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Representation and Integrability Theorems for Cauchy Transforms of Functions and Measures

Dissertation

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Preface

As part of this thesis, a preprint was written which appears as [24] in the bibliography. The content of this work can be found in the Sections 3 and 4 of Chapter 6.

Summary

> Men pass away, but their deeds abide.
>
> — Augustin-Louis Cauchy (see [21], p. 147)

Legend has it that these words were the last ones of Augustin-Louis Cauchy before his death. And most likely, there is hardly any mathematician in complex analysis for whom these words are so much true as for Cauchy. One of his greatest achievements in this topic was his integral formula, namely the representation of a holomorphic function f by means of a suitable curve integral:

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta.$$

The function which f is integrated against has therefore also got a special name, it is the Cauchy kernel $\zeta \mapsto (\zeta - z)^{-1}$. For this reason, integral transforms where this kernel is the integration kernel are named Cauchy transforms and for our thesis, they will be the main research subject. More precisely, we will investigate functions of the form

$$(C_\mu f)(z) = \frac{1}{2\pi i} \int \frac{f(\zeta)}{\zeta - z} \, d\mu(\zeta)$$

where μ is a suitable positive or complex Borel measure on $\mathbb{R}$ or $\mathbb{C}$ and f a suitable function of one real or complex variable.

Cauchy transforms have wide applications in several disciplines of mathematical analysis, for example in potential theory, approximation theory and partial differential equations (see [8], p. 9, cf. [5] and [76]). Moreover, the subject has also gained significant attention in other fields such as free probability theory and the theory of random matrices (see [71] and [6]). Beginning with Cauchy's integral formula in the middle of the 19th century (see [8, Preface]), the study of Cauchy transforms has involved a wide range of mathematicians.

Sokhotski, Plemelj and Priwalow investigated the boundary behavior of holomorphic functions with a special focus on Cauchy integrals. In 1873, Sokhotski presented in his PhD thesis (see [70]) a first version of an explicit formula for the boundary values of a function represented by a Cauchy integral, cf. [8], p. 56. This formula was refined by Plemelj ([61]) and Priwalow (see [63], p. 136) and is in literature also known as Sokhostki-Plemelj formula, see [8], p. 56. It is important to remark that these results were first formulated for functions on

the unit circle having bounded variation and afterwards generalized to measures on the unit circle, see [9], p. 84.

The boundary behavior of Cauchy-type integrals is one of the key ingredients for finding functions which can be represented by Cauchy transforms. Early results in this topic are from the Brothers Riesz, see [9], p. 85, at least for the unit circle. Later on, mathematicians like Markushevich (see [79]), Tumarkin (see [49]) and Havin ([32]) gave several conditions and characterizations for Cauchy transforms of measures with compact support with the aid of approximation and extremal problems. Havinson (see [33], cf. also [84], p. 187) presented sufficient criteria basing on analytic properties of the corresponding compact set while Isaev and Yulmukhametov investigated the Cauchy transforms of functionals on Bergman spaces (see [37] and cf. also [55]).

Even though proper research on Cauchy transforms has been done, there are still a lot of open questions. For example, in the case of representation theorems, i.e. the question when a function can be represented as a Cauchy transform, there is 'still no completely satisfactory answer' ([9], p. 84). There are characterizations for measures on the circle as presented in the monograph [7] and for general compactly supported measures on the complex plane as presented in [27]. However, there seems to exist no systematic treatise of the Cauchy transform as an operator on L_p spaces and weighted L_p spaces on the real axis.

This is the point where this thesis draws on and we are interested in developing several characterizations for the representability of a function by Cauchy transforms of L_p functions. Moreover, we will attack the issue of integrability of Cauchy transforms of functions and measures, a topic which is only partly explored (see [43]). We will develop different approaches involving Fourier transforms and potential theory and investigate into sufficient conditions and characterizations.

For our purposes, we shall need some notation and the concept of Hardy spaces which will be part of the preliminary Chapter 1. Moreover, we introduce Fourier transforms and their complex analogue, namely Fourier-Laplace transforms. This will be of extraordinary usage due to the close connection of Cauchy and Fourier(-Laplace) transforms.

In the second chapter we shall begin our research with a discussion of the Cauchy transformation on the classical (unweighted) L_p spaces. Therefore, we start with the boundary behavior of Cauchy transforms including an adapted version of the Sokhotski-Plemelj formula. This result will turn out helpful for the determination of the image of the Cauchy transformation under $L_p(\mathbb{R})$ for $p \in (1, \infty)$. The cases $p = 1$ and $p = \infty$ are playing special roles here which justifies a treatise in separate sections. For $p = 1$ we will involve the real Hardy space $H_1(\mathbb{R})$ whereas the case $p = \infty$ shall be attacked by an approach incorporating intersections of Hardy spaces and certain subspaces of $L_\infty(\mathbb{R})$.

The third chapter prepares ourselves for the study of the Cauchy transformation on subspaces of $L_p(\mathbb{R})$. We shall give a short overview of the basic facts about Cauchy transforms of measures and then proceed to Cauchy transforms of functions with support in a closed set $X \subset \mathbb{R}$. Our goal is to build up the main theory on which we can fall back in the subsequent chapters.

The fourth chapter deals with Cauchy transforms of functions and measures supported by an unbounded interval which is not the entire real axis. For convenience we restrict ourselves

to the interval $[0, \infty)$. Bringing once again the Fourier-Laplace transform into play, we deduce complex characterizations for the Cauchy transforms of functions in $L_2(0, \infty)$. Moreover, we analyze the behavior of Cauchy transform on several half-planes and shall use these results for a fairly general geometric characterization. In the second section of this chapter, we focus on Cauchy transforms of measures with support in $[0, \infty)$. In this context, we shall derive a reconstruction formula for these Cauchy transforms holding under pretty general conditions as well as results on the behaviur on the left half-plane. We close this chapter by rather technical real-type conditions and characterizations for Cauchy transforms of functions in $L_p(0, \infty)$ basing on an approach in [82].

The most common case of Cauchy transforms, those of compactly supported functions or measures, is the subject of Chapter 5. After complex and geometric characterizations originating from similar ideas as in the fourth chapter, we adapt a functional-analytic approach in [27] to special measures, namely those with densities to a given complex measure μ. The chapter is closed with a study of the Cauchy transformation on weighted L_p spaces. Here, we choose an ansatz through the finite Hilbert transform on $(-1, 1)$.

The sixth chapter is devoted to the issue of integrability of Cauchy transforms. Since this topic has no comprehensive treatise in literature yet, we start with an introduction of weighted Bergman spaces and general results on the interaction of the Cauchy transformation in these spaces. Afterwards, we combine the theory of Zen spaces with Cauchy transforms by using once again their connection with Fourier transforms. Here, we shall encounter general Paley-Wiener theorems of the recent past. Lastly, we attack the issue of integrability of Cauchy transforms by means of potential theory. Therefore, we derive a Fourier integral formula for the logarithmic energy in one and multiple dimensions and give applications to Fourier and hence Cauchy transforms.

Two appendices are annexed to this thesis. The first one covers important definitions and results from measure theory with a special focus on complex measures. The second appendix contains Cauchy transforms of frequently used measures and functions with detailed calculations.

Acknowledgements

Contents

Chapter 1

Preliminaries

> Someone is sitting in the shade today because someone planted a tree a long time ago.
>
> Warren Buffett

In this chapter, we will first introduce some notation and definitions that we will frequently use in the following work. Beyond this, we establish the concept of Hardy spaces and Fourier and Fourier-Laplace transforms.

1.1 Notation

In what follows, we need some notation that we briefly introduce in this section. We write $\mathbb{C}$ for the set of complex numbers which is always equipped with the euclidian metric induced by the euclidean length $|\cdot|$ and $\mathbb{C}_\infty$ for the Riemann sphere which will be always equipped with the chordal metric. For a set $A \subset \mathbb{C}_\infty$ we denote by $\bar{A}$ its closure and by ∂A its boundary which are both taken with respect to $\mathbb{C}_\infty$. If $\Omega \subset \mathbb{C}_\infty$ is open, then we mean by $H(\Omega)$ the set of all complex-valued holomorphic functions f on Ω. If $\infty \in \Omega$, we require in addition $f(\infty) = 0$. Here, a function f is said to be holomorphic at ∞ if $g(z) := f\left(\frac{1}{z}\right)$ is holomorphic at 0. We write $f^{(n)}$ for the n-th derivative of $f \in H(\Omega)$ and shortly f' instead of $f^{(1)}$ and f'' instead of $f^{(2)}$.

We shortly introduce some other important sets of numbers: $\mathbb{N}$ is the set of all natural numbers, $\mathbb{N}_0$ the set of all natural numbers including 0, $\mathbb{Z}$ is the set of all integers while $\mathbb{Q}$ and $\mathbb{R}$ stand for the rational and real numbers, respectively. Finally, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle in $\mathbb{C}$.

The space of all complex-valued, continuous functions on $\mathbb{R}$ is denoted by $C(\mathbb{R})$ and $C_0(\mathbb{R})$ is the subset of all continuous functions f which satisfy in addition $\lim_{|x|\to\infty} f(x) = 0$. If $I \subset \mathbb{R}$ is an open interval and $k \in \mathbb{N}_0 \cup \{\infty\}$ we write $C^k(I)$ for the set of all complex-valued, k times (real) differentiable functions on I. Again, $f^{(n)}$ stands for the n-th derivative of f.

For $p \in [1, \infty)$, let $\mathscr{L}_p(\mathbb{R}^n)$ be the space of all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|f\|_p := \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p} < +\infty.$$

Here we write dx for $d\lambda_n(x)$, where λ_n is the Lebesgue measure on $\mathbb{R}^n$. Unless stated otherwise, the term almost everywhere (a.e.) will in the context of $\mathbb{R}^n$ always correspond to λ_n. Moreover, $\mathscr{L}_\infty(\mathbb{R}^n)$ is the space of all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ which are essentially bounded, i.e.

$$\|f\|_\infty := \operatorname{ess\,sup} |f| := \inf_{\substack{N \in \mathcal{B}(\mathbb{R}^n) \\ \lambda_n(N)=0}} \sup_{x \in \mathbb{R}^n \setminus N} |f(x)| < +\infty.$$

If $f : \mathbb{R}^n \to \mathbb{C}$ is measurable and bounded, then

$$\|f\|_\infty = \max_{x \in \mathbb{R}^n} |f(x)|.$$

Upon identifying functions that equal almost everywhere, $\mathscr{L}_p(\mathbb{R}^n)$ becomes the Banach space $L_p(\mathbb{R}^n)$ which consists of all equivalence classes $[f]$ with $f \in \mathscr{L}_p(\mathbb{R}^n)$. Nevertheless, we follow the usual notation and often write f instead of $[f]$ if it does not matter which element of $[f]$ is chosen (cf. [68], p. 67). In these cases we therefore speak again of functions. For $X \in \mathcal{B}(\mathbb{R}^n)$ (see A.2) we set

$$\mathscr{L}_p(X) := \{ f \in \mathscr{L}_p(\mathbb{R}) : f \mathbb{1}_X = f \}$$
$$L_p(X) := \{ f \in L_p(\mathbb{R}) : f \mathbb{1}_X = f \}.$$

Here, $\mathbb{1}_X$ is the **indicator function** of X, i.e. $\mathbb{1}_X(x) = 1$ if $x \in X$ and $\mathbb{1}_X(x) = 0$ if $x \notin X$. Moreover, we write $L_{p,\mathrm{loc}}(X)$ for the set of all measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $f \mathbb{1}_X = f$ and $f \mathbb{1}_K \in L_p(X)$ for all compact $K \subset X$.

Next, we want to establish some abbreviations for certain subsets of the complex plane $\mathbb{C}$. We set

$$\Pi_+ := \{ z \in \mathbb{C} : \operatorname{Im}(z) > 0 \}$$
$$\Pi_- := \{ z \in \mathbb{C} : \operatorname{Im}(z) < 0 \}.$$

Beyond this, we write $\mathbb{C}_-$ for the cut plane, i.e. $\mathbb{C}_- := \mathbb{C} \setminus [0, \infty)$. For $\varepsilon > 0$ and $z \in \mathbb{C}$ we denote by $U_\varepsilon(z)$ the set of all $w \in \mathbb{C}$ satisfying $|z - w| < \varepsilon$. If $a, b \in \mathbb{C}$ we write $[a, b]$ for the **line segment** between a and b, i.e.

$$[a, b] := \{ (1 - \lambda)a + \lambda b : \lambda \in [0, 1] \}.$$

A piecewise differentiable mapping $\gamma : I \to \mathbb{C}$ where $I \subset \mathbb{R}$ is an interval (not necessarily bounded) is called a **path** in $\mathbb{C}$. The set $\gamma^* := \gamma(I)$ is called the **trace** of γ and γ is called **closed** if $I = [a, b]$ for $a, b \in \mathbb{R}$ and $\gamma(a) = \gamma(b)$. An important example for a closed path is the mapping $k_r(a) : [0, 2\pi] \to \mathbb{C}$, $k_r(a)(t) = a + re^{it}$ where $a \in \mathbb{C}$ and $r > 0$. If $f : \gamma^* \to \mathbb{C}$ is such that $(f \circ \gamma)\gamma' \in L_1(I)$, we set

$$\int_\gamma f := \int_\gamma f(\zeta) \, d\zeta := \int_I f(\gamma(t))\gamma'(t) \, dt$$

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and

$$\int_\gamma |f| := \int_\gamma |f(\zeta)|\, |d\zeta| := \int_I |f(\gamma(t))| \cdot |\gamma'(t)|\, dt.$$

Now, we want to consider generalizations of paths. If I is a finite set, we call a family of paths $\gamma = (\gamma_i)_{i \in I}$ a **chain** in $\mathbb{C}$ and again $\gamma^* := \bigcup_{i \in I} \gamma_i^*$ the **trace** of γ. If each γ_i is a closed curve, we call γ a **cycle** in $\mathbb{C}$.

For a function $f : \gamma^* \to \mathbb{C}$ such that $(f \circ \gamma_i)\gamma_i' \in L_1(I_i)$ for all $i \in I$ (here $\gamma_i : I_i \to \mathbb{C}$) we set

$$\int_\gamma f := \int_\gamma f(\zeta)\, d\zeta := \sum_{i \in I} \int_{\gamma_i} f$$

and

$$\int_\gamma |f| := \int_\gamma |f(\zeta)|\, |d\zeta| := \sum_{i \in I} \int_{\gamma_i} |f|.$$

If γ is a cycle in $\mathbb{C}$, then the function $\mathrm{ind}_\gamma : \mathbb{C} \setminus \gamma^* \to \mathbb{C}$, defined by

$$\mathrm{ind}_\gamma(z) := \frac{1}{2\pi i} \int_\gamma \frac{1}{\zeta - z}\, d\zeta \quad (z \in \mathbb{C} \setminus \gamma^*),$$

is called the **index** of γ. For a compact set $K \subset \mathbb{C}$ and an open set $U \supset K$ a cycle γ is called a **Cauchy cycle** if $\mathrm{ind}_\gamma(z) = 1$ ($z \in K$) and $\mathrm{ind}_\gamma(z) = 0$ ($z \in \mathbb{C} \setminus U$).

Sometimes, we want to consider integrals over (half-)lines in $\mathbb{C}$. If $I = (a, b)$ is an interval with $a = -\infty$ or $b = \infty$ and $\gamma(t) := c + tw$ ($t \in I$) with $c, w \in \mathbb{C}$, then γ^* is a (half-)line in $\mathbb{C}$. For $f : \gamma^* \to \mathbb{C}$ such that $(f \circ \gamma) \in \mathscr{L}_1(I)$, we set

$$\int_{c+a \cdot w}^{c+b \cdot w} f(\zeta)\, d\zeta := \int_\gamma f(\zeta)\, d\zeta = w \int_a^b f(c + tw)\, dt.$$

If U, V are vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ such that $U \cap V = \{0\}$, we write $U \oplus V$ for the **direct sum** of U and V. Moreover, if $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are (semi-)normed spaces and $S : X \to Y$ is a linear operator, we write $\|S\|_{\mathrm{op}}$ for the **operator norm** of S, i.e.

$$\|S\|_{\mathrm{op}} = \sup_{\|x\|_X \le 1} \|S(x)\|_Y.$$

1.2   Hardy spaces

Hardy spaces belong to the most important spaces of analytic functions and have been studied since the beginning of the last century. An early reference is [67] where the author considered analytic functions on the unit disk. Since the unit disk is conformally invariant to any half-plane it is no surprise that most of the theory transferred in an analogue way to the case of analytic functions on half-planes. This theory will also play a crucial role throughout our whole topic. Since we need Hardy spaces on multiple half-planes we use a general definition.

Definition 1.2.1 For an angle $\alpha \in (-\pi, \pi]$ and $p \in (0, \infty)$, we define the **Hardy space** $\mathcal{H}_p(e^{i\alpha}\Pi_+)$ as the set of all $F \in H(e^{i\alpha}\Pi_+)$ satisfying

$$\|F\|_{\mathcal{H}_p(e^{i\alpha}\Pi_+)} := \left(\sup_{y>0} \int_{\mathbb{R}} |F(e^{i\alpha}(x+iy))|^p \, dx \right)^{1/p} < +\infty.$$

Moreover, we set $\mathcal{H}_\infty(e^{i\alpha}\Pi_+)$ for the space of all $F \in H(e^{i\alpha}\Pi_+)$ such that

$$\|F\|_{\mathcal{H}_\infty(e^{i\alpha}\Pi_+)} := \sup_{z \in e^{i\alpha}\Pi_+} |F(z)| < +\infty.$$

Remark 1.2.2   1. The most important cases are $\alpha = 0, \alpha = \pi, \alpha = \pi/2$ and $\alpha = -\pi/2$, i.e.

$$\mathcal{H}_p(\Pi_+) = \left\{ F \in H(\Pi_+) : \|F\|_{\mathcal{H}_p(\Pi_+)} = \left(\sup_{y>0} \int_{\mathbb{R}} |F(x+iy)|^p \, dx \right)^{1/p} < +\infty \right\}$$

$$\mathcal{H}_p(\Pi_-) = \left\{ F \in H(\Pi_-) : \|F\|_{\mathcal{H}_p(\Pi_-)} = \left(\sup_{y<0} \int_{\mathbb{R}} |F(x+iy)|^p \, dx \right)^{1/p} < +\infty \right\}$$

$$\mathcal{H}_p(i\Pi_+) = \left\{ F \in H(i\Pi_+) : \|F\|_{\mathcal{H}_p(i\Pi_+)} = \left(\sup_{x<0} \int_{\mathbb{R}} |F(x+iy)|^p \, dy \right)^{1/p} < +\infty \right\}$$

$$\mathcal{H}_p(-i\Pi_+) = \left\{ F \in H(-i\Pi_+) : \|F\|_{\mathcal{H}_p(-i\Pi_+)} = \left(\sup_{x>0} \int_{\mathbb{R}} |F(x+iy)|^p \, dy \right)^{1/p} < +\infty \right\}$$

2. For each $\alpha \in (-\pi, \pi]$ and $p \in [1, \infty]$ the space $\mathcal{H}_p(e^{i\alpha}\Pi_+)$ is a Banach space. This follows from Montel's theorem, see [7], p. 48.

3. Sometimes we may also consider Hardy spaces on shifted half-planes: Let $a \in \mathbb{R}$. Then, we denote by $\mathcal{H}_p(a + i\Pi_+)$ the set of all $F \in H(a + i\Pi_+)$ such that $F(\cdot + a) \in \mathcal{H}_p(i\Pi_+)$ and in this case we define

$$\|F\|_{\mathcal{H}_p(a+i\Pi_+)} := \|F(\cdot + a)\|_{\mathcal{H}_p(i\Pi_+)} = \left(\sup_{x<a} \int_{\mathbb{R}} |F(x+iy)|^p \, dy \right)^{1/p}.$$

Analogously, $\mathcal{H}_p(ia + \Pi_+)$ denotes the set of those $F \in H(ia + \Pi_+)$ such that $F(\cdot + ia) \in \mathcal{H}_p(\Pi_+)$ and in this case we define

$$\|F\|_{\mathcal{H}_p(ia+\Pi_+)} := \|F(\cdot + ia)\|_{\mathcal{H}_p(\Pi_+)} = \left(\sup_{y>a} \int_{\mathbb{R}} |F(x+iy)|^p \, dy \right)^{1/p}.$$

The Hardy spaces $\mathcal{H}_p(a + i\Pi_-)$ and $\mathcal{H}_p(ia + \Pi_-)$ are defined in a similar way.

Our focus will only lie on Hardy spaces for exponents $p \in [1, \infty]$. The following proposition contains important and well-known results for functions in the Hardy space $\mathcal{H}_p(\Pi_+)$. These results hold in an analogue way for other (shifted) Hardy spaces. We refer to [16], p. 189-192, and [28, Chapter II, Theorem 4.4].

Proposition 1.2.3 ([16]) *Let $p \in [1, \infty]$ and $F \in \mathcal{H}_p(\Pi_+)$.*

1. *The limit*
$$F_+(x) := \lim_{y \to 0^+} F(x + iy)$$

   exists for almost every $x \in \mathbb{R}$, $F_+ \in L_p(\mathbb{R})$ and if $F \neq 0$, also
$$\int_\mathbb{R} \frac{\ln |F_+(x)|}{1 + x^2} \, dx > -\infty.$$

   Furthermore, $\|F\|_{\mathcal{H}_p(\Pi_+)} = \|F_+\|_p$ and in the case $p \in [1, \infty)$ also
$$\lim_{y \to 0^+} \int_\mathbb{R} |F(x + iy) - F_+(x)|^p \, dx = 0.$$

2. *In the case $p \in [1, \infty)$, the integral means*
$$\int_\mathbb{R} |F(x + iy)|^p \, dx$$

   are increasing as $y \downarrow 0$ and therefore
$$\|F\|_{\mathcal{H}_p(\Pi_+)} = \lim_{y \to 0^+} \left(\int_\mathbb{R} |F(x + iy)|^p \, dx \right)^{\frac{1}{p}}.$$

Remark and Definition 1.2.4 Let $p \in [1, \infty]$.

1. If $F \in \mathcal{H}_p(\Pi_+)$, we will call the function F_+ in Proposition 1.2.3 the **upper boundary function** of F.

2. If $F \in \mathcal{H}_p(\Pi_-)$, then one can show that
$$F_-(x) := \lim_{y \to 0^-} F(x + iy)$$

   exists for almost every $x \in \mathbb{R}$ and in this case we call F_- the **lower boundary function** of F.

3. By the first part of Proposition 1.2.3 a boundary function of a function belonging to $\mathcal{H}_p(\Pi_+)$ (or any other Hardy space) cannot vanish on a set of positive measure.

We shall now introduce an important class of measures which play a crucial role in interpolation theory.

Definition 1.2.5 A measure $\sigma \in \mathcal{M}_{\infty,+}(\Pi_+)$ is called a **Carleson measure** (for Π_+) if there is a constant $N(\sigma)$ such that
$$\sigma(\{x + iy : x_0 < x < x_0 + h, 0 < y < h\}) \leq N(\sigma)h$$

for all $x_0 \in \mathbb{R}$ and $h > 0$. The infimum over all such constants $N(\sigma)$ is called the **Carleson norm** of σ.

Remark 1.2.6 Carleson measures on the lower half-plane Π_- are defined in a similar way.

Remark 1.2.7   1. Let $\omega : (0, \infty) \to [0, \infty]$ be a measurable function. If ν is the Borel measure on Π_+ defined by

$$\nu(A) = \int_A \omega(y) \, dx \, dy \quad (A \in \mathcal{B}(\Pi_+)),$$

then ν is a Carleson measure if and only if $\omega \in L_1(0, \infty)$.

2. Let $\omega : \mathbb{R} \to [0, \infty]$ be a measurable function. If ν is the Borel measure on Π_+ defined by

$$\nu(A) = \int_A \omega(x) \, dx \, dy \quad (A \in \mathcal{B}(\Pi_+)),$$

then ν is a Carleson measure if and only if $\omega \in L_1(\mathbb{R})$.

3. The measure $\nu = \delta_0 \otimes (\mathbb{1}_{(0,\infty)} \lambda_1)$ is a Carleson measure with Carleson norm equal to 1.

The following theorem goes back to Carleson and is a fundamental result in the analysis of Hardy spaces. A proof can be found in [28, Chapter II, Theorem 3.9].

Theorem 1.2.8 (Carleson, [28]) *For a positive measure $\sigma \in \mathcal{M}_{\infty,+}(\Pi_+)$, the following statements are equivalent:*

a) σ is a Carleson measure.

b) For all $p \in (0, \infty)$ there exists a constant $K_p > 0$ only depending on p and σ such that

$$\int_{\Pi_+} |F|^p \, d\sigma \leq K_p \|F\|^p_{\mathcal{H}_p(\Pi_+)} \quad (F \in \mathcal{H}_p(\Pi_+)).$$

c) There exists $p \in (0, \infty)$ such that for all $F \in \mathcal{H}_p(\Pi_+)$ we have

$$\int_{\Pi_+} |F|^p \, d\sigma < +\infty.$$

Remark 1.2.9 The theorem holds mutatis mutandis for Carleson measures on the lower half-plane Π_-.

Remark and Definition 1.2.10 For $\alpha > 0$ and $t \in \mathbb{R}$ we consider the cones

$$\Gamma^+_\alpha(t) := \{z \in \Pi_+ : |\mathrm{Re}(z) - t| < \alpha \mathrm{Im}(z)\}$$
$$\Gamma^-_\alpha(t) := \{z \in \Pi_- : |\mathrm{Re}(z) - t| < -\alpha \mathrm{Im}(z)\}.$$

If g is a harmonic function on Π_+, then the function $g^*_{+,\alpha} : \mathbb{R} \to [0, \infty]$, defined by

$$g^*_{+,\alpha}(t) := \sup_{\Gamma^+_\alpha(t)} |g| \quad (t \in \mathbb{R}),$$

is called the **nontangential maximal function** of g. Similarly, if g is harmonic on Π_-, we set

$$g_{-,\alpha}^*(t) := \sup_{\Gamma_\alpha^-(t)} |g| \quad (t \in \mathbb{R}).$$

If $p \in (1,\infty)$ and $g \in \mathcal{H}_p(\Pi_+)$, then there is a constant $B_{p,\alpha} > 0$ only depending on p and α such that

$$\|g_{+,\alpha}^*\|_p^p \le B_{p,\alpha}\|g\|_{\mathcal{H}_p(\Pi_+)}^p,$$

see [28, Chapter I, Theorem 5.1]. If $g \in \mathcal{H}_1(\Pi_+)$, then there is a constant $B_{1,\alpha} > 0$ only depending on α such that

$$\lambda_1(\{t \in \mathbb{R} : g_{+,\alpha}^*(t) > M\}) \le \frac{B_{1,\alpha}}{M}\|g\|_{\mathcal{H}_1(\Pi_+)} \quad (M > 0).$$

Similar statements hold for $g \in \mathcal{H}_p(\Pi_-)$ and the constants $B_{p,\alpha}$ remain the same for symmetry reasons. We write $B_p := B_{p,1}$ for $p \in [1,\infty)$.

Remark 1.2.11 The constant $K_p > 0$ in Theorem 1.2.8 can be chosen as the product of the Carleson norm of the measure and the constant B_p.

1.3 Fourier and Fourier-Laplace transforms

In this section, we briefly introduce the concept of Fourier transforms, both real and analytic. For $\alpha \in [0,\pi], p \in [1,\infty)$ and $\mu \in \mathcal{M}_\infty(\mathbb{R})$ with $\mathrm{supp}(\mu) \subset [0,\infty)$, we consider the space

$$\mathscr{E}_{\alpha,p}(\mu) := \{f : e^{i\alpha} \cdot \mathbb{R} \to \mathbb{C} : f|_{e^{i\alpha}\cdot(-\infty,0)} = 0, f(e^{i\alpha}\cdot)e^{-a\cdot} \in \mathscr{L}_p(\mu) \text{ for all } a > 0\}.$$

We shall call this space also **exponential space of order p and angle α** (with respect to μ). For the sake of abbreviation, we write $\mathscr{E}_{\alpha,p}$ instead of $\mathscr{E}_{\alpha,p}\left(\mathbb{1}_{(0,\infty)}\lambda_1\right)$.

Remark and Definition 1.3.1 Let $\alpha \in [0,\pi], \mu \in \mathcal{M}_\infty(\mathbb{R})$ with $\mathrm{supp}(\mu) \subset [0,\infty)$ and $f \in \mathscr{E}_{\alpha,1}(\mu)$. Then, for each $z \in \mathbb{C}$ with $\mathrm{Im}(ze^{i\alpha}) > 0$ we have

$$\left|\int_0^\infty f(e^{i\alpha}t)e^{ize^{i\alpha}t}\,d\mu(t)\right| \le \int_0^\infty |f(e^{i\alpha}t)|e^{-\mathrm{Im}(ze^{i\alpha})t}\,d|\mu|(t) < +\infty.$$

Hence, the function $L_{\alpha,\mu}f : \{z \in \mathbb{C} : \mathrm{Im}(ze^{i\alpha}) > 0\} \to \mathbb{C}$, defined by

$$(L_{\alpha,\mu}f)(z) := \frac{1}{2\pi}\int_0^{e^{i\alpha}\cdot\infty} f(\zeta)e^{iz\zeta}\,d\mu(\zeta) = \frac{e^{i\alpha}}{2\pi}\int_0^\infty f(e^{i\alpha}t)e^{ize^{i\alpha}t}\,d\mu(t) \quad (\mathrm{Im}(ze^{i\alpha}) > 0)$$

is a holomorphic function. If $\mu = \mathbb{1}_{(0,\infty)}\lambda_1$, we simply write L_α instead of $L_{\alpha,\mathbb{1}_{(0,\infty)}\lambda_1}$.

Remark and Definition 1.3.2 The most important cases for us are $\alpha = 0$ and $\alpha = \pi$. If $f : \mathbb{R} \to \mathbb{C}$ is such that $f\mathbb{1}_{[0,\infty)} \in \mathscr{E}_{0,1}(\mu)$ and $f\mathbb{1}_{(-\infty,0]} \in \mathscr{E}_{\pi,1}(\mu)$, then we call the function $L_\mu f : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ defined by

$$(L_\mu f)(z) := \begin{cases} L_{0,\mu} f(z) = \frac{1}{2\pi} \int_0^\infty f(t) e^{izt} \, d\mu(t), & z \in \Pi_+ \\ L_{\pi,\mu} f(z) = -\frac{1}{2\pi} \int_0^\infty f(-t) e^{-izt} \, d\mu(t), & z \in \Pi_-. \end{cases}$$

the (μ-)**Fourier-Laplace transform** of f. One can show that $L_\mu f \in H(\mathbb{C} \setminus \mathbb{R})$ and we write Lf instead of $L_{\mathbb{1}_{(0,\infty)}\lambda_1} f$.

Remark 1.3.3 If $f : \mathbb{R} \to \mathbb{C}$ is such that $f\mathbb{1}_{[0,\infty)} \in \mathscr{E}_{0,1}$ and $f\mathbb{1}_{(-\infty,0]} \in \mathscr{E}_{\pi,1}$, then we have

$$(Lf)(z) = \begin{cases} \frac{1}{2\pi} \int_0^\infty f(t) e^{izt} \, dt, & z \in \Pi_+ \\ -\frac{1}{2\pi} \int_{-\infty}^0 f(t) e^{izt} \, dt, & z \in \Pi_-. \end{cases}$$

Remark and Definition 1.3.4 Let $f \in L_1(\mathbb{R})$. Then, $(L_0 f)(z)$ exists for every $z \in \overline{\Pi}_+$ and $(L_\pi f)(z)$ exists for every $z \in \overline{\Pi}_-$. The function $\widehat{f} : \mathbb{R} \to \mathbb{C}$ defined by

$$\widehat{f}(x) := 2\pi((L_0 f)(-x) - (L_\pi f)(-x)) = \int_{-\infty}^\infty f(t) e^{-ixt} \, dt \quad (x \in \mathbb{R})$$

is called the **Fourier transform** of f. By the dominated convergence theorem, one sees that $\widehat{f}$ is a uniformly continuous function and the Riemann-Lebesgue lemma (see e.g. [68, Theorem 9.6]) gives us that even $\widehat{f} \in C_0(\mathbb{R})$. Moreover, we have $\|\widehat{f}\|_\infty \leq \|f\|_1$. The mapping $\mathcal{F} : L_1(\mathbb{R}) \to C_0(\mathbb{R}), f \mapsto \widehat{f}$ is called the **Fourier transformation** (on $L_1(\mathbb{R})$). One can show that this mapping is injective, but not surjective.

Remark and Definition 1.3.5 Let $\mu \in \mathcal{M}(\mathbb{R}^n)$. Then, the function $\widehat{\mu} : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\widehat{\mu}(x) := \int_{\mathbb{R}^n} e^{-ix \cdot t} \, d\mu(t) \quad (x \in \mathbb{R}^n)$$

is called the **Fourier transform** of μ. Here, $\cdot$ denotes the usual inner product on $\mathbb{R}^n$, i.e. $x \cdot y = \sum_{k=1}^n x_k y_k$ whenever $x, y \in \mathbb{R}^n$. By the dominated convergence theorem, one sees again that $\widehat{\mu}$ is a uniformly continuous function. Moreover, we have $\|\widehat{\mu}\|_\infty \leq \|\mu\|$.

Our next step is to define the Fourier transform on $L_p(\mathbb{R})$ for $p \in (1, 2]$.

Remark and Definition 1.3.6 If $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, then we have $\widehat{f} \in L_2(\mathbb{R})$ and $\|\widehat{f}\|_2 = \|f\|_2$ (see [68, Theorem 9.13]). Since $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ is a dense subspace of $L_2(\mathbb{R})$, one can show that there is a unique isomorphism $T : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ such that $Tf = \widehat{f}$ for each $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$. This result is also known as **Plancherel's theorem** and in the following we always write $Tf = \widehat{f}$ for $f \in L_2(\mathbb{R})$ and call this mapping the **Fourier transformation**. Instead of T we also use the symbol $\mathcal{F}$.

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Remark and Definition 1.3.7 Let $p \in (1, 2)$ and $f \in L_p(\mathbb{R})$. Then, there are functions $f_1 \in L_1(\mathbb{R})$ and $f_2 \in L_2(\mathbb{R})$ such that $f = f_1 + f_2$. We therefore define the Fourier transform of f simply by $\widehat{f} = \widehat{f_1} + \widehat{f_2}$. Due to [50], p. 273, this definition does not depend on the particular representation of f, i.e. if $f = g_1 + g_2$ with $g_1 \in L_1(\mathbb{R})$ and $g_2 \in L_2(\mathbb{R})$, we have

$$\widehat{f_1} + \widehat{f_2} = \widehat{g_1} + \widehat{g_2}.$$

The Hausdorff-Young theorem (see, e.g., [50, Theorem 12.5]) tells us that $\widehat{f} \in L_q(\mathbb{R})$ where $q = \frac{p}{p-1}$ is the conjugate exponent of p and moreover $\|\widehat{f}\|_q \le \|f\|_p$. Again, we call the mapping $\mathcal{F} : L_p(\mathbb{R}) \to L_q(\mathbb{R}), f \mapsto \widehat{f}$ the Fourier transformation on $L_p(\mathbb{R})$.

Chapter 2

The Cauchy Transformation on the Classical L_p Spaces

> Nothing is brought about large-scale
> But is begun small-scale.
>
> ――――――――――――――――――
> Johann-Wolfgang von Goethe (cf. [66],
> p. 167.)

It seems natural to begin with the study of Cauchy transforms of functions belonging to $L_p(\mathbb{R})$ for some $p \in [1, \infty)$. In this setting, we will discuss both representation theorems and uniqueness and inversion theorems. An essential tool for these results is the boundary behavior of Cauchy transforms which will be the starting point of this chapter.

2.1 Cauchy transforms and their boundary behavior

We first give a general definition of the Cauchy transform.

Definition 2.1.1 Let $\mu \in \mathcal{M}_\infty(\mathbb{R})$. We say that a measurable function $f : \mathbb{R} \to \mathbb{C}$ is **Cauchy transformable** (with respect to μ) if

$$\int_{\mathbb{R}} \frac{|f(t)|}{|t| + 1} \, d|\mu|(t) < +\infty.$$

We write $\mathscr{C}(\mu)$ for the set of all functions which are Cauchy transformable with respect to μ. One easily sees that $\mathscr{C}(\mu)$ is a vector space over $\mathbb{C}$. For the sake of abbreviation we write $\mathscr{C}$ instead of $\mathscr{C}(\lambda_1)$.

Remark 2.1.2 Let $\mu \in \mathcal{M}_\infty(\mathbb{R})$.

1. Since $|t| + 1 \geq 1$ $(t \in \mathbb{R})$ one immediately sees that $L_1(\mu) \subset \mathscr{C}(\mu)$.

2. If $p \in (1, \infty]$ is such that

$$\int_{\mathbb{R}} \frac{1}{(|t| + 1)^q} \, d|\mu|(t) < +\infty$$

where q is the conjugate exponent of p, then we have $L_p(\mu) \subset \mathscr{C}(\mu)$. This follows from Hölder's inequality since

$$\int_{\mathbb{R}} \frac{|f(t)|}{|t| + 1} \, d|\mu|(t) \leq \|f\|_{L_p(\mu)} \cdot \left(\int_{\mathbb{R}} \frac{1}{(|t| + 1)^q} \, d|\mu|(t) \right)^{1/q} < +\infty.$$

In particular, $L_p(\mathbb{R}) \subset \mathscr{C}$ for all $p \in [1, \infty)$.

Remark and Definition 2.1.3 Let $\mu \in \mathcal{M}_\infty(\mathbb{R})$ and $f \in \mathscr{C}(\mu)$. If $z \in \mathbb{C} \setminus \mathbb{R}$, then we have

$$\int_{\mathbb{R}} \left| \frac{f(t)}{t - z} \right| d|\mu|(t) = \int_{\mathbb{R}} \frac{|f(t)|}{|t| + 1} \cdot \frac{|t| + 1}{|t - z|} \, d|\mu|(t) < +\infty.$$

and the function $C_\mu f : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$, with

$$(C_\mu f)(z) := \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t - z} \, d\mu(t) \quad (z \in \mathbb{C} \setminus \mathbb{R})$$

is called the (μ-)**Cauchy transform** of f. We clearly have $C_\mu f \in H(\mathbb{C} \setminus \mathbb{R})$ with

$$(C_\mu f)^{(n)}(z) = \frac{n!}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{(t - z)^{n+1}} \, d\mu(t) \quad (z \in \mathbb{C} \setminus \mathbb{R}, n \in \mathbb{N}_0),$$

which can be seen by differentiation under the integral sign. The map $C_\mu : \mathscr{C}(\mu) \to H(\mathbb{C} \setminus \mathbb{R})$ is clearly linear and is called the (μ-)**Cauchy transformation**. Again, we write Cf instead of $C_{\lambda_1} f$ and simply speak of the Cauchy transformation if we consider the map $C : \mathscr{C} \to H(\mathbb{C} \setminus \mathbb{R})$.

With respect to the theory of classical integral transforms, three major questions immediately arise:

1. Is a Cauchy transform uniquely determined by its initial function? That is to say: If $C_\mu f = 0$, then it follows that $f = 0$?

2. What conditions are necessary and sufficient for a function to be representable as a Cauchy transform?

3. How can one recover the initial function from its Cauchy transform?

The first question targets the injectivity of the Cauchy transformation and will be answered (positively) for $\mu = \lambda_1$ in this section. Later on, we shall relate this topic to the injectivity of the corresponding Fourier-Laplace transform L_μ. The second and the third question are closely related and turn towards the issue of inverting the Cauchy transformation. This task

will constitute a major part in this work and we start with the answer for $L_p(\mathbb{R})$ in the next section.

In order to prove the injectivity of the Cauchy transformation, we first establish the so called Plemelj formulas which lead to a decomposition theorem for L_p functions. We shall define another important integral transform which is crucially related to the Cauchy transform.

Remark and Definition 2.1.4 Let $f \in \mathscr{C}$. Then, the limit

$$(Hf)(x) := \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|t-x| \geq \varepsilon} \frac{f(t)}{x-t} \, dt$$

exists for almost every $x \in \mathbb{R}$ (see, e.g [50], p. 307). The function Hf is called the **Hilbert transform** of f. If $p \in (1, \infty)$ and $f \in L_p(\mathbb{R})$, then we have additionally $Hf \in L_p(\mathbb{R})$ (see, e.g. [40], p. 128). Moreover, the mapping $H : L_p(\mathbb{R}) \to L_p(\mathbb{R})$ is an isomorphism for $p \in (1, \infty)$ satisfying $HHf = -f$ ($f \in L_p(\mathbb{R})$).

In context of Cauchy transforms, one of the most important topics is their boundary behavior towards the real axis. We therefore introduce two integral transforms which together form the Cauchy transform.

Remark and Definition 2.1.5 Let $\mu \in \mathcal{M}_\infty(\mathbb{R})$ and $f : \mathbb{R} \to \mathbb{C}$ be measurable such that

$$\int_{\mathbb{R}} \frac{|f(t)|}{1+t^2} \, d|\mu|(t) < +\infty.$$

Then, the function $P_\mu f : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$, defined by

$$(P_\mu f)(z) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(x-t)^2 + y^2} f(t) \, d\mu(t) \quad (z = x + iy \in \mathbb{C} \setminus \mathbb{R})$$

is called the (μ-)**Poisson transform** of f. If $\mu = \lambda_1$, then we write shortly Pf instead of $P_{\lambda_1} f$ and in this case we always have (see [50, Corollary 11.11])

$$\lim_{y \to 0^+} (Pf)(x + iy) = f(x)$$

$$\lim_{y \to 0^-} (Pf)(x + iy) = -f(x)$$

for almost every $x \in \mathbb{R}$. Moreover, if $f \in L_p(\mathbb{R})$ for some $p \in [1, \infty)$, then

$$\sup_{y > 0} \int_{\mathbb{R}} |(Pf)(x + iy)|^p \, dx < +\infty$$

$$\sup_{y < 0} \int_{\mathbb{R}} |(Pf)(x + iy)|^p \, dx < +\infty,$$

see [50, Corollary 10.12].

Now, let $f \in \mathscr{C}(\mu)$. Then, the function $Q_\mu f : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$, defined by

$$(Q_\mu f)(z) := \frac{1}{\pi} \int_{\mathbb{R}} \frac{(x-t)}{(x-t)^2 + y^2} f(t) \, d\mu(t) \quad (z = x + iy \in \mathbb{C} \setminus \mathbb{R})$$

is called the (μ)-**conjugate Poisson transform** of f. Since

$$\frac{y}{(x-t)^2+y^2} = \frac{i}{2} \cdot \left(\frac{1}{t-\bar{z}} - \frac{1}{t-z}\right)$$

and

$$\frac{(x-t)}{(x-t)^2+y^2} = -\frac{1}{2}\left(\frac{1}{t-z} + \frac{1}{t-\bar{z}}\right)$$

for every $z = x + iy \in \mathbb{C} \setminus \mathbb{R}, t \in \mathbb{R}$ we see that

$$(C_\mu f)(z) = \frac{1}{2}((P_\mu f)(z) + i(Q_\mu f)(z)) \quad (z \in \mathbb{C} \setminus \mathbb{R}).$$

Again we simply write Qf instead of $Q_{\lambda_1} f$ and due to [50], p. 307, we have

$$\lim_{y \to 0^+} (Qf)(x + iy) = \lim_{y \to 0^-} (Qf)(x + iy) = (Hf)(x)$$

for almost every $x \in \mathbb{R}$.

The main result concerning the boundary behavior of Cauchy integrals is now an immediate consequence (see [61] and (even earlier) [70], cf. [50, Corollary 14.8]).

Proposition 2.1.6 (**Plemelj formulas**, [50]) *Let $f \in \mathscr{C}$. Then*

$$\lim_{y \to 0^+} (Cf)(x + iy) = \frac{1}{2}(f(x) + i(Hf)(x))$$

$$\lim_{y \to 0^-} (Cf)(x + iy) = \frac{1}{2}(-f(x) + i(Hf)(x))$$

for almost every $x \in \mathbb{R}$. In particular,

$$f(x) = \lim_{y \to 0^+} (Cf)(x + iy) - \lim_{y \to 0^-} (Cf)(x + iy)$$

for almost every $x \in \mathbb{R}$.

Since the formulas above hold in $\mathscr{C}$ we can deduce the following

Corollary 2.1.7 *If $f \in \mathscr{C}$ is such that $Cf = 0$ on $\mathbb{C} \setminus \mathbb{R}$, then $f = 0$ almost everywhere.*

Remark 2.1.8 If one considers the Cauchy transformation as an operator from $\mathscr{C}$ to $H(\Pi_+)$, i.e., $C : \mathscr{C} \to H(\Pi_+)$, then the statement of Corollary 2.1.7 is false. One reason is that any function $F \in \mathcal{H}_p(\Pi_+)$ satisfies

$$\int_{-\infty}^{\infty} \frac{F(t+ia)}{t-z} dt = 0 \quad (z \in \Pi_+)$$

for all $a > 0$, see [50, Lemma 13.1].

If a function $f \in \mathscr{C}$ has support in a closed set $X \subset \mathbb{R}$, then Cf has a holomorphic continuation along $\mathbb{C} \setminus X$. An application of the Plemelj formulas shows that the converse is true as well:

Lemma 2.1.9 *Let $f \in \mathscr{C}$. Then, for a closed set $X \subset \mathbb{R}$, the following assertions are equivalent:*

a) $Cf \in H(\mathbb{C} \setminus \mathbb{R})$ has a holomorphic continuation along $\mathbb{C} \setminus X$.

b) $f \mathbb{1}_X = f$.

Proof. The inclusion b) $\Rightarrow$ a) is clear. Furthermore, if Cf has an analytic continuation along $\mathbb{C} \setminus X$, then the Plemelj formulas imply that

$$f(x) = \lim_{y \to 0^+} (Cf)(x + iy) - \lim_{y \to 0^-} (Cf)(x + iy) = 0$$

for almost every $x \in \mathbb{R} \setminus X$. Therefore $f \mathbb{1}_X = f$. $\qquad \square$

2.2 Inversion of the Cauchy transformation on $L_p(\mathbb{R})$ for $p \in (1, \infty)$

One of the major tasks in this work is the determination and characterization of the image of the mapping C under certain subspaces of $\mathscr{C}$. This relates to the issue of solving the complex integral equation

$$F(z) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(t)}{t - z}\, dt \quad (z \in \mathbb{C} \setminus \mathbb{R})$$

for a given F which is holomorphic in $\mathbb{C} \setminus \mathbb{R}$ (or some open superset of $\mathbb{C} \setminus \mathbb{R}$). As we will see shortly, this equation will always have a solution $f \in L_p(\mathbb{R})$ for $p \in [1, \infty)$, provided that F belongs to the Hardy space on the upper and lower half plane. Since these functions will appear frequently, the corresponding space is worth a definition.

Remark and Definition 2.2.1 Let $p \in [1, \infty)$. Then we write $\mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})$ for the set of all functions $F \in H(\mathbb{C} \setminus \mathbb{R})$ which satisfy

$$\|F\|_{\mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})} := \left(\sup_{y \neq 0} \int_{\mathbb{R}} |F(x + iy)|^p\, dx \right)^{1/p} < +\infty.$$

Moreover, $\mathcal{H}_\infty(\mathbb{C} \setminus \mathbb{R})$ is the space of all bounded $F \in H(\mathbb{C} \setminus \mathbb{R})$ endowed with the norm

$$\|F\|_{\mathcal{H}_\infty(\mathbb{C} \setminus \mathbb{R})} := \sup_{z \in \mathbb{C} \setminus \mathbb{R}} |F(z)|.$$

It is easy to see that for each $F \in \mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})$, the following chain of inequalities is valid:

$$\max\{\|F\|_{\mathcal{H}_p(\Pi_+)}, \|F\|_{\mathcal{H}_p(\Pi_-)}\} \leq \|F\|_{\mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})} \leq \|F\|_{\mathcal{H}_p(\Pi_+)} + \|F\|_{\mathcal{H}_p(\Pi_-)}.$$

Therefore, a function $F \in H(\mathbb{C}\backslash\mathbb{R})$ belongs to $\mathcal{H}_p(\mathbb{C}\backslash\mathbb{R})$ if and only if $F|_{\Pi_+} \in \mathcal{H}_p(\Pi_+), F|_{\Pi_-} \in \mathcal{H}_p(\Pi_-)$. For $F \in \mathcal{H}_p(\mathbb{C} \backslash \mathbb{R})$ we can define almost everywhere the function

$$\tilde{F}(x) := F_+(x) + F_-(x) = \lim_{y \to 0^+} F(x + iy) + \lim_{y \to 0^-} F(x + iy).$$

From the completeness of Hardy spaces and the inequality above we deduce

Proposition 2.2.2 *Let $p \in [1, \infty]$. Then, the space $\mathcal{H}_p(\mathbb{C} \backslash \mathbb{R})$ is a Banach space.*

Remark 2.2.3 It should be mentioned that we can regard $\mathcal{H}_2(\mathbb{C}\backslash\mathbb{R})$ also as a Hilbert space. Therefore, we consider the inner product on $\mathcal{H}_2(\mathbb{C} \backslash \mathbb{R})$ defined by

$$\langle f, g \rangle := \frac{1}{2} \lim_{y \to 0^+} \int_{\mathbb{R}} f(x + iy)\overline{g(x + iy)} \, dx + \frac{1}{2} \lim_{y \to 0^-} \int_{\mathbb{R}} f(x + iy)\overline{g(x + iy)} \, dx$$

whenever $f, g \in \mathcal{H}_2(\mathbb{C}\backslash\mathbb{R})$. This inner product induces an equivalent norm to $\|\cdot\|_{\mathcal{H}_2(\mathbb{C}\backslash\mathbb{R})}$ since the integral means of functions in $\mathcal{H}_p(\Pi_+)$ ($\mathcal{H}_p(\Pi_-)$) are decreasing (increasing) as $y \to 0$, see Proposition 1.2.3.

Our considerations on the image of $L_p(\mathbb{R})$ under the Cauchy transformation C originate from a well-known result (see, e.g., [50, Corollary 14.8] or [45, Section 19.3, Corollary 1]) which tells that if $p \in (1, \infty)$ and $f \in L_p(\mathbb{R})$, then $Cf|_{\Pi_+} \in \mathcal{H}_p(\Pi_+)$ and $Cf|_{\Pi_-} \in \mathcal{H}_p(\Pi_-)$. In particular,

$$C(L_p(\mathbb{R})) \subset \mathcal{H}_p(\mathbb{C} \backslash \mathbb{R})$$

for $p \in (1, \infty)$.

In fact, $\mathcal{H}_p(\mathbb{C} \backslash \mathbb{R})$ is actually **equal** to $C(L_p(\mathbb{R}))$ in this case. This is what we want to show next. One of the main tools for the proof is the so called Riesz factorization theorem which gives a decomposition for $L_p(\mathbb{R})$ if $p \in (1, \infty)$. However, this result fails in the case $p = 1$! This suggests why the determination of $C(L_1(\mathbb{R}))$ is pretty much harder than the one of $C(L_p(\mathbb{R}))$ for $p > 1$.

We first introduce certain subspaces of $L_p(\mathbb{R})$.

Remark and Definition 2.2.4 For $p \in [1, \infty)$, we set

$$H_p^+(\mathbb{R}) := \left\{ f \in L_p(\mathbb{R}) : \int_{\mathbb{R}} \frac{f(t)}{t - z} \, dt = 0 \ (z \in \Pi_-) \right\}$$

$$H_p^-(\mathbb{R}) := \left\{ f \in L_p(\mathbb{R}) : \int_{\mathbb{R}} \frac{f(t)}{t - z} \, dt = 0 \ (z \in \Pi_+) \right\}.$$

Due to [16, Theorem 11.8], the space $H_p^+(\mathbb{R})$ consists exactly of the functions $f \in L_p(\mathbb{R})$ which are the boundary function of some $F \in \mathcal{H}_p(\Pi_+)$. Analogously, one sees that $H_p^-(\mathbb{R})$ is exactly the space of the functions $f \in L_p(\mathbb{R})$ which are the boundary function of some $F \in \mathcal{H}_p(\Pi_-)$.

Remark 2.2.5 Let $p \in [1, \infty)$ and $f \in L_p(\mathbb{R})$.

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1. It is clear that $f \in H_p^+(\mathbb{R})$ if and only if $(Cf)(z) = (Pf)(z)$ for all $z \in \Pi_+$. Hence, if $f \in H_p^+(\mathbb{R})$, then

$$\lim_{y \to 0^+} (Cf)(x + iy) = \lim_{y \to 0^+} (Pf)(x + iy) = f(x)$$

for almost every $x \in \mathbb{R}$ by Remark and Definition 2.1.5. On the other hand, $f \in H_p^-(\mathbb{R})$ if and only if $(Cf)(z) = (Pf)(z)$ for all $z \in \Pi_-$. This implies by a similar argumentation that if $f \in H_p^-(\mathbb{R})$, then
$$\lim_{y \to 0^-} (Cf)(x + iy) = -f(x)$$

for almost every $x \in \mathbb{R}$.

2. As a consequence we deduce that f belongs to $H_p^+(\mathbb{R})$ if and only if $Hf = -if$. Similar, f belongs to $H_p^-(\mathbb{R})$ if and only if $Hf = if$. This result appears in [64], p. 972, but without proof and we shall give one for completeness.

   Proof: We only prove the statement for $H_p^+(\mathbb{R})$. First notice that by 1. and the Plemelj formulas we must have $f = \frac{1}{2}(f + iHf)$ and therefore $Hf = -if$. Conversely, if $Hf = -if$, then $\lim_{y \to 0^-} (Cf)(x + iy) = 0$ for almost every $x \in \mathbb{R}$. Since $Hf \in L_p(\mathbb{R})$, [34, Theorem 3.1] implies that $(Cf)(z) = (Pf)(z)$ for all $z \in \Pi_+$ and hence $f \in H_p^+(\mathbb{R})$ by 1.

   This means

$$H_p^+(\mathbb{R}) = \{f \in L_p(\mathbb{R}) : Hf \in H_p^+(\mathbb{R})\}$$
$$H_p^-(\mathbb{R}) = \{f \in L_p(\mathbb{R}) : Hf \in H_p^-(\mathbb{R})\}.$$

3. We have $H_p^+(\mathbb{R}) \cap H_p^-(\mathbb{R}) = \{0\}$. For, if $f \in H_p^+(\mathbb{R}) \cap H_p^-(\mathbb{R})$, then $(Cf)(z) = 0$ for all $z \in \mathbb{C} \setminus \mathbb{R}$. Proposition 2.1.7 implies that $f = 0$.

By the third part of the previous remark, we can say that

$$H_p^+(\mathbb{R}) + H_p^-(\mathbb{R}) = H_p^+(\mathbb{R}) \oplus H_p^-(\mathbb{R})$$

for $p \in [1, \infty)$. If $p \neq 1$, then this direct sum is actually equal to $L_p(\mathbb{R})$. This is the statement of the **Riesz factorization theorem** for $L_p(\mathbb{R})$ (cf. [45, Section 19.3, Corollary 2]).

Proposition 2.2.6 (Riesz factorization theorem) *Let $p \in (1, \infty)$. Then,*

$$L_p(\mathbb{R}) = H_p^+(\mathbb{R}) \oplus H_p^-(\mathbb{R}).$$

Hence, any $f \in L_p(\mathbb{R})$ admits a unique decomposition

$$f = f_1 + f_2$$

where $f_1 \in H_p^+(\mathbb{R}), f_2 \in H_p^-(\mathbb{R})$.

Proof. If $f \in L_p(\mathbb{R})$, then $Cf|_{\Pi_+} \in \mathcal{H}_p(\Pi_+)$ and $Cf|_{\Pi_-} \in \mathcal{H}_p(\Pi_-)$, see the discussion before Remark and Definition 2.2.4. The Plemelj formulas tell us that $f = (Cf)_+ - (Cf)_-$, hence $f \in H_p^+(\mathbb{R}) \oplus H_p^-(\mathbb{R})$ by Remark and Definition 2.2.4. $\qquad\qquad\square$

The following result (see, e.g., [50, Theorem 13.2]) is not only crucial for our work, but also states that the spaces $H_p^+(\mathbb{R}), H_p^-(\mathbb{R})$ can be identified with the classical Hardy spaces $\mathcal{H}_p(\Pi_+), \mathcal{H}_p(\Pi_-)$.

Proposition 2.2.7 ([50]) *Let $p \in [1, \infty)$. Then the following statements hold:*

1. *The mapping $T : H_p^+(\mathbb{R}) \to \mathcal{H}_p(\Pi_+)$, $Tf = Cf|_{\Pi_+}$ is an isometric isomorphism with inverse $T^{-1} : \mathcal{H}_p(\Pi_+) \to H_p^+(\mathbb{R})$ defined by*

$$T^{-1}(g) = g_+ \quad (g \in \mathcal{H}_p(\Pi_+)).$$

2. *The mapping $T : H_p^-(\mathbb{R}) \to \mathcal{H}_p(\Pi_-)$, $Tf = Cf|_{\Pi_-}$ is an isometric isomorphism with inverse $T^{-1} : \mathcal{H}_p(\Pi_-) \to H_p^+(\mathbb{R})$ defined by*

$$T^{-1}(g) = -g_- \quad (g \in \mathcal{H}_p(\Pi_-)).$$

Remark 2.2.8 Let $p \in [1, \infty)$. By Proposition 2.2.7, the spaces $H_p^+(\mathbb{R})$ and $H_p^-(\mathbb{R})$ are Banach spaces. One could show this fact directly with the aid of Hölder's inequality: Given $f \in L_p(\mathbb{R})$ and a sequence $(f_n)_{n \in \mathbb{N}}$ in $H_p^+(\mathbb{R})$ with $f_n \to f$, we know that

$$\left| \int_{\mathbb{R}} \frac{f(t)}{t - z} \, dt \right| = \left| \int_{\mathbb{R}} \frac{f(t) - f_n(t)}{t - z} \, dt \right| \leq \int_{\mathbb{R}} \frac{1}{|t - z|} \cdot |f_n(t) - f(t)| \, dt \quad (z \in \Pi_-).$$

But Hölder's inequality and the fact $f_n \in H_p^+(\mathbb{R})$ for all $n \in \mathbb{N}$ imply now that

$$\int_{\mathbb{R}} \frac{1}{|t - z|} \cdot |f_n(t) - f(t)| \, dt \leq \|f_n - f\|_p \cdot \left\| \frac{1}{\cdot - z} \right\|_q \to 0 \quad (n \to \infty)$$

for any $z \in \Pi_-$, here q is the conjugate exponent of p. This implies $f \in H_p^+(\mathbb{R})$. The proof for $H_p^-(\mathbb{R})$ is in the same manner.

Remark 2.2.9 There is an analogue of Proposition 2.2.7 for harmonic functions which builds the fundament to prove these results. For $p \in [1, \infty)$ let $h_p(\Pi_+)$ be the space of all harmonic functions U on Π_+ such that

$$\|U\|_{h_p(\Pi_+)} := \left(\sup_{y > 0} \int_{\mathbb{R}} |U(x + iy)|^p \, dx \right)^{\frac{1}{p}} < +\infty$$

and $h_\infty(\Pi_+)$ be the space of all bounded harmonic functions U on Π_+ endowed with the norm

$$\|U\|_{h_\infty(\Pi_+)} := \sup_{z \in \Pi_+} |U(z)|.$$

The spaces $h_p(\Pi_-)$ are defined in an analogue way. Then, the Poisson representation theorem for Π_+ (see [50, Theorem 11.6, Theorem 11.7]) asserts the following:

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1. A harmonic function U belongs to $h_1(\Pi_+)$ if and only if there is $\mu \in \mathcal{M}(\mathbb{R})$ such that

$$U(z) = (P_\mu \mathbb{1}_\mathbb{R})(z) = \frac{1}{\pi} \int_\mathbb{R} \frac{y}{(x-t)^2 + y^2} \, d\mu(t) \quad (z = x + iy \in \Pi_+).$$

   In this case, μ is unique and satisfies $\|\mu\| = \|U\|_{h_1(\Pi_+)}$.

2. For $p \in (1, \infty]$, a harmonic function U belongs to $h_p(\Pi_+)$ if and only if there is $f \in L_p(\mathbb{R})$ such that

$$U(z) = (Pf)(z) = \frac{1}{\pi} \int_\mathbb{R} \frac{y}{(x-t)^2 + y^2} f(t) \, dt \quad (z = x + iy \in \Pi_+).$$

   In this case, f is unique and satisfies $\|f\|_p = \|U\|_{h_p(\Pi_+)}$.

If a Banach space X admits a direct sum representation, i.e. $X = U \oplus V$, then one can naturally define a complete norm on X by

$$\|x\|_\oplus := \|u\|_X + \|v\|_X \quad (x = u + v \in X).$$

For example, if we consider $L_p(\mathbb{R})$, then by Proposition 2.2.6 a norm arises by taking

$$\|f\|_\oplus := \|f_1\|_p + \|f_2\|_p \quad (f = f_1 + f_2 \in L_p(\mathbb{R})).$$

Proposition 2.2.7 asserts in a particular way that the Cauchy transformation is a bounded operator on $L_p(\mathbb{R})$ if we take this direct sum norm. However, our goal is that the Cauchy transformation is a bounded operator on $L_p(\mathbb{R})$ with respect to the usual norm $\|\cdot\|_p$. Hence, we have to show that it does not matter which norm we take and this is exactly the statement of

Lemma 2.2.10 *Let $(X, \|\cdot\|_X)$ be a Banach space and U, V be closed subspaces of X such that*

$$X = U \oplus V.$$

If

$$\|x\|_\oplus := \|u\|_X + \|v\|_X \quad (x = u + v \in X),$$

then $\|\cdot\|_\oplus$ defines a complete norm on X which is equivalent to $\|\cdot\|_X$.

Proof. We clearly have $\|x\|_X \leq \|x\|_\oplus$ $(x \in X)$ and therefore the identity mapping $I : (X, \|\cdot\|_\oplus) \to (X, \|\cdot\|_X)$ is continuous and bijective. Since $(X, \|\cdot\|_\oplus)$ is a Banach space, the Banach isomorphism theorem (see [68], p. 106) yields that I^{-1} is continuous. Hence, there is a constant $K > 0$ with

$$\|x\|_\oplus \leq K\|x\|_X \quad (x \in X)$$

and we are done when we combine this with the first inequality. $\square$

We are now in the position to formulate the representation theorem for the Cauchy transformation on $L_p(\mathbb{R}), p > 1$.

Theorem 2.2.11 *Let $p \in (1, \infty)$. Then, the Cauchy transformation $C : L_p(\mathbb{R}) \to H(\mathbb{C} \setminus \mathbb{R})$ is an isomorphism onto $\mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})$. Thus, a function $F \in H(\mathbb{C} \setminus \mathbb{R})$ belongs to $\mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})$ if and only if there is a $f \in L_p(\mathbb{R})$ such that*

$$F(z) = (Cf)(z) \quad (z \in \mathbb{C} \setminus \mathbb{R}).$$

In this case, f is unique and there are constants $K_{p,1}, K_{p,2} > 0$ only depending on p such that

$$\|F\|_{\mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})} \leq K_{p,1} \|f\|_p \leq K_{p,2} \|F\|_{\mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})}.$$

Proof. The injectivity has already been shown. For the surjectivity, fix $F \in \mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})$. By Proposition 2.2.7, there exist functions $f_1 \in H_p^+(\mathbb{R}), f_2 \in H_p^-(\mathbb{R})$ such that

$$F(z) = (Cf_1)(z) \quad (z \in \Pi_+),$$
$$F(z) = (Cf_2)(z) \quad (z \in \Pi_-).$$

But since

$$\int_{\mathbb{R}} \frac{f_1(t)}{t - z} \, dt = 0 \quad (z \in \Pi_-),$$
$$\int_{\mathbb{R}} \frac{f_2(t)}{t - z} \, dt = 0 \; (z \in \Pi_+)$$

we get

$$F(z) = C(f_1 + f_2)(z) \quad (z \in \mathbb{C} \setminus \mathbb{R}).$$

Therefore,

$$C(H_p^+(\mathbb{R}) \oplus H_p^-(\mathbb{R})) = \mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})$$

and the surjectivity is clear by the Riesz factorization theorem. It remains to show the continuity of C since then the Banach isomorphism theorem (see, e.g., [68, Theorem 5.10]) tells us that C^{-1} is continuous as well and C is an isomorphism. But for $f = f_1 + f_2 \in L_p(\mathbb{R})$, we know that

$$\|Cf\|_{\mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})} \leq \|Cf_1\|_{\mathcal{H}_p(\Pi_+)} + \|Cf_2\|_{\mathcal{H}_p(\Pi_-)} = \|f_1\|_p + \|f_2\|_p.$$

Together with Lemma 2.2.10, this implies that C is continuous and we are done. □

Remark 2.2.12 For $p \in (1, \infty)$, the inverse C^{-1} of the Cauchy transformation is given by the mapping $C^{-1} : \mathcal{H}_p(\mathbb{C} \setminus \mathbb{R}) \to L_p(\mathbb{R})$ with

$$C^{-1}(F) = F_+ - F_- \quad (F \in \mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})).$$

This is clear from the Plemelj formulas, since they tell us exactly that

$$C^{-1} \circ C = \mathrm{id}_{L_p(\mathbb{R})}.$$

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2.3 The case $p = 1$ and Paley-Wiener theorems

We now turn to the case $p = 1$. In this case, the Cauchy transformation does **not** map $L_1(\mathbb{R})$ into $\mathcal{H}_1(\mathbb{C} \setminus \mathbb{R})$. This is a consequence of

Proposition 2.3.1 *Let $p \in [1, \infty)$ and $f \in L_p(\mathbb{R})$. If $Cf \in \mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})$, then $Hf \in L_p(\mathbb{R})$.*

Proof. If $Cf \in \mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})$, then $Cf_+ \in L_p(\mathbb{R})$ and the Plemelj formulas imply that also $Hf \in L_p(\mathbb{R})$. $\qquad\qquad\square$

Hence, if the Riesz factorization theorem would hold for $p = 1$, we would have $Hf \in L_1(\mathbb{R})$ for any $f \in L_1(\mathbb{R})$. This is certainly not the case, since if we take

$$f(x) = \mathbb{1}_{[0,1]}(x) \quad (x \in \mathbb{R}),$$

we get

$$(Hf)(x) = \ln\left(\frac{|x|}{|x-1|}\right) \quad (x \in \mathbb{R} \setminus \{0, 1\})$$

which is a function which is not integrable.

The analogue of Theorem 2.2.11 reads for $p = 1$ as follows:

Proposition 2.3.2 *The Cauchy transformation $C : H_1^+(\mathbb{R}) \oplus H_1^-(\mathbb{R}) \to H(\mathbb{C} \setminus \mathbb{R})$ is an isomorphism onto $\mathcal{H}_1(\mathbb{C} \setminus \mathbb{R})$. Thus, a function $F \in H(\mathbb{C} \setminus \mathbb{R})$ belongs to $\mathcal{H}_1(\mathbb{C} \setminus \mathbb{R})$ if and only if there is a $f \in H_1^+(\mathbb{R}) \oplus H_1^-(\mathbb{R})$ such that*

$$F(z) = (Cf)(z) \quad (z \in \mathbb{C} \setminus \mathbb{R}).$$

In this case, f is unique and there are constants $K_1, K_2 > 0$ not depending on f and g such that

$$\|F\|_{\mathcal{H}_1(\mathbb{C}\setminus\mathbb{R})} \leq K_1 \|f\|_{H_1^+(\mathbb{R})\oplus H_1^-(\mathbb{R})} \leq K_2 \|F\|_{\mathcal{H}_1(\mathbb{C}\setminus\mathbb{R})}.$$

We omit the proof since it is similar to the one of Theorem 2.2.11. However, the factorization space $H_1^+ \oplus H_1^-(\mathbb{R})$ is somewhat unhandy and like in the case $p \in (1, \infty)$ we are interested in a precise description of this space. Proposition 2.3.1 suggests a necessary condition for a function $f \in L_1(\mathbb{R})$ to belong to $H_1^+(\mathbb{R}) \oplus H_1^-(\mathbb{R})$. Namely, Hf has to belong to $L_1(\mathbb{R})$ as well. The question arises if this condition is also sufficient. This will turn out to be the case.

Remark and Definition 2.3.3 The space

$$H_1(\mathbb{R}) := \{f \in L_1(\mathbb{R}) : Hf \in L_1(\mathbb{R})\}$$

is called the **real Hardy space**. It is a Banach space with respect to the norm

$$\|f\|_{H_1(\mathbb{R})} := \|f\|_1 + \|Hf\|_1 \quad (f \in H_1(\mathbb{R})),$$

see [26], p. 235.

Remark 2.3.4   1. The space $H_1(\mathbb{R})$ has multiple characterizations and applications, see [73]. A class of functions which belong to $H_1(\mathbb{R})$ can be obtained by taking decreasing, positive functions $g_0 \in \mathscr{L}_1(0, \infty)$ and considering functions of the form $g := \text{sign} \cdot g_0(|\cdot|)$ where sign is the signum function. This follows from the so called atomic decomposition of functions in $H_1(\mathbb{R})$, see [73], p. 178.

2. As remarked before we do not have $H(L_1(\mathbb{R})) \subset L_1(\mathbb{R})$ which explains in particular that there is no inversion formula for the Hilbert transformation on $L_1(\mathbb{R})$. However, the space $H_1(\mathbb{R})$ is here the natural substitute for $L_1(\mathbb{R})$ in the following sense: If $f \in H_1(\mathbb{R})$, then $HHf = -f$ (see [34], p. 344), and hence the Hilbert transformation is an isometric automorphism on $H_1(\mathbb{R})$.

The real Hardy space will turn out to be equal to the direct sum $H_1^+(\mathbb{R}) \oplus H_1^-(\mathbb{R})$. Clearly, by Proposition 2.2.7 and Proposition 2.3.1 we have $H_1^+(\mathbb{R}), H_1^-(\mathbb{R}) \subset H_1(\mathbb{R})$, hence also $H_1^+(\mathbb{R}) \oplus H_1^-(\mathbb{R}) \subset H_1(\mathbb{R})$ by the linearity of the Hilbert transformation. In order to prove the other inclusion we shall need

Lemma 2.3.5 *Let $f \in H_1(\mathbb{R})$. Then, $Cf \in \mathcal{H}_1(\mathbb{C} \setminus \mathbb{R})$.*

Proof. If $f \in H_1(\mathbb{R})$, then [34, Theorem 3.1] implies that $(Cf)(z) = (Pf)(z)$ for all $z \in \Pi_+$ and similar $(Cf)(z) = (Pf)(z)$ for all $z \in \Pi_-$. But now by Remark and Definition 2.1.5 we conclude that $Cf|_{\Pi_+} \in \mathcal{H}_1(\Pi_+), Cf|_{\Pi_-} \in \mathcal{H}_1(\Pi_-)$ and hence $Cf \in \mathcal{H}_1(\mathbb{C} \setminus \mathbb{R})$. $\square$

Proposition 2.3.6 *The following statements hold:*

1. $H_1^+(\mathbb{R}) \oplus H_1^-(\mathbb{R}) = H_1(\mathbb{R})$.

2. The Cauchy transformation establishes an isomorphism between $H_1(\mathbb{R})$ and $\mathcal{H}_1(\mathbb{C} \setminus \mathbb{R})$. In particular, a function $F \in H(\mathbb{C} \setminus \mathbb{R})$ belongs to $\mathcal{H}_1(\mathbb{C} \setminus \mathbb{R})$ if and only there is a $f \in H_1(\mathbb{R})$ such that
$$F(z) = (Cf)(z) \quad (z \in \mathbb{C} \setminus \mathbb{R}).$$
In this case, f is unique and there are constants $K_1, K_2 > 0$ independent of f such that

$$\|F\|_{\mathcal{H}_1(\mathbb{C} \setminus \mathbb{R})} \le K_1 \|f\|_{H_1(\mathbb{R})} \le K_2 \|F\|_{\mathcal{H}_1(\mathbb{C} \setminus \mathbb{R})}.$$

Proof. In view of Proposition 2.3.2 and Remark and Definition 2.3.3 it suffices to show that the first part holds. Since $H_1^+(\mathbb{R}) \oplus H_1^-(\mathbb{R}) \subset H_1(\mathbb{R})$ Lemma 2.3.5 implies that

$$C(H_1^+(\mathbb{R}) \oplus H_1^-(\mathbb{R})) \subset C(H_1(\mathbb{R})) \subset \mathcal{H}_1(\mathbb{C} \setminus \mathbb{R}).$$

By Proposition 2.3.2 we therefore must have $C(H_1^+(\mathbb{R}) \oplus H_1^-(\mathbb{R})) = C(H_1(\mathbb{R}))$ and this concludes the proof since C is injective on $\mathscr{C}$, see Proposition 2.1.7. $\square$

We briefly turn to another approach to characterize Cauchy transforms of functions. Therefore, we rely on a result which illustrates the close connection of Fourier and Cauchy transforms and which will be exploited several times in this work. We refer to [50, Theorem 12.6].

Proposition 2.3.7 *Let $p \in [1, 2]$. Then*

$$L\widehat{f} = Cf$$

for all $f \in L_p(\mathbb{R})$.

Remark 2.3.8 Basing on Proposition 2.3.7 it is a short way to prove that

$$H_p^+(\mathbb{R}) = \{f \in L_p(\mathbb{R}) : \widehat{f}\mathbb{1}_{(-\infty,0]} = 0\},$$
$$H_p^-(\mathbb{R}) = \{f \in L_p(\mathbb{R}) : \widehat{f}\mathbb{1}_{[0,\infty)} = 0\}$$

for each $p \in [1, 2]$, see [50], p. 287-288. In particular, if $f \in H_1(\mathbb{R})$, then f has a vanishing integral, i.e.

$$\int_{-\infty}^{\infty} f(t)\, dt = 0.$$

This follows from the continuity of Fourier transforms of L_1 functions. Conversely, not every function in $L_1(\mathbb{R})$ with a vanishing integral has to belong to $H_1(\mathbb{R})$. Take for example the function $g : \mathbb{R} \to \mathbb{R}$,

$$g(t) = \mathbb{1}_{[0,1]}(t) \quad (t \in \mathbb{R})$$

and a function $\varphi \in C^\infty(\mathbb{R})$ with support in $[0, 1]$ such that $\int \varphi = 1$. Then, the function $f := g - \varphi$ satisfies $f \in L_1(\mathbb{R})$, $\int f = 0$, but $Hg \notin L_1(\mathbb{R})$. This can be seen as follows: Since $H\varphi \in L_\infty(\mathbb{R})$ (see, e.g., [50, Theorem 14.1]) it suffices to show that $Hg \notin L_1(\mathbb{R})$ because

$$\int_{-\infty}^{\infty} |(Hf)(x)|\, dx \geq \int_{-\infty}^{\infty} |(Hg)(x)|\, dx - \|\varphi\|_\infty.$$

But this is clear since

$$(Hg)(x) = \ln\left(\frac{|x|}{|x-1|}\right) \quad (x \in \mathbb{R} \setminus \{0, 1\}).$$

Proposition 2.3.9 *The diagram*

is a commuting diagram of isomorphisms. Hence, for a function $F \in H(\mathbb{C} \setminus \mathbb{R})$, the following statements are equivalent:

a) There exists $f \in L_2(\mathbb{R})$ such that $F = Cf$.

b) There exists $g \in L_2(\mathbb{R})$ such that $F = Lg$.

c) $F \in \mathcal{H}_2(\mathbb{C} \setminus \mathbb{R})$.

In this case, both f and g are uniquely determined.

We can now deduce the Paley-Wiener theorems:

Proposition 2.3.10 *The following statements hold:*

1. *(Paley-Wiener for $p = 1$) If $F \in \mathcal{H}_1(\Pi_+)$, then there is a $f \in C_0(\mathbb{R})$ with $f\mathbb{1}_{(0,\infty)} = f$ such that*
$$F(z) = (Lf)(z) \quad (z \in \Pi_+).$$

2. *(Paley-Wiener for $p \in (1,2)$) Let $p \in (1,2)$. If $F \in \mathcal{H}_p(\Pi_+)$, then there is a $f \in L_q(0,\infty)$ such that*
$$F(z) = (Lf)(z) \quad (z \in \Pi_+).$$
   Here, q is the conjugate exponent of p.

3. *(Paley-Wiener for $p = 2$) A function $F \in H(\Pi_+)$ belongs to $\mathcal{H}_2(\Pi_+)$ if and only if there is some $f \in L_2(0,\infty)$ such that*
$$F(z) = (Lf)(z) \quad (z \in \Pi_+).$$

In this case, f is unique and

$$\|F\|_{\mathcal{H}_2(\Pi_+)} = \sqrt{2\pi}\|f\|_2.$$

Proof. We only have to prove 3. The Cauchy transform C establishes an isometric isomorphism between $(H_2^+(\mathbb{R}), \| \cdot \|_2)$ and $(\mathcal{H}_2(\Pi_+), \| \cdot \|_{\mathcal{H}_2(\Pi_+)})$. Moreover, the spaces $H_2^+(\mathbb{R})$ and $L_2(0,\infty)$ are isomorphic via the Fourier transform $\mathcal{F}$, see [50, Theorem 13.8]. Since

$$L = C \circ \mathcal{F}^{-1}$$

the mapping L has to be an isometric isomorphism as well. $\qquad\square$

2.4 Intersection spaces and a short treatise of the case $p = \infty$

Unfortunately, the previous results make in general no sense if $p = \infty$. This is because of the fact that the Cauchy transform of an arbitrary $f \in L_\infty(\mathbb{R})$ may not even be defined (take for example a constant function). Moreover, even if Cf is defined for some $f \in L_\infty(\mathbb{R})$, we will in general not get a bounded function on $\mathbb{C} \setminus \mathbb{R}$.

Example 2.4.1 Let $f = \mathbb{1}_{[0,1]}$. Then, clearly $f \in L_\infty \cap L_1(\mathbb{R})$ and hence Cf is defined and is even holomorphic on $\mathbb{C} \setminus [0, 1]$. But, since

$$(Cf)(x) = \frac{1}{2\pi i}(\ln(|1 - x|) - \ln(|x|)) \quad (x \in \mathbb{R} \setminus [0, 1]),$$

Cf is clearly not bounded on $\mathbb{C} \setminus \mathbb{R}$ or $\mathbb{C} \setminus [0, 1]$.

Hence, we can not expect to obtain a representation theorem for $\mathcal{H}_\infty(\mathbb{C}\setminus\mathbb{R})$ using Cauchy transforms. One idea is however to ask for a representation theorem for $\mathcal{H}_\infty(\mathbb{C}\setminus\mathbb{R})\cap\mathcal{H}_p(\mathbb{C}\setminus\mathbb{R})$. We shall see that this is indeed possible. The way to prove the corresponding result will incorporate again the theory of Poisson integrals. Recall that for a function $f \in H_p^+(\mathbb{R})$ we have

$$(Cf)(z) = (Pf)(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{y}{(x - t)^2 + y^2} f(t)\, dt \quad (z = x + iy \in \Pi_+).$$

In particular, Pf is analytic for each $f \in H_p^+(\mathbb{R}) \cup H_p^-(\mathbb{R})$. That the converse is also true tells us the following lemma (see [16, Theorem 11.2]).

Lemma 2.4.2 ([16]) *Let $p \in [1, \infty]$ and $f \in L_p(\mathbb{R})$.*

1. If the function $F : \Pi_+ \to \mathbb{C}$, defined by

$$F(x + iy) = (Pf)(z) \quad (z \in \Pi_+)$$

is analytic, then $F \in \mathcal{H}_p(\Pi_+)$ and $F_+ = f$.

2. If the function $F : \Pi_- \to \mathbb{C}$, defined by

$$F(z) = (Pf)(z) \quad (z \in \Pi_-)$$

is analytic, then $F \in \mathcal{H}_p(\Pi_-)$ and $F_- = -f$.

We now consider for $p_1 \in [1, \infty], p_2 \in [1, \infty)$ the intersection spaces

$$H_{p_1,p_2}^+(\mathbb{R}) := L_{p_1}(\mathbb{R}) \cap H_{p_2}^+(\mathbb{R})$$
$$H_{p_1,p_2}^-(\mathbb{R}) := L_{p_1}(\mathbb{R}) \cap H_{p_2}^-(\mathbb{R})$$

Endowed with the sum norm $\|\cdot\|_{p_1} + \|\cdot\|_{p_2}$, these spaces are Banach spaces. We shall ask for the image of C under the direct sum $H_{p_1,p_2}^+(\mathbb{R}) \oplus H_{p_1,p_2}^-(\mathbb{R})$.

Theorem 2.4.3 *Let $p_1 \in [1, \infty]$ and $p_2 \in [1, \infty)$. Then, the Cauchy transformation is an isomorphism between $H_{p_1,p_2}^+(\mathbb{R}) \oplus H_{p_1,p_2}^-(\mathbb{R})$ and $\mathcal{H}_{p_1}(\mathbb{C} \setminus \mathbb{R}) \cap \mathcal{H}_{p_2}(\mathbb{C} \setminus \mathbb{R})$. Thus, a function $F \in H(\mathbb{C} \setminus \mathbb{R})$ belongs to the intersection space $\mathcal{H}_{p_1}(\mathbb{C} \setminus \mathbb{R}) \cap \mathcal{H}_{p_2}(\mathbb{C} \setminus \mathbb{R})$ if and only if there is a $f = f_1 + f_2 \in H_{p_1,p_2}^+(\mathbb{R}) \oplus H_{p_1,p_2}^-(\mathbb{R})$ such that*

$$F(z) = (Cf)(z) \quad (z \in \mathbb{C} \setminus \mathbb{R}).$$

In this case, f is unique and there is a constant $K_{p_1,p_2} > 0$ only depending on p_1 and p_2 such that

$$\|F\|_{\mathcal{H}_{p_1}(\mathbb{C}\setminus\mathbb{R})} + \|F\|_{\mathcal{H}_{p_2}(\mathbb{C}\setminus\mathbb{R})} \leq 2(\|f_1\|_{p_1} + \|f_1\|_{p_2} + \|f_2\|_{p_1} + \|f_2\|_{p_2})$$
$$\leq 2K_{p_1,p_2}(\|F\|_{\mathcal{H}_{p_1}(\mathbb{C}\setminus\mathbb{R})} + \|F\|_{\mathcal{H}_{p_2}(\mathbb{C}\setminus\mathbb{R})}).$$

Proof. If $F = Cf$ where $f = f_1 + f_2 \in H^+_{p_1,p_2}(\mathbb{R}) \oplus H^-_{p_1,p_2}(\mathbb{R})$, then clearly $F \in \mathcal{H}_{p_2}(\mathbb{C} \setminus \mathbb{R})$. Moreover,

$$F(z) = (Pf_1)(z) \quad (z \in \Pi_+)$$

and therefore $F|_{\Pi_+} \in \mathcal{H}_{p_1}(\Pi_+)$ by Lemma 2.4.2. Similarly, one sees that $F|_{\Pi_-} \in \mathcal{H}_{p_1}(\Pi_-)$ and therefore $F \in \mathcal{H}_{p_1}(\mathbb{C} \setminus \mathbb{R}) \cap \mathcal{H}_{p_2}(\mathbb{C} \setminus \mathbb{R})$. Together with Remark 2.2.9 we get

$$\|F\|_{\mathcal{H}_{p_1}(\mathbb{C}\setminus\mathbb{R})} + \|F\|_{\mathcal{H}_{p_2}(\mathbb{C}\setminus\mathbb{R})} \le \|F\|_{\mathcal{H}_{p_1}(\Pi_+)} + \|F\|_{\mathcal{H}_{p_1}(\Pi_-)} + \|F\|_{\mathcal{H}_{p_2}(\Pi_+)} + \|F\|_{\mathcal{H}_{p_2}(\Pi_-)}$$
$$= 2(\|f_1\|_{p_1} + \|f_1\|_{p_2} + \|f_2\|_{p_1} + \|f_2\|_{p_2}).$$

Hence, we only have to show surjectivity since then the Banach isomorphism theorem yields that C is an isomorphism, as desired. If $F \in \mathcal{H}_{p_1}(\mathbb{C} \setminus \mathbb{R}) \cap \mathcal{H}_{p_2}(\mathbb{C} \setminus \mathbb{R})$, then there is a unique $f_1 \in L_{p_1}(\mathbb{R})$ such that

$$F(z) = (Pf_1)(z) \quad (z \in \Pi_+).$$

This follows from Remark 2.2.9 if $p_1 = +\infty$ and from Proposition 2.2.7 if $p_1 \in [1, \infty)$. On the other hand, the same proposition implies that there is a unique $g_1 \in H^+_{p_2}(\mathbb{R})$ such that

$$F(z) = (Cg_1)(z) = (Pg_1)(z) \quad (z \in \Pi_+).$$

But now, by Lemma 2.4.2, we know that

$$F_+ = f_1 = g_1$$

and hence, $f_1 \in H^+_{p_1,p_2}(\mathbb{R})$. Similarly, one can show that there is a unique $f_2 \in H^-_{p_1,p_2}(\mathbb{R})$ such that

$$F(z) = (Cf_2)(z) \quad (z \in \Pi_-),$$

so we conclude by setting $f := f_1 + f_2$. □

Corollary 2.4.4 *Let $p \in [1, \infty)$ and $f \in H^+_{\infty,p}(\mathbb{R}) \oplus H^-_{\infty,p}(\mathbb{R})$. Then, $Cf \in \mathcal{H}_\infty(\mathbb{C} \setminus \mathbb{R})$.*

We deduce two important corollaries which consider again the Riesz factorization theorem. The first corollary provides a decomposition for intersections of L_p spaces.

Corollary 2.4.5 *The following statements hold:*

1. *If $p_1, p_2 \in (1, \infty)$, then*

$$L_{p_1}(\mathbb{R}) \cap L_{p_2}(\mathbb{R}) = H^+_{p_1,p_2}(\mathbb{R}) \oplus H^-_{p_1,p_2}(\mathbb{R}).$$

2. *If $p \in [1, \infty)$, then*

$$H_1(\mathbb{R}) \cap L_p(\mathbb{R}) = H^+_{p,1}(\mathbb{R}) \oplus H^-_{p,1}(\mathbb{R}).$$

Proof. We only prove the first part since the proof of the second part is similar. Notice that the injectivity of the Cauchy transformation on $\mathscr{C}$ implies together with Theorem 2.2.11 that

$$C(L_{p_1}(\mathbb{R}) \cap L_{p_2}(\mathbb{R})) = \mathcal{H}_{p_1}(\mathbb{C} \setminus \mathbb{R}) \cap \mathcal{H}_{p_2}(\mathbb{C} \setminus \mathbb{R}).$$

But by Theorem 2.4.3

$$C(H^+_{p_1,p_2}(\mathbb{R}) \oplus H^-_{p_1,p_2}(\mathbb{R})) = \mathcal{H}_{p_1}(\mathbb{C} \setminus \mathbb{R}) \cap \mathcal{H}_{p_2}(\mathbb{C} \setminus \mathbb{R}).$$

Hence, again by the injectivity of the Cauchy transformation we conclude that 1. holds. □

Remark 2.4.6 1. The first part of Corollary 2.4.5 is false if we allow that $p_1 = 1$ or $p_1 = \infty$. For example, consider the function $f : \mathbb{R} \to \mathbb{R}, f(t) = \dfrac{1}{\pi(1 + t^2)}$. Then $f \in L_1(\mathbb{R}) \cap L_p(\mathbb{R})$ for every $p \in (1, \infty)$ but since (see Example B.5)

$$(Cf)(z) = \begin{cases} -\dfrac{1}{2\pi i (z + i)}, & z \in \Pi_+ \\ -\dfrac{1}{2\pi i (z - i)}, & z \in \Pi_-. \end{cases}$$

we see that $Cf \notin \mathcal{H}_1(\mathbb{C} \setminus \mathbb{R})$. Therefore, $f \notin H_{1,p}^+(\mathbb{R}) \oplus H_{1,p}^-(\mathbb{R})$.
On the other hand, if we take $f = \mathbb{1}_{[0,1]}$, then $f \in L_p(\mathbb{R})$ for all $p \in [1, \infty]$ but $f \notin H_{\infty,p}^+(\mathbb{R}) \oplus H_{\infty,p}^-(\mathbb{R})$ since $Cf \notin \mathcal{H}_\infty(\mathbb{C} \setminus \mathbb{R})$, see Example 2.4.1.

2. Similar, in the second part of Corollary 2.4.5 the statement is in general not true if $p = \infty$. This can be seen by taking for example $f = \mathbb{1}_{[0,1]} - \mathbb{1}_{[1,2]}$. Then,

$$(Hf)(x) = \ln\left(\frac{|x|}{|x - 1|}\right) - \ln\left(\frac{|x - 1|}{|x - 2|}\right) \quad (x \in \mathbb{R} \setminus \{0, 1, 2\})$$

and therefore $Hf \in L_1(\mathbb{R})$ which means $f \in H_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$. But $f \notin H_{\infty,1}^+(\mathbb{R}) \oplus H_{\infty,1}^-(\mathbb{R})$ since $Cf \notin \mathcal{H}_\infty(\mathbb{C} \setminus \mathbb{R})$.

Corollary 2.4.7 *Let $p \in (1, \infty), f \in L_p(\mathbb{R})$ and $f = f_1 + f_2$ the unique decomposition of f into $f_1 \in H_p^+(\mathbb{R})$ and $f_2 \in H_p^-(\mathbb{R})$. Then, the following statements are equivalent:*

a) $f_1, f_2 \in L_\infty(\mathbb{R})$.

b) $Cf \in \mathcal{H}_\infty(\mathbb{C} \setminus \mathbb{R})$.

Proof. If $f_1, f_2 \in L_\infty(\mathbb{R})$, then clearly $Cf \in \mathcal{H}_\infty(\mathbb{C} \setminus \mathbb{R})$ by Theorem 2.4.3. Conversely, if $Cf \in \mathcal{H}_\infty(\mathbb{C} \setminus \mathbb{R})$, then there is a unique $g = g_1 + g_2 \in H_{p,\infty}^+(\mathbb{R}) \oplus H_{p,\infty}^-(\mathbb{R})$ such that $Cg = Cf$ and due to the uniqueness of the decomposition we conclude that $f_1 = g_1$ and $f_2 = g_2$. $\quad\square$

We close this section with another approach to analyze Cauchy transforms of functions in $L_\infty(\mathbb{R})$. Instead of regarding intersection spaces one can also consider different subspaces of $L_\infty(\mathbb{R})$. This will lead to a certain symmetry relation of the Cauchy transform. Here is a preparing lemma which generalizes a result in [50, Lemma 13.1].

Lemma 2.4.8 *Let $U \subset \mathbb{C}$ be open with $\overline{\Pi}_+ \subset U$ and $\omega \in H(U)$ with $\omega(z) \neq 0$ $(z \in U)$. Moreover, suppose that for all $\beta > 0$ we have*

$$\lim_{R \to \infty} \int_{\theta_R}^{\pi - \theta_R} \frac{1}{|\omega(Re^{it} - i\beta)|} \, dt = 0$$

where $\theta_R := \arcsin(\beta/R)$. Then, for each $F \in \mathcal{H}_\infty(\Pi_+)$ and $\beta > 0$

$$\int_{-\infty}^\infty \frac{F(t + i\beta)}{(t - \overline{z})\omega(t)} \, dt = 0 \quad (z \in \Pi_+).$$

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Proof. Let us fix $\beta > 0$ and $z \in \Pi_+$. We choose $R > \beta + |z|$ and consider the path $\Gamma_{R,\beta}$ which consists of the two parts

$$\gamma_1 : [-R\cos(\theta_R), R\cos(\theta_R)] \to \mathbb{C}, \ \gamma_1(t) = t + i\beta$$
$$\gamma_2 : [\theta_R, \pi - \theta_R] \to \mathbb{C}, \ \gamma_2(t) = Re^{it}.$$

and is obviously closed. Since $1/\omega \in H(U)$, the Cauchy theorem implies that

$$\int_{\Gamma_{R,\beta}} \frac{F(\zeta)}{(\zeta - i\beta - \overline{z}) \cdot \omega(\zeta - i\beta)} \, d\zeta = 0,$$

hence

$$\int_{-R\cos(\theta_R)}^{R\cos(\theta_R)} \frac{F(t + i\beta)}{(t - \overline{z}) \cdot \omega(t)} \, dt = -\int_{\theta_R}^{\pi - \theta_R} \frac{F(Re^{it})}{(Re^{it} - i\beta - \overline{z}) \cdot \omega(Re^{it} - i\beta)} iRe^{it} \, dt.$$

But now,

$$\left| \int_{\theta_R}^{\pi - \theta_R} \frac{F(Re^{it})}{(Re^{it} - i\beta - \overline{z})\omega(Re^{it} - i\beta)} iRe^{it} \, dt \right|$$
$$\leq \frac{\|F\|_\infty R}{R - (\beta + |z|)} \int_{\theta_R}^{\pi - \theta_R} \frac{1}{|\omega(Re^{it} - i\beta)|} \, dt \to 0 \quad (R \to \infty)$$

and we conclude since $R\cos(\theta_R) \to \infty$ and $-R\cos(\theta_R) \to -\infty$ $(R \to \infty)$. $\qquad\square$

Remark 2.4.9 An important set of analytic functions which satisfy the condition in Lemma 2.4.8 is the set of all automorphisms on $\mathbb{C}$ whose root lies in the lower half-plane denoted by $\mathrm{Aut}_-(\mathbb{C})$. In particular, we consider functions $\omega : \mathbb{C} \to \mathbb{C}$, $\omega(z) = az + c$, where $a \in \mathbb{C}^*, c \in \mathbb{C}$ are such that $-\frac{c}{a} \in \Pi_-$. In this case, for fixed $\beta > 0$ and $R > 0$ sufficiently large

$$\int_{\theta_R}^{\pi - \theta_R} \frac{1}{|\omega(Re^{it} - i\beta)|} \, dt \leq \frac{2\pi}{|a|(R - \beta) - |c|} \to 0 \quad (R \to \infty).$$

If $\omega \in \mathrm{Aut}_-(\mathbb{C})$, then we write

$$\tilde{H}_{\infty,\omega}^+(\mathbb{R}) := \left\{ f \in L_\infty(\mathbb{R}) : \int_{\mathbb{R}} \frac{f(t)}{(t - z)\omega(t)} \, dt = 0 \ (z \in \Pi_-) \right\}.$$

Due to Hölder's inequality, this is a closed subspace of $L_\infty(\mathbb{R})$: If $(f_n)_{n \in \mathbb{N}}$ is a sequence in $\tilde{H}_{\infty,\omega}^+(\mathbb{R})$ such that $\|f_n - f\|_\infty \to 0$ for some $f \in L_\infty(\mathbb{R})$, then

$$\left| \int_{\mathbb{R}} \frac{f(t)}{(t - z)\omega(t)} \, dt \right| \leq \|f_n - f\|_\infty \cdot \left(\int_{\mathbb{R}} \frac{1}{|(t - z)\omega(t)|} \, dt \right) \to 0 \ (n \to \infty)$$

for all $z \in \Pi_-$.

The following proposition is a generalization of a result in [50], p. 284.

Proposition 2.4.10 *Let* $\omega \in \mathrm{Aut}_-(\mathbb{C})$. *Then, the mapping* $S : \tilde{H}^+_{\infty,\omega}(\mathbb{R}) \to \mathcal{H}_\infty(\Pi_+)$, *defined by*

$$(Sf)(z) := \omega(z) \cdot \left(C\frac{f}{\omega}\right)(z) \quad (z \in \Pi_+),$$

is an isomorphism.

Proof. The main tool is the identity

$$\frac{y}{(x-t)^2 + y^2} = \frac{1}{2i}\left(\frac{\omega(z)}{(t-z)\omega(t)} - \frac{\omega(\overline{z})}{(t-\overline{z})\omega(t)}\right) \quad (z = x + iy \in \mathbb{C}, t \in \mathbb{R}).$$

If $f \in \tilde{H}^+_{\infty,\omega}(\mathbb{R})$, then $Sf = Pf \in \mathcal{H}_\infty(\Pi_+)$. So, let us fix some $F \in \mathcal{H}_\infty(\Pi_+)$. Then, by Remark 2.2.9 there is some $f \in L_\infty(\mathbb{R})$ such that

$$F(x + iy) = (Pf)(z) \quad (z \in \Pi_+)$$

and

$$\lim_{\beta \to 0} \int_{\mathbb{R}} \varphi(t)F(t + i\beta)\,dt = \int_{\mathbb{R}} \varphi(t)f(t)\,dt$$

for all $\varphi \in L_1(\mathbb{R})$ by [50, Corollary 10.14]. In particular, if we put

$$\varphi(t) = \frac{1}{(t-\overline{z}) \cdot \omega(t)},$$

we obtain $f \in \tilde{H}^+_{\infty,\omega}(\mathbb{R})$ by Lemma 2.4.8. Therefore, $F = Sf$ and the surjectivity is clear. The injectivity of S follows from Proposition 2.1.7 and since $\|Sf\|_{\mathcal{H}_\infty(\Pi_+)} \le \|f\|_\infty$, the Banach isomorphism theorem gives us that S is indeed an isomorphism. $\qquad\square$

Remark 2.4.11 Let $\omega \in \mathrm{Aut}_-(\mathbb{C})$.

1. If we consider ω also as an element of $L_\infty(\mathbb{R})$, then Proposition 2.4.10 implies in particular the symmetric relation

$$C\left(\frac{1}{\omega} \cdot \tilde{H}^+_{\infty,\omega}(\mathbb{R})\right) = \frac{1}{\omega} \cdot \mathcal{H}_\infty(\Pi_+).$$

2. Let $f \in \tilde{H}^+_{\infty,\omega}(\Pi_+)$. Then, by Remark 2.2.9 and Proposition 2.4.10 one sees that $f \in \tilde{H}^+_{\infty,h}(\Pi_+)$ for **all** $h \in \mathrm{Aut}_-(\mathbb{C})$ (cf. [50], p. 284).

Chapter 3

The Cauchy Transformation on Subspaces of $L_p(\mathbb{R})$ $(1 \leq p < \infty)$

> To create was a fundament, to appreciate, a supplement.
>
> Jeanette Winterson, see [83], p. 35

In this chapter, we will build the fundament for our investigation of the Cauchy transformation on several subspaces of $L_p(\mathbb{R})$. Here, p belongs to $[1, \infty)$. Our main focus will be on spaces of the form $L_p(X)$ where $X \subset \mathbb{R}$ is closed. Therefore, we first introduce the most important facts about Cauchy transforms of measures and discuss their boundary behavior. Afterwards, we consider functions which belong to $L_p(X)$ for a closed set X. With the results of the previous chapter and a result about analytic continuation of Cauchy transforms we will be able to determine the image of C under $L_p(X)$ for $p \in (1, \infty)$ and to give also an answer in the case $p = 1$.

3.1 Complex measures and their Cauchy transforms

The Cauchy transform of a measure $\mu \in \mathcal{M}(\mathbb{R})$ has in some kind already been defined in 2.1.3. However, it is worth to define Cauchy transforms generally for Borel measures on $\mathbb{C}$ since this is the original setting for applications in approximation theory.

Remark and Definition 3.1.1 Let $\mu \in \mathcal{M}_\infty(\mathbb{C})$ such that

$$\int \frac{1}{|\zeta| + 1} \, d|\mu|(\zeta) < +\infty.$$

Then, the function $C\mu : \mathbb{C} \setminus \mathrm{supp}(\mu) \to \mathbb{C}$ defined by

$$(C\mu)(z) = \frac{1}{2\pi i} \int \frac{1}{\zeta - z} \, d\mu(\zeta) \quad (z \in \mathbb{C} \setminus \mathrm{supp}(\mu))$$

is called the **Cauchy transform** of μ. If $\mu \in \mathcal{M}_c(\mathbb{C})$, then we also regard $C\mu$ on $\mathbb{C}_\infty \backslash \operatorname{supp}(\mu)$. Clearly, we have $C\mu \in H(\mathbb{C} \backslash \operatorname{supp}(\mu))$ with

$$(C\mu)^{(n)}(z) = \frac{n!}{2\pi i} \int \frac{1}{(\zeta - z)^{n+1}} \, d\mu(\zeta) \quad (z \in \mathbb{C} \backslash \operatorname{supp}(\mu), n \in \mathbb{N}_0).$$

Remark 3.1.2   1. If $\mu \in \mathcal{M}_\infty(\mathbb{R})$ (compare with Remark A.3) is such that $\mathbb{1}_\mathbb{R} \in \mathscr{C}(\mu)$, then $C\mu = C_\mu \mathbb{1}_\mathbb{R}$.

2. If $f \in L_1(\mathbb{R})$ and $\mu = f\lambda_1$, then

$$(C\mu)(z) = \frac{1}{2\pi i} \int_\mathbb{R} \frac{f(t)}{t - z} \, dt = (Cf)(z) \quad (z \in \mathbb{C} \backslash \mathbb{R}).$$

A special case appears if we consider measures with compact support.

Remark and Definition 3.1.3 For $\mu \in \mathcal{M}_c(\mathbb{C})$, we define the function $|C|\mu : \mathbb{C} \to [0, \infty]$, with

$$(|C|\mu)(z) := \int \frac{1}{|\zeta - z|} \, d|\mu|(\zeta) \quad (z \in \mathbb{C}).$$

Since the function

$$\mathbb{C}^* \ni z \mapsto \frac{1}{z}$$

is in $L_{p,\mathrm{loc}}(\mathbb{C})$ for $p \in [1, 2)$, one can use Fubini's theorem and Jensen's inequality to show that $|C|\mu \in L_{p,\mathrm{loc}}(\mathbb{C})$ for $p \in [1, 2)$, see [27], p. 37. In particular, $(|C|\mu)(z)$ is finite for almost every $z \in \mathbb{C}$. Hence, the Cauchy transform

$$(C\mu)(z) = \frac{1}{2\pi i} \int \frac{1}{\zeta - z} \, d\mu(\zeta)$$

exists also for almost every $z \in \mathbb{C}$. We shall need this fact in the formulation of Carleson's theorem, see 3.1.15.

Let us collect some important properties of the Cauchy transform of compactly supported measures, see, e.g., [10, Section 18.5, Proposition 5.2], p. 193.

Proposition 3.1.4 ([10]) *For $\mu \in \mathcal{M}_c(\mathbb{C})$ the following statements hold:*

1. *$C\mu \in L_{p,\mathrm{loc}}(\mathbb{C})$ for $p \in [1, 2)$.*

2. *If $R > 0$ such that $\operatorname{supp}(\mu) \subset \overline{U_R(0)}$, then*

$$(C\mu)(z) = -\frac{1}{2\pi i} \sum_{n=0}^\infty m_n(\mu) \frac{1}{z^{n+1}} \quad (z \in \mathbb{C}_\infty \backslash \overline{U_R(0)}).$$

3. *$C\mu \in H(\mathbb{C}_\infty \backslash \operatorname{supp}(\mu))$ and $(C\mu)'(\infty) = -\mu(\mathbb{C})$.*

It is a natural question to ask when a given function F is the Cauchy transform of some compactly supported measure. Several conditions and characterizations can be found in [27]. In the next chapter, we shall put our focus only on the measures $\mu = f\nu$ (f having a compact support and $\nu \in \mathcal{M}(\mathbb{R})$) and give characterizations of the Cauchy transforms of these measures.

Similar to the case of functions, one can also show that Cauchy transforms of measures have a specific boundary behavior. In particular, an analogue of the Plemelj formulas holds in this case. We therefore extend the Hilbert transform to complex measures:

Remark and Definition 3.1.5 Let $\mu \in \mathcal{M}(\mathbb{R})$. Then, due to [46], the limit

$$(H\mu)(x) := \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-t| \geq \varepsilon} \frac{1}{x-t} \, d\mu(t)$$

exists for almost every $x \in \mathbb{R}$. The function $H\mu : \mathbb{R} \to \mathbb{C}$ is called the **Hilbert transform** of μ.

Remark and Definition 3.1.6 Let $\mu \in \mathcal{M}(\mathbb{R})$. Then, we write $P\mu$ instead of $P_\mu \mathbb{1}_\mathbb{R}$ and call this function the **Poisson transform** of μ. Due to [8, Theorem 10.4.2] we have

$$\lim_{y \to 0^+} (P\mu)(\cdot + iy) = \frac{d\mu}{d\lambda_1}$$

$$\lim_{y \to 0^-} (P\mu)(\cdot + iy) = -\frac{d\mu}{d\lambda_1}.$$

Similar, the function $Q\mu := Q_\mu \mathbb{1}_\mathbb{R}$ is called the **conjugate Poisson transform** of μ. It holds that

$$\frac{1}{2}(P\mu)(z) + \frac{i}{2}(Q\mu)(z) = (C\mu)(z) \quad (z \in \mathbb{C} \setminus \operatorname{supp}(\mu)).$$

Moreover,

$$\lim_{y \to 0^+} (Q\mu)(\cdot + iy) = \lim_{y \to 0^-} (Q\mu)(\cdot + iy) = (H\mu)$$

by [22, Theorem 1.2.7].

The following analogue of the Plemelj formulas holds, cf. [8, Proposition 10.4.2].

Proposition 3.1.7 *Let $\mu \in \mathcal{M}(\mathbb{R})$. Then,*

$$\lim_{y \to 0^+} (C\mu)(\cdot + iy) = \frac{1}{2} \left(\frac{d\mu}{d\lambda_1} + i(H\mu) \right)$$

$$\lim_{y \to 0^-} (C\mu)(\cdot + iy) = \frac{1}{2} \left(-\frac{d\mu}{d\lambda_1} + i(H\mu) \right).$$

In particular,

$$\frac{d\mu}{d\lambda_1} = \lim_{y \to 0^+} (C\mu)(\cdot + iy) - \lim_{y \to 0^-} (C\mu)(\cdot + iy).$$

Remark 3.1.8 There is a range of other results on the boundary behavior of Cauchy transforms:

1. An application of the dominated convergence shows that for any $\mu \in \mathcal{M}(\mathbb{R})$ one has

$$\lim_{y \to 0} y(C\mu)(x + iy) = i\mu(\{x\})$$

   for all $x \in \mathbb{R}$, see [22, Theorem 2.1.4]. In particular,

$$\lim_{y \to 0} y(C\mu)(x + iy) = 0$$

   for all $x \in \mathbb{R}$ if μ is absolutely continuous with respect to λ_1.

2. Another result addresses certain singular measures (see [22, Theorem 2.1.11]): Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{C}$ and $(b_n)_{n \in \mathbb{N}}$ a sequence in $(0, 1)$ such that

   i) $\sum\limits_{n=1}^{\infty} |a_n| < +\infty$,

   ii) $\sum\limits_{n=1}^{\infty} b_n = 1$,

   iii) $\sum\limits_{n=1}^{\infty} \dfrac{|a_n|}{b_n} < +\infty$.

   If $(c_n)_{n \in \mathbb{N}}$ is a sequence in $\mathbb{R}$ and $\mu = \sum\limits_{n=1}^{\infty} a_n \delta_{c_n}$, then

$$\lim_{y \to 0^+} (C\mu)(x + iy) = \sum_{n=1}^{\infty} \frac{a_n}{c_n - x}$$

   for almost every $x \in \mathbb{R}$.

It is important for us that an analogue of Proposition 2.3.7 holds. We refer to [50, Theorem 10.1] for a proof.

Proposition 3.1.9 ([50]) *Let $\mu \in \mathcal{M}(\mathbb{R})$. Then*

$$L\widehat{\mu} = C\mu.$$

Remark 3.1.10 The deep connection between Fourier-Laplace and Cauchy transforms gives a short proof for the injectivity of the Cauchy transformation on $\mathcal{M}(\mathbb{R})$: Let $\mu \in \mathcal{M}(\mathbb{R})$ with $C\mu = 0$. By Proposition 3.1.9 we know that

$$\int_0^{\infty} \widehat{\mu}(t) e^{-t} e^{ixt} \, dt = 0 = \int_{-\infty}^0 \widehat{\mu}(t) e^t e^{ixt} \, dt \quad (x \in \mathbb{R}).$$

Hence, $\mathcal{F}(\widehat{\mu}\mathbb{1}_{[0,\infty)} e^{-\cdot}) = 0 = \mathcal{F}(\widehat{\mu}\mathbb{1}_{(-\infty,0]} \exp)$ which implies $\widehat{\mu} = 0$. But this means $\mu = 0$.

Usually this fact is proven for positive, finite measures by means of the so called **Stieltjes inversion formula**, see [71, Chapter 3, Theorem 6] (there formulated for probability

measures, but a similar proof yields the formula for general finite positive measures): For all $a, b \in \mathbb{R}$ with $a < b$ we have

$$\lim_{y \to 0^+} \frac{1}{\pi} \int_a^b \operatorname{Im}((C\mu)(x + iy)) \, dx = \mu((a, b)) + \frac{1}{2}\mu(\{a, b\}).$$

A consequence of the uniqueness of Cauchy transforms is that there is no nontrivial, singular and finite measure whose Cauchy transform belongs to $\mathcal{H}_p(\mathbb{C} \backslash \mathbb{R})$ for some $p \in [1, \infty)$.

Corollary 3.1.11 *Let* $\mu \in \mathcal{M}(\mathbb{R})$ *such that* $\mu \perp \lambda_1$ *and* $p \in [1, \infty)$. *If* $C\mu \in \mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})$, *then* $\mu = 0$.

Proof. If $C\mu \in \mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})$, then $C\mu = Cg$ where

$$g = \lim_{y \to 0^+} (C\mu)(\cdot + iy) - (C\mu)(\cdot - iy) = \frac{d\mu}{d\lambda_1} = 0$$

by Proposition 3.1.7. Hence $C\mu = 0$ and this implies $\mu = 0$. $\qquad \square$

The latter result gives us together with the Lebesgue decomposition of complex measures, see Theorem A.11, a characterization when the Cauchy transform of a measure belongs to $\mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})$. It turns out that all these measures have already been considered before.

Proposition 3.1.12 *Let* $p \in (1, \infty)$ *and* $\mu \in \mathcal{M}(\mathbb{R})$. *Then, the following are equivalent:*

    a) $C\mu \in \mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})$.

    b) $\mu \ll \lambda_1$ *and* $\dfrac{d\mu}{d\lambda_1} \in L_p(\mathbb{R})$.

If $p = 2$, *then we also have the equivalent statement*

    c) $\widehat{\mu} \in L_2(\mathbb{R})$.

Proof. Suppose that $C\mu \in \mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})$ and write $\mu = \mu_a + \mu_s$ where $\mu_a \ll \lambda_1, \mu_s \perp \lambda_1$. By our assumption, we know that

$$\frac{d\mu}{d\lambda_1} = \lim_{y \to 0^+} (C\mu)(\cdot + iy) - (C\mu)(\cdot - iy) \in H_p^+(\mathbb{R}) \oplus H_p^-(\mathbb{R}) = L_p(\mathbb{R})$$

and therefore $C\mu_s = C\mu - C\mu_a \in \mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})$. Corollary 3.1.11 gives us that $\mu_s = 0$ and *b)* holds. Since $b) \Rightarrow a)$ is clear by Theorem 2.2.11 and Proposition 2.3.2, we have proven the equivalence of $a)$ and $b)$. The equivalence of $a)$ and $c)$ is clear by the Paley Wiener theorem 2.3.10 and Proposition 3.1.9. $\qquad \square$

Remark 3.1.13 Under the assumptions of Proposition 3.1.12 one similarly proves the following equivalence:

a) $C\mu \in \mathcal{H}_1(\mathbb{C} \setminus \mathbb{R})$.

b) $\mu \ll \lambda_1$ and $\dfrac{d\mu}{d\lambda_1} \in H_1(\mathbb{R})$.

However, one advantage of singular measures is that there is a simple inversion formula if their support has no accumulation point.

Proposition 3.1.14 *Let $\mu \in \mathcal{M}(\mathbb{R})$ such that $\mathrm{supp}(\mu)$ has no accumulation point. Then, for every $f \in \mathscr{C}(\mu)$ we have*

$$f(\omega)\mu(\{\omega\}) = -\int_{\gamma_\omega} (C_\mu f)(\zeta)\, d\zeta \quad (\omega \in \mathbb{R})$$

where γ_ω is a cycle in $\mathbb{C} \setminus \mathrm{supp}(\mu)$ with $\mathrm{ind}_{\gamma_\omega}(\omega) = 1, \mathrm{ind}_{\gamma_\omega}(z) = 0$ $(z \in \mathrm{supp}(\mu) \setminus \{\omega\})$.

Proof. We write $\mu = \sum\limits_{z \in A} \alpha_z \delta_z$ with $A \subset \mathbb{R}$ having no accumulation point. Let us fix $\omega \in \mathbb{R}$. Since $\mathrm{supp}(\mu)$ has no accumulation point, it is always possible to choose such a cycle γ_ω for each $\omega \in \mathbb{R}$. We have

$$-\int_{\gamma_\omega} (C_\mu f)(\zeta)\, d\zeta = \int_{\gamma_\omega} \sum_{z \in A} \frac{\alpha_z f(z)}{2\pi i(\zeta - z)}\, d\zeta = \sum_{z \in A} \alpha_z f(z) \frac{1}{2\pi i} \int_{\gamma_\omega} \frac{1}{\zeta - z}\, d\zeta.$$

If $\omega \notin \mathrm{supp}(\mu)$, then by Cauchy's integral formula

$$\sum_{z \in A} \alpha_z f(z) \frac{1}{2\pi i} \int_{\gamma_\omega} \frac{1}{\zeta - z}\, d\zeta = 0 = f(\omega)\mu(\{\omega\}).$$

If $\omega \in \mathrm{supp}(\mu)$, then

$$\sum_{z \in A} \alpha_z f(z) \frac{1}{2\pi i} \int_{\gamma_\omega} \frac{1}{\zeta - z}\, d\zeta = f(\omega)\alpha_\omega = f(\omega)\mu(\{\omega\}),$$

again by Cauchy's integral formula. $\qquad\square$

Clearly, if a measure $\mu \in \mathcal{M}(\mathbb{R})$ has support in a closed set X, then $C\mu$ has an analytic continuation along $\mathbb{C} \setminus X$. For our purposes, we shall need a converse of this statement. Therefore, we present one fundamental result on Cauchy transforms of compactly supported measures which goes back to Carleson. A proof can be found in [27, Chapter II, Corollary 1.3].

Theorem 3.1.15 (Carleson, [27]) *Let $\mu \in \mathcal{M}_c(\mathbb{C})$ and $\Omega \subset \mathbb{C}$ be open. Furthermore let $F \in H(\Omega)$. If $C\mu = F$ almost everywhere, then $|\mu|(\Omega) = 0$.*

Lemma 3.1.16 *Let $\mu \in \mathcal{M}(\mathbb{R})$. Then, for a closed set $X \subset \mathbb{R}$, the following statements are equivalent:*

a) $C\mu \in H(\mathbb{C} \setminus \mathbb{R})$ has a holomorphic continuation along $\mathbb{C} \setminus X$.

b) $\mathrm{supp}(\mu) \subset X$.

Proof. The inclusion b) $\Rightarrow$ a) is clear. Now, suppose that $C\mu$ has an analytic continuation G on $\mathbb{C} \setminus X$. For each $x \in \mathbb{R} \setminus X$, there is some open neighborhood U_x such that $U_x \subset \mathbb{C} \setminus X$. Let us pick $\rho \in \mathbb{N}$ such that $U_x \subset \rho \mathbb{D}$. If we now write

$$\mu_1(A) = \int_{A \cap \mathbb{R}} \mathbb{1}_{[-\rho, \rho]} \, d\mu \quad (A \in \mathcal{B}(\mathbb{C})),$$

and

$$\mu_2(A) = \int_{A \cap \mathbb{R}} \mathbb{1}_{(-\rho, \rho)^c} \, d\mu \quad (A \in \mathcal{B}(\mathbb{C})),$$

then $\mu(A) = \mu_1(A) + \mu_2(A)$ $(A \in \mathcal{B}(\mathbb{R}))$. If G denotes the analytic continuation of $C\mu$ onto $\mathbb{C} \setminus X$, we know that the function F defined by

$$F(z) := G(z) - \frac{1}{2\pi i} \int_{(-\rho, \rho)^c} \frac{1}{t - z} \, d\mu(t) \quad (z \in (\mathbb{C} \setminus X) \cap (\mathbb{C} \setminus (-\rho, \rho)^c))$$

is analytic in $(\mathbb{C} \setminus X) \cap (\mathbb{C} \setminus (-\rho, \rho)^c)$. Since $C\mu_1 = F$ on $\mathbb{C} \setminus \mathbb{R}$ and $\lambda_2(\mathbb{R}) = 0$, Theorem 3.1.15 gives us that

$$|\mu_1|(U_x) \leq |\mu_1|((\mathbb{C} \setminus X) \cap (\mathbb{C} \setminus (-\rho, \rho)^c)) = 0.$$

Clearly,

$$|\mu_2|(U_x) = 0$$

and hence

$$|\mu|(U_x) = 0.$$

But this means $x \notin \mathrm{supp}(\mu)$ and we are done. $\qquad\square$

3.2 Functions vanishing outside a closed set

Our goal in this section is to develop a basic theory for Cauchy transforms of functions whose support is not the entire real axis. Later, we will focus on the cases that the support lies in $[0, \infty)$ (see Chapter 4) or a compact set $K \subset \mathbb{R}$ (see Chapter 5). Therefore, we consider for a closed set $X \subset \mathbb{R}$ and $p \in [1, \infty]$ the space

$$L_p(X) := \{f \in L_p(\mathbb{R}) : f\mathbb{1}_X = f\}.$$

Since the Cauchy transform of a function $f \in L_p(X)$ (here $p \in [1, \infty)$) has always an analytic continuation along $\mathbb{C} \setminus X$, we will regard the Cauchy transformation on $L_p(X)$ as an operator into $H(\mathbb{C} \setminus X)$.

Our first task is to determine the image of C under $L_p(X)$. There is a natural candidate.

Remark and Definition 3.2.1 Let $X \subset \mathbb{R}$ be closed. Then, for $p \in [1, \infty)$, we set

$$\mathcal{H}_p(\mathbb{C} \setminus X) := \{F \in H(\mathbb{C} \setminus X) : F|_{\mathbb{C} \setminus \mathbb{R}} \in \mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})\}.$$

Moreover, we write

$$\|F\|_{\mathcal{H}_p(\mathbb{C} \setminus X)} := \|F\|_{\mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})}.$$

If X is in addition bounded, we set

$$\mathcal{H}_p(\mathbb{C}_\infty \setminus X) := \{F \in H(\mathbb{C}_\infty \setminus X) : F|_{\mathbb{C} \setminus \mathbb{R}} \in \mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})\}$$

and similar

$$\|F\|_{\mathcal{H}_p(\mathbb{C}_\infty \setminus X)} := \|F\|_{\mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})}.$$

It is clear that this defines a norm on $\mathcal{H}_p(\mathbb{C} \setminus X)$ and $\mathcal{H}_p(\mathbb{C}_\infty \setminus X)$, respectively (the definiteness is due to continuity if $X \neq \mathbb{R}$). However, at first sight it is not clear if this norm is complete if $X \neq \mathbb{R}$. We shall later see that this is indeed the case.

As in the case $X = \mathbb{R}$ it turns out that the Cauchy transformation provides still an isomorphism between $L_p(X)$ and $\mathcal{H}_p(\mathbb{C} \setminus X)$ for any closed $X \subset \mathbb{R}$ and $p \in (1, \infty)$. A similar statement holds for the case $p = 1$ if we consider for closed $X \subset \mathbb{R}$ the closed subspace

$$H_1(X) := \{f \in H_1(\mathbb{R}) : f\mathbb{1}_X = f\}.$$

One of the main tools is the identity theorem for holomorphic functions which applies here since $\mathbb{C} \setminus X$ is connected for every closed $X \subsetneq \mathbb{R}$.

Proposition 3.2.2 *Let $X \subset \mathbb{R}$ be closed.*

1. *If $p \in (1, \infty)$, then the Cauchy transformation establishes an isomorphism between $L_p(X)$ and $\mathcal{H}_p(\mathbb{C} \setminus X)$. Thus, a function $F \in H(\mathbb{C} \setminus X)$ belongs to $\mathcal{H}_p(\mathbb{C} \setminus X)$ if and only if there exists $f \in L_p(X)$ such that*

   $$F(z) = (Cf)(z) \quad (z \in \mathbb{C} \setminus X).$$

   In this case, f is unique and there exist constants $K_{p,1}, K_{p,2} > 0$ only depending on p such that
   $$\|F\|_{\mathcal{H}_p(\mathbb{C} \setminus X)} \leq K_{p,1} \|f\|_p \leq K_{p,2} \|F\|_{\mathcal{H}_p(\mathbb{C} \setminus X)}.$$

2. *The Cauchy transformation establishes an isomorphism between $H_1(X)$ and $\mathcal{H}_1(\mathbb{C} \setminus X)$. Thus, a function $F \in H(\mathbb{C} \setminus X)$ belongs to $\mathcal{H}_1(\mathbb{C} \setminus X)$ if and only if there exists $f \in H_1(X)$ such that*
   $$F(z) = (Cf)(z) \quad (z \in \mathbb{C} \setminus X).$$

   In this case, f is unique and there exist constants $K_1, K_2 > 0$ independent of f and F such that
   $$\|F\|_{\mathcal{H}_1(\mathbb{C} \setminus X)} \leq K_1 \|f\|_{H_1(\mathbb{R})} \leq K_2 \|F\|_{\mathcal{H}_1(\mathbb{C} \setminus X)}.$$

Proof. We only prove the first part since the second can be shown in a similar manner. If $f \in L_p(X)$, then clearly $Cf \in \mathcal{H}_p(\mathbb{C} \setminus X)$ by Theorem 2.2.11. Thus, suppose that $F \in \mathcal{H}_p(\mathbb{C} \setminus X)$. By Theorem 2.2.11 there exists $f \in L_p(\mathbb{R})$ such that

$$F(z) = (Cf)(z) \quad (z \in \mathbb{C} \setminus \mathbb{R}).$$

But this means that Cf has an analytic continuation along $\mathbb{C} \setminus X$ which implies that $f \in L_p(X)$ by Lemma 2.1.9. The uniqueness and the estimates follows from Theorem 2.2.11. $\square$

Remark 3.2.3 Let $X \subset \mathbb{R}$ be closed and $p \in [1, \infty)$. Then, $\mathcal{H}_p(\mathbb{C} \setminus X)$ is a Banach space by Proposition 3.2.2. If X is in addition bounded, then one can replace $\mathcal{H}_p(\mathbb{C}\setminus X)$ in Proposition 3.2.2 by $\mathcal{H}_p(\mathbb{C}_\infty \setminus X)$. Thus, also $\mathcal{H}_p(\mathbb{C}_\infty \setminus X)$ is a Banach space. In this case, we shall usually consider the Cauchy transformation as a mapping from $L_p(X)$ to $\mathcal{H}_p(\mathbb{C}_\infty \setminus X)$.

If $X \in \mathcal{B}(\mathbb{R})$ is such that $\lambda_1(X) = 0$, then $L_p(X) = \{0\}$. Thus:

Corollary 3.2.4 *Let $p \in [1, \infty)$ and $X \subset \mathbb{R}$ be a closed set with $\lambda_1(X) = 0$. Then, any function $F \in H(\mathbb{C} \setminus X)$ which satisfies*

$$\left(\sup_{y \neq 0} \int_{\mathbb{R}} |F(x + iy)|^p \, dx \right)^{1/p} < +\infty$$

has to be the zero function.

Remark 3.2.5 Let $X \subset \mathbb{R}$ be closed.

1. The inclusions
$$\mathcal{H}_q(\mathbb{C} \setminus X) \subset \mathcal{H}_p(\mathbb{C} \setminus X)$$
   hold for every $1 < p \leq q \leq \infty$ if and only if $\lambda_1(X) < +\infty$. This follows from the fact that
$$L_q(X) \subset L_p(X) \quad (1 \leq p \leq q \leq \infty)$$
   holds if and only if $\lambda_1(X) < +\infty$, see [80, Theorem 2] (notice that the Lebesgue measure is σ-finite).

2. Notice that for $p \in (1, \infty)$ we do not have $\mathcal{H}_p(\mathbb{C} \setminus X) \subset \mathcal{H}_1(\mathbb{C} \setminus X)$ if $\lambda_1(X) < +\infty$. Consider for example the function $f = \mathbb{1}_{[0,1]}$. Then, $f \in L_p(\mathbb{R})$ for all $p \in [1, \infty]$ and hence $Cf \in \mathcal{H}_p(\mathbb{C}\setminus[-1,1])$ for all $p \in (1, \infty)$. But clearly $Cf \notin \mathcal{H}_1(\mathbb{C}\setminus[-1,1])$ because $Hf \notin L_1(\mathbb{R})$, see Proposition 2.3.1.

3. If $\lambda_1(X) < +\infty$, then it follows from 1. that
$$\bigcup_{p \geq 1} \mathcal{H}_p(\mathbb{C} \setminus X) \subset C(L_1(X)).$$

We briefly turn to the problem of separating singularities of holomorphic functions. A famous theorem, first proved by Poincaré in 1892 (see [62]), says that if an open set $\Omega \subset \mathbb{C}$ is the intersection of two open sets Ω_1, Ω_2, then each $f \in H(\Omega)$ can be written as the sum of two functions $f_1 \in H(\Omega_1)$, $f_2 \in H(\Omega_2)$. We want to prove an analogue with $\mathcal{H}_p(\Omega_1)$ and $\mathcal{H}_p(\Omega_2)$ provided that Ω_1, Ω_2 contain the upper and lower half plane. Beyond this, we are interested in a characterization in which case this decomposition is unique.

Lemma 3.2.6 *Let $X_1, X_2 \subset \mathbb{R}$ be Borel sets and $p \in [1, \infty]$. Then, the following statements are equivalent:*

a) $L_p(X_1 \cup X_2) = L_p(X_1) \oplus L_p(X_2)$.

b) $\lambda_1(X_1 \cap X_2) = 0$.

Proof. First note that we always have

$$L_p(X_1 \cup X_2) = L_p(X_1) + L_p(X_2).$$

We first show that a) implies b). Suppose to the contrary that $\lambda_1(X_1 \cap X_2) > 0$. Without loss of generality, we may assume that $\lambda_1(X_1 \cap X_2) < +\infty$ (go to a suitable subset instead). Then $f = \mathbb{1}_{X_1 \cap X_2}$ is in $L_p(X_1) \cap L_p(X_2)$, but $f \neq 0$. If b) holds and $f \in L_p(X_1) \cap L_p(X_2)$, then

$$f = f\mathbb{1}_{X_2} = (f\mathbb{1}_{X_1})\mathbb{1}_{X_2} = f\mathbb{1}_{X_1 \cap X_2} = 0.$$

This implies a). $\qquad\square$

Proposition 3.2.7 *Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be open sets containing $\mathbb{C} \setminus \mathbb{R}$. Then every function $F \in \mathcal{H}_p(\Omega_1 \cap \Omega_2)$ can be written as the sum of $F_1 \in \mathcal{H}_p(\Omega_1)$ and $F_2 \in \mathcal{H}_p(\Omega_2)$. This decomposition is unique if and only if*

$$\lambda_1(\Omega_1^c \cap \Omega_2^c) = 0.$$

Proof. Choose $X_1, X_2 \subset \mathbb{R}$ be closed with $\Omega_i = \mathbb{C} \setminus X_i$ $(i = 1, 2)$, hence $\Omega_1 \cap \Omega_2 = \mathbb{C} \setminus (X_1 \cup X_2)$. If $g \in \mathcal{H}_p(\Omega_1 \cap \Omega_2)$, then $g = Cf$ for some $f \in L_p(X_1 \cup X_2)$. But since $L_p(X_1 \cup X_2) = L_p(X_1) + L_p(X_2)$ we know that there exist $f_1 \in L_p(X_1), f_2 \in L_p(X_2)$ with

$$f = f_1 + f_2.$$

The linearity of the Cauchy transformation implies together with Proposition 3.2.2 that $F_1 = Cf_1$ and $F_2 = Cf_2$ give the desired decomposition. Now, let $G_1 + G_2$ another decomposition of F. Then $G_1 = Cg_1$ for some $g_1 \in L_p(X_1)$ and $G_2 = Cg_2$ for some $g_2 \in L_p(X_2)$. Hence, $F_1 + F_2 = G_1 + G_2$ if and only if $f_1 + f_2 = g_1 + g_2$. Due to Lemma 3.2.6 the latter is equivalent to the fact that

$$\lambda_1(\Omega_1^c \cap \Omega_2^c) = \lambda_1(X_1 \cap X_2) = 0.$$

$\qquad\square$

Chapter 4

Cauchy Transforms of Functions and Measures with Support in an Unbounded Interval

> It is better to solve one problem five
> different ways, than to solve five
> problems one way.
>
> George Pólya

After we have set up a general theory for Cauchy transforms of functions supported in a closed set $X \subset \mathbb{R}$ in the last chapter, we will now treat special cases. One of the most important cases arises when we take X to be an unbounded interval which is not the entire real axis. Without loss of generality we will consider the interval $X = [0, \infty)$ and hence analytic functions on $\mathbb{C}_- := \mathbb{C} \setminus [0, \infty)$.

Our schedule is as follows: First, we will focus on complex characterizations that will once again involve the concept of Fourier-Laplace transforms. Moreover, we will present a geometric condition that involves special curves in the complex plane. In particular, we will discuss the behavior of a function $g \in \mathcal{H}_p(\mathbb{C}_-)$ on half-planes which are contained in $\mathbb{C}_-$. Next, we will transfer this task to the case of measures with support in $[0, \infty)$. With the aid of elliptic integrals we shall derive a reconstruction formula for Cauchy transforms of measures by its values on the imaginary axis and analyze as well the behavior on half-planes. After this, we will present an idea of Widder (see [82]) which leads to conditions and characterizations of real-type. Here, we only put requirements on the restriction of the analytic function to the negative half-axis $(-\infty, 0)$.

4.1 Complex and geometric characterizations

From a point of complex analysis, one is of course highly interested in having characterizations of the space $\mathcal{H}_p(\mathbb{C}_-)$ which are of complex nature. This is the point where the analyticity

of Cauchy transforms comes in. We shall start with a result which relates once again the concept of Fourier-Laplace transforms to our topic.

The following proposition can be regarded as an improvement of a result in [82, Chapter VIII, Theorem 4a] under certain additional conditions.

Proposition 4.1.1 *Let $\alpha \in (0, \pi)$ and $\mu \in \mathcal{M}_\infty(\mathbb{R})$ be such that $\mathrm{supp}(\mu) \subset [0, \infty)$ and $|\mu|$ is σ-finite. If $f \in \mathscr{C}(\mu)$ is such that $f\mathbb{1}_{[0,\infty)} \in \mathscr{E}_{0,1}(\mu)$, then*

$$(C_\mu f)(z) = -2\pi L_\alpha(L_\mu f)(-z)$$

for each $z \in \mathbb{C}$ with $\mathrm{Im}(ze^{i\alpha}) < 0$.

Proof. By Fubini's Theorem and our assumption we know that

$$
\begin{aligned}
\int_0^\infty |e^{-ize^{i\alpha}t}(L_\mu f)(e^{i\alpha}t)|\, dt
&\leq \int_0^\infty \int_0^\infty |f(x)e^{i(e^{i\alpha}xt)}e^{-ize^{i\alpha}t}|\, d|\mu|(x)\, dt \\
&= \int_0^\infty \int_0^\infty |f(x)|e^{-\mathrm{Im}(e^{i\alpha})xt}e^{\mathrm{Im}(ze^{i\alpha})t}\, d|\mu|(x)\, dt \\
&= \int_0^\infty |f(x)| \int_0^\infty e^{(\mathrm{Im}(ze^{i\alpha})-\sin(\alpha)x)t}\, dt\, d|\mu|(x) \\
&= \int_0^\infty \frac{|f(x)|}{\sin(\alpha)x - \mathrm{Im}(ze^{i\alpha})}\, d|\mu|(x) < +\infty.
\end{aligned}
$$

Therefore, we can apply again Fubini's Theorem to see that

$$
\begin{aligned}
\int_0^{e^{i\alpha}\cdot\infty} e^{-iz\zeta}(L_\mu f)(\zeta)\, d\zeta
&= \frac{e^{i\alpha}}{2\pi} \int_0^\infty e^{-ize^{i\alpha}t}(L_\mu f)(e^{i\alpha}t)\, dt \\
&= \frac{e^{i\alpha}}{(2\pi)^2} \int_0^\infty \int_0^\infty f(x)e^{ie^{i\alpha}(x-z)t}\, d\mu(x)\, dt \\
&= \frac{e^{i\alpha}}{(2\pi)^2} \int_0^\infty f(x) \int_0^\infty e^{ie^{i\alpha}(x-z)t}\, dt\, d\mu(x) \\
&= \frac{1}{(2\pi)^2 i} \int_0^\infty \frac{f(x)}{z-x}\, d\mu(x).
\end{aligned}
$$

This was what we had to prove. □

Remark 4.1.2 The proof of Proposition 4.1.1 shows that if $f \in L_1(\mu)$, then we can also allow $\alpha = 0$ or $\alpha = \pi$. In particular, upon choosing $\mu = \mathbb{1}_{(0,\infty)}\lambda_1$ and $f \in L_1(0, \infty)$, we obtain

$$(Cf)(z) = -2\pi(L_\pi(Lf))(-z) = \int_0^\infty (Lf)(-t)e^{izt}\, dt = (L\widehat{f})(z) \quad (z \in \Pi_+),$$

as well as

$$(Cf)(z) = -2\pi(L_0(Lf))(-z) = -\int_0^\infty (Lf)(t)e^{-izt}\, dt = (L\widehat{f})(z) \quad (z \in \Pi_-).$$

This shows that Proposition 4.1.1 contains the formulas in Proposition 2.3.7 for the case $p = 1$.

The formula above illustrates that the uniqueness of Cauchy transforms is deeply related to the uniqueness of Fourier-Laplace transforms. More precisely:

Corollary 4.1.3 *Let $\mu \in \mathcal{M}(\mathbb{R})$ be such that $\mathrm{supp}(\mu) \subset [0, \infty)$ and $|\mu|$ is σ-finite. For a set $A \subset \mathscr{C}(\mu) \cap \mathscr{E}_{0,1}(\mu)$ the following statements are equivalent:*

a) $C_\mu : A \to H(\mathbb{C}_-)$ is injective.

b) $L_\mu : A \to H(\Pi_+)$ is injective.

Proof. Suppose first that a) is valid. If $f, g \in A$ with $L_\mu f = L_\mu g$, then by Proposition 4.1.1 we have $C_\mu f = C_\mu g$ on $\mathbb{C}_-$ and hence $f = g$. Let now L_μ be injective on A and $f, g \in A$ such that $C_\mu f = C_\mu g$. Then, by Proposition 4.1.1 we know that $L_{\frac{\pi}{2}}(L_\mu f)(-z) = L_{\frac{\pi}{2}}(L_\mu g)(-z)$ for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) < 0$, i.e.

$$\int_0^\infty (L_\mu f)(it) e^{zt}\, dt = \int_0^\infty (L_\mu g)(it) e^{zt}\, dt \quad (\mathrm{Re}(z) < 0).$$

This implies in particular that

$$\widehat{((L_\mu f)(i\cdot) \exp(-\cdot) \mathbb{1}_{(0,\infty)})}(-y) = \widehat{((L_\mu g)(i\cdot) \exp(-\cdot) \mathbb{1}_{(0,\infty)})}(-y) \quad (y \in \mathbb{R})$$

and the uniqueness of Fourier transforms gives us that $(L_\mu f)(it) = (L_\mu g)(it)$ for all $t \in (0, \infty)$. By the identity theorem, we necessarily have $(L_\mu f)(z) = (L_\mu g)(z)$ for all $z \in \Pi_+$ and we conclude that $f = g$. $\qquad\square$

Remark and Definition 4.1.4 For $\beta \in \mathbb{R}$ we write $\mu_\beta = t^\beta \mathbb{1}_{(0,\infty)}(t)\, dt$ and consider the **weighted Dirichlet space** $\mathcal{D}_\beta(\Pi_+)$ which consists of all functions $F \in H(\Pi_+)$ satisfying

$$\int_{-\infty}^\infty \int_0^\infty |F'(x + iy)|\, d\mu_\beta(y)\, dx < +\infty.$$

Let $\beta > -1$. If $F \in \mathcal{D}_\beta(\Pi_+)$, then by [17, Theorem 3] there exists a unique $g \in L_2((0, \infty), \mu_{1-\beta})$ such that $F = Lg$. Setting $f(t) := t^{1-\beta} g(t)$ we see that

$$\int_0^\infty |f(t)|^2\, d\mu_{\beta-1}(t) = \int_0^\infty |g(t)|^2\, d\mu_{1-\beta}(t) < +\infty$$

and $F = L_{\mu_{\beta-1}} f$. Conversely, if $f \in L_2((0, \infty), \mu_{\beta-1})$ and $g(t) := t^{\beta-1} f(t)$, then

$$\int_0^\infty |g(t)|^2\, d\mu_{1-\beta}(t) = \int_0^\infty |f(t)|^2\, d\mu_{\beta-1}(t) < +\infty$$

and [17, Theorem 3] gives us that $L_{\mu_{\beta-1}} f \in \mathcal{D}_\beta(\Pi_+)$. To sum it up: For $\beta > -1$, the mapping $L_{\mu_{\beta-1}}$ is an isomorphism between $L_2((0, \infty), \mu_{\beta-1})$ and $D_\beta(\Pi_+)$.

If $\beta \in (0, 1)$ and $f \in L_2((0, \infty), \mu_{\beta-1})$, then

$$\int_0^\infty \frac{|f(t)|}{t+1}\, d\mu_{\beta-1}(t) \leq \left(\int_0^\infty |f(t)|^2\, d\mu_{\beta-1}(t) \right) \cdot \left(\int_0^\infty \frac{1}{(t+1)^2}\, d\mu_{\beta-1}(t) \right) < +\infty$$

by the Cauchy-Schwarz inequality. In particular, we have $L_2((0, \infty), \mu_{\beta-1}) \subset \mathscr{C}(\mu_{\beta-1})$. Therefore, Proposition 4.1.1 and Corollary 4.1.3 imply

Corollary 4.1.5 *Let $\beta \in (0,1)$ and $F \in H(\mathbb{C}_-)$. Then, the following statements are equivalent:*

a) *There exists a unique $f \in L_2((0,\infty), \mu_{\beta-1})$ such that $C_{\mu_{\beta-1}} f = F$.*

b) *There exists a unique $g \in \mathcal{D}_\beta(\Pi_+)$ such that for all $\alpha \in (0,\pi)$ we have $(L_\alpha g)(-z) = F(z)$ for all $z \in \mathbb{C}$ with $\operatorname{Im}(ze^{i\alpha}) < 0$.*

Remark 4.1.6 We remark that in general one does not have $L_2((0,\infty), \mu_{\beta-1}) \subset \mathscr{C}_{(\mu_{\beta-1})}$ for all $\beta \in (-1,\infty)$. For example, if $\beta = 0$, then then the function $f : (0,\infty) \to \mathbb{R}$, defined by

$$f(t) = \frac{1}{|\ln(t)|} \mathbb{1}_{(0, \frac{1}{2})}(t) \quad (t \in \mathbb{R}),$$

belongs to $L_2((0,\infty), \mu_{-1})$ but since

$$\int \frac{f(t)}{t+1} \, d\mu_{-1}(t) \geq \frac{2}{3} \int_0^{\frac{1}{2}} \frac{1}{t|\ln(t)|} \, dt = +\infty$$

clearly $f \notin \mathscr{C}(\mu_{-1})$.

From now on, we put our primary focus on the space $L_2(0,\infty)$. In this case, we have the famous Paley-Wiener theorem at hand. With this result in mind, we can now define the Fourier-Laplace transform on the Hardy space $\mathcal{H}_2(\Pi_+)$. We will exploit the fact that any $z \in \mathbb{C}_-$ lies in some half-plane $\{z \in \mathbb{C} : \operatorname{Im}(ze^{i\alpha}) < 0\}$ with $\alpha \in (0,\pi)$.

Remark and Definition 4.1.7 Let $\alpha \in (0,\pi)$. If $F \in H(\Pi_+)$ is such that $F(e^{i\alpha} \cdot) \mathbb{1}_{e^{i\alpha} \cdot [0,\infty)} \in \mathscr{E}_{\alpha,1}$, then

$$(\mathscr{L}_\alpha F)(z) := -2\pi (L_\alpha F)(-z) \left(= -\int_0^{e^{i\alpha} \cdot \infty} e^{-iz\zeta} F(\zeta) \, d\zeta \right)$$

exists for every $z \in \mathbb{C}$ with $\operatorname{Im}(ze^{i\alpha}) < 0$. Now let $F \in \mathcal{H}_2(\Pi_+)$. Then, by Proposition 2.3.10 we have $F = Lf$ with $f \in L_2(0,\infty)$ and Proposition 4.1.1 implies that

$$(\mathscr{L}_\alpha F)(z) = (Cf)(z) \quad (\operatorname{Im}(ze^{i\alpha}) < 0).$$

Therefore, the function $\mathscr{L}F : \mathbb{C}_- \to \mathbb{C}$, defined by

$$(\mathscr{L}F)(z) := (\mathscr{L}_\alpha F)(z) \quad (\operatorname{Im}(ze^{i\alpha}) < 0, \alpha \in (0,\pi)),$$

is well-defined and holomorphic in $\mathbb{C}_-$. We call $\mathscr{L}F$ the **Fourier-Laplace transform** of F. The mapping $\mathscr{L} : \mathcal{H}_2(\Pi_+) \to H(\mathbb{C}_-)$ is called the **Fourier-Laplace transformation** on $\mathcal{H}_2(\Pi_+)$.

What is the image of the Fourier-Laplace transformation on $\mathcal{H}_2(\Pi_+)$? Here is the answer:

Proposition 4.1.8 *The diagram*

is a commuting diagram of isomorphisms, i.e.

$$\mathscr{L}(Lf) = Cf \quad (f \in L_2(0, \infty)).$$

Hence,

$$\mathscr{L}(\mathcal{H}_2(\Pi_+)) = \mathcal{H}_2(\mathbb{C}_-).$$

We now turn to an intuitive geometric condition that characterizes the functions in $\mathcal{H}_p(\mathbb{C}_-)$. Therefore, it will be helpful to know more about the behavior of a function $g \in \mathcal{H}_p(\mathbb{C}_-)$ on other half-planes. The next proposition states not only that a function $g \in \mathcal{H}_p(\mathbb{C}_-)$ lies in the Hardy space of any half-plane contained in $\mathbb{C}_-$ but gives also an uniform estimation for the corresponding norms using the constants B_p from Remark 1.2.10.

Recall that a positive measure $\sigma \in \mathcal{M}_{\infty,+}(\Pi_+)$ is called a Carleson measure on Π_+ if there is a constant $N(\sigma)$ such that

$$\sigma(\{x + iy : x_0 < x < x_0 + h, 0 < y < h\}) \leq N(\sigma)h$$

for all $x_0 \in \mathbb{R}$ and $h > 0$.

Theorem 4.1.9 *Let $p \in [1, \infty)$ and $F \in H(\mathbb{C}_-)$. Then, the following statements are equivalent:*

a) $\displaystyle \sup_{\alpha \in (0,\pi)} \sup_{y>0} \int_{-\infty}^{\infty} |F(e^{i\alpha}(x+iy))|^p \, dx < +\infty.$

b) $\displaystyle \sup_{\alpha \in [0,\pi]} \sup_{y>0} \int_{-\infty}^{\infty} |F(e^{i\alpha}(x+iy))|^p \, dx < +\infty.$

c) $F \in \mathcal{H}_p(\mathbb{C}_-).$

Moreover, we have

$$\sup_{\alpha \in (0,\pi)} \sup_{y>0} \int_{-\infty}^{\infty} |F(e^{i\alpha}(x+iy))|^p \, dx \leq \sup_{\alpha \in [0,\pi]} \sup_{y>0} \int_{-\infty}^{\infty} |F(e^{i\alpha}(x+iy))|^p \, dx$$

$$\leq \sqrt{2} B_p (\|F\|_{\mathcal{H}_p(\Pi_+)}^p + \|F\|_{\mathcal{H}_p(\Pi_-)}^p)$$

$$\leq 2\sqrt{2} B_p \sup_{\alpha \in (0,\pi)} \sup_{y>0} \int_{-\infty}^{\infty} |F(e^{i\alpha}(x+iy))|^p \, dx.$$

Proof. Suppose that $a)$ holds. Then, for each $y > 0$ we have

$$\int_{-\infty}^{\infty} |F(x+iy)|^p \, dx \leq \liminf_{\alpha \to 0^+, \alpha < \pi} \int_{-\infty}^{\infty} |F(e^{i\alpha}(x+iy))|^p \, dx \leq \sup_{\alpha \in (0,\pi)} \sup_{y>0} \int_{-\infty}^{\infty} |F(e^{i\alpha}(x+iy))|^p \, dx$$

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by Fatou's lemma. Analogously, one sees that

$$\int_{-\infty}^{\infty} |F(x-iy)|^p \, dx \leq \liminf_{\alpha \to \pi^-, \alpha > 0} \int_{-\infty}^{\infty} |F(e^{i\alpha}(x+iy))|^p \, dx \leq \sup_{\alpha \in (0,\pi)} \sup_{y>0} \int_{-\infty}^{\infty} |F(e^{i\alpha}(x+iy))|^p \, dx.$$

Thus, $b)$ holds as well.

If $b)$ is true, then $c)$ is trivial (consider $\alpha = 0$ and $\alpha = \pi$).

So let us now suppose that $c)$ is true. Let $\alpha \in (0, \pi)$ and $\varepsilon > 0$. We set $\beta := -\frac{\cos(\alpha)}{\sin(\alpha)}\varepsilon$. Since

$$\operatorname{Im}(e^{i\alpha}(x+i\varepsilon)) = \sin(\alpha)x + \cos(\alpha)\varepsilon$$

the mapping $T_{\alpha,\varepsilon} : \mathbb{R} \to \mathbb{C}$ defined by $T_{\alpha,\varepsilon}(x) := e^{i\alpha}(x+i\varepsilon)$ $(t \in \mathbb{R})$ maps (β, ∞) into Π_+ and $(-\infty, \beta)$ into Π_-. We write $T_{\alpha,\varepsilon,1} = T_{\alpha,\varepsilon}\mathbb{1}_{(\beta,\infty)}$ and $T_{\alpha,\varepsilon,2} = T_{\alpha,\varepsilon}\mathbb{1}_{(-\infty,\beta)}$. Then, $\eta_{\alpha,\varepsilon,1} := \lambda_1^{T_{\alpha,\varepsilon,1}}$ is a Carleson measure on Π_+ since for each $x_0 \in \mathbb{R}, h > 0$ we have

$$\eta_{\alpha,\varepsilon,1}(\{z \in \Pi_+ : \operatorname{Re}(z) \in (x_0, x_0 + h), \operatorname{Im}(z) \in (0, h)\}) \leq \begin{cases} \frac{h}{\cos(\alpha)}, & \alpha \in [0, \frac{\pi}{4}) \cup (\frac{3}{4}\pi, \pi] \\ \frac{h}{\sin(\alpha)}, & \alpha \in [\frac{\pi}{4}, \frac{3}{4}\pi] \end{cases}$$

$$\leq \sqrt{2}h.$$

In particular, the Carleson norm of $\eta_{\alpha,\varepsilon,1}$ is at most $\sqrt{2}$. Similarly, one sees that $\eta_{\alpha,\varepsilon,2} := \lambda_1^{T_{\alpha,\varepsilon,2}}$ is a Carleson measure on Π_- with Carleson norm at most $\sqrt{2}$. Thus, by Theorem 1.2.8 and Remark 1.2.11 we know that

$$\int_{-\infty}^{\infty} |F(e^{i\alpha}(x+i\varepsilon))|^p \, dx = \int_{\Pi_+} |F(z)|^p \, d\eta_{\alpha,\varepsilon,1}(z) + \int_{\Pi_-} |F(z)|^p \, d\eta_{\alpha,\varepsilon,2}(z)$$

$$\leq \sqrt{2}B_p \|F\|^p_{\mathcal{H}_p(\Pi_+)} + \sqrt{2}B_p \|F\|^p_{\mathcal{H}_p(\Pi_-)}.$$

Thus,

$$\sup_{\alpha \in [0,\pi]} \sup_{y>0} \int_{-\infty}^{\infty} |F(e^{i\alpha}(x+iy))|^p \, dx < +\infty$$

and we conclude. $\square$

Remark 4.1.10 1. In particular, if $F \in \mathcal{H}_p(\mathbb{C}_-)$, then we have

$$\|F\|^p_{\mathcal{H}_p(i\Pi_+)} \leq B_p(\|F\|^p_{\mathcal{H}_p(\Pi_+)} + \|F\|^p_{\mathcal{H}_p(\Pi_-)}).$$

This follows from the fact that the Carleson norms of the measures $\eta_{\frac{\pi}{2},\varepsilon,1}$ and $\eta_{\frac{\pi}{2},\varepsilon,2}$ are equal to 1.

2. Theorem 4.1.9 is of course valid if we replace the assumption $F \in \mathcal{H}_p(\mathbb{C}_-)$ by $F \in \mathcal{H}_p(\mathbb{C} \setminus I)$ where I is an arbitrary unbounded interval which is not the entire real axis. In the case $I = [a, \infty)$, the result reads as

$$\sup_{\alpha \in [0,\pi]} \sup_{y>0} \int_{-\infty}^{\infty} |F(e^{i\alpha}(x+iy) - a)|^p \, dx < +\infty$$

and in the case $I = (-\infty, a]$ as

$$\sup_{\alpha \in [0,\pi]} \sup_{y>0} \int_{-\infty}^{\infty} |F(e^{i\alpha}(x+iy)+a)|^p \, dx < +\infty.$$

In particular, we have

$$\sup_{x \in \mathbb{R} \setminus I} \int_{-\infty}^{\infty} |F(x+iy)|^p \, dy < \infty.$$

This result also remains valid if we replace I by a union of two unbounded intervals and this union is not the entire real axis. This follows from Minkowki's inequality: If $I = I_1 \cup I_2$, we can assume without loss of generality that this union is disjoint (otherwise we have $I_1 \subset I_2$ or $I_2 \subset I_1$). If $F = Cf$, then we write

$$f = f\mathbb{1}_{I_1} + f\mathbb{1}_{I_2}.$$

By the linearity of the Cauchy transformation we obtain

$$\int_{-\infty}^{\infty} |F(x+iy)|^p \, dy \leq \frac{2^{p-1}}{(2\pi)^p} \cdot \left(\int_{-\infty}^{\infty} |(Cf\mathbb{1}_{I_1})(x+iy)|^p \, dy + \int_{-\infty}^{\infty} |(Cf\mathbb{1}_{I_2})(x+iy)|^p \, dy \right)$$

for $x \in \mathbb{R} \setminus I$. Therefore,

$$
\begin{aligned}
\sup_{x \in \mathbb{R} \setminus I} \int_{-\infty}^{\infty} |F(x+iy)|^p \, dy \quad &\leq \quad \sup_{x \in \mathbb{R} \setminus I_1} \int_{-\infty}^{\infty} |(Cf\mathbb{1}_{I_1})(x+iy)|^p \, dy \\
&+ \quad \sup_{x \in \mathbb{R} \setminus I_2} \int_{-\infty}^{\infty} |(Cf\mathbb{1}_{I_2})(x+iy)|^p \, dy \\
&< \quad \infty.
\end{aligned}
$$

Remark 4.1.11 It is important to note that it is not enough to require that $F \in \mathcal{H}_p(e^{i\alpha}\Pi_+)$ for some fixed $\alpha \in [0, \pi]$. Take for example

$$F(z) = \frac{\sqrt{-z}}{(z-1)^2} \quad (z \in \mathbb{C}_-).$$

If $x < 0$, then

$$
\begin{aligned}
\int_{-\infty}^{\infty} |F(x+iy)|^2 \, dy &= \int_{-\infty}^{\infty} \frac{\sqrt{x^2+y^2}}{((1-x)^2+y^2)^2} \, dy \\
&\leq \int_{-\infty}^{\infty} \frac{\sqrt{x^2+(1-x)^2+y^2}}{((1-x)^2+y^2)^2} \, dy \\
&\leq |x| \int_{-\infty}^{\infty} \frac{1}{((1-x)^2+y^2)^2} \, dy + \int_{-\infty}^{\infty} \frac{\sqrt{(1-x)^2+y^2}}{((1-x)^2+y^2)^2} \, dy \\
&= \frac{\pi|x|}{2(1-x)^3} + \frac{2}{(1-x)^2}
\end{aligned}
$$

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$$= \frac{4 - (4 + \pi)x}{2(1 - x)^3}.$$

Hence,

$$\sup_{x<0} \int_{-\infty}^{\infty} |F(x + iy)|^2 \, dy = 2 < \infty$$

and $F|_{i\Pi_+} \in \mathcal{H}_2(i\Pi_+)$. But for $G(x) := \lim_{y \to 0^+} F(x + iy)$ we have

$$\int_0^{\infty} |G(x)|^2 \, dx = \int_0^{\infty} \frac{x}{(x - 1)^4} \, dx = +\infty$$

and therefore $F \notin \mathcal{H}_2(\mathbb{C}_-)$ because $G \notin L_2(\mathbb{R})$.

Geometric characterizations or conditions for Cauchy transforms form a problem that has been attacked many ways in literature yet. There is a solution by Bourdon and Cima (see [8, Theorem 5.6.3]) for finite complex measures on the unit circle based on harmonic functions and conformal mappings. For general compactly supported, finite complex measures on the plane there is a characterization involving certain grids in the complex plane which satisfy a certain measure-theoretic condition, see [27, Chapter II, Theorem 2.3]. We shall come back to this approach in Chapter 5. Unfortunately, our situation of Cauchy transforms of functions in $L_p(0, \infty)$ or, more specific, functions in $\mathcal{H}_p(\mathbb{C}_-)$ does not fit into these cases and we have to come up with another approach.

The definition of the Hardy space $\mathcal{H}_p(\mathbb{C}_-)$ by integral means along horizontal lines has from a geometric point of view a line structure. Hence, if we are interested in a geometric condition or characterization of this space it is natural to search for something which reflects exactly this line structure. In terms of chains in the complex plane this leads to considering polygonal chains.

Definition 4.1.12 Let $\Omega \subset \mathbb{C}$ be open, $n \in \mathbb{N}$ and $\gamma = (\gamma_j)_{j=1,\dots,n}$ be a chain in Ω, i.e. $\gamma_j : I_j \to \Omega$ for each $1 \leq j \leq n$.

1. γ is called a **bounded polygonal chain** (with n parts) if there are $a_0, \dots, a_n \in \mathbb{C}$ such that $\gamma_j^* = [a_{j-1}, a_j]$ for each $1 \leq j \leq n$. We denote the set of all bounded polygonal chains in Ω with n parts by $\mathcal{P}_n(\Omega)$.

2. For $n \geq 2$ we call γ an **unbounded polygonal chain** with two unbounded parts if there are $k, l \in \{1, \dots, n\}$ such that

   i) $(\gamma_j)_{j \in \{1,\dots,n\} \setminus \{k,l\}}$ is a bounded polygonal chain.

   ii) I_k, I_l are unbounded intervals and there are $\alpha_k, \alpha_l \in [0, \pi], y_k, y_l \in \mathbb{R}$ such that

   $$\begin{aligned} \gamma_k(t) &= e^{i\alpha_k}(t + iy_k) \quad (t \in I_k) \\ \gamma_l(t) &= e^{i\alpha_l}(t + iy_l) \quad (t \in I_l). \end{aligned}$$

   We denote the set of all unbounded polygonal chains in Ω with n parts from which two are unbounded by $\mathcal{P}_{n,\infty,2}(\Omega)$.

Remark 4.1.13 Let $a, b > 0$. An important example for an unbounded polygonal chain in $\mathbb{C}_-$ with two unbounded parts is given by the chain $R_{a,b} = (R_{a,b,1}, R_{a,b,2}, R_{a,b,3})$ where

$$R_{a,b,1} : [-a, \infty) \to \mathbb{C}, \ R_{a,b,1}(t) = t + ib$$
$$R_{a,b,2} : [-b, b] \to \mathbb{C}, \ R_{a,b,2}(t) = -a + it$$
$$R_{a,b,3} : [-a, \infty) \to \mathbb{C}, \ R_{a,b,3}(t) = t - ib.$$

Here, $R^*_{a,b,2} = [-a - ib, a + ib]$ is the bounded part and $R^*_{a,b,1}$ and $R^*_{a,b,3}$ are the unbounded parts. For a function $g : R^*_{a,b} \to \mathbb{C}$ such that $(g \circ R_{a,b,1}), (g \circ R_{a,b,3}) \in L_1(-a, \infty)$ and $(g \circ R_{a,b,2}) \in L_1(-b, b)$, we have

$$\int_{R_{a,b}} g(\zeta) \, d\zeta = \int_{-a}^{\infty} g(t + ib) \, dt + \int_{-a}^{\infty} g(t - ib) \, dt + i \int_{-b}^{b} g(-a + it) \, dt.$$
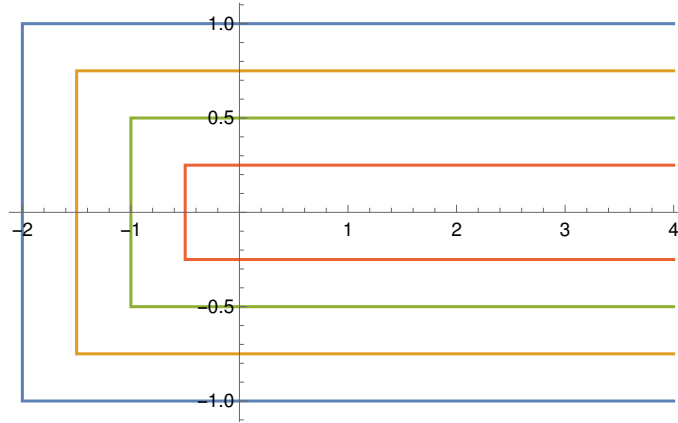
Figure 4.1: Plots of $R_{2,1}$ (blue), $R_{\frac{3}{2},\frac{3}{4}}$ (yellow), $R_{1,\frac{1}{2}}$ (green) and $R_{\frac{1}{2},\frac{1}{4}}$ (orange).

The chains $R_{a,b}$ can be seen in some kind as a model for unbounded polygonal chains in $\mathbb{C}_-$. More precisely, the following result is valid:

Theorem 4.1.14 *Let $p \in [1, \infty)$ and $F \in H(\mathbb{C}_-)$. Then, the following assertions are equivalent:*

a) $F \in \mathcal{H}_p(\mathbb{C}_-)$.

b) $\|F\|_{\mathbb{C}_-, p} := \sup \left\{ \left(\int_{R_{a,b}} |F(\zeta)|^p \, |d\zeta| \right)^{\frac{1}{p}} : a \geq b > 0 \right\} < +\infty.$

c) For each $n \geq 2$ we have

$$\|F\|_{\mathbb{C}_-, p, n} := \sup \left\{ \left(\int_{\gamma} |F(\zeta)|^p \, |d\zeta| \right)^{\frac{1}{p}} : \gamma \in \mathcal{P}_{n,\infty,2}(\mathbb{C}_-) \right\} < +\infty.$$

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In this case, each line in b) and c) defines a norm which is equivalent to $\|\cdot\|_{\mathcal{H}_p(\mathbb{C}_-)}$, i.e. we have for each $n \geq 2$

$$\|F\|_{\mathbb{C}_-,p} \leq (2+2B_p)^{\frac{1}{p}}\|F\|_{\mathcal{H}_p(\mathbb{C}_-)} \leq 2(2+2B_p)^{\frac{1}{p}}\|F\|_{\mathbb{C}_-,p}$$

and

$$\|F\|_{\mathbb{C}_-,p,n} \leq (2n\sqrt{2}B_p)^{\frac{1}{p}}\|F\|_{\mathcal{H}_p(\mathbb{C}_-)} \leq 2(2n\sqrt{2}B_p)^{\frac{1}{p}}\|F\|_{\mathbb{C}_-,p,n}.$$

Proof. Let us first suppose that $a)$ holds. Let $n \in \mathbb{N}$ and $\gamma \in \mathcal{P}_{n,\infty,2}(\mathbb{C}_-)$. Then, there are intervals $I_j \subset \mathbb{R}, \alpha_j \in [-\pi,\pi], a_j \in \mathbb{R}$ such that $\gamma_j : I_j \to \mathbb{C}$, $\gamma_j(t) = e^{i\alpha_j}(t + ia_j)$ for $1 \leq j \leq n$. Since $\gamma^* \subset \mathbb{C}_-$ it is always possible to choose the intervals such that either $\alpha_j \in (0,\pi]$ and $a_j > 0$ or $\alpha_j \in [-\pi,0)$ and $a_j < 0$. If we write $J_> := \{j \in \{1,\dots,n\} : \alpha_j > 0\}$ and $J_< := \{j \in \{1,\dots,n\} : \alpha_j < 0\}$, then

$$\int_\gamma |F(\zeta)|^p \, |d\zeta| = \sum_{j=1}^n \int_{I_j} |F(e^{i\alpha_j}(t + ia_j))|^p \, dt$$

$$\leq \sum_{j=1}^n \int_{-\infty}^\infty |F(e^{i\alpha_j}(t + ia_j))|^p \, dt$$

$$= \sum_{j\in J_>} \int_{-\infty}^\infty |F(e^{i\alpha_j}(t + ia_j))|^p \, dt + \sum_{j\in J_<} \int_{-\infty}^\infty |F(e^{i\alpha_j}(t + ia_j))|^p \, dt$$

$$= \sum_{j\in J_>} \int_{-\infty}^\infty |F(e^{i\alpha_j}(t + ia_j))|^p \, dt + \sum_{j\in J_<} \int_{-\infty}^\infty |F(-e^{i\alpha_j}(t - ia_j))|^p \, dt$$

$$= \sum_{j\in J_>} \int_{-\infty}^\infty |F(e^{i\alpha_j}(t + ia_j))|^p \, dt + \sum_{j\in J_<} \int_{-\infty}^\infty |F(e^{i\pi+i\alpha_j}(t - ia_j))|^p \, dt$$

$$\leq \sum_{j\in J_>} \|F\|_{\mathcal{H}_p(e^{i\alpha_j}\Pi_+)} + \sum_{j\in J_<} \|F\|_{\mathcal{H}_p(e^{i(\pi+\alpha_j)}\Pi_+)}$$

$$\leq n \cdot \sup_{\alpha\in[0,\pi]} \|F\|_{\mathcal{H}_p(e^{i\alpha}\Pi_+)}$$

$$\leq n\sqrt{2}B_p(\|F\|_{\mathcal{H}_p(\Pi_+)}^p + \|F\|_{\mathcal{H}_p(\Pi_-)}^p)$$

by Theorem 4.1.9. Hence, $c)$ holds. By Remark 4.1.13 it is clear that $c)$ implies $b)$. Hence, suppose that $b)$ holds and fix $y > 0$. Then, by the monotone convergence theorem

$$\int_{-\infty}^\infty |F(x + iy)|^p \, dx = \lim_{a\to\infty} \int_{-a}^\infty |F(x + iy)|^p \, dx$$

$$\int_{-\infty}^\infty |F(x - iy)|^p \, dx = \lim_{a\to\infty} \int_{-a}^\infty |F(x - iy)|^p \, dx.$$

Let us fix $\delta \geq y$. By our assumption,

$$\int_{-\delta}^\infty |F(x + iy)|^p \, dx \leq \int_{R_{\delta,y}} |F(\zeta)|^p \, |d\zeta| \leq \|F\|_{\mathbb{C}_-,p}$$

$$\int_{-\delta}^\infty |F(x - iy)|^p \, dx \leq \int_{R_{\delta,y}} |F(\zeta)|^p \, |d\zeta| \leq \|F\|_{\mathbb{C}_-,p}$$

and we conclude by letting $\delta \to \infty$ that $a)$ is true.

Finally, for $F \in \mathcal{H}_p(\mathbb{C}_-)$, we have

$$
\begin{aligned}
\|F\|_{\mathbb{C}_-,p} &\leq \left(\|F\|^p_{\mathcal{H}_p(\Pi_+)} + \|F\|^p_{\mathcal{H}_p(\Pi_-)} + \|F\|^p_{\mathcal{H}_p(i\Pi_+)} \right)^{\frac{1}{p}} \\
&\leq (1+B_p)^{\frac{1}{p}} \left(\|F\|^p_{\mathcal{H}_p(\Pi_+)} + \|F\|^p_{\mathcal{H}_p(\Pi_-)} \right)^{\frac{1}{p}} \\
&\leq (2+2B_p)^{\frac{1}{p}} \|F\|_{\mathcal{H}_p(\mathbb{C}_-)} \\
&\leq (2+2B_p)^{\frac{1}{p}} (\|F\|_{\mathcal{H}_p(\Pi_+)} + \|F\|_{\mathcal{H}_p(\Pi_-)}) \\
&\leq 2(2+2B_p)^{\frac{1}{p}} \|F\|_{\mathbb{C}_-,p}
\end{aligned}
$$

and

$$
\begin{aligned}
\|F\|_{\mathbb{C}_-,p,n} &\leq (n\sqrt{2}B_p)^{\frac{1}{p}} \left((\|F\|^p_{\mathcal{H}_p(\Pi_+)} + \|F\|^p_{\mathcal{H}_p(\Pi_-)}) \right)^{\frac{1}{p}} \\
&\leq (2n\sqrt{2}B_p)^{\frac{1}{p}} \|F\|_{\mathcal{H}_p(\mathbb{C}_-)} \\
&\leq 2(2n\sqrt{2}B_p)^{\frac{1}{p}} \|F\|_{\mathbb{C}_-,p} \\
&\leq 2(2n\sqrt{2}B_p)^{\frac{1}{p}} \|F\|_{\mathbb{C}_-,p,n}.
\end{aligned}
$$

This concludes the proof. $\qquad\square$

Remark 4.1.15 Theorem 4.1.9 is of course valid for $p = \infty$ if we replace the conditions b) and c) by

b') $\|F\|_{\mathbb{C}_-,\infty} := \sup \left\{ \|F\|_{\infty,(R_{a,b})^*} : a \geq b > 0 \right\} < +\infty.$

c') For each $n \geq 2$ we have

$$
\|F\|_{\mathbb{C}_-,\infty,n} := \sup \left\{ \|F\|_{\infty,\gamma^*} : \gamma \in \mathcal{P}_{n,\infty,2}(\mathbb{C}_-) \right\} < +\infty.
$$

4.2   Measures supported on a half-axis

The question arises which parts of the theory in the previous section can be adjusted to the case of measures with support in $[0,\infty)$. In Chapter 3 we have already seen that the measures μ (with support in $[0,\infty)$) which satisfy $C\mu \in \mathcal{H}_p(\mathbb{C}_-)$ are exactly of the form $\mu = f\lambda_1$ where $f \in L_p(0,\infty)$, cf. Proposition 3.1.12. However, in view of Theorem 4.1.9 one could also ask for a condition when $C\mu$ belongs to the Hardy space of any other half-plane in $\mathbb{C}_-$. For the sake of simplicity, we will put our focus on the left half-plane $i\Pi_+$. Under additional assumptions on the measure μ we shall derive a characterization which reflects in some kind a similarity between Cauchy transforms of measures and functions. One key element for the proof will be the theory of elliptic integrals.

Definition 4.2.1 The **incomplete elliptic integral of the first kind** is defined by the relation

$$F(x, m) = \int_0^x \frac{1}{\sqrt{(1 - mt^2) \cdot (1 - t^2)}} \, dt \quad (x \in (0, \infty), m \in (-\infty, 1)).$$

We set $K(m) := F(1, m)$ and call this function K the **complete elliptic integral of the first kind**. Hence,

$$K(m) = \int_0^1 \frac{1}{\sqrt{(1 - mt^2) \cdot (1 - t^2)}} \, dt \quad (m \in (-\infty, 1)).$$

For our purposes, we shall need a shifted version of K, namely the function $\mathcal{K}_a : \mathbb{R} \to [0, \infty)$, defined by

$$\mathcal{K}_a(s) := \begin{cases} K\left(1 - \dfrac{s^2}{a^2}\right), & s > 0 \\ 0, & s \leq 0 \end{cases}$$

where $a > 0$. The following integrability result involves the asymptotic expansions of K at 1 and $-\infty$.
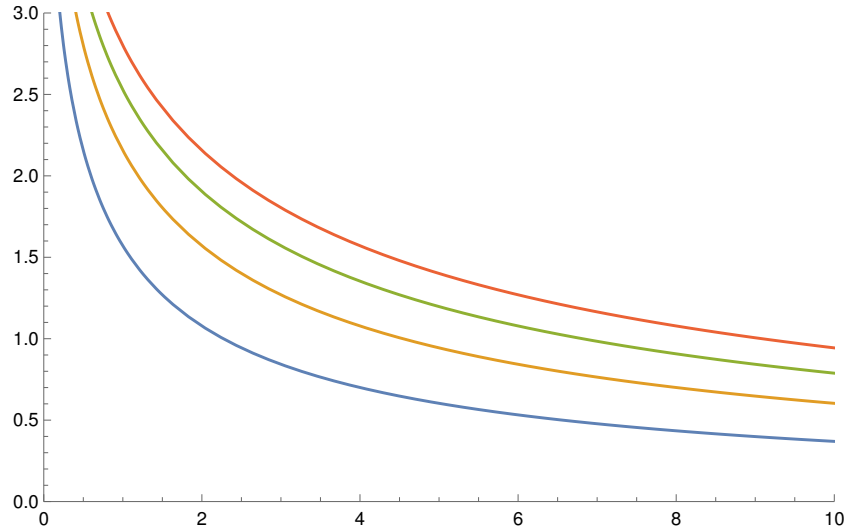
Figure 4.2: Plots of $\mathcal{K}_1$ (blue), $\mathcal{K}_2$ (yellow), $\mathcal{K}_3$ (green) and $\mathcal{K}_4$ (orange) on $(0, \infty)$.

Lemma 4.2.2 *Let $a \in (0, \infty)$ and $p \in (1, \infty)$. Then, $\mathcal{K}_a \in L_p(0, \infty)$.*

Proof. Since $\mathcal{K}_a$ is obviously continuous on $(0, \infty)$ by the dominated convergence theorem, we only have to show that $\mathcal{K}_a$ is locally p-integrable at ∞ and at 0. First, by [1, 17.3.29], we know that

$$\mathcal{K}_a(s) = \frac{2}{1 + \frac{s}{a}} K\left(\left(\frac{1 - \frac{s}{a}}{1 + \frac{s}{a}}\right)^2\right) \quad (s \in (0, \infty)).$$

Hence, if we put $k_a(s) = \left(\dfrac{1 - \frac{s}{a}}{1 + \frac{s}{a}}\right)^2$ $(s \in (0, \infty))$, we know that $k_a(s) \in (0,1)$ and therefore

$$\mathcal{K}_a(s) = \frac{2}{1 + \frac{s}{a}} K(k_a(s)) \leq \left(\frac{\pi}{2} - \ln(\sqrt{1 - k_a(s)})\right) \frac{2}{1 + \frac{s}{a}} \quad (s \in (0, \infty))$$

by [58, 19.9.1]. But now,

$$\ln(\sqrt{1 - k_a(s)}) \geq C_{1,a} \ln(s) \quad (s \in (0, \varepsilon_a))$$
$$\ln(\sqrt{1 - k_a(s)}) \geq -C_{2,a} \ln(s) \quad (s \in (\Theta_a, \infty))$$

for some $C_{1,a}, C_{2,a} > 0, \varepsilon_a \in (0,1), \Theta_a > 1$. We conclude that for $p \in (1, \infty)$

$$\|\mathcal{K}_a\|_p^p = \left(\int_0^{\varepsilon_a} + \int_{\varepsilon_a}^{\Theta_a} + \int_{\Theta_a}^{\infty}\right) (\mathcal{K}_a(s))^p \, ds$$

$$\leq C_{a,p} + \int_0^{\varepsilon_a} \left(\frac{\pi}{2} - C_{1,a} \ln(s)\right)^p \frac{2^p}{(1 + \frac{s}{a})^p} \, ds + \int_{\Theta_a}^{\infty} \left(\frac{\pi}{2} + C_{2,a} \ln(s)\right)^p \frac{2^p}{(1 + \frac{s}{a})^p} \, ds$$

$$< +\infty$$

where the constant $C_{a,p}$ only depends on ε_a, Θ_a and p. Hence, the function belongs to $L_p(0, \infty)$ for all $p \in (1, \infty)$. $\qquad\square$

Remark 4.2.3 Let $a > 0$. Then, clearly $\mathcal{K}_a \notin L_\infty(0, \infty)$ since $\lim_{\substack{s \to 0 \\ s > 0}} \mathcal{K}_a(s) = +\infty$. Moreover, again by [58, 19.9.1], we have

$$\mathcal{K}_a(s) \geq \frac{2(\ln(4) - \ln(\sqrt{1 - k_a(s)}))}{1 + \frac{s}{a}} \quad (s \in (0, \infty)).$$

Since

$$\ln(\sqrt{1 - k_a(s)}) \leq -C_{3,a} \ln(s) \quad (s \in (\Delta_a, \infty))$$

for some $C_{3,a} > 0, \Delta_a > 1$, we see that

$$\int_0^{\infty} \mathcal{K}_a(s) \, ds \geq 2 \int_{\Delta_a}^{\infty} \frac{\ln(4) + C_{3,a} \ln(s)}{1 + \frac{s}{a}} \, ds = +\infty.$$

Therefore, $\mathcal{K}_a \notin L_1(0, \infty)$.

Lemma 4.2.4 *Let $s, a > 0$. Then, the following statements hold:*

1. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\sqrt{(s^2 + t^2)(t^2 + a^2)}} \, dt = \frac{2}{a} \mathcal{K}_a(s).$

2. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(s - it)(t - ia)} \, dt = \frac{2\pi i}{s + a}.$

Proof. We start with 1. If $s < a$, then the formula is valid by [1, 17.4.41] and the same result implies in the case $s > a$ that

$$K\left(1 - \frac{a^2}{s^2}\right) = \frac{s}{2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{(s^2 + t^2)(t^2 + a^2)}} \, dt.$$

But now, the substitution $\varphi(t) = \sqrt{1 - t^2}$ gives us that whenever $s, a > 0$, then

$$
\begin{aligned}
\frac{1}{a}\mathcal{K}_a(s) &= \frac{1}{a} \int_0^1 \frac{1}{\sqrt{\left(1 - \left(1 - \frac{s^2}{a^2}\right) t^2\right)(1 - t^2)}} \, dt \\
&= \int_0^1 \frac{1}{\sqrt{(a^2(1 - t^2) + s^2 t^2)(1 - t^2)}} \, dt \\
&= \int_0^1 \frac{1}{\sqrt{(a^2 \varphi(t)^2 + s^2(1 - \varphi(t)^2))\varphi(t)^2}} \, dt \\
&= \int_0^1 \frac{1}{\sqrt{(a^2 u^2 + s^2(1 - u^2))(1 - u^2)}} \, du \\
&= \frac{1}{s} \int_0^1 \frac{1}{\sqrt{\left(1 - \left(1 - \frac{a^2}{s^2}\right) u^2\right)(1 - u^2)}} \, du \\
&= \frac{1}{s} K\left(1 - \frac{a^2}{s^2}\right).
\end{aligned}
$$

This concludes the proof of 1.

For 2. we consider the set $\Omega := \{z \in \mathbb{C} : \operatorname{Im}(z) > -s\} \setminus \{ia\}$ and the function $f : \Omega \to \mathbb{C}$ defined by

$$f(z) = \frac{1}{(s - iz)(z - ia)} \quad (z \in \Omega).$$

Then, $f \in H(\Omega)$ and $f(z) = O\left(1/|z|^2\right)$ $(|z| \to +\infty, z \in \Omega)$. Hence, we can apply [54, Theorem 5.2.12] and get

$$\int_{-\infty}^{\infty} \frac{1}{(s - it)(t - ia)} \, dt = 2\pi i \operatorname{Res}(f, ia) = \frac{2\pi i}{s + a}.$$

Here, $\operatorname{Res}(f, ia)$ denotes the residuum of f at ia. $\qquad \square$

In Section 4.1, we already saw that if $F \in \mathcal{H}_p(\mathbb{C}_-)$, then $F|_{i\Pi_+} \in \mathcal{H}_p(i\Pi_+)$. The theory of Chapter 2 gives that if one knows the values of F on the imaginary axis, then one can reconstruct the function F on the whole of $i\Pi_+$, namely by taking the Cauchy integral of $g(i\cdot)$. It is not clear if an analogue holds for Cauchy transforms of measures supported in $[0, \infty)$ since it may happen that the Cauchy integral of the function $(C\mu)(i\cdot)$ does not even exist.

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Basing on the theory of elliptic integrals and under further assumptions on μ we will be able to show two things. First, the Cauchy integral of $(C\mu)(i\cdot)$ is defined, i.e. $(C\mu)(i\cdot) \in \mathscr{C}$. And second: If one knows the values of $C\mu$ on the imaginary axis, one can reconstruct $C\mu$ on the whole of $i\Pi_+$!

Theorem 4.2.5 *Let $\mu \in \mathcal{M}_\infty(\mathbb{R})$ with $\operatorname{supp}(\mu) \subset [0, \infty)$ be such that $|\mu|$ is locally finite and $\ln \in \mathscr{C}(\mu)$. Then $(C\mu)(i\cdot) \in \mathscr{C}$ and*

$$(C\mu)(iz) = C(C\mu(i\cdot))(z) \quad (z \in \Pi_+).$$

Proof. We choose $a > 0, \Theta > 1$ such that $A|\ln(t)| \geq 1$ for all $t \in (\Theta, \infty)$. Then,

$$\int \frac{1}{t+1}\, d|\mu|(t) \leq |\mu|([0, \Theta]) + A \int_\Theta^\infty \frac{|\ln(t)|}{t+1}\, d|\mu|(t) < +\infty$$

which shows that $C\mu$ is well-defined. In the following, we shortly write $h : \mathbb{R} \to \mathbb{C}$, $h(t) = (C\mu)(it)$. If $a > 0$, then the first part of Lemma 4.2.4 gives us that

$$
\begin{aligned}
\int_0^\infty \int_{-\infty}^\infty \frac{1}{|s - it| \cdot |t - ia|}\, dt\, d|\mu|(s) &= \int_0^\infty \int_{-\infty}^\infty \frac{1}{\sqrt{(s^2 + t^2) \cdot (t^2 + a^2)}}\, dt\, d|\mu|(s) \\
&= \frac{2}{a} \int_0^\infty \mathcal{K}_a(s)\, d|\mu|(s).
\end{aligned}
$$

Let $\varepsilon_a, \Theta_a, C_1, C_2 > 0$ be the constants in the proof of Lemma 4.2.2. Since $\ln \in \mathscr{C}(\mu)$, we know that there is some $\varepsilon > 0$ such that

$$\int_0^\varepsilon |\ln(t)|\, d|\mu|(t) < +\infty.$$

If we write $\tilde{\varepsilon} := \min\{\varepsilon, \varepsilon_a\} < 1$ and $\tilde{\Theta} := \max\{\Theta, \Theta_a\} > 1$, then we have

$$
\begin{aligned}
\int_0^\infty \int_{-\infty}^\infty \frac{1}{|s - it| \cdot |t - ia|}\, dt\, d|\mu|(s) &\leq \frac{2}{a} \cdot \left(\int_0^{\tilde{\varepsilon}} \left(\frac{\pi}{2} - C_1 \ln(s) \right) d|\mu|(s) \right. \\
&\quad + |\mu|([\tilde{\varepsilon}, \tilde{\Theta}]) \cdot \max_{s \in [\tilde{\varepsilon}, \tilde{\Theta}]} \left| K\left(1 - \frac{s^2}{a^2} \right) \right| \\
&\quad + \left. \int_{\tilde{\Theta}}^\infty \left(\frac{\pi}{2} + C_2 \ln(s) \right) \frac{1}{(1 + \frac{s}{a})}\, d|\mu|(s) \right) \\
&< +\infty.
\end{aligned}
$$

Hence, the function $h : \mathbb{R} \to \mathbb{C}$, $h(t) = (C\mu)(it)$ belongs to $\mathscr{C}$. An application of Fubini's theorem and the second part of Lemma 4.2.4 now show that

$$(Ch)(ia) = \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{(C\mu)(it)}{t - ia}\, dt = \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{1}{t - ia} \cdot \left(\frac{1}{2\pi i} \int_0^\infty \frac{1}{s - it}\, d\mu(s) \right) dt$$

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$$= \frac{1}{2\pi i} \int_0^\infty \left(\frac{1}{2\pi i} \int_{-\infty}^\infty \frac{1}{(s-it)\cdot(t-ia)} \, dt \right) d\mu(s)$$

$$= \frac{1}{2\pi i} \int_0^\infty \frac{1}{s+a} \, d\mu(s)$$

$$= (C\mu)(i^2 a).$$

Since $a > 0$ was arbitrary we conclude by the identity theorem that $Ch = (C\mu)(i\cdot)$ on Π_+. $\square$

Corollary 4.2.6 *Let $\mu \in \mathcal{M}_\infty(\mathbb{R})$ with $\mathrm{supp}(\mu) \subset [0, \infty)$ be such that $|\mu|$ is locally finite and $\ln \in \mathscr{C}(\mu)$. Then for $p \in (1, \infty)$, the following statements are equivalent:*

a) $(C\mu)(i\cdot)|_{\mathbb{R}} \in L_p(\mathbb{R})$.

b) $C\mu|_{i\Pi_+} \in \mathcal{H}_p(i\Pi_+)$.

Proof. If *b)* holds, then *a)* is valid by Proposition 1.2.3.
Now, suppose that *a)* is fulfilled. Then, by Theorem 4.2.5, we have $(C\mu)(i\cdot) = C(C\mu(i\cdot))$ on Π_+. Theorem 2.2.11 therefore gives us that $(C\mu)(i\cdot) \in \mathcal{H}_p(\Pi_+)$ which is equivalent to $C\mu|_{i\Pi_+} \in \mathcal{H}_p(i\Pi_+)$. $\square$

Remark 4.2.7 Under the assumptions of Corollary 4.2.6 one similarly proves the following equivalence:

a) $(C\mu)(i\cdot)|_{\mathbb{R}} \in H_1(\mathbb{R})$.

b) $C\mu|_{i\Pi_+} \in \mathcal{H}_1(i\Pi_+)$.

An application of Minkowski's and Jensen's inequality gives simple sufficient conditions when the Cauchy transform of a measure belongs to $\mathcal{H}_p(i\Pi_+)$.

Proposition 4.2.8 *Let $\mu \in \mathcal{M}_\infty(\mathbb{R})$ with $\mathrm{supp}(\mu) \subset [0, \infty)$ and $p \in (1, \infty)$.*

1. *If $|\mu|$ is σ-finite and*
$$\int \frac{1}{t^{\frac{p-1}{p}}} \, d|\mu|(t) < +\infty,$$
   then $C\mu|_{i\Pi_+} \in \mathcal{H}_p(i\Pi_+)$.

2. *If μ is finite and*
$$\int \frac{1}{t^{p-1}} \, d|\mu|(t) < +\infty,$$
   then $C\mu|_{i\Pi_+} \in \mathcal{H}_p(i\Pi_+)$.

Proof. Fix $x \notin \mathrm{supp}(\mu)$. For 1. notice that by Minkowski's inequality for integrals (see, e.g., [75, A.1])

$$\left(\int_{-\infty}^\infty |(C\mu)(x+iy)|^p \, dy \right)^{\frac{1}{p}} \leq \left(\int_{-\infty}^\infty \left(\int \frac{1}{((x-t)^2 + y^2)^{\frac{1}{2}}} \, d|\mu|(t) \right)^p dy \right)^{\frac{1}{p}}$$

$$\leq \int \left(\int_{-\infty}^{\infty} \frac{1}{((x-t)^2 + y^2)^{\frac{p}{2}}} \, dy \right)^{\frac{1}{p}} d|\mu|(t)$$

$$= K_p \int \frac{1}{|x-t|^{\frac{p-1}{p}}} \, d|\mu|(t)$$

with some constant $K_p > 0$ only depending on p. Therefore,

$$\sup_{x<0} \int_{-\infty}^{\infty} |(C\mu)(x+iy)|^p \, dy \leq \sup_{x<0} K_p \int \frac{1}{|x-t|^{\frac{p-1}{p}}} \, d|\mu|(t) = \int \frac{1}{t^{\frac{p-1}{p}}} \, d|\mu|(t)$$

by the monotone convergence theorem. Part 2. is an application of Jensen's inequality. For $x \notin \operatorname{supp}(\mu)$ it holds by Fubini's theorem that

$$\int_{-\infty}^{\infty} |(C\mu)(x+iy)|^p \, dy \quad \leq \quad (|\mu|(\mathbb{R}))^{p-1} \int_{-\infty}^{\infty} \int \frac{1}{((x-t)^2+y^2)^{\frac{p}{2}}} \, d|\mu|(t) \, dy$$

$$= \quad (|\mu|(\mathbb{R}))^{p-1} \int \int_{-\infty}^{\infty} \frac{1}{((x-t)^2+y^2)^{\frac{p}{2}}} \, dy \, d|\mu|(t)$$

$$= \quad (|\mu|(\mathbb{R}))^{p-1} K_p \int \frac{1}{|x-t|^{p-1}} \, d|\mu|(t)$$

with some constant $K_p > 0$ only depending on p. Again, we conclude by the monotone convergence theorem. $\qquad \square$

Remark 4.2.9   1. Proposition 4.2.8 holds mutatis mutandis if we replace the interval $[0, \infty)$ by any other unbounded interval which is not the entire real axis.

   2. Proposition 4.2.8 is false for $p = 1$. Because then, the stated conditions reduce to the condition that μ is finite. We may take $\mu = f\lambda_1$ where f is defined by

$$f(t) = \frac{1}{\sqrt{t}(t+1)} \mathbb{1}_{(0,\infty)}(t) \quad (t \in \mathbb{R}).$$

Then, we have (see Example B.7)

$$(C\mu)(z) = \frac{1}{2i(-z + \sqrt{-z})} \quad (z \in \mathbb{C}_-).$$

If $C\mu|_{i\Pi_+}$ belonged to the space $\mathcal{H}_1(i\Pi_+)$, then we would have

$$\lim_{x \to 0} \int_{-\infty}^{\infty} |(C\mu)(x+iy)| \, dy = \sup_{x<0} \int_{-\infty}^{\infty} |(C\mu)(x+iy)| \, dy < +\infty.$$

Note that the limit and the supremum in general only coincide if the function belongs to $\mathcal{H}_1(i\Pi_+)$. But by Fatou's lemma, we know that

$$\lim_{x \to 0} \int_{-\infty}^{\infty} |(C\mu)(x+iy)| \, dy \geq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{|-iy + \sqrt{-iy}|} \, dy.$$

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Since $|\sqrt{-iy}| = \sqrt{y}$ for $y > 0$, we see that

$$\lim_{x \to 0} \int_{-\infty}^{\infty} |(C\mu)(x + iy)| \, dy \geq \int_{0}^{\infty} \frac{1}{(y + \sqrt{y})} \, dy = +\infty$$

which is a contradiction.

3. Note that the conditions in Proposition 4.2.8 are only sufficient, but not necessary. Take for example $\mu = f\lambda_1$ where

$$f(t) = \frac{1}{\sqrt{t}|\ln(t)|} \mathbb{1}_{(0,\frac{1}{2})}(t) \quad (t \in \mathbb{R}).$$

Then, μ is finite since $f \in L_1(\mathbb{R})$, the function $C\mu|_{i\Pi_+}$ belongs to $\mathcal{H}_2(i\Pi_+)$ by Theorem 4.1.9 since $f \in L_2(0, \infty)$, but clearly

$$\int \frac{1}{\sqrt{t}} \, d\mu(t) = \int_{0}^{\frac{1}{2}} \frac{1}{t|\ln(t)|} \, dt = +\infty.$$

4. The second part of Proposition 4.2.8 is false if the measure is not finite. Take for example $\mu = f\lambda_1$ where f is defined by

$$f(t) = \frac{\sqrt{t}}{t+1} \mathbb{1}_{(0,\infty)}(t) \quad (t \in \mathbb{R}).$$

Then,

$$\int \frac{1}{t} \, d|\mu|(t) = \int_{0}^{\infty} \frac{1}{\sqrt{t}(t+1)} \, dt < +\infty.$$

One calculates (see Example B.8) that

$$(C\mu)(z) = \frac{1 - \sqrt{-z}}{2i(z+1)} \quad (z \in \mathbb{C}_-).$$

If $C\mu|_{i\Pi_+}$ would belong to $\mathcal{H}_2(i\Pi_+)$, then its boundary function towards the imaginary axis must belong to $L_2(\mathbb{R})$. But

$$F(y) := \lim_{x \to 0}(C\mu)(x + iy) = \frac{1 - \sqrt{-iy}}{2i(iy + 1)} \quad (y \in \mathbb{R} \setminus \{0\})$$

and thus

$$\int_{-\infty}^{\infty} |F(y)|^2 \, dy \geq \int_{0}^{\infty} \frac{(1 - \frac{\sqrt{y}}{\sqrt{2}})^2 + \frac{y}{2}}{4(y^2 + 1)} \, dy = +\infty.$$

Hence $C\mu|_{i\Pi_+} \notin \mathcal{H}_2(i\Pi_+)$. Notice that

$$\int \frac{1}{\sqrt{t}} \, d\mu(t) = \int_{0}^{\infty} \frac{1}{t+1} \, dt = +\infty.$$

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Example 4.2.10   1. Consider the measure $\mu = f\lambda_1$ where

$$f(t) = \frac{t}{\sqrt{1-t^2}} \mathbb{1}_{(0,1)}(t) \quad (t \in \mathbb{R}).$$

Since $f \in L_1(\mathbb{R})$, we know that μ is finite and because

$$\int \frac{1}{t} \, d\mu(t) = \int_0^1 \frac{1}{\sqrt{1-t^2}} \, dt = \frac{\pi}{2},$$

we know that $C\mu|_{i\Pi_+} \in \mathcal{H}_2(i\Pi_+)$. Note that $f \notin L_2(0,1)$ and hence Theorem 4.1.9 does not apply.

2. Let $\mu = f\#$ where $\#$ is the counting measure on $\mathbb{R}$ and

$$f(t) = \frac{t}{(t+1)^3} \mathbb{1}_{\mathbb{N}}(t) \quad (t \in \mathbb{R}).$$

Then,

$$\int \frac{1}{t^{p-1}} \, d\mu(t) = \sum_{k=1}^{\infty} \frac{k^{2-p}}{(k+1)^3}$$

and hence $C\mu|_{i\Pi_+} \in \mathcal{H}_p(i\Pi_+)$ for all $p \in (1, \infty)$.

4.3   Real conditions and characterizations

The main idea that we present in this section goes back to Widder, see [82]. It relies on particular differential operators on the negative half-axis and will deliver a characterization for Cauchy transforms of functions in $L_p(0, \infty)$ by only regarding the restriction of Cf to $(-\infty, 0)$. Before stating this result, we need some further notation. For $n \in \mathbb{N}_0$, let $P_n(x) := x^n$ $(x \in \mathbb{R})$ and for $n \in -\mathbb{N}$, let $P_n(x) := x^n$ $(x \in (-\infty, 0))$.

Definition 4.3.1 Let $k \in \mathbb{N}$ with $k \geq 2$. Then, we define the **Widder differential operator** W_k by

$$W_k g := \frac{(-P_1)^{k-1}}{k!(k-2)!} [P_k g]^{(2k-1)} \quad (g \in C^{\infty}(-\infty, 0)).$$

Clearly, each W_k is a linear operator from $C^{\infty}(-\infty, 0)$ to $C^{\infty}(-\infty, 0)$. As a matter of fact, its kernel given by $\mathrm{span}(\{P_n : n \in \{-k, \ldots, k-2\}\})$ for each $k \geq 2$, cf. [82], p. 381. Hence, W_k is not injective for $k \geq 2$.

In his monograph on the Laplace Transform (see [82]), Widder actually defined these operators on the space $C^{\infty}(0, \infty)$. However, these operators only fit for the Stieltjes transform which is the Cauchy transform with a change of variable. This is why we use a modified version of these original operators.

Our first proposition is kind of a real-nature recipe how to reconstruct a function from its Cauchy transform. It should be remarked that Widder stated an analogue of this result with respect to the original operators. Since it is not clear at first sight if the result holds for our modified operators, we give a detailed proof.

Proposition 4.3.2 *Let $f \in \mathcal{L}_{1,\text{loc}}(0,\infty) \cap \mathcal{C}$. Then*

$$\lim_{k \to \infty} [W_k(Cf)](-t) = f(t)$$

for almost every positive t.

Proof. Due to [82, Chapter VIII, Theorem 9 and Corollary 9.1] and the product rule it holds for almost every positive t that

$$
\begin{aligned}
f(t) &= \lim_{k \to \infty} \frac{(-t)^{k-1}}{k!(k-2)!} \sum_{l=0}^{2k-1} \binom{2k-1}{l} (P_k)^{(l)}(t) \cdot [Cf(-\cdot)]^{(2k-1-l)}(t) \\
&= \lim_{k \to \infty} \frac{(-t)^{k-1}}{k!(k-2)!} \sum_{l=0}^{2k-1} (-1)^{2k-1-l} \cdot \binom{2k-1}{l} (P_k)^{(l)}(t) \cdot (Cf)^{(2k-1-l)}(-t)
\end{aligned}
$$

But now, as

$$(P_k)^{(l)}(x) = (-1)^{k+l}(P_k)^{(l)}(-x) \quad (x \in \mathbb{R}, k, l \in \mathbb{N}_0),$$

we see that

$$
\begin{aligned}
f(t) &= \lim_{k \to \infty} (-1)^{k-1} \frac{(-t)^{k-1}}{k!(k-2)!} \sum_{l=0}^{2k-1} \binom{2k-1}{l} (P_k)^{(l)}(-t) \cdot (Cf)^{(2k-1-l)}(-t) \\
&= \lim_{k \to \infty} \frac{t^{k-1}}{k!(k-2)!} \sum_{l=0}^{2k-1} \binom{2k-1}{l} (P_k)^{(l)}(-t) \cdot (Cf)^{(2k-1-l)}(-t) \\
&= \lim_{k \to \infty} [W_k(Cf)](-t)
\end{aligned}
$$

for almost every positive t. $\qquad\square$

Remark and Definition 4.3.3 Let $p \in (1,\infty)$. We write $C_p^\infty(-\infty,0)$ for the set of all functions $g \in C^\infty(-\infty,0)$ satisfying

a) $\lim_{x \to 0^-} xg(x) = 0$,

b) $\sup_{k \geq 2} \int_{-\infty}^0 |W_k g|^p < +\infty$.

In contrast, we set $C_1^\infty(-\infty,0)$ as the space of all functions $g \in C^\infty(-\infty,0)$ satisfying

a) $\lim_{x \to 0^-} xg(x) = 0$,

b) $W_k g \in L_1(-\infty,0) \quad (k \in \mathbb{N})$,

c) $\lim_{k,l \to \infty} \int_{-\infty}^0 |W_k g - W_l g| = 0$.

We are now ready to extend our characterization of Cauchy transforms, basing on the results in [82].

Proposition 4.3.4 *Let $p \in (1, \infty)$. Then*

$$\mathcal{H}_p(\mathbb{C}_-) = \{F \in H(\mathbb{C}_-) : F|_{(-\infty,0)} \in C_p^\infty(-\infty, 0)\}.$$

In other words, for a function $F \in H(\mathbb{C}_-)$ the following are equivalent:

   a) There exists some $f \in L_p(0, \infty)$ such that $F = Cf$.

   b) $F|_{(-\infty,0)} \in C_p^\infty(-\infty, 0)$.

   c) $F \in \mathcal{H}_p(\mathbb{C}_-)$.

Proof. The equivalence of *a)* and *c)* is exactly Proposition 3.2.2. By [82, Chapter VIII, Theorem 9a], condition *b)* is necessary and sufficient that

$$F(x) = (Cf)(x) \quad (x \in (-\infty, 0))$$

for some $f \in L_p(0, \infty)$. Hence, it is clear that *a)* implies *b)* and conversely *b)* implies *a)* by the identity theorem since both functions are analytic in $\mathbb{C}_-$. $\qquad\square$

Proposition 4.3.5 *For a function $F \in H(\mathbb{C}_-)$ the following statements are equivalent:*

   a) There exists some $f \in L_1(0, \infty)$ such that $F = Cf$.

   b) $F|_{(-\infty,0)} \in C_1^\infty(-\infty, 0)$.

Proof. By [82, Chapter VIII, Theorem 21], condition *b)* is necessary and sufficient that

$$F(x) = (Cf)(x) \quad (x \in (-\infty, 0))$$

for some $f \in L_1(0, \infty)$. Hence, it is clear that *a)* implies *b)* and conversely *b)* implies *a)* by the identity theorem since both functions are analytic in $\mathbb{C}_-$. $\qquad\square$

We saw earlier that the Cauchy transform is injective on $L_p(0, \infty)$ for each $p \in [1, \infty)$. Due to [82, Chapter VIII, Theorem 19b and Corollary 21] we even have

Proposition 4.3.6 (Widder, [82]) *Let $p \in [1, \infty)$. Then,*

$$\left(\lim_{k \to \infty} \int_{-\infty}^0 |W_k(Cf)|^p \right)^{\frac{1}{p}} = \|f\|_p \quad (f \in L_p(0, \infty)).$$

Remark and Definition 4.3.7 Let $p \in (1, \infty)$. Then for $F \in \mathcal{H}_p(\mathbb{C}_-)$, we set

$$\||F\||_{\mathcal{H}_p(\mathbb{C}_-)} := \lim_{k \to \infty} \left(\int_{-\infty}^0 |W_k F|^p \right)^{\frac{1}{p}}.$$

One easily verifies that this defines a norm on $\mathcal{H}_p(\mathbb{C}_-)$: Proposition 3.2.2 and Proposition 4.3.6 assure that $\||F\||_{\mathcal{H}_p(\mathbb{C}_-)} < +\infty$. Moreover, again by Proposition 3.2.2 we have $\||F\||_{\mathcal{H}_p(\mathbb{C}_-)} = 0$ if and only if $F = 0$. Hence, for $p \in (1, \infty)$, the Cauchy transformation is an isometric isomorphism between $L_p(0, \infty)$ and $(\mathcal{H}_p(\mathbb{C}_-), \||\cdot\||_{\mathcal{H}_p(\mathbb{C}_-)})$. Therefore, there are constants $K_{p,1}, K_{p,2} > 0$ only depending on p such that

$$\|F\|_{\mathcal{H}_p(\mathbb{C}_-)} \le K_{p,1} \||F\||_{\mathcal{H}_p(\mathbb{C}_-)} \le K_{p,2} \|F\|_{\mathcal{H}_p(\mathbb{C}_-)} \quad (F \in \mathcal{H}_p(\mathbb{C}_-)).$$

In particular, both norms are equivalent and complete.

Chapter 5

Cauchy Transforms of Functions and Measures with Compact Support

> Mathematics is the science which uses easy words for hard ideas.
>
> ———————————————
>
> Edward Kasner

The research on Cauchy transforms of compactly supported measure has a long history and wide applications. For example, in polynomial and rational approximation one can use Cauchy transforms in combination with the Hahn-Banach theorem to show the famous theorem of Hartogs-Rosenthal, see [10, Chapter 18, Theorem 6.3]. In this chapter, we will not treat the general case of compactly supported measures since this has been discussed in literature (see [27]). Instead we shall focus on particular measures with compact support, namely measures of the form $f\mu$, where f is a compactly supported measurable function and μ is a Borel measure on $\mathbb{R}$.

A natural case is of course $\mu = \lambda_1$ as it will be in the first section where we establish a complex and a geometric condition similar to Chapter 4. Afterwards, we turn towards a functional-analytic approach for a representation theorem. Unsurprisingly, the Hahn-Banach theorem will come into play here and by using Runge's theorem we will also connect Cauchy transforms and approximation theory in another way. In the last section of this chapter, we will derive a representation theorem for weighted L_p spaces where we consider measures which are generalizations of the arcsine distribution. Together with the theory of Jacobi polynomials and orthonormal bases we will also give another theorem of Paley-Wiener type basing on Cauchy transforms.

5.1 Complex and geometric characterizations

Since functions of compact support appear as a special case of functions supported on an unbounded interval, one could expect that some elements and results of Chapter 4 should transfer in a suitable way to the compact case. For the complex condition regarding Fourier-

Laplace transforms and for the geometric condition in Theorem 4.1.14 basing on unbounded polygonal chains this is indeed the case. We shall start with the approach using Fourier-Laplace transforms. Therefore, we put our focus first on special compact sets on $\mathbb{R}$, namely symmetric intervals of the form $[-A, A]$ where $A > 0$. In this setting, certain entire functions play an important role.

Remark and Definition 5.1.1 Let $A > 0$. An entire function F is said to be of class $\mathrm{Exp}(A)$ if there is a constant $C > 0$ such that

$$|F(z)| \leq Ce^{A|z|} \quad (z \in \mathbb{C}).$$

If $F \in \mathrm{Exp}(A)$ for some $A > 0$, then F is also called a function of **exponential type**. Moreover, for $p \in [1, \infty)$, let $\mathrm{Exp}_p(A)$ denote the class of all $F \in \mathrm{Exp}(A)$ such that $F|_{\mathbb{R}} \in L_p(\mathbb{R})$. The space $\mathrm{Exp}_p(A)$ is a Banach space when equipped with the norm $\|\cdot\|_p$ (see [2], p. 202).

One of the main proceedings in the theory of functions of exponential type are the following results (see, e.g., [40], p. 132-135, and [48, Theorem 2]).

Theorem 5.1.2 (Paley-Wiener, [40]) *Let $A > 0$.*

1. *An entire function F belongs to the class $\mathrm{Exp}_2(A)$ if and only if there is some $f \in L_2(-A, A)$ such that*

$$F(z) = (Lf)(z) = \frac{1}{2\pi} \int_{-A}^{A} f(t)e^{izt}\, dt \quad (z \in \mathbb{C}).$$

In this case, f is unique.

2. *An entire function F belongs to the class $\mathrm{Exp}_1(A)$ if and only if there is some $f \in L_1(-A, A) \cap \{f \in L_1(\mathbb{R}) : \widehat{f} \in L_1(\mathbb{R})\}$ such that*

$$F(z) = (Lf)(z) \quad (z \in \mathbb{C}).$$

In all cases, f is unique.

Theorem 5.1.3 (Paley-Wiener, [48]) *Let $A > 0$ and $p \in (1, 2)$. Then an entire function F belongs to the class $\mathrm{Exp}_p(A)$ if and only if*

$$F(z) = (Lf)(z) \quad (z \in \mathbb{C})$$

with $f \in L_q(-A, A)$ where q is the conjugate exponent of p and

$$\sum_{n=-\infty}^{\infty} \left| \frac{1}{2A} \int_{-A}^{A} f(t)e^{-in\pi t/A}\, dt \right|^q < +\infty.$$

In this case, f is unique.

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Since every $F \in \text{Exp}_p(A)$ is bounded on the real axis (see [2], p. 217) we can consider the Fourier-Laplace transform $L(F|_\mathbb{R})$ of $F|_\mathbb{R}$. Analogously, to the case of unbounded intervals this leads to another characterization of Cauchy transforms.

Proposition 5.1.4 *Let $A > 0$ and $F \in H(\mathbb{C} \setminus [-A, A])$. Then $F \in \mathcal{H}_2(\mathbb{C} \setminus [-A, A])$ if and only if there is $G \in \text{Exp}_2(A)$ such that*

$$F(z) = L(G|_\mathbb{R})(z) \quad (z \in \Pi_+).$$

In this case, G is unique.

Proof. This follows from Proposition 2.3.7 together with Theorem 5.1.2. $\qquad\square$

Remark 5.1.5 *Let $p \in (1, 2)$ and $F \in \text{Exp}_p(A)$. Then, by Proposition 2.3.7 and Theorem 5.1.3 we know that LF has an extension to a function in $\mathcal{H}_q(\mathbb{C} \setminus [-A, A])$ where q is the conjugate exponent of p.*

Similarly to Proposition 5.1.4

Proposition 5.1.6 *Let $A > 0$ and $F \in H(\mathbb{C} \setminus [-A, A])$. Then, the following statements are equivalent:*

a) *There exists $f \in L_1(-A, A) \cap \{f \in L_1(\mathbb{R}) : \hat{f} \in L_1(\mathbb{R})\}$ such that $F = Cf$.*

b) *There exists $G \in \text{Exp}_1(A)$ such that $F = L(G|_\mathbb{R})$ on Π_+.*

In this case, both f and G are unique.

We are now looking for an analogue of Theorem 4.1.14 where we considered unbounded polygonal chains surrounding the unbounded interval $[0, \infty)$. In the case of compact sets it seems more natural to consider bounded polygonal chains since the set of singularities of the Cauchy transform is now bounded itself.

Remark 5.1.7 For $A, B \in \mathbb{R}$ with $A \leq B$ and $C \in [0, \infty)$ we consider the chain $\tilde{R}_{A,B,C} = (\tilde{R}_{A,B,C,1}, \tilde{R}_{A,B,C,2}, \tilde{R}_{A,B,C,3}, \tilde{R}_{A,B,C,4})$ where

$$\tilde{R}_{A,B,C,1} : [A, B] \to \mathbb{C}, \ \tilde{R}_{A,B,C,1}(t) = t + iC$$
$$\tilde{R}_{A,B,C,2} : [-C, C] \to \mathbb{C}, \ \tilde{R}_{A,B,C,2}(t) = A + it$$
$$\tilde{R}_{A,B,C,3} : [A, B] \to \mathbb{C}, \ \tilde{R}_{A,B,C,3}(t) = t - iC$$
$$\tilde{R}_{A,B,C,4} : [-C, C] \to \mathbb{C}, \ \tilde{R}_{A,B,C,4}(t) = B + it.$$

Then, $\tilde{R}_{A,B,C}$ belongs to $\mathcal{P}_4(\mathbb{C} \setminus [A, B])$. For a function $g : \tilde{R}^*_{A,B,C} \to \mathbb{C}$ such that $(g \circ \tilde{R}_{A,B,C,1}), (g \circ \tilde{R}_{A,B,C,3}) \in L_1(A, B)$ and $(g \circ \tilde{R}_{A,B,C,2}), g \circ \tilde{R}_{A,B,C,4} \in L_1(-C, C)$, we have

$$\int_{\tilde{R}_{A,B,C}} g(\zeta) \, d\zeta = \int_A^B g(t + iC) + g(t - iC) \, dt + i \int_{-C}^C g(A + it) + g(B + it) \, dt.$$
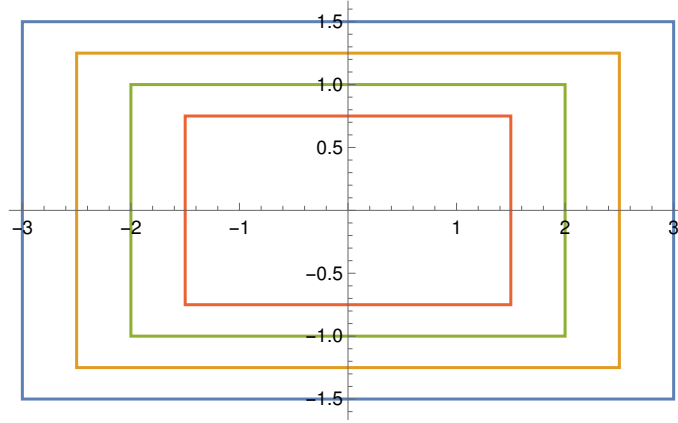
70

Figure 5.1: Plots of $\tilde{R}_{-3,3,\frac{3}{2}}$ (blue), $\tilde{R}_{-\frac{5}{2},\frac{5}{2},\frac{5}{4}}$ (yellow), $\tilde{R}_{-2,2,1}$ (green) and $\tilde{R}_{-\frac{3}{2},\frac{3}{2},\frac{3}{4}}$ (orange).

These chains $\tilde{R}_{A,B,C}$ can be seen as model for bounded polygonal chains in $\mathbb{C}_\infty \setminus K$ where $K \subset \mathbb{R}$ is compact. This result is an analogue of Theorem 4.1.14.

Theorem 5.1.8 *Let $p \in [1, \infty), K \subset \mathbb{R}$ be compact and $M := \min\limits_{x \in K} x, N := \max\limits_{x \in K} x$. For a function $F \in H(\mathbb{C}_\infty \setminus K)$, the following statements are equivalent:*

a) $F \in \mathcal{H}_p(\mathbb{C}_\infty \setminus K)$.

b) $\|F\|_{\mathbb{C}_\infty \setminus K, p} := \sup \left\{ \left(\int_{\tilde{R}_{M-a, N+a, b}} |F(\zeta)|^p \, |d\zeta| \right)^{\frac{1}{p}} : a \geq b > 0 \right\} < +\infty.$

c) For each $n \geq 2$ we have

$$\|F\|_{\mathbb{C}_\infty \setminus K, p, n} := \sup \left\{ \left(\int_\gamma |F(\zeta)|^p \, |d\zeta| \right)^{\frac{1}{p}} : \gamma \in \mathcal{P}_n(\mathbb{C} \setminus K) \right\} < +\infty.$$

In this case, each line in b) and c) defines a norm which is equivalent to $\| \cdot \|_{\mathcal{H}_p(\mathbb{C}_\infty \setminus K)}$. More precisely, we have for each $n \geq 2$

$$\|F\|_{\mathbb{C}_\infty \setminus K, p} \leq (2 + 4B_p)^{\frac{1}{p}} \|F\|_{\mathcal{H}_p(\mathbb{C}_\infty \setminus K)} \leq 2(2 + 4B_p)^{\frac{1}{p}} \|F\|_{\mathbb{C}_\infty \setminus K, p}$$

and

$$\|F\|_{\mathbb{C}_\infty \setminus K, p, n} \leq (2n\sqrt{2}B_p)^{\frac{1}{p}} \|F\|_{\mathcal{H}_p(\mathbb{C}_\infty \setminus K)} \leq 2(2n\sqrt{2}B_p)^{\frac{1}{p}} \|F\|_{\mathbb{C}_\infty \setminus K, p, n}.$$

Proof. If *a)* holds and $\gamma \in \mathcal{P}_n(\mathbb{C} \setminus K)$, then we have

$$\int_\gamma |F(\zeta)|^p \, |d\zeta| \leq n\sqrt{2}B_p(\|F\|_{\mathcal{H}_p(\Pi_+)}^p + \|F\|_{\mathcal{H}_p(\Pi_-)}^p)$$

as in the proof of Theorem 4.1.14. Clearly, $c)$ implies $b)$ by Remark 5.1.7. Let us suppose that $b)$ is true and fix $y > 0$. Then, by the monotone convergence theorem

$$\int_{-\infty}^{\infty} |F(x + iy)|^p \, dx = \lim_{a \to \infty} \int_{M-a}^{N+a} |F(x + iy)|^p \, dx$$

$$\int_{-\infty}^{\infty} |F(x - iy)|^p \, dx = \lim_{a \to \infty} \int_{M-a}^{N+a} |F(x - iy)|^p \, dx.$$

Let us fix $a \geq y$. By our assumption,

$$\int_{M-a}^{N+a} |F(x + iy)|^p \, dx \leq \int_{\tilde{R}_{M-a,N+a,y}} |F(\zeta)|^p \, |d\zeta| \leq \|F\|_{\mathbb{C}_\infty \setminus K, p}$$

$$\int_{M-a}^{N+a} |F(x - iy)|^p \, dx \leq \int_{\tilde{R}_{M-a,N+a,y}} |F(\zeta)|^p \, |d\zeta| \leq \|F\|_{\mathbb{C}_\infty \setminus K, p}$$

and we conclude by letting $a \to \infty$ that $a)$ is true.

Finally, for $F \in \mathcal{H}_p(\mathbb{C}_\infty \setminus K)$, we have

$$
\begin{aligned}
\|F\|_{\mathbb{C}_\infty \setminus K, p} &\leq \left(\|F\|^p_{\mathcal{H}_p(\Pi_+)} + \|F\|^p_{\mathcal{H}_p(\Pi_-)} + \|F\|^p_{\mathcal{H}_p(M+i\Pi_+)} + \|F\|^p_{\mathcal{H}_p(N+i\Pi_-)} \right)^{\frac{1}{p}} \\
&\leq (1 + 2B_p)^{\frac{1}{p}} \left(\|F\|^p_{\mathcal{H}_p(\Pi_+)} + \|F\|^p_{\mathcal{H}_p(\Pi_-)} \right)^{\frac{1}{p}} \\
&\leq (2 + 4B_p)^{\frac{1}{p}} \|F\|_{\mathcal{H}_p(\mathbb{C}_\infty \setminus K)} \\
&\leq (2 + 4B_p)^{\frac{1}{p}} (\|F\|_{\mathcal{H}_p(\Pi_+)} + \|F\|_{\mathcal{H}_p(\Pi_-)}) \\
&\leq 2(2 + 4B_p)^{\frac{1}{p}} \|F\|_{\mathbb{C}_\infty \setminus K, p}
\end{aligned}
$$

and

$$
\begin{aligned}
\|F\|_{\mathbb{C}_\infty \setminus K, p, n} &\leq (n\sqrt{2}B_p)^{\frac{1}{p}} \left((\|F\|^p_{\mathcal{H}_p(\Pi_+)} + \|F\|^p_{\mathcal{H}_p(\Pi_-)}) \right)^{\frac{1}{p}} \\
&\leq (2n\sqrt{2}B_p)^{\frac{1}{p}} \|F\|_{\mathcal{H}_p(\mathbb{C}_\infty \setminus K)} \\
&\leq 2(2n\sqrt{2}B_p)^{\frac{1}{p}} \|F\|_{\mathbb{C}_\infty \setminus K, p} \\
&\leq 2(2n\sqrt{2}B_p)^{\frac{1}{p}} \|F\|_{\mathbb{C}_\infty \setminus K, p, n}.
\end{aligned}
$$

This concludes the proof. $\qquad\square$

Remark 5.1.9 Theorem 5.1.8 is of course valid for $p = \infty$ if we replace the conditions b) and c) by

b) $\|F\|_{\mathbb{C}_\infty \setminus K, \infty} := \sup \left\{ \|F\|_{\infty, (\tilde{R}_{a,b})^*} : a \geq b > 0 \right\} < +\infty.$

c) For each $n \geq 2$ we have

$$\|F\|_{\mathbb{C}_\infty \setminus K, \infty, n} := \sup \{ \|F\|_{\infty, \gamma^*} : \gamma \in \mathcal{P}_n(\mathbb{C}_\infty \setminus K) \} < +\infty.$$

Remark 5.1.10 In Section 4.1 we already mentioned a result in [27] which characterizes Cauchy transforms of compactly supported measures. More precisely, if $K \subset \mathbb{C}$ is compact, then a function $F \in H(\mathbb{C}_\infty \setminus K)$ is the Cauchy transform of some $\mu \in \mathcal{M}(K)$ if and only if there is a (so-called) admissible grid $\mathcal{R}$ (a collection of open rectangles satisfying a certain measure-theoretic condition) such that

$$\int_{\partial R} |F(\zeta)| \, |d\zeta| < +\infty, \quad -\frac{1}{2\pi i} \int_{\partial R} F(\zeta) \, d\zeta = \mu(R)$$

for all $R \in \mathcal{R}$. In Theorem 5.1.8 we can consider Cauchy transforms of measures of the form $f\lambda_1$ where $f \in L_p(K)$ (if $p > 1$) or $f \in H_1(K)$ for a compact set $K \subset \mathbb{R}$. These special cases lead to far more restrictive characterizations of the form that there exists a constant $M > 0$ with

$$\int_{\partial R} |F(\zeta)|^p \, |d\zeta| \leq M < +\infty$$

for **all** rectangles R in $\mathbb{C} \setminus K$. In particular, if $f \in H_1(K)$, then

$$\sup_{\substack{R \subset \mathbb{C} \setminus K \\ R \text{ rectangle}}} \int_{\partial R} |(Cf)(\zeta)| \, |d\zeta| < +\infty.$$

In general, Cauchy transforms of compactly supported measures do not need to satisfy this condition. Take for example μ as the arcsine distribution, i.e. $\mu = f\lambda_1$ with

$$f(t) = \frac{1}{\pi\sqrt{1 - t^2}} \mathbb{1}_{(-1,1)}(t) \quad (t \in \mathbb{R}).$$

Then, we have (see Example B.4)

$$(C\mu)(z) = -\frac{1}{2\pi i \sqrt{z^2 - 1}} \quad (z \in \mathbb{C} \setminus [-1, 1]).$$

If $C\mu$ would satisfy this uniform estimation above, then we would especially have

$$\lim_{x \to 1} \int_{-\infty}^{\infty} |(C\mu)(x + iy)| \, dy = \sup_{x > 1} \int_{-\infty}^{\infty} |(C\mu)(x + iy)| \, dy < +\infty.$$

But by Fatou's lemma,

$$\lim_{x \to 1} \int_{-\infty}^{\infty} |(C\mu)(x + iy)| \, dy \geq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{|y^2 - 2iy|^{\frac{1}{2}}} \, dy = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{|y|^{\frac{1}{2}} (y^2 + 4)^{\frac{1}{4}}} \, dy = +\infty$$

which is a contradiction. Notice that $f \notin H_1([-1, 1])$ since $\widehat{f}(0) = \int f = 1 \neq 0$.

The question arises if one could replace the bounded polygonal chains in the previous theorem by other cycles, for example circles or ellipses. In general, this is not possible.

Proposition 5.1.11 *Let $K \subset \mathbb{R}$ be compact, $\mu \in \mathcal{M}(K)$ and $M := \max_{x \in K} |x|$. Then, the following statements are equivalent:*

a) $\sup\limits_{R>M} \int_{k_R(0)} |(C\mu)(\zeta)|^2 \, |d\zeta| = \lim\limits_{R\to M} \int_{k_R(0)} |(C\mu)(\zeta)|^2 \, |d\zeta| < +\infty.$

b) $\sum\limits_{k=0}^{\infty} \dfrac{|m_k(\mu)|^2}{M^{2k}} < +\infty.$

Proof. Fix $R > M$. Then, by Proposition 3.1.4

$$\int_{k_R(0)} |(C\mu)(\zeta)|^2 \, |d\zeta| = \frac{R}{4\pi^2} \int_{-\pi}^{\pi} \left| \sum_{k=0}^{\infty} m_k(\mu) \cdot (Re^{it})^{-(k+1)} \right|^2 dt$$

$$= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \left| \sum_{k=0}^{\infty} m_k(\mu) R^{-k} e^{ikt} \right|^2 dt.$$

If we set

$$c_k(R) := \begin{cases} R^{-k} m_k(\mu), & k \in \mathbb{N}_0 \\ 0, & k < 0 \end{cases}$$

then

$$\sum_{k=-\infty}^{\infty} |c_k(R)|^2 = \sum_{k=0}^{\infty} |c_k(R)|^2 \leq \|\mu\| \sum_{k=0}^{\infty} \left(\frac{M}{R}\right)^k < +\infty.$$

By the Riesz-Fischer theorem (see [68, 4.26]) we know that there is a unique $g_R \in L_2(-\pi, \pi)$ such that

$$\widehat{g_R}(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{-ikt} \, dt = c_k(R) \quad (k \in \mathbb{Z}).$$

Hence, by Parseval's formula

$$\int_{k_R(0)} |(C\mu)(\zeta)|^2 \, |d\zeta| = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \left| \sum_{k=0}^{\infty} \widehat{g_R}(k) e^{ikt} \right|^2 dt = \frac{1}{4\pi^2} \sum_{k=0}^{\infty} |c_k(R)|^2$$

$$= \frac{1}{4\pi^2} \sum_{k=0}^{\infty} \frac{|m_k(\mu)|^2}{R^{2k}}.$$

This concludes the proof. $\qquad\square$

Example 5.1.12 If $K = [-1, 1]$ and $f(t) = \dfrac{1}{\pi\sqrt{1 - t^2}} \mathbb{1}_{(-1,1)}(t)$, then

$$\sum_{n=0}^{\infty} \left| \int_{-1}^{1} x^n f(x) \, dx \right|^2 = \sum_{n=0}^{\infty} \binom{2n}{n}^2 \left(\frac{1}{2}\right)^{4n} = \infty,$$

hence

$$\int_{k_R(0)} |(Cf)(\zeta)|^2 \, |d\zeta| = \int_{k_R(0)} \frac{1}{|\zeta^2 - 1|} \, |d\zeta| \to \infty \quad (R \to 1).$$

5.2   A functional-analytic approach

Next, we want to set up for a functional analytic approach to characterize Cauchy transforms. This leads to a pretty general characterization of Cauchy transforms using certain linear mappings.

Definition 5.2.1 For $A \subset \mathbb{C}_\infty$ we set

$$H_A := \{(F, U) : F \in H(U), U \text{ is an open neighborhood of } A\}.$$

If $(F, U), (G, V) \in H_A$, we write $(F, U) \sim_A (G, V)$ if there is an open set $W \subset \mathbb{C}_\infty$ with $A \subset W \subset U \cap V$ and $F|_W = G|_W$. Then $\sim_A$ defines an equivalence relation on H_A and the set

$$\mathscr{H}(A) := H_A/_{\sim_A}$$

is called the space of **germs of holomorphic functions** on A. In the following, we shall write $[(F, U)]_A$ instead of $[(F, U)]_{\sim_A}$.

Now consider a compact set $K \subset \mathbb{C}$ and a function $F \in H(\mathbb{C}_\infty \setminus K)$. Recall that a Cauchy cycle for K in an open neighborhood Ω is a cycle in Ω satisfying $\mathrm{ind}_\gamma(z) = 1$ for all $z \in K$ and $\mathrm{ind}_\gamma(z) = 0$ for all $z \in \mathbb{C} \setminus \Omega$.

If $(G_1, U) \sim_K (G_2, V)$, then we know that $G_1 = G_2$ on an open set $W \subset \mathbb{C}_\infty$ satisfying $K \subset W \subset U \cap V$. Let γ_1, γ_2 and γ_3 be Cauchy cycles for K in U, V and W, respectively. An application of Cauchy's theorem (cf., e.g., [68, Theorem 10.35]) implies that

$$\int_{\gamma_1} F(\zeta)G_1(\zeta) \, d\zeta = \int_{\gamma_3} F(\zeta)G_1(\zeta) \, d\zeta = \int_{\gamma_3} F(\zeta)G_2(\zeta) \, d\zeta = \int_{\gamma_2} F(\zeta)G_2(\zeta) \, d\zeta.$$

Therefore, the mapping $\varphi_F : \mathscr{H}(K) \to \mathbb{C}$, defined by

$$\varphi_F([(G, U)]_K) = -\frac{1}{2\pi i} \int_\gamma F(\zeta)G(\zeta) \, d\zeta \quad ([(G, U)]_K \in \mathscr{H}(K)),$$

with γ being a Cauchy cycle for K in U, is well-defined since its values do neither depend on the particular representative nor on the chosen Cauchy cycle. Moreover, φ_F is linear.

A famous theorem by Havin (see, e.g. [27, Chapter II, Theorem 3.5]) states that a function $F \in H(\mathbb{C}_\infty \setminus K)$ is the Cauchy transform of some $\mu \in \mathcal{M}(\mathbb{C})$ with $\mathrm{supp}(\mu) \subset K$ if and only if the mapping φ_F is continuous with respect to $\| \cdot \|_{\infty, K}$, i.e.

$$|\varphi_F([(G, U)]_K)| \leq K_F \|G\|_{\infty, K} \quad ([(G, U)]_K \in \mathscr{H}(K))$$

for some constant $K_F > 0$.

We are interested in a refinement of this statement for compact sets $K \subset \mathbb{R}$. In particular, we want to characterize the μ-Cauchy transforms of functions $f \in L_p(\mu)$ where $\mu \in \mathcal{M}(\mathbb{R})$ is such that $\mathrm{supp}(\mu) \subset K$. Let us start with the following observations. Suppose for $p \in [1, \infty]$, a compact set $K \subset \mathbb{R}$ and $\mu \in \mathcal{M}(\mathbb{R})$ with $\mathrm{supp}(\mu) \subset K$ that a function $F \in H(\mathbb{C}_\infty \setminus K)$ admits a representation

$$F(z) = (C_\mu f)(z) \quad (z \in \mathbb{C}_\infty \setminus K),$$

where $f \in L_p(\mu)$. In this case, we can apply the Fubini theorem and Cauchy's integral formula to get

$$\varphi_F([(G,U)]_K) = -\frac{1}{2\pi i} \int_\gamma F(\zeta)G(\zeta)\,d\zeta$$

$$= -\frac{1}{2\pi i} \int_\gamma \left(\frac{1}{2\pi i} \int_K \frac{f(t)}{t - \zeta}\,d\mu(t)\right) G(\zeta)\,d\zeta$$

$$= \frac{1}{2\pi i} \int_K f(t) \left(\frac{1}{2\pi i} \int_\gamma \frac{G(\zeta)}{\zeta - t}\,d\zeta\right) d\mu(t)$$

$$= \frac{1}{2\pi i} \int_K f(t)G(t)\,d\mu(t)$$

for all $[(G,U)]_K \in \mathscr{H}(K)$. By Hölder's inequality, we conclude that

$$2\pi \cdot |\varphi_F([(G,U)]_K)| \leq \|f\|_{L_p(\mu)} \cdot \|G\|_{L_q(\mu)} \quad ([(G,U)]_K \in \mathscr{H}(K)),$$

where q is the conjugate exponent of p. But this simply means that φ_F is continuous with respect to $\|\cdot\|_{L_q(\mu)}$. We shall now show that the converse is true as well if $p > 1$.

Theorem 5.2.2 *Let $p \in (1, \infty], K \subset \mathbb{R}$ be compact and $\mu \in \mathcal{M}(\mathbb{R})$ with $\mathrm{supp}(\mu) \subset K$. For a function $F \in H(\mathbb{C}_\infty \setminus K)$ the following are equivalent:*

a) There is some $f \in L_p(\mu)$ such that $F = C_\mu f$.

b) φ_F is continuous with respect to $\|\cdot\|_{L_q(\mu)}$, where q is the conjugate exponent of p.

In this case, f is unique and $2\pi\|\varphi_F\|_{\mathrm{op}} = \|f\|_{L_p(\mu)}$.

Proof. Necessity has already been shown above. So, suppose that there is a constant K_F such that

$$|\varphi_F([(G,U)]_K)| \leq K_F\|G\|_{L_q(\mu)} \quad ([(G,U)]_K \in \mathscr{H}(K)).$$

Hence, if $[(G_1,U)]_K, [(G_2,V)]_K$ satisfy $G_1|_K = G_2|_K$, then we must have $\varphi_F([(G_1,U)]_K) = \varphi_F([(G_2,V)]_K)$. Upon setting

$$H(K) := \{h\mathbb{1}_K : [(h,U)]_K \in \mathscr{H}(K)\},$$

we see that the mapping $\varphi : H(K) \to \mathbb{C}$, defined by

$$\varphi(h) = \varphi_F([(H,U)]_K) \quad (h \in H(K)),$$

where U is an open neighborhood of K and $H \in H(U)$ satisfies $H\mathbb{1}_K = h$, is well-defined, linear and continuous. In particular, $\|\varphi\|_{\mathrm{op}} = \|\varphi_F\|_{\mathrm{op}}$. Since $H(K) \subset L_q(\mu)$, we can apply the Hahn-Banach theorem and get a linear and continuous extension $\psi : L_q(\mu) \to \mathbb{C}$ of φ of the same norm. But now, by standard L_p duality theory (see, e.g. [68, Theorem 6.16] and notice that $q \in [1, \infty)$), we know that there is some $f \in L_p(\mu)$ such that

$$\psi(h) = \frac{1}{2\pi i} \int_K f(t)h(t)\,d\mu(t) \quad (h \in L_q(\mu)).$$

Moreover, $\|f\|_{L_p(\mu)} = 2\pi\|\psi\|_{\mathrm{op}} = 2\pi\|\varphi_F\|_{\mathrm{op}}$.

Now, let $z \in \mathbb{C} \setminus K$. We choose an open neighborhood U of K such that $z \notin U$. If we consider the function $G : U \to \mathbb{C}$, $G(\zeta) = (\zeta - z)^{-1}$, then $[(G, U)]_K \in \mathcal{H}(K)$. On the one hand, we clearly have

$$\psi(G|_K) = (C_\mu f)(z).$$

But, on the other hand, we also have

$$\psi(G|_K) = \varphi(G|_K) = \varphi_F([(G, U)]_K) = \frac{1}{2\pi i} \int_\gamma \frac{F(\zeta)}{z - \zeta} \, d\zeta = F(z)$$

by Cauchy's integral formula, see [47, Theorem 9.7], cf. [4, Lemma 1.11]. $\qquad\square$

If $K \subset \mathbb{R}$ is compact, $F \in H(\mathbb{C}_\infty \setminus K)$ and $G \in H_K$, i.e. $(G, U) \in H(U)$ where $U \supset K$ is open, then one can always restrict G to an open neighborhood of K with connected complement without changing the value of $\varphi_F(G)$. This is the key observation when we now want to bring in Runge's theorem. The corresponding result, see Theorem 5.2.3 below, holds at least partly for the unit circle (see [7, Theorem 5.5.1]) and for Cauchy transforms of compactly supported measures (see [27, Chapter II, Corollary 3.6]). One advantage that we encounter in our situation is that due to the fact that our measures have compact support in $\mathbb{R}$ we can use polynomial approximation. The associated condition c) in this theorem is therefore much easier to apply than condition b) relying on rational approximation.

If $r := \max_{x \in K} |x|$, then F has a Laurent expansion

$$F(z) = \sum_{k=-\infty}^{0} c_k z^k \quad (|z| > r).$$

Here,

$$c_k := \frac{1}{2\pi i} \int_{k_R(0)} F(\zeta)\zeta^{-k-1} \, d\zeta \quad (k \in \mathbb{Z})$$

where $R > r$. Notice that if $\mu \in \mathcal{M}(\mathbb{R})$ with $\mathrm{supp}(\mu) \subset K$ and $F = C_\mu f$ for some $f \in L_1(\mu)$, then Proposition 3.1.4 implies that

$$c_k = \begin{cases} 0, & k \geq 0 \\ -\int f(t)t^{-k-1} \, d\mu(t), & k \leq -1. \end{cases}$$

In particular, $c_{-k-1} = m_k(f\mu)$ whenever $k \geq 0$.

Theorem 5.2.3 *Let $p \in (1, \infty]$, q the conjugate exponent of p, $K \subset \mathbb{R}$ be compact and $\mu \in \mathcal{M}(\mathbb{R})$ with $\mathrm{supp}(\mu) \subset K$. For a function $F \in H(\mathbb{C}_\infty \setminus K)$, the following statements are equivalent:*

a) There is some $f \in L_p(\mu)$ with $F = C_\mu f$.

b) *There is a constant $C > 0$ such that*

$$\left|\sum_{k=1}^{n} \alpha_k F(w_k)\right| \leq C \cdot \left\|\sum_{k=1}^{n} \frac{\alpha_k}{\cdot - w_k}\right\|_{L_q(\mu)},$$

whenever $n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in \mathbb{C}$ and $w_1, \ldots, w_n \in \mathbb{C} \setminus K$.

c) *There is a constant $C > 0$ such that*

$$\left|\sum_{k=0}^{n} \alpha_k c_{-k-1}\right| \leq C \cdot \left\|\sum_{k=0}^{n} \alpha_k (\cdot)^k\right\|_{L_q(\mu)},$$

whenever $n \in \mathbb{N}, \alpha_1, \ldots, \alpha_n \in \mathbb{C}$.

Moreover, f is unique and satisfies $\|f\|_{L_p(\mu)} \leq 2\pi C$.

Proof. We show $a) \Leftrightarrow b)$ and $a) \Leftrightarrow c)$. Let us start with the equivalence of $a)$ and $b)$. First note that if $\alpha_1, \ldots, \alpha_n \in \mathbb{C}, w_1, \ldots, w_n \in \mathbb{C} \setminus K$, we can choose an open neighborhood U of K such that $w_1, \ldots, w_n \in \mathbb{C} \setminus U$. If we set

$$G(z) := \sum_{k=1}^{n} \frac{\alpha_k}{z - w_k} \quad (z \in U),$$

then

$$\varphi_F([(G, U)]_K) = \sum_{k=1}^{n} \alpha_k G(w_k)$$

by [47, Theorem 9.7]. With this observation in mind, necessity of the condition is clear. So let us turn to sufficiency. Let us fix some function $G \in H(U)$ where U is an open neighborhood of K. If γ is a Cauchy cycle for K in U, we choose a compact set $V \subset U$ such that $K \cup \gamma^* \subset V$. But then by a variant of Runge's theorem (cf. [47, Theorem 10.2] and [68, Theorem 13.9]), there is a sequence $(G_m)_{m \in \mathbb{N}}$ of rational functions with simple poles only in $\mathbb{C} \setminus V$ such that $G_m \to G$ uniformly on V. In particular, $G_m \to G$ uniformly on γ^* which means $\varphi_F(G_m) \to \varphi_F(G)$. Certainly, $\|G_m - G\|_{L_q(\mu)} \to 0$ since μ is finite, hence by our assumption and the observation above, we see that φ_F is continuous with $\|\varphi_F\| \leq C$. For the proof of the equivalence of $a)$ and $c)$, first note that for each polynomial P

$$\varphi_F(p) = \frac{1}{2\pi i} \int_{k_R(0)} F(\zeta) P(\zeta) \, d\zeta \quad (k \in \mathbb{Z}),$$

where $R > 0$ is sufficiently large. Therefore, necessity of the condition is clear. So, let us again prove sufficiency. Let us fix some function $G \in H(U)$ with an open neighborhood U of K. Without loss of generality, we can choose U to be such that $\mathbb{C}_\infty \setminus U$ is connected (for example let $\varepsilon := \inf\{|z - w| : z \in K, w \in \partial U\}$ and replace U by $\bigcup_{z \in K} U_{\varepsilon/2}(z)$). Hence, there

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is a sequence of polynomials $(P_m)_{m \in \mathbb{N}}$ such that $P_m \to G$ locally uniformly on U. But this means that for any sufficiently large $R > 0$ and any Cauchy cycle γ for K in U we have

$$\varphi_F(P_m) = \frac{1}{2\pi i} \int_{k_R(0)} F(\zeta) P_m(\zeta) \, d\zeta = \frac{1}{2\pi i} \int_{\gamma} F(\zeta) P_m(\zeta) \, d\zeta \to \varphi_F(G).$$

On the other hand, clearly $\|P_m - G\|_{L_q(\mu)} \to 0$ and hence φ_F is continuous with $\|\varphi_F\|_{\mathrm{op}} \le C$.
□

As a consequence we get conditions for the summability of moments of certain measures. Recall that for a measure $\mu \in \mathcal{M}(X)$ where $X \subset \mathbb{R}$ and $y \in [0, \infty)$

$$|m_y|(|\mu|) = \int |x|^y \, d|\mu|(x) \in [0, +\infty].$$

We remark that in Theorem 5.2.3 the necessity of condition $c)$ holds also true in the case $p = 1$.

Corollary 5.2.4 *Let $K \subset \mathbb{R}$ be compact, $\mu \in \mathcal{M}(\mathbb{R})$ with $\mathrm{supp}(\mu) \subset K$ and $(a_n)_{n \in \mathbb{N}}$ a sequence in $\mathbb{C}$. If $p \in [1, \infty]$ and $f \in L_p(\mu)$, the following statements hold:*

i) If $p > 1$ and

$$\sum_{k=1}^{\infty} |a_k m_k(f\mu)|^q \cdot |m_{kq}|(|\mu|) < +\infty,$$

where q is the conjugate exponent of p, then

$$\sum_{k=1}^{\infty} |a_k| |m_k(f\mu)|^2 < +\infty.$$

ii) If $p = 1$ and

$$\sum_{k=1}^{\infty} |a_k| |m_k(f\mu)| M^k < +\infty,$$

where $M := \max_{x \in K} |x|$, then also

$$\sum_{k=1}^{\infty} |a_k| |m_k(f\mu)|^2 < +\infty.$$

Proof. Fix $n \in \mathbb{N}$. If we pick $\alpha_k = |a_k| \overline{m_k(f\mu)}$ $(k = 1, \ldots, n)$ in Theorem 5.2.3 (choose $F = C_\mu f$ and notice that $c_{-k-1} = m_k(f\mu)$), then

$$\sum_{k=1}^{n} |a_k| |m_k(f\mu)|^2 \le C \left\| \sum_{k=1}^{n} |a_k| \overline{m_k(f\mu)} (\cdot)^k \right\|_{L_q(\mu)}$$

If $p > 1$, then

$$\int_K \left| \sum_{k=1}^{n} |a_k| \overline{m_k(f\mu)} t^k \right|^q d|\mu|(t) \le n^q \int_K \left(\sum_{k=1}^{n} \frac{1}{n} |a_k m_k(f\mu)| \cdot |t|^k \right)^q d|\mu|(t)$$

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$$\leq \sum_{k=1}^{n} |a_k m_k(f\mu)|^q \int_K |t|^{kq} \, d|\mu|(t)$$

$$= \sum_{k=1}^{n} |a_k m_k(f\mu)|^q \cdot |m_{kq}|(|\mu|)$$

by Jensen's inequality. If $p = 1$, then $q = \infty$ and hence

$$\sum_{k=1}^{n} |a_k||m_k(f\mu)|^2 \leq |\mu|(K) \sum_{k=1}^{n} |a_k||m_k(f\mu)|M^k.$$

$\square$

We now turn to the case $p = 1$. In this case, the argumentation in Theorem 5.2.2 fails since the dual of $L_\infty(K)$ is **not** isomorphic to $L_1(K)$. This illustrates that we need a stronger condition to characterize the Cauchy transforms by properties of induced linear mappings. However, there is a solution:

Theorem 5.2.5 *Let $K \subset \mathbb{R}$ be compact, $\mu \in \mathcal{M}(\mathbb{R})$ with $\mathrm{supp}(\mu) \subset K$ and $F \in H(\mathbb{C}_\infty \setminus K)$. Then the following statements are equivalent:*

a) There is some $f \in L_1(\mu)$ with $F = C_\mu f$.

b) There is a linear and continuous mapping $L : L_\infty(\mu) \to \mathbb{C}$ such that

b1) $L(H \mathbb{1}_K) = \varphi_F([(H, U)]_K)$ for all $[(H, U)]_K \in \mathscr{H}(K)$.

b2) $L(h_n) \to 0$ for all bounded sequences $(h_n)_{n \in \mathbb{N}}$ in $L_\infty(\mu)$ satisfying $h_n \to 0$ $|\mu|$-a.e.

In this case, f is unique and satisfies $2\pi \|L\|_{\mathrm{op}} = 2\pi \|\varphi_F\|_{\mathrm{op}} = \|f\|_{L_1(\mu)}$.

Proof. If $F = C_\mu f$ for some $f \in L_1(\mu)$, then $L : L_\infty(\mu) \to \mathbb{C}$, defined by

$$L(h) = \frac{1}{2\pi i} \int_K f(t)h(t) \, d\mu(t) \quad (h \in L_\infty(\mu)),$$

is obviously linear and continuous. If $(h_n)_{n \in \mathbb{N}}$ is a bounded sequence in $L_\infty(\mu)$ with $h_n \to 0$ a.e. (with respect to μ), then $L(h_n) \to 0$ by the dominated convergence theorem (use the polar decomposition of μ).

On the other side, suppose that there is a linear and continuous functional L on $L_\infty(\mu)$ with the properties in b). Then, by [15, Chapter IV, Section 8, Theorem 16], or [69, Proposition 4.13], there is an additive mapping $\nu : \mathcal{B}(\mathbb{R}) \to \mathbb{C}$ such that

$$L(h) = \frac{1}{2\pi i} \int_K h(t) \, d\nu(t) \quad (h \in L_\infty(\mu)).$$

For the integration with respect to a content we refer to [69], p. 17-21. Moreover, $\nu(A) = 0$ for every $A \in \mathcal{B}(\mathbb{R})$ with $|\mu|(A) = 0$. In particular, $\nu(A) = \nu(A \cap K)$ for all $A \in \mathcal{B}(\mathbb{R})$. Our goal is to show that ν is even countably-additive, hence a finite complex measure. In

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this case, we can apply the Radon-Nikodym theorem to conclude the proof. So, let us fix a sequence $(A_n)_{n \in \mathbb{N}}$ of pairwise disjoint Borel sets on $\mathbb{R}$ and set

$$
\begin{aligned}
E_n &:= \bigcup_{m=1}^{n} A_m \ (n \in \mathbb{N}) \\
E &:= \bigcup_{m=1}^{\infty} A_m.
\end{aligned}
$$

If we consider the sequence $(h_n)_{n \in \mathbb{N}}$ with $h_n := \mathbb{1}_{E \setminus E_n}$, then clearly $h_n \to 0$ pointwise and $\|h_n\|_{L_\infty(\mu)} \leq 1$. Therefore,

$$
\nu(E) - \sum_{m=1}^{n} \nu(A_m) = \int h_n(t) \, d\nu(t) = 2\pi i L(h_n) \to 0 \quad (n \to \infty).
$$

Hence, $\nu \in \mathcal{M}(\mathbb{R})$ and $\|\nu\| = 2\pi \|L\|_{\mathrm{op}}$. Clearly, $\mathrm{supp}(\nu) \subset K$ and ν is absolutely continuous to $|\mu|$. Therefore, a variant of the Radon-Nikodym theorem (see [68, 6.10]) guarantees the existence of a unique $r \in L_1(\mu)$ such that $\nu = r|\mu|$, so we can say that

$$
L(h) = \frac{1}{2\pi i} \int_K r(t) h(t) \, d|\mu|(t) \quad (h \in L_\infty(\mu)).
$$

If $\mu = s|\mu|$ is a polar decomposition of μ, we get upon setting $f = \frac{r}{s}$ that

$$
L(h) = \frac{1}{2\pi i} \int_K f(t) h(t) \, d\mu(t) \quad (h \in L_\infty(\mu)).
$$

Clearly, we have $f \in L_1(\mu)$ with $\|f\|_{L_1(\mu)} = \|\nu\|$. The rest of the proof is exactly the same than in the proof for $p > 1$. $\qquad \square$

5.3 Weighted L_p spaces and another theorem of Paley-Wiener type

The determination of the image of weighted L_p spaces under the Cauchy transformation C is a far more delicate question than for the unweighted ones. If the corresponding weight is integrable and has compact support, one can apply the results in Section 5.2. However, we also want to attack this problem in a similar matter as for the unweighted L_p spaces. Turning back to Chapter 2, we see that the key element was the boundary behavior of Cauchy transforms: If $p \in [1, \infty)$ and $f \in L_p(\mathbb{R})$, then $(Cf)_+ - (Cf)_- = f$ and $\widetilde{Cf} = (Cf)_+ + (Cf)_- = iHf$. In particular, if $p > 1$, then $\widetilde{Cf} \in L_p(\mathbb{R})$. Therefore, also the function $\widetilde{Cf}$ reflects the behavior of the original function f.

In this section, we shall consider certain weighted L_p spaces where the corresponding weight has support in the unit interval $I = (-1, 1)$. Moreover, we will always have the situation that these weighted spaces are contained in $L_2(I)$. This means that the image of these weighted L_p spaces under C will be a space of functions $F \in \mathcal{H}_2(\mathbb{C} \setminus [-1, 1])$ where only

the behavior of $\tilde{F} = F_+ + F_-$ is crucial. Similar to the unweighted case, it shall turn out that properties from the corresponding weighted L_p space are transferred to the function $\tilde{F}\mathbb{1}_I$.

If $F = Cf$ where $f \in L_2(I)$, then $\tilde{F}\mathbb{1}_I = (iHf)\mathbb{1}_I$. This means that our subject is to analyze the restriction of the Hilbert transform to the interval I. Since we also want f to belong to some weighted L_p space, the question arises for which spaces this restriction is defined.

Definition 5.3.1 Let $p \in (1, \infty)$. A weight $\omega : \mathbb{R} \to [0, \infty]$ is said to satisfy the $A_{p,I}$-**condition** with constant $M > 0$ if $\omega = \omega\mathbb{1}_I$ and

$$\sup_{\substack{J \subset I \\ J \text{ interval}}} \left(\frac{1}{\lambda_1(J)} \int_J \omega(x)\,dx \right) \cdot \left(\frac{1}{\lambda_1(J)} \int_J (\omega(x))^{-\frac{1}{p-1}}\,dx \right) \leq M < +\infty.$$

The results in the following remark can be found in [36], p. 231.

Remark 5.3.2   1. From Hölder's inequality it follows that if a weight $\omega : \mathbb{R} \to [0, \infty]$ satisfies the $A_{p,I}$-condition for some $p \in (1, \infty)$ with constant $M > 0$, then it also satisfies the $A_{q,I}$-condition with constant $M > 0$ whenever $q > p > 1$.

2. From the definition it follows directly that if a weight $\omega : \mathbb{R} \to [0, \infty]$ satisfies the $A_{p,I}$-condition with constant $M > 0$ for some $p \in (1, \infty)$, then the weight $\omega^{-\frac{1}{p-1}}$ satisfies the $A_{q,I}$-condition with constant $M^{\frac{1}{p-1}}$ where q is the conjugate exponent of p.

Remark 5.3.3 An important class of weights satisfying the $A_{p,I}$-condition arises when considering weights ω of the form

$$\omega_{\alpha,\beta}(x) = \begin{cases} (1-x)^\alpha(1+x)^\beta, & x \in (-1, 1), \\ 0, & x \in \mathbb{R} \setminus (-1, 1). \end{cases}$$

Due to [3], p. 1164, this weight $\omega_{\alpha,\beta}$ satisfies the $A_{p,I}$-condition if and only if $\alpha, \beta \in (-1, p-1)$. In the following, we may write $\mu_{\alpha,\beta} = \omega_{\alpha,\beta}\lambda_1$.

Remark and Definition 5.3.4 Let $p \in (1, \infty)$ and ω satisfy the $A_{p,I}$-condition. Then, $L_p(I, \omega\lambda_1) \subset \mathscr{C}$. To see this, we consider the weight $\omega^* : \mathbb{R} \to [0, \infty]$ defined by

$$\omega^*(x) = \omega((-1)^n(x - 2n)) \quad (x \in (2n-1, 2n+1), n \in \mathbb{Z}).$$

Due to [3], p. 1159, ω satisfies the $A_{p,I}$-condition if and only if $\omega^* \in A_p$, i.e.

$$\sup_{J \subset \mathbb{R}} \left(\frac{1}{\lambda_1(J)} \int_J \omega^*(x)\,dx \right) \cdot \left(\frac{1}{\lambda_1(J)} \int_J (\omega^*(x))^{-\frac{1}{p-1}}\,dx \right) \leq M < +\infty.$$

Hence, we can apply [36, Lemma 2] to see that indeed $L_p(I, \omega\lambda_1) \subset \mathscr{C}$. Therefore, if $f \in L_p(I, \omega\lambda_1)$, we can define the function $H_I f = (Hf)\mathbb{1}_I$ and call this function the **finite Hilbert transform** of f. Since ω satisfies the $A_{p,I}$-condition, [3, Proposition 2.2] tells us that the mapping $H_I : L_p(I, \omega\lambda_1) \to L_p(I, \omega\lambda_1)$ is well-defined and bounded.

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There is a wide theory on the finite Hilbert transform which goes back to Carleman. One reason for the importance of the finite Hilbert transform lies in its appearance in the airfoil equation. We shall put our focus on the special weights $\omega_{\alpha,\beta}$ and in this case, the following result whose proof can be found in [3, Corollary 2.3] is valid.

Theorem 5.3.5 (Astala, Päivärinta, Saksman, [3]) *Let $p \in (1, \infty)$, $A, B \in \left(-\frac{p}{2}, \frac{p}{2}\right)$ and put $\alpha = A + \frac{p}{2} - 1, \beta = B + \frac{p}{2} - 1$. For the mapping $H_I : L_p(I, \mu_{\alpha,\beta}) \to L_p(I, \mu_{\alpha,\beta})$ the following statement holds:*

1. *H_I is an isomorphism if and only if $AB < 0$.*

In the case $AB \geq 0$, the following statements hold:

2. *H_I is surjective if and only if $A, B > 0$.*

3. *H_I is injective if and only if $A, B \leq 0$.*

Remark 5.3.6 For the simplest case, namely $p = 2$ and $\alpha = \beta = 0$, Proposition 5.3.5 says that the mapping $H_I : L_2(I) \to L_2(I)$ is bounded and injective, but **not** surjective.

One of the most important functions in context with the finite Hilbert transform is the density of the arcsine distribution

$$\omega(x) = \omega_{-\frac{1}{2},-\frac{1}{2}}(x) = \begin{cases} \dfrac{1}{\sqrt{1 - x^2}}, & x \in (-1, 1) \\ 0, & x \in \mathbb{R} \setminus (-1, 1) \end{cases}.$$

The reason is that $H_I\omega = 0$ and moreover, if $p \in (1, \infty)$ and a function $g \in L_p(I)$ satisfies $H_I g = 0$, then we must have $g = c\omega$ for some constant $c \in \mathbb{C}$ (see, e.g., [78], p. 176, or [11], p. 1840). In particular, if $p > 1$ and $\alpha, \beta < p - 1$, then the kernel of the finite Hilbert transform $H_I : L_p(I, \mu_{\alpha,\beta}) \to L_p(I, \mu_{\alpha,\beta})$ is either the null space or the one dimensional vector space $\{c\omega : c \in \mathbb{C}\}$.

One of the major problems arising in the study of the finite Hilbert transform is the determination of its image. This corresponds to solving the integral equation

$$H_I f = g$$

where f and g must belong to some (weighted) L_p space. Usually this integral equation is called the airfoil equation, see, e.g., [78], p. 175.

Even for the simplest case $p = 2, \alpha = \beta = 0$ this issue was unsolved for many years until in 1991, the authors in [57] presented a characterization and closed this gap. Here is the corresponding result (see [57, Theorem 4.2]).

Theorem 5.3.7 (Okada, Elliott, [57]) *Let $g \in L_2(I) \cap L_1(I, \omega\lambda_1)$. Then, the following statements are equivalent:*

a) *There is $f \in L_2(I)$ such that $H_I f = g$.*

b) $\frac{1}{\omega} H_I(g\omega) \in L_2(I)$ *and*

$$\int_I g(t)\omega(t)\,dt = 0.$$

In this case, we have $f = -\frac{1}{\omega} H_I(g\omega)$.

Remark 5.3.8 In the recent past, several generalizations have arisen. For example, the first author in [57] et al. consider in [11] a more general situation which involves so-called **rearrangement-invariant (r.i.) Banach function spaces** on I. These spaces include for example $L_p(I)$ for $p \in [1, \infty]$, cf. [11], p. 1139. A Banach function space X on I is a Banach space of functions on I such that if $g \in X$ and $|f| \leq |g|$ a.e., then $f \in X$ and $\|f\|_X \leq \|g\|_X$. The rearrangement invariance means that if $g \in X$ and $f^* \leq g^*$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$. Here, $f^* : [0, 2] \to [0, \infty]$ is the decreasing rearrangement of f, i.e. the right continuous inverse of its distribution function $x \mapsto \lambda_1(|f|^{-1}(x, \infty))$. One of the main results in the corresponding paper was that under certain circumstances the image of an r.i. Banach function space X under the finite Hilbert transformation is the space of all functions $f \in X$ satisfying

$$\int_I f(t)\omega(t)\,dt = 0,$$

see [11, Theorem 3.3].

Unfortunately, the spaces $L_p(I, \mu_{\alpha,\beta})$ are in general **not** r.i. This is because of the fact that any two indicator functions $\mathbb{1}_A, \mathbb{1}_B$ where A, B are any two measurable subsets of $(-1, 1)$ satisfying $\lambda_1(A) = \lambda_1(B)$ have the same decreasing rearrangement but of course not the same norm.

Nevertheless, the result in [11] is still valid for these weighted spaces. This is what we want to prove next. For $p \in [1, \infty), \alpha, \beta \in \mathbb{R}$ we therefore consider the subspace

$$L_{p,0}(I, \mu_{\alpha,\beta}) := \left\{ f \in L_p(I, \mu_{\alpha,\beta}) : \int_I f(t)\omega(t)\,dt = 0 \right\}.$$

In order to apply Theorem 5.3.7 to our situation we need to know when $L_p(I, \mu_{\alpha,\beta})$ is contained in $L_2(I) \cap L_1(I, \omega\lambda_1)$. The next two lemmas give sufficient conditions when this is the case.

Lemma 5.3.9 *Let* $p \in [1, \infty)$ *and* $\alpha, \beta < \frac{p}{2} - 1$. *Then, the following statements hold:*

1. $L_p(I, \mu_{\alpha,\beta}) \subset L_1(I, \omega\lambda_1)$.

2. $L_{p,0}(I, \mu_{\alpha,\beta})$ *is continuously embedded into* $L_p(I, \mu_{\alpha,\beta})$.

Proof. The first statement is an application of Hölder's inequality: If $p > 1$ and $f \in L_p(I, \mu_{\alpha,\beta})$, then

$$\int_I |f(t)|\omega(t)\,dt = \int_I |f(t)|\omega(t) \cdot \frac{\omega_{\frac{\alpha}{p}, \frac{\beta}{p}}(t)}{\omega_{\frac{\alpha}{p}, \frac{\beta}{p}}(t)}\,dt$$

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$$\leq \left(\int_I |f(t)|^p \, d\mu_{\alpha,\beta}(t) \right)^{\frac{1}{p}} \cdot \left(\int_I \left(\frac{1}{\omega_{\frac{\alpha}{p}+\frac{1}{2},\frac{\beta}{p}+\frac{1}{2}}(t)} \right)^{\frac{p}{p-1}} dt \right)^{\frac{p-1}{p}}$$

$$= \left(\int_I |f(t)|^p \, d\mu_{\alpha,\beta}(t) \right)^{\frac{1}{p}} \cdot \left(\int_I \frac{1}{\omega_{\frac{2\alpha+p}{2(p-1)},\frac{2\beta+p}{2(p-1)}}(t)} \, dt \right)^{\frac{p-1}{p}}$$

$$< +\infty$$

since $\alpha, \beta < \frac{p}{2} - 1$. If $p = 1$, then by Hölder's inequality

$$\int_I |f(t)| \omega(t) \, dt \leq \int_I |f(t)| \, d\mu_{\alpha,\beta}(t) \cdot \max_{t \in I} \frac{1}{\omega_{\alpha+\frac{1}{2},\beta+\frac{1}{2}}(t)} < +\infty.$$

For the second statement fix a sequence $(f_n)_{n \in \mathbb{N}}$ in $L_{p,0}(I, \mu_{\alpha,\beta})$. Then, there is $f \in L_p(I, \mu_{\alpha,\beta})$ with $f_n \to f$ in $L_p(I, \mu_{\alpha,\beta})$. But now

$$\left| \int f(t) \omega(t) \, dt \right| = \left| \int f_n(t) - f(t) \omega(t) \, dt \right|$$

$$\leq \left(\int_I |f_n(t) - f(t)|^p \, d\mu_{\alpha,\beta}(t) \right)^{\frac{1}{p}} \cdot \left(\int_I \frac{1}{\omega_{\frac{2\alpha+p}{2(p-1)},\frac{2\beta+p}{2(p-1)}}(t)} \, dt \right)^{\frac{p-1}{p}}$$

$$\to 0.$$

This concludes the proof. $\qquad\square$

Lemma 5.3.10 *Let $p \in [2, \infty)$ and $\alpha, \beta < \frac{p}{2} - 1$. Then, $L_p(I, \mu_{\alpha,\beta}) \subset L_2(I)$.*

Proof. This is again an application of Hölder's inequality. If $f \in L_p(I, \mu_{\alpha,\beta})$, then

$$\int_I |f(t)|^2 \, dt = \int_I |f(t)|^2 \frac{\omega_{\frac{2\alpha}{p},\frac{2\beta}{p}}(t)}{\omega_{\frac{2\alpha}{p},\frac{2\beta}{p}}(t)} \, dt$$

$$\leq \left(\int_I |f(t)|^p \, d\mu_{\alpha,\beta}(t) \right)^{\frac{2}{p}} \cdot \left(\int_I \frac{1}{\omega_{\frac{2\alpha}{p-2},\frac{2\beta}{p-2}}(t)} \, dt \right)^{\frac{p-2}{p}} < +\infty$$

since $\alpha, \beta < \frac{p}{2} - 1$. $\qquad\square$

Remark 5.3.11 It is important to notice that Lemma 5.3.10 is not valid for $p \in [1, 2)$. We even have $L_{p,0}(I, \mu_{\alpha,\beta}) \not\subset L_2(I)$ for $p \in [1, 2)$.
For example, take a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ such that $\varepsilon_1 = 1, \varepsilon_n \to 0 \ (n \to \infty)$ and

$$\int_{\varepsilon_{n+1}}^{\varepsilon_n} \frac{1}{t} \, dt \geq (n+1)^3 \quad (n \in \mathbb{N}).$$

Then, choose $g : [0,1] \to \mathbb{R}$ continuous such that $g|_{[\varepsilon_{n+1}, \varepsilon_n]}$ is linear for each $n \in \mathbb{N}$ with $g(\varepsilon_{n+1}) = \dfrac{1}{(n+1)^2}, g(\varepsilon_n) = \dfrac{1}{n^2}$. We extend g onto $[-1,1]$ by setting $g(x) = -g(-x)$ $(x \in [-1,0])$. Now, let $r \in \left(\frac{1}{2}, \frac{1}{p}\right), \alpha, \beta \in (-1, p/2 - 1)$ with $\alpha = \beta$ and set

$$f(t) = \frac{\omega_{-\frac{\alpha}{p}, -\frac{\beta}{p}}(t)g(t)}{|t|^r} \quad (t \in I).$$

Then,

$$\int_I |f(t)|^p \, d\mu_{\alpha,\beta}(t) = \int_I \frac{|g(t)|^p}{|t|^{rp}} \, dt = 2 \sum_{n=1}^{\infty} \int_{\varepsilon_{n+1}}^{\varepsilon_n} \frac{(g(t))^p}{t^{rp}} \, dt \le \frac{4}{1 - rp} \sum_{n=1}^{\infty} \frac{1}{n^{2p}} < +\infty$$

while

$$\int_I |f(t)|^2 \, dt = \int_I \frac{(1 - t^2)^{-\frac{2\alpha}{p}} |g(t)|^2}{|t|^{2r}} \, dt \ge \int_0^{\frac{1}{2}} \frac{(1 - t^2)^{-\frac{2\alpha}{p}} |g(t)|^2}{t^{2r}} \, dt$$

$$\ge \left(\frac{3}{4}\right)^{\frac{-2\alpha}{p}} \int_0^{\frac{1}{2}} \frac{(g(t))^2}{t} \, dt \ge \left(\frac{3}{4}\right)^{\frac{-2\alpha}{p}} \sum_{n=N}^{\infty} \int_{\varepsilon_{n+1}}^{\varepsilon_n} \frac{(g(t))^2}{t} \, dt$$

$$\ge \left(\frac{3}{4}\right)^{\frac{-2\alpha}{p}} \sum_{n=N}^{\infty} \frac{1}{n+1} = +\infty$$

where $N \in \mathbb{N}$ is chosen such that $\varepsilon_N \le \frac{1}{2}$. Moreover, by Lemma 5.3.9 we have $f \in L_1(I, \mu_{-\frac{1}{2}, -\frac{1}{2}})$ and since f is odd (because g is) we have

$$\int_I f(t)\omega(t) \, dt = 0.$$

We are now in a position to determine the image of the finite Hilbert transform under the weighted spaces $L_p(I, \mu_{\alpha,\beta})$. The corresponding result is a generalization of a theorem in [81], p. 5.

Proposition 5.3.12 *Let $p \in [2, \infty)$ and $\alpha, \beta \in (-1, \frac{p}{2}-1)$. Then, the finite Hilbert transform $H_I : L_p(I, \mu_{\alpha,\beta}) \to L_{p,0}(I, \mu_{\alpha,\beta})$ is an isomorphism.*

Proof. First, if $f \in L_p(I, \mu_{\alpha,\beta})$, then $H_I f \in L_p(I, \mu_{\alpha,\beta})$ by Remark and Definition 5.3.4. Thus, Lemma 5.3.9 implies that $H_I f \in L_1(I, \omega\lambda_1)$ and Theorem 5.3.7 gives us that the mapping is well-defined because $L_p(I, \mu_{\alpha,\beta}) \subset L_2(I)$ by Lemma 5.3.10. Choosing $A = \alpha - \frac{p}{2} + 1, B = \beta - \frac{p}{2} + 1$ in Theorem 5.3.5 (here we use $\alpha, \beta > -1$), we see that H_I is injective. Since it is bounded by Remark and Definition 5.3.4 we only have to show surjectivity. The continuity of the inverse follows then from the Banach isomorphism theorem because both domain and range are Banach spaces.

So let $g \in L_{p,0}(I, \mu_{\alpha,\beta})$. Then, $g_1 := g\omega$ belongs to $L_p(I, \mu_{\frac{p}{2}+\alpha, \frac{p}{2}+\beta})$ since

$$\int_I |g_1(t)|^p \, d\mu_{\frac{p}{2}+\alpha, \frac{p}{2}+\beta}(t) = \int_I |g(t)|^p \, d\mu_{\alpha,\beta}(t)$$

and the same holds for $H_I g_1$ by Remark and Definition 5.3.4. Hence, if $p = 2$, then

$$\int_I (1 - t^2)|(H_I g_1)(t)|^2 \, dt \le \int_I (1 - t)^{1+\alpha}(1 + t)^{1+\beta}|(H_I g_1)(t)|^2 \, dt < +\infty$$

where we have used the inequality $(1 - t)^{-\alpha}(1 + t)^{-\beta} \le 1$ for $t \in I$ and $\alpha, \beta \in (-1, 0)$. If $p > 2$, then

$$
\begin{aligned}
\int_I (1 - t^2) \cdot |(H_I g_1)(t)|^2 \, dt &= \int_I |(H_I g_1)(t)|^2 \frac{\omega_{1+\frac{2\alpha}{p}, 1+\frac{2\beta}{p}}(t)}{\omega_{\frac{2\alpha}{p}, \frac{2\beta}{p}}(t)} \, dt \\
&\le \left(\int_I |(H_I g_1)(t)|^p \, d\mu_{\frac{p}{2}+\alpha, \frac{p}{2}+\beta}(t) \right)^{\frac{2}{p}} \cdot \left(\int_{-1}^1 \left(\frac{1}{\omega_{\frac{2\alpha}{p}, \frac{2\beta}{p}}(t)} \right)^{\frac{p}{p-2}} dt \right)^{\frac{p-2}{p}} \\
&= \left(\int_I |(H_I g_1)(t)|^p \, d\mu_{\frac{p}{2}+\alpha, \frac{p}{2}+\beta}(t) \right)^{\frac{2}{p}} \cdot \left(\int_I \frac{1}{\omega_{\frac{2\alpha}{p-2}, \frac{2\beta}{p-2}}(t)} \, dt \right)^{\frac{p-2}{p}} \\
&< +\infty
\end{aligned}
$$

by Hölder's inequality. Theorem 5.3.7 guarantees us the existence of a unique $f \in L_2(I)$ such that $H_I f = g$ and moreover, $f = -\frac{1}{\omega} H_I g_1$. Thus, we have

$$\int_I |f(t)|^p \, d\mu_{\alpha,\beta}(t) = \int_I |(H_I g_1)(t)|^p \, d\mu_{\frac{p}{2}+\alpha, \frac{p}{2}+\beta}(t) < +\infty$$

and therefore $f \in L_p(I, \mu_{\alpha,\beta})$. $\qquad\square$

Recall that for $F \in \mathcal{H}_p(\mathbb{C} \setminus [-1, 1])$ the function $\tilde{F}$ is defined almost everywhere by

$$\tilde{F}(x) := F_+(x) + F_-(x) = \lim_{y \to 0^+} F(x + iy) + \lim_{y \to 0^-} F(x + iy).$$

From the Plemelj formulas, it is clear that if $F = Cf$ with $f \in L_p([-1, 1])$, then $\tilde{F} = iHf$.

Lemma 5.3.13 *Let $p, r \in [2, \infty)$ and $\alpha, \beta \in (-1, \frac{p}{2} - 1)$. Then, the space*

$$\mathcal{H}_{r,p,\alpha,\beta}(\mathbb{C} \setminus [-1, 1]) := \{F \in \mathcal{H}_r(\mathbb{C} \setminus [-1, 1]) : \tilde{F}\mathbb{1}_I \in L_{p,0}(I, \mu_{\alpha,\beta})\}$$

is a Banach space with respect to the norm

$$\|F\|_{\mathcal{H}_r(\mathbb{C}\setminus[-1,1])} + \|\tilde{F}\|_{L_p(I, \mu_{\alpha,\beta})}.$$

Proof. Let $(F_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{H}_{r,p,\alpha,\beta}(\mathbb{C} \setminus [-1, 1])$ with respect to the given norm. Since $\mathcal{H}_r(\mathbb{C}\setminus[-1, 1])$ is a Banach space, we know that $\|F_n - F\|_{\mathcal{H}_r(\mathbb{C}\setminus[-1,1])} \to 0$ $(n \to \infty)$ for some $F \in \mathcal{H}_r(\mathbb{C} \setminus [-1, 1])$. Since $L_{p,0}(I, \mu_{\alpha,\beta})$ is a Banach space (see Lemma 5.3.9), we know that there is some $h \in L_{p,0}(I, \mu_{\alpha,\beta})$ such that $\|\tilde{F}_n - h\|_{L_p(I,\mu_{\alpha,\beta})} \to 0$ $(n \to \infty)$. Let us now show that $\tilde{F}\mathbb{1}_I = h$. We have

$$\|\tilde{F} - h\|_{L_2(I)} \le \|h - \tilde{F}_n\|_{L_2(I)} + \|\tilde{F}_n - \tilde{F}\|_{L_2(I)}$$

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$$
\begin{aligned}
&\leq\ K_{p,\alpha,\beta}\|h - \tilde{F}_n\|_{L_p(I,\mu_{\alpha,\beta})} + K_{r,1}\|\tilde{F}_n - \tilde{F}\|_{L_r(I)}\\
&=\ K_{p,\alpha,\beta}\|h - \tilde{F}_n\|_{L_p(I,\mu_{\alpha,\beta})} + K_{r,1}\|H_I C^{-1}(F_n - F)\|_{L_r(I)}\\
&\leq\ K_{p,\alpha,\beta}\|h - \tilde{F}_n\|_{L_p(I,\mu_{\alpha,\beta})} + K_{r,2}\|C^{-1}(F_n - F)\|_{L_r(I)}\\
&\to\ 0\ (n \to \infty)
\end{aligned}
$$

where the constant $K_{p,\alpha,\beta} > 0$ is chosen from Lemma 5.3.10, the constant $K_{r,1} > 0$ from the fact that $L_r(I) \subset L_2(I)$ for $r \geq 2$ and $K_{r,2} > 0$ from the continuity of the finite Hilbert transformation H_I on $L_r(I)$ (see Remark and Definition 5.3.4). Notice that the last term approaches 0 since the inverse Cauchy transformation is continuous on $L_r(I)$. $\qquad\square$

It is now a short way to prove the following theorem.

Theorem 5.3.14 *Let $p \in [2,\infty)$ and $\alpha,\beta \in (-1,\frac{p}{2}-1)$. Then, the Cauchy transformation establishes an isomorphism between $L_p(I,\mu_{\alpha,\beta})$ and $\mathcal{H}_{2,p,\alpha,\beta}(\mathbb{C}\backslash[-1,1])$. In particular, a function $F \in H(\mathbb{C}\backslash[-1,1])$ belongs to $\mathcal{H}_{2,p,\alpha,\beta}(\mathbb{C}\backslash[-1,1])$ if and only if there exists $f \in L_p(I,\mu_{\alpha,\beta})$ such that*

$$F(z) = (Cf)(z) \quad (z \in \mathbb{C} \setminus [-1,1]).$$

In this case, f is unique and there are constants $K_{p,\alpha,\beta,1}, K_{p,\alpha,\beta,2} > 0$ only depending on p,α,β such that

$$
\begin{aligned}
\|F\|_{\mathcal{H}_p(\mathbb{C}\backslash[-1,1])} + \|\tilde{F}\|_{L_r(I,\mu_{\alpha,\beta})} &\leq K_{p,\alpha,\beta,1}\|f\|_{L_p(I,\mu_{\alpha,\beta})}\\
&\leq K_{p,\alpha,\beta,2} \cdot (\|F\|_{\mathcal{H}_p(\mathbb{C}\backslash[-1,1])} + \|\tilde{F}\|_{L_r(I,\mu_{\alpha,\beta})}).
\end{aligned}
$$

Proof. If $f \in L_p(I,\mu_{\alpha,\beta}) \subset L_2(I)$, then $Cf \in \mathcal{H}_2(\mathbb{C} \setminus [-1,1])$ and $\widetilde{Cf}\mathbb{1}_I = iH_I f$. Therefore, the mapping is well-defined by Proposition 5.3.12 and also injective since $L_p(I,\mu_{\alpha,\beta}) \subset \mathscr{C}$. Moreover, Proposition 3.2.2 and Proposition 5.3.12 imply that

$$
\begin{aligned}
\|Cf\|_{\mathcal{H}_2(\mathbb{C}\backslash[-1,1])} + \|\widetilde{Cf}\|_{L_p(I,\mu_{\alpha,\beta})} &\leq\ K_p\|f\|_{L_2(I)} + \|H_I f\|_{L_p(I,\mu_{\alpha,\beta})}\\
&\leq\ K_{p,\alpha,\beta} \cdot \|f\|_{L_p(I,\mu_{\alpha,\beta})}
\end{aligned}
$$

from which we conclude that C is continuous. Thus, we only have to show surjectivity since then C must be an isomorphism by Lemma 5.3.13 and the Banach isomorphism theorem. Therefore, we pick $F \in \mathcal{H}_{2,p,\alpha,\beta}(\mathbb{C} \setminus [-1,1])$. Then, there is some $f_1 \in L_2(I)$ such that $Cf_1 = F$ and hence $\tilde{F} = iH_I f_1$. Moreover, by Proposition 5.3.12 we know that there is some $f_2 \in L_p(I,\mu_{\alpha,\beta})$ such that $\tilde{F} = H_I f_2$. Since the finite Hilbert transformation is injective on $L_2(I)$ by Remark 5.3.6 we conclude that $f_1 = -if_2$ and therefore $f_1 \in L_p(I,\mu_{\alpha,\beta})$. This concludes the proof. $\qquad\square$

Let us turn back to our magic triangle in Proposition 2.3.7. Our goal is to deduce a related Paley-Wiener theorem from the results above.

Remark and Definition 5.3.15 Let $m \in \mathbb{N}_0$ and $\alpha,\beta > -1$. Then, the function $P_m^{(\alpha,\beta)} : I \to \mathbb{R}$ defined by

$$P_m^{(\alpha,\beta)}(x) := \frac{1}{2^m}\sum_{k=0}^{m}\binom{\alpha+m}{k}\binom{\beta+m}{m-k}(x-1)^{m-k}(x+1)^k$$

is called the m-th **Jacobi polynomial** (with parameters α and β). If $\alpha = \beta = -\frac{1}{2}$, then the Jacobi polynomials become the **Chebyshev polynomials** $(T_m)_{m\in\mathbb{N}_0}$ where $T_m(x) = \cos(m \arccos(x))$ $(x \in (-1,1))$. It is a well-known fact that the sequence $(P_m^{(\alpha,\beta)})_{m\in\mathbb{N}_0}$ forms an orthogonal basis in $L_2(I, \mu_{\alpha,\beta})$, see [1], p. 774.

Remark 5.3.16 Let $\alpha, \beta \in (-1, 0)$. Then, we have $\|f\|_2 \leq \|f\|_{L_2(I,\mu_{\alpha,\beta})}$ for all $f \in L_2(I, \mu_{\alpha,\beta})$. Since $(P_m^{(\alpha,\beta)})_{m\in\mathbb{N}_0}$ forms an orthogonal basis in $L_2(I, \mu_{\alpha,\beta})$ we see by Plancherel's theorem that

$$\mathcal{F}(L_2(I, \mu_{\alpha,\beta})) \subset \overline{\operatorname{span}}\left\{\widehat{P_m^{(\alpha,\beta)}} : m \in \mathbb{N}_0\right\}$$

where the closure is taken with respect to the norm $\|\cdot\|_2$.

One question still remains: What are the Fourier transforms of the Jacobi polynomials? The answer is given by the following result which can be found in [14, Theorem 3.1]. For convenience we write

$$E_n(x) = \sum_{j=0}^{n} \frac{x^j}{j!} \quad (x \in \mathbb{R})$$

for the n-th partial sum of the exponential function. Moreover, for $z \in \mathbb{C}$ and $k \in \mathbb{N}$ the number

$$(z)_k := \prod_{n=0}^{k-1}(z - n)$$

is called the **falling factorial** of z.

Proposition 5.3.17 (Dixit, Jiu, Moll, Vignat, [14]) *Let $m \in \mathbb{N}_0$ and $\alpha, \beta > -1$. Then,*

$$
\begin{aligned}
\widehat{P_m^{(\alpha,\beta)}}(t) &= 2e^{-it}(\alpha+1)_m \sum_{k=0}^{m} \frac{(m+\alpha+\beta+1)_k}{(m-k)!(\alpha+1)_k}\left[\frac{e^{2it}E_k(-2it) - 1}{(2it)^{k+1}}\right] \\
&= 2\sum_{k=0}^{m} \frac{(m+\alpha+\beta+1)_k}{(2it)^{k+1}(m-k)!}[(-1)^{m-k}e^{it}(\beta+k+1)_{m-k} - e^{-it}(\alpha+k+1)_{m-k}]
\end{aligned}
$$

for $t \in \mathbb{R} \setminus \{0\}$ and

$$\widehat{P_m^{(\alpha,\beta)}}(0) = \frac{m+\alpha+\beta+1}{2}\left[\binom{\alpha+m}{m-1} + (-1)^{m-1}\binom{\beta+m}{m-1}\right].$$

Example 5.3.18 For the Chebyshev polynomials $(T_m)_{m\in\mathbb{N}_0}$ we get

$$\widehat{T_m}(t) = \sum_{k=0}^{m} \frac{m2^k(m+k)!k!}{(m-k)!(2k)!(m+k)}\frac{(-1)^{m-k}e^{it} - e^{-it}}{(it)^{k+1}} \quad (t \in \mathbb{R} \setminus \{0\})$$

and

$$\widehat{T_m}(0) = \int_{-1}^{1} T_m(x)\, dx = \begin{cases} 2, & m = 0 \\ 0, & m = 1 \\ \dfrac{1 + (-1)^m}{1 - m^2}, & m > 1. \end{cases}$$

Now, we can deduce a Paley-Wiener theorem which brings everything together.

Proposition 5.3.19 *Let $\alpha, \beta \in (-1, 0)$ and $F \in H(\Pi_+)$. If F extends to a function $G \in \mathcal{H}_{2,2,\alpha,\beta}(\mathbb{C} \setminus [-1, 1])$, then*

$$F(z) = (Lf)(z) \quad (z \in \Pi_+)$$

for some $f \in \overline{\mathrm{span}} \left\{ 2e^{-it}(\alpha + 1)_m \sum_{k=0}^{m} \frac{(m + \alpha + \beta + 1)_k}{(m - k)!(\alpha + 1)_k} \left[\frac{e^{2it}E_k(-2it) - 1}{(2it)^{k+1}} \right] : m \in \mathbb{N}_0 \right\}$. In this case, f is unique and we have

$$\|f\|_2 \leq K(\|G\|_{\mathcal{H}_2(\mathbb{C} \setminus [-1,1])} + \|\tilde{G}\|_{L_2(I, \mu_{\alpha,\beta})})$$

where the constant $K > 0$ does not depend on either f or G.

Proof. Due to Theorem 5.3.14 there is a unique $h \in L_2(I, \mu_{\alpha,\beta})$ such that $Ch = G$. But now, by Proposition 2.3.7 it holds that $L\hat{h} = G = F$ on Π_+ and we conclude by Proposition 5.3.17 that $f = \hat{h}$ is as desired. The estimate follows from Plancherel's theorem and and Theorem 5.3.14. $\qquad\square$
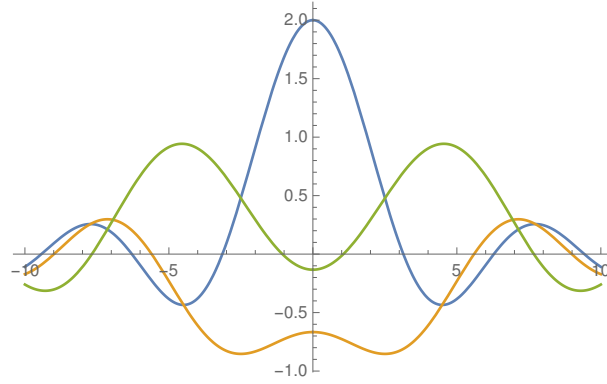

Figure 5.2: Plots of the real part of $\widehat{T_0}$ (blue), $\widehat{T_2}$ (yellow) and $\widehat{T_4}$ (green).
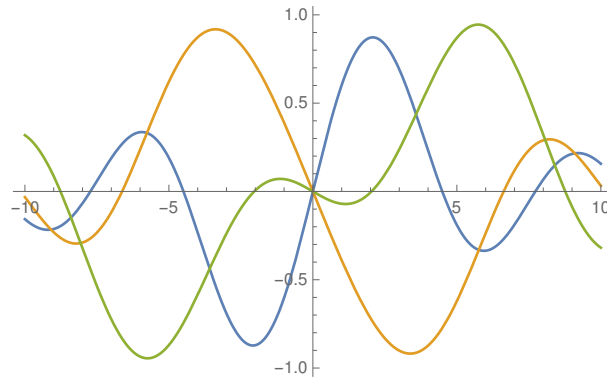

Figure 5.3: Plots of the imaginary part of $\widehat{T_1}$ (blue), $\widehat{T_3}$ (yellow) and $\widehat{T_5}$ (green).

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Chapter 6

Integrability of Cauchy Transforms

> Pure mathematics is, in its way, the
> poetry of logical ideas.

<div align="right">

Albert Einstein
</div>

Beside the Hardy spaces, there is another class of Banach spaces of analytic functions which plays an important in complex analysis. There is talk of the Bergman spaces. These spaces consist of analytic functions which are integrable on a certain open subset of the complex plane subject to a certain measure on this set. If one considers Hardy and Bergman spaces on the unit disk, then one sees close connections between these spaces, see [18]. However, the situation changes a lot if one switches to the upper half-plane or to $\mathbb{C} \setminus \mathbb{R}$. While the Cauchy transform of a function $f \in L_p(\mathbb{R})$ where $p \in (1, \infty)$ always belongs to the Hardy space $\mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})$, it may happen that it does not belong to the (unweighted) Bergman space on $\mathbb{C} \setminus \mathbb{R}$. Hence, it is a very delicate question whether and under which conditions the Cauchy transform of a measure or a function is integrable with respect to a certain measure on the upper (or lower) half-plane.

In order to answer this question, we shall first introduce weighted and unweighted Bergman spaces and derive basic conditions when the Cauchy transform belongs to these spaces. Furthermore, we will establish so-called Zen spaces which are kind of a refinement of Bergman spaces. For our case of Cauchy transforms, Bergman and Zen spaces will however essentially be the same. The reason why we put our focus on these spaces is that there has been a deep investigation of Zen spaces in context with Fourier-Laplace transforms. In particular, the authors in [31] or [60] have used the concept of Zen spaces to formulate generalized Paley-Wiener theorems. Since Fourier transforms and Cauchy transforms are closely related we shall see that the integrability of the Fourier transform will imply the integrability of the corresponding Cauchy transform and vice versa. Finally, we will draw a connection between Bergman spaces and potential theory. Therefore, we shall first show a formula for the logarithmic energy of a measure by means of its Fourier transform. Using Bessel functions, one can generalize this formula to the multi-dimensional case which we will do in the last section of this chapter. We will close with a result which gives a simple sufficient condition

for integrability.

It should be remarked that the integrability of Cauchy transforms has been discussed before, namely in [43]. Here, the authors considered the Cauchy transform C_μ on the space $L_2(\mathbb{R}, \mu)$ for a locally finite, positive Borel measure μ on $\mathbb{R}$. The task was to determine all measures σ on Π_+ such that C_μ is a bounded operator from $L_2(\mathbb{R}, \mu)$ to $L_2(\Pi_+, \sigma)$. Their approach based on so-called A_2-conditions, dyadic grids and decomposition theorems. Our research will differ in two main points. First, we will generally discuss the case of measures and of functions in $L_p(\mathbb{R})$. The results in [43] do not apply to this situation. And second, we will also put our focus on the integrability of derivatives of Cauchy transforms which was also not part of [43].

6.1 Weighted Bergman and Zen spaces

We start with a general definition of Bergman spaces.

Definition 6.1.1 Let Ω be an open set in the complex plane and $p \in [1, \infty)$. For $\nu \in \mathcal{M}_\infty(\Omega)$, we define the **weighted Bergman space** $\mathcal{A}_\nu^p(\Omega)$ which is the set of all $F \in H(\Omega)$ that fulfill

$$\|F\|_{\mathcal{A}_\nu^p(\Omega)} := \left(\int_\Omega |F(z)|^p \, d|\nu|(z) \right)^{\frac{1}{p}} < +\infty.$$

Here, $\mathcal{A}^p(\Omega) := \mathcal{A}_{\lambda_2}^p(\Omega)$ is the (unweighted) **Bergman space** on Ω.

Remark 6.1.2   1. If $p \in (1, \infty)$ and $f \in L_p(\mathbb{R})$, then $Cf \in \mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})$ by Theorem 2.2.11. Therefore, we have $Cf \in \mathcal{A}_{\lambda_1 \otimes \nu}^p(\mathbb{C} \setminus \mathbb{R})$ for all $\nu \in \mathcal{M}(\mathbb{R})$ and in this case

$$\|Cf\|_{\mathcal{A}_{\lambda_1 \otimes \nu}^p(\mathbb{C} \setminus \mathbb{R})} \leq (|\nu|(\mathbb{R}))^{\frac{1}{p}} \|Cf\|_{\mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})}.$$

A similar argumentation shows that if $f \in H_1(\mathbb{R})$, then $Cf \in \mathcal{A}_{\lambda_1 \otimes \nu}^1(\mathbb{C} \setminus \mathbb{R})$ for all $\nu \in \mathcal{M}(\mathbb{R})$ and in this case

$$\|Cf\|_{\mathcal{A}_{\lambda_1 \otimes \nu}^1(\mathbb{C} \setminus \mathbb{R})} \leq |\nu|(\mathbb{R}) \|Cf\|_{\mathcal{H}_1(\mathbb{C} \setminus \mathbb{R})}.$$

2. Let $p, r \in [1, \infty)$. If $f \in H_{p,\infty}^+(\mathbb{R}) \oplus H_{p,\infty}^-(\mathbb{R})$, then $Cf \in \mathcal{A}_\nu^r(\mathbb{C} \setminus \mathbb{R})$ for all $\nu \in \mathcal{M}(\mathbb{C} \setminus \mathbb{R})$. This follows from the fact that Cf is bounded on $\mathbb{C} \setminus \mathbb{R}$, see Corollary 2.4.4.

We shall now give a sufficient condition for the boundedness of derivatives of the Cauchy transformation, both on $\mathcal{M}(\mathbb{R})$ and on $L_p(\mathbb{R})$.

Proposition 6.1.3 *Let $p, r \in [1, \infty)$ and $n \in \mathbb{N}_0$.*

1. If $\mu \in \mathcal{M}(\mathbb{R})$, then $(C\mu)^{(n)} \in \mathcal{A}_\nu^r(\mathbb{C} \setminus \mathbb{R})$ for each $\nu \in \mathcal{M}_\infty(\mathbb{C} \setminus \mathbb{R})$ satisfying

$$\int \frac{1}{|\mathrm{Im}(z)|^{r(n+1)}} \, d|\nu|(z) < +\infty.$$

2. If $f \in L_p(\mathbb{R})$, then $(Cf)^{(n)} \in \mathcal{A}_\nu^r(\mathbb{C} \setminus \mathbb{R})$ for all $\nu \in \mathcal{M}_\infty(\mathbb{C} \setminus \mathbb{R})$ satisfying

$$\int \left(\frac{1}{|\mathrm{Im}(z)|}\right)^{\frac{r}{p}+rn} d|\nu|(z) < +\infty.$$

Proof. For the first statement, notice that

$$|(C\mu)^{(n)}(z)|^r \le \frac{n!^r}{(2\pi)^r} \left(\int \frac{1}{|t-z|^{n+1}} d|\mu|(t)\right)^r \le \frac{n!^r \|\mu\|^r}{(2\pi)^r |\mathrm{Im}(z)|^{r(n+1)}}$$

and the first statement follows. Hence, the second statement is clear for $p = 1$ and we only have to treat the case $p > 1$. Here, Hölder's inequality gives us

$$|(Cf)^{(n)}(z)|^r \le \frac{n!^r}{(2\pi)^r} \left(\int \frac{|f(t)|}{|t-z|^{n+1}} dt\right)^r \le \left(\frac{n!\|f\|_p}{2\pi}\right)^r \cdot \left(\int_{\mathbb{R}} \frac{1}{|t-z|^{q(n+1)}} dt\right)^{\frac{r}{q}},$$

where q is the conjugate exponent of p. But by the substitution $t = \mathrm{Re}(z) - \mathrm{Im}(z)s$ we see that

$$\int_{\mathbb{R}} \frac{1}{|t-z|^{q(n+1)}} dt = \int_{\mathbb{R}} \left(\frac{1}{|\mathrm{Im}(z) \cdot (s+i)|}\right)^{q(n+1)} |\mathrm{Im}(z)| \, ds$$

$$= \frac{1}{|\mathrm{Im}(z)|^{q(n+1)-1}} \int_{\mathbb{R}} \left(\frac{1}{s^2+1}\right)^{\frac{q}{2}(n+1)} ds$$

$$= \frac{K_{q,n}}{|\mathrm{Im}(z)|^{q(n+1)-1}}$$

with some constant $K_{q,n} > 0$ only depending on q and n. This concludes the proof since $\frac{r}{q}(q(n+1)-1) = \frac{r}{p} + rn$. $\qquad\square$

Remark 6.1.4 The converse of Proposition 6.1.3 is false. Consider for example $f : \mathbb{R} \to \mathbb{C}$ with

$$f(t) = \frac{1}{\pi(1+t^2)} \quad (t \in \mathbb{R}).$$

Then, we have (see Example B.5)

$$(Cf)(z) = \begin{cases} -\dfrac{1}{2\pi i(z+i)}, & z \in \Pi_+ \\ -\dfrac{1}{2\pi i(z-i)}, & z \in \Pi_-. \end{cases}$$

Now if $\nu = \lambda_1 \otimes (g\lambda_1)$ where $g(y) = \frac{1}{y+1}\mathbb{1}_{(0,\infty)}(y)$ $(y \in \mathbb{R})$, then

$$\int |(Cf)(z)|^2 d\nu(z) = \int_0^\infty \frac{1}{4\pi(y+1)^2} dy < +\infty.$$

Therefore, $Cf \in \mathcal{A}_\nu^2(\mathbb{C} \setminus \mathbb{R})$ but clearly

$$\int \frac{1}{|\mathrm{Im}(z)|^2} \, d\nu(z) = \int_{-\infty}^\infty \int_0^\infty \frac{1}{y^2(y+1)} \, dy \, dx = +\infty$$

$$\int \frac{1}{|\mathrm{Im}(z)|} \, d\nu(z) = \int_{-\infty}^\infty \int_0^\infty \frac{1}{y^2 + y} \, dy \, dx = +\infty.$$

We notice that Cf does not belong to the unweighted Bergman space $\mathcal{A}^2(\mathbb{C} \setminus \mathbb{R})$. But the following is true: If $\nu \in \mathcal{M}_\infty(\mathbb{R})$ is such that

$$\int \frac{1}{|t| + 1} \, d|\nu|(t) < +\infty,$$

then $Cf \in \mathcal{A}_{\lambda_1 \otimes \nu}^2(\mathbb{C} \setminus \mathbb{R})$. The same holds if $\nu \in \mathcal{M}_\infty(\mathbb{R})$ satisfies

$$\int_{-\infty}^\infty |t|^{-1} \, d|\nu|(t) < +\infty.$$

We show that the observations in the latter remark are no coincidence.

Proposition 6.1.5 *Let* $\mu \in \mathcal{M}(\mathbb{R}), p \in [1, \infty)$ *and* $n > \frac{1}{p} - 1$. *If* $\nu \in \mathcal{M}_\infty(\mathbb{R})$ *is* σ-*finite and satisfies*

$$\int_{-\infty}^\infty |t|^{1-p(n+1)} \, d|\nu|(t) < +\infty,$$

then $(C\mu)^{(n)} \in \mathcal{A}_{\lambda_1 \otimes \nu}^p(\mathbb{C} \setminus \mathbb{R})$.

Proof. First notice that

$$|(C\mu)^{(n)}(x+iy)|^p \leq \|\mu\|^{p-1} \left(\frac{n!}{2\pi}\right)^p \cdot \int_{\mathbb{R}} \frac{1}{|t - (x+iy)|^{p(n+1)}} \, d|\mu|(t),$$

by Jensen's inequality. Now, notice that for fixed $t \in \mathbb{R}, y \in \mathbb{R} \setminus \{0\}$, we have

$$\int_{\mathbb{R}} \frac{1}{|t - (x+iy)|^{p(n+1)}} \, dx = \frac{K_{p,n}}{|y|^{p(n+1)-1}}$$

with some constant $K_{p,n} > 0$ only depending on p and n. If we put everything together and apply the Fubini theorem, we get

$$\int |(C\mu)^{(n)}(x+iy)|^p \, dx \, d|\nu|(y) \leq K_{p,n} \cdot \left(\frac{n!\|\mu\|}{2\pi}\right)^p \cdot \int_{-\infty}^\infty \frac{1}{|y|^{p(n+1)-1}} \, d|\nu|(y) < +\infty.$$

$\square$

Corollary 6.1.6 *Let* $p \in [1, \infty), n > \frac{1}{p} - 1$ *and* $\nu \in \mathcal{M}_\infty(\mathbb{R})$ *be* σ-*finite. Then, the following are equivalent:*

a) *The mapping* $C_n : \mathcal{M}(\mathbb{R}) \to \mathcal{A}_{\lambda_1 \otimes \nu}^p(\mathbb{C} \setminus \mathbb{R}), \ \mu \mapsto (C\mu)^{(n)}$ *is bounded.*

b) $(C\mu)^{(n)} \in \mathcal{A}^p_{\lambda_1 \otimes \nu}(\mathbb{C} \setminus \mathbb{R})$ *for all* $\mu \in \mathcal{M}(\mathbb{R})$.

c) $\int_{-\infty}^{\infty} t^{1-p(n+1)}\, d|\nu|(t) < +\infty$.

Proof. a) clearly implies b). Now, suppose that b) holds. If we take $\mu = \delta_0$ we have

$$(C\mu)(z) = -\frac{1}{2\pi i z} \quad (z \in \mathbb{C}^*)$$

and hence

$$
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |(C\mu)^{(n)}(x+iy)|^p\, dx\, d|\nu|(y) \;=\; \frac{n!^p}{(2\pi)^p}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(x^2+y^2)^{\frac{p(n+1)}{2}}}\, dx\, d|\nu|(y)
$$

$$
=\; K_{p,n}\int_{-\infty}^{\infty} \frac{1}{|y|^{p(n+1)-1}}\, d|\nu|(y)
$$

where the constant $K_{p,n}$ only depends on p and n. Therefore, c) holds.
If c) holds, then a) is valid by Proposition 6.1.5. $\qquad\square$

Remark 6.1.7 It is not possible to formulate a characterization for the measures $\nu \in \mathcal{M}_\infty(\mathbb{R})$ such that $C\mu \in \mathcal{A}^p_{\nu \otimes \lambda_1}(\mathbb{C} \setminus \mathbb{R})$ for all $\mu \in \mathcal{M}(\mathbb{R})$. Because such a measure must satisfy

$$\int_{\mathbb{R}} \frac{1}{|x|^{p-1}}\, d|\nu|(x) < +\infty$$

which can again be seen by taking the Dirac measure δ_0. But on the other side, the measure $\nu = f\lambda_1$ where $f(x) = \dfrac{|x|^{p-1}}{|x-1|^r(x^2+1)}$ where $r = 2-p$ for $p \in (1,2)$ and $r = 0$ else satisfies this condition but if $\mu = \delta_1$, then

$$
\int_{\Pi_+} |(C\mu)(z)|^p\, d(\nu \otimes \lambda_1)(z) = K_p \begin{cases} \displaystyle\int_{-\infty}^{\infty} \frac{|x|^{p-1}}{|x-1|^{p-1+r}(x^2+1)}\, dx = +\infty, & p < 2 \\[2ex] \displaystyle\int_{-\infty}^{\infty} \frac{|x|^{p-1}}{|x-1|^{p-1}(x^2+1)}\, dx = +\infty, & p \geq 2 \end{cases}
$$

where the constant $K_p > 0$ only depends on p.

In what follows, we will introduce a refined version of Bergman spaces. Since we bring once again the Fourier transformation into play, it is convenient to consider functions on the upper half-plane (however modified results also hold for the lower half-plane). Therefore, we shall only consider measures which satisfy a certain growth condition.

Definition 6.1.8 Let $\nu \in \mathcal{M}_{\infty,+}(\mathbb{R})$ with $\operatorname{supp}(\nu) \subset [0,\infty)$. We say that ν satisfies the Δ_2**-condition** if

$$\sup_{t>0} \frac{\nu([0,2t))}{\nu([0,t))} < +\infty.$$

We denote by $\mathcal{M}_{\Delta_2}$ the set of all positive measures which satisfy the Δ_2-condition. Moreover, $\mathcal{M}_{\Delta_2,0}$ is the subset of all $\nu \in \mathcal{M}_{\Delta_2}$ with the property that $\nu(\{0\}) = 0$.

Example 6.1.9 1. The Dirac measure δ_0 is a measure which satisfies the Δ_2-condition.

2. The measure $\omega\lambda_1$ where $\omega(x) = \frac{1}{x+1}\mathbb{1}_{[1,\infty)}(x)$ $(x \in \mathbb{R})$ does not satisfy the Δ_2-condition.

3. The Lebesgue measure λ_1 satisfies the Δ_2-condition.

Remark 6.1.10 Note that a measure $\nu \in \mathcal{M}_{\Delta_2}$ is locally finite, hence (since we are dealing with a measure on $[0, \infty)$) σ-finite and the product measure $\lambda_1 \otimes \nu$ is a regular Borel measure on $\mathbb{R} \times [0, \infty) = \overline{\Pi}_+$.

Definition 6.1.11 Let $\nu \in \mathcal{M}_{\Delta_2}$. Then, for $p \in [1, \infty)$ a function $F \in H(\Pi_+)$ belongs to the **Zen space** $\mathcal{Z}_\nu^p(\Pi_+)$ if

$$\|F\|_{\mathcal{Z}_\nu^p(\Pi_+)} := \sup_{r>0} \left(\int_{\overline{\Pi}_+} |F(\cdot + ir)|^p \, d(\lambda_1 \otimes \nu) \right)^{\frac{1}{p}} < +\infty.$$

Remark 6.1.12 If ν is the Dirac measure concentrated at 0, i.e. $\nu = \delta_0$, then $\mathcal{Z}_\nu^p(\Pi_+)$ is the Hardy space on the upper half plane $\mathcal{H}_p(\Pi_+)$.

The following useful characterization of Zen spaces can be found in [60, Corollary 4.2]:

Lemma 6.1.13 (Peloso, Salvatori, [60]) *Let $\nu \in \mathcal{M}_{\Delta_2}$ and $p \in [1, \infty)$. Then, $\mathcal{Z}_\nu^p(\Pi_+)$ is a Banach space and the following statements hold:*

1. If $\nu(\{0\}) = 0$, then

$$\mathcal{Z}_\nu^p(\Pi_+) = \bigcap_{a>0} \mathcal{H}_p(ia + \Pi_+) \cap \mathcal{A}_{\lambda_1 \otimes \nu}^p(\Pi_+).$$

2. If $\nu(\{0\}) > 0$, then

$$\mathcal{Z}_\nu^p(\Pi_+) = \mathcal{H}_p(\Pi_+) \cap L_p(\overline{\Pi}_+, \lambda_1 \otimes \nu).$$

In both cases, we have

$$\|F\|_{\mathcal{Z}_\nu^p(\Pi_+)}^p = \int_{\overline{\Pi}_+} |F|^p \, d(\lambda_1 \otimes \nu).$$

Remark 6.1.14 Notice that by Lemma 6.1.13 we know that if $\nu(\{0\}) > 0$, then each $F \in \mathcal{Z}_\nu^p(\Pi_+)$ admits a boundary function on the real axis.

For our purposes which always fit into the case $\nu(\{0\}) = 0$ the Zen spaces can therefore be seen as refined Bergman spaces. Similar to the case of Bergman spaces also the Zen spaces appear in context with the Fourier-Laplace transform. In order to use these results we want to show that the Cauchy transform of a measure $\mu \in \mathcal{M}(\mathbb{R})$ or a function $f \in L_p(\mathbb{R})$ where $p \in [1, 2]$ belongs to the Zen space $\mathcal{Z}_\nu^2(\Pi_+)$ if and only if it belongs to the corresponding Bergman space $\mathcal{A}_{\lambda_1 \otimes \nu}^2(\Pi_+)$. We shall use the above characterization and start with a result on the Fourier-Laplace transform.

Recall that for $p \in [1, \infty]$ the space $\mathscr{E}_{0,p}$ consists of those measurable functions $f : \mathbb{R} \to \mathbb{C}$ satisfying $f\mathbb{1}_{[0,\infty)} = f$ and $fe^{-a\cdot} \in L_p(0, \infty)$ for all $a > 0$.

Lemma 6.1.15 *Let $g \in \mathcal{E}_{0,1}$ and $y > 0$. Then,*

$$\int_{-\infty}^{\infty} |(Lg)^{(n)}(x+iy)|^2 \, dx = \frac{1}{2\pi} \int_0^{\infty} |g(t)|^2 t^{2n} e^{-2yt} \, dt.$$

Proof. Let $y > 0$ and $G_y(t) = g(t)(it)^n e^{-yt}$. Then,

$$\int_{-\infty}^{\infty} |G_y(t)| \, dt \leq \max_{t \in (0,\infty)} t^n e^{-\frac{y}{2}t} \cdot \int_0^{\infty} |g(t)| e^{-\frac{y}{2}t} \, dt < +\infty$$

since $g \in \mathcal{E}_{0,1}$ and hence $G_y \in L_1(\mathbb{R})$. Furthermore,

$$\int_{-\infty}^{\infty} |(Lg)^{(n)}(x+iy)|^2 \, dx = \int_{-\infty}^{\infty} \left| \frac{1}{2\pi} \int_0^{\infty} g(t)(it)^n e^{-yt} e^{ixt} \, dt \right|^2 \, dx = \frac{1}{4\pi^2} \|\widehat{G_y}\|_2^2.$$

First, suppose that $G_y \in L_2(\mathbb{R})$. Then we can apply Plancherel's theorem and get

$$\int_{-\infty}^{\infty} |(Lg)^{(n)}(x+iy)|^2 \, dx = \frac{1}{2\pi} \|G_y\|_2^2 = \frac{1}{2\pi} \int_0^{\infty} |g(t)|^2 t^{2n} e^{-2yt} \, dt.$$

If $G_y \notin L_2(\mathbb{R})$, then also $\widehat{G_y} \notin L_2(\mathbb{R})$ (otherwise the fact that $G_y \in L_1(\mathbb{R})$ would imply $G_y \in L_2(\mathbb{R})$) and hence

$$\int_{-\infty}^{\infty} |(Lg)^{(n)}(x+iy)|^2 \, dx = +\infty.$$

$\square$

One interesting consequence of Lemma 6.1.15 is that it allows characterizations when a derivative of a Cauchy transform belongs to the Hardy space $\mathcal{H}_2(\Pi_+)$.

Corollary 6.1.16 *Let $n \in \mathbb{N}_0$.*

1. *If $\mu \in \mathcal{M}(\mathbb{R})$, then $(C\mu)^{(n)} \in \mathcal{H}_2(\Pi_+)$ if and only if*

$$\int_0^{\infty} |\widehat{\mu}(t)|^2 t^{2n} \, dt < +\infty$$

   and in this case

$$\|(C\mu)^{(n)}\|_{\mathcal{H}_2(\Pi_+)}^2 = \frac{1}{2\pi} \int_0^{\infty} |\widehat{\mu}(t)|^2 t^{2n} \, dt.$$

2. *If $p \in [1,2]$ and $f \in L_p(\mathbb{R})$, then $(Cf)^{(n)} \in \mathcal{H}_2(\Pi_+)$ if and only if*

$$\int_0^{\infty} |\widehat{f}(t)|^2 t^{2n} \, dt < +\infty$$

   and in this case

$$\|(Cf)^{(n)}\|_{\mathcal{H}_2(\Pi_+)}^2 = \frac{1}{2\pi} \int_0^{\infty} |\widehat{f}(t)|^2 t^{2n} \, dt.$$

Proof. Let $\mu \in \mathcal{M}(\mathbb{R})$. Since $\widehat{\mu}\mathbb{1}_{(0,\infty)} \in \mathscr{E}_{0,1}$ (notice that $\widehat{\mu}$ is bounded), Proposition 3.1.9, Lemma 6.1.15 and the monotone convergence theorem imply that

$$\sup_{y>0} \int_{-\infty}^{\infty} |(C\mu)^{(n)}(x+iy)|^2 \, dx = \sup_{y>0} \frac{1}{2\pi} \int_0^{\infty} |\widehat{\mu}(t)|^2 t^{2n} e^{-2yt} \, dt = \frac{1}{2\pi} \int_0^{\infty} |\widehat{\mu}(t)|^2 t^{2n} \, dt.$$

The second statement is therefore clear for $p = 1$. If $f \in L_2(\mathbb{R})$, then $\widehat{f}\mathbb{1}_{(0,\infty)} \in \mathscr{E}_{0,1}$ by Hölder's inequality. If $f \in L_p(\mathbb{R})$ where $p \in (1,2)$, then we write $f = f_1 + f_2$ with $f_1 \in L_1(\mathbb{R}), f_2 \in L_2(\mathbb{R})$. Since $\mathscr{E}_{0,1}$ is a vector space over $\mathbb{C}$ we see that $\widehat{f}\mathbb{1}_{(0,\infty)} \in \mathscr{E}_{0,1}$ as well. The rest of the proof is similar to the case of measures. $\square$

Proposition 6.1.17 *Let $p \in (1,\infty), r \in [1,2]$ and $n \in \mathbb{N}_0$. Then, the following statements hold:*

1. *If $\mu \in \mathcal{M}(\mathbb{R})$, then $(C\mu)^{(n)} \in \mathcal{H}_p(ia + \Pi_+)$ for all $a > 0$.*

2. *If $f \in L_r(\mathbb{R})$, then $(Cf)^{(n)} \in \mathcal{H}_2(ia + \Pi_+)$ for all $a > 0$.*

Proof. Let $y > 0$. For the first statement recall that

$$\int |(C\mu)^{(n)}(x+iy)|^p \, dx \leq K_{p,n} \frac{(n! \|\mu\|)^p}{(2\pi)^p |y|^{p(n+1)-1}}$$

with some constant $K_{p,n} > 0$. This implies that the first statement is valid. The second statement is therefore clear for $p = 1$. If $f \in L_2(\mathbb{R})$, then we use Proposition 2.3.7 and Lemma 6.1.15 and see that for $a > 0$ we have

$$\sup_{y>a} \int_{-\infty}^{\infty} |(Cf)^{(n)}(x+iy)|^2 \, dx \leq \frac{1}{2\pi} \int_0^{\infty} |\widehat{f}(t)|^2 t^{2n} e^{-2at} \, dt.$$

Thus, $(Cf)^{(n)} \in \mathcal{H}_2(ia + \Pi_+)$ for each $n \in \mathbb{N}_0$ and $a > 0$. If $f \in L_r(\mathbb{R})$ where $r \in (1,2)$, then there are functions $f_1 \in L_1(\mathbb{R})$ and $f_2 \in L_2(\mathbb{R})$ such that $f = f_1 + f_2$ and we conclude by the linearity of the Cauchy transform that $(Cf)^{(n)} \in \mathcal{H}_2(ia + \Pi_+)$ for each $a > 0$. $\square$

The following corollary is crucial for the subsequent section.

Corollary 6.1.18 *Let $p \in (1,\infty), r \in [1,2], n \in \mathbb{N}_0$ and $\nu \in \mathcal{M}_{\Delta_2,0}$. Then, the following statements hold:*

1. *If $\mu \in \mathcal{M}(\mathbb{R})$, then $(C\mu)^{(n)} \in \mathcal{Z}_\nu^p(\Pi_+)$ if and only if $(C\mu)^{(n)} \in \mathcal{A}_{\lambda_1 \otimes \nu}^p(\Pi_+)$.*

2. *If $f \in L_r(\mathbb{R})$, then $(Cf)^{(n)} \in \mathcal{Z}_\nu^2(\Pi_+)$ if and only if $(Cf)^{(n)} \in \mathcal{A}_{\lambda_1 \otimes \nu}^2(\Pi_+)$.*

6.2 Integrability connections between Cauchy and Fourier transforms

In the following, we say that a measure $\nu \in \mathcal{M}_{\infty,+}(\mathbb{R})$ belongs to the set $\mathcal{M}_{\exp}$ if it has support in $[0, \infty)$ and

$$\frac{1}{2\pi} \int_0^\infty e^{-\alpha y} \, d\nu(y) < +\infty$$

for every $\alpha \in (0, \infty)$. This condition is also referred to as the **exponential condition**. In this case, we write

$$w_{\nu,n}(t) := \begin{cases} \dfrac{t^{2n}}{2\pi} \displaystyle\int_0^\infty e^{-2yt} \, d\nu(y), & t \in (0, \infty) \\ 0, & t \in (-\infty, 0] \end{cases}$$

for $n \in \mathbb{N}_0$ and call this function $w_{\nu,n}$ the **exponential weight of ν of order** n. We shall consider two important subsets of $\mathcal{M}_{\exp}$. The first one is the set $\mathcal{M}_{\mathrm{eps}}$ consisting of all measures $\nu = \omega\lambda_1$ where $\omega : (0, \infty) \to [0, \infty]$ satisfies the ε-**condition**, that is $\omega > 0$ almost everywhere and for each $0 < a < b < \infty$ there exists some $\varepsilon(a,b) > 0$ such that

$$\int_a^b \omega(y)^{-\varepsilon(a,b)} \, dy < +\infty$$

The second subset of $\mathcal{M}_{\exp}$ that we consider is the set $\mathcal{M}_{\Delta_2}$. It is important to notice that we neither have $\mathcal{M}_{\Delta_2} \subset \mathcal{M}_{\mathrm{eps}}$ nor $\mathcal{M}_{\mathrm{eps}} \subset \mathcal{M}_{\Delta_2}$. For example, if $\omega(x) = x\mathbb{1}_{[0,1]}(x)$ $(x \in [0, \infty))$, then $\omega\lambda_1$ satisfies the Δ_2-condition, but not the ε-condition. Conversely, the function $\omega(x) = \frac{1}{x}\mathbb{1}_{(0,\infty)}(x)$ $(x \in [0, \infty))$ satisfies the ε-condition, but the measure $\omega\lambda_1$ does not fulfill the Δ_2-condition since it is not locally finite.

We start with a lemma which is an analogue of a result in [42, Theorem 1].

Lemma 6.2.1 *Let $\nu \in \mathcal{M}_{\exp}$ and $n \in \mathbb{N}_0$. For $g \in \mathscr{E}_{0,1}$ the following statements are equivalent:*

a) $g \in L_2((0, \infty), w_{\nu,n}\lambda_1)$.

b) $(Lg)^{(n)} \in \mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)$.

In this case,

$$\|(Lg)^{(n)}\|_{\mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)} = \|g\|_{L_2((0,\infty), w_{\nu,n}\lambda_1)}.$$

Proof. By Lemma 6.1.15 we have

$$\int_0^\infty \int_{-\infty}^\infty |(Lg)^{(n)}(x+iy)|^2 \, dx \, d\nu(y) \;=\; \frac{1}{2\pi} \int_0^\infty \left(\int_0^\infty |g(t)|^2 t^{2n} e^{-2yt} \, dt \right) d\nu(y)$$

$$=\; \|g\|^2_{L_2((0,\infty), w_{\nu,n}\lambda_1)}.$$

$\square$

Remark 6.2.2 Notice that in general for a measure $\nu \in \mathcal{M}_{\exp}$ it does not hold that $L_2((0,\infty), w_{\nu,n}\lambda_1) \subset \mathcal{E}_{0,1}$. Take for example $\nu = \delta_1$, the Dirac measure concentrated at 1. Then, $w_{\nu,n}(t) = \frac{t^{2n}e^{-2t}}{2\pi}\mathbb{1}_{(0,\infty)}(t)$ and thus, the function $f(t) = e^{\frac{t}{2}}\mathbb{1}_{(0,\infty)}(t)$ belongs to $L_2((0,\infty), w_{\nu,n}\lambda_1)$ but not to $\mathcal{E}_{0,1}$.

Corollary 6.2.3 *Let $\nu \in \mathcal{M}_{\exp}, p \in [1,2]$ and $n \in \mathbb{N}_0$.*

1. *If $\mu \in \mathcal{M}(\mathbb{R})$ is such that $\widehat{\mu}\mathbb{1}_{(0,\infty)} \in L_2((0,\infty), w_{\nu,n}\lambda_1)$, then $(C\mu)^{(n)} \in \mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)$ and in this case*

$$\|(C\mu)^{(n)}\|_{\mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)} = \|\widehat{\mu}\|_{L_2((0,\infty), w_{\nu,n}\lambda_1)}.$$

2. *If $f \in L_p(\mathbb{R})$ is such that $\widehat{f}\mathbb{1}_{(0,\infty)} \in L_2((0,\infty), w_{\nu,n}\lambda_1)$, then $(Cf)^{(n)} \in \mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)$ and in this case*

$$\|(Cf)^{(n)}\|_{\mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)} = \|\widehat{\mu}\|_{L_2((0,\infty), w_{\nu,n}\lambda_1)}.$$

Proof. This follows immediately from Proposition 2.3.7, Proposition 3.1.9 and Lemma 6.2.1.
$\square$

The latter observations are crucial to illustrate the relation between Cauchy transforms and real Fourier transforms. One could ask for a converse of the previous result. If we put extra conditions on the measure, then a converse statement holds.

One of the main ingredients of the proof is a theorem of Paley-Wiener type which is due to Harper and can be found in [31, Theorem 2.1].

Proposition 6.2.4 (Harper, [31]) *Let $\nu \in \mathcal{M}_{\mathrm{eps}}$. A function $F \in H(\Pi_+)$ belongs to $\mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)$ if and only if there is some $g : \mathbb{R} \to \mathbb{C}$ such that*

$$\int_{-\infty}^{\infty} |g(t)|^2 \cdot \left(\int_0^{\infty} e^{-2yt}\, d\nu(y)\right) dt < +\infty$$

and

$$F(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(t)e^{izt}\, dt \quad (z \in \Pi_+).$$

In this case, g is unique and

$$\|F\|^2_{\mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} |g(t)|^2 \cdot \left(\int_0^{\infty} e^{-2yt} d\nu(y)\right) dt.$$

Remark 6.2.5 By Proposition 6.2.4 the space $\mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)$ is a Hilbert space for each $\nu \in \mathcal{M}_{\mathrm{eps}}$.

Remark 6.2.6 For $\alpha \in \mathbb{R}$ we write $\mu_\alpha = t^\alpha \mathbb{1}_{(0,\infty)}(t)\, dt$. Then, Proposition 6.2.4 implies that for $\alpha > -1$ we have an isomorphism between the space of all measurable functions $g : \mathbb{R} \to \mathbb{C}$ satisfying

$$\int_{-\infty}^{\infty} |g(t)|^2 \cdot \left(\int_0^{\infty} y^\alpha e^{-2yt}\, dy\right) dt < +\infty$$

and $\mathcal{A}^2_{\lambda_1 \otimes \mu_\alpha}(\Pi_+)$. But since such a function g has to satisfy $\mathrm{supp}(g) \subset (0,\infty)$, we deduce in particular that the Fourier-Laplace transformation is an isomorphism between $L_2((0,\infty), \mu_{-\alpha-1})$ and $\mathcal{A}^2_{\lambda_1 \otimes \mu_\alpha}(\Pi_+)$ which was exactly what the authors in [17, Theorem 1] proved.

In particular, the Fourier-Laplace transformation establishes an isomorphism between the space $L_2((0, \infty), dt/t)$ and the unweighted Bergman space $\mathcal{A}^2(\Pi_+)$. Applying similar techniques for the lower half-plane Π_- one sees that the Fourier-Laplace transformation is also an isomorphism between $L_2(\mathbb{R}, dt/|t|)$ and $\mathcal{A}^2(\mathbb{C} \setminus \mathbb{R})$. Combining this once again with Proposition 2.3.7 and Proposition 3.1.9 we deduce

Corollary 6.2.7 *The following statements hold:*

1. *Let $\mu \in \mathcal{M}(\mathbb{R})$. Then, $C\mu \in \mathcal{A}^2(\mathbb{C} \setminus \mathbb{R})$ if and only if $\widehat{\mu} \in L_2(\mathbb{R}, dt/|t|)$.*

2. *Let $p \in [1, 2]$ and $f \in L_p(\mathbb{R})$. Then, $Cf \in \mathcal{A}^2(\mathbb{C} \setminus \mathbb{R})$ if and only if $\widehat{f} \in L_2(\mathbb{R}, dt/|t|)$.*

Corollary 6.2.8 *Let $\mu \in \mathcal{M}(\mathbb{R})$ with $\mu(\mathbb{R}) \neq 0$. Then, $C\mu \notin \mathcal{A}^2(\mathbb{C} \setminus \mathbb{R})$.*

Proof. Since $\widehat{\mu}(0) \neq 0$ and $\widehat{\mu}$ is continuous on $\mathbb{R}$ we know that $|\widehat{\mu}|^2 \geq K$ on some interval $(-\varepsilon, \varepsilon)$ for some constant $K > 0$. But this implies $\widehat{\mu} \notin L_2(\mathbb{R}, dt/|t|)$ and hence $C\mu \notin \mathcal{A}^2(\mathbb{C} \setminus \mathbb{R})$ by Corollary 6.2.7. $\qquad\square$

The following theorem of Paley-Wiener type is due to Peloso and Salvatori and can be found in [60, Theorem 4.3].

Proposition 6.2.9 (Peloso, Salvatori, [60]) *Let $\nu \in \mathcal{M}_{\Delta_2}$. A function $F \in H(\Pi_+)$ belongs to $\mathcal{Z}_\nu^2(\Pi_+)$ if and only if there exists some $g \in L_2((0, \infty), w_{\nu,0}\lambda_1)$ such that*

$$F(z) = (Lg)(z) \quad (z \in \Pi_+).$$

In this case, g is unique and

$$\|F\|_{\mathcal{Z}_\nu^2(\Pi_+)} = \|g\|_{L_2((0,\infty), w_{\nu,0}\lambda_1)}.$$

Remark 6.2.10 Proposition 6.2.9 states in particular that for $\alpha > -1$

$$L(L_2((0, \infty), \mu_{-\alpha-1})) = \mathcal{Z}_{\mu_\alpha}^2(\Pi_+).$$

But due to the injectivity of L and Remark 6.2.6 we conclude that $\mathcal{A}_{\lambda_1 \otimes \mu_\alpha}^2(\Pi_+) = \mathcal{Z}_{\mu_\alpha}^2(\Pi_+)$ for all $\alpha > -1$. In particular, the unweighted Bergman space coincides with the unweighted Zen space.

Now, we are ready to prove converse statements to Corollary 6.2.3.

Proposition 6.2.11 *Let $\nu \in \mathcal{M}_{\mathrm{eps}} \cup \mathcal{M}_{\Delta_2,0}, p \in [1, 2]$ and $n \in \mathbb{N}_0$.*

1. *If $\mu \in \mathcal{M}(\mathbb{R})$ is such that $(C\mu)^{(n)} \in \mathcal{A}_{\lambda_1 \otimes \nu}^2(\Pi_+)$, then $\widehat{\mu}\mathbb{1}_{(0,\infty)} \in L_2((0, \infty), w_{\nu,n}\lambda_1)$ and*

$$\|(C\mu)^{(n)}\|_{\mathcal{A}_{\lambda_1 \otimes \nu}^2(\Pi_+)} = \|\widehat{\mu}\|_{L_2((0,\infty), w_{\nu,n}\lambda_1)}.$$

2. If $f \in L_p(\mathbb{R})$ is such that $(Cf)^{(n)} \in \mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)$, then $\widehat{f}\mathbb{1}_{(0,\infty)} \in L_2((0,\infty), w_{\nu,n}\lambda_1)$ and

$$\|(Cf)^{(n)}\|_{\mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)} = \|\widehat{f}\|_{L_2((0,\infty), w_{\nu,n}\lambda_1)}.$$

Proof. We only prove 1. since 2. is similar. At first we remark that, according to Proposition 3.1.9, we have

$$(C\mu)^{(n)}(z) = \frac{1}{2\pi} \int_0^\infty \widehat{\mu}(t)(it)^n e^{izt}\, dt \quad (z \in \Pi_+).$$

Suppose first that $\nu \in \mathcal{M}_{\mathrm{eps}}$. Since $(C\mu)^{(n)} \in \mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)$, we know by Proposition 6.2.4 that there is a unique g such that

$$(C\mu)^{(n)}(z) = (Lg)(z) \quad (z \in \Pi_+)$$

and

$$\|(C\mu)^{(n)}\|^2_{\mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)} = \frac{1}{2\pi} \int_{-\infty}^\infty |g(t)|^2 \cdot \left(\int_0^\infty e^{-2yt}\, d\nu(y) \right)\, dt < +\infty.$$

Hence, the Fourier transforms of the functions $t \mapsto \widehat{\mu}(t)\mathbb{1}_{(0,\infty)}(t)(it)^n \exp(-yt)$ and $g \exp(-y \cdot)$ coincide for all $y \in (0,\infty)$, and by the uniqueness of Fourier transforms, we must have

$$g(t) = \widehat{\mu}(t)(it)^n \mathbb{1}_{(0,\infty)}(t) \quad (t \in \mathbb{R}).$$

In particular, $\mathrm{supp}(g) \subset (0,\infty)$ and

$$\int_0^\infty |\widehat{\mu}(t)|^2 w_{\nu,n}(t)\, dt = \int_0^\infty |g(t)|^2 w_{\nu,0}(t)\, dt.$$

If $\nu \in \mathcal{M}_{\Delta_2,0}$, then $(C\mu)^{(n)} \in \mathcal{Z}^2_\nu(\Pi_+)$ by Lemma 6.1.17. Hence, Proposition 6.2.9 gives us a unique $g : (0,\infty) \to \mathbb{C}$ such that $(C\mu)^{(n)} = Lg$ on Π_+ and moreover

$$\|(C\mu)^{(n)}\|_{\mathcal{Z}^2_\nu(\Pi_+)} = \|(C\mu)^{(n)}\|_{\mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)} = \|g\|_{L_2((0,\infty), w_{\nu,0}\lambda_1)} < +\infty.$$

The rest of the proof is similar to the case $\nu \in \mathcal{M}_{\mathrm{eps}}$. $\qquad\square$

Corollary 6.2.12 Let $p \in [1,2]$, $n \in \mathbb{N}_0$ and $\nu \in \mathcal{M}_{\mathrm{eps}} \cup \mathcal{M}_{\Delta_2,0}$. Moreover, let $F = Cf \in \mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})$. Then $F^{(n)}|_{\Pi_+}$ belongs to $\mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)$ if and only if $\widehat{f}\mathbb{1}_{(0,\infty)} \in L_2((0,\infty), w_{\nu,n}\lambda_1)$ and in this case

$$\|F^{(n)}\|_{\mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)} = \|\widehat{f}\|_{L_2((0,\infty), w_{\nu,n}\lambda_1)}.$$

Proof. If $\widehat{f}\mathbb{1}_{(0,\infty)} \in L_2((0,\infty), w_{\nu,n}\lambda_1)$, then $F^{(n)} = (Cf)^{(n)} \in \mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)$ by Lemma 6.2.3 and

$$\|F^{(n)}\|_{\mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)} = \|\widehat{f}\|_{L_2((0,\infty), w_{\nu,n}\lambda_1)}.$$

Conversely, if $F^{(n)} \in \mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)$, then Proposition 6.2.11 implies $\widehat{f}\mathbb{1}_{(0,\infty)} \in L_2((0,\infty), w_{\nu,n}\lambda_1)$. $\square$

Due to Corollary 6.2.12, for $p \in (1,2]$, the Cauchy transform is also a bijective map between the space

$$\left\{ f \in L_p(\mathbb{R}) : \widehat{f}\mathbb{1}_{(0,\infty)} \in L_2((0,\infty), w_{\nu,n}\lambda_1) \right\}$$

and
$$\{F \in \mathcal{H}_p(\mathbb{C} \setminus \mathbb{R}) : F^{(n)} \in \mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)\}.$$

A similar statement can be said for the case $p = 1$. It is natural to ask if it is an isomorphism as well, i.e., if both spaces are complete (and in which sense). This is indeed the case.

Lemma 6.2.13 *Let $p \in [1, \infty), n \in \mathbb{N}_0$ and $\nu \in \mathcal{M}_{\mathrm{eps}} \cup \mathcal{M}_{\Delta_2,0}$. Then, the space $\{F \in \mathcal{H}_p(\mathbb{C} \setminus \mathbb{R}) : F^{(n)} \in \mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)\}$ is a Banach space with respect to the norm $\|F\|_{\mathcal{H}_p(\mathbb{C}\setminus\mathbb{R})} + \|F^{(n)}\|_{\mathcal{A}^2_{\lambda_1\otimes\nu}(\Pi_+)}$.*

Proof. We pick a Cauchy sequence $(F_k)_{k \in \mathbb{N}}$ which is Cauchy with respect to the given norm. Then, it is Cauchy with respect to $\|\cdot\|_{\mathcal{H}_p(\mathbb{C}\setminus\mathbb{R})}$ and hence there is some $F \in \mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})$ such that $\|F_k - F\|_{\mathcal{H}_p(\mathbb{C}\setminus\mathbb{R})} \to 0$ $(k \to \infty)$. In particular, $F_k^{(n)} \to F^{(n)}$ local uniformly on $\mathbb{C} \setminus \mathbb{R}$, see [50, Lemma 11.5]. On the other hand, since $(F_k)_{n \in \mathbb{N}}$ is Cauchy with respect to $\|\cdot\|_{\mathcal{A}^2_{\lambda_1\otimes\nu}(\Pi_+)}$, we know that there is $\tilde{F} \in \mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)$ such that $\|F_k^{(n)} - \tilde{F}\|_{\mathcal{A}^2_{\lambda_1\otimes\nu}(\Pi_+)} \to 0$ (here we used Lemma 6.1.13 for the case that $\nu \in \mathcal{M}_{\Delta_2,0}$). In particular, $F_k^{(n)} \to \tilde{F}$ almost everywhere on Π_+, at least for a subsequence, and we conclude that $F^{(n)}|_{\Pi_+} \in \mathcal{A}^2_{\lambda_1 \otimes \nu}(\Pi_+)$. $\square$

Remark 6.2.14 Let $n \in \mathbb{N}_0$ and $\nu \in \mathcal{M}_{\mathrm{eps}} \cup \mathcal{M}_{\Delta_2,0}$. Then, Theorem 6.2.12 implies in particular that the following assertions hold:

1. For $p \in (1, 2]$ the space $\left\{f \in L_p(\mathbb{R}) : \widehat{f}\mathbb{1}_{(0,\infty)} \in L_2((0, \infty), w_{\nu,n}\lambda_1)\right\}$ is a Banach space with respect to the norm $\|f\|_p + \|\widehat{f}\|_{L_2((0,\infty),w_{\nu,n}\lambda_1)}$.

2. The space $\left\{f \in H_1(\mathbb{R}) : \widehat{f}\mathbb{1}_{(0,\infty)} \in L_2((0, \infty), w_{\nu,n}\lambda_1)\right\}$ is a Banach space with respect to the norm $\|f\|_{H_1(\mathbb{R})} + \|\widehat{f}\|_{L_2((0,\infty),w_{\nu,n}\lambda_1)}$.

Moreover, the Cauchy transformation is an isomorphism in Theorem 6.2.12 when the spaces are equipped with the associated norms.

Remark 6.2.15 It can be proved by L_p theory and and the monotone convergence theorem that for a measurable function $\omega : \mathbb{R} \to [0, \infty]$ and $\nu = \omega\lambda_1$ the following assertions are equivalent:

a) The mapping
$$\iota : L_2(\mathbb{R}) \to L_2((0, \infty), w_{\nu,0}\lambda_1), \ f \mapsto f\mathbb{1}_{(0,\infty)}$$
is bounded.

b) $f\mathbb{1}_{(0,\infty)} \in L_2((0, \infty), w_{\nu,0}\lambda_1)$ for each $f \in L_2(\mathbb{R})$.

c) $\omega \in L_1(0, \infty)$.

Corollary 6.2.12 gives another proof of this equivalence. We only have to prove that $b)$ implies $c)$. If $b)$ holds and $f \in L_2(\mathbb{R})$, then $\widehat{f}\mathbb{1}_{(0,\infty)} \in L_2((0, \infty), w_{\nu,0}\lambda_1)$ by our assumption and Plancherel's theorem. But by Lemma 6.2.3, we know that $Cf \in \mathcal{A}^2_{\lambda_1\otimes\nu}(\Pi_+)$. Corollary 6.2.12 (or Lemma 2.2.7) now implies that $\mathcal{H}_2(\Pi_+) \subset \mathcal{A}^2_{\lambda_1\otimes\nu}(\Pi_+)$ and hence $\omega \in L_1(0, \infty)$ by Carleson's Theorem, see Theorem 1.2.8.

6.3 An energy formula on the real line

We have seen earlier (Corollary 6.2.8) that if $\mu \in \mathcal{M}(\mathbb{R})$ is such that $\mu(\mathbb{R}) \neq 0$, then $C\mu \notin \mathcal{A}^2(\mathbb{C} \setminus \mathbb{R})$. In particular, our argument was that $\widehat{\mu} \notin L_2(\mathbb{R}, dt/|t|)$ since

$$\int_{|t| \leq 1} \frac{|\widehat{\mu}(t)|^2}{|t|} = +\infty.$$

It is natural to ask what can be said about the integral

$$\int_{|t| \geq 1} \frac{|\widehat{\mu}(t)|^2}{|t|}\, dt.$$

Let us first take a look at the discrete analogue of this integral, i.e. if $\mu \in \mathcal{M}(\mathbb{T})$ (here $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$) we consider

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{|a_k|^2}{|k|}$$

where

$$a_k = \frac{1}{2\pi} \int_{\mathbb{T}} \zeta^{-k}\, d\mu(\zeta) \quad (k \in \mathbb{Z})$$

are the Fourier coefficients of μ. A basic result (see [39], p. 35) states that if $\mu \in \mathcal{M}(\mathbb{T})$ is such that $\mu \geq 0$ or

$$\int \int |\ln(|z - w|)|\, d|\mu|(z)\, d|\mu|(w) < +\infty,$$

then

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{|a_k|^2}{|k|} = 2 \int \int \ln\left(\frac{1}{|z - w|}\right)\, d\mu(z)\, d\mu(w). \tag{6.3.1}$$

This gives especially a characterization when $\sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{|a_k|^2}{|k|}$ is finite.

In the following, we shall derive a similar characterization for the finiteness of our integral. It will hold under fairly general conditions and in particular for positive measures in $\mathcal{M}_c(\mathbb{R})$. Moreover, we will also answer the question when $C\mu$ belongs to $\mathcal{A}^2(\mathbb{C} \setminus \mathbb{R})$.

Due to the continuity of $\widehat{\mu}$ it is necessary that $\mu(\mathbb{R}) = 0$. Unfortunately, this condition is not sufficient. For example, take $\mu = \delta_1 - \delta_{-1}$. Then, $\mu(\mathbb{R}) = 1 - 1 = 0$ but

$$\int_{\mathbb{R}} \frac{|\widehat{\mu}(t)|^2}{|t|}\, dt = 4 \int_{\mathbb{R}} \frac{\sin^2(t)}{|t|}\, dt = +\infty.$$

This suggests that one needs some additional properties of the measure μ in order to formulate a converse statement.

We write $\mathcal{M}_{\ln}(\mathbb{R}^n)$ for the set of all $\mu \in \mathcal{M}(\mathbb{R}^n)$ such that one of the following two conditions is fulfilled:

i) $\mu \in \mathcal{M}_+(\mathbb{R}^n)$ and

$$\int \int \ln\left(\frac{1}{|x-y|}\right) d\mu(y) \, d\mu(x)$$

exists in $[-\infty, +\infty]$,

ii) $\mu \in \mathcal{M}(\mathbb{R}^n)$ and

$$\int \int \left| \ln\left(\frac{1}{|x-y|}\right) \right| d|\mu|(y) \, d|\mu|(x) < +\infty.$$

Definition 6.3.1 Let $\mu \in \mathcal{M}_{\ln}(\mathbb{R}^n)$. Then,

$$p_\mu(x) := \int \ln\left(\frac{1}{|x-y|}\right) d\mu(y)$$

exists in $\mathbb{C} \cup \{\pm\infty\}$ for μ-almost every $x \in \mathbb{R}^n$ and p_μ is called the **logarithmic potential** of μ. Moreover,

$$I(\mu) := \int \int \ln\left(\frac{1}{|x-y|}\right) d\mu(y) \, d\bar{\mu}(x) = \int p_\mu(x) \, d\bar{\mu}(x)$$

is called the **logarithmic energy** of μ.

Remark 6.3.2   1. If $\mu \in \mathcal{M}_+(\mathbb{R}^n)$ satisfies

$$\int \ln(1 + |x|) \, d\mu(x) < +\infty,$$

then $I(\mu) \in (-\infty, +\infty]$ thanks to the inequality $|x-y| \le (1+|x|)(1+|y|)$. In particular, this shows that $\mathcal{M}_{c,+}(\mathbb{R}^n) \subset \mathcal{M}_{\ln}(\mathbb{R}^n)$.

2. Let $\mu, \nu \in \mathcal{M}_{\ln}(\mathbb{R}^n)$ be positive measures such that $I(\mu), I(\nu) \in (-\infty, +\infty]$ or $I(\mu), I(\nu) \in [-\infty, +\infty)$. Then, due to [52, Theorem 2.2], we have

$$2 \int \int \ln\left(\frac{1}{|x-y|}\right) d\mu(x) \, d\nu(y) \le I(\mu) + I(\nu).$$

Moreover, if $I(\mu), I(\nu) \in \mathbb{R}$, then the left side is finite as well. In particular, this implies that if $\mu, \nu \in \mathcal{M}_{\ln}(\mathbb{R}^n)$ are complex measures, then

$$\int \int \left| \ln\left(\frac{1}{|x-y|}\right) \right| d|\mu|(x) \, d|\nu|(y) < +\infty.$$

However, there exist $\mu, \nu \in \mathcal{M}_+(\mathbb{R}) \setminus \mathcal{M}_{\ln}(\mathbb{R})$ such that $\int p_\mu \, d\nu = -\infty$, see [52, Remark 2.3].

Remark and Definition 6.3.3 A subset $E \subset \mathbb{R}^n$ is called **polar** if $I(\mu) = +\infty$ for every $0 \ne \mu \in \mathcal{M}_{c,+}(\mathbb{R}^n)$ with support in E. For compact $K \subset \mathbb{C}(= \mathbb{R}^2)$, one can shows that K is polar if and only if $\mathcal{A}^2(\mathbb{C} \setminus K) = \{0\}$, see [10, Chapter 21, Theorem 9.5].

Remark and Definition 6.3.4 Let $K \subset \mathbb{R}^n$ be compact and non-polar. Then, see [65, Theorem 3.3.2 and Theorem 3.7.6], there exists a unique probability measure ν with

$$I(\nu) = \inf\{I(\mu) : \mu \in \mathcal{M}_+(\mathbb{R}^n), \operatorname{supp}(\mu) \subset K, \mu(\mathbb{R}^n) = 1\}.$$

This measure is called the **equilibrium measure** for K. In this case, the number $c_K := e^{-I(\nu)}$ is called the **logarithmic capacity** of K.

Example 6.3.5   1. If $K = \{x \in \mathbb{R}^2 : |x| = 1\}$, then the equilibrium measure for K is (see [6, Example 4.2.9]) the normalized arc length measure on ∂K, i.e. the measure $2\pi m = \lambda_1^g$, where $g : \mathbb{R} \to \mathbb{R}^2$, $g(t) = (\cos(t), \sin(t))\mathbb{1}_{[-\pi,\pi]}(t)$.

2. If $K = [-1, 1]$, then the equilibrium measure for K is (see [6, Example 4.2.9]) the arcsine distribution,. i.e. the measure $f\lambda_1$, where $f : \mathbb{R} \to \mathbb{R}$, $f(t) = \dfrac{1}{\pi\sqrt{1-t^2}}\mathbb{1}_{(-1,1)}(t)$.

In order to derive a formula which connects the Fourier transform of a measure and its logarithmic energy we shall need some preparation. Our goal is that this formula holds both in the case that the energy is finite and in the case that the energy is not a finite number. We start with a lemma that will turn out quite helpful for the finite case.

Lemma 6.3.6 Let $g \in L_{1,\mathrm{loc}}(0, \infty)$ and $u \in \mathbb{C}$. If $t \mapsto g(t)/t$ is improperly integrable at ∞ and $t \mapsto (g(t) - u)/t$ is improperly integrable at 0, then for arbitrary $a > 0$

$$\lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_\varepsilon^N \frac{g(ra) - g(r)}{r}\, dr = u \ln\left(\frac{1}{a}\right).$$

If, in addition, $g \in L_\infty(\mathbb{R})$, then

$$\left| \int_\varepsilon^N \frac{g(ra) - g(r)}{r}\, dr \right| \leq 2(\|g\|_\infty + |u|)|\ln(a)|.$$

Proof. We may assume that $0 < a < 1$. For fixed $0 < \varepsilon < N < +\infty$ we have

$$
\int_\varepsilon^N \frac{g(ra) - g(r)}{r}\, dr \; = \; \int_{\varepsilon a}^{Na} \frac{g(r)}{r}\, dr - \int_\varepsilon^N \frac{g(r)}{r}\, dr = \left(\int_{\varepsilon a}^\varepsilon - \int_{Na}^N \right) \frac{g(r)}{r}\, dr
$$
$$
= \; u \int_{\varepsilon a}^\varepsilon \frac{dr}{r} + \int_{\varepsilon a}^\varepsilon \frac{g(r) - u}{r}\, dr - \int_{Na}^N \frac{g(r)}{r}\, dr \;.
$$

The first term is $u \ln(1/a)$. According to Cauchy's criterion, the second term vanishes as ε tends to 0 and the third as N tends to ∞. Moreover, if g is essentially bounded we obtain that

$$\left| \int_\varepsilon^N \frac{g(ra) - g(r)}{r}\, dr \right| \leq 2(\|g\|_\infty + |u|)\, |\ln(a)| \;.$$

This concludes the proof. $\qquad\qquad\square$

Corollary 6.3.7 *Let* $a \in \mathbb{R} \setminus \{0\}$. *Then, for* $0 < \varepsilon < N < \infty$ *we have*

$$\lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_\varepsilon^N \frac{\cos(ra) - \cos(r)}{r} \, dr = \ln\left(\frac{1}{|a|}\right)$$

$$\left| \int_\varepsilon^N \frac{\cos(ra) - \cos(r)}{r} \, dr \right| \leq 2 \left| \ln\left(|a|\right) \right|.$$

Proof. Take $g = \cos, u = 1$ in Lemma 6.3.6 and use the symmetry of $\cos$. $\square$

Remark 6.3.8 Due to [30, Lemma 4.2.5], we also have for $a \in \mathbb{R} \setminus \{0\}$ and $0 < \varepsilon < N < \infty$

$$\lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_\varepsilon^N \frac{e^{-ira} - \cos(r)}{r} \, dr = \ln\left(\frac{1}{|a|}\right) - i\frac{\pi}{2}\text{sign}(a),$$

$$\left| \int_\varepsilon^N \frac{e^{-ira} - \cos(r)}{r} \, dr \right| \leq 2 \left| \ln\left(|a|\right) \right| + 4.$$

We take a short look to the situation on the unit circle. If $\mu \in \mathcal{M}(\mathbb{S})$ satisfies $\sum_{k=1}^\infty \frac{|a_k|^2}{k} < +\infty$, then also $\sum_{k=-\infty}^{-1} \frac{|a_k|^2}{|k|} < +\infty$, see [41, Lemma 2]. This is of course clear for real-valued measures since then one always has $a_{-k} = \overline{a_k}$ for all $k \in \mathbb{Z}$ but the point of the matter is that the result also holds for complex measures. The question suggests oneself if an analogue may be true for the continuous situation on the real axis. The answer is positive.

Proposition 6.3.9 *Let* $\mu \in \mathcal{M}(\mathbb{R})$. *Then, the following are equivalent:*

a) $\int_0^\infty \frac{|\widehat{\mu}(t)|^2}{t} \, dt < +\infty.$

b) $\int_{-\infty}^0 \frac{|\widehat{\mu}(t)|^2}{|t|} \, dt < +\infty.$

Proof. For fixed $0 < \varepsilon < N < +\infty$ Fubini's theorem gives us that

$$\int_\varepsilon^N \frac{|\widehat{\mu}(t)|^2}{t} \, dt - \int_{-N}^{-\varepsilon} \frac{|\widehat{\mu}(t)|^2}{-t} \, dt$$

$$= \int_\varepsilon^N \left(\int e^{-ixt} \, d\mu(x) \right) \cdot \left(\int e^{iyt} \, d\bar{\mu}(y) \right) \frac{dt}{t} - \int_{-N}^{-\varepsilon} \left(\int e^{-ixt} \, d\mu(x) \right) \cdot \left(\int e^{iyt} \, d\bar{\mu}(y) \right) \frac{dt}{-t}$$

$$= \int \int \left(\int_\varepsilon^N \frac{e^{i(y-x)t}}{t} \, dt - \int_{-N}^{-\varepsilon} \frac{e^{i(y-x)t}}{-t} \, dt \right) d\mu(x) \, d\bar{\mu}(y)$$

$$= 2i \int \int \int_\varepsilon^N \frac{\sin((y-x)t)}{t} \, dt \, d\mu(x) \, d\bar{\mu}(y)$$

If a) holds, then

$$\left| \int_0^\infty \frac{|\widehat{\mu}(t)|^2}{t} \, dt - \int_{-\infty}^0 \frac{|\widehat{\mu}(t)|^2}{|t|} \, dt \right| = 2 \left| \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int \int \int_\varepsilon^N \frac{\sin((x-y)t)}{t} \, dt \, d\mu(x) \, d\bar{\mu}(y) \right|.$$

Now, let $(x, y) \in \mathbb{R}^2$. By Remark 6.3.8 we know that

$$\lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_\varepsilon^N \frac{\sin((y - x)t)}{t}\, dt = \frac{\pi}{2}\mathrm{sign}(y - x)$$

and that

$$\left| \int_\varepsilon^N \frac{\sin((y - x)t)}{t}\, dt \right| \leq 4$$

for all $0 < \varepsilon < N < +\infty$. The dominated convergence theorem implies that

$$\left| \int_0^\infty \frac{|\widehat{\mu}(t)|^2}{t}\, dt - \int_{-\infty}^0 \frac{|\widehat{\mu}(t)|^2}{|t|}\, dt \right| \leq \pi \left| \int \int \mathrm{sign}(y - x)\, d\mu(x)\, d\bar{\mu}(y) \right| \leq 2\pi \|\mu\|^2.$$

Therefore, also b) holds and the reverse statement is true for symmetry reasons. □

Corollary 6.3.10 *Let $\mu \in \mathcal{M}(\mathbb{R})$. Then, the following are equivalent:*

a) $C\mu \in \mathcal{A}^2(\Pi_+)$.

b) $C\mu \in \mathcal{A}^2(\Pi_-)$.

Remark 6.3.11 *Let $\mu \in \mathcal{M}(\mathbb{R})$ and $n \in \mathbb{N}_0$. Then,*

$$\frac{|\widehat{\mu}(t)|^2 t^{2n}}{|t|^{2n+1}} = \frac{|\widehat{\mu}(t)|^2}{|t|} \quad (t \in \mathbb{R} \setminus \{0\})$$

and by Proposition 3.1.9 we have $(C\mu)^{(n)} = L(\widehat{\mu}(i\,\cdot)^n)$. Combining these facts with Proposition 6.3.9 and Remark 6.2.6 we deduce that the following statements are equivalent:

a) $\int_0^\infty \int_{-\infty}^\infty |(C\mu)^{(n)}(x + iy)|^2 y^{2n}\, dx\, dy < +\infty.$

b) $\int_{-\infty}^0 \int_{-\infty}^\infty |(C\mu)^{(n)}(x + iy)|^2 y^{2n}\, dx\, dy < +\infty.$

This is a generalization of the Corollary 6.3.10.

We are now ready for a first integrability result for Cauchy transforms:

Proposition 6.3.12 *Let $K \subset \mathbb{R}$ be compact and polar. For a measure $\mu \in \mathcal{M}(\mathbb{R})$ with $\mathrm{supp}(\mu) \subset K$, the following statements are equivalent:*

a) $C\mu \in \mathcal{A}^2(\Pi_+)$.

b) $\mu = 0$.

Proof. If $C\mu \in \mathcal{A}^2(\Pi_+)$, then we also have $C\mu \in \mathcal{A}^2(\mathbb{C} \setminus K)$ by Corollary 6.3.10. Therefore, Remark and Definition 6.3.3 implies $C\mu = 0$ and hence $\mu = 0$. $\qquad\square$

As mentioned before, so as not to exclude constantly any cases we want our formula to hold when the energy of a measure is equal to $+\infty$ or $-\infty$. Actually, these cases will make up the greater part of the proof. For the sake of readability we shall therefore outsource two crucial facts into two lemmas.

Lemma 6.3.13 *Let* $g \in L_{1,\mathrm{loc}}(0,\infty)$ *be real-valued with* $u := \mathrm{ess}\sup|g| < \infty$ *and such that* $t \mapsto (u - g(t))/t$ *is integrable at* 0. *Then, for all* $N > 0$ *we have*

$$\int_0^N \frac{g(ra) - g(r)}{r}\, dr \begin{cases} \geq 0, & a \in [0,1] \\ \leq 0, & a \in [1,\infty) \end{cases}.$$

Proof. Fix $N > 0$. Then,

$$\begin{aligned}
\int_0^N \frac{g(ra) - g(r)}{r}\, dr &= -\int_0^N \frac{g(0) - g(ra)}{r}\, dr + \int_0^N \frac{g(0) - g(r)}{r}\, dr \\
&= -\int_0^{aN} \frac{g(0) - g(r)}{r}\, dr + \int_0^N \frac{g(0) - g(r)}{r}\, dr.
\end{aligned}$$

If $a \in [0,1]$, then

$$\int_0^N \frac{g(ra) - g(r)}{r}\, dr = \int_{aN}^N \frac{g(0) - g(r)}{r}\, dr \geq 0.$$

If $a \in [1,\infty)$, then

$$\int_0^N \frac{g(ra) - g(r)}{r}\, dr = -\int_N^{aN} \frac{g(0) - g(r)}{r}\, dr \leq 0.$$

$\qquad\square$

Lemma 6.3.14 *Let* g *satisfy the assumptions of Lemma 6.3.6 and Lemma 6.3.13 with the same* u. *Then, there are* $R, c > 0$ *such that for each* $a > R$ *and each* $0 < \varepsilon < 1 < N$

$$\int_\varepsilon^N \frac{g(at) - g(t)}{t}\, dt \leq \int_0^1 \frac{u - g(t)}{t}\, dt + c.$$

Proof. We choose $R > 0$ such that

$$\left| \int_x^y \frac{g(t)}{t}\, dt \right| \leq 1$$

whenever $x, y \geq R$. If $a \in [R,\infty)$ and $N > 1 > \varepsilon > 0$, then

$$\begin{aligned}
\int_\varepsilon^N \frac{g(at) - g(t)}{t}\, dt &= \int_\varepsilon^1 \frac{g(at) - g(t)}{t}\, dt + \int_1^N \frac{g(at)}{t}\, dt - \int_1^N \frac{g(t)}{t}\, dt \\
&= \int_\varepsilon^1 \frac{g(at) - u}{t}\, dt + \int_\varepsilon^1 \frac{u - g(t)}{t}\, dt
\end{aligned}$$

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$$+ \int_a^{aN} \frac{g(t)}{t} \, dt - \int_1^N \frac{g(t)}{t} \, dt$$

$$\leq \int_0^1 \frac{u - g(t)}{t} \, dt + c$$

where the constant $c > 0$ is chosen such that $\sup\limits_{N>1} \int_1^N t^{-1} g(t) \, dt \geq -c + 1$. $\qquad \square$

We have now all tools together to prove the energy formula for the logarithmic energy. In this section, we will formulate the basic version which is a special case of the multidimensional case in the next section. This version will give a general information on the logarithmic integral of general product measures and will however need more preliminaries.

Theorem 6.3.15 *Let $\mu, \nu \in \mathcal{M}(\mathbb{R})$ with*

$$\int \int |\ln(|x - y|)| \, d|\mu|(y) \, d|\nu|(x) < \infty$$

or $\mu, \nu \in \mathcal{M}_+(\mathbb{R})$ with the property that $\int p_\mu \, d\nu$ exists in $[-\infty, +\infty]$. Then,

$$2 \int p_\mu \, d\bar{\nu} = \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\varepsilon \leq |t| \leq N} \frac{\widehat{\mu}(t) \cdot \overline{\widehat{\nu}(t)} - \mu(\mathbb{R})\bar{\nu}(\mathbb{R}) \cos(t)}{|t|} \, dt.$$

In particular, for every $\mu \in \mathcal{M}_{\ln}(\mathbb{R})$ we have

$$2I(\mu) = \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\varepsilon \leq |t| \leq N} \frac{|\widehat{\mu}(t)|^2 - |\mu(\mathbb{R})|^2 \cos(t)}{|t|} \, dt.$$

Proof. Fix $0 < \varepsilon < N < \infty$. Then, by Fubini's theorem

$$\int_{\varepsilon \leq |t| \leq N} \frac{\widehat{\mu}(t) \cdot \overline{\widehat{\nu}(t)} - \mu(\mathbb{R})\bar{\nu}(\mathbb{R}) \cos(t)}{|t|} \, dt$$

$$= \int_{\varepsilon \leq |t| \leq N} \left[\left(\int e^{-ixt} \, d\mu(x) \right) \cdot \left(\int e^{iyt} \, d\bar{\nu}(y) \right) - \mu(\mathbb{R})\bar{\nu}(\mathbb{R}) \cos(t) \right] \frac{dt}{|t|^n}$$

$$= \int_{\mathbb{R} \times \mathbb{R}} \int_{\varepsilon \leq |t| \leq N} \frac{e^{it(y-x)} - \cos(t)}{|t|} \, dt \, d(\mu \otimes \bar{\nu})(x, y)$$

$$= 2 \int \left(\int_\varepsilon^N \frac{\cos((x - y)t) - \cos(t)}{t} \, dt \right) d(\mu \otimes \bar{\nu})(x, y).$$

First, suppose that

$$\int \int |\ln(|x - y|)| \, d|\mu|(x) \, d|\nu|(y) < +\infty.$$

This implies that $(|\mu| \otimes |\nu|)(\{(x, y) \in \mathbb{R}^2 : x = y\}) = 0$. If $(x, y) \in \mathbb{R}^2$ with $x \neq y$, then

$$\lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_\varepsilon^N \frac{\cos((x - y)t) - \cos(t)}{t} \, dt \;=\; \ln\left(\frac{1}{|x - y|} \right)$$

$$\left| \int_{\varepsilon}^{N} \frac{\cos((x-y)t) - \cos(t)}{t} \, dt \right| \leq 2 \left| \ln \left(\frac{1}{|x-y|} \right) \right|$$

by Corollary 6.3.7. But now

$$\int 2 \left| \ln \left(\frac{1}{|x-y|} \right) \right| d(|\mu| \otimes |\nu|)(x,y) < +\infty$$

because $I(\mu) \in \mathbb{C}$. Therefore, we can apply the dominated convergence theorem and get

$$\lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\varepsilon \leq |t| \leq N} \frac{\widehat{\mu}(t) \cdot \overline{\widehat{\nu}(t)} - \mu(\mathbb{R})\bar{\nu}(\mathbb{R}) \cos(t)}{|t|} \, dt = 2 \int p_\mu \, d\bar{\nu}.$$

If μ and ν are positive measures such that

$$\int \int \ln \left(\frac{1}{|x-y|} \right) d\mu(x) \, d\nu(y) = +\infty,$$

then

$$\int_{\{|x-y|<1\}} \ln \left(\frac{1}{|x-y|} \right) d(\mu \otimes \nu)(x,y) = +\infty$$

since

$$\int_{\{|x-y|\geq 1\}} \ln \left(\frac{1}{|x-y|} \right) d(\mu \otimes \nu)(x,y) \in (-\infty, 0].$$

The same argument implies together with Corollary 6.3.7 and the dominated convergence theorem that

$$\lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\{|x-y|\geq 1\}} \left(\int_{\varepsilon}^{N} \frac{\cos((x-y)t) - \cos(t)}{t} \, dt \right) d(\mu \otimes \nu)(x,y)$$

exists and is finite. Therefore, we only have to show that

$$\liminf_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\{|x-y|<1\}} \left(\int_{\varepsilon}^{N} \frac{\cos((x-y)t) - \cos(t)}{t} \, dt \right) d(\mu \otimes \nu)(x,y) = +\infty.$$

If $(x,y) \in \mathbb{R}^2$ with $|x-y| \in [0,1)$ and $N > 1 > \varepsilon > 0$, then

$$\int_{\varepsilon}^{N} \frac{\cos((x-y)t) - \cos(t)}{t} \, dt = \int_{\varepsilon}^{N} \frac{\cos(|x-y|t) - \cos(t)}{t} \, dt \geq - \int_0^\varepsilon \frac{\cos(|x-y|t) - \cos(t)}{t} \, dt$$

$$\geq - \int_0^1 \frac{\cos(|x-y|t) - \cos(t)}{t} \, dt \geq - \int_{|x-y|}^1 \frac{1 - \cos(t)}{t} \, dt$$

$$\geq - \int_0^1 \frac{1 - \cos(t)}{t} \, dt$$

by Lemma 6.3.13 and since cos is an even function. Therefore, we can apply Fatou's lemma since $\mu \otimes \nu$ is finite and get

$$\liminf_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\{|x-y|<1\}} \left(\int_\varepsilon^N \frac{\cos((x-y)t) - \cos(t)}{t} \, dt \right) d(\mu \otimes \nu)(x,y)$$

$$\geq \int_{\{x=y\}} \liminf_{\substack{\varepsilon \to 0 \\ N \to \infty}} \left(\int_\varepsilon^N \frac{1 - \cos(t)}{t} \, dt \right) d(\mu \otimes \nu)(x,y) + \int_{\{0<|x-y|<1\}} \ln \left(\frac{1}{|x-y|} \right) d(\mu \otimes \nu)(x,y)$$

If $(\mu \otimes \nu)(\{(x,y) \in \mathbb{R}^2 : x = y\}) > 0$, then the first integral is equal to $+\infty$ since

$$\liminf_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_\varepsilon^N \frac{1 - \cos(t)}{t} \, dt = +\infty.$$

If $(\mu \otimes \nu)(\{(x,y) \in \mathbb{R}^2 : x = y\}) = 0$, then the second integral is equal to $+\infty$.
Suppose finally that μ and ν are positive measures with

$$\int \int \left| \ln \left(\frac{1}{|x-y|} \right) \right| d\mu(x) \, d\nu(y) = -\infty.$$

Then,

$$\int_{\{|x-y| \geq T\}} \ln \left(\frac{1}{|x-y|} \right) d(\mu \otimes \nu)(x,y) = -\infty$$

for all $T \geq 1$ since

$$\int_{\{|x-y| \leq T\}} \ln \left(\frac{1}{|x-y|} \right) d(\mu \otimes \nu)(x,y) \in \mathbb{R}.$$

The same argument implies together with Corollary 6.3.7 and the dominated convergence theorem that

$$\lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\{|x-y| \leq T\}} \left(\int_\varepsilon^N \frac{\cos((x-y)t) - \cos(t)}{t} \, dt \right) d(\mu \otimes \nu)(x,y)$$

exists and is finite for all $T \geq 1$. If we pick $g = \cos$ in Lemma 6.3.14, then there is some $R > 1$ such that for all $|x-y| \geq R$ and $0 < \varepsilon < 1 < N$

$$\int_\varepsilon^N \frac{\cos((x-y)t) - \cos(t)}{t} \, dt = \int_\varepsilon^N \frac{\cos(|x-y|t) - \cos(t)}{t} \, dt \leq \int_0^1 \frac{1 - \cos(t)}{t} \, dt + c$$

for some constant $c \in \mathbb{R}$ independent of x, y, ε and N. Therefore, we can apply Fatou's lemma since $\mu \otimes \nu$ is finite and get

$$\limsup_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\{|x-y| \geq R\}} \left(\int_\varepsilon^N \frac{\cos((x-y)t) - \cos(t)}{t} \, dt \right) d(\mu \otimes \nu)(x,y)$$

$$\leq \int_{\{|x-y| \geq R\}} \ln \left(\frac{1}{|x-y|} \right) d(\mu \otimes \nu)(x,y) = -\infty.$$

This concludes the proof. $\qquad \square$

Remark 6.3.16   1. If $\widehat{\mu}\,\overline{\widehat{\nu}} - \mu(\mathbb{R})\bar{\nu}(\mathbb{R})$ is locally integrable at the origin with respect to $dt/|t|$, then the same holds for $\widehat{\mu}\,\overline{\widehat{\nu}} - \mu(\mathbb{R})\bar{\nu}(\mathbb{R})\cos$ since $\cos(0) = 1$. In this case, the double-sided limit reduces to a one-sided $\lim_{N\to\infty}$. This holds if $\widehat{\mu}\,\overline{\widehat{\nu}}$ is Dini continuous at 0, which is, in particular, the case if μ and ν have compact support.

2. In particular, the proof of Theorem 6.3.15 shows together with Remark 6.3.8 that if $\mu \in \mathcal{M}_{\ln}(\mathbb{R})$ is real-valued with $I(\mu) \in \mathbb{C}$, then

$$\int \mu(\{x \in \mathbb{R} : x > y\}) - \mu(\{x \in \mathbb{R} : x < y\})\,d\mu(y) = \int\int \operatorname{sign}(x-y)\,d\mu(x)\,d\mu(y) = 0.$$

3. Let $\mu = f\lambda_1$ where $f(t) := \frac{1}{2}e^{-|t|}$ $(t \in \mathbb{R})$. Then,

$$\widehat{\mu}(t) = \frac{1}{1+t^2} \quad (t \in \mathbb{R}),$$

see [20, 1.4 (1)], and due to symmetry we therefore have

$$I(\mu) = \lim_{\substack{\varepsilon\to0\\N\to\infty}} \int_\varepsilon^N \frac{1}{t(1+t^2)^2} - \frac{\cos(t)}{t}\,dt$$

Upon setting $g(t) := \frac{1}{t(1+t^2)^2} - \frac{\cos(t)}{t}$ and

$$\operatorname{Ci}(t) := -\int_t^\infty \frac{\cos(x)}{x}\,dx \quad (t \in (0,\infty)),$$

one sees that the function $G : (0,\infty) \to \mathbb{R}$, defined by

$$G(t) := \ln(t) - \ln(\sqrt{t^2+1}) + \frac{1}{2(t^2+1)} - \operatorname{Ci}(t) \quad (t \in (0,\infty)),$$

satisfies $G' = g$ and $\lim_{t\to\infty} G(t) = 0$. Since $\lim_{t\to0} G(t) = -\gamma + \frac{1}{2}$ (see [58], p. 150) where

$$\gamma = \lim_{n\to\infty} \sum_{k=1}^n \frac{1}{k} - \ln(n)$$

is the **Euler-Mascheroni constant** we deduce that $I(\mu) = \gamma - \frac{1}{2}$.

Remark 6.3.17 By [35], p. 278, we also have

$$\lim_{\substack{\varepsilon\to0\\N\to\infty}} \int_\varepsilon^N \frac{J_0(at) - \cos(bt)}{t}\,dt = \ln\left(\frac{2b}{a}\right)$$

for all $a, b > 0$. This implies that the function $\cos$ in Theorem 6.3.15 may be replaced by $J_0(2\,\cdot)$.

We now turn back to the question at the beginning of this section: What conditions assure the finiteness of the integral

$$\int_{|t|\geq 1} \frac{|\widehat{\mu}(t)|^2}{|t|}\, dt$$

for a given $\mu \in \mathcal{M}(\mathbb{R})$? Under certain circumstances this task is related to the energy of the measure:

Corollary 6.3.18 *Let $\mu \in \mathcal{M}_{\ln}(\mathbb{R})$ be a positive measure such that*

$$\lim_{\varepsilon \to 0} \int_{\varepsilon \leq |t| \leq 1} \frac{|\widehat{\mu}(t)|^2 - |\mu(\mathbb{R})|^2}{|t|}\, dt$$

exists in $\mathbb{C}$. Then, the following are equivalent:

a) $I(\mu) \in \mathbb{C}$.

b) $\displaystyle\int_{|t|\geq 1} \frac{|\widehat{\mu}(t)|^2}{|t|}\, dt < +\infty.$

Proof. By Theorem 6.3.15 we know that

$$2I(\mu) = \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\varepsilon \leq |t| \leq N} \frac{|\widehat{\mu}(t)|^2 - |\mu(\mathbb{R})|^2 \cos(t)}{|t|}\, dt.$$

According to our assumption and the smoothness of cos the limit in the equation above exists if and only if

$$\lim_{N \to \infty} \int_{1 \leq |t| \leq N} \frac{|\widehat{\mu}(t)|^2 - |\mu(\mathbb{R})|^2 \cos(t)}{|t|}\, dt$$

exists. Now,

$$\lim_{N \to \infty} \int_{1 \leq |t| \leq N} \frac{\cos(t)}{|t|}\, dt = 2 \lim_{N \to \infty} \int_1^N \frac{\cos(t)}{t}\, dt$$

is finite and we conclude that $I(\mu) \in \mathbb{C}$ if and only if

$$\lim_{N \to \infty} \int_{1 \leq |t| \leq N} \frac{|\widehat{\mu}(t)|^2}{|t|}\, dt = \int_{|t|\geq 1} \frac{|\widehat{\mu}(t)|^2}{|t|}\, dt$$

exists in $\mathbb{R}$. $\qquad\square$

Another important energy, often used in higher dimensions, is the Riesz energy. Let $\alpha \geq 0$. Then, we write $\mathcal{M}_\alpha(\mathbb{R}^n)$ for the set of all measures $\mu \in \mathcal{M}(\mathbb{R}^n)$ such that $\mu \in \mathcal{M}_+(\mathbb{R}^n)$ or

$$\int \int \frac{1}{|x - y|^\alpha}\, d|\mu|(x)\, d|\mu|(y) < +\infty.$$

Definition 6.3.19 Let $\alpha \geq 0$ and $\mu \in \mathcal{M}_\alpha(\mathbb{R}^n)$. Then,

$$p_{\mu,\alpha}(x) := \int \frac{1}{|x - y|^\alpha}\, d\mu(y)$$

exists in $\mathbb{C} \cup \{+\infty\}$ for μ-almost every $x \in \mathbb{R}^n$ and $p_{\mu,\alpha}$ is called the **Riesz potential** of order α of μ. Moreover,

$$I_\alpha(\mu) := \int\int \frac{1}{|x - y|^\alpha}\, d\mu(y)\, d\bar{\mu}(x) = \int p_{\mu,\alpha}(x)\, d\bar{\mu}(x)$$

is called the **Riesz energy** of order α of μ.

Remark 6.3.20 By [52, 5.1] there exists a constant $K > 0$ such that whenever $\mu, \nu \in \mathcal{M}_+(\mathbb{R}^n)$, then

$$\left(\int p_{\mu,\alpha}\, d\nu\right)^2 \leq K \cdot \left(\int\int \frac{1}{|x - y|^\alpha}\, d\mu(y)\, d\mu(x)\right) \cdot \left(\int\int \frac{1}{|x - y|^\alpha}\, d\nu(y)\, d\nu(x)\right).$$

This implies in particular that

$$\int\int \frac{1}{|x - y|^\alpha}\, d|\mu|(x)\, d|\nu|(y)$$

exists in $\mathbb{C} \cup \{+\infty\}$ whenever $\mu, \nu \in \mathcal{M}_\alpha(\mathbb{R}^n)$.

It is a well-known fact (see [44], p. 353) that if $\alpha \in (0, n)$, and $\mu \in \mathcal{M}_\alpha(\mathbb{R}^n)$, then

$$\gamma_{n,\alpha} I_\alpha(\mu) = \int \frac{|\widehat{\mu}(t)|^2}{|t|^{n-\alpha}}\, dt$$

where $\gamma_{n,\alpha} = \frac{2^\alpha \pi^{\frac{n}{2}} \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}$. The proof mainly uses the concept of approximate identities. We present another proof of this formula for $n = 1$ and $\mu \in \mathcal{M}_\alpha(\mathbb{R})$ based on similar calculations as for the logarithmic energy.

Theorem 6.3.21 *Let $\alpha \in (0, 1)$ and $\mu, \nu \in \mathcal{M}_+(\mathbb{R})$ or $\mu, \nu \in \mathcal{M}(\mathbb{R})$ with*

$$\int\int \frac{1}{|x - y|^\alpha}\, d|\mu|(x)\, d|\nu|(y) < +\infty.$$

Then,

$$2\Gamma(\alpha) \cos\left(\frac{\pi}{2}\alpha\right) \int p_{\mu,\alpha}\, d\bar{\nu} = \lim_{N \to \infty} \int_{|t| \leq N} |t|^{\alpha-1} \widehat{\mu}(t)\overline{\widehat{\nu}(t)}\, dt.$$

In particular, for every $\mu \in \mathcal{M}_\alpha(\mathbb{R})$ we have

$$2\Gamma(\alpha) \cos\left(\frac{\pi}{2}\alpha\right) I_\alpha(\mu) = \int |t|^{\alpha-1} |\widehat{\mu}(t)|^2\, dt.$$

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Proof. Let $N > 0$. Then, by Fubini's theorem

$$\int_{|t| \leq N} |t|^{\alpha-1} \widehat{\mu}(t)\overline{\widehat{\nu}(t)} \, dt = \int_{|t| \leq N} \left(\int e^{-ixt} \, d\mu(x) \right) \cdot \left(\int e^{iyt} \, d\bar{\nu}(y) \right) |t|^{\alpha-1} \, dt$$

$$= \int \int \left(\int_{|t| \leq N} e^{i(y-x)t} |t|^{\alpha-1} \, dt \right) d\mu(x) \, \bar{\nu}(y)$$

$$= 2 \int \left(\int_0^N \cos(|x-y|t) t^{\alpha-1} \, dt \right) d(\mu \otimes \bar{\nu})(x, y).$$

First, suppose that

$$\int \int \frac{1}{|x-y|^\alpha} \, d|\mu|(x) \, d|\nu|(y) < +\infty.$$

Then, we have $(|\mu| \otimes |\nu|)(\{(x, y) \in \mathbb{R}^2 : x = y\}) = 0$. If $(x, y) \in \mathbb{R}^2$ with $x \neq y$, then by [20, 6.5 (21)]

$$\lim_{N \to \infty} \int_0^N \cos(|x-y|t) t^{\alpha-1} \, dt = \lim_{N \to \infty} \frac{1}{|x-y|^\alpha} \int_0^{|x-y|N} \cos(t) t^{\alpha-1} \, dt = \frac{\Gamma(\alpha) \cos\left(\frac{\pi}{2}\alpha\right)}{|x-y|^\alpha}$$

and

$$\left| \int_0^N \cos(|x-y|t) t^{\alpha-1} \, dt \right| \leq \frac{1}{|x-y|^\alpha} \cdot \sup_{M > 0} \left| \int_0^M \cos(t) t^{\alpha-1} \, dt \right|.$$

Therefore, the dominated convergence theorem implies that

$$\lim_{N \to \infty} \int_{|t| \leq N} |t|^{\alpha-1} \widehat{\mu}(t)\overline{\widehat{\nu}(t)} \, dt = \lim_{N \to \infty} 2 \int \left(\int_0^N \cos(|x-y|t) t^{\alpha-1} \, dt \right) d(\mu \otimes \bar{\nu})(x, y)$$

$$= 2\Gamma(\alpha) \cos\left(\frac{\pi}{2}\alpha\right) \int \int \frac{1}{|x-y|^\alpha} \, d\mu(x) \, d\bar{\nu}(y).$$

Now, let μ and ν be positive measures such that

$$\int \frac{1}{|x-y|^\alpha} \, d(\mu \otimes \nu)(x, y) = +\infty.$$

Then, we necessarily have

$$\int_{\{|x-y|<1\}} \frac{1}{|x-y|^\alpha} \, d(\mu \otimes \nu)(x, y) = +\infty$$

since

$$\int_{\{|x-y|\geq 1\}} \frac{1}{|x-y|^\alpha} \, d(\mu \otimes \nu)(x, y) \in [0, \infty).$$

The same argument implies together with the dominated convergence theorem that

$$\lim_{N \to \infty} \int_{\{|x-y|\geq 1\}} \left(\int_0^N \cos(|x-y|t) t^{\alpha-1} \, dt \right) d(\mu \otimes \nu)(x, y)$$

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exists and is finite. Therefore, we only have to show that

$$\liminf_{N\to\infty}\int_{\{|x-y|<1\}}\left(\int_0^N\cos(|x-y|t)t^{\alpha-1}\,dt\right)d(\mu\otimes\nu)(x,y)=+\infty.$$

But since

$$\int_0^N\cos(|x-y|t)t^{\alpha-1}\,dt=\frac{1}{|x-y|^\alpha}\int_0^{|x-y|N}\cos(t)t^{\alpha-1}\,dt\geq\inf_{M>0}\int_0^M\cos(t)t^{\alpha-1}\,dt$$

whenever $|x-y|<1$, Fatou's lemma (notice that $\mu\otimes\nu$ is finite) gives us that

$$\liminf_{N\to\infty}\int_{\{|x-y|<1\}}\left(\int_0^N\cos(|x-y|t)t^{\alpha-1}\,dt\right)d(\mu\otimes\nu)(x,y)$$

$$\geq\int_{\{x=y\}}\liminf_{N\to\infty}\left(\int_0^N t^{\alpha-1}\,dt\right)d(\mu\otimes\nu)(x,y)+\int_{\{0<|x-y|<1\}}\frac{1}{|x-y|^\alpha}\,d(\mu\otimes\nu)(x,y).$$

If $(\mu\otimes\nu)(\{(x,y)\in\mathbb{R}^2:x=y\})>0$, then the first integral is equal to $+\infty$ since

$$\liminf_{N\to\infty}\int_0^N t^{\alpha-1}\,dt=+\infty.$$

If $(\mu\otimes\nu)(\{(x,y)\in\mathbb{R}^2:x=y\})=0$, then the second integral is equal to $+\infty$. $\qquad\square$

Remark 6.3.22 Theorem 6.3.21, Proposition 3.1.9 and Remark 6.2.6 imply that if $\alpha\in(0,1)$ and $\mu\in\mathcal{M}_\alpha(\mathbb{R})$ satisfies $I_\alpha(\mu)\in\mathbb{C}$, then

$$\int_{-\infty}^\infty\int_0^\infty|(C\mu)(x+iy)|^2y^{-\alpha}\,dy\,dx<+\infty.$$

6.4 An energy formula in higher dimensions

We are interested in a generalization of our above results to the multidimensional case. Therefore, we start with a lemma whose proof can be found in [23, Theorem 2.49]. We write

$$\mathbb{S}^{n-1}:=\{x\in\mathbb{R}^n:|x|=1\}$$

for the **unit sphere** in $\mathbb{R}^n$ and σ_{n-1} for the **surface measure** on $\mathbb{S}^{n-1}$. Here, $\sigma_0=\delta_1+\delta_{-1}$. Moreover, $\omega_{n-1}=\sigma_{n-1}(\mathbb{S}^{n-1})=\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ denotes the area of $\mathbb{S}^{n-1}$.

Lemma 6.4.1 Let $f\in L_1(\mathbb{R}^n)$. Then,

$$\int_{\mathbb{R}^n}f(x)\,dx=\int_0^\infty\int_{\mathbb{S}^{n-1}}f(r\zeta)\,d\sigma_{n-1}(\zeta)\,r^{n-1}\,dr.$$

Remark 6.4.2 Let $0<\varepsilon<N+\infty$. Replacing f by $x\mapsto f(x)\mathbb{1}_{\varepsilon\leq|x|\leq N}(x)$ in Lemma 6.4.1 the formula reads then as

$$\int_{\varepsilon\leq|x|\leq N}f(x)\,dx=\int_\varepsilon^N\int_{\mathbb{S}^{n-1}}f(r\zeta)\,d\sigma_{n-1}(\zeta)\,r^{n-1}\,dr.$$

We shall need the substitution formula in this version.

With this formula in mind, integrating with respect to the unit sphere $\mathbb{S}^{n-1}$ will play a central role. In particular, in view of the proof of the one-dimensional case we need to know the Fourier transform of the spherical measure σ_{n-1}. Therefore, we introduce an important class of functions.

Remark and Definition 6.4.3 Let $\alpha \in \mathbb{C}$. Then, the function J_α, defined by

$$J_\alpha(x) := \sum_{k=0}^\infty \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+\alpha}}{\Gamma(\alpha + k + 1)k!} \quad (x \in (0, \infty))$$

is called **Bessel function** of the first kind and order α. It can be shown (see [1], p. 360) that J_α is a solution for **Bessel's differential equation**, i.e.

$$x^2 y''(x) + xy'(x) + (x^2 - \alpha^2)y(x) = 0.$$

With the aid of the Bessel functions, one can express the Fourier transform of the spherical measure σ_{n-1}. We refer for the corresponding result to [74], p. 154, or [30, B. 4].

Lemma 6.4.4 *Let $n \in \mathbb{N}$. Then,*

$$\int_{\mathbb{S}^{n-1}} e^{i\xi x} \, d\sigma_{n-1}(\xi) = (2\pi)^{\frac{n}{2}} J_{\frac{n}{2}-1}(|x|) \cdot |x|^{1-\frac{n}{2}} \quad (x \in \mathbb{R}^n \setminus \{0\}).$$

Remark and Definition 6.4.5   1. Notice that the formula is also valid for $n = 1$ since (see [58, 10.16.1])

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos(x) \quad (x \in (0, \infty))$$

and cos is an even function. For $n = 2$ we get

$$\int_{\mathbb{S}^1} e^{i\xi x} \, d\sigma_1(\xi) = 2\pi J_0(|x|)$$

and for $n = 3$ we have by [58, 10.16.1]

$$\int_{\mathbb{S}^2} e^{i\xi x} \, d\sigma_2(\xi) = (2\pi)^{\frac{3}{2}} J_{\frac{1}{2}}(|x|) \cdot |x|^{-\frac{1}{2}} = 4\pi \frac{\sin(|x|)}{|x|}.$$

2. We consider the **generalized hypergeometric function** $K_n : \mathbb{C} \to \mathbb{C}$, defined by

$$K_n(z) = \sum_{k=0}^\infty \frac{(-1)^k}{(n/2, k)k!} \left(\frac{z}{2}\right)^{2k} \quad (z \in \mathbb{C})$$

with the Pochhammer symbol $(\alpha, k) := \Gamma(\alpha + k)/\Gamma(\alpha)$. In particular, one obtains $K_1 = \cos$ and in the case $n = 2$, in which the logarithmic potential coincides with the Newtonian, $K_2 = J_0$. More generally, K_n can be expressed in terms of $J_{n/2-1}$ as

$$K_n(t) = \Gamma(n/2)(|t|/2)^{1-n/2} J_{n/2-1}(|t|) \quad (t \in \mathbb{R}).$$

This formula shows especially that $K_n|_{\mathbb{R}}$ is the normalized Fourier transform of σ_{n-1}.

3. Due to the fact that $J_\alpha(x) = O(x^{-\frac{1}{2}})$ ($\mathbb{R} \ni x \to \infty$) for $\alpha \geq 0$ (see [58, 10.17.3]) one sees directly that $K_n|_{[0,\infty)}$ satisfies the assumptions of Lemma 6.3.6. In particular,

$$K_n(0) = \frac{\sigma_{n-1}(\mathbb{S}^{n-1})}{\omega_{n-1}} = 1.$$

But since

$$-1 \leq K_n(t) \leq \frac{1}{\omega_{n-1}} \int_{\mathbb{S}^{n-1}} 1 \, d\sigma_{n-1}(\xi) \leq 1$$

and $K_n(x) = K_n(0) + O(x^2)$ ($x \to 0$) by Remark and Definition 6.4.3 we see also that the assumptions of Lemma 6.3.13 are fulfilled if we take $g = K_n$.

4. Because $K_n'(0) = 0$ and $K_n''(0) < 0$ we know that K_n is decreasing in some interval $[0, \delta]$. Writing $\Delta_a(t) := K_n(at) - K(t)$, for $a \in [0,1)$ and $N > 1 > \delta > \varepsilon > 0$ we thus obtain by Lemma 6.3.13

$$
\begin{aligned}
\int_\varepsilon^N \frac{K_n(at) - K_n(t)}{t} \, dt &= \int_0^N \frac{\Delta_a(t)}{t} \, dt - \int_0^\varepsilon \frac{\Delta_a(t)}{t} \, dt \geq -\int_0^\varepsilon \frac{\Delta_a(t)}{t} \, dt \\
&\geq -\int_0^\delta \frac{\Delta_a(t)}{t} \, dt \geq -\int_{a\delta}^a \frac{\Delta_a(t)}{t} \, dt \\
&\geq -\int_0^1 \frac{1 - K_n(t)}{t} \, dt \, .
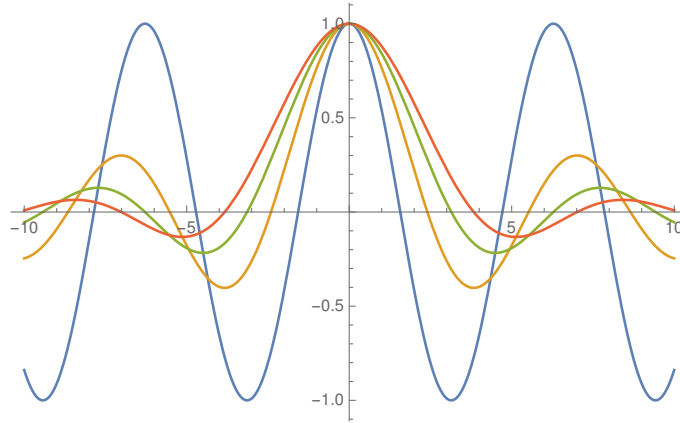\end{aligned}
$$

Figure 6.1: Plots of K_1 (blue), K_2 (yellow), K_3 (green) and K_4 (orange)

Now, we are in a position to formulate a generalized version of the energy formula in one dimension.

Theorem 6.4.6 *Let $\mu, \nu \in \mathcal{M}(\mathbb{R}^n)$ with*

$$\int \int |\ln(|x - y|)| \, d|\mu|(y) \, d|\nu|(x) < \infty$$

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or $\mu, \nu \in \mathcal{M}_+(\mathbb{R}^n)$ with the property that $\int p_\mu \, d\nu$ exists in $[-\infty, +\infty]$. Then,

$$\omega_{n-1} \int p_\mu \, d\bar\nu = \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\varepsilon \leq |t| \leq N} \left(\widehat\mu(t) \cdot \overline{\widehat\nu(t)} - \mu(\mathbb{R}^n)\bar\nu(\mathbb{R}^n) K_n(|t|) \right) \frac{dt}{|t|^n}.$$

In particular, for every $\mu \in \mathcal{M}_{\ln}(\mathbb{R}^n)$ we have

$$\omega_{n-1} I(\mu) = \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\varepsilon \leq |t| \leq N} \left(|\widehat\mu(t)|^2 - |\mu(\mathbb{R}^n)|^2 K_n(|t|) \right) \frac{dt}{|t|^n}.$$

Proof. Fix $0 < \varepsilon < N < +\infty$. Then, by Fubini's theorem and Lemma 6.4.1

$$\int_{\varepsilon \leq |t| \leq N} \left(\widehat\mu(t) \cdot \overline{\widehat\nu(t)} - \mu(\mathbb{R}^n)\bar\nu(\mathbb{R}^n) K_n(|t|) \right) \frac{dt}{|t|^n}$$

$$= \int_{\varepsilon \leq |t| \leq N} \left[\left(\int e^{-ixt} \, d\mu(x) \right) \cdot \left(\int e^{iyt} \, d\bar\nu(y) \right) - \mu(\mathbb{R}^n)\bar\nu(\mathbb{R}^n) K_n(|t|) \right] \frac{dt}{|t|^n}$$

$$= \int_{\mathbb{R}^n \times \mathbb{R}^n} \int_{\varepsilon \leq |t| \leq N} \frac{e^{it(y-x)} - K_n(|t|)}{|t|^n} \, dt \, d(\mu \otimes \bar\nu)(x,y)$$

$$= \int_{\mathbb{R}^n \times \mathbb{R}^n} \int_{\mathbb{S}^{n-1}} \int_\varepsilon^N \frac{e^{ir\zeta(y-x)} - K_n(r)}{r} \, dr \, d\sigma_{n-1}(\zeta) \, d(\mu \otimes \bar\nu)(x,y).$$

Another application of Fubini's theorem and Lemma 6.4.4 give us

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \int_{\mathbb{S}^{n-1}} \int_\varepsilon^N \frac{e^{ir\zeta(y-x)} - K_n(r)}{r} \, dr \, d\sigma_{n-1}(\zeta) \, d(\mu \otimes \bar\nu)(x,y)$$

$$= \int_{\mathbb{R}^n \times \mathbb{R}^n} \int_\varepsilon^N \left[\left(\int_{\mathbb{S}^{n-1}} e^{ir(y-x)} \, d\sigma_{n-1}(\zeta) \right) - \omega_{n-1} K_n(r) \right] \frac{dr}{r} \, d(\mu \otimes \bar\nu)(x,y)$$

$$= \omega_{n-1} \int_{\mathbb{R}^n \times \mathbb{R}^n} \int_\varepsilon^N \frac{K_n(r|x-y|) - K_n(r)}{r} \, dr \, d(\mu \otimes \bar\nu)(x,y).$$

First, suppose that

$$\int \int \left| \ln\left(\frac{1}{|x-y|} \right) \right| \, d|\mu|(x) \, d|\nu|(y) < +\infty.$$

Then, we necessarily have $(|\mu| \otimes |\nu|)(\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}) = 0$. Therefore, if we take $g = K_n$ in Lemma 6.3.6 (see Remark 6.4.5), then the dominated convergence theorem yields that

$$\lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \omega_{n-1} \int_{\mathbb{R}^n \times \mathbb{R}^n} \int_\varepsilon^N \frac{K_n(r|x-y|) - K_n(r)}{r} \, dr \, d(\mu \otimes \bar\nu)(x,y)$$

$$= \omega_{n-1} \int_{\mathbb{R}^n \times \mathbb{R}^n} \ln\left(\frac{1}{|x-y|} \right) \, d(\mu \otimes \bar\nu)(x,y) = \omega_{n-1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \ln\left(\frac{1}{|x-y|} \right) \, d\mu(x) \, d\bar\nu(y).$$

Now, let $\mu, \nu \in \mathcal{M}_+(\mathbb{R}^n)$ be such that

$$\int \int \ln\left(\frac{1}{|x-y|} \right) \, d\mu(x) \, d\nu(y) = +\infty.$$

In this case,

$$\int_{\{|x-y|<1\}} \ln\left(\frac{1}{|x-y|}\right) d(\mu \otimes \nu)(x,y) = +\infty$$

since

$$\int_{\{|x-y|\geq 1\}} \ln\left(\frac{1}{|x-y|}\right) d(\mu \otimes \nu)(x,y) \in (-\infty, 0].$$

The same argument implies together with Lemma 6.3.6, Remark 6.4.5 and the dominated convergence theorem that

$$\lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\{|x-y|\geq 1\}} \left(\int_{\varepsilon}^{N} \frac{K_n(|x-y|r) - K_n(r)}{r} dr\right) d(\mu \otimes \nu)(x,y)$$

exists and is finite. Therefore, we only have to show that

$$\liminf_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\{|x-y|<1\}} \left(\int_{\varepsilon}^{N} \frac{K_n(|x-y|r) - K_n(r)}{r} dr\right) d(\mu \otimes \nu)(x,y) = +\infty.$$

By Remark 6.4.5, we know that

$$\int_{\varepsilon}^{N} \frac{K_n(|x-y|r) - K_n(r)}{r} dr \geq -\int_{0}^{1} \frac{1 - K_n(r)}{r} dr$$

for $|x-y| < 1$ and sufficiently large N and small ε. Therefore, we can apply Fatou's lemma (notice that $\mu \otimes \nu$ is finite) and get with $K_n(0) = 1$

$$\liminf_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\{|x-y|<1\}} \left(\int_{\varepsilon}^{N} \frac{K_n(|x-y|r) - K_n(r)}{r} dr\right) d(\mu \otimes \nu)(x,y)$$

$$\geq \int_{\{x=y\}} \liminf_{\substack{\varepsilon \to 0 \\ N \to \infty}} \left(\int_{\varepsilon}^{N} \frac{1 - K_n(r)}{r} dr\right) d(\mu \otimes \nu)(x,y) + \int_{\{0<|x-y|<1\}} \ln\left(\frac{1}{|x-y|}\right) d(\mu \otimes \nu)(x,y)$$

If $(\mu \otimes \nu)(\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}) > 0$, then the first integral is equal to $+\infty$ since

$$\liminf_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\varepsilon}^{N} \frac{1 - K_n(t)}{t} dt = +\infty.$$

If $(\mu \otimes \nu)(\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}) = 0$, then the second integral is equal to $+\infty$. Suppose finally that $\mu, \nu \in \mathcal{M}_+(\mathbb{R}^n)$ and

$$\int \int \ln\left(\frac{1}{|x-y|}\right) d\mu(x) \, d\nu(y) = -\infty.$$

Then,

$$\int_{\{|x-y|\geq T\}} \ln\left(\frac{1}{|x-y|}\right) d(\mu \otimes \nu)(x,y) = -\infty$$

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for all $T \geq 1$ since

$$\int_{\{|x-y| \leq T\}} \ln \left(\frac{1}{|x-y|} \right) d(\mu \otimes \nu)(x,y) \in \mathbb{R}.$$

The same argument implies together with Lemma 6.3.6 and the dominated convergence theorem that

$$\lim_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\{|x-y| \leq T\}} \left(\int_{\varepsilon}^{N} \frac{K_n(|x-y|r) - K_n(r)}{r} dr \right) d(\mu \otimes \nu)(x,y)$$

exists and is finite for all $T \geq 1$. If we pick $g = K_n$ in Lemma 6.3.14, then there is some $R > 1$ such that for all $|x-y| \geq R$ and $0 < \varepsilon < 1 < N$

$$\int_{\varepsilon}^{N} \frac{K_n(|x-y|r) - K_n(r)}{r} dr \leq \int_{0}^{1} \frac{1 - K_n(r)}{r} dr + c$$

for some constant $c \in \mathbb{R}$ independent of x, y, ε and N. Therefore, we can apply Fatou's lemma since $\mu \otimes \nu$ is finite and get

$$\limsup_{\substack{\varepsilon \to 0 \\ N \to \infty}} \int_{\{|x-y| \geq R\}} \left(\int_{\varepsilon}^{N} \frac{K_n(|x-y|r) - K_n(r)}{r} dr \right) d(\mu \otimes \nu)(x,y)$$

$$\leq \int_{\{|x-y| \geq R\}} \ln \left(\frac{1}{|x-y|} \right) d(\mu \otimes \nu)(x,y) = -\infty.$$

This concludes the proof. $\qquad \square$

Remark 6.4.7   1. Similar to the one-dimensional case, the double-sided limit reduces to a one-sided $\lim_{N \to \infty}$ if $\widehat{\mu} \, \overline{\widehat{\nu}} - \mu(\mathbb{R}^n) \bar{\nu}(\mathbb{R}^n)$ is locally integrable at the origin with respect to $dt/|t|^n$. This is a consequence of $K_n(0) = 1$ and holds if $\widehat{\mu} \, \overline{\widehat{\nu}}$ is Dini continuous at 0, which is, in particular, the case if μ and ν have compact support.

2. According to Lemma 6.4.4, Theorem 6.4.6 and Lemma 6.4.1 imply that

$$I(\sigma_{n-1}) = \omega_{n-1}^2 \int_0^\infty (K_n^2(r) - K_n(r)) \frac{dr}{r} \qquad (n \in \mathbb{N}).$$

Since $K_2 = J_0$ and since $I(\sigma_1) = 0$ (see [53, Lemma 4.20]), we obtain

$$0 = \int_0^\infty (J_0^2(r) - J_0(r)) \frac{dr}{r}. \tag{6.4.1}$$

Applying Lemma 6.3.6 with $g = J_0^2$, we get, more generally,

$$\int_0^\infty (J_0^2(ar) - J_0(r)) \frac{dr}{r} = \ln \left(\frac{1}{a} \right) \tag{6.4.2}$$

for $a > 0$. Also, taking $n = 1$ in Theorem 6.4.6 and using (6.4.1) leads to

$$\ln 2 = \int_0^\infty (J_0^2(r) - \cos r) \frac{dr}{r} = \int_0^\infty (J_0(r) - \cos r) \frac{dr}{r}.$$

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According to Lemma 6.3.6, we end at

$$\int_0^\infty (J_0(ar) - \cos r)\frac{dr}{r} = \ln\left(\frac{2}{a}\right)$$

for arbitrary $a > 0$ (see e.g. [35], p. 278). In particular, this implies that for $n = 1$ the function $K_0 = \cos$ in Theorem 6.4.6 may be replaced by $\xi \mapsto J_0(2|\xi|)$.

3. The standard one-dimensional example is the arcsine distribution $\mu = f\lambda_1$ where

$$f(t) = \frac{1}{\pi\sqrt{1-t^2}}\mathbb{1}_{(-1,1)}(t) \quad (t \in \mathbb{R}).$$

Here is $J_0 = \widehat{\mu}$, see [20, 1.3 (8)]. Upon setting $\nu := \mu \otimes \delta_0$ we see that

$$\widehat{\nu}(x_1, x_2) = \widehat{\mu}(x_1) = J_0(x_1) \qquad (x_1, x_2 \in \mathbb{R}).$$

Choosing $n = 2$ in Theorem 6.4.6 and using (6.4.2), we have

$$I(\nu) = I(\mu) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^\infty (J_0^2(r\cos\theta) - J_0(r))\frac{dr}{r}\,d\theta = \frac{2}{\pi}\int_0^{\pi/2}\ln\left(\frac{1}{\cos\theta}\right)d\theta = \ln 2,$$

which is of course folklore but usually proved in a quite different way by exploiting some amount of potential theory in the plane. More precisely, one shows that μ is the equilibrium measure of $[-1,1]$, that is, μ minimizes logarithmic energy among all Borel probability measures on $[-1,1]$, and that the minimum is $\ln 2$ or, in other words, that the logarithmic capacity of $[-1,1]$ is $1/2 = e^{-\ln 2}$.

Similar to the case of one dimension, one consequence of the energy formula is a characterization for the finiteness of the logarithmic energy.

Corollary 6.4.8 *Let $\mu \in \mathcal{M}_{\ln}(\mathbb{R}^n)$ a positive measure such that*

$$\lim_{\varepsilon \to 0}\int_{\varepsilon \leq |t| \leq 1}\frac{|\widehat{\mu}(t)|^2 - |\mu(\mathbb{R}^n)|^2}{|t|^n}\,dt$$

exists in $\mathbb{R}$. Then, the following are equivalent:

a) $I(\mu) \in \mathbb{C}$.

b) $\displaystyle\int_{|t| \geq 1}\frac{|\widehat{\mu}(t)|^2}{|t|^n}\,dt < +\infty.$

Proof. By Theorem 6.4.6 we know that

$$\omega_{n-1}I(\mu) = \lim_{\substack{\varepsilon \to 0 \\ N \to \infty}}\int_{\varepsilon \leq |t| \leq N}\left(|\widehat{\mu}(t)|^2 - |\mu(\mathbb{R}^n)|^2 K_n(|t|)\right)\frac{dt}{|t|^n}.$$

By our assumption and the smoothness of K_n the limit in the equation above exists if and only if

$$\lim_{N\to\infty} \int_{1\le|t|\le N} \left(|\widehat{\mu}(t)|^2 - |\mu(\mathbb{R}^n)|^2 K_n(|t|) \right) \frac{dt}{|t|^n}$$

exists. Now, by Lemma 6.4.1

$$\lim_{N\to\infty} \int_{1\le|t|\le N} \frac{K_n(|t|)}{|t|^n} \, dt = \omega_{n-1} \lim_{N\to\infty} \int_1^N \frac{K_n(r)}{r} \, dr$$

is finite and we conclude that $I(\mu) \in \mathbb{C}$ if and only if

$$\lim_{N\to\infty} \int_{1\le|t|\le N} \frac{|\widehat{\mu}(t)|^2}{|t|^n} \, dt = \int_{|t|\ge 1} \frac{|\widehat{\mu}(t)|^2}{|t|^n} \, dt < +\infty.$$

$\square$

Let us now turn to another consequence of our formula for the logarithmic energy. We consider therefore the space $\mathcal{M}_{\ln,0}(\mathbb{R}^n)$ of all $\mu \in \mathcal{M}_{\ln}(\mathbb{R}^n)$ with vanishing total mass, i.e. $\mu(\mathbb{R}^n) = 0$. For $\mu \in \mathcal{M}_{\ln,0}(\mathbb{R}^n)$ and arbitrary $\nu \in \mathcal{M}_{\ln}(\mathbb{R}^n)$, Theorem 6.4.6 says that

$$\omega_{n-1} \int p_\mu \, d\bar{\nu} = \lim_{\substack{\varepsilon\to 0 \\ N\to\infty}} \int_{\varepsilon\le|\xi|\le N} (\widehat{\mu}\,\overline{\widehat{\nu}})(\xi) \frac{d\xi}{|\xi|^n}.$$

In particular, for $\nu = \mu$ we get

$$\omega_{n-1}I(\mu) = \int |\widehat{\mu}(\xi)|^2 \frac{d\xi}{|\xi|^n} \in [0,\infty) .$$

This may be seen as a continuous version of (6.3.1). It implies in particular (see Corollary 6.4.9 below) that the logarithmic energy integral is positive definite on the space $\mathcal{M}_{\ln,0}(\mathbb{R}^n)$. This fact has already been proved in the case of compactly supported, signed (real) measures in $\mathcal{M}_{\ln,0}(\mathbb{R}^n)$ (see [29, Chapter III, Theorem 3.1], [25], [12, 5, III.5]). Moreover (see [52, 3.3]), if μ is a (not necessarily finite) signed (real) measure with $\mu(\mathbb{R}^n) = 0$ and

$$\int\int \ln\left(\frac{1}{|x-y|}\right) d\mu(x)\, d\mu(y)$$

exists in $[-\infty, +\infty]$, then this integral is either ≥ 0 or equal to $+\infty$. The following corollary can be seen as a generalization and completion of these known results which are now also valid for complex measures.

Corollary 6.4.9 *Let $\mu, \nu \in \mathcal{M}_{\ln}(\mathbb{R}^n)$ or $\mu, \nu \in \mathcal{M}_+(\mathbb{R}^n)$ such that $\int p_\mu \, d\nu$ exists in $[-\infty, +\infty]$. If $\widehat{\mu}\,\overline{\widehat{\nu}} \ge 0$ almost everywhere and $\mu(\mathbb{R}^n) \cdot \bar{\nu}(\mathbb{R}^n) = 0$, then*

$$\int p_\mu \, d\bar{\nu} \in [0, +\infty].$$

In particular, $I(\mu) \ge 0$ for all $\mu \in \mathcal{M}_{\ln,0}(\mathbb{R}^n)$. Moreover, if $I(\mu) = 0$, then $\mu = 0$.

We now answer the question when the Cauchy transform of a measure belongs to the unweighted Bergman space $\mathcal{A}^2(\mathbb{C} \setminus \mathbb{R})$. As seen before, it is necessary that $\mu(\mathbb{R}) = 0$ but unfortunately this condition is in general not sufficient, see the discussion at the beginning of the previous section. However, for the class of complex measures with finite logarithmic energy this gives in fact a characterization and this result follows immediately from

Corollary 6.4.10 *Let $\mu \in \mathcal{M}_{\ln}(\mathbb{R}^n)$ with $I(\mu) \in \mathbb{C}$. Then, the following are equivalent:*

    a) $\mu(\mathbb{R}^n) = 0$.

    b) $\displaystyle\int_{\mathbb{R}^n} \frac{|\widehat{\mu}(t)|^2}{|t|^n} \, dt < +\infty$.

In this case,

$$\omega_{n-1} I(\mu) = \int_{\mathbb{R}^n} \frac{|\widehat{\mu}(t)|^2}{|t|^n} \, dt.$$

Corollary 6.4.11 *Let $\mu \in \mathcal{M}_{\ln}(\mathbb{R})$ with $I(\mu) \in \mathbb{C}$. Then, the following are equivalent:*

    a) $\mu(\mathbb{R}) = 0$.

    b) $C\mu \in \mathcal{A}^2(\mathbb{C} \setminus \mathbb{R})$.

Similar to the case of one dimension, the above calculations hold in an analogue way for the Riesz energy. However, our method only applies for exponents in the interval $\left(0, \frac{n+1}{2}\right)$ due to the fact that K_n has to be (at least improperly) integrable at ∞ with respect to $t^{\alpha-1} dt$.

Theorem 6.4.12 *Let $\alpha \in \left(0, \frac{n+1}{2}\right)$ and $\mu, \nu \in \mathcal{M}_+(\mathbb{R}^n)$ or $\mu, \nu \in \mathcal{M}(\mathbb{R}^n)$ with*

$$\int \int \frac{1}{|x-y|^\alpha} \, d|\mu|(x) \, d|\nu|(y) < +\infty.$$

Then,

$$\gamma_{n,\alpha} \int \int \frac{1}{|x-y|^\alpha} \, d\mu(x) \, d\bar{\nu}(y) = \lim_{N \to \infty} \int_{|t| \leq N} |t|^{\alpha-1} \widehat{\mu}(t) \overline{\widehat{\nu}(t)} \, dt.$$

In particular,

$$\gamma_{n,\alpha} I_\alpha(\mu) = \int |t|^{\alpha-n} |\widehat{\mu}(t)|^2 \, dt.$$

Proof. Let $N > 0$. Then, by Fubini's theorem, Lemma 6.4.1 and Lemma 6.4.4

$$
\begin{aligned}
\int_{|t| \leq N} |t|^{\alpha-1} \widehat{\mu}(t) \overline{\widehat{\nu}(t)} \, dt &= \int_{|t| \leq N} \left(\int e^{-ixt} \, d\mu(x) \right) \cdot \left(\int e^{iyt} \, d\bar{\nu}(y) \right) |t|^{\alpha-1} \, dt \\
&= \int \int \left(\int_{|t| \leq N} e^{i(y-x)t} |t|^{\alpha-1} \, dt \right) d\mu(x) \, \bar{\nu}(y) \\
&= \int \int \left(\int_{\mathbb{S}^{n-1}} \int_0^N e^{ir\zeta(y-x)} r^{\alpha-1} \, dr \, d\sigma_{n-1}(\zeta) \right) d\mu(x) \, \bar{\nu}(y)
\end{aligned}
$$

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$$= \omega_{n-1} \int \left(\int_0^N K_n(r|x-y|) r^{\alpha-1} \, dr \right) d(\mu \otimes \bar{\nu})(x,y).$$

First, suppose that

$$\int \int \frac{1}{|x-y|^\alpha} \, d|\mu|(x) \, d|\nu|(y) < +\infty.$$

Then, we have $(|\mu| \otimes |\nu|)(\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}) = 0$. If $(x,y) \in \mathbb{R}^n$ with $x \neq y$, then by definition of K_n and [58, 10.22.43]

$$\lim_{N\to\infty} \int_0^N K_n(|x-y|t) t^{\alpha-1} \, dt = \lim_{N\to\infty} \frac{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}-1}}{|x-y|^\alpha} \int_0^{|x-y|N} J_{\frac{n}{2}-1}(t) t^{\alpha-\frac{n}{2}} \, dt$$

$$= \frac{2^{\alpha-1} \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)} \frac{1}{|x-y|^\alpha}$$

and

$$\left| \int_0^N K_n(|x-y|t) t^{\alpha-1} \, dt \right| \leq \frac{1}{|x-y|^\alpha} \cdot \sup_{M>0} \left| \int_0^M K_n(t) t^{\alpha-1} \, dt \right|.$$

Therefore, the dominated convergence theorem implies that

$$\lim_{N\to\infty} \int_{|t|\leq N} |t|^{\alpha-n} \widehat{\mu}(t) \overline{\widehat{\nu}(t)} \, dt = \lim_{N\to\infty} \omega_{n-1} \int \left(\int_0^N K_n(r|x-y|) r^{\alpha-1} \, dr \right) d(\mu \otimes \bar{\nu})(x,y)$$

$$= \frac{2^{\alpha-1} \omega_{n-1} \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)} \int \int \frac{1}{|x-y|^\alpha} \, d\mu(x) \, d\bar{\nu}(y)$$

$$= \gamma_{n,\alpha} \int \int \frac{1}{|x-y|^\alpha} \, d\mu(x) \, d\bar{\nu}(y).$$

Now, let μ and ν be positive measures such that

$$\int \frac{1}{|x-y|^\alpha} \, d(\mu \otimes \nu)(x,y) = +\infty.$$

Then, we necessarily have

$$\int_{\{|x-y|<1\}} \frac{1}{|x-y|^\alpha} \, d(\mu \otimes \nu)(x,y) = +\infty$$

since

$$\int_{\{|x-y|\geq 1\}} \frac{1}{|x-y|^\alpha} \, d(\mu \otimes \nu)(x,y) \in [0,\infty).$$

The same argument implies together with the dominated convergence theorem that

$$\lim_{N\to\infty} \int_{\{|x-y|\geq 1\}} \left(\int_0^N K_n(r|x-y|) r^{\alpha-1} \, dr \right) d(\mu \otimes \nu)(x,y)$$

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exists and is finite. Therefore, we only have to show that

$$\liminf_{N\to\infty} \int_{\{|x-y|<1\}} \left(\int_0^N K_n(r|x-y|)r^{\alpha-1}\, dr \right) d(\mu \otimes \nu)(x,y) = +\infty.$$

But since

$$\int_0^N K_n(r|x-y|)r^{\alpha-1}\, dr = \frac{1}{|x-y|^\alpha} \int_0^{|x-y|N} K_n(r)r^{\alpha-1}\, dr \geq \inf_{M>0} \int_0^M K_n(r)r^{\alpha-1}\, dr$$

whenever $|x-y| < 1$, Fatou's lemma (notice that $\mu \otimes \nu$ is finite) gives us that

$$\liminf_{N\to\infty} \int_{\{|x-y|<1\}} \left(\int_0^N K_n(r|x-y|)r^{\alpha-1}\, dr \right) d(\mu \otimes \nu)(x,y)$$

$$\geq \int_{\{x=y\}} \liminf_{N\to\infty} \left(\int_0^N r^{\alpha-1}\, dr \right) d(\mu \otimes \nu)(x,y) + \gamma_{n,\alpha} \int_{\{0<|x-y|<1\}} \frac{1}{|x-y|^\alpha}\, d(\mu \otimes \nu)(x,y).$$

If $(\mu \otimes \nu)(\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}) > 0$, then the first integral is equal to $+\infty$ since

$$\liminf_{N\to\infty} \int_0^N r^{\alpha-1}\, dr = +\infty.$$

If $(\mu \otimes \nu)(\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}) = 0$, then the second integral is equal to $+\infty$. $\qquad\square$

We deduce a corollary that illustrates a similarity between the logarithmic and the Riesz energy. As a special case, it implies that the Riesz energy is positive definite on the space $\mathcal{M}_\alpha(\mathbb{R}^n)$, see [44], p. 353.

Corollary 6.4.13 *Let $\alpha \in \left(0, \frac{n+1}{2}\right)$ and $\mu, \nu \in \mathcal{M}_\alpha(\mathbb{R}^n)$ with $\widehat{\mu}\widehat{\nu} \geq 0$ almost everywhere. Then,*

$$\int p_{\mu,\alpha}\, d\bar{\nu} \in [0, +\infty].$$

In particular, $I_\alpha(\mu) \geq 0$ for all $\mu \in M_\alpha(\mathbb{R}^n)$ and if $I_\alpha(\mu) = 0$, then $\mu = 0$.

Appendix A

Results from Measure and Integration Theory

In this appendix we give a rudimentary introduction into the topic of complex measures. We therefore gather the most important definitions and results that will be of frequent use in this work.

Definition A.1 Let (X, Σ) be a measurable space.

1. A function $\mu : \Sigma \to [0, +\infty]$ is called a **positive measure** (on (X, Σ)), if

   $\mu 1)$ $\mu(\emptyset) = 0$.

   $\mu 2)$ $\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$ for each pairwise disjoint family $(A_n)_{n \in \mathbb{N}} \in \Sigma^{\mathbb{N}}$.

   We shall write $\mathcal{M}_{\infty, +}(X, \Sigma)$ for the set of all positive measures on (X, Σ) and $\mathcal{M}_+(X, \Sigma)$ for the set of all **finite positive measures**, i.e. those $\mu \in \mathcal{M}_{\infty, +}(X, \Sigma)$ such that $\infty \notin \mu(\Sigma)$.

2. A function $\mu : \Sigma \to \mathbb{C}$ is called a **complex measure** (on (X, Σ)), if

   $\mu 1)$ $\mu(\emptyset) = 0$.

   $\mu 2)$ $\mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \sum_{n \in \mathbb{N}} \mu(A_n)$ for each pairwise disjoint family $(A_n)_{n \in \mathbb{N}} \in \Sigma^{\mathbb{N}}$.

   In this case, the series in $\mu 2)$ are absolutely convergent. We shall write $\mathcal{M}(X, \Sigma)$ for the set of all complex measures on (X, Σ). Moreover, let $\mathcal{M}_\infty(X, \Sigma)$ be the union of $\mathcal{M}_{\infty, +}(X, \Sigma)$ and $\mathcal{M}(X, \Sigma)$.

Remark and Definition A.2 If X is a topological space and $\mathcal{B}(X)$ is the Borel σ algebra on X, then we call a complex measure on $(X, \mathcal{B}(X))$ also a complex Borel measure on X and similar a (finite) positive measure on $(X, \mathcal{B}(X))$ also a (finite) positive Borel measure on X. Moreover, we write $\mathcal{M}_\infty(X)$ instead of $\mathcal{M}_\infty(X, \mathcal{B}(X)), \mathcal{M}(X)$ instead of $\mathcal{M}(X, \mathcal{B}(X)), \mathcal{M}_{\infty, +}(X)$ instead of $\mathcal{M}_{\infty, +}(X, \mathcal{B}(X))$ and $\mathcal{M}_+(X)$ instead of $\mathcal{M}_+(X, \mathcal{B}(X))$.

Remark A.3 Let X be a topological space and $A \in \mathcal{B}(X)$. If $\mu \in \mathcal{M}_\infty(A)$, then there is a unique $\nu \in \mathcal{M}_\infty(X)$ such that $\nu|_{\mathcal{B}(A)} = \mu$, given by $\nu(B) = \mu(B \cap A)$ $(B \in \mathcal{B}(X))$. We therefore identify μ and ν and therefore consider Borel measures which are defined on A also as Borel measures defined on X.

Definition A.4 Let $X \subset \mathbb{C}$ and $\mu \in \mathcal{M}_{\infty,+}(X)$. Then, μ is called **locally finite** if $\mu(K) < +\infty$ for all compact $K \subset X$.

Our main focus in this appendix will be on finite complex measures.

Remark and Definition A.5 Let (X, Σ) be a measurable space.

1. For $\mu \in \mathcal{M}_\infty(X, \Sigma)$ we set

$$|\mu|(A) := \sup \left\{ \sum_{n \in \mathbb{N}} |\mu(A_n)| : A = \bigcup_{n \in \mathbb{N}} A_n, A_n \cap A_m = \emptyset \ (n \neq m) \right\}.$$

Then, this defines a positive measure on (X, Σ), the so called **total variation measure of** $|\mu|$, see [68, Theorem 6.2]. One can show that $|\mu|$ is finite if $\mu \in \mathcal{M}(X, \Sigma)$ (cf. [68, Theorem 6.4]) and in this case there is a $|\mu|$- almost unique function $h : X \to \mathbb{T}$ such that

$$\mu(A) = \int_A h \, d|\mu| \quad (A \in \Sigma),$$

see [68, Theorem 6.12]. We write $\mu = h|\mu|$.

2. For $\mu \in \mathcal{M}(X, \Sigma)$, we write

$$\|\mu\| := |\mu|(X).$$

Then, this defines a norm on $\mathcal{M}(X, \Sigma)$, the so called **total variation norm**.

Remark A.6 Let $f \in L_1(\mathbb{R})$. Then, this function naturally induces a complex measure by

$$\mu_f(A) := \int_{A \cap \mathbb{R}} f(t) \, dt \quad (A \in \mathcal{B}(\mathbb{C})).$$

Since

$$\mu_f(A) = \int_A f(t) \, d\nu \quad (A \in \mathcal{B}(\mathbb{C}))$$

where

$$\nu(B) := \lambda_1(B \cap \mathbb{R}) \quad (B \in \mathcal{B}(\mathbb{C})),$$

it follows from [68, Theorem 6.13], that

$$|\mu_f|(A) = \int_{A \cap \mathbb{R}} |f(t)| \, dt \quad (A \in \mathcal{B}(\mathbb{C})).$$

In particular, $\|\mu_f\| = \|f\|_1$.

Definition A.7 Let (X, Σ) be a measurable space and $p \in [1, \infty]$. For $\mu \in \mathcal{M}_\infty(X, \Sigma)$ we set $L_p(X, \Sigma, \mu) := L_p(X, \Sigma, |\mu|)$ and for $f \in L_p(\mu)$ we define $\|f\|_{L_p(X, \Sigma, \mu)} := \|f\|_{L_p(X, \Sigma, |\mu|)}$. If X is a topological space and $\mu \in \mathcal{M}_\infty(X)$, then we write shortly $L_p(X, \mu)$ or $L_p(\mu)$. Finally, in this case, if $A \in \mathcal{B}(X)$, we denote by $L_p(A, \mu)$ the subspace of all $f \in L_p(\mu)$ satisfying $f\mathbb{1}_A = f$.

Definition A.8 Let $X \subset \mathbb{C}$ and $\mu \in \mathcal{M}_\infty(X)$. For $k \in \mathbb{N}_0$ we call

$$|m_k|(|\mu|) := \int |z|^k \, d|\mu|(z) \in [0, +\infty]$$

the k-th **absolute moment** of μ. If $|m_k|(|\mu|) \in [0, \infty)$, then we call

$$m_k(\mu) := \int z^k \, d\mu(z)$$

the k-th **moment** of μ. If $X \subset \mathbb{R}$ and $y \in [0, \infty)$, we additionally set

$$|m_y|(|\mu|) := \int |x|^y \, d|\mu|(x) \in [0, +\infty]$$

and in the case where $|m_y|(|\mu|)$ is finite also

$$m_y(\mu) := \int x^y \, d\mu(x).$$

Remark and Definition A.9 Let X be a separable metric space and $\mu \in \mathcal{M}_{\infty,+}(X)$. If we set

$$N := \{x \in X : \text{there is some neighborhood } U_x \text{ of } x \text{ such that } \mu(U_x) = 0\}.$$

then N is an open subset of X. For fixed $x \in X$, let us fix a neigborhood U_x of zero measure. Then $(U_x)_{x \in X}$ is an open cover of N and since X is separable, we know that there is some open subcover $(U_k)_{k \in \mathbb{N}}$. But now,

$$\mu(N) \le \sum_{k \in \mathbb{N}} \mu(U_k) = 0.$$

The set

$$\operatorname{supp}(\mu) := X \setminus N$$

is called the **support** of μ and is the smallest closed set $F \subset X$ such that $\mu(X \setminus F) = 0$. One can show that

$$\operatorname{supp}(\mu) = X \setminus V$$

where V is the union of all open sets of zero measure with respect to μ.
Lastly, if $\mu \in \mathcal{M}_\infty(X)$, then we set

$$\operatorname{supp}(\mu) := \operatorname{supp}(|\mu|)$$

and call this set again the **support** of μ. We write $\mathcal{M}_c(X)$ for the finite complex Borel measures on X with compact support and $\mathcal{M}_{c,+}(X)$ for the subset of its positive elements.

Remark and Definition A.10 Let (X, Σ) be a measurable space and $\mu, \nu \in \mathcal{M}_\infty(X, \Sigma)$.

1. We say that μ is **concentrated** on a set $A \in \Sigma$, if $\mu(E) = \mu(A \cap E)$ for all $E \in \Sigma$.

2. The measures μ and ν are called **mutually singular** if there exist disjoint sets $A, B \in \Sigma$ such that μ is concentrated on A and ν is concentrated on B. In this case we write $\mu \perp \nu$.

3. The measure μ is called **absolutely continuous** with respect to ν if $\mu(N) = 0$ for all $N \in \Sigma$ with $\nu(N) = 0$. In this case we write $\mu \ll \nu$.

4. If there exists a measurable function $h : X \to \mathbb{C}$ such that

$$\mu(A) = \int_A h \, d\nu \quad (A \in \Sigma),$$

   then $\mu \ll \nu$ and we say that μ has the ν-**density** h. In this case, we also write $\mu = h\nu$.

We close this part of the appendix with 'probably the most important theorem in measure theory' (see [68, Theorem 6.9]). It is the main result about absolute continuity and delivers a decomposition of arbitrary measures.

Theorem A.11 (Lebesgue-Radon-Nikodym, [68]) *Let (X, Σ) be a measurable space, $\mu \in \mathcal{M}(X, \Sigma)$ and $\nu \in \mathcal{M}_{\infty,+}(X, \Sigma)$ be σ-finite.*

1. *There is unique pair $(\mu_a, \mu_s) \in \mathcal{M}(X, \Sigma) \times \mathcal{M}(X, \Sigma)$ such that $\mu_a \ll \nu, \mu_s \perp \nu$ and*

$$\mu = \mu_a + \mu_s.$$

   Moreover, $\mu_a \perp \mu_s$.

2. *There is unique $h \in L_1(\nu)$ such that*

$$\mu_a(A) = \int_A h \, d\nu \quad (A \in \Sigma).$$

Definition A.12 The unique function h in Theorem A.11 is called the **Radon-Nikodym derivative** of μ and we often write $\dfrac{d\mu}{d\nu}$ instead of h.

Appendix B

Cauchy Transforms of Frequently Used Functions and Measures

In this part of the appendix, we gather the Cauchy transforms of frequently used functions and measures. We give a detailed calculation and present different techniques that can be used to determine the Cauchy transform of a function or measure.

Example B.1 If $a \in \mathbb{C}$ and δ_a is the **Dirac measure** on $\mathbb{C}$ with respect to a, i.e.

$$\delta_a(A) = \begin{cases} 1, & a \in A \\ 0, & a \notin A. \end{cases}$$

for $A \in \mathcal{B}(\mathbb{C})$. Then,

$$(C\delta_a)(z) = \frac{1}{2\pi i(a-z)} \quad (z \in \mathbb{C} \setminus \{a\}).$$

We now want to consider absolutely continuous measures and calculate their Cauchy transform. For this purpose, we shall need the concept of complex roots. First notice that there exists a unique holomorphic function $g : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ such that

i) $g^2(z) = z \quad (z \in \mathbb{C} \setminus (-\infty, 0])$,

ii) $g(z) > 0 \quad (z \in (0, \infty))$.

In this case, g is called the **principal branch** of the square root and we often write $\sqrt{z}$ instead of $g(z)$. One can also define square roots of other polynomials as the following lemma shows:

Lemma B.2 *There is a unique holomorphic function $g : \mathbb{C} \setminus [-1, 1] \to \mathbb{C}$ such that*

i) $g^2(z) = z^2 - 1 \quad (z \in \mathbb{C} \setminus [-1, 1])$.

ii) $g(z) > 0 \quad (z \in (1, \infty))$.

Proof. Let $G := \mathbb{C} \setminus ((-\infty, -1] \cup [1, \infty))$ and consider the mapping $h : G \to \mathbb{C}, h(z) = 1 - z^2$. Then, h is a holomorphic function on a simply connected domain and $0 \notin h(G)$. Hence, there is a unique holomorphic function $\varphi : G \to \mathbb{C}$ such that $\varphi^2(z) = h(z)$ and $\varphi(0) = 1$, see [68, Theorem 13.11]. If we put $\psi : \mathbb{C} \setminus [-1, 1] \to \mathbb{C}, \psi(z) = \varphi\left(\frac{1}{z}\right)$, then clearly

$$\psi^2(z) = 1 - \frac{1}{z^2} = \frac{z^2 - 1}{z^2} \quad (z \in \mathbb{C} \setminus [-1, 1]).$$

Therefore, the function $g : \mathbb{C} \setminus [-1, 1] \to \mathbb{C}, g(z) = z\psi(z)$, satisfies

$$g^2(z) = z^2 - 1 \quad (z \in \mathbb{C} \setminus [-1, 1])$$

and moreover $g(z) > 0$ for $z \in (1, \infty)$ by the choice of φ. The uniqueness is clear since any holomorphic function f which satisfies both conditions has to fulfill the equation

$$f(x) = \sqrt{x^2 - 1} \quad (x \in (1, \infty)),$$

from which the uniqueness follows together with the identity theorem. $\qquad\square$

Remark B.3 For the function g in the previous lemma, we often write $g(z) = \sqrt{z^2 - 1}$.

Example B.4 Let μ be the arcsine distribution, i.e. the measure with the λ_1 density f defined by

$$f(t) = \frac{1}{\pi\sqrt{1 - t^2}} \mathbb{1}_{(-1,1)}(t) \quad (t \in \mathbb{R}).$$

We want to calculate the Cauchy transform of μ. Since μ has compact support, we can use Proposition 3.1.4:

$$(C\mu)(z) = -\frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{m_n(\mu)}{z^{n+1}} \quad (|z| > 1),$$

where

$$m_n(\mu) = \int_{-1}^{1} f(t)t^n \, dt \quad (n \in \mathbb{N}_0)$$

are the moments of the arcsine distribution. But since (see [56], p. 306)

$$m_{2n}(\mu) = \binom{2n}{n} \cdot \left(\frac{1}{2}\right)^{2n}, m_{2n+1}(\mu) = 0$$

we see that

$$(C\mu)(z) = -\frac{1}{2\pi i z} \sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{1}{2}\right)^{2n} z^{-2n} = -\frac{1}{2\pi i z \varphi\left(\frac{1}{z}\right)} \quad (|z| > 1)$$

where φ is the function in the proof of B.2. This can be seen by the equality

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1 - 4x}} \quad \left(x \in \mathbb{R}, |x| < \frac{1}{4}\right),$$

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see [13], p. 449, and the identity theorem. If $|z| > 1$, then the equality

$$\sqrt{z^2 - 1} = z\varphi\left(\frac{1}{z}\right)$$

holds. This is again a consequence of the identity theorem since the equation is satisfied if $z \in (1, \infty)$. But this means

$$(C\mu)(z) = -\frac{1}{2\pi i \sqrt{z^2 - 1}} \quad (|z| > 1)$$

and again, by the identity theorem, for all $z \in \mathbb{C} \setminus [-1, 1]$.

Example B.5 Let μ be the Cauchy measure, i.e. $\mu = f\lambda_1$ where

$$f(t) = \frac{1}{\pi(1 + t^2)} \quad (t \in \mathbb{R}).$$

Since $\text{supp}(\mu)$ is not compact, we can not use Proposition 3.1.4 and have to try a direct calculation. First, for $z \in \Pi_+ \setminus \{i\}$ we consider the set $\Omega := \{z \in \mathbb{C} : \text{Im}(z) > -1\} \setminus \{z, i\}$. Then, the function $F : \Omega \to \mathbb{C}$ defined by

$$f(z) = \frac{1}{\pi(w^2 + 1)(w - z)} \quad (w \in \Omega)$$

satisfies $f(w) = O\left(1/|w|^3\right)$ $(|w| \to +\infty, w \in \Omega)$. Hence, we can apply [54, Theorem 5.2.12] and get

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{\pi(t^2 + 1)(t - z)} \, dt = \text{Res}(f, z) + \text{Res}(f, i) = \frac{1}{\pi}\left(\frac{1}{z^2 + 1} + \frac{1}{2i(i - z)}\right) = -\frac{1}{2\pi i(z + i)}.$$

A similar argument gives

$$(Cf)(z) = -\frac{1}{2\pi i(z - i)} \quad (z \in \Pi_- \setminus \{-i\})$$

and therefore by the identity theorem

$$(C\mu)(z) = \begin{cases} -\dfrac{1}{2\pi i(z + i)}, & z \in \Pi_+ \\ -\dfrac{1}{2\pi i(z - i)}, & z \in \Pi_-. \end{cases}$$

Example B.6 We consider the function f defined by

$$f(t) = \frac{1}{\sqrt{t}}\mathbb{1}_{(0,\infty)}(t) \quad (t \in \mathbb{R}).$$

Then, the substitution $u = \sqrt{t}$ gives

$$\int_0^\infty \frac{1}{\sqrt{t}(t+1)} \, dt = 2 \int_0^\infty \frac{1}{u^2+1} \, du = \pi$$

and therefore $f \in \mathscr{C}$. If $a \in (-\infty, 0)$, then again the substitution $u = \sqrt{t}$ gives us that

$$(Cf)(a) = \frac{1}{2\pi i} \int_0^\infty \frac{1}{\sqrt{t}(t-a)} \, dt = \frac{1}{\pi i} \int_0^\infty \frac{1}{u^2-a} \, du = \frac{1}{2i\sqrt{-a}}.$$

The identity theorem gives us that

$$(Cf)(z) = \frac{1}{2i\sqrt{-z}} \quad (z \in \mathbb{C}_-).$$

Example B.7 We consider the function f defined by

$$f(t) = \frac{1}{\sqrt{t} \cdot (t+1)} \mathbb{1}_{(0,\infty)}(t) \quad (t \in \mathbb{R}).$$

Since

$$\int_0^\infty \frac{1}{\sqrt{t} \cdot (t+1)} \, dt = 2 \int_0^\infty \frac{1}{u^2+1} \, du = \pi$$

we see that $f \in \mathscr{C}$. Moreover, by substituting $u = \sqrt{t}$ we have for $a \in (-\infty, 0) \setminus \{-1\}$:

$$
\begin{aligned}
(Cf)(a) &= \frac{1}{2\pi i} \int_0^\infty \frac{1}{\sqrt{t}(t+1)(t-a)} \, dt \\
&= \frac{1}{\pi i} \int_0^\infty \frac{1}{(u^2+1)(u^2-a)} \, du \\
&= \frac{1}{\pi i(-a-1)} \int_0^\infty \frac{1}{u^2+1} - \frac{1}{u^2-a} \, du \\
&= \frac{1}{i(-a-1)} - \frac{1}{i(-a-1)\sqrt{-a}} \\
&= \frac{1}{i(-a+\sqrt{-a})}.
\end{aligned}
$$

By the identity theorem, we conclude that

$$(Cf)(z) = \frac{1}{i(-z+\sqrt{-z})} \quad (z \in \mathbb{C}_-).$$

Example B.8 We consider the function f defined by

$$f(t) = \frac{\sqrt{t}}{t+1} \mathbb{1}_{(0,\infty)}(t) \quad (t \in \mathbb{R}).$$

The substitution $u = \sqrt{t}$ and [72], p. 110, imply that

$$\int_0^\infty \frac{\sqrt{t}}{(t+1)^2}\, dt = 2\int_0^\infty \frac{u}{(u^2+1)^2}\, du = 2\int_0^\infty \frac{1}{u^2+1}\, du - 2\int_0^\infty \frac{1}{(u^2+1)^2}\, du = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

and therefore $f \in \mathscr{C}$. Moreover, by substituting $u = \sqrt{t}$ we have for $a \in (-\infty, 0) \setminus \{-1\}$:

$$
\begin{aligned}
(Cf)(a) &= \frac{1}{2\pi i}\int_0^\infty \frac{\sqrt{t}}{(t+1)(t-a)}\, dt \\
&= \frac{1}{\pi i}\int_0^\infty \frac{u^2}{(u^2+1)(u^2-a)}\, du \\
&= \frac{-a}{\pi i(-a-1)}\int_0^\infty \frac{1}{u^2-a}\, du - \frac{1}{\pi i(-a-1)}\int_0^\infty \frac{1}{u^2+1}\, du \\
&= \frac{1}{2\pi i(-a-1)}\left(\pi\sqrt{-a} - \pi\right) \\
&= \frac{1 - \sqrt{-a}}{2i(a+1)}.
\end{aligned}
$$

By the identity theorem, we conclude that

$$(Cf)(z) = \frac{1 - \sqrt{-z}}{2i(z+1)} \quad (z \in \mathbb{C}_-).$$

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Nomenclature

$U_\varepsilon(z)$ open disk around z with radius ε, page 10

$\mathbb{C}$ set of all complex numbers, page 9

$\mathbb{C}_-$ cut plane, i.e. $\mathbb{C} \setminus [0, \infty)$, page 10

$\mathbb{C}_\infty$ Riemann sphere, page 9

$\mathbb{N}$ set of all natural numbers, page 9

$\mathbb{N}_0$ set of all natural numbers including 0, page 9

Π_+ open upper half-plane in $\mathbb{C}$, page 10

Π_- open lower half-plane in $\mathbb{C}$, page 10

$\mathbb{Q}$ set of all rational numbers, page 9

$\mathbb{R}$ set of all real numbers, page 9

$\mathbb{Z}$ set of all integer numbers, page 9

$\mathbb{S}^{n-1}$ unit sphere in $\mathbb{R}^n$, page 117

$\mathbb{T}$ unit circle in $\mathbb{C}$, page 9

$\mathbb{1}_X$ indicator function of X, page 10

$I(\mu)$ logarithmic energy of μ, page 105

$I_\alpha(\mu)$ Riesz energy of order α of μ, page 115

$\|\mu\|$ total variation norm of μ, page 129

δ_a Dirac measure with respect to a, page 132

$\frac{d\mu}{d\nu}$ Radon-Nikodym derivative of μ with respect to ν, page 131

λ_n Lebesgue measure on $\mathbb{R}^n$, page 10

$\mu \ll \nu$ μ is absolutely continuous with respect to ν, page 131

$\mu \perp \nu$ μ and ν are mutually singular, page 131

$\mu_{\alpha,\beta}$ measure with λ_1 density $t \mapsto (1-t)^{\alpha}(1+t)^{\beta}\mathbb{1}_{(-1,1)}(t)$, page 82

μ_{β} measure with λ_1 density $t \mapsto t^{\beta}\mathbb{1}_{(0,\infty)}(t)$, page 49

ω_{n-1} area of $\mathbb{S}^{n-1}$, page 117

σ_{n-1} surface measure on $\mathbb{S}^{n-1}$, page 117

$|m_k|(\mu)$ k-th absolute moment of μ, page 130

$|\mu|$ total variation measure of μ, page 129

$h\nu$ measure with ν-density h, page 131

$m_k(\mu)$ k-th moment of μ, page 130

$p_{\mu,\alpha}$ Riesz potential of order α of μ, page 115

p_{μ} logarithmic potential of μ, page 105

$w_{\nu,n}$ exponential weight of ν of order n, page 99

B_p constant for the nontangential maximal function, page 15

$\gamma_{n,\alpha}$ normalization constant for the Riesz kernel of order α, page 115

$\mathcal{B}(X)$ Borel σ algebra on a topological space X, page 128

$\mathcal{M}(X)$ set of all finite complex Borel measures on a topological space X, page 128

$\mathcal{M}_+(X)$ set of all finite, positive Borel measures on a topological space X, page 128

$\mathcal{M}_{\Delta_2,0}$ set of all $\nu \in \mathcal{M}_{\Delta_2}$ with $\nu(\{0\}) = 0$, page 95

$\mathcal{M}_{\Delta_2}$ set of all positive Borel measures on $\mathbb{R}$ satisfying the Δ_2-condition, page 95

$\mathcal{M}_{\alpha}(\mathbb{R}^n)$ set of all $\mu \in \mathcal{M}(\mathbb{R}^n)$ for which the Riesz energy $I_{\alpha}(\mu)$ is defined, page 114

$\mathcal{M}_{\infty,+}(X)$ set of all positive Borel measures on a topological space X, page 128

$\mathcal{M}_{\infty}(X)$ set of all Borel measures on a topological space X which are either positive or complex, page 128

$\mathcal{M}_{\mathrm{eps}}$ set of all positive Borel measures on $\mathbb{R}$ satisfying the ε-condition, page 99

$\mathcal{M}_{\mathrm{exp}}$ set of all positive Borel measures on $\mathbb{R}$ satisfying the exponential condition, page 99

$\mathcal{M}_{\ln,0}(\mathbb{R}^n)$ set of all $\mu \in \mathcal{M}_{\ln}(\mathbb{R}^n)$ with $\mu(\mathbb{R}^n) = 0$, page 124

$\mathcal{M}_{\ln}(\mathbb{R}^n)$ set of all $\mu \in \mathcal{M}(\mathbb{R}^n)$ for which the logarithmic energy $I(\mu)$ is defined, page 104

$\mathcal{M}_{c,+}(X)$ set of the positive elements of $\mathcal{M}_c(X)$, page 130

$\mathcal{M}_c(X)$ set of all compactly supported, finite complex Borel measures on a separable topological space X, page 130

$\mathrm{Aut}_-(\mathbb{C})$ set of all automorphisms on $\mathbb{C}$ with root in Π_-, page 35

$H(\Omega)$ set of all functions holomorphic on Ω, page 9

$H_1(\mathbb{R})$ real Hardy space, page 28

H_A set of all functions which are holomorphic in an open neighborhood of A, page 75

$H_p^+(\mathbb{R})$ set of all boundary functions of functions in $\mathcal{H}_p(\Pi_+)$, page 23

$H_p^-(\mathbb{R})$ set of all boundary functions of functions in $\mathcal{H}_p(\Pi_-)$, page 23

$H_1(X)$ set of all $f \in H_1(\mathbb{R})$ with $f\mathbb{1}_X = f$, page 44

$H_{p_1,p_2}^+(\mathbb{R})$ $= L_{p_1}(\mathbb{R}) \cap H_{p_2}^+(\mathbb{R})$, page 32

$H_{p_1,p_2}^-(\mathbb{R})$ $= L_{p_1}(\mathbb{R}) \cap H_{p_2}^-(\mathbb{R})$, page 32

$U \oplus V$ direct sum of U and V, page 11

$\mathcal{A}^p(\Omega)$ $= \mathcal{A}_{\lambda_2}^p(\Omega)$, page 92

$\mathcal{A}_\nu^p(\Omega)$ weighted Bergman space on Ω of order p with respect to ν, page 92

$\mathcal{H}_p(\mathbb{C} \setminus X)$ Hardy space of order p on $\mathbb{C} \setminus X$, page 44

$\mathcal{H}_p(\mathbb{C} \setminus \mathbb{R})$ Hardy space of order p on $\mathbb{C} \setminus \mathbb{R}$, page 22

$\mathcal{H}_p(\mathbb{C}_\infty \setminus X)$ Hardy space of order p on $\mathbb{C}_\infty \setminus X$, page 44

$\mathcal{H}_p(e^{i\alpha}\Pi_+)$ Hardy space of order p on the half-plane $e^{i\alpha}\Pi_+$, page 12

$\mathcal{H}_{r,p,\alpha,\beta}(\mathbb{C} \setminus [-1,1])$ set of all $F \in \mathcal{H}_r(\mathbb{C} \setminus [-1,1])$ satisfying $\tilde{F}\mathbb{1}_{(-1,1)} \in L_{p,0}(I, \mu_{\alpha,\beta})$, page 87

$\mathcal{Z}_\nu^p(\Omega)$ Zen space of order p on Ω with respect to ν, page 96

$\mathscr{C}$ $= \mathscr{C}(\lambda_1)$, page 18

$\mathscr{C}(\mu)$ set of all functions which are Cauchy transformable with respect to μ, page 18

$\mathscr{E}_{\alpha,p}$ $= \mathscr{E}_{\alpha,p}(\lambda_1)$, page 15

$\mathscr{E}_{\alpha,p}(\mu)$ exponential space of order p and angle α (with respect to μ), page 15

$\mathrm{Exp}(A)$ space of all entire functions of exponential type at most A, page 69

$\mathrm{Exp}_p(A)$ set of all $F \in \mathrm{Exp}(A)$ with $F \in L_p(\mathbb{R})$, page 69

$\tilde{H}^+_{\infty,\omega}(\mathbb{R})$ space of all $f \in L_\infty(\mathbb{R})$ with $C\frac{f}{\omega} = 0$ on Π_-, page 35

$C_\mu f$ μ-Cauchy transform of f, page 19

Cf $= C_{\lambda_1} f$, page 19

F_+ upper boundary function of $F \in \mathcal{H}_p(\Pi_+)$, page 13

F_- lower boundary function of $F \in \mathcal{H}_p(\Pi_-)$, page 13

$H_I f$ finite Hilbert transform of f, page 82

Hf Hilbert transform of f, page 20

$L_{\alpha,\mu} f$ μ-Fourier-Laplace transform of f on the ray $e^{i\alpha}(0,\infty)$, page 15

$L_\alpha f$ $= L_{\alpha,\mathbb{1}_{(0,\infty)}\lambda_1} f$, page 15

$L_\mu f$ $= L_{0,\mu} f \mathbb{1}_{\Pi_+} + L_{\pi,\mu} f \mathbb{1}_{\Pi_-}$, μ-Fourier-Laplace transform of f, page 16

Lf Fourier-Laplace transform of f, page 16

$P_\mu f$ μ-Poisson transform of f, page 20

Pf $= P_{\lambda_1} f$, page 20

$Q_\mu f$ μ-conjugate Poisson transform of f, page 20

Qf $= Q_{\lambda_1} f$, page 21

$\mathscr{L}F$ Fourier-Laplace transform of $F \in \mathcal{H}_2(\Pi_+)$, page 50

$\widehat{f}$ Fourier transform of f, page 16

$\widetilde{F}$ $= F_+ + F_-$ for $F \in \mathcal{H}_p(\mathbb{C} \setminus X)$, page 23

J_α Bessel function of the first kind of order α, page 118

$K(m)$ complete elliptic integral of the first kind at m, page 58

K_n generalized hypergeometric function, page 118

$P_m^{(\alpha,\beta)}$ m-th Jacobi polynomial with parameters α and β, page 88

T_m m-th Chebyshev polynomial, page 89

$\mathcal{K}_a$ shifted version of the complete elliptic integral of the first kind, page 58

$R_{A,B,C}$ model bounded polygonal chain in $\mathbb{C}$, page 70

$R_{a,b}$ model unbounded polygonal chain in $\mathbb{C}_-$, page 55

γ^* trace of γ, page 11

$\mathcal{P}_{n,\infty,2}(\Omega)$ set of all unbounded polygonal chains in Ω with n parts from which two are unbounded, page 54

$\mathcal{P}_n(\Omega)$ set of all bounded polygonal chains in Ω with n parts, page 54

$\operatorname{ind}_\gamma$ index of γ, page 11

$k_r(a)$ path parametrizing the circle around a with radius r, page 10

W_k Widder differential operator of order k, page 65

φ_F linear functional induced by g, page 75

$\|S\|_{\mathrm{op}}$ operator norm of S, page 11

$\|\cdot\|_{\mathbb{C}_\infty \setminus K, p, n}$ norm on $\mathcal{H}_p(\mathbb{C}_\infty \setminus K)$ induced by $\mathcal{P}_n(\mathbb{C} \setminus K)$, page 71

$\|\cdot\|_{\mathbb{C}_\infty \setminus K, p}$ norm on $\mathcal{H}_p(\mathbb{C}_\infty \setminus K)$ induced by cycles $R_{A,B,C}$, page 71

$\|f\|_p$ p-norm of f, page 10

$\|\cdot\|_{\mathbb{C}_-, p, n}$ norm on $\mathbb{C}_-$ generated by $\mathcal{P}_{n,\infty,2}(\mathbb{C}_-)$, page 55

$\|\cdot\|_{\mathbb{C}_-, p}$ norm on $\mathbb{C}_-$ generated by $R_{a,b}$ where $a \geq b > 0$, page 55

$\interleave\cdot\interleave_{\mathcal{H}_p(\mathbb{C}_-)}$ norm on $\mathcal{H}_p(\mathbb{C}_-)$ induced by $(W_k)_{k\in\mathbb{N}}$, page 67

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