.
\end{aligned}$$

□

Now, we want to characterize the trace space. By taking a closer look at the proof of Theorem 6.1, we see:

Theorem 6.2.
Let $\Omega \subset \mathbb{R}^d$ be a bounded, nonempty, open, and $\gamma \in \mathcal{K}$ with $\operatorname{ess\,sup}_{x\in\Omega} \int_\Gamma \gamma(y,x)\mathrm{d}y < \infty$. Define the function $w\colon \Gamma \to (0,\infty]$ by

$$w(y) = \int_\Omega \gamma(y,x)\,\mathrm{d}x.$$

Then, $\{y \in \Gamma\colon\ w(y) = \infty\}$ is a null set,

$$\mathrm{Tr}\colon \mathrm{V}(\Omega;\gamma) \to \mathrm{L}^2(\Gamma;w),\ \mathrm{Tr}(v) = v|_\Gamma \text{ is a bounded, linear, and surjective operator,}$$

and

$$\mathrm{Ext}\colon \mathrm{L}^2(\Gamma;w) \to \mathrm{V}(\Omega;\gamma),\ \mathrm{Ext}(c) = c\chi_\Gamma \text{ is a bounded, injective, and linear operator.}$$

Furthermore, for all $c \in \mathrm{L}^2(\Gamma;w)$, we have $\mathrm{Tr}(\mathrm{Ext}(c)) = c$.

Proof.

Because $w \in \mathrm{L}^1(\Gamma)$, we see that $\{y \in \Gamma\colon w(y) = \infty\}$ is a null set. For $u \in \mathrm{V}(\Omega;\gamma)$, Fubini's Theorem and the Jensen's inequality yield

$$\begin{aligned}
&\|\mathrm{Tr}(u)\|^2_{\mathrm{L}^2(\Gamma;w)}\\
=&\int_\Gamma u^2(y)w(y)\,\mathrm{d}y\\
=&\int_\Omega\int_\Gamma (u(x)-u(y)-u(x))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x\\
\leqslant&2\operatorname*{ess\,sup}_{x\in\Omega}\int_\Gamma \gamma(y,x)\,\mathrm{d}y\int_\Omega u^2(x)\,\mathrm{d}x + 2\int_\Omega\int_{\mathbb{R}^d}(u(x)-u(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x\\
\leqslant&2\max\{\operatorname*{ess\,sup}_{x\in\Omega}\int_\Gamma\gamma(y,x)\,\mathrm{d}y, 1\}\|u\|^2_{\mathrm{V}(\Omega;\gamma)}.
\end{aligned}$$

Let $c \in \mathrm{L}^2(\Gamma;w)$. Then, we have

$$\|\operatorname{Ext}(c)\|^2_{\mathrm{V}(\Omega;\gamma)} = \int_\Omega\int_{\mathbb{R}^d}(c(y)\chi_\Gamma(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x = \|c\|_{\mathrm{L}^2(\Gamma;w)}.$$

□

Corollary 6.3.
Let the assumptions of either Theorem 6.1 or Theorem 6.2 be satisfied and set $w\colon \Gamma \to (0,\infty]$ accordingly. Then, a measurable function $g\colon \Gamma \to \mathbb{R}$ satisfies the continuous functional condition if $\frac{g}{\sqrt{w}} \in \mathrm{L}^2(\Gamma)$ holds, in other words if we have

$$\left\|\frac{g}{\sqrt{w}}\right\|^2_{\mathrm{L}^2(\Gamma)} = \int_\Gamma \frac{g^2(y)}{w(y)}\mathrm{d}y < \infty.$$

Moreover, if we assume

$$\operatorname*{ess\,inf}_{y\in\Gamma} w(y) > 0,$$

then every $g \in \mathrm{L}^2(\Gamma)$ satisfies the continuous functional condition. In particular, if we have

$$0 < \operatorname*{ess\,inf}_{y\in\Gamma} w(y) \leqslant \operatorname*{ess\,sup}_{y\in\Gamma} w(y) < \infty,$$

then we obtain $u|_{\Omega\cup\Gamma} \in \mathrm{L}^2(\Omega\cup\Gamma)$ for all $u \in \mathrm{V}(\Omega;\gamma)$.

Proof.

By the Hölder inequality, we have, for all $u \in \mathrm{V}(\Omega;\gamma)$

$$\int_\Gamma g(y)u(y)\mathrm{d}y \leqslant \left\|\frac{g}{\sqrt{w}}\right\|_{\mathrm{L}^2(\Gamma)}\|\mathrm{Tr}(u)\|_{\mathrm{L}^2(\Gamma;w)}.$$

The rest is a direct consequence of the bounds of w and either Theorem 6.1 or Theorem 6.2. □

Recalling the characterization of the local trace space (see [19, Theorem 18.40]), we obtain:

Theorem 6.4.
Let $\Omega \subset \mathbb{R}^d$ be a bounded, nonempty, open subset, and $\gamma \in \mathcal{K}$ such that for a.e. $x \in \Omega$, we have

$$\int_\Gamma \gamma(z,x)\,\mathrm{d}z < \infty.$$

Furthermore, let $c \in [0,\infty)$ satisfy $\operatorname*{ess\,inf}_{x\in\Omega} \int_\Gamma \gamma(y,x)\,\mathrm{d}y + c > 0$ and define the function $w\colon \Gamma \to (0,\infty]$ by

$$w(y) = \int_\Omega \frac{\gamma(y,x)}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c}\,\mathrm{d}x.$$

Finally, set $\mathrm{W}(\Gamma;\gamma) := \{u : \Gamma \to \mathbb{R}$ measurable with $\|u\|_{\mathrm{W}(\Gamma;\gamma)} < \infty\}$ where

$$\|u\|^2_{\mathrm{W}(\Gamma;\gamma)} := \int_\Gamma u^2(y)w(y)\,\mathrm{d}y + \int_\Gamma\int_\Gamma (u(y)-u(z))^2 \int_\Omega \frac{\gamma(y,x)\gamma(z,x)}{\int_\Gamma \gamma(s,x)\,\mathrm{d}s + c}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Then:

(i) $\mathrm{Tr}\colon \mathrm{V}(\Omega;\gamma) \to \mathrm{W}(\Gamma;\gamma)$, $\mathrm{Tr}(v) = v|_\Gamma$ *is a linear operator such that there is a $C > 0$ with*

$$\|\mathrm{Tr}(u)\|_{\mathrm{W}(\Gamma;\gamma)} \leqslant C\|u\|_{\mathrm{V}(\Omega;\gamma)} \text{ for } u \in \mathrm{V}(\Omega;\gamma).$$

(ii) $\mathrm{E}\colon \mathrm{W}(\Gamma;\gamma) \to \mathrm{L}^2(\Omega)$,

$$\mathrm{E}(v) = \int_\Gamma v(y)\frac{\gamma(y,\cdot)}{\int_\Gamma \gamma(s,\cdot)\,\mathrm{d}s + c}\,\mathrm{d}y$$

is a linear operator such that there is a $C > 0$ with

$$\|\mathrm{E}(v)\|^2_{\mathrm{L}^2(\Omega)} + \int_\Omega\int_\Gamma (E(v)(x) - v(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \leqslant C\|v\|^2_{\mathrm{W}(\Gamma;\gamma)}$$

for $v \in \mathrm{W}(\Gamma;\gamma)$.

(iii) If there is a $C > 0$ with

$$\int_\Omega\int_\Omega (E(v)(x) - E(v)(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \leqslant C\|v\|^2_{\mathrm{W}(\Gamma;\gamma)} \quad \text{for } v \in \mathrm{W}(\Gamma;\gamma),$$

then Tr *is surjective.*

Proof.

Because $\int_\Gamma w(y)\,\mathrm{d}y \leqslant \lambda(\Omega)$ holds, w is finite almost everywhere. Now, let $u \in \mathrm{V}(\Omega;\gamma)$. Then, we have already shown in Theorem 6.1 that

$$\|\mathrm{Tr}(u)\|^2_{\mathrm{L}^2(\Gamma;w)} = \int_\Gamma u^2(y)w(y)\,\mathrm{d}y \leqslant 2\max\left\{\frac{1}{\operatorname*{ess\,inf}_{x\in\Omega}\int_\Gamma \gamma(y,x)\,\mathrm{d}y + c}, 1\right\}\|u\|^2_{\mathrm{V}(\Omega;\gamma)}.$$

By using Jensen's inequality and Fubini's Theorem, we get

$$\begin{aligned}
&\int_\Gamma \int_\Gamma (u(y)-u(z))^2 \int_\Omega \frac{\gamma(y,x)\gamma(z,x)}{\int_\Gamma \gamma(s,x)\,\mathrm{d}s + c}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
=&\int_\Gamma \int_\Gamma \int_\Omega (u(y)-u(x)+u(x)-u(z))^2 \frac{\gamma(y,x)\gamma(z,x)}{\int_\Gamma \gamma(s,x)\,\mathrm{d}s + c}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
\leqslant&2\int_\Gamma \int_\Omega (u(y)-u(x))^2 \frac{\gamma(y,x)\int_\Gamma \gamma(z,x)\,\mathrm{d}z}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c}\,\mathrm{d}x\,\mathrm{d}y \\
&+2\int_\Gamma \int_\Omega (u(y)-u(x))^2 \frac{\gamma(z,x)\int_\Gamma \gamma(y,x)\,\mathrm{d}y}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c}\,\mathrm{d}x\,\mathrm{d}z \\
\leqslant&4\|u\|^2_{\mathrm{V}(\Omega;\gamma)}.
\end{aligned}$$

Now, let $v \in \mathrm{W}(\Omega;\gamma)$. Then, the Hölder inequality yields

$$\begin{aligned}
\|\,\mathrm{E}(v)\|^2_{\mathrm{L}^2(\Omega)} &= \int_\Omega \left(\int_\Gamma v(y) \frac{\gamma(y,x)}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c}\,\mathrm{d}y \right)^2 \mathrm{d}x \\
&\leqslant \int_\Gamma v^2(y) \int_\Omega \frac{\gamma(y,x)}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega \int_\Gamma \left(E(v)(x) - v(y)\right)^2 \gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
=&\int_\Omega \int_\Gamma \left(\int_\Gamma v(s) \frac{\gamma(s,x)}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c}\,\mathrm{d}s - v(y) \frac{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c} \right)^2 \gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
\leqslant&\int_\Omega \int_\Gamma 2 \left(\int_\Gamma v(s) - v(y) \frac{\gamma(s,x)}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c}\,\mathrm{d}s \right)^2 \gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
&+ \int_\Omega \int_\Gamma 2v^2(y) \frac{c^2 \gamma(y,x)}{(\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c)^2}\,\mathrm{d}y\,\mathrm{d}x \\
\leqslant&2\int_\Gamma \int_\Gamma (v(s)-v(y))^2 \int_\Omega \frac{\gamma(s,x)\gamma(y,x)}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c}\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y \\
&+2\int_\Gamma v^2(y) \int_\Omega \frac{\gamma(y,x)}{(c^{-1}\int_\Gamma \gamma(z,x)\,\mathrm{d}z + 1)^2}\,\mathrm{d}x\,\mathrm{d}y \\
\leqslant&2\int_\Gamma \int_\Gamma (v(s)-v(y))^2 \int_\Omega \frac{\gamma(s,x)\gamma(y,x)}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c}\,\mathrm{d}x\,\mathrm{d}s\,\mathrm{d}y \\
&+2\int_\Gamma v^2(y) \int_\Omega \frac{c\,\gamma(y,x)}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

□

Corollary 6.5.
Let $\Omega \subset \mathbb{R}^d$ be a bounded, nonempty, open subset, and $\gamma \in \mathcal{K}$ such that for a.e. $x \in \Omega$, we have

$$\int_\Gamma \gamma(z,x)\,\mathrm{d}z < \infty$$

and such that there is a $c \in [0,\infty)$ *with* $\operatorname{ess\,inf}_{x\in\Omega} \int_\Gamma \gamma(z,x)\,\mathrm{d}z + c > 0$,

$$\begin{aligned} &\operatorname*{ess\,sup}_{x\in\Omega} \int_\Omega \int_\Gamma \frac{(k(s,x)-k(s,y))^2}{k(s,x)+k(s,y)} \chi_{\{k(s,x)+k(s,y)>0\}}\,\mathrm{d}s\,\gamma(y,x)\,\mathrm{d}y < \infty, \\ \textit{and} \quad &\operatorname*{ess\,sup}_{y\in\Omega} \int_\Omega \int_\Gamma \frac{(k(s,x)-k(s,y))^2}{k(s,x)+k(s,y)} \chi_{\{k(s,x)+k(s,y)>0\}}\,\mathrm{d}s\,\gamma(y,x)\,\mathrm{d}x < \infty \end{aligned}$$

where $k(s,x) = \dfrac{\gamma(s,x)}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c}$ *for* $(s,x) \in \Gamma\times\Omega$. *Define the function* $w\colon \Gamma \to (0,\infty]$ *by*

$$w(y) := \int_\Omega k(y,x)\,\mathrm{d}x = \int_\Omega \frac{\gamma(y,x)}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c}\,\mathrm{d}x$$

and set $\mathrm{W}(\Gamma;\gamma) := \{u : \Gamma \to \mathbb{R}$ *measurable with* $\|u\|_{\mathrm{W}(\Gamma;\gamma)} < \infty\}$ *where*

$$\|u\|^2_{\mathrm{W}(\Gamma;\gamma)} := \int_\Gamma u^2(y)w(y)\,\mathrm{d}y + \int_\Gamma \int_\Gamma (u(y)-u(z))^2 \int_\Omega \frac{\gamma(y,x)\gamma(z,x)}{\int_\Gamma \gamma(s,x)\,\mathrm{d}s + c}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Then, $\mathrm{Tr}\colon \mathrm{V}(\Omega;\gamma) \to \mathrm{W}(\Gamma;\gamma)$, $\mathrm{Tr}(v) = v|_\Gamma$ *is a bounded, linear, and surjective operator.*

Proof.

Due to Theorem 6.4, it remains to show that there is a $C > 0$ with

$$\int_\Omega \int_\Omega (E(v)(x) - E(v)(y))^2 \gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \leqslant C\|v\|_{\mathrm{W}(\Gamma;\gamma)}.$$

First, we mention that k is nonnegative. Hence, we obtain for a.e. $x,y \in \Omega$ and a.e. $s \in \Gamma$ that $k(s,x)+k(s,y) = 0$ holds if and only if $k(s,x) = k(s,y) = 0$ is satisfied. Choose $C > 0$ with

$$\begin{aligned} &\operatorname*{ess\,sup}_{x\in\Omega} \int_\Omega \int_\Gamma \frac{(k(s,x)-k(s,y))^2}{k(s,x)+k(s,y)} \chi_{\{k(s,x)+k(s,y)>0\}}\,\mathrm{d}s\,\gamma(y,x)\,\mathrm{d}y < \frac{C}{2}, \\ \text{and} \quad &\operatorname*{ess\,sup}_{y\in\Omega} \int_\Omega \int_\Gamma \frac{(k(s,x)-k(s,y))^2}{k(s,x)+k(s,y)} \chi_{\{k(s,x)+k(s,y)>0\}}\,\mathrm{d}s\,\gamma(y,x)\,\mathrm{d}x < \frac{C}{2}. \end{aligned}$$

Let $v \in \mathrm{W}(\Gamma;\gamma)$. Then, we get by Hölder's inequality

$$\begin{aligned} &(E(v)(x) - E(v)(y))^2 \\ &= \left(\int_\Gamma v(s) \left(\frac{\gamma(s,x)}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c} - \frac{\gamma(s,y)}{\int_\Gamma \gamma(z,y)\,\mathrm{d}z + c}\right)\mathrm{d}s\right)^2 \\ &= \left(\int_\Gamma v(s)\sqrt{\frac{k(s,x)+k(s,y)}{k(s,x)+k(s,y)}}(k(s,x)-k(s,y))\chi_{\{k(s,x)+k(s,y)>0\}}\,\mathrm{d}s\right)^2 \\ &\leqslant \int_\Gamma v^2(t)(k(t,x)+k(t,y))\,\mathrm{d}t \int_\Gamma \frac{(k(s,x)-k(s,y))^2}{k(s,x)+k(s,y)}\chi_{\{k(s,x)+k(s,y)>0\}}\,\mathrm{d}s \end{aligned}$$

for a.e. $x, y \in \Omega$. Therefore, we conclude

$$\begin{aligned}
&\int_\Omega \int_\Omega (E(v)(x) - E(v)(y))^2 \gamma(y,x) \,\mathrm{d}y \,\mathrm{d}x \\
\leqslant &\int_\Gamma v^2(t) \int_\Omega k(t,x) \int_\Omega \int_\Gamma \frac{(k(s,x)-k(s,y))^2}{k(s,x)+k(s,y)} \chi_{\{k(s,x)+k(s,y)>0\}} \,\mathrm{d}s \gamma(y,x) \,\mathrm{d}y \,\mathrm{d}x \,\mathrm{d}t \\
&+ \int_\Gamma v^2(t) \int_\Omega k(t,y) \int_\Omega \int_\Gamma \frac{(k(s,x)-k(s,y))^2}{k(s,x)+k(s,y)} \chi_{\{k(s,x)+k(s,y)>0\}} \,\mathrm{d}s \gamma(y,x) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}t \\
\leqslant &C \int_\Gamma v^2(s) \int_\Omega \frac{\gamma(s,x)}{\int_\Gamma \gamma(z,x) \,\mathrm{d}z + c} \,\mathrm{d}x \,\mathrm{d}s \\
\leqslant &C \|v\|^2_{\mathrm{W}(\Gamma;\gamma)}.
\end{aligned}$$

□

Corollary 6.6.
Let $\Omega \subset \mathbb{R}^d$ be a bounded, convex, nonempty, open subset, and $\gamma \in \mathcal{K}$ such that for a.e. $x \in \Omega$, we have

$$\int_\Gamma \gamma(z,x) \,\mathrm{d}x < \infty$$

and such that $\frac{\gamma(y,\cdot)}{\int_\Gamma \gamma(z,\cdot) \,\mathrm{d}z}$ is differentiable a.e. in Ω for a.e. $y \in \Gamma$. Furthermore, let there be a measurable function $\varphi \colon \mathbb{R}^d \to [0,\infty]$ and a constant $k > 0$ with $\gamma(y,x) \leqslant k\varphi(y-x)$ for a.e. $x, y \in \Omega$ and

$$\int_{\mathbb{R}^d} \min\{1, \|z\|^2\} \varphi(z) \,\mathrm{d}z < \infty.$$

Let $c \in [0,\infty)$ satisfy $\operatorname*{ess\,inf}_{x\in\Omega} \int_\Gamma \gamma(y,x) \,\mathrm{d}y + c > 0$, define the function $w \colon \Gamma \to (0,\infty]$ by

$$w(y) = \int_\Omega \frac{\gamma(y,x)}{\int_\Gamma \gamma(z,x) \,\mathrm{d}z + c} \,\mathrm{d}x$$

and set $\mathrm{W}(\Gamma;\gamma) := \{u : \Gamma \to \mathbb{R}$ measurable with $\|u\|_{\mathrm{W}(\Gamma;\gamma)} < \infty\}$ where

$$\|u\|^2_{\mathrm{W}(\Gamma;\gamma)} := \int_\Gamma u^2(y) w(y) \,\mathrm{d}y + \int_\Gamma \int_\Gamma (u(y)-u(z))^2 \int_\Omega \frac{\gamma(y,x)\gamma(z,x)}{\int_\Gamma \gamma(s,x) \,\mathrm{d}s + c} \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z.$$

If there is a constant $C > 0$ with

$$\|\frac{\partial}{\partial x}\left(\frac{\gamma(y,\cdot)}{\int_\Gamma \gamma(z,\cdot) \,\mathrm{d}z + c}\right)\| \leqslant C \left(\frac{\gamma(y,\cdot)}{\int_\Gamma \gamma(z,\cdot) \,\mathrm{d}z + c}\right) \quad \text{a.e. in } \Omega \text{ for a.e. } y \in \Gamma,$$

then $\mathrm{Tr} \colon \mathrm{V}(\Omega;\gamma) \to \mathrm{W}(\Gamma;\gamma)$, $\mathrm{Tr}(v) = v|_\Gamma$ is a bounded, linear, and surjective operator.

Proof.

For a.e. $s \in \Gamma$ and a.e. $x, y \in \Omega$, we get

$$\begin{aligned}
&\left| \frac{\gamma(s,x)}{\int_\Gamma \gamma(z,x) \,\mathrm{d}z + c} - \frac{\gamma(s,y)}{\int_\Gamma \gamma(z,y) \,\mathrm{d}z + c} \right| \\
= &\left| \int_{(0,1)} \langle \frac{\partial}{\partial x} \frac{\gamma(s,x+t(y-x))}{\int_\Gamma \gamma(z,x+t(y-x)) \,\mathrm{d}z + c}, y-x \rangle \,\mathrm{d}t \right| \\
\leqslant &C \int_{(0,1)} \frac{\gamma(s,x+t(y-x))}{\int_\Gamma \gamma(z,x+t(y-x)) \,\mathrm{d}z + c} \|y-x\| \,\mathrm{d}t.
\end{aligned}$$

Let $v \in \mathrm{W}(\Gamma;\gamma)$. By the Hölder inequality, we estimate

$$\begin{aligned}&\int_\Omega\int_\Omega (E(v)(x)-E(v)(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x\\ \leqslant &C\int_\Gamma v^2(s)\int_\Omega\int_\Omega\int_{(0,1)}\frac{\gamma(s,x+t(y-x))}{\int_\Gamma\gamma(z,x+t(y-x))\,\mathrm{d}z+c}\|y-x\|^2\gamma(y,x)\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Because Ω is bounded, there is a $R>1$ with $\|x-y\|\leqslant R$ for all $x,y\in\Omega$ and substitution yields for a.e $s\in\Gamma$

$$\begin{aligned}&\int_\Omega\int_\Omega\int_{(0,1)}\frac{\gamma(s,x+t(y-x))}{\int_\Gamma\gamma(z,x+t(y-x))\,\mathrm{d}z+c}\|y-x\|^2\gamma(y,x)\,\mathrm{d}t\,\mathrm{d}y\,\mathrm{d}x\\ \leqslant &k\int_\Omega\frac{\gamma(s,a)}{\int_\Gamma\gamma(z,a)\,\mathrm{d}z+c}\int_\Omega\int_{(0,1)}\min\{\|\frac{a-x}{t}\|^2,R\}\varphi(\frac{a-x}{t})\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}a\\ \leqslant &kR\int_{\mathbb{R}^d}\min\{1,\|x\|^2\}\varphi(x)\,\mathrm{d}z\int_\Omega\frac{\gamma(s,a)}{\int_\Gamma\gamma(z,a)\,\mathrm{d}z+c}\,\mathrm{d}a.\end{aligned}$$

All in all, we conclude there is a $\alpha>0$ with

$$\int_\Omega\int_\Omega (E(v)(x)-E(v)(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x\leqslant\alpha\|v\|^2_{\mathrm{W}(\Gamma;\gamma)}\text{ for all }v\in\mathrm{W}(\Gamma;\gamma).$$

□

Remark 6.7.
The definition of $w\colon\Gamma\to(0,\infty]$ and $\mathrm{W}(\Gamma;\gamma)$ in Theorem 6.1 can be generalized. Let $\widetilde{\Omega}\subset\Omega$ be open and let a measurable functions $\alpha\in\mathrm{L}^\infty(\widetilde{\Omega})$ satisfy $\operatorname{ess\,inf}_{x\in\widetilde{\Omega}}\left(\int_\Gamma\gamma(z,x)\,\mathrm{d}z+\alpha(x)\right)>0$. Instead of assuming that $\int_\Gamma\gamma(z,x)\,\mathrm{d}z<\infty$ holds for a.e. $x\in\Omega$, we just assume that this inequality holds for a.e. on $\widetilde{\Omega}$. Then, our statements remain true if we define $w\colon\Gamma\to(0,\infty]$ by

$$w(y)=\int_{\widetilde{\Omega}}\frac{\gamma(y,x)}{\int_\Gamma\gamma(z,x)\,\mathrm{d}z+\alpha(x)}\,\mathrm{d}x\qquad\text{for } y\in\Gamma$$

and if we set $\mathrm{W}(\Gamma;\gamma):=\{u:\Gamma\to\mathbb{R}$ measurable with $\|u\|_{\mathrm{W}(\Gamma,\gamma)}<\infty\}$ where

$$\|u\|_{\mathrm{W}(\Gamma;\gamma)}:=\int_\Gamma u^2(y)w(y)\,\mathrm{d}y+\int_\Gamma\int_\Gamma(u(y)-u(z))^2\int_{\widetilde{\Omega}}\frac{\gamma(y,x)\gamma(z,x)}{\int_\Gamma\gamma(s,x)\,\mathrm{d}s+\alpha(x)}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Furthermore, we highlight that we assume $\Omega\subset\mathbb{R}^d$ to be bounded so that w is integrable on Γ, and therefore, a.e. finite. Even if $\lambda(\Omega)=\infty$ holds, Theorem 6.1 and Theorem 6.4 remain valid if w is finite a.e. on Γ. This is for example the case if $\int_\Omega\gamma(y,x)\,\mathrm{d}x<\infty$ holds for a.e. $y\in\Gamma$.

Theorem 6.8.
Let $\Omega\subset\mathbb{R}^d$ be a bounded, nonempty, open subset, and $\gamma\in\mathcal{K}$ such that for a.e. $x\in\Omega$, we have

$$\int_\Gamma\gamma(z,x)\,\mathrm{d}x<\infty.$$

For a given $c\in(0,\infty)$, define the function $w\colon\Gamma\to(0,\infty]$ by

$$w(y)=\int_\Omega\frac{\gamma(y,x)}{\int_\Gamma\gamma(z,x)\,\mathrm{d}z+c}\,\mathrm{d}x.$$

Furthermore, let $\Gamma_0\subset\Gamma$ be measurable. Then, $\mathrm{Tr}|_{\Gamma_0}\colon\mathrm{V}(\Omega;\gamma)\to\mathrm{L}^2(\Gamma_0,w)$, $u\mapsto u|_{\Gamma_0}$ is a bounded linear operator and $\{u\in\mathrm{V}(\Omega;\gamma)$ such that $\mathrm{Tr}|_{\Gamma_0}u=0\}$ is a closed subspace of $\mathrm{V}(\Omega;\gamma)$ with respect to $\|\cdot\|_{\mathrm{V}(\Omega;\gamma)}$.

Proof.

Because of Theorem 6.1, we see that $\mathrm{Tr}|_{\Gamma_0}$ is a bounded and linear operator. Now, let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $\{u \in \mathrm{V}(\Omega;\gamma) \colon \mathrm{Tr}|_{\Gamma_0} u = 0\}$ converging to $v \in \mathrm{V}(\Omega;\gamma)$ with respect to $\|\cdot\|_{\mathrm{V}(\Omega;\gamma)}$. Then

$$\|\mathrm{Tr}|_{\Gamma_0}(v)\|_{\mathrm{L}^2(\Gamma_0;w)} = \|\mathrm{Tr}|_{\Gamma_0}(v - v_n)\|_{\mathrm{L}^2(\Gamma_0;w)} \leqslant \|v - v_n\|_{\mathrm{V}(\Omega;\gamma)} \to 0, \quad \text{for } n \to \infty.$$

Therefore, $v(y)w(y) = 0$ holds for a.e. $y \in \Gamma_0$ and because w is positive on $\Gamma_0 \subset \Gamma$, we get $\mathrm{Tr}|_{\Gamma_0}(v) = 0$. $\square$

Chapter 7

A Regional Problem

In this section, we consider the nonlocal Robin problem

$$\begin{cases} \mathcal{L}u(x) = f(x) & \text{for } x \in \Omega, \\ \alpha(y)u(y)w(y) + (1-\alpha(y))\mathcal{N}u(y) = g(y)w(y) & \text{for } y \in \widehat{\Gamma}, \end{cases} \tag{RP}$$

for $\gamma \in \mathcal{K}$. Recalling Definition 4.1 and Theorem 2.1, we define our weak solution as follows.

Definition 7.1.
Let $\Omega \subset \mathbb{R}^d$ be bounded, nonempty, and open and let $\gamma \in \mathcal{K}$ be symmetric with $\int_\Gamma \gamma(x,y)\,\mathrm{d}y < \infty$ for a.e. $x \in \Omega$. Set

$$w\colon \Gamma \to [0,\infty),\ w(y) = \int_\Omega \frac{\gamma(y,x)}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c}\,\mathrm{d}x$$

where the constant $c \geqslant 0$ satisfies $\operatorname{ess\,inf}_{x\in\Omega} \int_\Gamma \gamma(z,x)\,\mathrm{d}z + c > 0$. *Furthermore, let $\alpha\colon \Gamma \to [0,1]$ be measurable and set $\Gamma_1 := \{y \in \Gamma\colon \alpha(y) < 1\}$. Given measurable functions $f\colon \Omega \to \mathbb{R}$ and $g\colon \Gamma \to \mathbb{R}$, a function $u \in \{v \in \mathrm{V}(\Omega;\gamma)\colon v = g$ a.e. on $\Gamma \setminus \Gamma_1\}$ is called a weak solution to the nonlocal Robin problem* (RP) *if*

$$\int_\Omega f(x)v(x)\,\mathrm{d}x + \int_{\Gamma_1} \frac{g(y)\,v(y)\,w(y)}{1-\alpha(y)}\,\mathrm{d}y = \mathfrak{B}(u,v) + \int_{\Gamma_1} \frac{\alpha(y)\,u(y)\,v(y)\,w(y)}{1-\alpha(y)}\,\mathrm{d}y$$

holds for all $v \in \{v \in \mathrm{V}(\Omega;\gamma)\colon v = 0$ a.e. on $\Gamma \setminus \Gamma_1\}$.

Following the proof of Theorem 4.7, we can easily obtain sufficient assumptions for the existence of a weak solution. However, we want to present a different approach.

As first shown in [1] for the fractional Laplacian, it is possible to incorporate the following nonlocal Robin boundary condition

$$\alpha(y)u(y) + (1-\alpha(y))\left(\int_\Omega \frac{1}{\|x-y\|^{d+2s}}\,\mathrm{d}x\right)^{-1} \int_\Omega \frac{u(x)-u(y)}{\|x-y\|^{d+2s}}\,\mathrm{d}x = 0 \quad \text{for } y \in \Gamma, \tag{7.1}$$

into the nonlocal operator $\mathcal{L}$ by rearranging (7.1) into

$$u(y) = (1-\alpha(y))\left(\int_\Omega \frac{1}{\|x-y\|^{d+2s}}\,\mathrm{d}x\right)^{-1} \int_\Omega \frac{u(x)}{\|x-y\|^{d+2s}}\,\mathrm{d}x \quad \text{for } y \in \Gamma,$$

and then inserting this representation into $\mathcal{L}u$. In this section, we follow along the same lines and reformulate our nonlocal Robin problem (RP). For simplicity, we consider problem (P), i.e.,

$$\begin{cases} \mathcal{L}u(x) &= \int_{\mathbb{R}^d} u(x)\gamma(x,y) - u(y)\gamma(y,x)\,\mathrm{d}y &= f(x) \quad \text{for } x \in \Omega, \\ & \alpha(y)u(y) + (1-\alpha(y))\mathcal{N}u(y) &= g(y) \quad \text{for } y \in \widehat{\Gamma}, \end{cases} \tag{P}$$

to be our Robin problem. Note that boundary conditions on $\widehat{\Gamma} \setminus \Gamma$ are irrelevant in the evaluation of $\mathcal{L}u$ (see remark 4.10).

Theorem 7.2.
Let $\Omega \subset \mathbb{R}^d$ be an open, nonempty subset, and $\gamma \in \mathcal{K}$ with

$$\begin{aligned} \|\gamma(\cdot,x)\|_{\mathrm{L}^\infty(\widehat{\Gamma})} + \int_{\widehat{\Gamma}} \gamma(x,y)\,\mathrm{d}y < \infty \quad &\text{for a.e. } x \in \Omega \\ \text{and} \quad \|\gamma(\cdot,y)\|_{\mathrm{L}^\infty(\Omega)} + \int_{\Omega} \gamma(y,x)\,\mathrm{d}x < \infty \quad &\text{for a.e. } y \in \widehat{\Gamma}. \end{aligned}$$

Furthermore, let the measurable function $u\colon \mathbb{R}^d \to \mathbb{R}$ satisfy

$$\int_\Omega |u(x)|\,\mathrm{d}x < \infty \quad \text{and} \quad \int_\Omega \int_{\mathbb{R}^d} |u(x)\gamma(x,y) - u(y)\gamma(y,x)|\,\mathrm{d}y\,\mathrm{d}x < \infty.$$

Let $\alpha\colon \widehat{\Gamma} \to [0,1]$ be measurable and $g \in \mathrm{L}^1(\widehat{\Gamma})$ be given such that for all $y \in \widehat{\Gamma}$

$$\alpha(y)u(y) - (1-\alpha(y)) \int_\Omega u(x)\gamma(x,y) - u(y)\gamma(y,x)\,\mathrm{d}x = g(y) \tag{RBC}$$

holds. Then, defining $\gamma_\alpha\colon \Omega \times \Omega \to [0,\infty]$ by

$$\gamma_\alpha(x,z) = \gamma(x,z) + \int_\Gamma \frac{(1-\alpha(y))\gamma(x,y)\gamma(y,z)}{(1-\alpha(y))\int_\Omega \gamma(y,v)\,\mathrm{d}v + \alpha(y)}\,\mathrm{d}y, \quad (x,z) \in \Omega \times \Omega,$$

we have for a.e. $x \in \Omega$ that

$$\begin{aligned} \mathcal{L}_\gamma u(x) &= \int_{\mathbb{R}^d} u(x)\gamma(x,y) - u(y)\gamma(y,x)\,\mathrm{d}y \\ &= \int_\Omega u(x)\gamma_\alpha(x,z) - u(z)\gamma_\alpha(z,x)\,\mathrm{d}z \\ &\quad + u(x)\left(\int_{\widehat{\Gamma}\setminus\Gamma} \gamma(x,y)\,\mathrm{d}y + \int_\Gamma \frac{\alpha(y)\gamma(x,y)}{(1-\alpha(y))\int_\Omega \gamma(y,z)\,\mathrm{d}z + \alpha(y)}\,\mathrm{d}y\right) \\ &\quad - \int_\Gamma \frac{g(y)\gamma(y,x)}{(1-\alpha(y))\int_\Omega \gamma(y,v)\,\mathrm{d}v + \alpha(y)}\,\mathrm{d}y. \end{aligned}$$

Proof.

First of all, by Fubini's Theorem, γ_α is measurable and for a.e. $x \in \Omega$, we see

$$0 \leqslant \int_\Omega \int_\Gamma \frac{(1-\alpha(y))\gamma(x,y)\gamma(y,z)}{(1-\alpha(y))\int_\Omega \gamma(y,v)\,\mathrm{d}v + \alpha(y)}\,\mathrm{d}y\,\mathrm{d}z \leqslant \int_\Gamma \frac{\int_\Omega \gamma(y,z)\,\mathrm{d}z}{\int_\Omega \gamma(y,v)\,\mathrm{d}v}\gamma(x,y)\,\mathrm{d}y < \infty.$$

So γ_α is finite a.e. on $\Omega \times \Omega$. Without loss of generality, we assume

$$\int_{\mathbb{R}^d} |u(x)\gamma(x,y) - u(y)\gamma(y,x)| \, \mathrm{d}y < \infty$$

for all $x \in \Omega$. Now, let $x \in \Omega$, and for $y \in \widehat{\Gamma}$ set

$$c(y) := \frac{1}{(1-\alpha(y)) \int_\Omega \gamma(y,v) \, \mathrm{d}v + \alpha(y)}.$$

Then, we have to show that

$$\begin{aligned}
\mathcal{L}_\gamma u(x) = &\int_\Omega u(x)\gamma_\alpha(x,z) - u(z)\gamma_\alpha(z,x) \, \mathrm{d}z \\
&+ u(x) \left(\int_{\widehat{\Gamma}\setminus\Gamma} \gamma(x,y) \, \mathrm{d}y + \int_\Gamma c(y)\alpha(y)\gamma(x,y) \, \mathrm{d}y \right) \\
&- \int_\Gamma c(y) g(y) \, \mathrm{d}y.
\end{aligned}$$

First, we observe

$$\begin{aligned}
&\int_\Omega |u(x)|\gamma(x,y) + |u(y)|\gamma(y,x) \, \mathrm{d}x \\
\leqslant &\|\gamma(\cdot,y)\|_{\mathrm{L}^\infty(\Omega)} \int_\Omega |u(x)| \, \mathrm{d}x + |u(y)| \int_\Omega \gamma(y,x) \, \mathrm{d}x < \infty, \quad \text{for } y \in \Gamma,
\end{aligned}$$

and by rearranging (RBC) we therefore obtain for $y \in \Gamma$

$$c(y) \left(g(y) + (1-\alpha(y)) \int_\Omega u(z)\gamma(z,y) \, \mathrm{d}z \right) = u(y). \tag{7.2}$$

Linearity of the integral yields

$$\begin{aligned}
\mathcal{L}_\gamma u(x) &= \int_\Omega u(x)\gamma(x,y) - u(y)\gamma(y,x) \, \mathrm{d}y + \int_{\widehat{\Gamma}} u(x)\gamma(x,y) - u(y)\gamma(y,x) \, \mathrm{d}y \\
&= \int_\Omega u(x)\gamma(x,y) - u(y)\gamma(y,x) \, \mathrm{d}y + \int_\Gamma u(x)\gamma(x,y) - u(y)\gamma(y,x) \, \mathrm{d}y \\
&\quad + u(x) \int_{\widehat{\Gamma}\setminus\Gamma} \gamma(x,y) \, \mathrm{d}y
\end{aligned}$$

and (7.2) yields

$$\begin{aligned}
&\int_\Gamma u(x)\gamma(x,y) - u(y)\gamma(y,x) \, \mathrm{d}y \\
= &\int_\Gamma c(y) \left(\frac{1}{c(y)} u(x)\gamma(x,y) - \left(g(y) + (1-\alpha(y)) \int_\Omega u(z)\gamma(z,y) \, \mathrm{d}z \right) \gamma(y,x) \right) \, \mathrm{d}y.
\end{aligned}$$

For $y \in \Gamma$, we have

$$\begin{aligned}
&\frac{1}{c(y)} u(x)\gamma(x,y) - \left((1-\alpha(y)) \int_\Omega u(z)\gamma(z,y) \, \mathrm{d}z \right) \gamma(y,x) \\
= &\left((1-\alpha(y)) \int_\Omega u(x)\gamma(x,y)\gamma(y,z) - u(z)\gamma(z,y)\gamma(y,x) \, \mathrm{d}z \right) + u(x)\alpha(y)\gamma(x,y).
\end{aligned}$$

Therefore, our statement is valid if

$$\begin{aligned}
&\int_\Gamma |c(y)g(y)\gamma(y,x)|\,\mathrm{d}y,\\
&\int_\Gamma c(y)\left((1-\alpha(y))\int_\Omega |u(x)\gamma(x,y)\gamma(y,z)-u(z)\gamma(z,y)\gamma(y,x)|\,\mathrm{d}z\right)\mathrm{d}y,\\
\text{and } &\int_\Gamma |c(y)\alpha(y)\gamma(x,y)|\,\mathrm{d}y
\end{aligned}$$

are all finite. Because of $\frac{1}{c(y)} \geqslant \min\{(1-\alpha(y))\int_\Omega \gamma(y,v)\,\mathrm{d}v,\ \alpha(y)\}$ for $y \in \Gamma$, we get

$$\int_\Gamma c(y)\alpha(y)\gamma(x,y)\,\mathrm{d}y \leqslant \int_\Gamma \gamma(x,y)\,\mathrm{d}y < \infty.$$

And since $\int_\Omega \int_\Gamma \frac{|g(y)|\gamma(y,x)}{\int_\Omega \gamma(y,v)\,\mathrm{d}v}\,\mathrm{d}y\,\mathrm{d}x = \int_\Gamma |g(y)|\mathrm{d}y < \infty$ holds, we obtain

$$\begin{aligned}
&\int_\Gamma |c(y)g(y)\gamma(y,x)|\,\mathrm{d}y\\
\leqslant &\int_\Gamma c(y)(1-\alpha(y))|g(y)|\gamma(y,x)\,\mathrm{d}y + \int_\Gamma c(y)\alpha(y)|g(y)|\gamma(y,x)\,\mathrm{d}y\\
\leqslant &\int_\Gamma \frac{|g(y)|\gamma(y,x)}{\int_\Omega \gamma(y,v)\,\mathrm{d}v}\,\mathrm{d}y + \int_\Gamma |g(y)|\gamma(y,x)\,\mathrm{d}y\\
\leqslant &\int_\Gamma \frac{|g(y)|\gamma(y,x)}{\int_\Omega \gamma(y,v)\,\mathrm{d}v}\,\mathrm{d}y + \int_\Gamma |g(y)|\,\mathrm{d}y \|\gamma(\cdot,x)\|_{\mathrm{L}^\infty(\Gamma)}\\
< &\infty.
\end{aligned}$$

Finally, we have for a.e. $(y,z) \in \Gamma \times \Omega$ that

$$\begin{aligned}
&|u(x)\gamma(x,y)\gamma(y,z) - u(z)\gamma(z,y)\gamma(y,x)|\\
=&|u(x)\gamma(x,y)\gamma(y,z) - u(y)\gamma(y,x)\gamma(y,z) + u(y)\gamma(y,x)\gamma(y,z) - u(z)\gamma(z,y)\gamma(y,x)|\\
\leqslant&|u(x)\gamma(x,y) - u(y)\gamma(y,x)|\,\gamma(y,z) + |u(y)\gamma(y,z) - u(z)\gamma(z,y)|\,\gamma(y,x)
\end{aligned}$$

and that

$$c(y)(1-a(y)) \leqslant \frac{1}{\int_\Omega \gamma(y,v)\,\mathrm{d}v}$$

holds. Due to

$$\begin{aligned}
&\int_\Omega \int_\Gamma c(y)\left((1-\alpha(y))\int_\Omega |u(x)\gamma(x,y)\gamma(y,z)-u(z)\gamma(z,y)\gamma(y,x)|\,\mathrm{d}z\right)\mathrm{d}y\,\mathrm{d}x\\
\leqslant&2\int_\Omega \int_\Gamma |u(x)\gamma(x,y)-u(y)\gamma(y,x)|\,\mathrm{d}y\,\mathrm{d}x\\
<&\infty,
\end{aligned}$$

we, without loss of generality, conclude

$$\int_\Gamma c(y)\left((1-\alpha(y))\int_\Omega |u(x)\gamma(x,y)\gamma(y,z)-u(z)\gamma(z,y)\gamma(y,x)|\,\mathrm{d}z\right)\mathrm{d}y < \infty.$$

Then, the rest follows then by Fubini's Theorem. □

We now exploit the result of Theorem 7.2 to reformulate the nonlocal Robin problem (RP) into the equivalent *regional problem*

$$\mathcal{L}_{\gamma_\alpha} u(x) + \gamma_{\alpha,\Omega}(x)u(x) = f(x) + g_\Gamma(x) \quad \text{for } x \in \Omega \tag{REG}$$

where for $(x, z) \in \Omega \times \Omega$, we set

$$\begin{aligned}
g_\Gamma(x) &:= \int_\Gamma \frac{g(y)\gamma(y,x)}{(1-\alpha(y))\int_\Omega \gamma(y,v)\,\mathrm{d}v + \alpha(y)}\,\mathrm{d}y,\\
\gamma_\alpha(x,z) &:= \gamma(x,z) + \int_\Gamma \frac{(1-\alpha(y))\gamma(x,y)\gamma(y,z)}{(1-\alpha(y))\int_\Omega \gamma(y,v)\,\mathrm{d}v + \alpha(y)}\,\mathrm{d}y,\\
\text{and}\quad \gamma_{\alpha,\Omega}(x) &:= \int_\Gamma \frac{\alpha(y)\gamma(x,y)}{(1-\alpha(y))\int_\Omega \gamma(y,z)\,\mathrm{d}z + \alpha(y)}\,\mathrm{d}y + \int_{\widehat{\Gamma}\setminus\Gamma} \gamma(x,y)\,\mathrm{d}y.
\end{aligned}$$

We note that the regional problem (REG) is of the form

$$\mathcal{L}_\eta u(x) + \lambda(x)u(x) = \widetilde{f}(x) \quad \text{for } x \in \Omega,$$

where $\eta \in \mathcal{K}$ vanishes identically outside $\Omega \times \Omega$ and both $\lambda\colon \Omega \to [0,\infty)$ and $\widetilde{f}\colon \Omega \to \mathbb{R}$ are measurable. We recapitulate:

Definition 7.3.
Let $\Omega \subset \mathbb{R}^d$ be an open, nonempty subset and $\gamma \in \mathcal{K}$ with $\widehat{\Gamma} = \Gamma$,

$$\begin{aligned}
\|\gamma(\cdot,x)\|_{\mathrm{L}^\infty(\Gamma)} + \int_\Gamma \gamma(x,y)\,\mathrm{d}y < \infty \quad &\textit{for } x \in \Omega\\
\textit{and}\quad \|\gamma(\cdot,y)\|_{\mathrm{L}^\infty(\Omega)} + \int_\Omega \gamma(y,x)\,\mathrm{d}x < \infty \quad &\textit{for } y \in \Gamma.
\end{aligned}$$

Let $f\colon \Omega \to \mathbb{R}$ and $g\colon \Gamma \to \mathbb{R}$ be given and for $(x,z) \in \Omega \times \Omega$, set

$$\begin{aligned}
g_\Gamma(x) &:= \int_\Gamma \frac{g(y)\gamma(y,x)}{(1-\alpha(y))\int_\Omega \gamma(y,v)\,\mathrm{d}v + \alpha(y)}\,\mathrm{d}y,\\
\gamma_\alpha(x,z) &:= \gamma(x,z) + \int_\Gamma \frac{(1-\alpha(y))\gamma(x,y)\gamma(y,z)}{(1-\alpha(y))\int_\Omega \gamma(y,v)\,\mathrm{d}v + \alpha(y)}\,\mathrm{d}y,\\
\textit{and}\quad \gamma_{\alpha,\Omega}(x) &:= \int_\Gamma \frac{\alpha(y)\gamma(x,y)}{(1-\alpha(y))\int_\Omega \gamma(y,z)\,\mathrm{d}z + \alpha(y)}\,\mathrm{d}y.
\end{aligned}$$

Then, we call a measurable function $u\colon \Omega \to \mathbb{R}$ regional solution *to the Robin problem* (P) *if u is a solution of* (REG).

We call a function $u \in \mathrm{V}(\Omega;\gamma_\alpha)$ weak regional solution *to the Robin problem* (P) *if u is a weak solution (in the sense of Definition 4.1) of the regional problem* (REG).

In Remark 4.4 and Theorem 4.11, well-posedness results regarding problem (REG) are given. Furthermore, for the time–dependent case the regional solution has also been studied by Cortazar et al. [4]. Because $\gamma_{\alpha,\Omega}$ is in general not an element of $\mathrm{L}^2(\Omega)$, we now study a slightly different test function space.

Theorem 7.4.
Let $\Omega \subset \mathbb{R}^d$ be an open, nonempty subset, $\lambda\colon \Omega \to [0,\infty)$ be a measurable function, and let $\gamma \in \mathcal{K}$ be regional. Then, $\{v \in \mathrm{V}(\Omega;\gamma)\colon \int_\Omega v^2(x)\lambda(x)\,\mathrm{d}x < \infty\}$ is a Hilbert space with respect to the inner product

$$\langle u, v\rangle_1 = \int_\Omega u(x)v(x)(1+\lambda(x))\,\mathrm{d}x + \int_\Omega\int_\Omega (u(x)-u(y))(v(x)-v(y))\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x.$$

Proof.

We show that the norm induced by the inner product is complete. So let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\{v \in \mathrm{V}(\Omega;\gamma)\colon \int_\Omega v^2(x)\lambda(x)\,\mathrm{d}x < \infty\}$. Then, $(u_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathrm{V}(\Omega;\gamma)$, and by Corollary 3.2 converges to $u \in \mathrm{V}(\Omega;\gamma)$ with respect to $\|\cdot\|_{\mathrm{V}(\Omega;\gamma)}$. Without loss of generality, we assume that u_n converges a.e. in Ω to u. Therefore, the Lemma of Fatou yields

$$\begin{aligned} &\int_\Omega u^2(x)\lambda(x)\,\mathrm{d}x \leqslant \liminf_{n\to\infty}\int_\Omega u_n^2(x)\lambda(x)\,\mathrm{d}x \leqslant \sup_{n\in\mathbb{N}}\int_\Omega u_n^2(x)\lambda(x)\,\mathrm{d}x < \infty \\ \text{and}\quad &\lim_{n\to\infty}\int_\Omega (u(x)-u_n(x))^2\lambda(x)\,\mathrm{d}x \leqslant \lim_{n\to\infty}\liminf_{m\to\infty}\int_\Omega (u_m(x)-u_n(x))^2\lambda(x)\,\mathrm{d}x = 0. \end{aligned}$$

All in all, we get that $\{v \in \mathrm{V}(\Omega;\gamma)\colon \int_\Omega v^2(x)\lambda(x)\,\mathrm{d}x < \infty\}$ is a Hilbert space with respect to the inner product

$$\langle u, v\rangle_1 = \int_\Omega u(x)v(x)(1+\lambda(x))\,\mathrm{d}x + \int_\Omega\int_\Omega (u(x)-u(y))(v(x)-v(y))\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x$$

for $u, v \in \{v \in \mathrm{V}(\Omega;\gamma)\colon \int_\Omega v^2(x)\lambda(x)\,\mathrm{d}x < \infty\}$. □

Finally, we compare our new test function space with $\mathrm{V}(\Omega;\gamma)$.

Theorem 7.5.
Let $\Omega \subset \mathbb{R}^d$ be an open, nonempty subset, and $\gamma \in \mathcal{K}$ satisfy $\widehat{\Gamma} = \Gamma$,

$$\begin{aligned} \|\gamma(\cdot,x)\|_{\mathrm{L}^\infty(\Gamma)} + \int_\Gamma \gamma(x,y)\,\mathrm{d}y < \infty \quad &\textit{for } x \in \Omega \\ \textit{and } \|\gamma(\cdot,y)\|_{\mathrm{L}^\infty(\Omega)} + \int_\Omega \gamma(y,x)\,\mathrm{d}x < \infty \quad &\textit{for } y \in \Gamma. \end{aligned}$$

Furthermore, let $\alpha\colon \Gamma \to [0,1]$ be measurable and set

$$\mathrm{V}_{1-\alpha}(\Omega;\gamma) := \{v \in \mathrm{V}(\Omega;\gamma_\alpha)\colon \int_\Omega v^2(x)\gamma_{\alpha,\Omega}(x)\,\mathrm{d}x < \infty\}$$

where

$$\begin{aligned} &\gamma_\alpha\colon \Omega\times\Omega \to [0,\infty),\ \gamma_\alpha(x,z) := \gamma(x,z) + \int_\Gamma \frac{(1-\alpha(y))\gamma(x,y)\gamma(y,z)}{(1-\alpha(y))\int_\Omega \gamma(y,v)\,\mathrm{d}v + \alpha(y)}\,\mathrm{d}y \\ \textit{and}\quad &\gamma_{\alpha,\Omega}\colon \Omega \to [0,\infty),\ \gamma_{\alpha,\Omega}(x) := \int_\Gamma \frac{\alpha(y)\gamma(x,y)}{(1-\alpha(y))\int_\Omega \gamma(y,z)\,\mathrm{d}z + \alpha(y)}\,\mathrm{d}y \end{aligned}$$

Then, the following assertions hold.

(i) *We have* $\gamma_1 = \gamma$ *in* $\Omega \times \Omega$ *and* $\gamma_{0,\Omega} = 0$ *in* Ω.

(ii) $\mathrm{V}_{1-\alpha}(\Omega;\gamma)$ *is a Hilbert space with respect to the inner product*

$$\begin{aligned}\langle u, v\rangle_{\mathrm{V}_{1-\alpha}(\Omega;\gamma)} := &\int_\Omega u(x)v(x)(1+\gamma_{\alpha,\Omega}(x))\,\mathrm{d}x\\ &+\int_\Omega\int_\Omega (u(x)-u(y))(v(x)-v(y))\gamma_\alpha(y,x)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

(iii) $\mathrm{V}_0(\Omega;\gamma)$ *and* $\{v \in \mathrm{V}(\Omega;\gamma)\colon \mathrm{Tr}(v) = 0\}$ *are isomorphic, that is, there exists a bijective and bounded operator* $E\colon \mathrm{V}_0(\Omega;\gamma) \to \{v \in \mathrm{V}(\Omega;\gamma)\colon \mathrm{Tr}(v) = 0\}$ *with*

$$\langle u, v\rangle_{\mathrm{V}_0(\Omega;\gamma)} = \langle Eu, Ev\rangle_{\mathrm{V}(\Omega;\gamma)}, \qquad \textit{for } u, v \in \mathrm{V}_0(\Omega;\gamma).$$

(iv) *There is a constant* $C > 0$ *such that for any* $u \in \mathrm{V}_1(\Omega;\gamma)$, *there exists* $\widetilde{u} \in \mathrm{V}(\Omega;\gamma)$ *with* $u = \widetilde{u}$ *in* Ω *and*

$$\|\widetilde{u}\|_{\mathrm{V}(\Omega;\gamma)} \leqslant C\|u\|_{\mathrm{V}_1(\Omega;\gamma)}.$$

(v) *There is a constant* $C > 0$ *such that for any* $u \in \mathrm{V}(\Omega;\gamma)$, *we have* $u|_\Omega \in \mathrm{V}_1(\Omega;\gamma)$ *and*

$$\|u|_\Omega\|_{\mathrm{V}_1(\Omega;\gamma)} \leqslant C\|u\|_{\mathrm{V}(\Omega;\gamma)}.$$

Proof.

While (i) follows by definition, we obtain (ii) as a consequence of Theorem 7.4.

In order to show (iii), we define the zero extension operator outside Ω by

$$E\colon \mathrm{V}_0(\Omega;\gamma) \to \{v \in \mathrm{V}(\Omega;\gamma)\colon \mathrm{Tr}(v) = 0\},\ Eu(x) = \begin{cases} u(x) & \text{for } x \in \Omega,\\ 0 & \text{for } x \in \Gamma.\end{cases}$$

Then, E is a bijective operator with

$$\begin{aligned}&\langle u, v\rangle_{\mathrm{V}_0(\Omega;\gamma)}\\ =&\int_\Omega u(x)v(x)(1+\gamma_{1,\Omega}(x))\,\mathrm{d}x + \int_\Omega\int_\Omega (u(x)-u(y))(v(x)-v(y))\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x\\ =&\int_\Omega Eu(x)Ev(x)\,\mathrm{d}x + \int_\Omega\int_{\mathbb{R}^d} (Eu(x)-Eu(y))(Ev(x)-Ev(y))\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x\\ =&\langle Eu, Ev\rangle_{\mathrm{V}(\Omega;\gamma)}\end{aligned}$$

for all $u, v \in \mathrm{V}_0(\Omega;\gamma)$. This implies (iii) and we now proceed by showing (iv). For $u \in \mathrm{V}_1(\Omega;\gamma)$, we define

$$\widetilde{u}(x) = \begin{cases} u(x) & \text{for } x \in \Omega,\\ \displaystyle\int_\Omega \frac{u(z)\gamma(z,x)}{\int_\Omega \gamma(v,x)\,\mathrm{d}v}\,\mathrm{d}z & \text{for } x \in \Gamma.\end{cases}$$

Because for a.e. $x \in \Gamma$, we obtain

$$\int_\Omega \frac{|u(z)|\gamma(z,x)}{\int_\Omega \gamma(v,x)\,\mathrm{d}v}\,\mathrm{d}z \leqslant \int_\Omega \frac{u^2(z)\gamma(z,x)}{\int_\Omega \gamma(v,x)\,\mathrm{d}v}\,\mathrm{d}z \leqslant \frac{\|u\|_{\mathrm{L}^2(\Omega)}\|\gamma(\cdot,x)\|_{\mathrm{L}^\infty(\Omega)}}{\int_\Omega \gamma(v,x)\,\mathrm{d}v} < \infty,$$

the extension $\widetilde{u}$ is well defined. By Hölder's inequality and Fubini's Theorem, we see that

$$\begin{aligned}
&\int_\Omega \int_\Gamma (\widetilde{u}(x) - \widetilde{u}(y))^2 \gamma(y,x) \,\mathrm{d}y \,\mathrm{d}x \\
=& \int_\Omega \int_\Gamma \left(u(x) - \int_\Omega \frac{u(z)\gamma(z,y)}{\int_\Omega \gamma(v,y)\,\mathrm{d}v} \,\mathrm{d}z \right)^2 \gamma(y,x) \,\mathrm{d}y \,\mathrm{d}x \\
=& \int_\Omega \int_\Gamma \left(\int_\Omega \frac{(u(x) - u(z))\gamma(z,y)}{\int_\Omega \gamma(v,y)\,\mathrm{d}v} \,\mathrm{d}z \right)^2 \gamma(y,x) \,\mathrm{d}y \,\mathrm{d}x \\
\leqslant& \int_\Omega \int_\Omega (u(x) - u(z))^2 \int_\Gamma \frac{\gamma(y,x)\gamma(z,y)}{\int_\Omega \gamma(v,y)\,\mathrm{d}v} \,\mathrm{d}y \,\mathrm{d}z \,\mathrm{d}x
\end{aligned}$$

holds and, therefore, also

$$\begin{aligned}
\|\widetilde{u}\|^2_{\mathrm{V}(\Omega;\gamma)} &= \int_\Omega \widetilde{u}^2(x) \,\mathrm{d}x + \int_\Omega \int_{\mathbb{R}^d} (\widetilde{u}(x) - \widetilde{u}(y))^2 \gamma(y,x) \,\mathrm{d}y \,\mathrm{d}x \\
&\leqslant \int_\Omega u^2(x) \,\mathrm{d}x + \int_\Omega \int_\Omega (u(x) - u(y))^2 \gamma_0(y,x) \,\mathrm{d}y \,\mathrm{d}x = \|u\|^2_{\mathrm{V}_1(\Omega;\gamma)}.
\end{aligned}$$

For $v \in \mathrm{V}(\Omega;\gamma)$, we have by Jensen's inequality

$$\begin{aligned}
&\int_\Omega \int_\Omega (v(x) - v(z))^2 \int_\Gamma \frac{\gamma(y,x)\gamma(z,y)}{\int_\Omega \gamma(v,y)\,\mathrm{d}v} \,\mathrm{d}y \,\mathrm{d}z \,\mathrm{d}x \\
=& \int_\Omega \int_\Omega \int_\Gamma (v(x) - v(y) + v(y) - v(z))^2 \frac{\gamma(y,x)\gamma(z,y)}{\int_\Omega \gamma(v,y)\,\mathrm{d}v} \,\mathrm{d}y \,\mathrm{d}z \,\mathrm{d}x \\
\leqslant& 2 \int_\Omega \int_\Gamma (v(x) - v(y))^2 \gamma(y,x) \,\mathrm{d}y \,\mathrm{d}x
\end{aligned}$$

such that $\|v\|_{\mathrm{V}_1(\Omega;\gamma)} \leqslant 2\|v\|_{\mathrm{V}(\Omega;\gamma)}$. □

Chapter 8

Nonlocal Energy-based Coupling

This section aims to study the (energy-based) coupling of $n \in \mathbb{N}$ nonlocal problems by following the approach of Capodaglio et al. [10]. However, we first show that the weak solution of Problem (NP) solves a minimization problem.

For a bounded, nonempty and open set $\Omega \subset \mathbb{R}^d$, $\Lambda \in \mathrm{V}(\Omega;\gamma)^*$, and a symmetric $\gamma \in \mathcal{K}$ find a $u \in \mathrm{V}(\Omega;\gamma)$ such that

$$\mathcal{E}(u;\Omega,\gamma,\Lambda) \leqslant \mathcal{E}(v;\Omega;\gamma,\Lambda)$$

holds for all $v \in \mathrm{V}(\Omega;\gamma)$ where $\mathcal{E}(\cdot;\Omega,\gamma,\Lambda)\colon \mathrm{V}(\Omega;\gamma) \to \mathbb{R}$ is the nonlocal energy of the system given by

$$\mathcal{E}(v;\Omega,\gamma,\Lambda) := \frac{1}{2}\mathfrak{B}_\gamma(v,v) - \Lambda(v).$$

The following theorem connects this minimization problem to problem (NP).

Theorem 8.1.
Let $\Omega \subset \mathbb{R}^d$ be a bounded, nonempty, and open subset, and $\gamma \in \mathcal{K}$ be symmetric. Moreover, let $f \in \mathrm{L}^2(\Omega)$, and $g\colon \Gamma \to \mathbb{R}$ and assume that a bounded linear operator $\Lambda\colon \mathrm{V}(\Omega;\gamma) \to \mathbb{R}$ is given by

$$\Lambda(v) = \int_\Omega f(x)v(x)\,\mathrm{d}x + \int_\Gamma g(y)v(y)\,\mathrm{d}y \quad \text{for } v \in \mathrm{V}(\Omega;\gamma).$$

Then, a function in $\mathrm{V}(\Omega;\gamma)$ is a weak solution of the problem (NP) if and if only it is a stationary point of $\mathcal{E}(\cdot;\Omega,\gamma,\Lambda)$

Furthermore, every stationary point of $\mathcal{E}(\cdot;\gamma,\Lambda)$ minimizes $\mathcal{E}(\cdot;\Omega,\gamma,\Lambda)$ and every minimizer of $\mathcal{E}(\cdot;\Omega,\gamma,\Lambda)$ is a stationary point of $\mathcal{E}(\cdot;\Omega,\gamma,\Lambda)$.

Proof.

Let $u, v \in \mathrm{V}(\Omega;\gamma)$ be given arbitrarily. We have

$$\begin{aligned}\mathcal{E}(u-v;\Omega;\gamma,\Lambda) &= \frac{1}{2}\left(\mathfrak{B}_\gamma(u,u) - 2\mathfrak{B}_\gamma(u,v) + \mathfrak{B}_\gamma(v,v)\right) - \Lambda(u-v)\\ &= \mathcal{E}(u;\Omega;\gamma,\Lambda) + \frac{1}{2}\mathfrak{B}_\gamma(v,v) + \Lambda(v) - \mathfrak{B}_\gamma(u,v)\\ &\geqslant \mathcal{E}(u;\Omega;\gamma,\Lambda) + \Lambda(v) - \mathfrak{B}_\gamma(u,v).\end{aligned}$$

The Gateaux derivative of $\mathcal{E}(\cdot;\Omega,\gamma,\Lambda)$ at u in the direction v is given by

$$\begin{aligned}\mathrm{d}(\mathcal{E}(\cdot;\Omega,\gamma,\Lambda))(u;v) &:= \lim_{t\to 0}\frac{1}{t}\left(\mathcal{E}(u-t\,v;\Omega,\gamma,\Lambda) - \mathcal{E}(u;\Omega,\gamma,\Lambda)\right)\\ &= \lim_{t\to 0} t\mathfrak{B}_\gamma(v,v) + \Lambda(v) - \mathfrak{B}_\gamma(u,v)\\ &= \Lambda(v) - \mathfrak{B}_\gamma(u,v).\end{aligned}$$

By comparing $t \downarrow 0$ and $t \uparrow 0$ in $\mathrm{d}(\mathcal{E}(\cdot;\Omega,\gamma,\Lambda))$, we see that every minimizer of $\mathcal{E}(\cdot;\Omega,\gamma,\Lambda)$ is a stationary point. Recalling that $u \in \mathrm{V}(\Omega;\gamma)$ is a weak solution if we have

$$\mathfrak{B}_\gamma(u,v) = \int_\Omega f(x)v(x)\,\mathrm{d}x + \int_\Gamma g(y)v(y)\,\mathrm{d}y \quad \text{for all } v \in \mathrm{V}(\Omega;\gamma),$$

we obtain our statement. □

Instead of decomposing $\mathcal{E}(\cdot;\gamma,\Lambda)$ into an equivalent energy functional of coupled systems, we will consider the energy functional of coupled systems and reformulate it into a joint problem.

For the remainder of this section, let $n \in \mathbb{N}$ be given arbitrarily. For all $i = 1,\ldots,n$, let $\Omega_i \subset \mathbb{R}^d$ be bounded, nonempty, and open sets and let $\gamma_i \in \mathcal{K}$ be symmetric in $\Omega_i \times \Omega_i$. For $i = 1,\ldots,n$, define the nonlocal boundary of Ω_i with respect to γ_i as

$$\Gamma_i := \{y \in \mathbb{R}^d \setminus \Omega_i \colon \int_{\Omega_i} \gamma_i(y,x)\,\mathrm{d}x > 0\}$$

and set

$$\Omega := \bigcup_{i=1}^n \Omega_i, \quad \Gamma := \bigcup_{i=1}^n \Gamma_i \setminus \Omega \qquad \text{and} \quad \widetilde{\Gamma}_i := \Gamma_i \cap \Omega.$$

Further, set, for $i = 1,\ldots,n$,

$$\begin{aligned}\mathcal{L}_i u(x) &:= \int_{\mathbb{R}^d} (u(x)-u(y))\gamma_i(y,x)\,\mathrm{d}y && \text{for } x \in \Omega_i,\\ \text{and} \quad \mathcal{N}_i u(y) &:= \int_{\Omega_i} (u(y)-u(x))\gamma_i(y,x)\,\mathrm{d}x && \text{for } y \in \Gamma_i.\end{aligned}$$

Then, we consider the following coupled nonlocal problem

$$\begin{cases} \sum_{i=1}^n \Big((\mathcal{L}_i u)\chi_{\Omega_i} + (\mathcal{N}_i u)\chi_{\Gamma_i}\Big) = f & \text{on } \Omega,\\ \sum_{i=1}^n \Big((\mathcal{N}_i u)\chi_{\Gamma_i}\Big) = g & \text{on } \Gamma. \end{cases} \tag{C}$$

If we set

$$\gamma(y,x) := \sum_{i=1}^{n} \left(\gamma_i(y,x)\chi_{\mathbb{R}^d\times\Omega_i}(y,x) + \gamma_i(x,y)\chi_{\Omega_i\times\widetilde{\Gamma}_i}(y,x)\right) \quad \text{for } (y,x) \in \mathbb{R}^d \times \mathbb{R}^d,$$

then for $x \in \Omega$, we obtain

$$\begin{aligned}
&\mathcal{L}_\gamma u(x) \\
&= \int_{\mathbb{R}^d} (u(x)-u(y))\gamma(y,x)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d} (u(x)-u(y)) \left(\sum_{i=1}^{n} \gamma_i(y,x)\chi_{\Omega_i}(x)\right) \mathrm{d}y + \int_{\mathbb{R}^d} (u(x)-u(y)) \left(\sum_{i=1}^{n} \gamma_i(x,y)\chi_{\Omega_i\times\widetilde{\Gamma}_i}(y,x)\right) \mathrm{d}y \\
&= \sum_{i=1}^{n} \left(\int_{\mathbb{R}^d} (u(x)-u(y))\gamma_i(y,x)\chi_{\Omega_i}(x)\,\mathrm{d}y\right) + \sum_{i=1}^{n} \left(\int_{\Omega_i} (u(x)-u(y))\gamma_i(x,y)\chi_{\widetilde{\Gamma}_i}(x)\,\mathrm{d}y\right) \\
&= \sum_{i=1}^{n} \left((\mathcal{L}_i u(x))\chi_{\Omega_i}(x) + (\mathcal{N}_i u(x))\chi_{\Gamma_i}(x)\right)
\end{aligned}$$

and for $y \in \Gamma$, we get

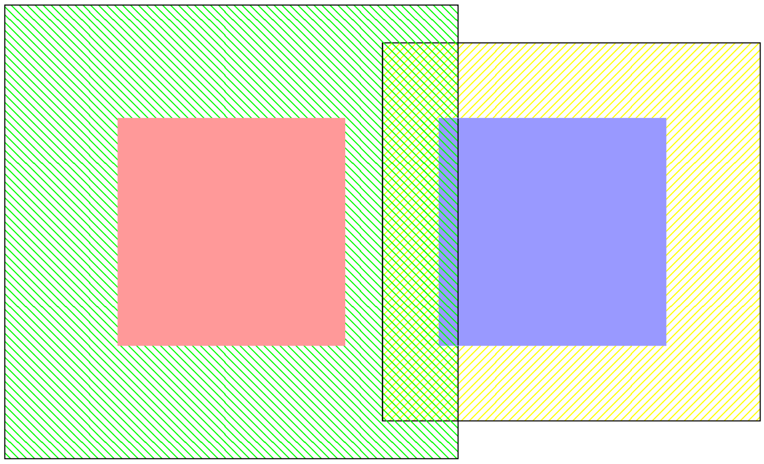
$$\begin{aligned}
\mathcal{N}_\gamma u(y) &= \int_{\Omega} (u(y)-u(x))\gamma(y,x)\,\mathrm{d}x \\
&= \int_{\Omega} (u(y)-u(x)) \left(\sum_{i=1}^{n} (\gamma_i(y,x)\chi_{\mathbb{R}^d\times\Omega_i}(y,x)\right) \mathrm{d}x \\
&= \sum_{i=1}^{n} \left(\int_{\Omega_i} (u(y)-u(x))\gamma_i(y,x)\,\mathrm{d}x\right) \chi_{\Gamma_i}(y) \\
&= \sum_{i=1}^{n} (\mathcal{N}_i u(y))\chi_{\Gamma_i}(y).
\end{aligned}$$

Hence, problem (C) can be reformulated into

$$\begin{cases} \mathcal{L}u &= f \quad \text{on } \Omega, \\ \mathcal{N}u &= g \quad \text{on } \Gamma, \end{cases}$$

Remark 8.2.
We now briefly interpret Problem (C) *in the case of $n = 2$, i.e.,*

$$\begin{cases}
\mathcal{L}_1 u(x) + \mathcal{N}_2 u(x)\chi_{\Gamma_2}(x) &= f(x), \quad \textit{for } x \in \Omega_1, \\
\mathcal{L}_2 u(x) + \mathcal{N}_1 u(x)\chi_{\Gamma_1}(x) &= f(x), \quad \textit{for } x \in \Omega_2, \\
\mathcal{N}_1 u(y) &= 0, \quad \textit{for } y \in \Gamma_1 \setminus (\Omega_2 \cup \Gamma_2), \\
\mathcal{N}_2 u(y) &= 0, \quad \textit{for } y \in \Gamma_2 \setminus (\Omega_1 \cup \Gamma_1), \\
\mathcal{N}_1 u(y) + \mathcal{N}_2 u(y) &= 0, \quad \textit{for } y \in \Gamma_1 \cap \Gamma_2.
\end{cases}$$



Coupled problem

In the coupled system, not only the diffusion $\mathcal{L}_1 u$ is relevant in Ω_1 but also the flux from Ω_2 to Ω_1 with respect to γ_2, meaning that $\mathcal{N}_2 u$ is potentially of significance in Ω_1. Roughly said, for $x \in \widehat{\Gamma}_2 \subset \Omega_1$, the total diffusion in x is the diffusion with respect to γ_1 in x added with the nonlocal flux potentially coming from Ω_2 to x according to γ_2, so the total diffusion is given by $\mathcal{L}_1 u(x) + \mathcal{N}_2 u(x)\chi_{\Gamma_2}(x)$. In the same manner, we have to consider the total diffusion in Ω_2. Finally, Γ are the points outside of Ω, where we only have to consider the flux from Ω.

Lemma 8.3.
Let $n \in \mathbb{N}$. For all $i = 1, \dots, n$ let $\Omega_i \subset \mathbb{R}^d$ be bounded, nonempty, and open and let $\gamma_i \in \mathcal{K}$ be symmetric in $\Omega_i \times \Omega_i$. Set $\Omega = \bigcup_{i=1}^n \Omega_i$ and for $(y,x) \in \mathbb{R}^d \times \mathbb{R}^d$, define

$$\gamma(y,x) := \sum_{i=1}^n \big(\gamma_i(y,x)\chi_{\mathbb{R}^d \times \Omega_i}(y,x) + \gamma_i(x,y)\chi_{\Omega_i \times \widetilde{\Gamma}_i}(y,x)\big).$$

Further, set

$$\begin{aligned}\mathfrak{B}_i(u,v) := &\frac{1}{2}\int_{\Omega_i}\int_{\Omega_i}(u(x)-u(y))(v(x)-v(y))\gamma_i(y,x)\,\mathrm{d}y\,\mathrm{d}x \\ &+ \int_{\Omega_i}\int_{\Gamma_i}(u(x)-u(y))(v(x)-v(y))\gamma_i(y,x)\,\mathrm{d}y\,\mathrm{d}x,\end{aligned}$$

for $i = 1, \dots, n$ and $u, v \in \mathrm{V}(\Omega_i;\gamma_i)$. Then, the norms $\|\cdot\|_{\mathrm{V}(\Omega;\gamma)}$ and $\sum_{i=1}^n \|\cdot\|_{\mathrm{V}(\Omega_i;\gamma_i)}$ are equivalent in $\mathrm{V}(\Omega;\gamma)$. Furthermore, we obtain

$$\begin{aligned}\mathfrak{B}(u,v) := &\frac{1}{2}\int_{\Omega}\int_{\Omega}(u(x)-u(y))(v(x)-v(y))\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\ &+ \int_{\Omega}\int_{\Gamma}(u(x)-u(y))(v(x)-v(y))\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\ = &\sum_{i=1}^n \mathfrak{B}_i(u,v)\end{aligned}$$

for all $u, v \in \mathrm{V}(\Omega;\gamma)$.

Proof.

For all $i = 1, \dots, n$, the definition of Γ_i and $\widetilde{\Gamma}_i$ yield

$$\begin{aligned}\int_{\Omega_i}\int_{\Gamma_i\setminus\Omega}(u(x)-u(y))^2\gamma_i(y,x)\,\mathrm{d}y\,\mathrm{d}x &= \int_{\Omega_i}\int_{\mathbb{R}^d\setminus\Omega}(u(x)-u(y))^2\gamma_i(y,x)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{\Omega_i}\int_{\Gamma}(u(x)-u(y))^2\gamma_i(y,x)\,\mathrm{d}y\,\mathrm{d}x\end{aligned} \tag{8.1}$$

and

$$\begin{aligned}&\int_{\Omega_i}\int_{\widetilde{\Gamma}_i}(u(x)-u(y))^2\gamma_i(y,x)\,\mathrm{d}y\,\mathrm{d}x + \int_{\Omega_i}\int_{\Omega_i}(u(x)-u(y))^2\gamma_i(y,x)\,\mathrm{d}y\,\mathrm{d}x \\ =&\int_{\Omega_i}\int_{\Omega}(u(x)-u(y))^2\gamma_i(y,x)\,\mathrm{d}y\,\mathrm{d}x\end{aligned} \tag{8.2}$$

holds. Then, we can conclude by Fubini's Theorem, (8.1), and (8.2) that

$$\begin{aligned}&\sum_{i=1}^{n}\mathfrak{B}_i(u,u) \\ =&\sum_{i=1}^{n}\Big(\frac{1}{2}\int_{\Omega_i}\int_{\Omega_i}(u(x)-u(y))^2\gamma_i(y,x)\,\mathrm{d}y\,\mathrm{d}x \\ &+\frac{1}{2}\int_{\Omega_i}\int_{\widetilde{\Gamma}_i}(u(x)-u(y))^2\gamma_i(y,x)\,\mathrm{d}y\,\mathrm{d}x \\ &+\frac{1}{2}\int_{\widetilde{\Gamma}_i}\int_{\Omega_i}(u(x)-u(y))^2\gamma_i(y,x)\,\mathrm{d}x\,\mathrm{d}y \\ &+\int_{\Omega_i}\int_{\Gamma_i\setminus\Omega}(u(x)-u(y))^2\gamma_i(y,x)\,\mathrm{d}y\,\mathrm{d}x\Big) \\ =&\sum_{i=1}^{n}\Big(\frac{1}{2}\int_{\Omega_i}\int_{\Omega}(u(x)-u(y))^2\gamma_i(y,x)\,\mathrm{d}y\,\mathrm{d}x \\ &+\frac{1}{2}\int_{\widetilde{\Gamma}_i}\int_{\Omega_i}(u(x)-u(y))^2\gamma_i(y,x)\,\mathrm{d}x\,\mathrm{d}y \\ &+\int_{\Omega_i}\int_{\Gamma}(u(x)-u(y))^2\gamma_i(y,x)\,\mathrm{d}y\,\mathrm{d}x\Big) \\ =&\frac{1}{2}\int_{\Omega}\int_{\Omega}(u(x)-u(y))^2\gamma(x,y)\,\mathrm{d}y\,\mathrm{d}x \\ &+\int_{\Omega}\int_{\Gamma}(u(x)-u(y))^2\gamma(x,y)\,\mathrm{d}y\,\mathrm{d}x\end{aligned}$$

holds for all $\in \mathrm{V}(\Omega;\gamma)$. Because of $\frac{1}{n}\sum_{i=1}^{n}\|u\|_{\mathrm{L}^2(\Omega_i)} \leqslant \|u\|_{\mathrm{L}^2(\Omega)} \leqslant \sum_{i=1}^{n}\|u\|_{\mathrm{L}^2(\Omega_i)}$ and (3.2), we see that our statement is valid. □

Therefore, for a given $\Lambda \in \mathrm{V}(\Omega;\gamma)^*$ we want to show there exist a function $u \in \mathrm{V}(\Omega;\gamma)$ such that

$$\mathcal{E}(u;\prod_{i=1}^{n}(\Omega_i,\gamma_i),\Lambda) \leqslant \mathcal{E}(v;\prod_{i=1}^{n}(\Omega_i,\gamma_i),\Lambda) \text{ holds for all } v \in \mathrm{V}(\Omega;\gamma) \tag{E}$$

where the nonlocal energy of the coupled system $\mathcal{E}(\cdot;\prod_{i=1}^{n}(\Omega_i,\gamma_i),\Lambda)\colon \mathrm{V}(\Omega;\gamma)\to\mathbb{R}$ is given by

$$\mathcal{E}(v;\prod_{i=1}^{n}(\Omega_i,\gamma_i),\Lambda)=\frac{1}{2}(\underbrace{\sum_{i=1}^{n}\mathfrak{B}_i(v,v)}_{=\mathfrak{B}(v,v)})-\Lambda(v).$$

As mentioned before, coupled nonlocal problems are also studied by Capodaglio et al. [10]. Furthermore, Glusa et al. [14] also analyze coupled nonlocal problems. While, both of them derive the same nonlocal energy of the coupled system

$$u\mapsto\int_{\Omega\cup\Omega_I}\int_{\Omega\cup\Omega_I}(u(y)-u(x))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x-\Lambda(u)$$

where

$$\Omega_I=\left\{y\in\mathbb{R}^d\setminus\Omega \text{ such that } \gamma(y,x)\neq 0 \text{ for some } x\in\Omega\right\},$$

their strong formulations are different. Given constants $\delta_1,\delta_2,C_{1,1},C_{2,1},C_{1,2},C_{2,2}>0$ the strong formulation in [10] is given by

$$\begin{cases} 2\displaystyle\int_{\Omega_1\cup\Gamma_C}(u(x)-u(y))\gamma_{1,1}(y,x)\,\mathrm{d}y+\int_{\Omega_2}(u(x)-u(y))\overline{\gamma_{1,2}}(y,x)\,\mathrm{d}y &= f(x)\quad \text{for } x\in\Omega_1,\\ 2\displaystyle\int_{\Omega_2}(u(x)-u(y))\gamma_{2,2}(y,x)\,\mathrm{d}y+\int_{\Omega_1}(u(x)-u(y))\overline{\gamma_{2,1}}(y,x)\,\mathrm{d}y &= f(x)\quad \text{for } x\in\Omega_2,\\ u(y) &= 0\qquad \text{for } y\in\Gamma_C, \end{cases}$$

where $\Omega_1,\Omega_2\subset\mathbb{R}^d$ are disjoint, open sets,

$$\Gamma_C=\{z\in\mathbb{R}^d\setminus(\Omega_1\cup\Omega_2)\colon \inf_{s\in\Omega_1}\|z-s\|\leqslant\delta_1\},$$

and for $y,x\in\mathbb{R}^d$, we have

$$\begin{aligned}
\gamma_{1,1}(y,x)&:=C_{1,1}\chi_{\mathrm{B}_{\delta_1}(x)}(y)\chi_{\Omega_1\cup\Gamma_C}(y)\chi_{\Omega_1\cup\Gamma_C}(x)\\
\gamma_{1,2}(y,x)&:=C_{1,2}\chi_{\mathrm{B}_{\delta_1}(x)}(y)\chi_{\Omega_2}(y)\chi_{\Omega_1\cup\Gamma_C}(x),\\
\gamma_{2,1}(y,x)&:=C_{2,1}\chi_{\mathrm{B}_{\delta_2}(x)}(y)\chi_{\Omega_1\cup\Gamma_C}(y)\chi_{\Omega_2}(x),\\
\gamma_{2,2}(y,x)&:=C_{2,2}\chi_{\mathrm{B}_{\delta_2}(x)}(y)\chi_{\Omega_2}(y)\chi_{\Omega_2}(x),\\
\overline{\gamma_{1,2}}(y,x)&:=\gamma_{1,2}(y,x)+\gamma_{2,1}(y,x),\\
\overline{\gamma_{2,1}}(y,x)&:=\gamma_{2,1}(y,x)+\gamma_{1,2}(y,x).
\end{aligned}$$

Note that $\gamma_{1,1}$ represents the interaction between $\Omega_1\cup\Gamma_C$ with $\Omega_1\cup\Gamma_C$, $\gamma_{1,2}$ represents the interaction between $\Omega_1\cup\Gamma_C$ with Ω_2, $\gamma_{2,1}$ represents the interaction between Ω_2 with $\Omega_1\cup\Gamma_C$, and $\gamma_{2,2}$ represents the interaction between Ω_2 with Ω_2.

Let now $\delta_1,\delta_2>0$ with $\delta_1<\delta_2$ be given. Then, the strong formulation in [14] is given by

$$\begin{cases} 2\displaystyle\int_{\mathbb{R}^d}(u(x)-u(y))\gamma_1(y,x)\,\mathrm{d}y &= f(x)\quad \text{for } x\in\Omega_1,\\ 2\displaystyle\int_{\mathbb{R}^d}(u(x)-u(y))\gamma_2(y,x)\,\mathrm{d}y &= f(x)\quad \text{for } x\in\Omega_2,\\ u(y) &= 0\qquad \text{for } y\in\Gamma,\\ (\mathcal{F}(u))(y) &= g(y)\quad \text{for } y\in\Gamma_G, \end{cases}$$

where $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ are disjoint, open sets, and $\gamma_1, \gamma_2, \gamma_1^J, \gamma_2^J \in \mathcal{K}$ such that for $y, x \in \mathbb{R}^d$

$$\gamma_i(y,x) = \gamma_i(y,x)\chi_{\mathrm{B}_{\delta_i}(y-x)} \text{ and } \gamma_i^J(y,x) = \gamma_i^j(y,x)\chi_{\mathrm{B}_{\delta_i}(y-x)}$$

holds for $i = 1, 2$. Further, we have

$$\Gamma_G := \widetilde{\Gamma}_1 \cup \widetilde{\Gamma}_i \cup \Omega_2^J$$

where $\Omega_2^J := \{y \in \Omega_2 \setminus \Gamma_1 \colon \|y - x\| \leqslant \delta_2 \text{ for some } x \in \widetilde{\Gamma}_2\}$. And the interface-flux operator $\mathcal{F}$ is given by

$$\begin{aligned}&(\mathcal{F}(u))(y)\\ &:= \begin{cases} 2\int_{\widetilde{\Gamma}_2}(u(y)-u(x))(\gamma_2^J(y,x)-\gamma_2(y,x))\,\mathrm{d}x & \text{for } y \in \widetilde{\Gamma}_1\\ 2\int_{\Omega_2^J}(u(y)-u(x))\gamma_2^J(y,x)\,\mathrm{d}x + 2\int_{\widetilde{\Gamma}_1}(u(y)-u(x))(\gamma_1^J(y,x)-\gamma_1(y,x))\,\mathrm{d}x & \text{for } y \in \widetilde{\Gamma}_2\\ 2\int_{\widetilde{\Gamma}_2}(u(x)u(y))(\gamma_2^J(y,x)-\gamma_2(y,x))\,\mathrm{d}y & \text{for } y \in \Omega_2^J. \end{cases}.\end{aligned}$$

In our case, by following the proof of Theorem 8.1, we see that for a given $\Lambda \in \mathrm{V}(\Omega;\gamma)^*$, every minimizer $u \in \mathrm{V}(\Omega;\gamma)$ of $\mathcal{E}(\cdot; \prod_{i=1}^n(\Omega_i,\gamma_i),\Lambda)$ satisfies

$$\mathfrak{B}(u,v) = \Lambda(v) \quad \text{for all } v \in \mathrm{V}(\Omega;\gamma).$$

Conversely, every function $u \in \mathrm{V}(\Omega;\gamma)$ solving $\mathfrak{B}(u,v) = \Lambda(v)$ for all $v \in \mathrm{V}(\Omega;\gamma)$ is a minimizer of $\mathcal{E}(\cdot; \prod_{i=1}^n(\Omega_i,\gamma_i),\Lambda)$.

Assuming that the nonlocal integration by parts formula is applicable for the minimizer $u \in \mathrm{V}(\Omega;\gamma)$ of $\mathcal{E}(\cdot; \prod_{i=1}^n(\Omega_i,\gamma_i),\Lambda)$ on each $\mathfrak{B}_i$, we get

$$\int_{\Omega_i} \mathcal{L}_i u(x)v(x)\,\mathrm{d}x + \int_{\Gamma_i} \mathcal{N}_i(y)v(y)\,\mathrm{d}y = \mathfrak{B}_i(u,v) \text{ for all } v \in \mathrm{V}(\Omega;\gamma).$$

Let $\Lambda \in \mathrm{V}(\Omega;\gamma)^*$ be of the form

$$\Lambda(v) = \int_{\Omega} f(x)v(x)\,\mathrm{d}x + \int_{\Gamma} g(y)v(y)\,\mathrm{d}y \text{ for } v \in \mathrm{V}(\Omega;\gamma)$$

where $f \in \mathrm{L}^2(\Omega)$ and $g \in \mathrm{L}^1(\Gamma)$. Then, we obtain that

$$\begin{aligned}&\sum_{i=1}^n \left(\int_{\Omega_i} \mathcal{L}_i u(x)v(x)\,\mathrm{d}x + \int_{\Gamma_i} \mathcal{N}_i u(y)v(y)\,\mathrm{d}y\right)\\ &= \int_{\Omega} f(x)v(x)\,\mathrm{d}x + \int_{\Gamma} g(y)v(y)\,\mathrm{d}y\end{aligned}$$

for all $v \in \mathrm{V}(\Omega;\gamma)$. By this equation, we see that the strong formulation of the minimization principle (E) is given by problem (C). Furthermore, following the proof of Theorem 4.5, we obtain that if

$$\int_{\Omega} f(x)\,\mathrm{d}x + \int_{\Gamma} g(y)\,\mathrm{d}y = 0$$

is valid and if the nonlocal Poincaré inequality holds on $\mathrm{V}(\Omega;\gamma)$, then there exists an up to an additive constant unique function $u \in \mathrm{V}(\Omega;\gamma)$ solving

$$\mathfrak{B}(u,v) = \Lambda(v) \quad \text{for all } v \in \mathrm{V}(\Omega;\gamma).$$

However, if the nonlocal Poincaré inequality holds on $\mathrm{V}(\Omega_i;\gamma_i)$ for all $i = 1, \ldots, n$, then we obtain the following result.

Theorem 8.4.
Let $n \in \mathbb{N}$ be given arbitrarily. For all $i = 1, \dots, n$ let $\Omega_i \subset \mathbb{R}^d$ be bounded, nonempty and open and let $\gamma_i \in \mathcal{K}$ be symmetric in $\Omega_i \times \Omega_i$. Set $\Omega = \bigcup_{i=1}^n \Omega_i$ and for $(y,x) \in \mathbb{R}^d \times \mathbb{R}^d$ define

$$\gamma(y,x) := \sum_{i=1}^n \big(\gamma_i(y,x)\chi_{\mathbb{R}^d \times \Omega_i}(y,x) + \gamma_i(x,y)\chi_{\Omega_i \times \widetilde{\Gamma}_i}(y,x)\big).$$

We assume that $(\Omega_i)_{i\in\mathbb{N}, i\leqslant n}$ are pairwise disjoint and that

$$\sum_{i=1}^n \int_{\Omega_i}\int_{\Gamma_i} \gamma_i(y,x)\,\mathrm{d}y\,\mathrm{d}x < \infty$$

holds. If $f \in \mathrm{L}^2(\Omega)$ satisfies

$$\int_{\Omega_i} f(x)\,\mathrm{d}x = 0$$

for $i = 1, \dots, n$ and if the Poincaré inequality holds on $\mathrm{V}(\Omega_i;\gamma_i)$ for each $i = 1, \dots, n$, then there is a $u \in \mathrm{V}(\Omega;\gamma)$ solving

$$\mathfrak{B}(u, v - \sum_{i=1}^n v_{\Omega_i}\chi_{\Omega_i}) = \int_\Omega f(x)v(x)\,\mathrm{d}x \quad \textit{for all } v \in \mathrm{V}(\Omega;\gamma)$$

where $v_{\Omega_i} := \frac{1}{\lambda(\Omega_i)}\int_{\Omega_i} v(x)\,\mathrm{d}x$ for $i = 1, \dots, n$.

Proof.

We recall that for all $u, v \in \mathrm{V}(\Omega;\gamma)$ we have

$$\begin{aligned}\mathfrak{B}(u,v) = \sum_{i=1}^n \bigg(&\frac{1}{2}\int_{\Omega_i}\int_{\Omega_i}(u(x)-u(y))(v(x)-v(y))\gamma_i(y,x)\,\mathrm{d}y \\ &+ \int_{\Omega_i}\int_{\Gamma_i}(u(x)-u(y))(v(x)-v(y))\gamma_i(y,x)\,\mathrm{d}y\,\mathrm{d}x\bigg).\end{aligned}$$

For any sequence $(a_i)_{i\in\mathbb{N}}$ in $\mathbb{R}$, we get

$$\left\|\sum_{i=1}^n a_i\chi_{\Omega_i}\right\|_{\mathrm{L}^2(\Omega)} \leqslant \sup_{j=1,\dots,n} a_j^2\lambda(\Omega) < \infty$$

and

$$\mathfrak{B}(\sum_{i=1}^n a_i\chi_{\Omega_i}, \sum_{i=1}^n a_i\chi_{\Omega_i}) \leqslant n \sup_{j=1,\dots,n} a_j^2 \sum_{i=1}^n \int_{\Omega_i}\int_{\Gamma_i}\gamma_i(y,x)\,\mathrm{d}y\,\mathrm{d}x < \infty.$$

Therefore, we introduce the space

$$\widehat{V}(\Omega;\gamma) := \left\{ s \in \mathrm{V}(\Omega;\gamma)\colon \int_{\Omega_i} s(x)\,\mathrm{d}x = 0 \text{ for all } i = 1, \dots, n\right\}.$$

Due to the Poincaré inequality, we see that $\widehat{V}(\Omega;\gamma)$ is a Hilbert space with respect to the inner product $\mathfrak{B}$. Consider the linear functional

$$\Lambda(v) = \int_\Omega f(x)v(x)\,\mathrm{d}x \qquad \text{on } \widehat{V}(\Omega;\gamma).$$

Because of the Cauchy–Schwarz inequality, Λ is bounded and, therefore, the Riesz representation theorem gives us a unique $u \in \widehat{V}(\Omega;\gamma)$ such that

$$\Lambda(v) = \mathfrak{B}(u,v) \quad \text{for all } v \in \widehat{V}(\Omega;\gamma).$$

For all $v \in \mathrm{V}(\Omega;\gamma)$, we conclude

$$\Lambda(v) = \Lambda(v - \sum_{i=1}^{n} v_{\Omega_i}\chi_{\Omega_i}) = \mathfrak{B}(u, v - \sum_{i=1}^{n} v_{\Omega_i}\chi_{\Omega_i}).$$

□

Chapter 9

Optimal Control Governed By Nonlocal Equations

In this section, we consider an optimal control problem governed by problem (NP). The case that the state equation is given by a PDE is discussed, for example, by Lions [20] or Tröltzsch [35]. We follow the approaches of Lions [20] and Tröltzsch [35] in order to study the minimization problem

$$\min J(f, g, u) := \frac{1}{2}\|u - z\|^2_{\mathrm{L}^2(\Omega_0)} + \frac{1}{2}\|u - z\|^2_{\mathrm{L}^2(\Gamma_0;w)} + \frac{\alpha}{2}\|f\|^2_{\mathrm{L}^2(\Omega)} + \frac{\beta}{2}\|g\|^2_{\mathrm{L}^2(\Gamma;w)},$$

subject to the state equation

$$\begin{cases} \mathcal{L}u + \tau u &= f \qquad \text{on } \Omega, \\ \mathcal{N}u + \tau u w &= g w \qquad \text{on } \Gamma, \end{cases} \tag{C1}$$

and the pointwise constraints

$$\begin{cases} \underline{f} \leqslant f \leqslant \overline{f} \qquad \text{on } \Omega, \\ \underline{g} \leqslant g \leqslant \overline{g} \qquad \text{on } \Gamma, \end{cases} \tag{C2}$$

under the assumptions

- Ω_0 is an open subset of the bounded and open set $\Omega \subset \mathbb{R}^d$,
- $\gamma \in \mathcal{K}$ is a symmetric kernel such that the nonlocal Poincaré inequality holds on $\mathrm{V}(\Omega; \gamma)$,
- Γ_0 is an open subset of the nonlocal boundary Γ,
- $\int_\Gamma \gamma(y, x)\,\mathrm{d}y < \infty$ holds for a.e. $x \in \Omega$,
- $\alpha, \beta \geqslant 0$ are constant,
- $z \in \mathrm{L}^2(\Omega_0 \cup \Gamma_0; \chi_{\Omega_0} + w\chi_{\Gamma_0})$ and $\tau \in \mathrm{L}^\infty(\Omega \cup \Gamma)$ with $\tau \geqslant 0$ are given,
- $\underline{f}, \overline{f} \colon \Omega \to \mathbb{R} \cup \{-\infty, \infty\}$ with $\underline{f}\chi_{\{-\infty<\underline{f}\}} \in \mathrm{L}^2(\Omega)$ and $\overline{f}\chi_{\{\overline{f}<\infty\}} \in \mathrm{L}^2(\Omega)$ satisfy $\underline{f} \leqslant \overline{f}$,
- $\underline{g}, \overline{g} \colon \Gamma \to \mathbb{R} \cup \{-\infty, \infty\}$with $\underline{g}\chi_{\{-\infty<\underline{g}\}} \in \mathrm{L}^2(\Gamma; w)$ and $\overline{g}\chi_{\{\overline{g}<\infty\}} \in \mathrm{L}^2(\Gamma; w)$ satisfy $\underline{g} \leqslant \overline{g}$.

Recall that $w\colon \Gamma \to (0,\infty]$ is defined by $w(y) = \int_\Omega \frac{\gamma(y,x)}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c}\,\mathrm{d}x$ for $y \in \Gamma$ where $c \geqslant 0$ is chosen such that $\operatorname{ess\,inf}_{x\in\Omega} \int_\Gamma \gamma(y,x)\,\mathrm{d}y + c > 0$ holds. Then,

$$\mathrm{Tr}\colon \mathrm{V}(\Omega;\gamma) \to \mathrm{L}^2(\Gamma;w), \quad v \mapsto v|_\Gamma$$

is a continuous linear operator according to Theorem 6.1. Further, we want to highlight that the kernel $\gamma \in \mathcal{K}$ satisfies

(i) γ is symmetric

(ii) the nonlocal Poincaré inequality holds on $\mathrm{V}(\Omega;\gamma)$,

(iii) $\int_\Gamma \gamma(y,x)\,\mathrm{d}y < \infty$ holds for a.e. $x \in \Omega$.

Assumption (iii) is required in order to invoke Theorem 6.1. Moreover, assumptions (i) and (ii) are necessary for an existence result. In particular, we assume $\gamma \in \mathcal{K}$ to be symmetric so that we obtain $\mathfrak{B} = \widehat{\mathfrak{B}}$. Let

$$\widetilde{\mathcal{K}} := \widetilde{\mathcal{K}}(\Omega) \subset \mathcal{K},$$

denote the set of kernels satisfying all of these three assumptions.

In our objective function

$$J(f,g,u) = \frac{1}{2}\|u-z\|^2_{\mathrm{L}^2(\Omega_0)} + \frac{1}{2}\|u-z\|^2_{\mathrm{L}^2(\Gamma_0;w)} + \frac{\alpha}{2}\|f\|^2_{\mathrm{L}^2(\Omega)} + \frac{\beta}{2}\|g\|^2_{\mathrm{L}^2(\Gamma;w)},$$

z represents the desired function, u is the state associated with the control function (f,g). The relation between the state u and the control function (f,g) is given by (C1) and $\underline{f}, \overline{f}, \underline{g}, \overline{g}$ define the sets of admissible controls. By following [35], we study this optimal control problem.

We begin with an investigation of (C1). First, we remark that (C1) must be satisfied in the weak variational sense, i.e., the state $u \in \mathrm{V}(\Omega;\gamma)$ must solve

$$\begin{aligned}\mathfrak{A}(u,v) :=& \mathfrak{B}(u,v) + \int_\Omega \tau(x)u(x)v(x)\,\mathrm{d}x + \int_\Gamma \tau(y)u(y)v(y)w(y)\,\mathrm{d}y \\ =& \int_\Omega f(x)v(x)\,\mathrm{d}x + \int_\Gamma g(y)v(y)w(y)\,\mathrm{d}y\end{aligned}$$

for all $v \in \mathrm{V}(\Omega;\gamma)$. In the case that $\tau = 0$ a.e. on $\Omega \cup \Gamma$, the controls must satisfy the compatibility condition

$$\int_\Omega f(x)v(x)\,\mathrm{d}x + \int_\Gamma g(y)v(y)w(y)\,\mathrm{d}y = 0.$$

In the case that $\|\tau\|_{\mathrm{L}^\infty(\Omega\cup\Gamma)} > 0$ holds, we require a generalization of the nonlocal Poincaré inequality and nonlocal Friedrich's inequality in order to get an existence result for a state satisfying (C1).

Lemma 9.1.
Let $\Omega \subset \mathbb{R}^d$ be a bounded, open, and nonempty set and let $\gamma \in \mathcal{K}$. Then, the following statements are equivalent.

(i) The nonlocal Poincaré inequality holds.

(ii) *For any measurable subset $E \subset \Omega$ with $\lambda(E) > 0$, there exists a constant $C_E > 0$ such that for all $v \in \mathrm{V}(\Omega;\gamma)$, we get*

$$\|v\|^2_{\mathrm{L}^2(\Omega)} \leqslant C_E \left(\mathfrak{B}_\gamma(v,v) + \left(\int_E v(x)\,\mathrm{d}x \right)^2 \right).$$

Proof.

Let $v \in \mathrm{V}(\Omega;\gamma)$ and let $E \subset \Omega$ be measurable with $\lambda(E) > 0$. Then, we have

$$\begin{aligned}
&\lambda(E)\|v\|^2_{\mathrm{L}^2(\Omega)} \\
=&\int_\Omega \int_E (v(x) - v(y) + v(y))^2 \,\mathrm{d}y\,\mathrm{d}x \\
\leqslant&2 \int_\Omega \int_\Omega (v(x) - v(y))^2 \,\mathrm{d}y\,\mathrm{d}x + 2\lambda(\Omega) \left(\int_E v^2(y)\,\mathrm{d}y \right) \\
=&2\mathfrak{B}_{\chi_{\Omega\times\Omega}}(v,v) + \frac{\lambda(\Omega)}{\lambda(E)} \left(\int_E \int_E v^2(y) - 2v(y)v(x) + v^2(x) + 2v(y)v(x)\,\mathrm{d}y\,\mathrm{d}x \right) \\
\leqslant&\left(2 + \frac{\lambda(\Omega)}{\lambda(E)}\right) \left(\mathfrak{B}_{\chi_{\Omega\times\Omega}}(v,v) + \left(\int_E v(x)\,\mathrm{d}x \right)^2 \right).
\end{aligned}$$

Consequently, the equivalence follows by Lemma 5.2. □

Lemma 9.2.
Let $\Omega \subset \mathbb{R}^d$ be a bounded, open, and nonempty set and let $\gamma \in \mathcal{K}$ satisfy $\int_\Gamma \gamma(y,x)\,\mathrm{d}x < \infty$ for a.e. $x \in \Omega$. Set $w\colon \Gamma \to (0,\infty]$, $w(y) = \int_\Omega \frac{\gamma(y,x)}{\int_\Gamma \gamma(z,x)\,\mathrm{d}z + c}\,\mathrm{d}x$ for $y \in \Gamma$ where $c \geqslant 0$ is chosen such that $\operatorname{ess\,inf}_{x\in\Omega} \int_\Gamma \gamma(y,x)\,\mathrm{d}y + c > 0$ *holds. If the nonlocal Poincaré inequality holds, then for every measurable subset $E \subset \Gamma$ with $\lambda(E) > 0$, there exists a constant $C_E > 0$ such that for all $v \in \mathrm{V}(\Omega;\gamma)$, we get*

$$\|v\|^2_{\mathrm{L}^2(\Omega)} \leqslant C_E \left(\mathfrak{B}(v,v) + \left(\int_E v(y)w(y)\,\mathrm{d}y \right)^2 \right).$$

Proof.

Let $v \in \mathrm{V}(\Omega;\gamma)$ and $E \subset \Gamma$ be measurable with $\lambda(E) > 0$. Then, we have

$$0 < \int_E w(y)\,\mathrm{d}y = \int_\Omega \int_E \frac{\gamma(y,z)}{\int_\Gamma \gamma(s,z)\,\mathrm{d}s + c}\,\mathrm{d}y\,\mathrm{d}z \leqslant \lambda(\Omega)$$

and, therefore,

$$\begin{aligned}
\int_E w(y)\,\mathrm{d}y \|v\|^2_{\mathrm{L}^2(\Omega)} =& \int_\Omega \int_\Omega \int_E v^2(x) \frac{\gamma(y,z)}{\int_\Gamma \gamma(s,z)\,\mathrm{d}s + c}\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x \\
\leqslant& 4 \int_\Omega \int_\Omega (v(x) - v(z))^2 \,\mathrm{d}z\,\mathrm{d}x \\
&+ 4\lambda(\Omega) \int_\Omega \int_E (v(z) - v(y))^2 \frac{\gamma(y,z)}{\int_\Gamma \gamma(s,z)\,\mathrm{d}s + c}\,\mathrm{d}y\,\mathrm{d}z \\
&+ 4\lambda(\Omega) \int_E v^2(y)w(y)\,\mathrm{d}y.
\end{aligned}$$

The last term is estimated by

$$\begin{aligned}
&2\int_E v^2(y)w(y)\,\mathrm{d}y\\
=&\frac{1}{\int_E w(s)\,\mathrm{d}s}\int_E\int_E (v^2(y)+v^2(t))w(y)w(t)\,\mathrm{d}y\,\mathrm{d}t\\
=&\frac{1}{\int_E w(s)\,\mathrm{d}s}\int_E\int_E ((v(y)-v(t))^2+2v(t)v(y))w(y)w(t)\,\mathrm{d}y\,\mathrm{d}t\\
=&\frac{1}{\int_E w(s)\,\mathrm{d}s}\left(\int_E\int_E (v(y)-v(t))^2 w(y)w(t)\,\mathrm{d}y\,\mathrm{d}t+2\left(\int_E v(t)w(t)\,\mathrm{d}t\right)^2\right).
\end{aligned}$$

We conclude

$$\begin{aligned}
&\int_E\int_E (v(y)-v(t))^2 w(y)w(t)\,\mathrm{d}y\,\mathrm{d}t\\
=&\int_E\int_E\int_\Omega\int_\Omega (v(y)-v(t))^2\frac{\gamma(y,z)}{\int_\Gamma\gamma(s,z)\,\mathrm{d}s+c}\frac{\gamma(t,x)}{\int_\Gamma\gamma(s,x)\,\mathrm{d}s+c}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}t\\
\leqslant&4\int_E\int_E\int_\Omega\int_\Omega (v(y)-v(z))^2\frac{\gamma(y,z)}{\int_\Gamma\gamma(s,z)\,\mathrm{d}s+c}\frac{\gamma(t,x)}{\int_\Gamma\gamma(s,x)\,\mathrm{d}s+c}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}t\\
&+4\int_E\int_E\int_\Omega\int_\Omega (v(z)-v(x))^2\frac{\gamma(y,z)}{\int_\Gamma\gamma(s,z)\,\mathrm{d}s+c}\frac{\gamma(t,x)}{\int_\Gamma\gamma(s,x)\,\mathrm{d}s+c}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}t\\
&+4\int_E\int_E\int_\Omega\int_\Omega (v(x)-v(t))^2\frac{\gamma(y,z)}{\int_\Gamma\gamma(s,z)\,\mathrm{d}s+c}\frac{\gamma(t,x)}{\int_\Gamma\gamma(s,x)\,\mathrm{d}s+c}\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}t\\
\leqslant&8\lambda(\Omega)\int_\Omega\int_E (v(z)-v(y))^2\frac{\gamma(y,z)}{\int_\Gamma\gamma(s,z)\,\mathrm{d}s+c}\,\mathrm{d}y\,\mathrm{d}z\\
&+4\int_\Omega\int_\Omega (v(x)-v(z))^2\,\mathrm{d}z\,\mathrm{d}x.
\end{aligned}$$

By the nonlocal Poincaré inequality, there is a constant $C>0$ independent of v with

$$\int_\Omega\int_\Omega (v(x)-v(z))^2\,\mathrm{d}z\,\mathrm{d}x\leqslant C\mathfrak{B}(v,v)$$

and, by the properties of the constant c, we see

$$\int_\Omega\int_E (v(z)-v(y))^2\frac{\gamma(y,z)}{\int_\Gamma\gamma(s,z)\,\mathrm{d}s+c}\,\mathrm{d}y\,\mathrm{d}z\leqslant\frac{1}{\operatorname{ess\,inf}_{x\in\Omega}\int_\Gamma\gamma(s,x)\,\mathrm{d}s+c}\mathfrak{B}(v,v).$$

Therefore, there exists a constant C_E with

$$\|v\|^2_{\mathrm{L}^2(\Omega)}\leqslant C_E\left(\mathfrak{B}(v,v)+\left(\int_E v(y)w(y)\,\mathrm{d}y\right)^2\right)\quad\text{for all } v\in\mathrm{V}(\Omega;\gamma).$$

□

With Lemma 9.1 and Lemma 9.2, we obtain a new existence results.

Theorem 9.3.
Let $\Omega \subset \mathbb{R}^d$ be a bounded, nonempty, and open subset and $\gamma \in \widetilde{\mathcal{K}}$. Furthermore, assume that the nonnegative function $\tau \in \mathrm{L}^\infty(\Omega \cup \Gamma)$ satisfies $\lambda(\{x \in \Omega \cup \Gamma \colon \tau(x) > 0\}) > 0$. Then, for any $(f, g) \in \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Gamma; w)$ the problem

$$\begin{cases} \mathcal{L}u + \tau u &= f \quad \text{on } \Omega, \\ \mathcal{N}u + \tau u w &= g w \quad \text{on } \Gamma, \end{cases}$$

has a unique weak solution, i.e., there is a unique $u \in \mathrm{V}(\Omega; \gamma)$ such that

$$\begin{aligned} &\mathfrak{B}(u, v) + \int_\Omega \tau(x) u(x) v(x) \,\mathrm{d}x + \int_\Gamma \tau(y) u(y) v(y) w(y) \,\mathrm{d}y \\ &= \int_\Omega f(x) v(x) \,\mathrm{d}x + \int_\Gamma g(y) v(y) w(y) \,\mathrm{d}y \end{aligned}$$

holds for all $v \in \mathrm{V}(\Omega; \gamma)$.

Proof.

We show that $\mathrm{V}(\Omega; \gamma)$ is a Hilbert space with respect to

$$\mathfrak{A}(u, v) := \mathfrak{B}(u, v) + \int_\Omega \tau(x) u(x) v(x) \,\mathrm{d}x + \int_\Gamma \tau(y) u(y) v(y) w(y) \,\mathrm{d}y \quad \text{for } u, v \in \mathrm{V}(\Omega; \gamma).$$

By definition, $\mathfrak{A}$ is a semi-inner-product and the Hölder inequality yields

$$|\mathfrak{A}(u, v)| \leqslant (1 + \|\tau\|_{\mathrm{L}^\infty(\Omega \cup \Gamma)}) \|u\|_{\mathrm{V}(\Omega;\gamma)} \|v\|_{\mathrm{V}(\Omega;\gamma)} \quad \text{for all } u, v \in \mathrm{V}(\Omega; \gamma).$$

By continuity, there is a $\delta > 0$ with $\lambda(\{x \in \Omega \cup \Gamma \colon \tau(x) > \delta\}) > 0$. Set

$$E := \{x \in \Omega \colon \tau(x) > \delta\} \quad \text{and} \quad F := \{y \in \Gamma \colon \tau(y) > \delta\}.$$

Then, Lemma 9.1 and Lemma 9.2 provide the existence of a constant $C > 0$ such that for all $v \in \mathrm{V}(\Omega; \gamma)$, we have

$$\lambda(E) \|v\|^2_{\mathrm{V}(\Omega;\gamma)} \leqslant C \left(\mathfrak{B}(v, v) + \left(\int_E v(x) \,\mathrm{d}x \right)^2 \right)$$

and

$$\int_E w(y) \,\mathrm{d}y \|v\|^2_{\mathrm{V}(\Omega;\gamma)} \leqslant C \left(\mathfrak{B}(v, v) + \left(\int_F v(y) w(y) \,\mathrm{d}y \right)^2 \right).$$

Furthermore, $\lambda(E) + \lambda(F) = \lambda(\{x \in \Omega \cup \Gamma \colon \tau(x) > \delta\}) > 0$ holds and for all $v \in \mathrm{V}(\Omega; \gamma)$, the Hölder inequality yields

$$\left(\int_E v(x) \,\mathrm{d}x \right)^2 \leqslant \frac{\lambda(E)}{\delta} \int_E \delta v^2(x) \,\mathrm{d}x \leqslant \frac{\lambda(E)}{\delta} \int_\Omega \tau(x) v^2(x) \,\mathrm{d}x$$

and

$$\left(\int_F v(y) w(y) \,\mathrm{d}y \right)^2 \leqslant \frac{\lambda(\Omega)}{\delta} \int_F \delta v^2(y) w(y) \,\mathrm{d}y \leqslant \frac{\lambda(\Omega)}{\delta} \int_\Gamma \tau(y) v^2(y) w(y) \,\mathrm{d}y.$$

All in all, we have shown there are constants $c_1, c_2 > 0$ with

$$c_1 \|v\|_{\mathrm{V}(\Omega;\gamma)} \leqslant \mathfrak{A}(v,v) \leqslant c_2 \|v\|_{\mathrm{V}(\Omega;\gamma)} \quad \text{for all } v \in \mathrm{V}(\Omega;\gamma).$$

Hence, $\mathrm{V}(\Omega;\gamma)$ is a Hilbert space with respect to $\mathfrak{A}$. Furthermore, by the Hölder inequality and Theorem 6.1, there is a $C > 0$ with

$$\begin{aligned}&\int_\Omega f(x)v(x)\,\mathrm{d}x + \int_\Gamma g(y)v(y)w(y)\,\mathrm{d}y \\ \leqslant & C(\|f\|_{\mathrm{L}^2(\Omega)} + \|g\|_{\mathrm{L}^2(\Gamma;w)})\|v\|_{\mathrm{V}(\Omega;\gamma)} \quad \text{for all } v \in \mathrm{V}(\Omega;\gamma).\end{aligned}$$

Then, the Riesz representation theorem gives us the existence of a unique solution in $\mathrm{V}(\Omega;\gamma)$. □

We obtain that $\mathrm{V}(\Omega;\gamma)$ is the appropriate state space for our minimization problem. We first study the case where $\tau \in \mathrm{L}^\infty(\Omega \cup \Gamma)$ is nonnegative with $\lambda(\{x \in \Omega \cup \Gamma \colon \tau(x) > 0\}) > 0$.

Definition 9.4.
Let $\Omega \subset \mathbb{R}^d$ be a bounded, nonempty, and open subset, $\gamma \in \widetilde{\mathcal{K}}$, and let $\tau \in \mathrm{L}^\infty(\Omega \cup \Gamma)$ be nonnegative. Furthermore, consider the state equation (C1), *i.e.,*

$$\begin{cases} \mathcal{L}u + \tau u &= f \quad \text{on } \Omega, \\ \mathcal{N}u + \tau u w &= g w \quad \text{on } \Gamma. \end{cases}$$

Let

$$\mathcal{G} \colon \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Gamma;w) \to \mathrm{V}(\Omega;\gamma)$$

be a bounded linear operator such that for every given $(f,g) \in \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Gamma;w)$, we have

$$\mathfrak{A}(\mathcal{G}(f,g),v) = \int_\Omega f(x)v(x)\,\mathrm{d}x + \int_\Gamma g(y)v(y)w(y)\,\mathrm{d}y \quad \text{for all } v \in \mathrm{V}(\Omega;\gamma).$$

Then, we call $\mathcal{G}$ control-to-state operator and we define the bounded linear operator

$$\mathcal{S} \colon \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Gamma;w) \to \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Gamma;w)$$

by

$$\mathcal{S}(f,g) := (\mathcal{S}_1(f,g), \mathcal{S}_2(f,g)) := (\mathcal{G}(f,g)|_\Omega, \mathcal{G}(f,g)|_\Gamma) \text{ for } (f,g) \in \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Gamma;w).$$

Because $J \colon \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Gamma;w) \to \mathrm{L}^2(\Omega \cup \Gamma; \chi_\Omega + w\chi_\Gamma)$, $(u,v) \mapsto u\chi_\Omega + v\chi_\Gamma$ is an (isometric) isomorphism, we identify the elements of $\mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Gamma;w)$ with their corresponding functions in $\mathrm{L}^2(\Omega \cup \Gamma; \chi_\Omega + w\chi_\Gamma)$. By this identification, we now study the adjoint operator of $\mathcal{S}$.

Lemma 9.5.
Let $\Omega \subset \mathbb{R}^d$ be a bounded, nonempty, and open subset, $\gamma \in \widetilde{\mathcal{K}}$, and let $\tau \in \mathrm{L}^\infty(\Omega \cup \Gamma)$ be nonnegative with $\lambda(\{x \in \Omega \cup \Gamma \colon \tau(x) > 0\}) > 0$. Then, the control-to-state operator

$$\mathcal{G} \colon \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Gamma;w) \to \mathrm{V}(\Omega;\gamma)$$

exists and is unique. Furthermore, the unique operator $\mathcal{S} \colon \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Gamma;w) \to \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Gamma;w)$ given by

$$\mathcal{S}(f,g) := (\mathcal{S}_1(f,g), \mathcal{S}_2(f,g)) := (\mathcal{G}(f,g)|_\Omega, \mathcal{G}(f,g)|_\Gamma) \text{ for } (f,g) \in \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Gamma;w)$$

is self-adjoint, i.e., for all $(f,g),(s,t) \in \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Gamma;w)$, we have

$$\langle s, \mathcal{S}_1(f,g)\rangle_{\mathrm{L}^2(\Omega)} + \langle t, \mathcal{S}_2(f,g)\rangle_{\mathrm{L}^2(\Gamma;w)} = \langle \mathcal{S}_1(s,t), f\rangle_{\mathrm{L}^2(\Omega)} + \langle \mathcal{S}_2(s,t), g\rangle_{\mathrm{L}^2(\Gamma;w)}.$$

Proof.

The existence and uniqueness of $\mathcal{G}$ follows by Theorem 9.3.

Moreover, for all $(f, g), (s, t) \in \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Gamma; w)$, we obtain

$\langle s, \mathcal{S}_1(f, g)\rangle_{\mathrm{L}^2(\Omega)} + \langle t, \mathcal{S}_2(f, g)\rangle_{\mathrm{L}^2(\Gamma;w)} = \mathfrak{A}(\mathcal{G}(f, g), \mathcal{G}(s, t)) = \langle \mathcal{S}_1(s, t), f\rangle_{\mathrm{L}^2(\Omega)} + \langle \mathcal{S}_2(s, t), g\rangle_{\mathrm{L}^2(\Gamma;w)}$.

□

Because the set of admissible controls C_{ad} depends on the state equation (C1), it is determined by τ. If τ satisfies $\lambda(\{x \in \Omega \cup \Gamma \colon \tau(x) > 0\}) > 0$, then the admissible set C_{ad} is given by

$$\{(f, g) \in \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Gamma; w) \colon \underline{f} \leqslant f \leqslant \overline{f} \text{ a.e on } \Omega \text{ and } \underline{g} \leqslant g \leqslant \overline{g} \text{ a.e. on } \Gamma\}$$

and our optimal control problem reduces to the quadratic optimization problem

$$\min_{(f,g)\in \mathrm{C}_{ad}} \mathcal{J}(f, g) := J(f, g, \mathcal{G}(f, g)). \tag{Q1}$$

For $(f, g) \in \mathrm{C}_{ad}$, we have

$$\mathcal{J}(f, g) = \frac{1}{2}\|\mathcal{S}_1(f, g) - z\|^2_{\mathrm{L}^2(\Omega_0)} + \frac{1}{2}\|\mathcal{S}_2(f, g) - z\|^2_{\mathrm{L}^2(\Gamma_0;w)} + \frac{\alpha}{2}\|f\|^2_{\mathrm{L}^2(\Omega)} + \frac{\beta}{2}\|g\|^2_{\mathrm{L}^2(\Gamma;w)}.$$

To sum it up, we want to find an optimal control $(f_{opt}, g_{opt}) \in \mathrm{C}_{ad}$ such that

$$\mathcal{J}(f_{opt}, g_{opt}) \leqslant \mathcal{J}(f, g) \quad \text{holds for all } (f, g) \in \mathrm{C}_{ad}.$$

An existence result is derived by applying the following well-known result.

Theorem 9.6.
Let $(B, \|\cdot\|_B)$ be a reflexive Banach space and let $C \subset B$ be a closed, convex, and nonempty subset. Let $F \colon C \to \mathbb{R} \cup \infty$ be a convex function such that

(i) $F \neq \infty$,

(ii) for every $\tau \in \mathbb{R}$ the set $\{c \in C \colon F(c) \leqslant \tau\}$ is closed,

(iii) $\lim_{c\in C, \|c\|_B\to\infty} F(c) = \infty$.

Then, there exists a $c_{opt} \in A$ with $F(c_{opt}) = \min_{c\in C} F(c)$,

Proof.

The proof can be found in [16, Corollary 3.23.]. □

Therefore, we obtain an existence result.

Theorem 9.7.
Let

- Ω_0 *be an open subset of the bounded and open set* $\Omega \subset \mathbb{R}^d$,
- $\gamma \in \widetilde{\mathcal{K}}$,
- Γ_0 *be an open subset of the nonlocal boundary* Γ,
- $\alpha, \beta \geqslant 0$ *be constants,*
- $z \in \mathrm{L}^2(\Omega_0 \cup \Gamma_0; \chi_{\Omega_0} + w\chi_{\Gamma_0})$,
- $\tau \in \mathrm{L}^\infty(\Omega \cup \Gamma)$ *with* $\tau \geqslant 0$ *and* $\lambda(\{x \in \Omega \cup \Gamma \colon \tau(x) > 0\}) > 0$ *be given,*
- $\underline{f}, \overline{f} \colon \Omega \to \mathbb{R} \cup \{-\infty, \infty\}$ *with* $\underline{f}\chi_{\{-\infty<\underline{f}\}} \in \mathrm{L}^2(\Omega)$ *and* $\overline{f}\chi_{\{\overline{f}<\infty\}} \in \mathrm{L}^2(\Omega)$ *satisfy* $\underline{f} \leqslant \overline{f}$,
- $\underline{g}, \overline{g} \colon \Gamma \to \mathbb{R} \cup \{-\infty, \infty\}$*with* $\underline{g}\chi_{\{-\infty<\underline{g}\}} \in \mathrm{L}^2(\Gamma; w)$ *and* $\overline{g}\chi_{\{\overline{g}<\infty\}} \in \mathrm{L}^2(\Gamma; w)$ *satisfy* $\underline{g} \leqslant \overline{g}$.

Then, the following assertions are valid.

(i) If C_{ad} *is bounded, then there exists a solution to the minimization problem* (Q1).

(ii) If $\alpha\beta > 0$ *holds, then there exists a unique solution to the minimization problem* (Q1).

(iii) Furthermore, $(f_{opt}, g_{opt}) \in \mathrm{C}_{ad}$ *is a solution to the minimization problem* (Q1) *if and only if* $(f_{opt}, g_{opt}) \in \mathrm{C}_{ad}$ *solves the variational inequality*

$$
\begin{aligned}
&\langle \mathcal{S}_1(f_{opt}, g_{opt}) - z, \mathcal{S}_1(f - f_{opt}, g - g_{opt})\rangle_{\mathrm{L}^2(\Omega_0)} + \alpha\langle f_{opt}, f - f_{opt}\rangle_{\mathrm{L}^2(\Omega)} \\
\geqslant &\langle \mathcal{S}_2(f_{opt}, g_{opt}) - z, \mathcal{S}_2(f_{opt} - f, g_{opt} - g)\rangle_{\mathrm{L}^2(\Gamma_0; w)} + \beta\langle g_{opt}, g_{opt} - g\rangle_{\mathrm{L}^2(\Gamma; w)}
\end{aligned}
$$

for all $(f, g) \in \mathrm{C}_{ad}$.

Proof.

By definition, the set C_{ad} is closed, convex, and nonempty. Therefore, Theorem 9.6 gives us the existence of a solution if C_{ad} is bounded.

If $\alpha\beta > 0$ then fix a $(s, t) \in \mathrm{C}_{ad}$ and consider

$$\mathrm{C} := \{(f, g) \in \mathrm{C}_{ad} \colon \|f\|^2_{\mathrm{L}^2(\Omega)} + \|g\|^2_{\mathrm{L}^2(\Gamma; w)} \leqslant \frac{2}{\min\{\alpha, \beta\}} \mathcal{J}(s, t)\}.$$

Since $\mathcal{J}(f, g) \geqslant \frac{1}{2}\min\{\alpha, \beta\}(\|f\|^2_{\mathrm{L}^2(\Omega)} + \|g\|^2_{\mathrm{L}^2(\Gamma; w)})$ holds for $(f, g) \in \mathrm{C}_{ad}$, we obtain

$$\min_{(f,g)\in \mathrm{C}_{ad}} \mathcal{J}(f, g) = \min_{(f,g)\in \mathrm{C}} \mathcal{J}(f, g).$$

Then, the existence of a unique optimal control follows by Theorem 9.6 and the fact that $\mathcal{J}$ is strictly convex if $\alpha\beta > 0$.

We recall C_{ad} and $\mathcal{J}$ is convex. Therefore, if $x \in \mathrm{C}_{ad}$ is optimal, we obtain

$$\lim_{\tau \to 0} \frac{1}{\tau}(\mathcal{J}(x + \tau(y - x)) - \mathcal{J}(x)) \geqslant 0 \quad \text{for all } y \in \mathrm{C}_{ad}.$$

Conversely, for all $x, y \in \mathrm{C}_{ad}$, we have

$$\lim_{\tau\to 0} \frac{1}{\tau}\mathcal{J}(x+\tau(y-x)) - \mathcal{J}(x) \leqslant \mathcal{J}(y) - \mathcal{J}(x).$$

Hence, $x \in \mathrm{C}_{ad}$ is optimal if and only if

$$\lim_{\tau\to 0} \frac{1}{\tau}(\mathcal{J}(x+\tau(y-x)) - \mathcal{J}(x)) \geqslant 0 \quad \text{for all } y \in \mathrm{C}_{ad}.$$

Since, the polarization identity yields

$$\begin{aligned}&\lim_{\tau\to 0} \frac{\mathcal{J}((f,g)+\tau(s-f,g-t)) - \mathcal{J}(f,g)}{\tau}\\ =&\langle \mathcal{S}_1(f,g)-z, \mathcal{S}_1(s-f,t-g)\rangle_{\mathrm{L}^2(\Omega_0)} + \langle \mathcal{S}_2(f,g)-z, \mathcal{S}_2(s-f,t-g)\rangle_{\mathrm{L}^2(\Gamma_0;w)}\\ &+ \alpha\langle f, s-f\rangle_{\mathrm{L}^2(\Omega} + \beta\langle g, t-g\rangle_{\mathrm{L}^2(\Gamma;w)}\end{aligned}$$

for all $(f,g),(s,t) \in \mathrm{C}_{ad}$, we conclude our assertion. □

Corollary 9.8.
Let

- Ω_0 *be an open subset of the bounded and open set* $\Omega \subset \mathbb{R}^d$,
- $\gamma \in \widetilde{\mathcal{K}}$,
- Γ_0 *be an open subset of the nonlocal boundary* Γ,
- $\alpha, \beta \geqslant 0$ *be constants,*
- $z \in \mathrm{L}^2(\Omega_0 \cup \Gamma_0; \chi_{\Omega_0} + w\chi_{\Gamma_0})$,
- $\tau \in \mathrm{L}^\infty(\Omega \cup \Gamma)$ *with* $\tau \geqslant 0$ *and* $\lambda(\{x \in \Omega\cup\Gamma\colon \tau(x) > 0\}) > 0$ *be given,*
- $\underline{f}, \overline{f}\colon \Omega \to \mathbb{R}\cup\{-\infty,\infty\}$ *with* $\underline{f}\chi_{\{-\infty<\underline{f}\}} \in \mathrm{L}^2(\Omega)$ *and* $\overline{f}\chi_{\{\overline{f}<\infty\}} \in \mathrm{L}^2(\Omega)$ *satisfy* $\underline{f} \leqslant \overline{f}$,
- $\underline{g}, \overline{g}\colon \Gamma \to \mathbb{R}\cup\{-\infty,\infty\}$*with* $\underline{g}\chi_{\{-\infty<\underline{g}\}} \in \mathrm{L}^2(\Gamma;w)$ *and* $\overline{g}\chi_{\{\overline{g}<\infty\}} \in \mathrm{L}^2(\Gamma;w)$ *satisfy* $\underline{g} \leqslant \overline{g}$.

Then, $(f_{opt}, g_{opt}) \in \mathrm{C}_{ad}$ *satisfies* $\mathcal{J}(f_{opt}, g_{opt}) \leqslant \mathcal{J}(f,g)$ *for all* $(f,g) \in \mathrm{C}_{ad}$ *if and only if there is a unique adjoint state* $p \in \mathrm{V}(\Omega;\gamma)$ *with*

$$p = \mathcal{G}((\mathcal{S}_1(f_{opt}, g_{opt}) - z)\chi_{\Omega_0}, (\mathcal{S}_2(f_{opt}, g_{opt}) - z)\chi_{\Gamma 0})$$

such that the variational inequality

$$\langle p + \alpha f_{opt}, f - f_{opt}\rangle_{\mathrm{L}^2(\Omega)} + \langle p + \beta g_{opt}, g - g_{opt}\rangle_{\mathrm{L}^2(\Gamma;w)} \geqslant 0 \ \ \textit{holds for all } (f,g) \in \mathrm{C}_{ad}.$$

Proof.

Follows by Theorem 9.7 and Lemma 9.5. □

Lemma 9.9 (Projection formulas)**.**
Let

- Ω_0 *be an open subset of the bounded and open set* $\Omega \subset \mathbb{R}^d$,
- $\gamma \in \widetilde{\mathcal{K}}$,
- Γ_0 *be an open subset of the nonlocal boundary* Γ,
- $\alpha, \beta \geqslant 0$ *be constants,*
- $z \in \mathrm{L}^2(\Omega_0 \cup \Gamma_0; \chi_{\Omega_0} + w\chi_{\Gamma_0})$,
- $\tau \in \mathrm{L}^\infty(\Omega \cup \Gamma)$ *with* $\tau \geqslant 0$ *and* $\lambda(\{x \in \Omega \cup \Gamma \colon \tau(x) > 0\}) > 0$ *be given,*
- $\underline{f}, \overline{f} \colon \Omega \to \mathbb{R} \cup \{-\infty, \infty\}$ *with* $\underline{f}\chi_{\{-\infty < \underline{f}\}} \in \mathrm{L}^2(\Omega)$ *and* $\overline{f}\chi_{\{\overline{f} < \infty\}} \in \mathrm{L}^2(\Omega)$ *satisfy* $\underline{f} \leqslant \overline{f}$,
- $\underline{g}, \overline{g} \colon \Gamma \to \mathbb{R} \cup \{-\infty, \infty\}$*with* $\underline{g}\chi_{\{-\infty < \underline{g}\}} \in \mathrm{L}^2(\Gamma; w)$ *and* $\overline{g}\chi_{\{\overline{g} < \infty\}} \in \mathrm{L}^2(\Gamma; w)$ *satisfy* $\underline{g} \leqslant \overline{g}$.

Let control $(f_{opt}, g_{opt}) \in \mathrm{C}_{ad}$ *and* $p \in \mathrm{V}(\Omega; \gamma)$ *be given such that*

$$\begin{aligned} &p = \mathcal{G}((\mathcal{S}_1(f_{opt}, g_{opt}) - z)\chi_{\Omega_0}, (\mathcal{S}_2(f_{opt}, g_{opt}) - z)\chi_{\Gamma 0}), \\ \text{and} \quad &\langle p + \alpha f_{opt}, f - f_{opt}\rangle_{\mathrm{L}^2(\Omega)} + \langle p + \beta g_{opt}, g - g_{opt}\rangle_{\mathrm{L}^2(\Gamma; w)} \geqslant 0 \quad \text{holds for all } (f, g) \in \mathrm{C}_{ad}. \end{aligned}$$

For $a, b \in \mathbb{R}$ *with* $a \leqslant b$, *let* $P_{[a,b]}$ *denote the projection of* $\mathbb{R}$ *onto* $[a, b]$, *i.e.,*

$$P_{[a,b]}(z) := \begin{cases} b & \text{if } b < z, \\ z & \text{if } a \leqslant z \leqslant b, \\ a & \text{if } z < a, \end{cases} \qquad \text{for } z \in \mathbb{R}$$

Then

(i) If $\alpha = 0$ *holds, then for a.e.* $x \in \Omega$, *we have*

$$f_{opt}(x) = \begin{cases} \overline{f}(x) & \text{if } p(x) < 0, \\ \underline{f}(x) & \text{if } p(x) > 0. \end{cases}$$

(ii) If $\beta = 0$ *holds, then for a.e.* $y \in \Gamma$, *we have*

$$g_{opt}(y) = \begin{cases} \overline{g}(y) & \text{if } p(y) < 0, \\ \underline{g}(y) & \text{if } p(y) > 0. \end{cases}$$

(iii) If $\alpha > 0$ *holds, then for a.e.* $x \in \Omega$, *we have*

$$f_{opt}(x) = P_{[\underline{f}(x), \overline{f}(x)]}\left(-\frac{1}{\alpha}p(x)\right).$$

(iv) If $\beta > 0$ *holds, then for a.e.* $y \in \Gamma$, *we have*

$$g_{opt}(y) = P_{[\underline{g}(y), \overline{g}(y)]}\left(-\frac{1}{\beta}p(y)\right).$$

(v) *Furthermore if $\alpha > 0$ and $\beta > 0$ holds, then any control $(s,t) \in \mathrm{C}_{ad}$ and adjoint state $q \in \mathrm{V}(\Omega;\gamma)$ with*

$$\begin{aligned} q &= \mathcal{G}((\mathcal{S}_1(s,t)-z)\chi_{\Omega_0}, (\mathcal{S}_2(s,t)-z)\chi_{\Gamma 0}), \\ s(x) &= P_{[\underline{f}(x),\overline{f}(x)]}\left(-\frac{1}{\alpha}q(x)\right) \text{ for a.e. } x \in \Omega, \\ \text{and} \quad t(y) &= P_{[\underline{g}(y),\overline{g}(y)]}\left(-\frac{1}{\beta}q(y)\right) \text{ for a.e. } y \in \Gamma, \end{aligned}$$

we obtain $(f_{opt}, g_{opt}) = (s,t)$ and $p = q$.

Proof.

First, we set

$$\begin{aligned} \Omega_1 &:= \{x \in \Omega \colon p(x) + \alpha f_{opt}(x) < 0\}, & \Gamma_1 &:= \{y \in \Gamma \colon p(y) + \alpha g_{opt}(y) < 0\}, \\ \Omega_2 &:= \{x \in \Omega \colon p(x) + \alpha f_{opt}(x) > 0\}, & \Gamma_2 &:= \{y \in \Gamma \colon p(y) + \alpha f_{opt}(y) > 0\}, \\ \Omega_3 &:= \{x \in \Omega \colon p(x) + \alpha f_{opt}(x) = 0\}, & \Gamma_3 &:= \{y \in \Gamma \colon p(y) + \alpha f_{opt}(y) = 0\}. \end{aligned}$$

Then, we decompose the variational inequality for all $(f,g) \in \mathrm{C}_{ad}$ into

$$\begin{aligned} 0 \leqslant & \langle p + \alpha f_{opt}, f - f_{opt}\rangle_{\mathrm{L}^2(\Omega)} + \langle p + \beta g_{opt}, g - g_{opt}\rangle_{\mathrm{L}^2(\Gamma;w)} \\ = & \sum_{i=1}^{3} (\langle p + \alpha f_{opt}, f - f_{opt}\rangle_{\mathrm{L}^2(\Omega_i)} + \langle p + \beta g_{opt}, g - g_{opt}\rangle_{\mathrm{L}^2(\Gamma_i;w)}) \\ = & \sum_{i=1}^{2} (\langle p + \alpha f_{opt}, f - f_{opt}\rangle_{\mathrm{L}^2(\Omega_i)} + \langle p + \beta g_{opt}, g - g_{opt}\rangle_{\mathrm{L}^2(\Gamma_i;w)}) \end{aligned}$$

Then, we show (i),(ii),(iii) and (iv) by contradiction. Assume that there are functions $\widetilde{f} \in \mathrm{L}^2(\Omega)$ and $\widetilde{g} \in \mathrm{L}^2(\Gamma;w)$ and measurable sets $\widetilde{\Omega}_1 \subset \Omega_1$, $\widetilde{\Omega}_2 \subset \Omega_2$ $\widetilde{\Gamma}_1 \subset \Gamma_1$, $\widetilde{\Gamma}_2 \subset \Gamma_2$ such that

$$\sum_{i=1}^{2} \lambda(\lambda_i \cup \Gamma_i) > 0$$

holds and

$$\begin{aligned} f_{opt}(x) &< \widetilde{f}(x) \leqslant \overline{f}(x) \text{ a.e. on } \widetilde{\Omega}_1, & g_{opt}(y) &< \widetilde{g}(y) \leqslant \overline{g}(y) \text{ a.e. on } \widetilde{\Gamma}_1, \\ \underline{f}(x) &\leqslant \widetilde{f}(x) < f_{opt}(x) \text{ a.e. on } \widetilde{\Omega}_2, & \overline{g}(y) &\leqslant \widetilde{g}(y) < g_{opt}(y) \text{ a.e. on } \widetilde{\Gamma}_2, \end{aligned}$$

Then, we have $(\widetilde{f}, g_{opt}) \in \mathrm{C}_{ad}$ and $(f_{opt}, \widetilde{g}) \in \mathrm{C}_{ad}$. Consequently, the variational inequality yields

$$0 \leqslant \sum_{i=1}^{2} \langle p + \alpha f_{opt}, \widetilde{f} - f_{opt}\rangle_{\mathrm{L}^2(\Omega_i)} < 0 \quad \text{and} \quad 0 \leqslant \sum_{i=1}^{2} \langle p + \beta g_{opt}, g - g_{opt}\rangle_{\mathrm{L}^2(\Gamma_i;w)} < 0.$$

This is evidently a contradiction.

Let the control $(s,t) \in \mathrm{C}_{ad}$ and the adjoint state $q \in \mathrm{V}(\Omega;\gamma)$ satisfy

$$\begin{aligned} q &= \mathcal{G}((\mathcal{S}_1(s,t)-z)\chi_{\Omega_0}, (\mathcal{S}_2(s,t)-z)\chi_{\Gamma 0}), \\ s(x) &= P_{[\underline{f}(x),\overline{f}(x)]}\left(-\frac{1}{\alpha}q(x)\right) \text{ for a.e. } x \in \Omega, \\ \text{and} \quad t(y) &= P_{[\underline{g}(y),\overline{g}(y)]}\left(-\frac{1}{\beta}q(y)\right) \text{ for a.e. } y \in \Gamma. \end{aligned}$$

In order to show (v), it is sufficient by Corollary 9.8 to verify that

$$\langle q+\alpha s, f-s\rangle_{\mathrm{L}^2(\Omega)} + \langle q+\beta t, g-t\rangle_{\mathrm{L}^2(\Gamma;w)} \geqslant 0 \quad \text{holds for all } (f,g) \in \mathrm{C}_{ad}.$$

However, this follows by considering the decomposition

$$\begin{aligned} \Omega &= \{x \in \Omega\colon s(x) = \overline{f}(x)\} \cup \{x \in \Omega\colon s(x) = \underline{f}(x)\} \cup \{x \in \Omega\colon s(x) = -\frac{1}{\alpha}q(x)\} \\ \text{and} \quad \Gamma &= \{y \in \Gamma\colon t(x) = \overline{g}(y)\} \cup \{y \in \Gamma\colon t(y) = \underline{g}(y)\} \cup \{y \in \Gamma\colon t(y) = -\frac{1}{\beta}q(y)\}. \end{aligned}$$

□

If $\tau = 0$ a.e. on $\Omega \cup \Gamma$, then state equation (C1) becomes

$$\begin{cases} \mathcal{L}u &= f \quad \text{on } \Omega, \\ \mathcal{N}u &= gw \quad \text{on } \Gamma. \end{cases} \tag{C1*}$$

Even if there exists a weak solution to this state equation, it is only unique up to an additive constant (see Remark 4.4). We avoid this problem by adding a volume constraint, but now the question is how to choose this volume constraint. Taking a look at our objective function, we see that only

$$\frac{1}{2}\|u-z\|^2_{\mathrm{L}^2(\Omega_0)} + \frac{1}{2}\|u-z\|^2_{\mathrm{L}^2(\Gamma_0;w)}$$

depends on the state u. Furthermore,

$$\begin{aligned} &\arg\min_{c\in\mathbb{R}} \frac{1}{2}\|u+c-z\|^2_{\mathrm{L}^2(\Omega_0)} + \frac{1}{2}\|u+c-z\|^2_{\mathrm{L}^2(\Gamma_0;w)} \\ &= -\frac{\left(\int_{\Omega_0}(u(x)-z(x))\,\mathrm{d}x + \int_{\Gamma_0}(u(y)-z(y))w(y)\,\mathrm{d}y\right)}{2(\lambda(\Omega_0) + \int_{\Gamma_0} w(y)\,\mathrm{d}y)} \end{aligned}$$

holds such that we can assume, without loss of generality, that

$$\int_{\Omega_0} z(x)\,\mathrm{d}x + \int_{\Gamma_0} z(y)w(y)\,\mathrm{d}y = 0$$

is satisfied. Finally, we set our volume constraint to be

$$\int_{\Omega_0} u(x)\,\mathrm{d}x + \int_{\Gamma_0} u(y)w(y)\,\mathrm{d}y = 0.$$

Definition 9.10.
Let $\Omega \subset \mathbb{R}^d$ be a bounded, nonempty, and open subset, $\gamma \in \widetilde{\mathcal{K}}$. Furthermore, consider the state equation (C1*)*, i.e.,*

$$\begin{cases} \mathcal{L}u &= f \qquad \text{on } \Omega, \\ \mathcal{N}u &= gw \qquad \text{on } \Gamma, \end{cases}$$

and let $\Omega_0 \subset \Omega$ and $\Gamma_0 \subset \Gamma$ both be open. We set

$$\begin{aligned} \widehat{C} &:= \{(f,g) \in \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Gamma; w) \colon \int_\Omega f(x)\,\mathrm{d}x + \int_\Gamma g(y)w(y)\,\mathrm{d}y = 0\} \\ \widehat{C}_1 &:= \{(s,t) \in \mathrm{L}^2(\Omega) \times \mathrm{L}^2(\Gamma; w) \colon \int_{\Omega_0} s(x)\,\mathrm{d}x + \int_{\Gamma_0} t(y)w(y)\,\mathrm{d}y = 0\} \\ \text{and} \quad \widehat{\mathrm{V}}(\Omega;\gamma) &:= \{v \in \mathrm{V}(\Omega;\gamma) \colon \int_{\Omega_0} u(x)\,\mathrm{d}x + \int_{\Gamma_0} u(y)w(y)\,\mathrm{d}y = 0\}, \end{aligned}$$

and let

$$\mathcal{H} \colon \widehat{C} \to \widehat{\mathrm{V}}(\Omega;\gamma)$$

be a bounded linear operator such that for every given $(f,g) \in \widehat{C}$, we have

$$\mathfrak{B}(\mathcal{H}(f,g), v) = \int_\Omega f(x)v(x)\,\mathrm{d}x + \int_\Gamma g(y)v(y)w(y)\,\mathrm{d}y \qquad \text{for all } v \in \mathrm{V}(\Omega;\gamma).$$

Then, we call $\mathcal{H}$ control-to-state operator and we define the bounded linear operator

$$\mathcal{T} \colon \widehat{C} \to \widehat{C}_1$$

by

$$\mathcal{T}(f,g) := (\mathcal{T}_1(f,g), \mathcal{T}_2(f,g)) := (\mathcal{H}(f,g)|_\Omega, \mathcal{H}(f,g)|_\Gamma) \text{ for } (f,g) \in \widehat{C}.$$

Similar to before, we now study the adjoint operators of $\mathcal{T}$.

Lemma 9.11.
Let $\Omega \subset \mathbb{R}^d$ be a bounded, nonempty, and open subset and let $\gamma \in \widetilde{\mathcal{K}}$. Furthermore, let $\Omega_0 \subset \Omega$ and $\Gamma_0 \subset \Gamma$ both be open. Then, the control-to-state operator

$$\mathcal{H} \colon \widehat{C} \to \widehat{\mathrm{V}}(\Omega;\gamma)$$

exists and is unique. Therefore, $\mathcal{T} \colon \widehat{C} \to \widehat{C}_1$ given by

$$\mathcal{T}(f,g) := (\mathcal{T}_1(f,g), \mathcal{T}_2(f,g)) := (\mathcal{H}(f,g)|_\Omega, \mathcal{H}(f,g)|_\Gamma) \text{ for } (f,g) \in \widehat{C}.$$

is unique. Moreover, the adjoint $\mathcal{T}^* \colon \widehat{C}_1 \to \widehat{C}$ of $\mathcal{T}$ is the unique bounded linear operator such that for every given $(s,t) \in \widehat{C}_1$, we have

$$\mathfrak{B}(\mathcal{T}_1^*(s,t)\chi_\Omega + \mathcal{T}_2^*(s,t)\chi_\Gamma, v) = \int_\Omega s(x)v(x)\,\mathrm{d}x + \int_\Gamma t(y)v(y)w(y)\,\mathrm{d}y \qquad \text{for all } v \in \widehat{\mathrm{V}}(\Omega;\gamma).$$

Proof.

The existence and uniqueness of $\mathcal{T}$ is a consequence of Theorem 4.5. Following the proof of Theorem 4.5, we obtain that $\mathcal{T}^*$ exists and is unique.

Moreover, the definitions of $\mathcal{T}^*$ and $\mathcal{T}$ yields

$$\langle s, \mathcal{T}_1(f,g)\rangle_{\mathrm{L}^2(\Omega)} + \langle t, \mathcal{T}_2(f,g)\rangle_{\mathrm{L}^2(\Gamma;w)} = \langle \mathcal{T}_1^*(s,t), f\rangle_{\mathrm{L}^2(\Omega)} + \langle \mathcal{T}_2^*(s,t), g\rangle_{\mathrm{L}^2(\Gamma;w)}$$

for all $(f,g) \in \widehat{C}$ and all $(s,t) \in \widehat{C}_1$. □

For simplicity, we henceforth make the following assumptions $\Omega_0 = \Omega$ and $\Gamma_0 = \Gamma$. If $\tau = 0$ holds a.e. on $\Omega \cup \Gamma$, then our admissible set $\widehat{\mathrm{C}}_{ad}$ is given by

$$\{(f,g) \in \widehat{C} \colon \underline{f} \leqslant f \leqslant \overline{f} \text{ a.e. on } \Omega \text{ and } \underline{g} \leqslant g \leqslant \overline{g} \text{ a.e. on } \Gamma\}.$$

Therefore, our reduced quadratic optimization problem is given by

$$\min_{(f,g)\in\widehat{\mathrm{C}}_{ad}} \mathcal{J}(f,g) := J(f,g,\mathcal{H}(f,g)). \tag{Q2}$$

Theorem 9.12.
Let

- $\Omega \subset \mathbb{R}^d$ *be a bounded, nonempty, and open set,*
- $\gamma \in \widetilde{\mathcal{K}}$,
- $\alpha, \beta \geqslant 0$ *be constants,*
- $z \in \mathrm{L}^2(\Omega \cup \Gamma; \chi_\Omega + w\chi_\Gamma)$ *with* $\int_\Omega z(x)\,\mathrm{d}x + \int_\Gamma z(y)w(y)\,\mathrm{d}y = 0$,
- $\underline{f}, \overline{f} \colon \Omega \to \mathbb{R} \cup \{-\infty, \infty\}$ *with* $\underline{f}\chi_{\{-\infty<\underline{f}\}} \in \mathrm{L}^2(\Omega)$ *and* $\overline{f}\chi_{\{\overline{f}<\infty\}} \in \mathrm{L}^2(\Omega)$ *satisfy* $\underline{f} \leqslant \overline{f}$,
- $\underline{g}, \overline{g} \colon \Gamma \to \mathbb{R} \cup \{-\infty, \infty\}$*with* $\underline{g}\chi_{\{-\infty<\underline{g}\}} \in \mathrm{L}^2(\Gamma; w)$ *and* $\overline{g}\chi_{\{\overline{g}<\infty\}} \in \mathrm{L}^2(\Gamma; w)$ *satisfy* $\underline{g} \leqslant \overline{g}$.

Assuming $\widehat{\mathrm{C}}_{ad}$ *to be nonempty, then following statements are valid.*

(i) If $\widehat{\mathrm{C}}_{ad}$ *is bounded, then there exists a solution to the minimization problem* (Q2).

(ii) If $\alpha\beta > 0$ *holds, then there exists a unique solution to the minimization problem* (Q2).

(iii) Furthermore, $(f_{opt}, g_{opt}) \in \widehat{\mathrm{C}}_{ad}$ *is a solution to the minimization problem* (Q2) *if and only if* $(f_{opt}, g_{opt}) \in \widehat{\mathrm{C}}_{ad}$ *solves the variational inequality*

$$\begin{aligned}&\langle \mathcal{T}_1(f_{opt}, g_{opt}) - z, \mathcal{T}_1(f - f_{opt}, g - g_{opt})\rangle_{\mathrm{L}^2(\Omega)} + \alpha\langle f_{opt}, f - f_{opt}\rangle_{\mathrm{L}^2(\Omega)}\\ \geqslant&\langle \mathcal{T}_2(f_{opt}, g_{opt}) - z, \mathcal{T}_2(f_{opt} - f, g_{opt} - g)\rangle_{\mathrm{L}^2(\Gamma;w)} + \beta\langle g_{opt}, g_{opt} - g\rangle_{\mathrm{L}^2(\Gamma;w)}\end{aligned}$$

for all $(f,g) \in \widehat{\mathrm{C}}_{ad}$.

Proof.

By definition the set $\widehat{\mathrm{C}}_{ad}$ is closed and convex and by assumption $\widehat{\mathrm{C}}_{ad}$ is nonempty. Therefore, the proof is analogous to the proof of Theorem 9.7. □

Corollary 9.13.
Let

- $\Omega \subset \mathbb{R}^d$ *be a bounded, nonempty, and open set,*
- $\gamma \in \widetilde{\mathcal{K}}$,
- $\alpha, \beta \geqslant 0$ *be constants,*
- $z \in \mathrm{L}^2(\Omega \cup \Gamma; \chi_\Omega + w\chi_\Gamma)$ *with* $\int_\Omega z(x)\,\mathrm{d}x + \int_\Gamma z(y)w(y)\,\mathrm{d}y = 0$,
- $\underline{f}, \overline{f} \colon \Omega \to \mathbb{R} \cup \{-\infty, \infty\}$ *with* $\underline{f}\chi_{\{-\infty<\underline{f}\}} \in \mathrm{L}^2(\Omega)$ *and* $\overline{f}\chi_{\{\overline{f}<\infty\}} \in \mathrm{L}^2(\Omega)$ *satisfy* $\underline{f} \leqslant \overline{f}$,
- $\underline{g}, \overline{g} \colon \Gamma \to \mathbb{R} \cup \{-\infty, \infty\}$*with* $\underline{g}\chi_{\{-\infty<\underline{g}\}} \in \mathrm{L}^2(\Gamma; w)$ *and* $\overline{g}\chi_{\{\overline{g}<\infty\}} \in \mathrm{L}^2(\Gamma; w)$ *satisfy* $\underline{g} \leqslant \overline{g}$.

Assume that $\widehat{\mathrm{C}}_{ad}$ *is nonempty. Then,* $(f_{opt}, g_{opt}) \in \widehat{\mathrm{C}}_{ad}$ *satisfies* $\mathcal{J}(f_{opt}, g_{opt}) \leqslant \mathcal{J}(f, g)$ *for all* $(f, g) \in \widehat{\mathrm{C}}_{ad}$, *if and only if there is a unique adjoint state* $p \in \widehat{\mathrm{V}}(\Omega; \gamma)$ *with*

$$p = \mathcal{H}(\mathcal{T}(f_{opt}, g_{opt}) - z, \mathcal{T}(f_{opt}, g_{opt}) - z)$$

such that the variational inequality

$$\langle p + \alpha f_{opt}, f - f_{opt}\rangle_{\mathrm{L}^2(\Omega)} + \langle p + \beta g_{opt}, g - g_{opt}\rangle_{\mathrm{L}^2(\Gamma; w)} \geqslant 0 \text{ holds for all } (f, g) \in \widehat{\mathrm{C}}_{ad}.$$

Proof.

This is a consequence of Theorem 9.12 and Lemma 9.11. □

Lemma 9.14.
Let

- $\Omega \subset \mathbb{R}^d$ *be a bounded, nonempty, and open set,*
- $\gamma \in \widetilde{\mathcal{K}}$,
- $\alpha, \beta \geqslant 0$ *be constants,*
- $z \in \mathrm{L}^2(\Omega \cup \Gamma; \chi_\Omega + w\chi_\Gamma)$ *with* $\int_\Omega z(x)\,\mathrm{d}x + \int_\Gamma z(y)w(y)\,\mathrm{d}y = 0$,
- $\underline{f}, \overline{f} \colon \Omega \to \mathbb{R} \cup \{-\infty, \infty\}$ *with* $\underline{f}\chi_{\{-\infty<\underline{f}\}} \in \mathrm{L}^2(\Omega)$ *and* $\overline{f}\chi_{\{\overline{f}<\infty\}} \in \mathrm{L}^2(\Omega)$ *satisfy* $\underline{f} \leqslant \overline{f}$,
- $\underline{g}, \overline{g} \colon \Gamma \to \mathbb{R} \cup \{-\infty, \infty\}$*with* $\underline{g}\chi_{\{-\infty<\underline{g}\}} \in \mathrm{L}^2(\Gamma; w)$ *and* $\overline{g}\chi_{\{\overline{g}<\infty\}} \in \mathrm{L}^2(\Gamma; w)$ *satisfy* $\underline{g} \leqslant \overline{g}$.

Assume that $\widehat{\mathrm{C}}_{ad}$ *is nonempty and let the control* $(f_{opt}, g_{opt}) \in \widehat{\mathrm{C}}_{ad}$ *and the adjoint state* $p \in \widehat{\mathrm{V}}(\Omega; \gamma)$ *be given such that*

$$\begin{aligned} &p = \mathcal{H}(\mathcal{T}(f_{opt}, g_{opt}) - z, \mathcal{T}(f_{opt}, g_{opt}) - z), \\ \text{and} \quad &\langle p + \alpha f_{opt}, f - f_{opt}\rangle_{\mathrm{L}^2(\Omega)} + \langle p + \beta g_{opt}, g - g_{opt}\rangle_{\mathrm{L}^2(\Gamma; w)} \geqslant 0 \quad \text{holds for all } (f, g) \in \widehat{\mathrm{C}}_{ad}. \end{aligned}$$

For $a, b \in \mathbb{R}$ with $a \leqslant b$ let $P_{[a,b]}$ denote the projection of $\mathbb{R}$ onto $[a, b]$, i.e.,

$$P_{[a,b]}(z) := \begin{cases} b & \text{if } b < z, \\ z & \text{if } a \leqslant z \leqslant b, \\ a & \text{if } z < a, \end{cases} \qquad \text{for } z \in \mathbb{R}.$$

If we have

$$\int_\Omega P_{[\underline{f}(x),\overline{f}(x)]}\left(-\frac{1}{\alpha}p(x)\right)\,\mathrm{d}x + \int_\Gamma P_{[\underline{g}(y),\overline{g}(y)]}\left(-\frac{1}{\beta}p(y)\right)w(y)\,\mathrm{d}y = 0,$$

then we obtain

$$\begin{aligned} f_{opt}(x) &= P_{[\underline{f}(x),\overline{f}(x)]}\left(-\frac{1}{\alpha}p(x)\right) \text{ for a.e. } x \in \Omega \\ \text{and} \quad g_{opt}(y) &= P_{[\underline{g}(y),\overline{g}(y)]}\left(-\frac{1}{\beta}p(y)\right) \text{ for a.e. } y \in \Gamma. \end{aligned}$$

Proof.

Set

$$\begin{aligned} s(x) &= P_{[\underline{f}(x),\overline{f}(x)]}\left(-\frac{1}{\alpha}p(x)\right) \text{ for a.e. } x \in \Omega \\ \text{and} \quad t(y) &= P_{[\underline{g}(y),\overline{g}(y)]}\left(-\frac{1}{\beta}p(y)\right) \text{ for a.e. } y \in \Gamma. \end{aligned}$$

Then, by definition we have $(s, t) \in \widehat{\mathrm{C}}_{ad}$. By Theorem 9.12 and Corollary 9.13, it is enough to show that

$$\begin{aligned} &\langle p + \alpha s, f - s\rangle_{\mathrm{L}^2(\Omega)} + \langle p + \beta t, g - t\rangle_{\mathrm{L}^2(\Gamma;w)} \\ =&\langle p - \alpha + \alpha s, f - s\rangle_{\mathrm{L}^2(\Omega)} + \langle p - \beta + \beta t, g - t\rangle_{\mathrm{L}^2(\Gamma;w)} \\ \geqslant& 0 \quad \text{holds for all } (f, g) \in \widehat{\mathrm{C}}_{ad}. \end{aligned}$$

However, this follows by considering the decomposition

$$\Omega = \{x \in \Omega \colon s(x) = \overline{f}(x)\} \cup \{x \in \Omega \colon s(x) = \underline{f}(x)\} \cup \{x \in \Omega \colon s(x) = -\frac{1}{\alpha}p(x) +\}$$

and

$$\Gamma = \{y \in \Gamma \colon t(x) = \overline{g}(y)\} \cup \{y \in \Gamma \colon t(y) = \underline{g}(y)\} \cup \{y \in \Gamma \colon t(y) = -\frac{1}{\beta}p(y)\}$$

in the variational inequality. $\square$

Chapter 10

Parabolic Nonlocal Equations

This section is dedicated to the nonlocal time-dependent Neumann problem. We begin by stating our nonlocal time-dependent Neumann problem.

For $T \in (0, \infty)$ and for a bounded, nonempty, and open set $\Omega \subset \mathbb{R}^d$, we consider the following evolution equation

$$\begin{cases} \frac{\partial}{\partial t} u(t,x) + \mathcal{L}_{\gamma_t} u(t,x) & = f(t,x) \quad \text{for } (t,x) \in (0,T) \times \Omega, \\ \mathcal{N}_{\gamma_t} u(t,y) & = 0 \qquad\quad \text{for } (t,y) \in (0,T) \times \Gamma(\Omega, \gamma_t), \\ u(0) & = u_0 \end{cases} \tag{E}$$

where for each $t \in (0,T)$, we have $u_0 \in \mathrm{L}^2(\Omega)$, $f(t,\cdot) \in \mathrm{L}^2(\Omega)$, and $\gamma_t \in \mathcal{K}$.

For simplicity, we assume that γ_t is symmetric for each $t \in (0,T)$. Furthermore, let the measurable function $u\colon (0,T) \times \mathbb{R}^d \to \mathbb{R}$ be a solution of (E) such that for $s \in (0,T)$ we have

$$\begin{aligned} &\int_\Omega \left(\frac{\partial}{\partial t} u(s,x) \right)^2 \mathrm{d}x + \int_\Omega \int_{\mathbb{R}^d} \left(u(s,x) - u(s,y) \right)^2 \gamma_s(y,x) | \,\mathrm{d}y \,\mathrm{d}x < \infty, \\ \text{and} \quad &\int_\Omega \left(\int_{\mathbb{R}^d} |u(s,x) - u(s,y)| \gamma_s(y,x) \,\mathrm{d}y \right)^2 \mathrm{d}x < \infty. \end{aligned}$$

Then, for all $s \in (0,T)$, the nonlocal integration by parts formula (see Theorem 2.1) yields

$$\langle f(s,\cdot), v \rangle_{\mathrm{L}^2(\Omega)} = \langle \frac{\partial u}{\partial t}(s,\cdot), v \rangle_{\mathrm{L}^2(\Omega)} + \mathfrak{B}_s(u(s,\cdot), v) \quad \text{for all } v \in \mathrm{V}(\Omega; \gamma_s)$$

where for $w, v \in \mathrm{V}(\Omega; \gamma_s)$, we set

$$\begin{aligned} \mathfrak{B}_s(w,v) := \mathfrak{B}_{\gamma_s}(w,v) :=& \frac{1}{2} \int_\Omega \int_\Omega (w(x) - w(y))(v(x) - v(y)) \gamma_s(y,x) \,\mathrm{d}y \,\mathrm{d}x \\ &+ \int_\Omega \int_{\Gamma(\Omega, \gamma_s)} (w(x) - w(y))(v(x) - v(y)) \gamma_s(y,x) \,\mathrm{d}y \,\mathrm{d}x. \end{aligned}$$

As mentioned before, we consider $(\gamma_t)_{t \in (0,T)}$ to be a family of symmetric kernels in $\mathcal{K}$. Furthermore, we henceforth assume that there are constants $\alpha, \beta > 0$ and a kernel $\gamma \in \mathcal{K}$ with $\alpha\gamma \leqslant \gamma_s \leqslant \beta\gamma$ for a.e. $s \in (0,T)$ and that for all $u, v \in \mathrm{V}(\Omega;\gamma)$ the function $(0,T) \to \mathbb{R}, s \mapsto \mathfrak{B}_s(u,v)$ is measurable. Then, for every $v \in \mathrm{V}(\Omega;\gamma)$ and for all $\varphi \in \mathrm{C}_0^\infty(0,T)$, we have

$$\int_{(0,T)} (\langle \frac{\partial u}{\partial t}(s,\cdot), v\rangle_{\mathrm{L}^2(\Omega)} + \mathfrak{B}_s(u(s,\cdot),v))\varphi(s)\,\mathrm{d}s = \int_{(0,T)} (\langle f(s,\cdot), v\rangle_{\mathrm{L}^2(\Omega)}\varphi(s)\,\mathrm{d}s.$$

or, equivalently,

$$-\int_{(0,T)} \langle u(s,\cdot), v\rangle_{\mathrm{L}^2(\Omega)}\varphi'(s)\,\mathrm{d}s + \int_{(0,T)} \mathfrak{B}_s(u(s,\cdot),v)\varphi(s)\,\mathrm{d}s = \int_{(0,T)} (\langle f(s,\cdot), v\rangle_{\mathrm{L}^2(\Omega)}\varphi(s)\,\mathrm{d}s. \quad (10.1)$$

However, if now follow the approach used in [41, Chapter 23] or [34, Chapter III], then we require a continuous, injective, and linear operator $\iota\colon \mathrm{V}(\Omega;\gamma) \to \mathrm{L}^2(\Omega)$. To the best of our knowledge such an operator does not exist in general. We avoid this problem by considering the regional weak formulation (see Definition 7.3).

Instead of considering Lebesgue-integrals of real-valued functions, we now consider integrals of functions taking values in a Banach space. We now present a short introduction to integrable functions with values in a Banach space following [19], [33], [41] and [34].

Definition 10.1.
Let $(B, \|\cdot\|_B)$ be a Banach space. Recall that

$$B^* := \{p\colon B \to \mathbb{R} \text{ linear with } \sup_{b\in B, \|b\|_B \leqslant 1} |p(b)| < \infty\}$$

is the (continuous) dual space *of $(B, \|\cdot\|_B)$.*

A function $f\colon (0,T) \to B$ is weakly measurable *if for every $\tau \in B^*$ the function $\tau(f)\colon (0,T) \to \mathbb{R}$ is measurable.*

A function $f\colon (0,T) \to B$ is strongly measurable *if there is a sequence $(s_n)_{n\in\mathbb{N}}$ of simple functions, i.e., functions $s\colon (0,T) \to B$ of the form*

$$s = \sum_{i=1}^{\ell} b_i \chi_{I_i}$$

where $\ell \in \mathbb{N}$, $b_i \in B$ and I_i are disjoint measurable subsets of $(0,T)$ with $\bigcup_{i=1}^{\ell} I_i = (0,T)$, such that

$$\lim_{n\to\infty} \|s_n(x) - f(x)\|_B = 0 \quad \text{holds for a.e. } x \in (0,T).$$

The next theorem shows the connection between weakly and strongly measurable functions.

Theorem 10.2 (Pettis)**.**
Let $T \in (0,\infty)$ and a Banach space $(B, \|\cdot\|_B)$ be given. Then, $f\colon (0,T) \to B$ is strongly measurable if and only if f is weakly measurable and if there is a set $N \subset (0,T)$ with $\lambda(N) = 0$ such that

$$\{f(x)\colon x \in (0,T) \setminus N\} \subset B$$

is separable.

Proof.

See [33, Theorem 1.1.6] for the proof. □

As a consequence, we see that the definition of strongly measurable and weakly measurable functions coincide in a separable Banach space.

Corollary 10.3.
Let $T \in (0, \infty)$ and a separable Banach space $(B, \|\cdot\|_B)$ be given. For a function $f : (0,T) \to B$ the following equivalence.

(i) f is strongly measurable.

(ii) f is weakly measurable.

Moreover for a Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$ a function $g : (0,T) \to H$ is weakly measurable if and only if $\langle g(\cdot), h\rangle_H : (0,T) \to H$ is measurable for all $h \in H$.

Proof.

As mentioned before, this follows by Theorem 10.2 and the Riesz representation Theorem. □

Definition 10.4.
Let $T \in (0, \infty)$be given and let $(B, \|\cdot\|_B)$ be a reflexive and separable Banach space. Then, we consider a function $f : (0,T) \to B$ to be measurable if f is weakly measurable.

For $p \in [1, \infty)$, let $\mathrm{L}^p(0,T;B)$ consist of all measurable functions $f : (0,T) \to B$ with

$$\|f\|^p_{\mathrm{L}^p(0,T;B)} := \int_{(0,T)} \|f(x)\|^p_B \,\mathrm{d}x < \infty.$$

We then call $\mathrm{L}^p(0,T;B)$ Bochner space. *Furthermore, we define the* (Pettis-)integral *over the measurable set $I \subset (0,T)$ to be the unique element $\int_I f(x)\,\mathrm{d}x \in B$ solving*

$$\tau\left(\int_I f(x)\,\mathrm{d}x\right) = \int_I \tau(f(x))\,\mathrm{d}x \quad \textit{for all } \tau \in B^*.$$

Let $\mathrm{H}^1(0,T;B)$ be the set of all functions $f \in \mathrm{L}^2(0,T;B)$ such that there exists a unique function $g \in \mathrm{L}^2(0,T;B)$ satisfying

$$\int_{(0,T)} f(x)\varphi'(x)\,\mathrm{d}x = -\int_{(0,T)} g(x)\varphi(x)\,\mathrm{d}x,$$

for all $\varphi \in \mathrm{C}^\infty_0(0,T)$. In this case, we set $f' := g$.

The existence of the integral defined in Definition 10.4 is a consequence of the the following lemma.

Lemma 10.5 (Dunford)**.**
Let $T \in (0,\infty)$ be given and let $(B, \|\cdot\|_B)$ be a Banach space. Assume that $f\colon (0,T) \to B$ is weakly measurable and that for each $\tau \in B^*$ the function $\tau(f(\cdot))\colon (0,T) \to \mathbb{R}$ is an element of $\mathrm{L}^1(0,T)$, i.e., $\int_{(0,T)} |\tau(f(x))| \,\mathrm{d}x < \infty$. Then, for each measurable $I \subset (0,T)$ there exists a unique $\xi_{I,f}$ in the bidual $B^{**} = (B^*)^*$ of B, such that

$$\xi_{I,f}(\tau) = \int_I \tau(f(x)) \,\mathrm{d}x \quad \text{for all } \tau \in B^*.$$

Moreover, $\xi_{I,f}$ is called the Dunford-integral *of f over the measurable set I.*

Proof.

See [33, Lemma 2.1.1. (Dunford)]. □

Corollary 10.6.
Let $T \in (0,\infty)$ and $p \in [1,\infty)$ be given and further let $(B, \|\cdot\|_B)$ be a reflexive Banach space. Then, for every $f \in \mathrm{L}^p(0,T;B)$ and every measurable subset $I \subset (0,T)$, the integral $\int_I f(x)\,\mathrm{d}x$ is well-defined in B.

Proof.

For every $\tau \in B^*$ there is a $C > 0$ such that

$$\int_{(0,T)} |\tau(f(x))| \,\mathrm{d}x \leqslant C \int_{(0,T)} |\|f(x)\|_B \,\mathrm{d}x \leqslant CT^{\frac{p-1}{p}} \left(\int_{(0,T)} \|f(x)\|_B^p \,\mathrm{d}x \right)^{\frac{1}{p}} < \infty$$

holds for all $f \in \mathrm{L}^p(0,T;B)$ and $x \in (0,T)$. The rest follows by Lemma 10.5 and the fact that B is reflexive. □

In general, the Pettis-integral is a Dunford-integral. An even stronger integral, the so-called *Bochner-integral* is defined by using simple functions approximating the integrand. If a Bochner-integral can be assigned to a function, then a Pettis-integral can be assigned to this function as well and the integrals coincide (see [33, Proposition 2.3.1.]). However, every function in $\mathrm{L}^2(0,T;H)$ is, in fact, Bochner-integrable (see [33, Theorem 1.4.3.]). Therefore, simple functions are dense in $\mathrm{L}^2(0,T;H)$.

Similar to the case of $B = \mathbb{R}$, we have $\|f\|_{\mathrm{L}^2(0,T;B)} = 0$ if and only if $\|f\|_B = 0$ holds a.e. in $(0,T)$ for $f \in \mathrm{L}^2(0,T;B)$. So we identify each $f \in \mathrm{L}^2(0,T;B)$ with it's respective equivalence class

$$[f] := \{g \in \mathrm{L}^2(0,T;B)\colon g = f \text{ a.e. in } (0,T)\}.$$

As we mostly consider $\mathrm{L}^2(0,T;H)$ where $(H, \langle\cdot,\cdot\rangle_H)$ is a separable Hilbert space, we will in the following theorem see that, in this case, $\mathrm{L}^2(0,T;H)$ is also a separable Hilbert space. However, due to the fact that the proof is analogous to the case that $H = \mathbb{R}$, we omit the proof. Nonetheless, we want to highlight that many results regarding $\mathrm{L}^2(0,T)$ can be extended to $\mathrm{L}^2(0,T;H)$.

Theorem 10.7.
Let $T \in (0, \infty)$, $p \in [1, \infty)$ and a separable Banach space $(B, \|\cdot\|_B)$ be given. Then, $\mathrm{L}^p(0, T; B)$ is a separable Banach space with respect to the norm

$$\|f\|_{\mathrm{L}^p(0,T;B)}^p := \int_{(0,T)} \|f(x)\|_B^p \,\mathrm{d}x \quad \text{for } f \in \mathrm{L}^p(0, T; B).$$

If $(H, \langle\cdot,\cdot\rangle_H)$ is a separable Hilbert space, then $\mathrm{L}^2(0, T; H)$ is a separable Hilbert space with respect to the inner product

$$\langle f, g\rangle_{\mathrm{L}^2(0,T;H)} := \int_{(0,T)} \langle f(x), g(x)\rangle_H \,\mathrm{d}x \quad \text{for } f, g \in \mathrm{L}^2(0, T; H).$$

Furthermore,

$$\mathrm{H}^1(0, T; H) := \{f \in \mathrm{L}^2(0, T; H) \colon f' \in \mathrm{L}^2(0, T; H)\}$$

is a separable Hilbert space with respect to the inner product

$$\langle f, g\rangle_{\mathrm{H}^1(0,T;H)} := \langle f, g\rangle_{\mathrm{L}^2(0,T;H)} + \langle f', g'\rangle_{\mathrm{L}^2(0,T;H)} \quad \text{for } f, g \in \mathrm{H}^1(0, T; H).$$

Now, we formulate a generalization of the Fundamental Theorem of Calculus for the Lebesgue Integral. For this reason, we recall that a $f \colon (0, T) \to H$ is called *absolutely continuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every sequence of disjoint intervals $(s_n, t_n)_{n\in\mathbb{N}}$ in $[0, T]$ with $\sum_{i\in\mathbb{N}} |t_i - s_i| \leqslant \delta$ there follows $\sum_{i\in\mathbb{N}} \|f(t_i) - f(s_i)\|_H \leqslant \varepsilon$.

Theorem 10.8.
Let $T \in (0, \infty)$ and a separable Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$ be given. A function is $f \colon (0, T) \to H$ is absolutely continuous if and only if

(i) *f is continuous in $[0, T]$,*

(ii) *f is differentiable a.e. in $(0, T)$ with $f' \in \mathrm{L}^1(0, T; H)$,*

(iii) *for $x \in [0, T]$, we have*

$$f(x) = f(0) + \int_{(0,x)} f'(t) \,\mathrm{d}t.$$

Proof.

Taking into account that a Hilbert space is a reflexive Banach space, we refer to Theorem 8.38 in [19] for the proof. □

Using this characterization, we obtain another characterization for $\mathrm{H}^1(0, T; H)$.

Theorem 10.9.
Let $T \in (0, \infty)$ and a separable Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$ be given. For $f \in \mathrm{L}^2(0, T; H)$, we have $f \in \mathrm{H}^1(0, T; H)$ if and only if there is a absolutely continuous function $u \colon [0, T] \to H$ with $u = f$ a.e. in $(0, T)$ and $u' \in \mathrm{L}^2(0, T; H)$. Moreover, we have $f' = u'$ a.e. in $(0, T)$.

Proof.

Taking into account that a Hilbert space is a reflexive Banach space, we refer to Theorem 8.57 in [19] and Theorem 10.8 for the proof. □

In order to generalize the chain rule, we require the following definition.

Definition 10.10.
Let $T \in (0,\infty)$ and a separable Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$ be given. A family $(\mathcal{A}_t)_{t\in[0,T]}$ of linear continuous operators $\mathcal{A}_t \colon H \to H$ with $\operatorname{ess\,sup}_{t\in[0,T]} |\langle \mathcal{A}_t u, v\rangle_H| < \infty$ *for $u, v \in H$ is called* regular *if for each $u, v \in H$ the function $\langle \mathcal{A}_{(\cdot)} u, v\rangle_H \colon [0,T] \to \mathbb{R}$ is absolutely continuous and if there is a $K \in \mathrm{L}^\infty(0,T)$ such that*

$$\left|\frac{\partial}{\partial t}\langle \mathcal{A}_t u, v\rangle_H\right| \leqslant K(t)\|u\|_H\|v\|_H \quad \text{holds for } u, v \in H \text{ and a.e. } t \in (0,T).$$

Furthermore, for a.e. $t \in (0,T)$, we define $\mathcal{A}'_t \colon H \to H$ as the linear operator satisfying

$$\langle \mathcal{A}'_t(u), v\rangle_H = \frac{\partial}{\partial t}\langle \mathcal{A}_t u, v\rangle_H \quad \text{for all } u, v \in H.$$

Then, we obtain the generalized chain rule.

Theorem 10.11.
Let $T \in (0,\infty)$ and a separable Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$ be given. Let $(\mathcal{A}_t)_{t\in[0,T]}$ be a family of linear continuous operators $\mathcal{A}_t \colon H \to H$ with $\operatorname{ess\,sup}_{t\in[0,T]} |\langle \mathcal{A}_t u, v\rangle_H| < \infty$ *for $u, v \in H$. Then, for $u, v \in \mathrm{H}^1(0,T;H)$, the function $\langle \mathcal{A}_{(\cdot)} u(\cdot), v(\cdot)\rangle_H \colon [0,T] \to \mathbb{R}$ is absolutely continuous with*

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\langle \mathcal{A}_t u(t), v(t)\rangle_H\Big) &= \langle \frac{\partial}{\partial t}\Big(\mathcal{A}_t u(t)\Big), v(t)\rangle_H + \langle \mathcal{A}_t u(t), v'(t)\rangle_H) \\ &= \langle \mathcal{A}_t u'(t), v(t)\rangle_H + \langle \mathcal{A}'_t(u(t)), v(t)\rangle_H + \langle \mathcal{A}_t u(t), v'(t)\rangle_H\end{aligned}$$

for a.e. $t \in (0,T)$. Moreover, if there is a $\lambda > 0$ with

$$\langle (\mathcal{A}'_t) u, v\rangle_H + \lambda\langle \mathcal{A}_t u, v\rangle_H \geqslant 0 \quad \text{for all } u, v \in H \text{ and a.e. } t \in (0,T),$$

then the Grönwall's inequality holds, i.e.,

$$\langle \mathcal{A}_t u(t), v\rangle_H \geqslant \exp(-\lambda t)\langle \mathcal{A}_0 u, v\rangle_H \quad \text{for all } u, v \in H \text{ and a.e. } t \in (0,T).$$

Proof.

For the proof, we refer to Proposition 3.1 in [34, Chapter III]. □

Now, we formulate our weak solution.

Definition 10.12.
Let $T \in (0,\infty)$ and a bounded, open, and nonempty set $\Omega \subset \mathbb{R}^d$ be given. Further, let $\gamma \in \mathcal{K}$ be a

regional kernel, and $(\gamma_t)_{t\in(0,T)}$ *be a family of regional symmetric kernels in* $\mathcal{K}$ *such that there is a constant* $C > 0$ *with*

$$\operatorname*{ess\,sup}_{s\in(0,T)} |\mathfrak{B}_s(u,u)| \leqslant C\|u\|^2_{\mathrm{V}(\Omega;\gamma)} \qquad \textit{for all } u \in \mathrm{V}(\Omega;\gamma).$$

For given $f \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$ *and* $u_0 \in \mathrm{L}^2(\Omega)$, *the function* $u \in \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ *is a* (weak) solution *of the nonlocal evolution equation* (E), *i.e.,*

$$\begin{cases} (u'(t))(x) + (\mathcal{L}_{\gamma_t}u(t))(x) &= (f(t))(x) \quad \textit{for } (t,x) \in (0,T)\times\Omega, \\ u(0) &= u_0, \end{cases}$$

if $u(0) = u_0$ *and if for each* $v \in \mathrm{V}(\Omega;\gamma)$, *we have*

$$\frac{\partial}{\partial t}\langle u(t), v\rangle_{\mathrm{L}^2(\Omega)} = \langle f(t), v\rangle_{\mathrm{L}^2(\Omega)} - \mathfrak{B}_t(u(t),v) \qquad \textit{for a.e.}\, t \in (0,T). \tag{WF}$$

Remark 10.13.
Assume that $(\gamma_s)_{s\in(0,T)}$ *is a family of symmetric kernels in* $\mathcal{K}$ *such that for a.e.* $s \in (0,T)$ *we have*

$$\begin{aligned} &\|\gamma_s(\cdot,x)\|_{\mathrm{L}^\infty(\Gamma(\Omega,\gamma_s))} + \int_{\Gamma(\Omega,\gamma_s)} \gamma_s(x,y)\,\mathrm{d}y \leqslant C < \infty && \textit{for } x \in \Omega \\ \textit{and} \quad &\|\gamma_s(\cdot,y)\|_{\mathrm{L}^\infty(\Omega)} + \int_\Omega \gamma_s(y,x)\,\mathrm{d}x \leqslant C < \infty && \textit{for } y \in \Gamma(\Omega,\gamma_s). \end{aligned}$$

Then, we apply Theorem 7.2 and reformulate

$$\begin{cases} (u'(t))(x) + (\mathcal{L}_{\gamma_t}u(t))(x) &= (f(t))(x) \quad \textit{for } (t,x) \in (0,T)\times\Omega, \\ (\mathcal{N}_{\gamma_t}u(t))(y) &= 0 \qquad\qquad\;\; \textit{for } (t,y) \in (0,T)\times\Gamma(\Omega,\gamma_t), \\ u(0) &= u_0, \end{cases}$$

into

$$\begin{cases} (u'(t))(x) + (\mathcal{L}_{\eta t}u(t))(x) &= (f(t))(x) \quad \textit{for } (t,x) \in (0,T)\times\Omega, \\ u(0) &= u_0, \end{cases}$$

where the regional kernels $\eta_t \in$ *kernel is defined by*

$$\eta_t(x,z) := \left(\gamma_t(x,z) + \int_{\Gamma(\Omega,\gamma_t)} \frac{\gamma_t(x,y)\gamma_t(y,z)}{\int_\Omega \gamma_t(y,v)\,\mathrm{d}}\,\mathrm{d}y\right)\chi_{\Omega\times\Omega}(x,z) \qquad \textit{for } x,z \in \mathbb{R}^d$$

Taking a closer look at (WF), we immediately obtain the subsequent assertion.

Theorem 10.14.
Let $T \in (0,\infty)$ *and a bounded, open, and nonempty set* $\Omega \subset \mathbb{R}^d$ *be given. Further, let* $\gamma \in \mathcal{K}$ *be a regional kernel, and* $(\gamma_s)_{s\in(0,T)}$ *be a family of regional symmetric kernels in* $\mathcal{K}$ *such that there is a constant* $C > 0$ *with*

$$\operatorname*{ess\,sup}_{s\in(0,T)} |\mathfrak{B}_s(u,u)| \leqslant C\|u\|_{\mathrm{V}(\Omega;\gamma)} \qquad \textit{for all } u \in \mathrm{V}(\Omega;\gamma).$$

For given $f \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$*, and* $u_0 \in \mathrm{L}^2(\Omega)$*, let the function* $u \in \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ *be a solution of the nonlocal evolution equation* (E)*. Then,*

$$\int_\Omega (u(t))(x)\,\mathrm{d}x = \int_\Omega u_0(x)\,\mathrm{d}x + \int_{(0,t)}\int_\Omega (f(s))(x)\,\mathrm{d}x\,\mathrm{d}s \qquad \textit{holds for a.e.}\ t \in (0,T).$$

Proof.

Let $f \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$ be given and let $u \in \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ be a solution of the nonlocal evolution equation (E), i.e., for each $v \in \mathrm{V}(\Omega;\gamma)$, we have

$$\frac{\partial}{\partial t}\langle u(t), v\rangle_{\mathrm{L}^2(\Omega)} = \langle f(t), v\rangle_{\mathrm{L}^2(\Omega)} - \mathfrak{B}_t(u(t),v) \qquad \text{for a.e.}\ t \in (0,T).$$

Then, we have

$$\frac{\partial}{\partial t}\langle u(s), \chi_{\mathbb{R}^d}\rangle_{\mathrm{L}^2(\Omega)} = \langle f(s), \chi_{\mathbb{R}^d}\rangle_{\mathrm{L}^2(\Omega)} \qquad \text{for a.e.}\ s \in (0,T)$$

and Theorem 10.8 and Theorem 10.9 yield

$$\begin{aligned}\int_\Omega (u(t))(x)\,\mathrm{d}x - \int_\Omega u_0(x)\,\mathrm{d}x =& \langle u(t), \chi_{\mathbb{R}^d}\rangle_{\mathrm{L}^2(\Omega)} - \langle u_0, \chi_{\mathbb{R}^d}\rangle_{\mathrm{L}^2(\Omega)} \\ =& \int_{(0,t)} \frac{\mathrm{d}}{\mathrm{d}t}\langle u(s), \chi_{\mathbb{R}^d}\rangle_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s \\ =& \int_{(0,t)} \langle f(s), \chi_{\mathbb{R}^d}\rangle_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s \\ =& \int_{(0,t)}\int_\Omega (f(s))(x)\,\mathrm{d}x\,\mathrm{d}s \qquad \text{for a.e.}\ t \in (0,T).\end{aligned}$$

□

In other words, Theorem 10.14 states that the total mass in Ω at a time t is given by

$$\int_\Omega u_0(x)\,\mathrm{d}x + \int_{(0,t)}\int_\Omega (f(s))(x)\,\mathrm{d}x\,\mathrm{d}s.$$

In order to study the existence of problem (WF), we require a refinement of the Lax-Milgram Theorem.

Theorem 10.15 (Lions)**.**
Let $(H, \langle\cdot,\cdot\rangle_H)$ *be a Hilbert space and let* $(X, \|\cdot\|_X)$ *be a normed linear space such that* X *is continuously embedded in* H*, i.e., there is a* $C > 0$ *with* $\|x\|_H \leqslant C\|x\|_X$ *for* $x \in X$*. Further, suppose* $\mathfrak{E}\colon H \times X \to \mathbb{R}$ *is bilinear such that* $\mathfrak{E}(\cdot, x)\colon H \to \mathbb{R}$ *is a continuous linear operator for each* $x \in X$*. Then, the following statements are equivalent.*

(i) There is a $c > 0$ *with*

$$\inf_{\|x\|_X=1}\ \sup_{\|h\|_H\leqslant 1} |\mathfrak{E}(h,x)| \geqslant c.$$

(ii) *For each continuous linear operator* $F\colon X \to \mathbb{R}$, *there exists a* $h \in H$ *with*

$$\mathfrak{E}(h, x) = F(x) \quad \text{for } x \in X$$

and

$$\|h\|_H \leqslant \frac{1}{c}\|f\|_{X^*}.$$

Proof.

For the proof, we refer to Theorem 2.1 (Lions) and Corollary 2.1 in [34, III]. □

Recalling the product rule, our aim now is to reformulate (WF) in order to apply Theorem 10.15. For this reason, we require a new function space.

Definition 10.16.
We call "$B \subset H \subset B^*$" *an* evolution triple *if*

(i) $(B, \|\cdot\|_B)$ *is a separable and reflexive Banach space,*

(ii) $(H, \langle\cdot,\cdot\rangle_H)$ *is a separable Hilbert space*

(iii) *there is an injective linear operator* $\iota\colon B \to H$ *and a constant* $C > 0$ *such that*

$$\|\iota(b)\|_H \leqslant C\|b\|_B \quad \text{holds for all } b \in B$$

and that $\{\iota(b)\colon b \in B\}$ *is dense in* H.

Furthermore, we set

$$\mathrm{H}^1(0,T;B;H) := \{u \in \mathrm{L}^2(0,T;B)\colon \Phi(u) \in \mathrm{H}^1(0,T;B^*)\}$$

where $\Phi\colon \mathrm{L}^2(0,T;B) \to \mathrm{L}^2(0,T;B^*)$ *is defined by* $(\Phi(u))(v) = \langle u, v\rangle_H$ *for all* $u, v \in \mathrm{L}^2(0,T;B)$. *For* $u \in \mathrm{H}^1(0,T;B;H)$, *we set* $u' = (\Phi(u))'$, *i.e.,* u' *is the unique element in* $\mathrm{L}^2(0,T;B^*)$ *solving*

$$\int_{(0,T)} \langle u(t), v\rangle_H \varphi'(t)\,\mathrm{d}t = \int_{(0,T)} (u'(t))(v)\varphi(t)\,\mathrm{d}t$$

for all $v \in B$ *and all* $\varphi \in \mathrm{C}_0^\infty(0,T)$.

Theorem 10.17.
Let $T \in (0,\infty)$ *and let* "$B \subset H \subset B^*$" *be an evolution triple. For each* $u \in \mathrm{H}^1(0,T;B;H)$, *there is a unique element in* $u' \in \mathrm{L}^2(0,T;B^*)$. *Furthermore,* $\mathrm{H}^1(0,T;B;H)$ *is a Banach space with respect to*

$$\|u\|_{\mathrm{H}^1(0,T;B;H)} := \|u\|_{\mathrm{L}^2(0,T;B)} + \|u'\|_{\mathrm{L}^2(0,T;B^*)}.$$

Proof.

We refer to Zeidler [41, Proposition 23.20 and Proposition 23.23] for the proof. □

Remark 10.18.
Let $T \in (0,\infty)$, a bounded, open, and nonempty set $\Omega \subset \mathbb{R}^d$, a regional kernel $\gamma \in \mathcal{K}$, and $f \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$ be given. Further let $(\gamma_s)_{s\in(0,T)}$ be a family of regional symmetric kernels in $\mathcal{K}$ such that there is a constant $C > 0$ with

$$\operatorname*{ess\,sup}_{s\in(0,T)} |\mathfrak{B}_s(u,u)| \leqslant C\|u\|^2_{\mathrm{V}(\Omega;\gamma)} \quad \textit{for all } u \in \mathrm{V}(\Omega;\gamma).$$

Then, "$\mathrm{V}(\Omega;\gamma) \subset \overline{\mathrm{V}(\Omega;\gamma)}^{\|\cdot\|_{\mathrm{L}^2(\Omega)}} \subset (\mathrm{V}(\Omega;\gamma))^*$" is an evolution triple, because

(i) $(\mathrm{V}(\Omega;\gamma), \|\cdot\|_{\mathrm{V}(\Omega;\gamma)})$ *is a separable and reflexive Banach space,*

(ii) $(\mathrm{L}^2(\Omega), \langle\cdot,\cdot\rangle_{\mathrm{L}^2(\Omega)})$ *is a separable Hilbert space,*

(iii) the inclusion map $\iota\colon \mathrm{V}(\Omega;\gamma) \to \mathrm{L}^2(\Omega)$ is injective and the inequality $\|v\|_{\mathrm{L}^2(\Omega)} \leqslant \|v\|_{\mathrm{V}(\Omega;\gamma)}$ holds for all $v \in \mathrm{V}(\Omega;\gamma)$.

For simplicity, we set $\mathrm{S} := \overline{\mathrm{V}(\Omega;\gamma)}^{\|\cdot\|_{\mathrm{L}^2(\Omega)}}$.

Lemma 10.19.
Let $T \in (0,\infty)$ and let "$B \subset H \subset B^*$" be an evolution triple. Then, for each $u \in \mathrm{H}^1(0,T;B;H)$, there is a continuous function $f\colon (0,T) \to H$ with $f(t) = \iota(u(t))$ for a.e. $t \in (0,T)$. Furthermore, there is a $C > 0$ such that

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|\iota(u(t))\|_H \leqslant C(\|u\|_{\mathrm{L}^2(0,T;B)}) + \|u'\|_{\mathrm{L}^2(0,T;B^*)}) \quad \textit{for all } u \in \mathrm{H}^1(0,T;B;H)$$

and for all $u, v \in \mathrm{H}^1(0,T;B;H)$, the function $\langle\iota(v(\cdot)), \iota(u(\cdot))\rangle_H\colon (0,T) \to \mathbb{R}$ is absolutely continuous with

$$\frac{\partial}{\partial t}(\langle\iota(v(t)), \iota(u(t))\rangle_H) = (v'(t))(u(t)) + (u'(t))(v(t)) \quad \textit{for a.e. } t \in (0,T).$$

Proof.

We refer to Theorem 8.60 in [19] for the proof where we want to mention that the polarization identity on $(H, \langle\cdot,\cdot\rangle_H)$ yields

$$\langle x, y\rangle_H = \|x+y\|_H^2 - \|x-y\|_H^2 \quad \text{for all } x, y \in H.$$

$\square$

All in all, we now again reformulate our evolution equation (E).

Theorem 10.20.
Let $T \in (0,\infty)$ and a bounded, open, and nonempty set $\Omega \subset \mathbb{R}^d$ be given. Further, let $\gamma \in \mathcal{K}$ be a regional kernel, and $(\gamma_s)_{s\in(0,T)}$ be a family of regional symmetric kernels in $\mathcal{K}$ such that there is a constant $C > 0$ with

$$\operatorname*{ess\,sup}_{s\in(0,T)} |\mathfrak{B}_s(u,u)| \leqslant C\|u\|^2_{\mathrm{V}(\Omega;\gamma)} \quad \textit{for all } u \in \mathrm{V}(\Omega;\gamma).$$

Let $u_0 \in \mathrm{S}$ and $f \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$. Then, the following statements are equivalent.

(i) There is a $u \in \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ *with* $u(0) = u_0$ *and equation* (WF) *is satisfied, i.e., for each* $v \in \mathrm{V}(\Omega;\gamma)$*, we have*

$$\frac{\partial}{\partial t}\langle u(t), v\rangle_{\mathrm{L}^2(\Omega)} = \langle f(t), v\rangle_{\mathrm{L}^2(\Omega)} - \mathfrak{B}_t(u(t), v) \quad \textit{for a.e. } t \in (0,T).$$

(ii) There is a $u \in \mathrm{H}^1(0,T;\mathrm{V}(\Omega;\gamma);S)$ *with* $u(0) = u_0$ *and*

$$\int_{(0,T)} (u'(s))(v(s)) + \mathfrak{B}_s(u(s), v(s))\,\mathrm{d}s = \int_{(0,T)} \langle f(s), v(s)\rangle_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s$$

for all $v \in \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$*.*

(iii) There is a $u \in \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ *such that for every* $v \in \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ *with* $v \in \mathrm{H}^1(0,T;\mathrm{S})$ *and* $v(T) = 0$*, we have*

$$-\int_{(0,T)} \langle u(s), v'(s)\rangle_{\mathrm{L}^2(\Omega)} + \mathfrak{B}_s(u(s), v(s))\,\mathrm{d}s = \int_{(0,T)} \langle f(s), v(s)\rangle_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s + \langle u_0, v(0)\rangle_{\mathrm{L}^2(\Omega)}.$$

The problem of finding a function u satisfying on of these equivalent conditions is called Cauchy problem. Furthermore, we call (i) the weak formulation, (ii) the strong formulation and (iii) the variational formulation.

Proof.

This Theorem is an application of Proposition 2.1 in [34, III]. However, we still present the proof. By Remark 10.18, "$\mathrm{V}(\Omega;\gamma) \subset \mathrm{S} \subset (\mathrm{V}(\Omega;\gamma))^*$" is an evolution triple.

Assume that (*i*) holds. By definition, we obtain $u \in \mathrm{H}^1(0,T;\mathrm{V}(\Omega;\gamma);\mathrm{S})$. As mentioned before, simple functions are dense in $\mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$. Hence, we conclude (*ii*) is valid.

For every $v \in \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ with $v \in \mathrm{H}^1(0,T;\mathrm{S})$, we get $v \in \mathrm{H}^1(0,T;\mathrm{V}(\Omega;\gamma);\mathrm{S})$ with

$$(v'(t))(w) = \langle v'(t), w|_\Omega\rangle_{\mathrm{L}^2(\Omega)} \quad \text{for every } w \in \mathrm{V}(\Omega;\gamma) \text{ and a.e. } t \in (0,T).$$

By Lemma 10.19, we see that (*iii*) follows by (*ii*).

Finally, assume that (*iii*) holds. Then, we have

$$-\int_{(0,T)} \langle u(s), v\rangle_{\mathrm{L}^2(\Omega)}\varphi'(s)\,\mathrm{d}s + \int_{(0,T)} \mathfrak{B}_s(u(s), v)\varphi(s)\,\mathrm{d}s = \int_{(0,T)} \langle f(s), v\rangle_{\mathrm{L}^2(\Omega)}\varphi(s)\,\mathrm{d}s$$

for all $\varphi \in \mathrm{C}_0^\infty(0,T)$ and all $v \in \mathrm{V}(\Omega;\gamma)$. Hence, for each $v \in \mathrm{V}(\Omega;\gamma)$, we see

$$\frac{\partial}{\partial t}(\langle u(t), v\rangle_{\mathrm{L}^2(\Omega)}) = \langle f(t), v\rangle_{\mathrm{L}^2(\Omega)} - \mathfrak{B}_t(u(t), v) \quad \text{for a.e. } t \in (0,T). \tag{10.2}$$

Let $\varrho \in \mathrm{C}^1(0,T)$ with $\varrho(T) = 0$ and $v \in \mathrm{V}(\Omega;\gamma)$ be given. Then, (*iii*) and (10.2) imply

$$\begin{aligned}\int_{(0,T)} (\frac{\partial}{\partial t}\langle u(s), v\rangle_{\mathrm{L}^2(\Omega)})\varrho(s)\,\mathrm{d}s &= \int_{(0,T)} \langle f(s), v\rangle_{\mathrm{L}^2(\Omega)}\varrho(s)\,\mathrm{d}s - \int_{(0,T)} \mathfrak{B}_s(u(s), v)\varrho(s)\,\mathrm{d}s \\ &= -\int_{(0,T)} \langle u(s), v\varrho'(s)\rangle_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s - \langle u_0, v\rangle_{\mathrm{L}^2(\Omega)}\varrho(0).\end{aligned}$$

Therefore, Lemma 10.19 yield

$$
\begin{aligned}
&- \langle u_0, v\rangle_{\mathrm{L}^2(\Omega)}\varrho(0)\\
&= \int_{(0,T)} (\frac{\partial}{\partial t}\langle u(s), v\rangle_{\mathrm{L}^2(\Omega)})\varrho(s)\,\mathrm{d}s + \int_{(0,T)} \langle u(s), v\rangle_{\mathrm{L}^2(\Omega)}\varrho'(s)\,\mathrm{d}s\\
&= \int_{(0,T)} \frac{\partial}{\partial s}(\langle u(s), v\rangle_{\mathrm{L}^2(\Omega)}\varrho(s))\,\mathrm{d}s\\
&= - \langle u(0), v\rangle_{\mathrm{L}^2(\Omega)}\varrho(0).
\end{aligned}
$$

Thus, we conclude, $u(0) = u_0$. □

Before we apply Lion's Theorem (see Theorem 10.15) to obtain an existence result, we generalize problem (E). This makes it possible to consider even more general nonlocal evolution equations, e.g.,

$$
\begin{cases}
\frac{\partial u}{\partial t}(t,x) + \mathcal{L}_{\gamma_t}u(t,x) & = f(t,x) \quad \text{for } (t,x) \in (0,T)\times\Omega,\\
u(0) & = u(T),
\end{cases}
$$

or

$$
\begin{cases}
\frac{\partial}{\partial t}(b(t,x)u(t,x)) + \mathcal{L}_{\gamma_t}u(t,x) & = f(t,x) \quad \text{for } (t,x) \in (0,T)\times\Omega,\\
u(0) & = u_0.
\end{cases}
$$

As before, we first define a weak formulation and then reformulate the weak formulation into equivalent formulations. Afterwards, we apply Lion's Theorem (Theorem 10.15)) in order to get an existence result.

Definition 10.21.
Let $T \in (0,\infty)$ and a bounded, open, and nonempty set $\Omega \subset \mathbb{R}^d$ be given. Further, let $\gamma \in \mathcal{K}$ be a regional kernel, and $(\gamma_s)_{s\in(0,T)}$ be a family of regional symmetric kernels in $\mathcal{K}$ such that there is a constant $C > 0$ with

$$
\operatorname*{ess\,sup}_{s\in(0,T)} |\mathfrak{B}_s(u,u)| \leqslant C\|u\|_{\mathrm{V}(\Omega;\gamma)} \qquad \textit{for all } u \in \mathrm{V}(\Omega;\gamma).
$$

For given $f \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$, $u_0 \in \mathrm{S}$ and a linear operator $\mathcal{B}\colon \mathrm{S} \to \mathrm{S}$ with an adjoint $\mathcal{B}^*$, the function $u \in \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ is a solution of the nonlocal evolution equation

$$
\begin{cases}
\frac{\partial}{\partial t}u(t,x) + \mathcal{L}_{\gamma_t}u(t,x) & = f(t,x) \quad \textit{for } (t,x) \in (0,T)\times\Omega,\\
-\mathcal{B}^*u(T) + u(0) & = u_0,
\end{cases}
\tag{E1}
$$

if $-\mathcal{B}^*u(T) + u(0) = u_0$ and if for each $v \in \mathrm{V}(\Omega;\gamma)$, we have

$$
\frac{\partial}{\partial t}\langle u(t), v\rangle_{\mathrm{L}^2(\Omega)} = \langle f(t), v\rangle_{\mathrm{L}^2(\Omega)} - \mathfrak{B}_t(u(t), v) \qquad \textit{for a.e. } t \in (0,T).
$$

Theorem 10.22.
Let $T \in (0,\infty)$ and a bounded, open, and nonempty set $\Omega \subset \mathbb{R}^d$ be given. Further, let $\gamma \in \mathcal{K}$ be a

regional kernel, and $(\gamma_s)_{s\in(0,T)}$ *be a family of regional symmetric kernels in* $\mathcal{K}$ *such that there is a constant* $C > 0$ *with*

$$\operatorname*{ess\,sup}_{s\in(0,T)} |\mathfrak{B}_s(u,u)| \leqslant C\|u\|^2_{\mathrm{V}(\Omega;\gamma)} \quad \text{for all } u \in \mathrm{V}(\Omega;\gamma).$$

Assume that $\mathcal{B}\colon \mathrm{S} \to \mathrm{S}$ *is a linear operator with an adjoint* $\mathcal{B}^*$*. Then, for given* $u_0 \in \mathrm{S}$ *and* $f \in \mathrm{L}^2(0,T;S)$*, the following statements are equivalent.*

(i) There is a $u \in \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ *with* $-\mathcal{B}^*u(T) + u(0) = u_0$ *and equation* (WF) *is satisfied, i.e., for each* $v \in \mathrm{V}(\Omega;\gamma)$*, we have*

$$\frac{\partial}{\partial t}\langle u(t), v\rangle_{\mathrm{L}^2(\Omega)} = \langle f(t), v\rangle_{\mathrm{L}^2(\Omega)} - \mathfrak{B}_t(u(t),v) \quad \text{for a.e. } t \in (0,T).$$

(ii) There is a $u \in \mathrm{H}^1(0,T;\mathrm{V}(\Omega;\gamma);S)$ *with* $-\mathcal{B}^*u(T) + u(0) = u_0$ *and*

$$\int_{(0,T)} (u'(s))(v(s)) + \mathfrak{B}_s(u(s),v(s))\,\mathrm{d}s = \int_{(0,T)} \langle f(s), v(s)\rangle_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s$$

for all $v \in \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$*.*

(iii) There is a $u \in \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ *such that for every* $v \in \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ *with* $v \in \mathrm{H}^1(0,T;S)$ *and* $v(T) = \mathcal{B}(v(0))$*, we have*

$$-\int_{(0,T)} \langle u(s), v'(s)\rangle_{\mathrm{L}^2(\Omega)} + \mathfrak{B}_s(u(s),v(s))\,\mathrm{d}s = \int_{(0,T)} \langle f(s), v(s)\rangle_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s + \langle u_0, v(0)\rangle_{\mathrm{L}^2(\Omega)}.$$

Proof.

The proof is analogous to the proof of Theorem 10.20. □

Theorem 10.23.
Let $T \in (0,\infty)$ *and a bounded, open, and nonempty set* $\Omega \subset \mathbb{R}^d$ *be given. Further, let* $\gamma \in \mathcal{K}$ *be regional and let* $(\gamma_s)_{s\in(0,T)}$ *be a family of regional symmetric kernels in* $\mathcal{K}$ *such that there are constants* $c, C > 0$ *with* $c\gamma \leqslant \gamma_s \leqslant C\gamma$ *for a.e.* $s \in (0,T)$*. Let* $u_0 \in \mathrm{S}$*,* $f \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$ *and* $\mathcal{B}\colon \mathrm{S} \to \mathrm{S}$ *be a linear nonexpansive map, i.e.,*

$$\|\mathcal{B}(u) - \mathcal{B}(v)\|_{\mathrm{L}^2(\Omega)} \leqslant \|u - v\|_{\mathrm{L}^2(\Omega)} \quad \text{for } u, v \in \mathrm{S},$$

with an adjoint $\mathcal{B}^*$*. Then, there is a* $u \in \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ *such that for every* $v \in \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ *with* $v \in \mathrm{H}^1(0,T;S)$ *and* $v(T) = \mathcal{B}(v(0))$*, we have*

$$\int_{(0,T)} -\langle u(s), v'(s)\rangle_{\mathrm{L}^2(\Omega)} + \mathfrak{B}_s(u(s),v(s))\,\mathrm{d}s = \int_{(0,T)} \langle f(s), v(s)\rangle_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s + \langle u_0, v(0)\rangle_{\mathrm{L}^2(\Omega)}. \tag{10.3}$$

If $\mathcal{B}$ *is a linear contraction, i.e., there is a* $c < 1$ *with*

$$\|\mathcal{B}(u) - \mathcal{B}(v)\|_{\mathrm{L}^2(\Omega)} \leqslant c\|u - v\|_{\mathrm{L}^2(\Omega)} \quad \text{for } u, v \in \mathrm{L}^2(\Omega),$$

then we even obtain that there is at most one u *solving the variational formulation.*

Proof.

Follows by proposition 2.4 in [34] but we nevertheless present the proof. Without loss of generality, we assume that the nonlocal Poincaré inequality holds on $\mathrm{V}(\Omega;\gamma)$. If this is not the case we consider an exponential shift. To be more precise, we recall that for every $u \in \mathrm{H}^1(0,T;\mathrm{V}(\Omega;\gamma),\mathrm{S})$ and every $\lambda > 0$, we have $z \in \mathrm{H}^1(0,T;\mathrm{V}(\Omega;\gamma),\mathrm{S})$ where $z(t) = \exp(-\lambda t)u(t)$ for a.e. $t \in (0,T)$. Furthermore, for each $v \in \mathrm{V}(\Omega;\gamma)$, Lemma 10.19 yields

$$\begin{aligned}\frac{\partial}{\partial t}\langle z(t), v\rangle_{\mathrm{L}^2(\Omega)} =& \frac{\partial}{\partial t}\langle u(t), \exp(-\lambda t)v\rangle_{\mathrm{L}^2(\Omega)} \\ =& \exp(-\lambda t)\frac{\partial}{\partial t}\langle u(t), v\rangle_{\mathrm{L}^2(\Omega)} + -\lambda\langle z(t), v\rangle_{\mathrm{L}^2(\Omega)}\end{aligned}$$

for a.e. $t \in (0,T)$. Set $H := \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ and

$$\langle u, v\rangle_1 := \int_{(0,T)} \mathfrak{B}_s(u(s), v(s))\,\mathrm{d}s \quad \text{for } u, v \in H.$$

Then, $(H, \langle\cdot,\cdot\rangle_1)$ is a Hilbert space due to fact that the nonlocal Poincaré inequality on $\mathrm{V}(\Omega;\gamma)$ and the fact that $c\gamma \leqslant \gamma_s \leqslant C\gamma$ holds for a.e. $s \in (0,T)$. We further set

$$X := \{v \in \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))\colon v \in \mathrm{H}^1(0,T;\mathrm{S}) \text{ and } v(T) = \mathcal{B}(v(0))\}$$

and recall that for $v \in X$, we have

$$\int_{(0,T)} \langle v(s), g\rangle_{\mathrm{L}^2(\Omega)}\varphi'(s)\,\mathrm{d}s = -\int_{(0,T)} \langle v'(s), g\rangle_{\mathrm{L}^2(\Omega)}\varphi(s)\,\mathrm{d}s \quad \text{for all } g \in \mathrm{V}(\Omega;\gamma) \text{ and } \varphi \in \mathrm{C}_0^\infty(0,T).$$

Therefore, $v \in \mathrm{H}^1(0,T;(\mathrm{V}(\Omega;\gamma),\mathfrak{B}(\cdot,\cdot));\mathrm{S})$ holds and we set $\|v\|_X = \|v\|_{\mathrm{L}^2(0,T;(\mathrm{V}(\Omega;\gamma),\mathfrak{B}(\cdot,\cdot));\mathrm{S})}$. Then, $F_f\colon X \to \mathbb{R}$ defined by

$$\int_{(0,T)} \langle f(s), x(s)\rangle_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s + \langle u_0, x(0)\rangle_{\mathrm{L}^2(\Omega)} \quad \text{for } x \in X$$

is a continuous linear operator by Theorem 10.8. We further set

$$\mathfrak{E}(h, x) = \int_{(0,T)} -\langle h(s), x'(s)\rangle_{\mathrm{L}^2(\Omega)} + \mathfrak{B}_s(h(s), x(s))\,\mathrm{d}s \quad \text{for } h \in H \text{ and } x \in X.$$

Then, for each $x \in X$, the operator $\mathfrak{E}(\cdot, x)\colon H \to \mathbb{R}$ is continuous. Moreover, for each $x \in C$, the Riesz representation Theorem yields the existence of a $h_x \in H$ with

$$\int_{(0,T)} \langle h(s), x'(s)\rangle_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s = \langle h_x, h\rangle_1 \quad \text{for all } h \in H. \tag{10.4}$$

Hence, we get

$$\sup_{\|h\|\leqslant 1} |\mathfrak{E}(h, x)| = \sup_{\|h\|\leqslant 1} \langle -h_x + x, h\rangle_1 = \| -h_x + x\|_1 \quad \text{for all } x \in X.$$

By (10.4) and Theorem 10.8, we therefore obtain, for each $x \in X$,

$$
\begin{aligned}
\| - h_x + x\|_1^2 &= \langle -h_x + x, -h_x + x\rangle_1 \\
&= -\langle h_x, -h_x + x\rangle_1 + \langle -h_x + x, x\rangle_1 \\
&= \int_{(0,T)} -\langle -h_x(s) + x(s), x'(s)\rangle_{\mathrm{L}^2(\Omega)} + \mathfrak{B}_s(-h_x(s) + x(s), x(s))\,\mathrm{d}s \\
&= \int_{(0,T)} \langle h_x(s), x'(s)\rangle_{\mathrm{L}^2(\Omega)} - \langle x(s), x'(s)\rangle_{\mathrm{L}^2(\Omega)} - \mathfrak{B}_s(h_x(s), x(s)) + \mathfrak{B}_s(x(s), x(s))\,\mathrm{d}s \\
&= \|x\|_X^2 - \int_{(0,T)} 2\langle x(s), x'(s)\rangle_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s \\
&= \|x\|_X^2 - \int_{(0,T)} \frac{\partial}{\partial t}\langle x(s), x(s)\rangle_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s \\
&= \|x\|_X^2 - \|B(x(0))\|_{\mathrm{L}^2(\Omega)}^2 + \|x(0)\|_{\mathrm{L}^2(\Omega)}^2 \\
&\geqslant \|x\|_X^2.
\end{aligned}
$$

By Lion's Theorem (see Theorem 10.15), there is a solution.

Assume that both $u_1 \in \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ and $u_2 \in \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ solve equation (10.3). Then, by following the proof of Theorem 10.20, we get $u_1, u_2 \in \mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma);\mathrm{S})$ and

$$-\mathcal{B}^*u_1(T) + u_1(0) = -\mathcal{B}^*u_2(T) + u_2(0) = u_0.$$

Further, Lemma 10.19 implies

$$
\begin{aligned}
&\frac{\partial}{\partial t}\|u_1(t) - u_2(t)\|_{\mathrm{L}^2(\Omega)}^2 \\
=&\frac{\partial}{\partial t}\langle u_1(t) - u_2(t), u_1(t) - u_2(t)\rangle_{\mathrm{L}^2(\Omega)} \\
=&2(u_1'(t) - u_2'(t))(u_1(t) - u_2(t)) \\
=&2\langle f(t), u_1(t) - u_2(t)\rangle_{\mathrm{L}^2(\Omega)} - 2\mathfrak{B}_t(u_1(t) - u_2(t), u_1(t) - u_2(t)) \\
\leqslant&2\langle f(t), u_1(t) - u_2(t)\rangle_{\mathrm{L}^2(\Omega)}
\end{aligned}
$$

for a.e. $t \in (0,T)$. Hence, we obtain

$$\frac{\partial}{\partial t}\|u_1(t) - u_2(t)\|_{\mathrm{L}^2(\Omega)}^2 \leqslant \min\{2\langle f(t), u_1(t) - u_2(t)\rangle_{\mathrm{L}^2(\Omega)}, 2\langle f(t), u_2(t) - u_1(t)\rangle_{\mathrm{L}^2(\Omega)}\} \leqslant 0$$

for a.e. $t \in (0,T)$. Therefore, there is a $c \leqslant 1$ with

$$\|u_1(T) - u_2(T)\|_{\mathrm{L}^2(\Omega)}^2 \leqslant \|u_1(0) - u_2(0)\|_{\mathrm{L}^2(\Omega)}^2 = \|\mathcal{B}^*(u_1(T) - u_2(T))\|_{\mathrm{L}^2(\Omega)}^2 \leqslant c\|u_1(T) - u_2(T)\|_{\mathrm{L}^2(\Omega)}^2$$

implying that $\|u_1(\cdot) - u_2(\cdot)\|_{\mathrm{L}^2(\Omega)}^2 \colon (0,T) \to [0,\infty)$ is a.e. constant. And if for a.e. $t \in (0,T)$, we even have $\|u_1(t) - u_2(t)\|_{\mathrm{L}^2(\Omega)}^2 = 0$, then $\mathfrak{B}_t(u_1(t) - u_2(t), u_1(t) - u_2(t)) = 0$ and $u_1 = u_2$. □

Definition 10.24.
Let $T \in (0,\infty)$ and a bounded, open and nonempty set $\Omega \subset \mathbb{R}^d$ be given. Further, $\gamma \in \mathcal{K}$ be a regional kernel, and $(\gamma_s)_{s\in(0,T)}$ be a family of regional symmetric kernels in $\mathcal{K}$ such that there is a constant $C > 0$ with

$$\operatorname*{ess\,sup}_{s\in(0,T)} |\mathfrak{B}_s(u,u)| \leqslant C\|u\|_{\mathrm{V}(\Omega;\gamma)} \quad \text{for all } u \in \mathrm{V}(\Omega;\gamma).$$

Assume that $(\mathcal{B}_t)_{t\in(0,T)}$ *is a family of bounded linear operators* $\mathcal{B}_t\colon \mathrm{S}\to\mathrm{S}$ *with*

$$\operatorname*{ess\,sup}_{t\in(0,T)}\langle\mathcal{B}_t u, v\rangle_{\mathrm{L}^2(\Omega)} < \infty \quad \text{for all } u, v\in\mathrm{S}.$$

Then, for given $f\in\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$ *and* $u_0\in\mathrm{S}$*, the function* $u\in\mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ *is a solution of the nonlocal evolution equation*

$$\begin{cases} \frac{\partial}{\partial t}(\mathcal{B}_t(u(t)))(x) + (\mathcal{L}_{\gamma_t}u(t))(x) &= (f(t))(x) \quad \text{for } (t,x)\in(0,T)\times\Omega,\\ \mathcal{B}_0(u(0)) &= \mathcal{B}_0(u_0), \end{cases} \tag{E2}$$

if $\mathcal{B}_0(u(0)) = \mathcal{B}_0(u_0)$ *and if for each* $v\in\mathrm{V}(\Omega;\gamma)$*, we have*

$$\frac{\partial}{\partial t}\langle\mathcal{B}_t(u(t)), v\rangle_{\mathrm{L}^2(\Omega)} = \langle f(t), v\rangle_{\mathrm{L}^2(\Omega)} - \mathfrak{B}_t(u(t),v) \quad \text{for a.e. } t\in(0,T).$$

Theorem 10.25.
Let $T\in(0,\infty)$ *and a bounded, open, and nonempty set* $\Omega\subset\mathbb{R}^d$ *be given. Further, let* $\gamma\in\mathcal{K}$ *be a regional kernel, and* $(\gamma_s)_{s\in(0,T)}$ *be a family of regional symmetric kernels in* $\mathcal{K}$ *such that there is a constant* $C>0$ *with*

$$\operatorname*{ess\,sup}_{s\in(0,T)}|\mathfrak{B}_s(u,u)| \leqslant C\|u\|^2_{\mathrm{V}(\Omega;\gamma)} \quad \text{for all } u\in\mathrm{V}(\Omega;\gamma).$$

Assume that $(\mathcal{B}_t)_{t\in(0,T)}$ *is a family of bounded linear operators* $\mathcal{B}_t\colon \mathrm{S}\to\mathrm{S}$ *with*

$$\operatorname*{ess\,sup}_{t\in(0,T)}\langle\mathcal{B}_t u, v\rangle_{\mathrm{L}^2(\Omega)} < \infty \quad \text{for all } u, v\in\mathrm{S}.$$

Then, for given $f\in\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))$ *and* $u_0\in\mathrm{S}$*, the following statements are equivalent.*

(i) There is a $u\in\mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ *with* $\mathcal{B}_0(u(0)) = \mathcal{B}_0(u_0)$ *and for each* $v\in\mathrm{V}(\Omega;\gamma)$*, we have*

$$\frac{\partial}{\partial t}\langle\mathcal{B}_t(u(t)), v\rangle_{\mathrm{L}^2(\Omega)} = \langle f(t), v\rangle_{\mathrm{L}^2(\Omega)} - \mathfrak{B}_t(u(t),v) \quad \text{for a.e. } t\in(0,T).$$

(ii) There is a $u\in\mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ *with* $\mathcal{B}_\cdot(u(\cdot))\in\mathrm{H}^1(0,T;\mathrm{V}(\Omega;\gamma);S)$*,* $\mathcal{B}_0(u(0)) = \mathcal{B}_0(u_0)$ *and*

$$\int_{(0,T)}\left(\frac{\partial}{\partial t}(\mathcal{B}_s(u(s))\right)(v(s)) + \mathfrak{B}_s(u(s),v(s))\,\mathrm{d}s = \int_{(0,T)}\langle f(s), v(s)\rangle_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s$$

for all $v\in\mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$*.*

(iii) There is a $u\in\mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ *such that for every* $v\in\mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))$ *with* $v\in\mathrm{H}^1(0,T;S)$ *and* $v(T)=0$*, we have*

$$\begin{aligned}&-\int_{(0,T)}\langle\mathcal{B}_s(u(s)), v'(s)\rangle_{\mathrm{L}^2(\Omega)} + \mathfrak{B}_s(u(s),v(s))\,\mathrm{d}s\\ &= \int_{(0,T)}\langle f(s), v(s)\rangle_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s + \langle\mathcal{B}_0(u_0), v(0)\rangle_{\mathrm{L}^2(\Omega)}.\end{aligned}$$

Proof.

The proof is analogous to the proof of Theorem 10.20 where we apply Theorem 10.11. $\square$

Theorem 10.26.
Let $T \in (0, \infty)$ and a bounded, open, and nonempty set $\Omega \subset \mathbb{R}^d$ be given. Further, let $\gamma \in \mathcal{K}$ be a regional kernel, and $(\gamma_s)_{s\in(0,T)}$ be a family of regional symmetric kernels in $\mathcal{K}$ such that there is a constant $C > 0$ with

$$\operatorname*{ess\,sup}_{s\in(0,T)} |\mathfrak{B}_s(u, u)| \leqslant C\|u\|^2_{\mathrm{V}(\Omega;\gamma)} \quad \textit{for all } u \in \mathrm{V}(\Omega; \gamma).$$

Furthermore, assume that $(\mathcal{B}_t)_{t\in(0,T)}$ is a regular family of continuous self-adjoint linear operators $\mathcal{B}_t\colon \mathrm{S} \to \mathrm{S}$ such that there are constants $a, b, c > 0$ with

$$\langle \mathcal{B}'_t(v), v\rangle_{\mathrm{L}^2(\Omega)} + a\langle \mathcal{B}_t(v), v\rangle_{\mathrm{L}^2(\Omega)} \geqslant 0 \quad \textit{and} \quad \mathfrak{B}_t(u, u) + b\langle \mathcal{B}_t(v), v\rangle_{\mathrm{L}^2(\Omega)} \geqslant c\|v\|^2_{\mathrm{V}(\Omega;\gamma)}$$

for all $v \in \mathrm{V}(\Omega; \gamma)$ and a.e. $t \in (0, T)$. Then, for every $u_0 \in$ and $f \in \mathrm{L}^2(0, T; \mathrm{L}^2(\Omega))$, there is a $u \in \mathrm{L}^2(0, T; \mathrm{V}(\Omega; \gamma))$ and a $\alpha > 0$ such that for every $v \in \mathrm{L}^2(0, T; \mathrm{V}(\Omega; \gamma))$ with $v \in \mathrm{H}^1(0, T; S)$ and $v(T) = \mathcal{B}(v(0))$, we have

$$-\int_{(0,T)} \langle \mathcal{B}_s(u(s)), v'(s)\rangle_{\mathrm{L}^2(\Omega)} + \mathfrak{B}_s(u(s), v(s))\,\mathrm{d}s = \int_{(0,T)} \langle f(s), v(s)\rangle_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s + \langle \mathcal{B}_0(u_0), v(0)\rangle_{\mathrm{L}^2(\Omega)}.$$

and

$$\|u\|_{\mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))} \leqslant \alpha(\|f\|_{\mathrm{L}^2(0,T;\mathrm{V}(\Omega;\gamma))} + \langle \mathcal{B}_0 u_0, u_0\rangle_{\mathrm{L}^2(\Omega)}).$$

If we further assume that $\gamma.(y, x)\colon (0, T) \to [0, \infty)$ is absolutely continuous for a.e. $(y, x) \in \Omega \times \Omega$ and that there is a $k \in \mathrm{L}^1(0, T)$ with $\frac{\partial}{\partial t}\gamma_t \leqslant k(t)\gamma$ for a.e. $t \in (0, T)$, then there is at most one u solving the variational formulation.

Proof.

This is a consequence of Proposition 3.2 and Proposition 3.3 in [34, Chapter III]. $\square$

Chapter 11

Vanishing Nonlocality

As it is shown in [25], [26], [3] and [11], if $\Omega \subset \mathbb{R}^d$ is a smooth bounded domain, then there are families $(\gamma_\varepsilon)_{\varepsilon>0}$ in $\mathcal{K}$ such that $\lim_{\varepsilon\to 0} \|u\|_{\mathrm{V}(\Omega;\gamma_\varepsilon)} = \|u\|_{\mathrm{H}^1(\Omega)}$ for $u \in \mathrm{H}^1(\Omega)$ and

$$\lim_{\varepsilon\to 0} \mathcal{L}_\varepsilon v(x) = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} v(x)\gamma_\varepsilon(x,y) - v(y)\gamma_\varepsilon(y,x)\,\mathrm{d}y = -\Delta v(x) \quad \text{for } v \in \mathrm{C}_0^2(\mathbb{R}^d) \text{ and } x \in \Omega.$$

For example, consider $\gamma_\varepsilon(y,x) := \frac{2(d+2)}{\varepsilon^2|\mathrm{B}_\varepsilon(0)|}\chi_{\mathrm{B}_\varepsilon(0)}(y-x)$ for $\varepsilon > 0$ and $y, x \in \mathbb{R}^d$. Then, it is a well-known result that for every $u \in \mathrm{C}_0^2(\mathbb{R}^d)$ we have

$$\lim_{\varepsilon\to 0} \frac{2(d+2)}{\varepsilon^2|\mathrm{B}_\varepsilon(0)|} \int_{\mathrm{B}_\varepsilon(x)} (u(x) - u(y))\,\mathrm{d}y = -\Delta u(x) \quad \text{for } x \in \mathbb{R}^d.$$

Furthermore, for every $u \in \mathrm{C}_0^\infty(\mathbb{R}^d)$ by [5, Proposition 4.4] we obtain

$$\lim_{\varepsilon\to 0} \frac{c_{d,1-\varepsilon}}{2} \int_{\mathbb{R}^d} \frac{u(x) - u(y)}{\|y-x\|^{d+2(1-\varepsilon)}}\,\mathrm{d}y = -\Delta u(x) \quad \text{for } x \in \mathbb{R}^d.$$

In the following, we define the families in $\mathcal{K}$ we consider.

Definition 11.1.
Let $\Omega \subset \mathbb{R}^d$ be open. Henceforth, we define $\overline{\mathcal{K}}(\Omega)$ to be the families $(\gamma_\varepsilon)_{\varepsilon>0}$ in $\mathcal{K}$ such that there is a family $(\varphi_\varepsilon)_{\varepsilon>0}$ in $\mathrm{L}^1(\mathbb{R}^d)$ with

$$\begin{cases} \min\{1, \|y-x\|^2\} \max\{\gamma_\varepsilon(y,x), \gamma_\varepsilon(x,y)\} \leqslant \varphi_\varepsilon(y-x) & \textit{for a.e.}\ (y,x) \in (\mathbb{R}^d \times \Omega), \\ \lim\limits_{\varepsilon\to 0} \displaystyle\int_{\mathbb{R}^d \setminus \mathrm{B}_\delta(0)} \varphi_\varepsilon(z)\,\mathrm{d}z = 0 & \textit{for all}\ \delta > 0, \\ \sup\limits_{\varepsilon>0} \|\varphi_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^d)} < \infty. \end{cases}$$

Because we are, in general, only interested in the limit, we henceforth assume

$$\|y-x\|^2 \max\{\gamma_\varepsilon(y,x), \gamma_\varepsilon(x,y)\} \leqslant \varphi_\varepsilon(y-x) \quad \text{for a.e.}\ (y,x) \in (\mathbb{R}^d \times \Omega).$$

If we have a nonnegative function $\varphi \in \mathrm{L}^1(\mathbb{R}^d)$, then [3, Theorem 1] yields that there is a $C > 0$ such that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(u(x)-u(y))^2}{\|x-y\|^2}\varphi(y-x)\,\mathrm{d}y\,\mathrm{d}x \leqslant C\|u\|^2_{\mathrm{H}^1(\mathbb{R}^d)}\|\varphi\|_{\mathrm{L}^1(\mathbb{R}^d)} \quad \text{holds for all } u \in \mathrm{H}^1(\mathbb{R}^d).$$

Therefore, if we have a family $(\gamma_\varepsilon)_{\varepsilon>0} \in \overline{\mathcal{K}}(\Omega)$, then for every $\varepsilon > 0$, we have $\mathrm{H}^1(\mathbb{R}^d) \subset \mathrm{V}(\Omega;\gamma_\varepsilon)$.

In the following, our aim is to show that for a family in $\overline{\mathcal{K}}(\Omega)$ our nonlocal diffusion-convection operator converges weakly to a local diffusion-convection operator in $\mathrm{L}^2(\Omega)$. The coefficients of the local diffusion-convection operator are, in this case, given by the limits of

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{1}{2} y_i\, y_j \gamma_\varepsilon(y+x,x)\,\mathrm{d}y,\\
&\int_{\mathbb{R}^d} y_k \gamma_\varepsilon(y+x,x)\,\mathrm{d}y\\
\text{and}\quad &\int_{\mathbb{R}^d} \gamma_\varepsilon(x,y+x) - \gamma_\varepsilon(y+x,x)\,\mathrm{d}y
\end{aligned}$$

in some topological space. So, we either have to assume the existence of these limits or we require assumptions so that we at least have a convergent subsequence. We choose the latter.

So, before we present our first convergence result we give sufficient assumptions for the existence of a convergent subsequence. We recall the well known property that every bounded sequence in a Hilbert space admits a weakly convergent subsequence. To be more precise, a Banach space is reflexive if and only if every bounded sequence has a weakly convergent subsequence. This is a consequence of the Eberlein–Šmulian Theorem (see [38]) and the Kakutani Theorem (see [16, Theorem 3.17]).

Both the Eberlein–Šmulian Theorem and the Kakutani Theorem are consequences of the most essential property of the the weak*-topology, the compactness of the unit ball in the weak*-topology.

Definition 11.2.
Let $(B,\|\cdot\|_B)$ be a Banach space. Then, the weak-topology $\sigma(B^*,B)$ is the weakest topology on B^* such that for all $b \in B$, the map $J_b\colon B^* \to \mathbb{R}$ by $J_b(F) = F(b)$ for $F \in B^*$ is continuous.*

Furthermore, a sequence $(F_n)_{n\in\mathbb{N}}$ in B^* converges to $F \in B^*$ in the weak-topology $\sigma(B^*,B)$ if and only if*

$$F_n(b) \to F(b) \quad \text{as } n \to \infty \text{ holds for all } b \in B.$$

We now highkight and recall the most essential property of the weak*-topology.

Theorem 11.3 (Banach–Alaoglu–Bourbaki)**.**
Let $(B,\|\cdot\|_B)$ be a Banach space. Then, the closed unit ball

$$\{F \in B^*\colon \sup_{b\in B}\|F(b)\| \leqslant 1\}$$

is compact in the weak-topology $\sigma(B^*,B)$.*

Proof.

For the proof, we refer to [16, Theorem 3.16]. □

Now, we require the closed unit ball to metrizable in the weak*-topology. If there is a metric on the closed unit ball inducing the weak*-topology, then the compactness of the closed unit ball implies that every sequence in the closed unit ball has a convergent subsequence whose limit is in the closed unit ball. Fortunately, we have:

Theorem 11.4.
Let $(B, \|\cdot\|_B)$ be a Banach space. Then, the closed unit ball

$$\{F \in B^*\colon \sup_{b\in B} \|F(b)\| \leqslant 1\}$$

is metrizable in the weak-topology $\sigma(B^*, B)$ if and only if B is separable.*

Proof.

For the proof, we refer to [16, Theorem 3.28]. □

Corollary 11.5.
Let $(B, \|\cdot\|_B)$ be a separable Banach space and $(F_n)_{n\in\mathbb{N}}$ be a bounded sequence in B^*. Then, there exists a subsequence $(F_{n_\ell})_{\ell\in\mathbb{N}}$ and a $F \in B^*$ with

$$F_{n_\ell}(b) \to F(b) \quad \text{as } \ell \to \infty \text{ for all } b \in B.$$

Proof.

This is a consequence of the Banach–Alaoglu–Bourbaki Theorem, Theorem 11.4 and Definition 11.2. □

As a consequence, we obtain the following well-known theorem.

Theorem 11.6.
Let $\Omega \subset \mathbb{R}^d$ be measurable. Then, $\mathrm{L}^1(\Omega)$ is a separable Banach space and for every $F \in (\mathrm{L}^1(\Omega))^*$, there is a $f \in \mathrm{L}^\infty(\Omega)$ with

$$F(u) = \int_\Omega u(x)f(x)\,\mathrm{d}x \text{ for all } u \in \mathrm{L}^1(\Omega) \quad \text{and} \quad \|f\|_{\mathrm{L}^\infty(\Omega)} = \sup_{\|u\|_{\mathrm{L}^1(\Omega)}\leqslant 1} |F(u)|.$$

Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathrm{L}^\infty(\Omega)$. Then, there exists a subsequence $(f_{n_\ell})_{\ell\in\mathbb{N}}$ and a $f \in \mathrm{L}^\infty(\Omega)$ with

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega f_{n_\ell}(x)u(x)\,\mathrm{d}x = \int_\Omega f(x)u(x)\,\mathrm{d}x \quad \text{for all } u \in \mathrm{L}^1(\Omega) \\ \text{and} \quad &\|f\|_{\mathrm{L}^\infty(\Omega)} \leqslant \liminf_{\ell\to\infty} \|f_{n_\ell}\|_{\mathrm{L}^\infty(\Omega)}. \end{aligned}$$

Proof.

The proof that $\mathrm{L}^1(\Omega)$ is a separable Banach space and that $(\mathrm{L}^1(\Omega))^*$ and $\mathrm{L}^\infty(\Omega)$ are isometrically isomorphic can be found in [16, Theorem 4.8, Theorem 4.13 and Theorem 4.14]. The rest follows by Corollary 11.5 and

$$\|f\|_{\mathrm{L}^\infty(\Omega)} = \sup_{\|u\|_{\mathrm{L}^1(\Omega)}\leqslant 1} |\int_\Omega f(x)u(x)\,\mathrm{d}x| \leqslant \liminf_{\ell\to\infty} \|f_{n_\ell}\|_{\mathrm{L}^\infty(\Omega)}.$$

□

Finally, we need the following estimates for our convergence results which we obtain by following the proof of [16, Proposition 9.3].

Lemma 11.7.
If $u \in \mathrm{H}^2(\mathbb{R}^d)$, then for all $h \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} (u(x+h) - u(x) - \langle \nabla u(x), h\rangle)^2 \, \mathrm{d}x \leqslant \frac{d^2}{2}\|h\|^4 \int_{\mathbb{R}^d} \sum_{i,j=1}^d \left(\frac{\partial^2}{\partial x_i \partial x_j} u(x)\right)^2 \mathrm{d}x.$$

Proof.

Let $h \in \mathbb{R}^d$ be given. First, we assume $u \in \mathrm{C}_0^\infty(\mathbb{R}^d)$. Then, the Fundamental Theorem of Calculus yields

$$\begin{aligned}
u(x+h) - u(x) - \langle \nabla u(x), h\rangle &= \int_{(0,1)} \sum_{i=1}^d h_i \left(\frac{\partial}{\partial x_i} u(x+th) - \frac{\partial}{\partial x_i} u(x)\right) \mathrm{d}t \\
&= \sum_{i=1}^d \int_{(0,1)} h_i \left(\frac{\partial}{\partial x_i} u(x+th) - \frac{\partial}{\partial x_i} u(x)\right) \mathrm{d}t \\
&= \sum_{i=1}^d \int_{(0,1)} h_i \left(\sum_{j=1}^d \int_{(0,1)} \frac{\partial^2}{\partial x_i \partial x_j} u(x+sth) t h_j \, \mathrm{d}s\right) \mathrm{d}t \\
&= \int_{0,1} \int_{0,1} t \sum_{i,j=1}^d h_i \frac{\partial^2}{\partial x_i \partial x_j} u(x+sth) h_j \, \mathrm{d}s \, \mathrm{d}t.
\end{aligned}$$

Hence,

$$(u(x+h) - u(x) - \langle \nabla u(x), h\rangle)^2 \leqslant d^2 \|h\|^4 \int_{0,1} \int_{0,1} \sum_{i,j=1}^d \left(\frac{\partial^2}{\partial x_i \partial x_j} u(x+sth)\right)^2 \mathrm{d}s \, \mathrm{d}t$$

and, by substitution,

$$\begin{aligned}
\int_{\mathbb{R}^d} (u(x+h) - u(x) - \langle \nabla u(x), h\rangle)^2 \, \mathrm{d}x \leqslant & d^2 \|h\|^4 \int_{\mathbb{R}^d} \int_{0,1} \int_{0,1} t \sum_{i,j=1}^d \left(\frac{\partial^2}{\partial x_i \partial x_j} u(x+sth)\right)^2 \mathrm{d}s \, \mathrm{d}t \, \mathrm{d}x \\
= & \frac{d^2}{2} \|h\|^4 \int_{\mathbb{R}^d} \sum_{i,j=1}^d \left(\frac{\partial^2}{\partial x_i \partial x_j} u(x)\right)^2 \mathrm{d}x.
\end{aligned}$$

By Theorem 11.35 in [19]), we obtain that for every $u \in \mathrm{H}^2(\mathbb{R}^d)$, there is a sequence $(u_n)_{n \in \mathbb{N}}$ in $\mathrm{C}_0^\infty(\mathbb{R}^d)$ with

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \left(u_n(x+h) - u_n(x) - \nabla u_n(x)^\top h\right)^2 \mathrm{d}x = \int_{\mathbb{R}^d} \left(u(x+h) - u(x) - \nabla u(x)^\top h\right)^2 \mathrm{d}x$$

$$\text{and} \quad \lim_{n\to\infty} \int_{\mathbb{R}^d} \sum_{i,j=1}^d \left(\frac{\partial^2}{\partial x_i \partial x_j} u_n(x)\right)^2 \mathrm{d}x = \int_{\mathbb{R}^d} \sum_{i,j=1}^d \left(\frac{\partial^2}{\partial x_i \partial x_j} u(x)\right)^2 \mathrm{d}x.$$

Therefore, we conclude our statement. □

With this Lemma, we obtain our first convergence result.

Theorem 11.8.
Let $\Omega \subset \mathbb{R}^d$ be open and let $(\gamma_\varepsilon)_{\varepsilon>0} \in \overline{\mathcal{K}}(\Omega)$ satisfy

$$\sup_{\varepsilon>0} \sum_{k=1}^{d} \left\| \int_{\mathbb{R}^d} y_k \gamma_\varepsilon(y+\cdot,\cdot)\,\mathrm{d}y \right\|_{\mathrm{L}^\infty(\Omega)} + \sup_{\varepsilon>0} \left\| \int_{\mathbb{R}^d} \gamma_\varepsilon(\cdot, y+\cdot) - \gamma_\varepsilon(y+\cdot,\cdot)\,\mathrm{d}y \right\|_{\mathrm{L}^\infty(\Omega)} < \infty.$$

Then, there is a sequence $(\varepsilon_\ell)_{\ell\in\mathbb{N}}$ in $(0,\infty)$ with $\lim_{\ell\to\infty} \varepsilon_\ell = 0$ such that for $i,j,k = 1,\ldots,d$ there exist functions $a_{i,j}, \widetilde{b}_k, c \in \mathrm{L}^\infty(\Omega)$ with

$$\begin{aligned}
&\lim_{\ell\to\infty} \int_\Omega \left(\int_{\mathbb{R}^d} \frac{1}{2} y_i\, y_j \gamma_{\varepsilon_\ell}(y+x,x)\,\mathrm{d}y \right) v(x)\,\mathrm{d}x = \int_\Omega a_{i,j}(x) v(x)\,\mathrm{d}x,\\
&\lim_{\ell\to\infty} \int_\Omega \left(\int_{\mathbb{R}^d} y_k \gamma_{\varepsilon_\ell}(y+x,x)\,\mathrm{d}y \right) v(x)\,\mathrm{d}x = \int_\Omega \widetilde{b}_k(x) v(x)\,\mathrm{d}x\\
\text{and}\quad &\lim_{\ell\to\infty} \int_\Omega \left(\int_{\mathbb{R}^d} \gamma_{\varepsilon_\ell}(x, y+x) - \gamma_{\varepsilon_\ell}(y+x,x)\,\mathrm{d}y \right) v(x)\,\mathrm{d}x = \int_\Omega c(x) v(x)\,\mathrm{d}x
\end{aligned}$$

for all $v \in \mathrm{L}^1(\Omega)$. Furthermore, for all $u \in \mathrm{H}^2(\mathbb{R}^d)$, we have that $\mathcal{L}_{\varepsilon_\ell} u$ converges weakly to $\mathcal{E}u$ in $\mathrm{L}^2(\Omega)$ as $\ell \to \infty$ where for $x \in \Omega$, we set

$$\begin{aligned}
&\mathcal{E}u(x) := -\sum_{i,j=1}^{d} a_{i,j}(x) \frac{\partial u}{\partial x_i \partial x_j}(x) + \sum_{k=1}^{d} \widetilde{b}_k(x) \frac{\partial u}{\partial x_k}(x) + c(x)u(x)\\
\text{and}\quad &\mathcal{L}_\varepsilon u(x) := \int_{\mathbb{R}^d} u(x)\gamma_\varepsilon(x,y) - u(y)\gamma_\varepsilon(y,x)\,\mathrm{d}y.
\end{aligned}$$

To be more precise, for all $v \in \mathrm{L}^2(\Omega)$, we get

$$\lim_{\ell\to\infty} \int_\Omega \mathcal{L}_{\varepsilon_\ell} u(x)\, v(x)\,\mathrm{d}x = \int_\Omega \mathcal{E}u(x)\, v(x)\,\mathrm{d}x.$$

Proof.

We begin by showing that there is a $C > 0$ such that $\sup_{\varepsilon>0} \|\mathcal{L}_\varepsilon u\|_{\mathrm{L}^2(\Omega)} \leqslant C\|u\|_{\mathrm{H}^2(\mathbb{R}^d)}$ for all $u \in \mathrm{H}^2(\mathbb{R}^d)$. Note that this already implies that there is a sequence such that $\mathcal{L}_\varepsilon u$ converges weakly in $\mathrm{L}^2(\Omega)$. So let $u \in \mathrm{H}^2(\mathbb{R}^d)$ and $\varepsilon > 0$ be given arbitrarily. Now, Lemma 11.7 and the Hölder inequality yield

$$\begin{aligned}
&\int_\Omega \left(\int_{\mathbb{R}^d} (u(x) - u(y) - \langle \nabla u(x)), x-y\rangle \gamma_\varepsilon(y,x)\,\mathrm{d}y \right)^2 \mathrm{d}x\\
\leqslant &\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(y+x) - u(x) - \langle \nabla u(x)), y\rangle)^2}{\|y\|^2} \gamma_\varepsilon(y+x,x)\,\mathrm{d}y \int_{\mathbb{R}^d} \|y\|^2 \gamma_\varepsilon(y+x,x)\,\mathrm{d}y\,\mathrm{d}x\\
\leqslant &\int_{\mathbb{R}^d} \frac{\int_{\mathbb{R}^d} (u(y+x) - u(x) - \langle \nabla u(x)), y\rangle)^2\,\mathrm{d}x}{\|y\|^4} \varphi_\varepsilon(y)\,\mathrm{d}y \int_{\mathbb{R}^d} \varphi_\varepsilon(y)\,\mathrm{d}y\\
\leqslant &\frac{d^2}{2} \sup_{\varepsilon>0} \|\varphi_\varepsilon\|^2_{\mathrm{L}^1(\mathbb{R}^d)} \|u\|^2_{\mathrm{H}^2(\mathbb{R}^d)}.
\end{aligned} \tag{11.1}$$

Furthermore, we get

$$\begin{aligned}
&\int_\Omega \left(\int_{\mathbb{R}^d} \langle \nabla u(x), x-y \rangle \gamma_\varepsilon(y,x)\,\mathrm{d}y \right)^2 \mathrm{d}x \\
\leqslant& d \sum_{i=1}^d \int_\Omega \left(\left(\frac{\partial}{\partial x_i} u(x) \right) \int_{\mathbb{R}^d} y_i \gamma_\varepsilon(y+x,x)\,\mathrm{d}y \right)^2 \mathrm{d}x \\
\leqslant& d \sup_{\varepsilon>0} \sum_{i=1}^d \left\| \int_{\mathbb{R}^d} y_i \gamma_\varepsilon(y+\cdot,\cdot)\,\mathrm{d}y \right\|^2_{\mathrm{L}^\infty(\Omega)} \|u\|^2_{\mathrm{H}^2(\mathbb{R}^d)}
\end{aligned}$$

and with (11.1), we have

$$\begin{aligned}
&\int_\Omega \left(\int_{\mathbb{R}^d} (u(x)-u(y)) \gamma_\varepsilon(y,x)\,\mathrm{d}y \right)^2 \mathrm{d}x \\
=& \int_\Omega \left(\int_{\mathbb{R}^d} (u(x)-u(y) - \langle \nabla u(x) - \nabla u(x), x-y \rangle) \gamma_\varepsilon(y,x)\,\mathrm{d}y \right)^2 \mathrm{d}x \\
\leqslant& 2 \int_\Omega \left(\int_{\mathbb{R}^d} (u(x)-u(y) - \langle \nabla u(x), x-y \rangle) \gamma_\varepsilon(y,x)\,\mathrm{d}y \right)^2 \mathrm{d}x \\
&+ 2 \int_\Omega \left(\int_{\mathbb{R}^d} \langle \nabla u(x), x-y \rangle \gamma_\varepsilon(y,x)\,\mathrm{d}y \right)^2 \mathrm{d}x \\
\leqslant& 2d \left(\sup_{\varepsilon>0} \|\varphi_\varepsilon\|^2_{\mathrm{L}^1(\mathbb{R}^d)} + \sup_{\varepsilon>0} \sum_{i=1}^d \left\| \int_{\mathbb{R}^d} y_i \gamma_\varepsilon(y+\cdot,\cdot)\,\mathrm{d}y \right\|^2_{\mathrm{L}^\infty(\Omega)} \right) \|u\|^2_{\mathrm{H}^2(\mathbb{R}^d)}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\int_\Omega \left(\int_{\mathbb{R}^d} u(x)\gamma_\varepsilon(x,y) - u(y)\gamma_\varepsilon(y,x)\,\mathrm{d}y \right)^2 \mathrm{d}x \\
\leqslant& \int_\Omega \left(u(x) \int_{\mathbb{R}^d} \gamma_\varepsilon(x,y) - \gamma_\varepsilon(y,x)\,\mathrm{d}y \right)^2 \mathrm{d}x + \int_\Omega \left(\int_{\mathbb{R}^d} (u(x)-u(y)) \gamma_\varepsilon(y,x)\,\mathrm{d}y \right)^2 \mathrm{d}x \\
\leqslant& 2 \left(\sup_{\varepsilon>0} \left\| \int_{\mathbb{R}^d} \gamma_\varepsilon(\cdot,y) - \gamma_\varepsilon(y,\cdot)\,\mathrm{d}y \right\|^2_{\mathrm{L}^\infty(\Omega)} \right) \|u\|^2_{\mathrm{H}^2(\mathbb{R}^d)} \\
&+ 2d \left(\|\varphi\|^2_{\mathrm{L}^1(\mathbb{R}^d)} + \sup_{\varepsilon>0} \sum_{i=1}^d \left\| \int_{\mathbb{R}^d} y_i \gamma_\varepsilon(y+\cdot,\cdot)\,\mathrm{d}y \right\|^2_{\mathrm{L}^\infty(\Omega)} \right) \|u\|^2_{\mathrm{H}^2(\mathbb{R}^d)}.
\end{aligned}$$

Consequently, $\mathcal{L}_\varepsilon u \in \mathrm{L}^2(\Omega)$ with $\|\mathcal{L}_\varepsilon u\|_{\mathrm{L}^2(\Omega)} \leqslant C\|u\|_{\mathrm{H}^2(\mathbb{R}^d)}$ where $C > 0$ is independent of u. By Theorem 11.6, there is a sequence $(\varepsilon_\ell)_{\ell\in\mathbb{N}}$ in $(0,\infty)$ with $\varepsilon_\ell \to 0$ as $\ell \to \infty$ such that $a_{i,j}$, b_k, and c are all well-defined in $\mathrm{L}^\infty(\Omega)$. Now, we assume, without loss of generality, for the remainder of this proof that $u \in \mathrm{C}_0^\infty(\mathbb{R}^d)$. The rest follows then by the density argument given in [19, Theorem 11.35].

Let $v \in \mathrm{L}^2(\Omega)$ be given arbitrarily. Then, we obtain

$$\begin{aligned}
&\int_\Omega v(x) \int_{\mathbb{R}^d} u(x)\gamma_\varepsilon(x,y) - u(y)\gamma_\varepsilon(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
=& \int_\Omega v(x)u(x) \int_{\mathbb{R}^d} \gamma_\varepsilon(x,y+\cdot) - \gamma_\varepsilon(y+\cdot,x)\,\mathrm{d}y\,\mathrm{d}x + \int_\Omega v(x) \int_{\mathbb{R}^d} (u(x)-u(y))\gamma_\varepsilon(y,x)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

Due to

$$\begin{aligned}&\int_\Omega v(x)\int_{\mathbb{R}^d}\left(\langle\nabla u(x),x-y\rangle\right)\gamma_{\varepsilon_\ell}(y,x)\,\mathrm{d}y\,\mathrm{d}x\\&=\int_\Omega\sum_{k=1}^d\left(\left(\frac{\partial}{\partial x_k}u(x)\right)v(x)\left(\int_{\mathbb{R}^d}y_k\gamma_{\varepsilon_\ell}(y+x,x)\,\mathrm{d}y\right)\right)\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega v(x)\int_{\mathbb{R}^d}\left(\sum_{i,j=1}^d(x-y)_i\frac{\partial^2}{\partial x_i\partial x_j}u(x)(x-y)_j\right)\gamma_{\varepsilon_\ell}(y,x)\,\mathrm{d}y\,\mathrm{d}x\\&=\int_\Omega\sum_{i,j=1}^d\left(\int_{\mathbb{R}^d}y_iy_j\gamma_{\varepsilon_\ell}(y+x,x)\,\mathrm{d}y\right)\left(\frac{\partial^2}{\partial x_i\partial x_j}u(x)\right)v(x)\,\mathrm{d}x,\end{aligned}$$

it remains to show that

$$\int_\Omega\left(\int_{\mathbb{R}^d}\left(u(x)-u(y)-\langle\nabla u(x),x-y\rangle+\frac{1}{2}\sum_{i,j=1}^d(x-y)_i\frac{\partial^2}{\partial x_i\partial x_j}u(x)(x-y)_j\right)\gamma_{\varepsilon_\ell}(y,x)\,\mathrm{d}y\right)^2\mathrm{d}x$$

is a zero sequence. Let $x\in\Omega$ be given arbitrarily. Then, for all $y\in\mathbb{R}^d$, the Fundamental Theorem of Calculus gives us

$$\begin{aligned}\mathcal{T}u(x,y):=&u(x)-u(y+x)+\langle\nabla u(x),y\rangle+\frac{1}{2}\sum_{i,j=1}^d\left(y_i\frac{\partial^2}{\partial x_i\partial x_j}u(x)y_j\right)\\=&\int_0^1\int_0^1 t\left(\sum_{i,j=1}^d y_i\left(\frac{\partial^2}{\partial x_i\partial x_j}u(x)-\frac{\partial^2}{\partial x_i\partial x_j}u(x+sty)\right)y_j\right)\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

For all $\delta>0$, there is a $\eta>0$ such that all $y\in\mathbb{R}^d$ with $\|y\|\leqslant\eta$, we have

$$\int_{\mathbb{R}^d}\sum_{i,j=1}^d\left(\frac{\partial^2}{\partial x_i\partial x_j}u(x)-\frac{\partial^2}{\partial x_i\partial x_j}u(x+y)\right)^2\mathrm{d}x\leqslant\delta.$$

For every $\delta>0$, we get

$$\begin{aligned}&\int_\Omega\left(\int_{\mathbb{R}^d}\left(\mathcal{T}u(x,y-x)\right)\gamma_{\varepsilon_\ell}(y,x)\,\mathrm{d}y\right)^2\mathrm{d}x\\&=\int_\Omega\left(\int_{\mathbb{R}^d}\left(\mathcal{T}u(x,y-x)\right)\frac{\varphi_{\varepsilon_\ell}(y-x)}{\|y-x\|^2}\,\mathrm{d}y\right)^2\mathrm{d}x\\&\leqslant\frac{d^4}{4}\int_\Omega\left(\int_{\mathbb{R}^d}\int_0^1\int_0^1\sum_{i,j=1}^d\left|\frac{\partial^2}{\partial x_i\partial x_j}u(x)-\frac{\partial^2}{\partial x_i\partial x_j}u(x+sty)\right|\mathrm{d}s\,\mathrm{d}t\,\varphi_{\varepsilon_\ell}(y)\,\mathrm{d}y\right)^2\mathrm{d}x\\&\leqslant\frac{d^4}{16}\sup_{\varepsilon>0}\|\varphi_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^d)}\int_\Omega\int_{\mathbb{R}^d}\int_0^1\int_0^1\sum_{i,j=1}^d\left|\frac{\partial^2}{\partial x_i\partial x_j}u(x)-\frac{\partial^2}{\partial x_i\partial x_j}u(x+sty)\right|^2\mathrm{d}s\,\mathrm{d}t\,\varphi_{\varepsilon_\ell}(y)\,\mathrm{d}y\,\mathrm{d}x\\&\leqslant\delta\frac{d^4}{16}\sup_{\varepsilon>0}\|\varphi_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^d)}\int_{\mathrm{B}_\eta(0)}\varphi_{\varepsilon_\ell}(y)\,\mathrm{d}y+\frac{d^4}{8}\sup_{\varepsilon>0}\|\varphi_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^d)}\|u\|_{\mathrm{H}^2(\mathbb{R}^d)}\int_{\mathbb{R}^d\setminus\mathrm{B}_\eta(0)}\varphi_{\varepsilon_\ell}(y)\,\mathrm{d}y.\end{aligned}$$

By first letting $\ell\to\infty$ and then $\delta\to0$, we obtain the desired result. □

Now, we proceed with the convergence of our bilinarforms $\mathfrak{B}$ and $\widehat{\mathfrak{B}}$.

Theorem 11.9.
Let $\Omega \subset \mathbb{R}^d$ be open, $(\gamma_\varepsilon)_{\varepsilon>0}$ be a family in $\mathcal{K}$ with

$$\sup_{\varepsilon>0} \sum_{i=1}^{d} \left\| \int_{\mathbb{R}^d} y_i(\gamma_\varepsilon(\cdot, y+\cdot) - \gamma_\varepsilon(y+\cdot, \cdot)) \,\mathrm{d}y \right\|_{\mathrm{L}^\infty(\Omega)} < \infty,$$

and let k_ε in $\mathcal{K}$ satisfy $k_\varepsilon > 0$ and

$$\left\| \int_{\mathbb{R}^d} k_\varepsilon(y, \cdot) \,\mathrm{d}y \right\|_{\mathrm{L}^\infty(\Omega)} < \infty.$$

We set $\widetilde{\gamma_\varepsilon}(y,x) := \max\left\{ \gamma_\varepsilon(y,x), \dfrac{(\gamma_\varepsilon(x,y) - \gamma_\varepsilon(y,x))^2}{k_\varepsilon(y,x)} \right\}$ and assume $(\widetilde{\gamma_\varepsilon})_{\varepsilon>0} \in \overline{\mathcal{K}}(\Omega)$. For $\varepsilon > 0$, set

$$\Gamma_\varepsilon := \{ y \in \mathbb{R}^d \setminus \Omega : \int_\Omega \gamma_\varepsilon(y,x) + \gamma_\varepsilon(x,y) \,\mathrm{d}x > 0 \}.$$

Then, there is a sequence $(\varepsilon_\ell)_{\ell\in\mathbb{N}}$ in $(0,\infty)$ with $\varepsilon_\ell \to 0$ as $\ell \to \infty$ such that for $i,j,k = 1, \dots d$, there are functions $a_{i,j}, b_k \in \mathrm{L}^\infty(\Omega)$ with

$$\begin{aligned} &\lim_{\ell\to\infty} \int_\Omega \left(\int_{\mathbb{R}^d} \frac{1}{2} y_i\, y_j \gamma_{\varepsilon_\ell}(y+x, x) \,\mathrm{d}y \right) v(x) \,\mathrm{d}x = \int_\Omega a_{i,j}(x) v(x) \,\mathrm{d}x \\ \textit{and}\quad &\lim_{\ell\to\infty} \int_\Omega \left(\int_{\mathbb{R}^d} \frac{1}{2} y_k(\gamma_{\varepsilon_\ell}(x, y+x) - \gamma_{\varepsilon_\ell}(y+x, x)) \,\mathrm{d}y \right) v(x) \,\mathrm{d}x = \int_\Omega b_k(x) v(x) \,\mathrm{d}x \end{aligned}$$

for all $v \in \mathrm{L}^1(\Omega)$. Furthermore, for all $u, v \in \mathrm{H}^1(\mathbb{R}^d)$, we have

$$\lim_{\ell\to\infty} \widehat{\mathfrak{B}}_{\varepsilon_\ell}(u,v) = \int_\Omega \sum_{i,j=1}^{d} a_{i,j}(x) \frac{\partial}{\partial x_i} u(x) \frac{\partial}{\partial x_j} v(x) + \sum_{k=1}^{d} b_k(x) u(x) \frac{\partial}{\partial x_j} v(x) \,\mathrm{d}x$$

and in particular, we obtain

$$\lim_{\ell\to\infty} \mathfrak{B}_{\varepsilon_\ell}(u,v) = \int_\Omega \sum_{i,j=1}^{d} a_{i,j}(x) \frac{\partial}{\partial x_i} u(x) \frac{\partial}{\partial x_j} v(x) \,\mathrm{d}x.$$

Proof.

We only show the convergence of $\widehat{\mathfrak{B}}_{\varepsilon_\ell}(u,v)$ for all $u, v \in \mathrm{H}^1(\mathbb{R}^d)$, because the convergence of $\mathfrak{B}_{\varepsilon_\ell}(u,v)$ follows analogously. First, we show that there is a constant $C > 0$ such that

$$\sup_{\varepsilon>0} |\widehat{\mathfrak{B}}_\varepsilon(u,v)| \leqslant C \|u\|_{\mathrm{H}^1(\mathbb{R}^d)} \|v\|_{\mathrm{H}^1(\mathbb{R}^d)} \quad \text{holds for all } u, v \in \mathrm{H}^1(\mathbb{R}^d).$$

Following that, we show the convergence. Therefore, let $u, v \in \mathrm{H}^1(\mathbb{R}^d)$ and $\varepsilon > 0$ be given arbitrarily. Then, we obtain

$$\begin{aligned} |\widehat{\mathfrak{B}}_\varepsilon(u,v)| &\leqslant \left| \int_\Omega \int_{\mathbb{R}^d} u(x)\gamma_\varepsilon(x,y) - u(y)\gamma_\varepsilon(y,x)(v(x) - v(y)) \,\mathrm{d}y \,\mathrm{d}x \right| \\ &\leqslant \int_\Omega \int_{\mathbb{R}^d} |u(x) - u(y)| \gamma_\varepsilon(y,x) |v(x) - v(y)| \,\mathrm{d}y \,\mathrm{d}x \\ &\quad + \int_\Omega \int_{\mathbb{R}^d} |u(x)(\gamma_\varepsilon(x,y) - \gamma_\varepsilon(y,x))(v(x) - v(y))| \,\mathrm{d}y \,\mathrm{d}x \end{aligned}$$

Furthermore, Proposition 9.3 in Brezis [16] yields

$$
\begin{aligned}
\int_\Omega \int_{\mathbb{R}^d} (u(x)-u(y))^2 \widetilde{\gamma}_\varepsilon(y,x)\,\mathrm{d}y\,\mathrm{d}x \leqslant & \int_{\mathbb{R}^d} \frac{\int_{\mathbb{R}^d}(u(x)-u(y+x))^2\,\mathrm{d}x\varphi_\varepsilon(y)}{\|y\|^2}\,\mathrm{d}y \\
\leqslant & \sup_{\varepsilon>0} \|\varphi_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^d)} \|u\|^2_{\mathrm{H}^1(\mathbb{R}^d)}.
\end{aligned}
\tag{11.2}
$$

By Theorem 2.1, there is a constant $C>0$ with $\sup_{\varepsilon>0} |\widehat{\mathfrak{B}}_\varepsilon(u,v)| \leqslant C\|u\|_{\mathrm{H}^1(\mathbb{R}^d)}\|v\|_{\mathrm{H}^1(\mathbb{R}^d)}$ for all $u,v \in \mathrm{H}^1(\mathbb{R}^d)$. Because of [19, Theorem 11.35]) we without loss of generality henceforth assume $u,v \in \mathrm{C}_0^\infty(\mathbb{R}^d)$. By Theorem 11.6, there is a sequence $(\varepsilon_\ell)_{\ell\in\mathbb{N}}$ in $(0,\infty)$ with $\varepsilon_\ell \to 0$ as $\ell\to\infty$ such that $a_{i,j}$, $\widetilde{b}_k$ are well-defined. Now, let $\delta>0$ be given arbitrarily. Furthermore, choose $\eta>0$ such that for a.e. $y\in\mathbb{R}^d$ with $\|y\|\leqslant\eta$, we have

$$
\int_{\mathbb{R}^d} \|\nabla u(x+y)-\nabla u(x)\|^2 \mathrm{d}x + \int_{\mathbb{R}^d} \|\nabla v(x+y)-\nabla v(x)\|^2\mathrm{d}x \leqslant \delta.
$$

Then, we calculate

$$
\begin{aligned}
\widehat{\mathfrak{B}}_\varepsilon(u,v) =& \frac{1}{2}\int_\Omega\int_\Omega u(x)\gamma_\varepsilon(x,y)-u(y)\gamma_\varepsilon(y,x)(v(x)-v(y))\,\mathrm{d}y\,\mathrm{d}x \\
&+ \int_\Omega\int_{\Gamma_\varepsilon} u(x)\gamma_\varepsilon(x,y)-u(y)\gamma_\varepsilon(y,x)(v(x)-v(y))\,\mathrm{d}y\,\mathrm{d}x \\
=& \frac{1}{2}\int_\Omega\int_{\mathbb{R}^d} u(x)\gamma_\varepsilon(x,y)-u(y)\gamma_\varepsilon(y,x)(v(x)-v(y))\,\mathrm{d}y\,\mathrm{d}x \\
&+ \frac{1}{2}\int_\Omega\int_{\Gamma_\varepsilon} u(x)\gamma_\varepsilon(x,y)-u(y)\gamma_\varepsilon(y,x)(v(x)-v(y))\,\mathrm{d}y\,\mathrm{d}x \\
=& \frac{1}{2}\int_\Omega\int_{\mathbb{R}^d} (u(x)-u(y))(v(x)-v(y))\gamma_\varepsilon(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
&+ \frac{1}{2}\int_\Omega\int_{\Gamma_\varepsilon} (u(x)-u(y))(v(x)-v(y))\gamma_\varepsilon(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
&+ \frac{1}{2}\int_\Omega\int_{\mathbb{R}^d} u(x)(\gamma_\varepsilon(x,y)-\gamma_\varepsilon(y,x))(v(x)-v(y))\,\mathrm{d}y\,\mathrm{d}x \\
&+ \frac{1}{2}\int_\Omega\int_{\Gamma_\varepsilon} u(x)(\gamma_\varepsilon(x,y)-\gamma_\varepsilon(y,x))(v(x)-v(y))\,\mathrm{d}y\,\mathrm{d}x
\end{aligned}
$$

and proceed by showing show that

$$
\begin{aligned}
&\left|\int_\Omega\int_{\Gamma_\varepsilon} u(x)(\gamma_{\varepsilon_\ell}(x,y)-\gamma_{\varepsilon_\ell}(y,x))(v(x)-v(y))\,\mathrm{d}y\,\mathrm{d}x\right| \to 0, \\
\text{and}\quad &\left|\int_\Omega\int_{\Gamma_\varepsilon} (u(x)-u(y))(v(x)-v(y))\gamma_{\varepsilon_\ell}(y,x)\,\mathrm{d}y\,\mathrm{d}x\right| \to 0
\end{aligned}
\tag{11.3}
$$

holds as $\ell \to \infty$. We obtain

$$\begin{aligned}
&\lim_{\ell\to\infty} \int_\Omega \int_{\Gamma_{\varepsilon_\ell}} (u(x)-u(y))^2 \widetilde{\gamma}_{\varepsilon_\ell}(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
=&\lim_{\ell\to\infty} \int_\Omega \int_{\Gamma_{\varepsilon_\ell}} \frac{(u(x)-u(y+x))^2}{\|y\|^2} \varphi_{\varepsilon_\ell}(y)\,\mathrm{d}y\,\mathrm{d}x \\
\leqslant& \lim_{\ell\to\infty} \int_\Omega \int_{\Gamma_{\varepsilon_\ell}} \int_{(0,1)} \|\nabla u(x+ty)\|^2\,\mathrm{d}t \varphi_{\varepsilon_\ell}(y)\,\mathrm{d}y\,\mathrm{d}x \\
\leqslant& \sup_{z\in\mathbb{R}^d} \|\nabla u(z)\|^2 \int_\Omega \lim_{\ell\to\infty} \int_{\mathbb{R}^d\setminus \mathrm{B}_{\mathrm{dist}(x,\partial\Omega)}(x)} \varphi_{\varepsilon_\ell}(y)\,\mathrm{d}y\,\mathrm{d}x \\
=&0.
\end{aligned}$$

Because this holds for all $u \in \mathrm{C}_0^\infty(\mathbb{R}^d)$, the desired result follows. Due to our assumptions, we get

$$\begin{aligned}
&\int_\Omega \sum_{i,j=1}^d \frac{\partial}{\partial x_i} u(x) \frac{\partial}{\partial x_j} v(x) \int_{\mathbb{R}^d} \frac{1}{2} y_i\, y_j \gamma_{\varepsilon_\ell}(y+\cdot,\cdot)\,\mathrm{d}y\,\mathrm{d}x \to \int_\Omega \sum_{i,j=1}^d a_{i,j}(x) \frac{\partial}{\partial x_i} u(x) \frac{\partial}{\partial x_j} v(x)\,\mathrm{d}x \\
\text{and } &\int_\Omega \sum_{k=1}^d u(x) \frac{\partial}{\partial x_k} v(x) \int_{\mathbb{R}^d} \frac{1}{2} y_k (\gamma_{\varepsilon_\ell}(\cdot, y+\cdot) - \gamma_{\varepsilon_\ell}(y+\cdot,\cdot))\,\mathrm{d}y\,\mathrm{d}x \to \int_\Omega \sum_{k=1}^d b_k u(x) \frac{\partial}{\partial x_j} v(x)\,\mathrm{d}x
\end{aligned}$$

as $\ell \to \infty$. Therefore, it remains to show that

$$\begin{aligned}
&\int_\Omega \int_{\mathbb{R}^d} (\gamma_\varepsilon(x,y) - \gamma_\varepsilon(y,x)) |u(x)(v(x)-v(y)) - \sum_{k=1}^d y_k u(x) \frac{\partial}{\partial x_k} v(x)|\,\mathrm{d}y\,\mathrm{d}x \to 0 \\
\text{and} \quad &\int_\Omega \int_{\mathbb{R}^d} |(u(x)-u(y))(v(x)-v(y)) - \sum_{i,j=1}^d y_i y_j \frac{\partial}{\partial x_i} u(x) \frac{\partial}{\partial x_j} v(x)| \gamma_\varepsilon(y,x)\,\mathrm{d}y\,\mathrm{d}x \to 0
\end{aligned} \tag{11.4}$$

as $\ell \to \infty$. For a.e. $x, y \in \mathbb{R}^d$, we have by the Fundamental Theorem of Calculus that

$$\begin{aligned}
&\left| (v(x) - v(y+x)) - \sum_{k=1}^d y_k \frac{\partial}{\partial x_k} v(x) \right| \\
\leqslant& \sum_{k=1}^d \int_{(0,1)} \left| y_k \left(\frac{\partial}{\partial x_k} v(x+ty) - \frac{\partial}{\partial x_k} v(x) \right) \right| \mathrm{d}t \\
\leqslant& \|y\| \int_{(0,1)} \|\nabla v(x+ty) - \nabla v(x)\|\,\mathrm{d}t
\end{aligned}$$

and

$$
\begin{aligned}
&\left|(u(x)-u(y+x))(v(x)-v(y+x))-\sum_{i,j=1}^{d} y_i y_j \frac{\partial}{\partial x_i}u(x)\frac{\partial}{\partial x_j}v(x)\right| \\
\leqslant &\left|(u(x)-u(y+x))\left(v(x)-v(y+x)-\sum_{j=1}^{d} y_j \frac{\partial}{\partial x_j}v(x)\right)\right| \\
&+\left|(u(x)-u(y+x))\left(\sum_{j=1}^{d} y_j \frac{\partial}{\partial x_j}v(x)\right)-\sum_{i,j=1}^{d} y_i y_j \frac{\partial}{\partial x_i}u(x)\frac{\partial}{\partial x_j}v(x)\right| \\
\leqslant &\left|(u(x)-u(y+x))\left(v(x)-v(y+x)-\sum_{j=1}^{d} y_j \frac{\partial}{\partial x_j}v(x)\right)\right| \\
&+\left|\left(u(x)-u(y+x)-\sum_{i=1}^{d} y_i \frac{\partial}{\partial x_i}u(x)\right)\left(\sum_{j=1}^{d} y_j \frac{\partial}{\partial x_j}v(x)\right)\right|.
\end{aligned}
$$

The Hölder inequality and inequality (11.2) yield

$$
\begin{aligned}
&\int_{\Omega}\int_{\mathbb{R}^d}\left(u(x)-u(y+x)-\sum_{i=1}^{d} y_i \frac{\partial}{\partial x_i}u(x)\right)^2 \widetilde{\gamma}_{\varepsilon_\ell}(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
\leqslant &\int_{\Omega}\int_{\mathrm{B}_\eta(0)} \|y\|^2 \int_{(0,1)} \|\nabla v(x+ty)-\nabla v(x)\|^2\,\mathrm{d}t\widetilde{\gamma}_{\varepsilon_\ell}(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
&+\int_{\Omega}\int_{\mathbb{R}^d\setminus \mathrm{B}_\eta(0)} \|y\|^2 \int_{(0,1)} \|\nabla v(x+ty)-\nabla v(x)\|^2\,\mathrm{d}t\widetilde{\gamma}_{\varepsilon_\ell}(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
\leqslant &\int_{\Omega}\int_{\mathrm{B}_\eta(0)} \int_{(0,1)} \|\nabla v(x+ty)-\nabla v(x)\|^2\,\mathrm{d}t\varphi_{\varepsilon_\ell}(y)\,\mathrm{d}y\,\mathrm{d}x \\
&+\int_{\Omega}\int_{\mathbb{R}^d\setminus \mathrm{B}_\eta(0)} \int_{(0,1)} \|\nabla v(x+ty)-\nabla v(x)\|^2\,\mathrm{d}t\varphi_{\varepsilon_\ell}(y)\,\mathrm{d}y\,\mathrm{d}x \\
\leqslant &\delta + 2\|u\|_{\mathrm{H}^1(\mathbb{R}^d)}\int_{\mathbb{R}^d\setminus \mathrm{B}_\eta(0)} \varphi_{\varepsilon_\ell}(y)\,\mathrm{d}y
\end{aligned}
$$

By first letting $\ell \to \infty$ and then $\delta \to 0$, we obtain (11.4). □

Remark 11.10.
Let $\Omega \subset \mathbb{R}^d$ be open. Taking a look at Theorem 11.9, we would expect that we have

$$
\mathcal{E}u(x) := -\operatorname{div}(A(x)\nabla u(x) + b(x)u(x))
$$

as the limit of $\mathcal{L}_\varepsilon u(c)$ in Theorem 11.8, where the matrix $A(x) \in \mathbb{R}^{d\times d}$ and the vector $b(x) \in \mathbb{R}^d$ are chosen accordingly. This is not the case because for $u \in \mathrm{C}_0^\infty(\Omega)$, $-\operatorname{div}(A(x)\nabla u - b(x)u(x))$ is, to the best of our knowledge, in general not well-defined. However, if the coefficients are smooth *enough, we obtain*

$$
\begin{aligned}
&\mathcal{E}u(x) \\
=& -\operatorname{div}(A(x)\nabla u(x) + b(x)u(x)) \\
=& -\left(\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{d} a_{i,j}(x)\frac{\partial}{\partial x_j}u(x) + b_i(x)u(x)\right)\right) \\
=& -\sum_{i.j=1}^{d}\left(a_{i,j}(x)\frac{\partial^2}{\partial x_j \partial x_i}u(x) + \frac{\partial}{\partial x_i}a_{i,j}(x)\frac{\partial}{\partial x_j}u(x)\right) - \sum_{k=1}^{d}\left(b_k(x)\frac{\partial}{\partial x_k}u(x) + \frac{\partial}{\partial x_k}b_k(x)u(x)\right) \\
=& -\sum_{i,j=1}^{d} a_{i,j}(x)\frac{\partial u}{\partial x_i \partial x_j}(x) + \sum_{k=1}^{d}\widetilde{b}_k(x)\frac{\partial u}{\partial x_k}(x) + c(x)u(x).
\end{aligned}
$$

As we require our limits to be at least in $\mathrm{C}^1(\overline{\Omega})$*, we assume that the Arzelà–Ascoli Theorem is applicable such that there is a zero sequence* $(\varepsilon_\ell)_{\ell\in\mathbb{N}}$ *which satisfies*

$$
\begin{aligned}
& -b_i(\cdot) - \sum_{j=1}^{d}\frac{\partial}{\partial x_i}a_{i,j}(\cdot) \\
=& \lim_{\ell\to\infty}\left(\int_{\mathbb{R}^d}\frac{1}{2}y_i(\gamma_{\varepsilon_\ell}(y+\cdot,\cdot) - \gamma_{\varepsilon_\ell}(\cdot, y+\cdot))\,\mathrm{d}y - \sum_{j=1}^{d}\frac{\partial}{\partial x_j}\left(\int_{\mathbb{R}^d}\frac{1}{2}y_i\,y_j\gamma_{\varepsilon_\ell}(y+\cdot,\cdot)\,\mathrm{d}y\right)\right) \\
=& \lim_{\ell\to\infty}\int_{\mathbb{R}^d} y_i\gamma_{\varepsilon_\ell}(y+\cdot,\cdot)\,\mathrm{d}y \\
\\
=& \widetilde{b}_k(\cdot) \\
\textit{and}\quad & -\sum_{k=1}^{d}\frac{\partial}{\partial x_k}b_k(\cdot) \\
=& \lim_{\ell\to\infty}\sum_{k=1}^{d}\frac{\partial}{\partial x_k}\left(\int_{\mathbb{R}^d}\frac{1}{2}y_k(\gamma_{\varepsilon_\ell}(y+\cdot,\cdot) - \gamma_{\varepsilon_\ell}(\cdot,y+\cdot))\,\mathrm{d}y\right) \\
=& \lim_{\ell\to\infty}\left(\int_{\mathbb{R}^d}\gamma_{\varepsilon_\ell}(\cdot,y+\cdot) - \gamma_{\varepsilon_\ell}(y+\cdot,\cdot)\,\mathrm{d}y\right)
\end{aligned}
$$

uniformly in Ω *for* $i = 1, \ldots, d$*. Then, following the proof of Theorem 11.8, we in this case, conclude*

$$\lim_{\ell\to\infty}\mathcal{L}_{\varepsilon_\ell}u = -\operatorname{div}(A\nabla u - bu)$$

uniformly in Ω*. In particular, we obtain if our coefficients are smooth enough and if* Ω *is a Lipschitz domain by our convergence results, the local and nonlocal integration by parts formula, and the local*

Trace Theorem that for all $u \in \mathrm{H}^2(\mathbb{R}^d)$ and $v \in \mathrm{H}^1(\mathbb{R}^d)$, we have

$$\begin{aligned}
&\lim_{\ell\to\infty}\int_{\Gamma(\Omega,\gamma_{\varepsilon_\ell})} \mathcal{N}_{\varepsilon_\ell}u(y)v(y)\,\mathrm{d}y \\
&= \lim_{\ell\to\infty}\int_\Omega \mathcal{L}_{\varepsilon_\ell}u(x)v(x)\,\mathrm{d}x + \lim_{\ell\to\infty}\widehat{\mathfrak{B}}_{\varepsilon_\ell}(u,v) \\
&= \int_\Omega -\operatorname{div}(A(x)\nabla u(x) - b(x)u(x))v(x)\,\mathrm{d}x + \int_\Omega \langle A(x)\nabla u(x), \nabla v(x)\rangle\,\mathrm{d}x + \int_\Omega u(x)\langle b(x), \nabla v(x)\rangle\,\mathrm{d}x \\
&= \int_{\partial\Omega} \langle A(y)\nabla u(y) - b(y)u(y), \nu(y)\rangle\,\mathrm{d}y
\end{aligned}$$

where $\nu(y)$ is the outer normal direction on $y \in \partial\Omega$.

Remark 11.11.
By taking a closer look at the limit of our bilinear form, we want to see under which assumptions our weak solutions of nonlocal Neumann problems to converge to a weak solution of a local Neumann problem.

Theorem 11.6 implies that $A = (a_{i,j}) \in (\mathrm{L}^\infty(\Omega))^{d\times d}$ holds, i.e., there is a $C > 0$ with

$$\int_\Omega \langle A(x)\nabla u(x), \nabla u(x)\rangle\,\mathrm{d}x \leqslant C\int_\Omega \|\nabla u(x)\|^2\,\mathrm{d}x \quad \text{for all } u \in \mathrm{H}^1(\Omega).$$

In order to avoid a degenerate local Neumann problem, we require a $c > 0$ with

$$c\int_\Omega \|\nabla u(x)\|^2\,\mathrm{d}x \leqslant \int_\Omega \langle A(x)\nabla u(x), \nabla u(x)\rangle\,\mathrm{d}x \quad \text{for all } u \in \mathrm{H}^1(\Omega).$$

Let $\varrho\colon \mathbb{R}^d \to [0,\infty]$ be measurable with $\int_{\mathbb{R}^d}\|y\|^2\varrho(y)\,\mathrm{d}y < \infty$. If there is a $i = 1,\dots,d$ such that ϱ is invariant under rotations across the x_i-axis, i.e., $\varrho(y) = \varrho(Ry)$ for all $y \in \mathbb{R}^d$ where $R = (r_{i,j}) \in \mathbb{R}^{d\times d}$ is a diagonal matrix with $r_{i,i} = -1$ and $r_{j,j} = 1$ for $j \neq i$, then we have by substitution for all $j \neq i$ that

$$\int_{\mathbb{R}^d} y_iy_j\varrho(y)\,\mathrm{d}y = -\int_{\mathbb{R}^d} y_iy_j\varrho(Ry)\,\mathrm{d}y = -\int_{\mathbb{R}^d} y_iy_j\varrho(y)\,\mathrm{d}y = 0.$$

If ϱ is invariant under every permutation, i.e., $\varrho(y) = \varrho(Py)$ for all $y \in \mathbb{R}^d$ where $P \in \mathbb{R}^{d\times d}$ is a permutation matrix, then we have by substitution for all $i = 1,\dots,d$

$$\int_{\mathbb{R}^d} y_i^2\varrho(y)\,\mathrm{d}y = \frac{1}{d}\sum_{k=1}^d\int_{\mathbb{R}^d} y_k^2\varrho(y)\,\mathrm{d}y = \frac{1}{d}\int_{\mathbb{R}^d}\|y\|^2\varrho(y)\,\mathrm{d}y$$

holds and also

$$\int_{\mathbb{R}^d} y_iy_j\varrho(y)\,\mathrm{d}y = c \quad \text{for } j \neq i.$$

Let $(\gamma_\varepsilon)_{\varepsilon>0} \in \overline{\mathcal{K}}(\Omega)$ be given. Then, Theorem 11.9 yields the existence of a sequence $(\varepsilon_\ell)_{\ell\in\mathbb{N}}$ in $(0,\infty)$ with $\varepsilon_\ell \to 0$ as $\ell \to \infty$ such that

$$\int_\Omega\sum_{i,j=1}^d a_{i,j}(x)\frac{\partial}{\partial x_i}u(x)\frac{\partial}{\partial x_j}v(x)\,\mathrm{d}x = \lim_{\ell\to\infty}\frac{1}{2}\int_\Omega\int_{\mathbb{R}^d}(u(x)-u(y))(v(x)-v(y))\gamma_{\varepsilon_\ell}(y,x)\,\mathrm{d}y\,\mathrm{d}x$$

for all $u, v \in \mathrm{H}^1(\mathbb{R}^d)$ where for $i, j = 1, \ldots, d$ the functions $a_{i,j} \in \mathrm{L}^\infty(\Omega)$ are defined such that

$$\lim_{\ell\to\infty} \int_\Omega \left(\int_{\mathbb{R}^d} \frac{1}{2} y_i\, y_j \gamma_{\varepsilon_\ell}(y + x, x)\,\mathrm{d}y \right) v(x)\,\mathrm{d}x = \int_\Omega a_{i,j}(x) v(x)\,\mathrm{d}x \quad \textit{holds for all } v \in \mathrm{L}^1(\Omega).$$

Set $A = (a_{i,j}) \in (\mathrm{L}^\infty(\Omega))^{d\times d}$ and assume that there is a $c > 0$ with

$$c \int_\Omega \|\nabla u(x)\|^2\,\mathrm{d}x \leqslant \int_\Omega (A(x)\nabla u(x))^\top \nabla u(x)\,\mathrm{d}x \quad \textit{for all } u \in \mathrm{H}^1(\Omega)$$

Let $(\varrho_\varepsilon)_{\varepsilon>0}$ be a family of radial measurable functions $\mathbb{R}^d \to [0,\infty]$ with $\int_{\mathbb{R}^d} \|y\|^2 \varrho_\varepsilon(y)\,\mathrm{d}y = c$ for all $\varepsilon > 0$, then by Theorem 11.9, there is a sequence $(\varepsilon_\ell)_{\ell\in\mathbb{N}}$ in $(0,\infty)$ with $\varepsilon_\ell \to 0$ as $\ell \to \infty$ such that

$$\begin{aligned}
&\lim_{\ell\to\infty} \frac{1}{2} \int_\Omega \int_{\mathbb{R}^d} (u(x) - u(y))(v(x) - v(y))\gamma_{\varepsilon_\ell}(y, x)\,\mathrm{d}y\,\mathrm{d}x \\
=& \int_\Omega \sum_{i,j=1}^d a_{i,j}(x) \frac{\partial}{\partial x_i} u(x) \frac{\partial}{\partial x_j} v(x)\,\mathrm{d}x \\
\geqslant& c \int_\Omega \sum_{i=1}^d \frac{\partial}{\partial x_i} u(x) \frac{\partial}{\partial x_i} v(x)\,\mathrm{d}x \\
=& \lim_{\ell\to\infty} \frac{1}{2} \int_\Omega \int_{\mathbb{R}^d} (u(x) - u(y))(v(x) - v(y))\varrho_{\varepsilon_\ell}(y - x)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}$$

for all $u, v \in \mathrm{H}^1(\mathbb{R}^d)$. Consequently, there is a $L \in \mathbb{N}$ with $\ell \geqslant L$ for all $\ell \in \mathbb{N}$ and

$$\int_\Omega \int_{\mathbb{R}^d} (u(x) - u(y))^2 \varrho_{\varepsilon_\ell}(y - x)\,\mathrm{d}y\,\mathrm{d}x \leqslant \int_\Omega \int_{\mathbb{R}^d} (u(x) - u(y))^2 \gamma_{\varepsilon_\ell}(y, x)\,\mathrm{d}y\,\mathrm{d}x \quad \textit{for all } u \in \mathrm{H}^1(\mathbb{R}^d).$$

Now, we require the following two known compactness results.

Theorem 11.12.
Let $\Omega \subset \mathbb{R}^d$ with $d > 1$ be a bounded domain with Lipschitz boundary and let $(\varrho_n)_{n\in\mathbb{N}}$ be a sequence of radial functions in $\mathrm{L}^1(\mathbb{R}^d)$ with

$$\begin{cases} 0 \leqslant \varrho_n(y) & \textit{for a.e. } y \in \mathbb{R}^d, \\ \lim\limits_{n\to\infty} \displaystyle\int_{\mathbb{R}^d \setminus \mathrm{B}_\delta(0)} \varrho_n(z)\,\mathrm{d}z = 0 & \textit{for } \delta > 0, \\ \|\varrho_n\|_{\mathrm{L}^1(\mathbb{R}^d)} = 1 & \textit{for } n \in \mathbb{N}. \end{cases}$$

Then, an $N \in \mathbb{N}$ exists such that there is a $C > 0$ with

$$\int_\Omega \int_\Omega (u(x) - u(y))^2\,\mathrm{d}y\,\mathrm{d}x \leqslant C \inf_{n\in\mathbb{N}, n\geqslant N} \int_\Omega \int_\Omega \frac{(u(x) - u(y))^2}{\|x - y\|^2} \varrho_n(y - x)\,\mathrm{d}y\,\mathrm{d}x \quad \textit{for all } u \in \mathrm{L}^2(\Omega).$$

Furthermore, every bounded sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^2(\Omega)$ with

$$\sup_{n\in\mathbb{N}} \int_\Omega \int_\Omega \frac{(u_n(x) - u_n(y))^2}{\|x - y\|^2} \varrho_n(y - x)\,\mathrm{d}y\,\mathrm{d}x < \infty$$

is relatively compact in $\mathrm{L}^2(\Omega)$ and for every subsequence $(u_{n_\ell})_{\ell\in\mathbb{N}}$ converging to $u \in \mathrm{L}^2(\Omega)$, i.e., $\lim_{\ell\to\infty} \|u_{n_\ell} - u\|_{\mathrm{L}^2(\Omega)} = 0$, we have $u \in \mathrm{H}^1(\Omega)$ with

$$\int_\Omega \|\nabla u(x)\|^2\,\mathrm{d}x \leqslant 2d \sup_{n\in\mathbb{N}} \int_\Omega \int_\Omega \frac{(u_n(x) - u_n(y))^2}{\|x - y\|^2} \varrho_n(y - x)\,\mathrm{d}y.$$

Proof.

For the proof, we refer to [26, Theorem 1.1 and Theorem 1.2]. □

Theorem 11.13.
Let $(\varrho_n)_{n\in\mathbb{N}}$ be a sequence of radial functions in $\mathrm{L}^1(\mathbb{R})$ with

$$\begin{cases} 0 \leqslant \varrho_n(y) & \text{for a.e. } y \in \mathbb{R}, \\ \lim\limits_{n\to\infty} \displaystyle\int_{\mathbb{R}\setminus \mathrm{B}_\delta(0)} \varrho_n(z)\,\mathrm{d}z = 0 & \text{for } \delta > 0, \\ \|\varrho_n\|_{\mathrm{L}^1(\mathbb{R})} = 1 & \text{for } n \in \mathbb{N}, \end{cases}$$

and let there be a $\phi_0 \in (0,1)$ such that

$$\inf_{n\in\mathbb{N}} \int_{\mathbb{R}} \operatorname*{ess\,inf}_{\phi_0 \leqslant \phi \leqslant 1} \varrho_n(\phi x)\,\mathrm{d}x > 0.$$

Then, an $N \in \mathbb{N}$ exists such that there is a $C > 0$ with

$$\int_{(0,1)}\int_{(0,1)} (u(x)-u(y))^2\,\mathrm{d}y\,\mathrm{d}x \leqslant C \inf_{n\in\mathbb{N}, n\geqslant N} \int_{(0,1)}\int_{(0,1)} \frac{(u(x)-u(y))^2}{\|x-y\|^2}\varrho_n(y-x)\,\mathrm{d}y\,\mathrm{d}x$$

for all $u \in \mathrm{L}^2(\Omega)$. Furthermore, every bounded sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathrm{L}^2(\Omega)$ with

$$\sup_{n\in\mathbb{N}} \int_{(0,1)}\int_{(0,1)} \frac{(u_n(x)-u_n(y))^2}{\|x-y\|^2}\varrho_n(y-x)\,\mathrm{d}y\,\mathrm{d}x < \infty$$

is relatively compact in $\mathrm{L}^2(\Omega)$ and for every subsequence $(u_{n_\ell})_{\ell\in\mathbb{N}}$ converging to $u \in \mathrm{L}^2(\Omega)$, i.e., $\lim_{\ell\to\infty} \|u_{n_\ell} - u\|_{\mathrm{L}^2(\Omega)} = 0$, we have $u \in \mathrm{H}^1((0,1))$ and

$$\int_{(0,1)} \|\nabla u(x)\|^2\,\mathrm{d}x \leqslant 2d \sup_{n\in\mathbb{N}} \int_{(0,1)}\int_{(0,1)} \frac{(u_n(x)-u_n(y))^2}{\|x-y\|^2}\varrho_n(y-x)\,\mathrm{d}y.$$

Proof.

For the proof, we refer to [26, Theorem 1.3]. □

In accordance with Theorem 11.13 and Theorem 11.12, we make the following definition.

Definition 11.14.
Let $\Omega \subset \mathbb{R}^d$ be open. We set $\underline{\mathcal{K}}(\Omega)$ to be the families $(\gamma_\varepsilon)_{\varepsilon>0}$ in $\mathcal{K}$ such that there is a family of radial functions $(\varrho_\varepsilon)_{\varrho>0}$ in $\mathrm{L}^1(\mathbb{R}^d)$ with

$$\begin{cases} 0 \leqslant \varrho_\varepsilon(y-x) \leqslant \|y-x\|^2\gamma_\varepsilon(y,x) & \text{for a.e. } (y,x) \in (\mathbb{R}^d \times \Omega), \\ \lim\limits_{\varepsilon\to 0} \displaystyle\int_{\mathbb{R}^d\setminus \mathrm{B}_\delta(0)} \varrho_\varepsilon(z)\,\mathrm{d}z = 0 & \text{for } \delta > 0, \\ \inf\limits_{\varepsilon>0} \|\varrho_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^d)} > 0. \end{cases}$$

If $d = 1$, we further assume that there is a $\phi_0 \in (0,1)$ such that

$$\inf_{\varepsilon>0} \int_{\mathbb{R}} \operatorname*{ess\,inf}_{\phi_0 \leqslant \phi \leqslant 1} \varrho_\varepsilon(\phi x)\,\mathrm{d}x > 0.$$

We set $\mathcal{K}(\Omega) := \underline{\mathcal{K}}(\Omega) \cap \overline{\mathcal{K}}(\Omega)$.

We now require a generalization of the well-known result that every bounded sequence in a Hilbert space admits a weakly convergent subsequence.

Lemma 11.15.
Let $(H_n, \langle\cdot,\cdot\rangle_{H_n})_{n\in\mathbb{N}}$ be a family of Hilbert space and $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space such that $(H_n, \langle\cdot,\cdot\rangle_{H_n})$ converges to $(H, \langle\cdot,\cdot\rangle_H)$, i.e., there is a dense subspace C of H and a sequence of linear operators $T_n\colon C \to H_n$ with $\lim_{n\to\infty} \|T_n(u)\|_{H_n} = \|u\|_H$ for all $u \in C$. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence such that for $n \in \mathbb{N}$ we have both $h_n \in H_n$ and

$$\sup_{n\in\mathbb{N}} \|h_n\|_{H_n} < \infty.$$

Then, there is a subsequence $(h_{n_\ell})_{\ell\in\mathbb{N}}$ and a $h \in H$ with

$$\lim_{\ell\to\infty} \langle h_{n_\ell}, T_{n_\ell}(v)\rangle_{H_{n_\ell}} = \langle h, v\rangle_H \quad \text{for all } v \in H.$$

Proof.

This follows by [18, Lemma 2.2]. □

Finally, we return to our nonlocal Neumann problem.

Theorem 11.16.
Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $(\gamma_\varepsilon)_{\varepsilon>0}$ be a family in $\mathcal{K}$. For $\varepsilon > 0$, let k_ε in $\mathcal{K}$ satisfy

$$k_\varepsilon(y,x) > 0 \text{ and } \frac{(\gamma_\varepsilon(x,y) - \gamma_\varepsilon(y,x))^2}{k_\varepsilon(y,x)} \leqslant \gamma_\varepsilon(y,x) \text{ for a.e. } (y,x) \in \{\mathbb{R}^d \times \Omega\colon |\gamma(y,x) - \gamma(x,y)| > 0\}.$$

If we set

$$\Gamma_\varepsilon := \{y \in \mathbb{R}^d \setminus \Omega : \int_\Omega \gamma_\varepsilon(y,x) + \gamma_\varepsilon(x,y)\,\mathrm{d}x > 0\}$$
$$\text{and} \quad w_\varepsilon\colon \Gamma_\varepsilon \to \mathbb{R},\ w_\varepsilon(y) = \int_\Omega \frac{\gamma_\varepsilon(y,x)}{\int_{\Gamma_\varepsilon} \gamma_\varepsilon(z,x)\,\mathrm{d}z + c_\varepsilon}\,\mathrm{d}x,$$

where c_ε is chosen such that $\operatorname{ess\,inf}_{x\in\Omega} \int_{\Gamma_\varepsilon} \gamma_\varepsilon(z,x)\,\mathrm{d}z + c_\varepsilon > 0$, then the following statements are valid.

(i) For every $\varepsilon > 0$ with

$$\left\| \int_{\mathbb{R}^d} k_\varepsilon(y,\cdot)\,\mathrm{d}y \right\|_{\mathrm{L}^\infty(\Omega)} < \infty,$$

there exist $c_\varepsilon, C_\varepsilon, \kappa_\varepsilon > 0$ such that

$$c_\varepsilon \|v_\varepsilon\|^2_{\mathrm{V}(\Omega;\gamma_\varepsilon)} \leqslant \widehat{\mathfrak{B}}_\varepsilon(v_\varepsilon, v_\varepsilon) + \kappa_\varepsilon \int_\Omega v_\varepsilon^2(x)\,\mathrm{d}x \leqslant C_\varepsilon \|v_\varepsilon\|^2_{\mathrm{V}(\Omega;\gamma_\varepsilon)} \quad \text{for } v_\varepsilon \in \mathrm{V}(\Omega;\gamma_\varepsilon),$$

and such that for every $f_\varepsilon \in \mathrm{L}^2(\Omega)$ and $g_\varepsilon \in \mathrm{L}^2(\Gamma_\varepsilon, w_\varepsilon)$, there is a unique $u_\varepsilon \in \mathrm{V}(\Omega;\gamma_\varepsilon)$ solving

$$\widehat{\mathfrak{B}}_\varepsilon(u_\varepsilon, v_\varepsilon) + \kappa_\varepsilon \int_\Omega u_\varepsilon(x)v_\varepsilon(x)\,\mathrm{d}x = \int_\Omega f_\varepsilon(x)v_\varepsilon(x)\,\mathrm{d}x + \int_{\Gamma_\varepsilon} g_\varepsilon(y)v_\varepsilon(y)w_\varepsilon(y)\,\mathrm{d}x$$

for all $v_\varepsilon \in \mathrm{V}(\Omega;\gamma_\varepsilon)$*. Moreover,* u_ε *satisfies*

$$c_\varepsilon \|u_\varepsilon\|_{\mathrm{V}(\Omega;\gamma_\varepsilon)} \leqslant \|f_\varepsilon\|_{\mathrm{L}^2(\Omega)} + \|g_\varepsilon\|_{\mathrm{L}^2(\Gamma_\varepsilon,w_\varepsilon)}.$$

(ii) If

$$\sup_{\varepsilon>0} \left\| \int_{\mathbb{R}^d} k_\varepsilon(y,\cdot)\,\mathrm{d}y \right\|_{\mathrm{L}^\infty(\Omega)} < \infty,$$

then there exist $c, C, \kappa > 0$ *such that for every* $\varepsilon > 0$*, we have*

$$c\|v_\varepsilon\|^2_{\mathrm{V}(\Omega;\gamma_\varepsilon)} \leqslant \widehat{\mathfrak{B}}_\varepsilon(v_\varepsilon, v_\varepsilon) + \kappa \int_\Omega v_\varepsilon^2(x)\,\mathrm{d}x \leqslant C\|v_\varepsilon\|^2_{\mathrm{V}(\Omega;\gamma_\varepsilon)} \quad \text{for } v_\varepsilon \in \mathrm{V}(\Omega;\gamma_\varepsilon),$$

and such that for every $f_\varepsilon \in \mathrm{L}^2(\Omega)$ *and* $g_\varepsilon \in \mathrm{L}^2(\Gamma_\varepsilon, w_\varepsilon)$ *there is a unique* $u_\varepsilon \in \mathrm{V}(\Omega;\gamma_\varepsilon)$ *solving*

$$\widehat{\mathfrak{B}}_\varepsilon(u_\varepsilon, v_\varepsilon) + \kappa \int_\Omega u_\varepsilon(x)v_\varepsilon(x)\,\mathrm{d}x = \int_\Omega f_\varepsilon(x)v_\varepsilon(x)\,\mathrm{d}x + \int_{\Gamma_\varepsilon} g_\varepsilon(y)v_\varepsilon(y)w_\varepsilon(y)\,\mathrm{d}y$$

for all $v_\varepsilon \in \mathrm{V}(\Omega;\gamma_\varepsilon)$*. Moreover,* u_ε *satisfies*

$$c_\varepsilon \|u_\varepsilon\|_{\mathrm{V}(\Omega;\gamma_\varepsilon)} \leqslant \|f_\varepsilon\|_{\mathrm{L}^2(\Omega)} + \|g_\varepsilon\|_{\mathrm{L}^2(\Gamma_\varepsilon,w_\varepsilon)}.$$

(iii) If $(\gamma_\varepsilon)_{\varepsilon>0} \in \underline{\mathcal{K}}(\Omega)$ *is a family of symmetric functions, then there exist a zero sequence* $(\varepsilon_n)_{n\in\mathbb{N}}$ *in* $(0,\infty)$ *such that for every* $n \in \mathbb{N}$ *and for every* $f_{\varepsilon_n} \in \mathrm{L}^2(\Omega)$ *and* $g_{\varepsilon_n} \in \mathrm{L}^2(\Gamma_{\varepsilon_n}, w_{\varepsilon_n})$ *with*

$$\int_\Omega f_{\varepsilon_n}(x)\,\mathrm{d}x + \int_{\Gamma_{\varepsilon_n}} g_{\varepsilon_n}(y)w_{\varepsilon_n}(y)\,\mathrm{d}x = 0,$$

there is a $u_{\varepsilon_n} \in \mathrm{V}(\Omega;\gamma_{\varepsilon_n})$ *solving*

$$\mathfrak{B}_\varepsilon(u_{\varepsilon_n}, v_{\varepsilon_n}) = \int_\Omega f_{\varepsilon_n}(x)v_{\varepsilon_n}(x)\,\mathrm{d}x + \int_{\Gamma_{\varepsilon_n}} g_{\varepsilon_n}(y)v_{\varepsilon_n}(y)w_{\varepsilon_n}(y)\,\mathrm{d}x \quad \text{for } v_{\varepsilon_n} \in \mathrm{V}(\Omega;\gamma_{\varepsilon_n}),$$

Moreover, u_{ε_n} *is unique up to an additive constant and there is a constant* $c > 0$ *with*

$$c\|u - u_\Omega\|^2_{\mathrm{V}(\Omega;\gamma_{\varepsilon_n})} \leqslant \mathfrak{B}_\varepsilon(u_{\varepsilon_n}, u_{\varepsilon_n}) \leqslant \|f_\varepsilon\|_{\mathrm{L}^2(\Omega)} + \|g_\varepsilon\|_{\mathrm{L}^2(\Gamma_\varepsilon,w_\varepsilon)}.$$

Proof.

Let $\varepsilon > 0$ be given. For every $v_\varepsilon, u_\varepsilon \in \mathrm{V}(\Omega;\gamma_\varepsilon)$, we have by the Hölder inequality that

$$\begin{aligned}
\widehat{\mathfrak{B}}_\varepsilon(v_\varepsilon, v_\varepsilon) =& \frac{1}{2}\int_\Omega\int_\Omega (v_\varepsilon(x)\gamma_\varepsilon(x,y) - v_\varepsilon(y)\gamma_\varepsilon(y,x))(v_\varepsilon(x) - v_\varepsilon(y))\,\mathrm{d}y\,\mathrm{d}x \\
&+ \int_\Omega\int_{\Gamma_\varepsilon} (v_\varepsilon(x)\gamma_\varepsilon(x,y) - v_\varepsilon(y)\gamma_\varepsilon(y,x))(v_\varepsilon(x) - v_\varepsilon(y))\,\mathrm{d}y\,\mathrm{d}x \\
\geqslant& \frac{1}{2}\int_\Omega\int_{\mathbb{R}^d} (v_\varepsilon(x) - v_\varepsilon(y))^2\gamma_\varepsilon(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
&- \left|\int_\Omega\int_{\mathbb{R}^d} v_\varepsilon(x)(v_\varepsilon(x) - v_\varepsilon(y))(\gamma_\varepsilon(x,y) - \gamma_\varepsilon(y,x))\,\mathrm{d}y\,\mathrm{d}x\right| \\
\geqslant& \frac{1}{2}\int_\Omega\int_{\mathbb{R}^d} (v_\varepsilon(x) - v_\varepsilon(y))^2\gamma_\varepsilon(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
&- \int_\Omega\int_{\mathbb{R}^d} v_\varepsilon^2(x)k_\varepsilon(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
&- \frac{1}{4}\int_\Omega\int_{\mathbb{R}^d} (v_\varepsilon(x) - v_\varepsilon(y))^2\gamma_\varepsilon(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
\geqslant& \frac{1}{4}\int_\Omega\int_{\mathbb{R}^d} (v_\varepsilon(x) - v_\varepsilon(y))^2\gamma_\varepsilon(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
&- \left\|\int_{\mathbb{R}^d} k_\varepsilon(y,\cdot)\,\mathrm{d}y\right\|_{\mathrm{L}^\infty(\Omega)} \int_\Omega v_\varepsilon^2(x)\,\mathrm{d}x
\end{aligned} \tag{11.5}$$

and

$$\begin{aligned}
|\widehat{\mathfrak{B}}_\varepsilon(u_\varepsilon, v_\varepsilon)| \leqslant& \int_\Omega\int_{\mathbb{R}^d} (v_\varepsilon(x) - v_\varepsilon(y))(u_\varepsilon(x) - u_\varepsilon(y))\gamma_\varepsilon(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
&+ \left|\int_\Omega\int_{\mathbb{R}^d} u_\varepsilon(x)(v_\varepsilon(x) - v_\varepsilon(y))(\gamma_\varepsilon(x,y) - \gamma_\varepsilon(y,x))\,\mathrm{d}y\,\mathrm{d}x\right| \\
\leqslant& \left\|\int_{\mathbb{R}^d} k_\varepsilon(y,\cdot)\,\mathrm{d}y\right\|_{\mathrm{L}^\infty(\Omega)} \|u_\varepsilon\|_{\mathrm{V}(\Omega;\gamma_\varepsilon)}\|v_\varepsilon\|_{\mathrm{V}(\Omega;\gamma_\varepsilon)}.
\end{aligned}$$

Consequently, (i) and (ii) follow by (11.5) and the Lax-Milgram Theorem (see Evans [6, 6.2.1. Lax-Milgram Theorem.]). If $(\gamma_\varepsilon)_{\varepsilon>0} \in \underline{\mathcal{K}}(\Omega)$, then for $\varepsilon > 0$, we obtain

$$\begin{aligned}
\mathfrak{B}_\varepsilon(v_\varepsilon, v_\varepsilon) =& \frac{1}{2}\int_\Omega\int_\Omega (v_\varepsilon(x) - v_\varepsilon(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
&+ \int_\Omega\int_\Gamma (v_\varepsilon(x) - v_\varepsilon(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
\geqslant& \frac{1}{2}\int_\Omega\int_{\mathbb{R}^d} (v_\varepsilon(x) - v_\varepsilon(y))^2\gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x \\
\geqslant& \frac{1}{2}\int_\Omega\int_{\mathbb{R}^d} \frac{(v_\varepsilon(x) - v_\varepsilon(y))^2}{\|x-y\|^2}\varrho_\varepsilon(y-x)\,\mathrm{d}y\,\mathrm{d}x \\
\geqslant& \inf_{\varepsilon>0}\|\varrho_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^d)}\frac{1}{2}\int_\Omega\int_{\mathbb{R}^d} \frac{(v_\varepsilon(x) - v_\varepsilon(y))^2}{\|x-y\|^2}\frac{\varrho_\varepsilon(y-x)}{\|\varrho_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^d)}}\,\mathrm{d}y\,\mathrm{d}x \quad \text{holds for } v_\varepsilon \in \mathrm{V}(\Omega;\gamma_\varepsilon).
\end{aligned}$$

Then, (iii) is a consequence of either by Theorem 11.12 or Theorem 11.13 and Theorem 4.7. $\square$

Finally, we present our main results.

Theorem 11.17.
Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $(\gamma_\varepsilon)_{\varepsilon>0} \in \mathcal{K}(\Omega)$ satisfy

$$\sup_{\varepsilon>0} \sum_{i=1}^{d} \left\| \int_{\mathbb{R}^d} y_i(\gamma_\varepsilon(\cdot, y+\cdot) - \gamma_\varepsilon(y+\cdot,\cdot))\,\mathrm{d}y \right\|_{\mathrm{L}^\infty(\Omega)} < \infty.$$

For $\varepsilon > 0$, let k_ε in $\mathcal{K}$ satisfy $\sup_{\varepsilon>0} \left\| \int_{\mathbb{R}^d} k_\varepsilon(y,\cdot)\,\mathrm{d}y \right\|_{\mathrm{L}^\infty(\Omega)} < \infty$,

$k_\varepsilon(y,x) > 0$, and $\dfrac{(\gamma_\varepsilon(x,y) - \gamma_\varepsilon(y,x))^2}{k_\varepsilon(y,x)} \leqslant \gamma_\varepsilon(y,x)$ for a.e. $(y,x) \in \{\mathbb{R}^d \times \Omega \colon |\gamma(y,x) - \gamma(x,y)| > 0\}$.

If $(u_\varepsilon)_{\varepsilon>0}$ is a family with

$$u_\varepsilon \in \mathrm{V}(\Omega;\gamma_\varepsilon) \text{ for } \varepsilon > 0 \quad \text{and} \quad \sup_{\varepsilon>0} \|u_\varepsilon\|_{\mathrm{V}(\Omega;\gamma_\varepsilon)} < \infty,$$

then there is a zero sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ and a unique $u \in \mathrm{H}^1(\Omega)$ with

$$\lim_{\ell\to\infty} \|u_{\varepsilon_\ell} - u\|_{\mathrm{L}^2(\Omega)} = 0$$
$$\text{and} \quad \lim_{\ell\to\infty} \widehat{\mathfrak{B}}_{\varepsilon_\ell}(u_{\varepsilon_\ell}, v) = \int_\Omega \sum_{i,j=1}^{d} a_{i,j}(x) \frac{\partial}{\partial x_i} u(x) \frac{\partial}{\partial x_j} v(x) + \sum_{k=1}^{d} b_k(x) u(x) \frac{\partial}{\partial x_j} v(x)\,\mathrm{d}x$$

for all $v \in \mathrm{H}^1(\mathbb{R}^d)$ where for $i,j,k = 1,\dots,d$ the functions $a_{i,j}, b_k$ are the unique function in $\mathrm{L}^\infty(\Omega)$ solving

$$\lim_{\ell\to\infty} \int_\Omega \left(\int_{\mathbb{R}^d} \frac{1}{2} y_i\, y_j \gamma_{\varepsilon_\ell}(y+x,x)\,\mathrm{d}y \right) \tau(x)\,\mathrm{d}x = \int_\Omega a_{i,j}(x)\tau(x)\,\mathrm{d}x$$
$$\text{and} \quad \lim_{\ell\to\infty} \int_\Omega \left(\int_{\mathbb{R}^d} \frac{1}{2} y_k(\gamma_{\varepsilon_\ell}(\cdot, y+\cdot) - \gamma_{\varepsilon_\ell}(y+\cdot,\cdot))\,\mathrm{d}y \right) \tau(x)\,\mathrm{d}x = \int_\Omega b_k(x)\tau(x)\,\mathrm{d}x$$

for all $\tau \in \mathrm{L}^1(\Omega)$.

Proof.

First we recall that there is a bounded extension operator $E\colon \mathrm{H}^1(\Omega) \to \mathrm{H}^1(\mathbb{R}^d)$ (see [19, Theorem 13.17]) and for $\varepsilon > 0$ and for all $v_\varepsilon \in \mathrm{V}(\Omega;\gamma_\varepsilon)$, we obtain

$$\begin{aligned}\int_\Omega \int_{\mathbb{R}^d} (v_\varepsilon(x) - v_\varepsilon(y))^2 \gamma(y,x)\,\mathrm{d}y\,\mathrm{d}x &\geqslant \int_\Omega \int_{\mathbb{R}^d} \frac{(v_\varepsilon(x) - v_\varepsilon(y))^2}{\|x-y\|^2} \varrho_\varepsilon(y-x)\,\mathrm{d}y\,\mathrm{d}x \\ &\geqslant \inf_{\varepsilon>0} \|\varrho_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^d)} \int_\Omega \int_{\mathbb{R}^d} \frac{(v_\varepsilon(x) - v_\varepsilon(y))^2}{\|x-y\|^2} \frac{\varrho_\varepsilon(y-x)}{\|\varrho_\varepsilon\|_{\mathrm{L}^1(\mathbb{R}^d)}}\,\mathrm{d}y\,\mathrm{d}x.\end{aligned} \tag{11.6}$$

By Theorem 11.9 there is a zero sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ such that for $i,j,k = 1,\dots,d$ the functions $a_{i,j}, b_k \in \mathrm{L}^\infty(\Omega)$ are well-defined and such that for all $u \in \mathrm{H}^1(\Omega)$ and all $v \in \mathrm{H}^1(\mathbb{R}^d)$ we have

$$\begin{aligned}\lim_{\ell\to\infty}\Bigg(&\frac{1}{2}\int_\Omega\int_\Omega (Eu(x)-Eu(y))(v(x)-v(y))\gamma_{\varepsilon_\ell}(y,x)\,\mathrm{d}y\,\mathrm{d}x\\&+\int_\Omega\int_{\mathbb{R}^d\setminus\Omega}(u_{\varepsilon_\ell}(x)-u_{\varepsilon_\ell}(y))(v(x)-v(y))\gamma_{\varepsilon_\ell}(y,x)\,\mathrm{d}y\,\mathrm{d}x\Bigg)\\=\int_\Omega\sum_{i,j=1}^d a_{i,j}(x)\frac{\partial}{\partial x_i}u(x)\frac{\partial}{\partial x_j}v(x)\,\mathrm{d}x.\end{aligned}$$

and

$$\lim_{\ell\to\infty}\widehat{\mathfrak{B}}_{\varepsilon_\ell}(Eu,v)=\int_\Omega\sum_{i,j=1}^d a_{i,j}(x)\frac{\partial}{\partial x_i}u(x)\frac{\partial}{\partial x_j}v(x)+\sum_{k=1}^d b_k(x)u(x)\frac{\partial}{\partial x_j}v(x)\,\mathrm{d}x\quad\text{for all } v\in\mathrm{H}^1(\mathbb{R}^d).$$

Due to (11.6) either Theorem 11.12 or Theorem 11.13 is applicable and we further see that there are constants $c, C > 0$ with

$$c\int_\Omega\|\nabla u(x)\|^2\,\mathrm{d}x\leqslant\int_\Omega\sum_{i,j=1}^d a_{i,j}(x)\frac{\partial}{\partial x_i}u(x)\frac{\partial}{\partial x_j}u(x)\,\mathrm{d}x.\leqslant C\int_\Omega\|\nabla u(x)\|^2\,\mathrm{d}x \text{ for all } u\in\mathrm{H}^1(\Omega).$$

Then, by either Theorem 11.12 or Theorem 11.13 and Lemma 11.15, we without loss of generality assume that there is a function $u \in \mathrm{H}^1(\Omega)$ satisfying

$$\lim_{\ell\to\infty}\|u_{\varepsilon_\ell}-u\|_{\mathrm{L}^2(\Omega)}=0$$

and

$$\begin{aligned}\lim_{\ell\to\infty}\Bigg(&\frac{1}{2}\int_\Omega\int_\Omega (u_{\varepsilon_\ell}(x)-u_{\varepsilon_\ell}(y))(v(x)-v(y))\gamma_{\varepsilon_\ell}(y,x)\,\mathrm{d}y\,\mathrm{d}x\\&+\int_\Omega\int_{\mathbb{R}^d\setminus\Omega}(u_{\varepsilon_\ell}(x)-u_{\varepsilon_\ell}(y))(v(x)-v(y))\gamma_{\varepsilon_\ell}(y,x)\,\mathrm{d}y\,\mathrm{d}x\Bigg)\\=\int_\Omega\sum_{i,j=1}^d a_{i,j}(x)\frac{\partial}{\partial x_i}u(x)\frac{\partial}{\partial x_j}v(x)\,\mathrm{d}x.\end{aligned}$$

for all $v \in \mathrm{H}^1(\mathbb{R}^d)$. Therefore, it remains to show that

$$\begin{aligned}\lim_{\ell\to\infty}\Bigg(&\frac{1}{2}\int_\Omega\int_\Omega u_{\varepsilon_\ell}(x)(v(x)-v(y))(\gamma_{\varepsilon_\ell}(x,y)-\gamma_{\varepsilon_\ell}(y,x))\,\mathrm{d}y\,\mathrm{d}x\\&+\int_\Omega\int_{\mathbb{R}^d\setminus\Omega}u_{\varepsilon_\ell}(x)(v(x)-v(y))(\gamma_{\varepsilon_\ell}(x,y)-\gamma_{\varepsilon_\ell}(y,x))\,\mathrm{d}y\,\mathrm{d}x\Bigg)\\=\int_\Omega\sum_{k=1}^d b_k(x)u(x)\frac{\partial}{\partial x_j}v(x)\,\mathrm{d}x\end{aligned}$$

holds for all $v \in \mathrm{H}^1(\mathbb{R}^d)$. However, this follows by

$$\begin{aligned}
&\bigg(\frac{1}{2}\int_\Omega\int_\Omega |u_{\varepsilon_\ell}(x)-u(x)||v(x)-v(y)||\gamma_{\varepsilon_\ell}(x,y)-\gamma_{\varepsilon_\ell}(y,x)|\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad\qquad + \int_\Omega\int_{\mathbb{R}^d\setminus\Omega} |u_{\varepsilon_\ell}(x)-u(x)||v(x)-v(y)||\gamma_{\varepsilon_\ell}(x,y)-\gamma_{\varepsilon_\ell}(y,x)|\,\mathrm{d}y\,\mathrm{d}x\bigg)\\
&\leqslant \int_\Omega\int_{\mathbb{R}^d} |u_{\varepsilon_\ell}(x)-u(x)||v(x)-v(y)||\gamma_{\varepsilon_\ell}(x,y)-\gamma_{\varepsilon_\ell}(y,x)|\,\mathrm{d}y\,\mathrm{d}x\\
&\leqslant C\sup_{\varepsilon>0}\left\|\int_{\mathbb{R}^d} k_\varepsilon(y,\cdot)\,\mathrm{d}y\right\|_{\mathrm{L}^\infty(\Omega)}^{\frac{1}{2}} \|u_{\varepsilon_\ell}-u\|_{\mathrm{L}^2(\Omega)}\|v\|_{\mathrm{H}^1(\mathbb{R}^d)}
\end{aligned}$$

where $C>0$ only depends on $(\gamma_\varepsilon)_{\varepsilon>0} \in \mathcal{K}(\Omega)$. □

Corollary 11.18.
Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $(\gamma_\varepsilon)_{\varepsilon>0} \in \mathcal{K}(\Omega)$ satisfy

$$\sup_{\varepsilon>0}\sum_{i=1}^d \left\|\int_{\mathbb{R}^d} y_i(\gamma_\varepsilon(\cdot,y+\cdot)-\gamma_\varepsilon(y+\cdot,\cdot))\,\mathrm{d}y\right\|_{\mathrm{L}^\infty(\Omega)} < \infty.$$

For $\varepsilon>0$, let k_ε in $\mathcal{K}$ satisfy $\sup_{\varepsilon>0}\left\|\int_{\mathbb{R}^d} k_\varepsilon(y,\cdot)\,\mathrm{d}y\right\|_{\mathrm{L}^\infty(\Omega)} < \infty$,

$k_\varepsilon(y,x)>0$ and $\dfrac{(\gamma_\varepsilon(x,y)-\gamma_\varepsilon(y,x))^2}{k_\varepsilon(y,x)} \leqslant \gamma_\varepsilon(y,x)$ for a.e. $(y,x) \in \{\mathbb{R}^d\times\Omega\colon |\gamma(y,x)-\gamma(x,y)|>0\}$.

If $\kappa > 2\sup_{\varepsilon>0}\left\|\int_{\mathbb{R}^d} k_\varepsilon(y,\cdot)\,\mathrm{d}y\right\|_{\mathrm{L}^\infty(\Omega)}$ holds, then for every family $(f_\varepsilon)_{\varepsilon>0}$ in $\mathrm{L}^2(\Omega)$, there is a unique family $(u_\varepsilon)_{\varepsilon>0}$ where for $\varepsilon>0$ the function $u_\varepsilon \in \mathrm{V}(\Omega;\gamma_\varepsilon)$ solves

$$\widehat{\mathfrak{B}}_\varepsilon(u_\varepsilon,v_\varepsilon)+\kappa\int_\Omega u_\varepsilon(x)v_\varepsilon(x)\,\mathrm{d}x = \int_\Omega f_\varepsilon(x)v_\varepsilon(x)\,\mathrm{d}x \quad \text{for } v_\varepsilon \in \mathrm{V}(\Omega;\gamma_\varepsilon)$$

and satisfies

$$\|u_\varepsilon\|_{\mathrm{V}(\Omega;\gamma_\varepsilon)} \leqslant 4\|f_\varepsilon\|_{\mathrm{L}^2(\Omega)}.$$

If we further have

$$\sup_{\varepsilon>0}\|f_\varepsilon\|_{\mathrm{L}^2(\Omega)}<\infty,$$

then there exists a zero sequence $(\varepsilon_\ell)_{\ell\in\mathbb{N}}$ and a unique function $u \in \mathrm{H}^1(\Omega)$ with

$$\begin{aligned}
&\lim_{\ell\to\infty}\|u_{\varepsilon_\ell}-u\|_{\mathrm{L}^2(\Omega)} = 0,\\
&\lim_{\ell\to\infty}\widehat{\mathfrak{B}}_{\varepsilon_\ell}(u_{\varepsilon_\ell},v) = \int_\Omega \sum_{i,j=1}^d a_{i,j}(x)\frac{\partial}{\partial x_i}u(x)\frac{\partial}{\partial x_j}v(x)+\sum_{k=1}^d b_k(x)u(x)\frac{\partial}{\partial x_j}v(x)\,\mathrm{d}x,\\
\text{and}\quad &\int_\Omega \sum_{i,j=1}^d a_{i,j}(x)\frac{\partial}{\partial x_i}u(x)\frac{\partial}{\partial x_j}v(x)+\sum_{k=1}^d b_k(x)u(x)\frac{\partial}{\partial x_j}v(x)\,\mathrm{d}x+\kappa\int_\Omega u(x)v(x)\mathrm{d}x\\
&= \int_\Omega f(x)v(x)\,\mathrm{d}x
\end{aligned}$$

for all $v \in \mathrm{H}^1(\mathbb{R}^d)$ *where for* $i,j,k = 1,\dots,d$ *the functions* $a_{i,j}, b_k \in \mathrm{L}^\infty(\Omega)$ *and* $f \in \mathrm{L}^2(\Omega)$ *are defined by*

$$\begin{aligned}
&\lim_{\ell\to\infty}\int_\Omega \left(\int_{\mathbb{R}^d} \frac{1}{2} y_i\, y_j \gamma_{\varepsilon_\ell}(y+x,x)\,\mathrm{d}y\right)\tau(x)\,\mathrm{d}x = \int_\Omega a_{i,j}(x)\tau(x)\,\mathrm{d}x\\
&\lim_{\ell\to\infty}\int_\Omega \left(\int_{\mathbb{R}^d} \frac{1}{2} y_k(\gamma_{\varepsilon_\ell}(\cdot,y+\cdot) - \gamma_{\varepsilon_\ell}(y+\cdot,\cdot))\,\mathrm{d}y\right)\tau(x)\,\mathrm{d}x = \int_\Omega b_k(x)\tau(x)\,\mathrm{d}x\\
\textit{and}\quad &\lim_{\ell\to\infty}\int_\Omega f_{\varepsilon_\ell}(x)\psi(x)\,\mathrm{d}x = \int_\Omega f(x)\psi(x)\,\mathrm{d}x
\end{aligned}$$

for $\tau \in \mathrm{L}^1(\Omega)$ *and* $\psi \in \mathrm{L}^2(\Omega)$*.*

Proof.

The existence and uniqueness of the family $(u_\varepsilon)_{\varepsilon>0}$ is given by Theorem 11.16. Furthermore, there are constants $c, C > 0$ with

$$\begin{aligned}
&c\int_\Omega \|\nabla u(x)\|^2\,\mathrm{d}x \leqslant \int_\Omega \sum_{i,j=1}^d a_{i,j}(x)\frac{\partial}{\partial x_i}u(x)\frac{\partial}{\partial x_j}u(x)\,\mathrm{d}x \leqslant C\int_\Omega \|\nabla u(x)\|^2\,\mathrm{d}x\\
\text{and}\quad &c\int_\Omega \|\nabla u(x)\|^2\,\mathrm{d}x \leqslant \int_\Omega \sum_{i,j=1}^d a_{i,j}(x)\frac{\partial}{\partial x_i}u(x)\frac{\partial}{\partial x_j}u(x) + \sum_{k=1}^d b_k(x)u(x)\frac{\partial}{\partial x_j}u(x)\,\mathrm{d}x\\
&\qquad\leqslant C\int_\Omega \|\nabla u(x)\|^2\,\mathrm{d}x
\end{aligned}$$

for all $u \in \mathrm{H}^1(\Omega)$.

Then, the statement follows by Theorem 11.17, the existence of the a bounded extension operator $E\colon \mathrm{H}^1(\Omega) \to \mathrm{H}^1(\mathbb{R}^d)$ (see [19, Theorem 13.17]), and due to the fact that every bounded sequence in a Hilbert space has a weakly convergent subseqence. □

Corollary 11.19.
Let $\Omega \subset \mathbb{R}^d$ *be a bounded Lipschitz domain and* $(\gamma_\varepsilon)_{\varepsilon>0} \in \mathcal{K}(\Omega)$ *such that* γ_ε *is symmetric and that the nonlocal Poincaré inequality holds on* $\mathrm{V}(\Omega;\gamma_\varepsilon)$ *for every* $\varepsilon > 0$*. Assume that* $P > 0$ *is for every* $\varepsilon > 0$ *a Poincaré constant. Then, for every family* $(f_\varepsilon)_{\varepsilon>0}$ *in* $\mathrm{L}^2(\Omega)$ *with*

$$\int_\Omega f_\varepsilon(x)\,\mathrm{d}x = 0 \qquad \textit{for all } \varepsilon > 0$$

there is a unique family $(u_\varepsilon)_{\varepsilon>0}$ *where for* $\varepsilon > 0$ *the function* $u_\varepsilon \in \mathrm{V}(\Omega;\gamma_\varepsilon)$ *solves*

$$\mathfrak{B}_\varepsilon(u_\varepsilon, v_\varepsilon) = \int_\Omega f_\varepsilon(x)v_\varepsilon(x)\,\mathrm{d}x \qquad \textit{for } v_\varepsilon \in \mathrm{V}(\Omega;\gamma_\varepsilon)$$

and satisfies

$$\int_\Omega u_\varepsilon(x)\,\mathrm{d}x = 0 \qquad \textit{and} \qquad \|u_\varepsilon\|_{\mathrm{V}(\Omega;\gamma_\varepsilon)} \leqslant (P+1)\|f_\varepsilon\|_{\mathrm{L}^2(\Omega)}.$$

If we further have

$$\sup_{\varepsilon>0}\|f_\varepsilon\|_{\mathrm{L}^2(\Omega)} < \infty,$$

then there exists a zero sequence $(\varepsilon_\ell)_{\ell\in\mathbb{N}}$ *and a unique function* $u \in \mathrm{H}^1(\Omega)$ *with*

$$\begin{aligned}
&\int_\Omega u(x)\,\mathrm{d}x = 0,\\
&\lim_{\ell\to\infty} \|u_{\varepsilon_\ell} - u\|_{\mathrm{L}^2(\Omega)} = 0,\\
&\lim_{\ell\to\infty} \mathfrak{B}_{\varepsilon_\ell}(u_{\varepsilon_\ell}, v) = \int_\Omega \sum_{i,j=1}^d a_{i,j}(x)\frac{\partial}{\partial x_i}u(x)\frac{\partial}{\partial x_j}v(x)\,\mathrm{d}x,\\
\text{and}\quad &\int_\Omega \sum_{i,j=1}^d a_{i,j}(x)\frac{\partial}{\partial x_i}u(x)\frac{\partial}{\partial x_j}v(x)\,\mathrm{d}x = \int_\Omega f(x)v(x)\,\mathrm{d}x
\end{aligned}$$

for all $v \in \mathrm{H}^1(\mathbb{R}^d)$ *where for* $i,j = 1,\dots,d$ *the functions* $a_{i,j} \in \mathrm{L}^\infty(\Omega)$ *and* $f \in \mathrm{L}^2(\Omega)$ *are defined by*

$$\begin{aligned}
&\lim_{\ell\to\infty} \int_\Omega \left(\int_{\mathbb{R}^d} \frac{1}{2} y_i\, y_j \gamma_{\varepsilon_\ell}(y+x,x)\,\mathrm{d}y\right)\tau(x)\,\mathrm{d}x = \int_\Omega a_{i,j}(x)\tau(x)\,\mathrm{d}x\\
\text{and}\quad &\lim_{\ell\to\infty} \int_\Omega f_{\varepsilon_\ell}(x)\psi(x)\,\mathrm{d}x = \int_\Omega f(x)\psi(x)\,\mathrm{d}x
\end{aligned}$$

for $\tau \in \mathrm{L}^1(\Omega)$ *and* $\psi \in \mathrm{L}^2(\Omega)$.

Proof.

The existence and uniqueness of the family family $(u_\varepsilon)_{\varepsilon>0}$ is given by Theorem 4.5. Furthermore, there are constants $c, C > 0$ with

$$c\int_\Omega \|\nabla u(x)\|^2\,\mathrm{d}x \leqslant \int_\Omega \sum_{i,j=1}^d a_{i,j}(x)\frac{\partial}{\partial x_i}u(x)\frac{\partial}{\partial x_j}u(x)\,\mathrm{d}x \leqslant C\int_\Omega \|\nabla u(x)\|^2\,\mathrm{d}x$$

for all $u \in \mathrm{H}^1(\Omega)$.

Therefore. the statement follows by Theorem 11.17, the existence of the a bounded extension operator $E\colon \mathrm{H}^1(\Omega) \to \mathrm{H}^1(\mathbb{R}^d)$ (see [19, Theorem 13.17]) and due to the fact that every bounded sequence in a Hilbert space has a weakly convergent subseqence. □

We only obtain the existence of a subsequence which converges due to the coefficients. If these coefficients converge to a limit, then we obtain a stronger convergence result.

Corollary 11.20.
Let $\Omega \subset \mathbb{R}^d$ *be a bounded Lipschitz domain and and* $(\gamma_\varepsilon)_{\varepsilon>0} \in \mathcal{K}(\Omega)$ *satisfy*

$$\sup_{\varepsilon>0} \sum_{i=1}^d \left\| \int_{\mathbb{R}^d} y_i(\gamma_\varepsilon(\cdot, y+\cdot) - \gamma_\varepsilon(y+\cdot,\cdot))\,\mathrm{d}y \right\|_{\mathrm{L}^\infty(\Omega)} < \infty.$$

For $\varepsilon > 0$*, let* k_ε *in* $\mathcal{K}$ *satisfy* $\sup_{\varepsilon>0} \left\|\int_{\mathbb{R}^d} k_\varepsilon(y,\cdot)\,\mathrm{d}y\right\|_{\mathrm{L}^\infty(\Omega)} < \infty$,

$k_\varepsilon(y,x) > 0$, *and* $\dfrac{(\gamma_\varepsilon(x,y) - \gamma_\varepsilon(y,x))^2}{k_\varepsilon(y,x)} \leqslant \gamma_\varepsilon(y,x)$ *for a.e.* $(y,x) \in \{\mathbb{R}^d \times \Omega\colon |\gamma(y,x) - \gamma(x,y)| > 0\}$.

If $\kappa > 2\sup_{\varepsilon>0} \left\| \int_{\mathbb{R}^d} k_\varepsilon(y,\cdot)\,\mathrm{d}y \right\|_{\mathrm{L}^\infty(\Omega)}$ *holds, then for every family* $(f_\varepsilon)_{\varepsilon>0}$ *in* $\mathrm{L}^2(\Omega)$ *there is a unique family* $(u_\varepsilon)_{\varepsilon>0}$ *where for* $\varepsilon > 0$ *the function* $u_\varepsilon \in \mathrm{V}(\Omega;\gamma_\varepsilon)$ *solves*

$$\widehat{\mathfrak{B}}_\varepsilon(u_\varepsilon, v_\varepsilon) + \kappa \int_\Omega u_\varepsilon(x) v_\varepsilon(x)\,\mathrm{d}x = \int_\Omega f_\varepsilon(x) v_\varepsilon(x)\,\mathrm{d}x \qquad \text{for } v_\varepsilon \in \mathrm{V}(\Omega;\gamma_\varepsilon).$$

and satisfies

$$\|u_\varepsilon\|_{\mathrm{V}(\Omega;\gamma_\varepsilon)} \leqslant 4\|f_\varepsilon\|_{\mathrm{L}^2(\Omega)}.$$

And if we further assume that for $i,j,k = 1,\dots,d$ *there are functions* $a_{i,j}, b_k \in \mathrm{L}^\infty(\Omega)$ *and a function* $f \in \mathrm{L}^2(\Omega)$ *with*

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \int_\Omega \left(\int_{\mathbb{R}^d} \frac{1}{2} y_i\, y_j \gamma_\varepsilon(y+x,x)\,\mathrm{d}y \right) \tau(x)\,\mathrm{d}x = \int_\Omega a_{i,j}(x)\tau(x)\,\mathrm{d}x \\
&\lim_{\varepsilon\to 0} \int_\Omega \left(\int_{\mathbb{R}^d} \frac{1}{2} y_k (\gamma_\varepsilon(\cdot, y+\cdot) - \gamma_\varepsilon(y+\cdot,\cdot))\,\mathrm{d}y \right) \tau(x)\,\mathrm{d}x = \int_\Omega b_k(x)\tau(x)\,\mathrm{d}x \\
\text{and}\quad &\lim_{\varepsilon\to 0} \int_\Omega f_\varepsilon(x)\psi(x)\,\mathrm{d}x = \int_\Omega f(x)\psi(x)\,\mathrm{d}x
\end{aligned}$$

for all $\tau \in \mathrm{L}^1(\Omega)$ *and all* $\psi \in \mathrm{L}^2(\Omega)$*, then there is a unique function* $u \in \mathrm{H}^1(\Omega)$ *with*

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \|u_\varepsilon - u\|_{\mathrm{L}^2(\Omega)} = 0, \\
&\lim_{\varepsilon\to 0} \widehat{\mathfrak{B}}_\varepsilon(u_\varepsilon, v) = \int_\Omega \sum_{i,j=1}^d a_{i,j}(x) \frac{\partial}{\partial x_i} u(x) \frac{\partial}{\partial x_j} v(x) + \sum_{k=1}^d b_k(x) u(x) \frac{\partial}{\partial x_j} v(x)\,\mathrm{d}x, \\
\text{and}\quad &\int_\Omega \sum_{i,j=1}^d a_{i,j}(x) \frac{\partial}{\partial x_i} u(x) \frac{\partial}{\partial x_j} v(x) + \sum_{k=1}^d b_k(x) u(x) \frac{\partial}{\partial x_j} v(x)\,\mathrm{d}x + \kappa \int_\Omega u(x)v(x)\mathrm{d}x \\
&= \int_\Omega f(x)v(x)\,\mathrm{d}x.
\end{aligned}$$

Proof.

This statements is a consequence of Corollary 11.18 and the fact that every subsequence has the same unique limit. ☐

Notation

General notations

λ	Lebesgue measure
$\langle \cdot, \cdot \rangle$	Dot product on $\mathbb{R}^d$
$\lvert x \rvert$	Absolute value of $x \in \mathbb{R}$
$\lvert\lvert \cdot \rvert\rvert$	Euclidean norm on $\mathbb{R}^d$
$\mathrm{B}_r(x)$	Open ball of radius $r > 0$ centered at the point x
S	Closure of $\mathrm{V}(\Omega; \gamma)$ with respect to $\lvert\lvert \cdot \rvert\rvert_{\mathrm{L}^2(\Omega)}$, page 93
$\overline{\Omega}$	Closure of $\Omega \subset \mathbb{R}^d$ with respect to $\lvert\lvert \cdot \rvert\rvert$
$\partial\Omega$	Topological boundary of $\Omega \subset \mathbb{R}^d$ with respect to $\lvert\lvert \cdot \rvert\rvert$
u_Ω	Average of $u \in \mathrm{L}^1(\Omega)$, page 27

Alphabetical Index

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