Construction of optimal quantizers for Gaussian measures on Banach spaces

Dissertation

zur Erlangung des akademischen Grades eines
Doktors der Naturwissenschaften

Dem Fachbereich IV der Universität Trier
vorgelegt von

Benedikt Wilbertz

Trier, im Juli 2008

Contents

1 Introduction **2**

2 Preliminaries **5**
2.1 Gaussian Measures 5
2.2 Tensor products on Banach spaces 8
2.3 Spline interpolation and approximation 11
2.4 Additional notations and conventions 16

3 Optimal Quantization **18**
3.1 Definitions and problem description 18
3.2 Existence 21
3.3 Optimal Quantization rates and schemes 22
3.3.1 Finite dimensional results 22
3.3.2 Infinite dimensional results 23

4 A new upper Bound **34**
4.1 Constructive upper bound for the quantization error on Banach spaces 34
4.2 n-width 38
4.3 Constructive upper bound for the Kolmogorov n-width 40
4.4 Approximation of X on $C[0,1]$ by Spline-functions 43
4.5 Examples 54
4.6 Notes 59

5 New Optimal Schemes **60**
5.1 New schemes 60
5.2 Comparisons with the known schemes 63

6 Numerical results **65**

7 Open Problems / Future prospects **74**

Notation Index **75**

References **77**

1 Introduction

The quantization problem of a Radon random variable X on a Banach space $(E, \|\cdot\|)$, which satisfies $\mathbb{E}\|X\|^p \leq \infty$ for some $p \in [1, \infty)$, consists for $N \in \mathbb{N}$ in solving the optimization problem

$$\inf \left\{ \left(\mathbb{E} \min_{a \in \alpha} \|X - a\|^p \right)^{1/p} : \ \alpha \subset E, \ |\alpha| \leq N \right\}, \tag{1.1}$$

i.e. we are searching for those N elements in E which give the best approximation to the random variable X in the average sense.

In other words, a quantizer $\alpha \subset E$ with $|\alpha| \leq N$ which is a solution to (1.1) provides an optimal discretization of the random variable X on E. Therefore, let $C_a(\alpha)$ denote a Borel partition of E which satisfies

$$C_a(\alpha) \subset \left\{ x \in E : \|x - a\| = \min_{b \in \alpha} \|x - b\| \right\}. \tag{1.2}$$

Then

$$\hat{X} := \sum_{a \in \alpha} a \, \mathbb{1}_{C_a(\alpha)}(X) \tag{1.3}$$

defines a random variable, which is the best approximation to X taking only N values in E.

This problem has its origin in the field of signal processing of the late 40s, when, with the invention of Pulse-Code-Modulation techniques (PCM), there was need for an optimal strategy to transform a continuous signal into a discrete one, from which, moreover, the original signal could be reconstructed at a given error level.

A very comprehensive survey of the historical development of quantization in the engineering world is given in [GN98].

Since these continuous signals were modeled by probability distributions on $\mathbb{R}^d$, the abstract quantization problem of finding an optimal discretization for a random variable X found its way very quickly into mathematical literature. This process culminated in the publication of [GL00], which is still the standard reference for finite dimensional quantization.

Shortly afterwards, the establishment of a sharp asymptotic formula in [LP02] for the quantization of Gaussian processes on Hilbert spaces turned over a new leaf in quantization theory. In fact, the quantization of infinite dimensional random variables was highly investigated throughout the last years and this thesis is also located within this area of research.

Among many other applications, the use of quantization as a cubature formula, which is especially tailored to the distribution of the random variable X, became a promising tool in various fields of mathematical finance (cf. e.g. [PP03], [PPP04] or [PP05]).

Indeed, let $\alpha \subset E$ with $|\alpha| \leq N$ be a quantizer for X and denote by $C_a(\alpha)$ the Voronoi-partition from (1.2). Then, if $F : E \to \mathbb{R}$ is a measurable functional, we get a cubature formula for the expectation $\mathbb{E}F(X)$ by means of the weighted sum

$$\sum_{a\in\alpha} \mathbb{P}(X \in C_a(\alpha))\, F(a). \tag{1.4}$$

As a matter of fact, the optimal quantization error (1.1) for $p = 1$ provides a worst case error bound for numerical integration on the class of Lipschitz continuous functionals, i.e. let Lip_1 denote the set of all Lipschitz continuous functionals on E with Lipschitz constant ≤ 1. Then one may show (see e.g. [CDMGR08]) that we have for some $\alpha \subset E$ with $|\alpha| \leq N$ and any cubature formula $S^X_\alpha(F)$ which evaluates the functional F only at the points $a \in \alpha$

$$\mathbb{E}\min_{a\in\alpha}\|X - a\| \leq \sup_{F\in\mathrm{Lip}_1} \big|\mathbb{E}F(X) - S^X_\alpha(F)\big|.$$

Moreover, equality holds iff S^X_α is the cubature formula from (1.4).

Thus, choosing α as solution to the quantization problem (1.1) for $p = 1$, we arrive by means of formula (1.4) at a cubature formula, which is optimal in the worst case setting for integration on the class of Lipschitz continuous functionals.

A similar assertion holds in the case $p = 2$, if we additionally assume that F is continuously differentiable with 1-Lipschitz derivative (cf. [Pag08]). In that case, the left-hand side of the lower bound reads $\mathbb{E}\min_{a\in\alpha}\|X - a\|^2$.

Motivated by these applications and their demand for "good" solutions to quantization problem (1.1), we focus in this thesis on the constructive approaches to find (at least asymptotically) optimal solutions to the quantization problem for centered Gaussian random variables X with values in a Banach space.

The expression "constructive" needs to be explained a little more in detail. In this work, the term "constructive" refers to a method which can be implemented by computer algorithms. In particular, this means that all the infinite dimensional problems are reduced to finite dimensional ones on some l^d_q-spaces.

After some preliminary facts and definitions in section 2, we present in section 3 the known results for constructive quantization of Gaussian measures on Banach spaces.

This particularly includes the case of Gaussian X on Hilbert spaces, where the expansion of X in a basis consisting of the eigenvectors of the covariance operator is optimal to reduce the infinite dimensional quantization problem to finite dimensional ones in a constructive manner (cf. [LP02]).

Moreover, it is known from [LP08] that for mean regular processes with paths in some $L^q([0,1], dt)$-spaces for $1 \leq q < \infty$, an expansion in the Haar basis provides the proper method to construct asymptotically optimal quantizers in many cases.

Solely the case $q = \infty$, that is if we consider a Gaussian process with, e.g., continuous paths as random variable on the Banach space $(C[0,1], \|\cdot\|_\infty)$, seems to be different. In fact, the so far developed approaches, which are only based on linear series expansion of X, failed to achieve the optimal rate of the quantization error for $N \to \infty$ (cf. [LP07]).

The only matching asymptotics for this case are based on a non-constructive relation to the Small Ball Probability-Function of X (see [DFMS03]) or assume the existence of some non-constructive infinite dimensional quantizers ([DS06]).

Nevertheless, in section 4, we will be able to derive a constructive upper bound for the quantization error of Gaussian X on $(C[0,1], \|\cdot\|_\infty)$, which circumvents the problems that arise when transferring the methods from the case $q < \infty$ to $q = \infty$ by introducing a non-linear expansion of X. The reason for this different proceeding is actually deeply rooted in Banach space geometry and may cause a different approximation rate for linear and non-linear expansions in non B-convex Banach spaces. This is especially the case for $q = \infty$.

As a matter of fact, this non-linear transformation is based on a spline approximation of X and enables us to relate the quantization error of X to the smoothness of its covariance function by means of a constructive upper bound.

Moreover, this new upper bound can reproduce rates of any order, which in a way also generalizes the results for the Haar basis in the case $q < \infty$.

Finally, in section 6, we present some numerical results for quantizers of the Brownian Motion as random variable on $(C[0,1], \|\cdot\|_\infty)$, which are constructed from a new Quantization scheme introduced in section 5.

Acknowledgments. I would like to express my sincerest gratitude to Prof. Dr. Harald Luschgy and Prof. Dr. Gilles Pagès for their great support, guidance and interest in my work throughout the last years. They offered me the opportunity to spend several months at the Laboratoire de Probabilités et Modèles Aléatoires at Paris and to benefit from the stimulating atmosphere at that institute, which finally enabled the breakthrough of this thesis.

Moreover, I would like to thank Prof. Dr. Wolfgang Sendler for offering me a position at the University of Trier. It was a pleasure to work as his assistant.

Last but not least, I would like to express my gratitude to Dr. Anja Leist for proofreading and her caring and encouraging support during the completion of this work.

2 Preliminaries

2.1 Gaussian Measures

Let X be a Borel-measurable random variable on an abstract probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ with values in a Banach space $(E, \|\cdot\|)$, that is X is measurable with respect to the σ-field generated by the open sets in E. Then, for every $\varphi \in E^*$, where E^* denotes the topological dual of E, the linear action of φ on X, which will be written as (φ, X), is a Borel random variable with values in $\mathbb{R}$.

The random variable X is called *Gaussian*, iff for every $\varphi \in E^*$

$$(\varphi, X)$$

is normally distributed on $(\mathbb{R}, \mathfrak{B}(\mathbb{R}))$ or follows a Dirac distribution, that is either

$$\mathbb{P}\left((\varphi, X) \leq x\right) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy$$

for some $\mu \in \mathbb{R}$ and $\sigma > 0$ or

$$(\varphi, X) \stackrel{d}{=} \delta_\mu.$$

Moreover X is called *centered*, iff $\mu = 0$ for all $\varphi \in E^*$.

In the sequel, we will focus only on Radon random variables. On Banach spaces this is equivalent to X being tight, i.e. for every $\varepsilon > 0$, there exists a compact set $K \subset E$, such that

$$\mathbb{P}(X \in K) \geq 1 - \varepsilon.$$

This in turn implies that X is concentrated with probability one on some separable and closed subspace of E and we may assume from now on unless stated otherwise that the underlying Banach space E is separable. In the latter case, any Borel random variable on E is in fact Radon.

Note furthermore, that this restriction is in our context quite necessary, because only for Radon random variables the quantization error is well-behaved, i.e. it decreases asymptotically to zero (c.f. [Cre01], Prop 2.8.2).

By a theorem of Fernique, a Gaussian random variable has finite moments of any order and these moments are more or less equivalent. Indeed, if we define the *p-norm* of X as

$$\|X\|_p := \left(\mathbb{E}\|X\|^p\right)^{1/p},$$

the following is known:

Proposition 2.1. *([LT91], Cor 3.2) Let $p, q \in (0, \infty)$. Then there is a constant $C_{p,q} > 0$, such that for any Gaussian random variable X it holds that*

$$C_{p,q}^{-1}\|X\|_p \leq \|X\|_q \leq C_{p,q}\|X\|_p.$$

In accordance with the above definition of the p-norm, we denote the Banach space of all Bochner-integrable Radon random variables on E by

$$L^p(E) := L^p(\Omega, \mathfrak{A}, \mathbb{P}; E) := \{Y : (\Omega, \mathfrak{A}) \to (E, \mathfrak{B}(E)) \text{ Radon with } \|X\|_p < \infty\}.$$

In the case $E = \mathbb{R}$ we also will sometimes omit the space E in the above notation.

A centered Gaussian X is uniquely determined by its *covariance operator*

$$C_X : E^* \to E, \quad \varphi \mapsto \mathbb{E}(\varphi, X)X,$$

where the Banach space-valued integral $\mathbb{E}(\varphi, X)X$ is supposed to be of Bochner-type. Nevertheless, since one may assume that E is separable, it coincides with the Pettis-Integral, which is uniquely characterized through the identity

$$(\lambda, \mathbb{E}(\varphi, X)X) = \mathbb{E}(\varphi, X)(\lambda, X) \tag{2.1}$$

for every $\lambda \in E^*$.

If we also denote by $\|\cdot\|$ the usual operator norm, we may conclude that C_X is bounded, since we estimate

$$\begin{aligned}\|C_X\| &= \sup\{\|C_X\lambda\| : \lambda \in E^*, \|\lambda\| \le 1\} \\ &= \sup\{|\mathbb{E}(\varphi, X)(\lambda, X)| : \lambda, \varphi \in E^* \text{ with } \|\lambda\|, \|\varphi\| \le 1\} \\ &\le \mathbb{E}\|X\|^2\end{aligned}$$

and due to the finiteness of Gaussian moments. Hence, $C_X : E^* \to E$ is a linear and continuous operator, which we denote by $C_X \in \mathcal{L}(E^*, E)$.

Closely related to the covariance operator of a centered Gaussian is a densely in the support of X embedded Hilbert space

$$H_X := \{h \in E : \|h\|_{H_X} < \infty\}$$

with

$$\|h\|_{H_X} := \sup\left\{|(\varphi, h)| : \varphi \in E^* \text{ with } \mathbb{E}\varphi(X)^2 \le 1\right\},$$

which is called *Cameron-Martin space.*

Let $E_X^* := \overline{\{\varphi \in E^*\}}^{L^2(\mathbb{P}_X)}$ denote the closure of E^* in $L^2(\mathbb{P}_X)$. Then E_X^* is a Hilbert space equipped with the inner product from $L^2(\mathbb{P}_X)$.

Note that on this space E_X^*, the operator $\tilde{C}_X g = \mathbb{E}(g, X)X$ is also well-defined and establishes an isometric isomorphism between E_X^* and H_X. Indeed, if $\|h\|_{H_X} < \infty$, then

$$\hat{H}_h : E^* \to \mathbb{R}, \quad \varphi \mapsto \varphi(h)$$

can be extended to a bounded linear functional on E_X^* and by the Riesz theorem, we get the existence of a $g \in E_X^*$, such that

$$\varphi(h) = \langle \varphi, g \rangle_{L^2(\mathbb{P}_X)} \quad \forall \varphi \in E^*,$$

where $\langle \cdot, \cdot \rangle_{L^2(\mathbb{P}_X)}$ denotes the inner product in $L^2(\mathbb{P}_X)$. Thus, $h = \tilde{C}_X g$ by (2.1) and $\tilde{C}_X$ is surjective.

Conversely, for each $g \in E_X^*$ we may conclude

$$\begin{aligned} \|\tilde{C}_X g\|_{H_X} &= \sup \left\{ |(\varphi, \tilde{C}_X g)| : \varphi \in E^*, \, \mathbb{E}\varphi(X)^2 \leq 1 \right\} \\ &= \sup \left\{ |\langle \varphi, g \rangle_{L^2(\mathbb{P}_X)}| : \varphi \in E_X^*, \|\varphi(X)\|_{L^2(\mathbb{P}_X)} \leq 1 \right\} \\ &= \|g\|_{L^2(\mathbb{P}_X)} < \infty, \end{aligned}$$

which implies the fact that $\tilde{C}_X$ defines an isometric isomorphism.

Consequently, the inner product on H_X is given by

$$\langle h_1, h_2 \rangle_{H_X} = \langle g_1, g_2 \rangle_{L^2(\mathbb{P}_X)}, \quad h_1, h_2 \in H_X$$

with $g_1, g_2 \in E_X^*$ uniquely determined by the relation

$$(\varphi, h_i) = \langle \varphi, g_i \rangle_{L^2(\mathbb{P}_X)} \quad \forall \varphi \in E^*$$

and $i = 1, 2$.

To summarize the situation, we may draw the following diagram

$$E^* \subset E_X^* \cong H_X \subset E,$$

where the inclusion maps are continuous and the isomorphism between E_X^* and H_X is given by $\tilde{C}_X$. This way, we easily get a factorization of C_X through a Hilbert space, which will serve as a useful tool for the later analysis. For example we may set

$$S : E_X^* \to E, \quad g \mapsto \tilde{C}_X g.$$

Thus, its adjoint $S^* : E^* \to E_X^*$ is simply the inclusion map and we arrive at

$$C_X = SS^*,$$

since C_X and $\tilde{C}_X$ coincide on E^*.

If furthermore $(\xi)_{n \in \mathbb{N}}$ denotes a sequence of i.i.d. standard normals on $\mathbb{R}$ and $(e_n)_{n \in \mathbb{N}}$ is an orthonormal basis of the Hilbert space E_X^*, then

$$\sum_{n \in \mathbb{N}} \xi_n \, S e_n$$

converges a.s. in E and it holds

$$X \overset{d}{=} \sum_{n \in \mathbb{N}} \xi_n S e_n.$$

Moreover, we verify

$$\begin{aligned}
\|C_X\| &\leq \|S\|\|S^*\| = \|S^*\|^2 \\
&= \sup\left\{\|\varphi\|^2_{L^2(\mathbb{P}_X)} : \varphi \in E^*,\ \|\varphi\|_{E^*} \leq 1\right\} \\
&= \sup\left\{\mathbb{E}\varphi(X)^2 : \varphi \in E^*,\ \|\varphi\|_{E^*} \leq 1\right\} \\
&\leq \sup\left\{|\mathbb{E}\lambda(X)\varphi(X)| : \lambda, \varphi \in E^*,\ \|\lambda\|, \|\varphi\| \leq 1\right\} \\
&= \sup\left\{\|C_X\varphi\|_E : \varphi \in E^*,\ \|\varphi\| \leq 1\right\} \\
&= \|C_X\|,
\end{aligned} \tag{2.2}$$

which implies

$$\|C_X\|^{1/2} = \|S\| = \|S^*\|. \tag{2.3}$$

Note that due to the canonical isomorphism between separable Hilbert spaces, this factorization is valid for any separable Hilbert space. For more details on this topic, see e.g. [Bog98] or [VTC87].

If we consider a centered Gaussian X with values in a separable Hilbert space H, then we may regard due to Riesz Theorem C_X as operator on H, i.e. $C_X : H \to H$. This operator is self-adjoint and compact, thus by the Hilbert-Schmidt Theorem, there is a orthonormal basis of H consisting of eigenvectors of C_X. We will see later on that this orthonormal basis plays an important role in the Quantization of Gaussian measure on Hilbert spaces.

2.2 Tensor products on Banach spaces

Let E and F be Banach spaces. For $x_i \in E$ and $y_i \in F$, we associate the formal expression

$$\sum_{i=1}^{n} x_i \otimes y_i \tag{2.4}$$

for some $n \in \mathbb{N}$, with an operator $A : E^* \to F$,

$$A\varphi = \sum_{i=1}^{n} \varphi(x_i) \cdot y_i.$$

On the set of formal expressions of type (2.4), we introduce an equivalence relation

$$\sum_{i=1}^{n} x_i \otimes y_i \sim \sum_{i=1}^{n} a_i \otimes b_i,$$

iff these two formal expressions refer to the same operator $A : E^* \to F$.

The set of all these equivalence classes is denoted by $E \otimes F$, whereas in the following, we may identify a formal expression $\sum_{i=1}^{n} x_i \otimes y_i$, with the whole equivalence class generated by $\sum_{i=1}^{n} x_i \otimes y_i$, similar to the treatment of some function f as representative for its equivalence class in L^p.

The space $E \otimes F$, equipped with the scalar multiplication and addition defined on the associated operator A, is a vector space, which is called *algebraic tensor product.*

In addition, there exist various possible topologies on $E \otimes F$, which can be constructed from the underlying topologies on E resp. F.

For example, we can define a norm on $E \otimes F$ by assigning to $\sum_{i=1}^n x_i \otimes y_i$ the operator norm of the associated operator from E^* to F, i.e.

$$\lambda\left(\sum_{i=1}^n x_i \otimes y_i\right) := \left\|\sum_{i=1}^n x_i \otimes y_i\right\|_\lambda := \sup\left\{\left\|\sum_{i=1}^n \varphi(x_i) y_i\right\| : \varphi \in E^*, \|\varphi\| \leq 1\right\}.$$

Moreover, for the *dyad* $x \otimes y$ we have

$$\begin{aligned}\lambda(x \otimes y) &= \sup\{\|\varphi(x)y\| : \varphi \in E^*, \|\varphi\| \leq 1\} \\ &= \sup\{|\varphi(x)|\,\|y\| : \varphi \in E^*, \|\varphi\| \leq 1\} \\ &= \|x\|\|y\|.\end{aligned}$$

Motivated by this observation, any norm α on $E \otimes F$, which satisfies

$$\alpha(x \otimes y) = \|x\|\|y\|$$

is called a *crossnorm.*

Furthermore, it is possible to define a linear form on $E \otimes F$ by

$$\left(\sum_{i=1}^n \varphi_i \otimes \psi_i\right)\left(\sum_{j=1}^m x_j \otimes y_j\right) = \sum_{i=1}^n \sum_{j=1}^m \varphi_i(x_j)\psi_i(y_j).$$

We then say that α is a *reasonable norm* on $E \otimes F$, iff for every $\varphi \in E^*$ and $\psi \in F^*$ the linear form $\varphi \otimes \psi$ is bounded on $(E \otimes F, \alpha)$ with norm $\|\varphi\|\|\psi\|$.

For example, the above defined norm λ is a reasonable norm. Moreover, it can be shown that λ is the least reasonable crossnorm, i.e. it satisfies

$$\lambda(z) \leq \alpha(z), \quad z \in E \otimes F$$

for all reasonable crossnorms α on $E \otimes F$.

The dual norm to λ is defined as

$$\gamma(z) := \inf\left\{\sum_{i=1}^n \|x_i\|\|y_i\| : x_i \in E, y_i \in F,\ z = \sum_{i=1}^n x_i \otimes y_i\right\},$$

which is the greatest reasonable crossnorm on $E \otimes F$.

In general, the algebraic tensor product spaces are not complete with respect to a given reasonable crossnorm α. We will therefore denote the completion of $E \otimes F$ with respect to α by

$$E \otimes_\alpha F.$$

In this way, we are able to realize classical spaces as tensor products of more elementary ones, where the equality stands up to an isometric isomorphism.

Let K be any compact Hausdorff space and F an arbitrary Banach space. Then we will write

$$C(K, F)$$

for the space of all *continuous functions* from K to F equipped with the norm

$$\|f\|_\infty := \sup_{s\in K} \|f(s)\|_F.$$

and obtain this way

$$C(K) \otimes_\lambda F \cong C(K, F).$$

Indeed, we may associate each

$$u = \sum_{i=1}^{n} x_i \otimes y_i$$

in $C(K) \otimes F$ with an element f_z in $C(K, F)$ by

$$f_z(s) = \sum_{i=1}^{n} x_i(s) y_i.$$

Using

$$\begin{aligned} \lambda(z) &= \sup \left\{ \left\| \sum_{i=1}^{n} \varphi(y_i) x_i \right\|_E : \varphi \in F^*, \|\varphi\| \leq 1 \right\} \\ &= \sup_{\varphi} \sup_{s\in K} \left| \sum_{i=1}^{n} \varphi(y_i) x_i(s) \right| \\ &= \sup_{s} \sup_{\varphi} \left| \varphi\Big(\sum_{i=1}^{n} x_i(s) y_i \Big) \right| \\ &= \sup_{s} \|f_z\|_F = \|f_z\|_\infty \end{aligned} \tag{2.5}$$

the linear map $z \mapsto f_z$ defines an isometric isomorphism between $C(K) \otimes F$ and $C(K, F)$, which clearly extends to $C(K) \otimes_\lambda F$.

Hence, it remains to show that the image of $C(K) \otimes F$ under this map is dense in $C(K, F)$. But this can be easily accomplished by a partition of unity and the compactness of K.

If we especially choose $F := C(K')$, we may identify each $f \in C(K \times K')$ with a $\tilde{f} \in C(K, C(K'))$, for which it holds $\tilde{f}(s) = f_s$, where f_s is the section of f defined by $f_s(t) = f(s, t)$.

Consequently, we may get the following proposition

Proposition 2.2. *Let K, K' be compact Hausdorff spaces. Then, it holds with isometric isomorphisms*

$$C(K) \otimes_\lambda C(K') \cong C(K, C(K')) \cong C(K \times K').$$

More details on tensor products can be found, e.g., in [DF93] or [LC85].

2.3 Spline interpolation and approximation

Let $T := (t_i)_{1 \le i \le n}$ be an increasing knot-sequence in $(a,b) \subset \mathbb{R}$, i.e.

$$a < t_1 \le \ldots \le t_n < b$$

and $r \in \mathbb{N}$. A function S on $\mathbb{R}$ is called a *spline of order r with breakpoints T*, iff on each interval (t_i, t_{i+1}) and $(-\infty, t_1), (t_n, \infty)$ it is a polynomial of degree $< r$ and at least on one of them of exact degree $r-1$.

Moreover, S is assumed to have continuous derivatives of order $< r - k_i$, where k_i denotes the multiplicity of the breakpoint t_i in T, which is supposed to be $\le r$, and an order < 0 means a possible discontinuity of S in t_i.

In the latter case, the spline may not be defined in t_i. We then decide for the càdlàg-version and set $S(t_i) := S(t_i+)$.

Hence, a spline is a piecewise polynomial function and the splines of order $r = 1$ with simple knots are just the step functions, whereas the ones of order $r = 2$ are broken lines.

The set of all spline functions of order r with breakpoints T restricted to $[a,b]$ is called *Schoenberg space* and denoted by $\mathcal{S}_T^r := \mathcal{S}_T^r([a,b])$.

If we introduce auxiliary knots

$$t_{-r+1} \le \ldots \le t_0 = a \quad \text{and} \quad b = t_{n+1} \le \ldots \le t_{n+r}$$

we may define for

$$\Lambda := \{-r+1, \ldots, n\}$$

the *B-spline function*

$$N_j(x) := N(x; t_j, \ldots, t_{j+r}) := (t_{j+r} - t_j)[t_j, \ldots, t_{j+r}](\cdot - x)_+^{r-1}, \quad j \in \Lambda,$$

where the *truncated powers* x_+^k are defined as

$$x_+^k := \begin{cases} x^k & x \ge 0 \\ 0 & x < 0 \end{cases},$$

and the *n-th divided difference* of f is given by

$$[x_0, \ldots, x_n]f := A_n,$$

with A_n the coefficient of x^n in the Hermitian interpolation polynomial of f at points $x_0, \dots, x_n$.

Note that x_+^k is not uniquely defined for $x = 0$ and $k = 0$. In fact, any choice which preserves the normalization (2.7) of the B-Spline functions N_j within the possible discontinuities would be admissible, so we decide once more for the càdlàg-version complemented by a possible left-side limit in the boundary point b.

If the knots t_j are all simple, i.e. $t_j < t_{j+1}$ for all $1 \leq j \leq n$, there is a nice recurrence formula for the B-spline functions, which writes

$$\begin{aligned} N(x; t_0, t_1) &= (t_1 - x)_+^0 - (t_0 - x)_+^0 = \mathbb{1}_{[t_0, t_1)}(x) \\ N(x; t_0, \dots, t_r) &= \frac{x - t_0}{t_{r-1} - t_0} N(x; t_0, \dots, t_{r-1}) + \frac{t_r - x}{t_r - t_1} N(x; t_1, \dots, t_r) \end{aligned} \tag{2.6}$$

(see e.g. [DL93], Chapter 4 for more details on it).

These N_j are normalized such that they perform a partition of unity on $[a, b]$, i.e.

$$\sum_{j \in \Lambda} N_j(x) = 1, \quad x \in [a, b] \tag{2.7}$$

and have support $[t_j, t_{j+r}]$, which is the smallest possible support for a spline function in $\mathcal{S}_T^r$ (c.f. [DL93], Ch.5, §3).

As a consequence of (2.6), the B-spline functions N_j with simple knots t_j, $j \in \{-r+1, \dots, n+r\}$ of order $r = 1$ are indicator functions on $(t_j, t_{j+1}]$ and for $r = 2$ we arrive at overlapping tent-functions with spike at t_{j+1}.

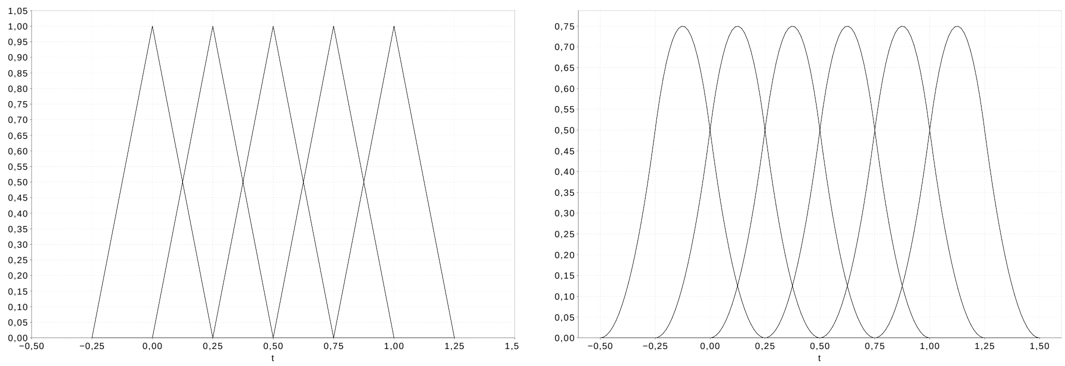

Figure 2.1: B-Splines N_j of order $r = 2, 3$ for the knot sequence $t_j = j/4$, $j \in \{-r+1, \dots, 3+r\}$.

An important result of Curry-Schoenberg ([CS66]) states that $(N_j)_{j \in \Lambda}$ forms a basis of $\mathcal{S}_T^r$, where the coefficients of this representation are given by the *de Boor-Fix functionals* c_j.

These are defined for $j \in \Lambda$ as

$$c_j(S) := \sum_{\nu=0}^{r-1} (-1)^\nu g_{j,r}^{(r-\nu-1)}(\xi_j) S^{(\nu)}(\xi_j), \quad S \in \mathcal{S}_T^r, \tag{2.8}$$

where ξ_j are arbitrary points from $(t_j, t_{j+r}) \cap [a,b]$ and

$$g_{j,1} \equiv 1, \quad g_{j,r}(x) = \frac{1}{(r-1)!}(x - t_{j+1}) \cdots (x - t_{j+r-1}), \quad r \geq 2.$$

Note that the derivative $S^{(\nu)}$ may not exist in some breakpoints $t_j < t_i < t_{j+1}$ for $r - k_i \leq \nu \leq r-1$. But then t_i is a root of $g_{j,r}$ of order k_i and we get $g_{j,r}^{(r-\nu-1)}(t_i) = 0$. Hence (2.8) is well defined.

In fact, the c_j's are independent of the choice of the ξ_j's, i.e. we have

Theorem 2.3. *([dBF73], c.f. [DL93], Ch.5, Thm 3.2)*
Each $S \in \mathcal{S}_T^r$ can be uniquely written as a B-spline series

$$S(x) = \sum_{j \in \Lambda} c_j(S) N_j(x), \quad x \in [a,b]. \tag{2.9}$$

So we might choose $\xi_j^1 := (t_j + t_{j+1})/2$ for $r = 1$ and $\xi_j^2 := t_{j+1}$ in the case $r = 2$ and arrive at

$$c_j(S) = \begin{cases} S\big((t_j + t_{j+1})/2\big) & r = 1 \\ S(t_{j+1}) & r = 2 \end{cases}. \tag{2.10}$$

Furthermore, the de Boor-Fix functionals, which map a spline function S from $L^q := L^q([a,b], dt)$ into the sequence spaces l_q for $1 \leq q \leq \infty$, are bounded from below and above in the following way:

Proposition 2.4. *([DL93], Ch.5, Thm 4.2)*
There is a constant $D_r > 0$, such that for each spline $S = \sum_{j \in \Lambda} c_j(S) N_j$ and each $1 \leq q \leq \infty$

$$D_r \|c'\|_{l_q} \leq \|S\|_{L^q} \leq \|c'\|_{l_q},$$

where $c' := \Big((\frac{t_{j+r} - t_j}{r})^{1/q} \, c_j(S) \Big)_{j \in \Lambda}$.

If we examine the case $q = \infty$ a little bit more in detail, we recognize that the functionals c_j are uniformly bounded by a constant, which is independent of the knot sequence T, i.e.

$$|c_j(S)| \leq D_r^{-1} \|S\|_\infty \tag{2.11}$$

for every knot sequence T, each spline $S \in \mathcal{S}_T^r$ and linear functionals c_j with $S = \sum_{j \in \Lambda} c_j(S) N_j$.

Since each Schoenberg space $\mathcal{S}_T^r$ is a subspace of the Banach space

$$\mathbb{D}([a,b]) := \{f : [a,b] \to \mathbb{R},\ f \text{ is càdlàg}\}$$

equipped with the $\|\cdot\|_\infty$-norm, there exists by the Hahn-Banach Theorem for each $c_j \in (\mathcal{S}_T^r)^*$ an extension γ_j to $\mathbb{D}([a,b])$, which is in the same way uniformly bounded as the c_j in (2.11).

Therefore, we define the *Quasi-Interpolant* Q_T as projection from $\mathbb{D}([a,b])$ to $\mathcal{S}_T^r$ by

$$Q_T(f) := \sum_{j\in\Lambda} (\gamma_j, f)\, N_j. \tag{2.12}$$

This linear operator is again bounded by the same general constant, which is independent of the knot sequence T, i.e.

$$\begin{aligned}
\|Q_T(f)\|_\infty &= \Big\|\sum_{j\in\Lambda} (\gamma_j, f)\, N_j\Big\|_\infty \\
&\le \max_{j\in\Lambda} |(\gamma_j, f)|\, \Big\|\sum_{j\in\Lambda} N_j\Big\|_\infty \\
&\le D_r^{-1}\, \|f\|_\infty,
\end{aligned}$$

since the N_j's are non-negative and form a partition of unity.

Note that these results clearly remain unaffected if we restrict Q_T to the subspace $C([a,b])$ of $\mathbb{D}([a,b])$.

For $q < \infty$, it is possible to derive the same result with γ_j being the Hahn-Banach Extension to $L^1([a,b], dt)$. We thus may state:

Proposition 2.5. *([DL93], Ch.5, Thm 4.4)*
For some constant C_r, each Schoenberg space $\mathcal{S}_T^r$ and each $f \in L^q([a,b], dt)$ with $1 \le q < \infty$ resp. $f \in C([a,b])$ for $q = \infty$, it holds true that

$$\|Q_T(f)\|_{L^q} \le C_r \|f\|_{L^q}.$$

This estimate is the main tool in the proof of the following upper bound for the approximation power of Quasi-Interpolants.

Theorem 2.6. *([DL93], Ch.7 Thm 7.3)*
For a Quasi-Interpolant Q_T of order r and each $f \in L^q([a,b], dt), 1 \le q < \infty, f \in C([a,b]), q = \infty$, one has with $\delta := \max_{0\le j\le n}(t_{j+1} - t_j)$

$$\|f - Q_T f\|_{L^q} \le C_r w_r(f, \delta)_q,$$

for a constant $C_r > 0$ depending only on r.

Here $w_r(f,\delta)_q$ stands for the modulus of smoothness of f, which we introduce in the sequel.

For $h \in \mathbb{R}$ let T_h denote the *translation operator*, i.e. $T_h f = f(\cdot + h)$ and define the *finite difference operator of order* r with $r \in \mathbb{N}$ as the polynomial

$$\Delta_h^r := (T_h - I)^r,$$

where I is the identity operator.

Applying the binomial theorem, we get with $(T_h)^k = T_{kh}$ the identity

$$\Delta_h^r f = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} f(\cdot + kh).$$

We then define the *r-th modulus of smoothness* for $f \in L^q([a,b],dt), 1 \le q < \infty$ and $f \in C([a,b]), q = \infty$ as

$$w_r(f,t)_q := \sup_{0<h\le t} \|\Delta_h^r f\|_{L^q}.$$

Note that f is defined on $[a,b]$, whereas $\Delta_h^r f$ is defined only on $[a, b-rh]$, hence $\|\cdot\|_{L^q}$ should be restricted to $[a, b-rh]$. Nevertheless, we will abuse notation and denote it as above.

The modulus of smoothness obeys the following algebraic properties:

Proposition 2.7. *(c.f. [DL93], Ch.2, §7)*
Let $f,g \in L^q([a,b],dt), 1 \le q < \infty$ or $f \in C([a,b])$ for $q = \infty$ and $r,k \in \mathbb{N}$. Then

(i) $w_r(f,t)_q < \infty \; \forall t \in \mathbb{R}_+$,

(ii) $w_r(f,t)_q \to 0$ *as* $t \to 0$,

(iii) $w_r(f+g,t)_q \le w_r(f,t)_q + w_r(g,t)_q$,

(iv) $w_{r+k}(f,t)_q \le 2^k w_r(f,t)_q$,

(v) $w_r(f,\lambda t)_q \le (1+\lambda)^r w_r(f,t)_q, \; \lambda > 0$,

(vi) $w_{r+k}(f,t)_q \le t^r w_k(f^{(r)},t)_q, \; f \in C^r([a,b])$.

Analogously to the one dimensional case, we define the bivariate differences operator as

$$\Delta_{(h_1,h_2)}^{r_1,r_2} := \left(T_{(h_1,0)} - I\right)^{r_1} \left(T_{(0,h_2)} - I\right)^{r_2}.$$

In the case $r_1 = r_2$, we will also write $\Delta^r_{(h_1,h_2)} := \Delta^{r,r}_{(h_1,h_2)}$. Consequently, the bivariate modulus of smoothness for a function $\Gamma \in L^q([a,b]^2, dt), 1 \leq q < \infty$ or $\Gamma \in C([a,b]^2), q = \infty$ now reads

$$w_{r_1,r_2}(\Gamma,t)_q := \sup_{0<h_1,h_2\leq t} \|\Delta^{r_1,r_2}_{(h_1,h_2)}\Gamma\|_{L^q}.$$

This modulus of smoothness has similar properties than its one dimensional counterpart. We only state here the following important relation:

Proposition 2.8. *([Sch81], Thm* 13.23*)*
For $r \in \mathbb{N}_0$ let $\Gamma \in C^{r,r}([a,b]^2)$ and $r_i, k_i \in \mathbb{N}_0$ with $r_i \leq r$, $i = 1, 2$. Then, it holds for $t > 0$

$$w_{k_1+r_1,k_2+r_2}(\Gamma,t)_q \leq Ct^{r_1 r_2}\, w_{k_1,k_2}(\Gamma^{(r_1,r_2)},t)_q.$$

with some constant $C > 0$ and $\Gamma^{(r_1,r_2)} := \partial\Gamma/\partial^{r_1}\partial^{r_2}$ denoting the partial derivative.

2.4 Additional notations and conventions

Since we will later on characterize in detail the rate of sequences converging to zero, we introduce the following notions:

Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ be two null sequences. We say that a_n has the same sharp asymptotics as b_n, iff $\lim_{n\to\infty} a_n/b_n = 1$ and denote it by $a_n \sim b_n$.

In case of the weak asymptotics, where we cannot say anything about the existence of the above limit nor the constant it reaches and only may ensure that $\liminf_{n\to\infty} a_n/b_n$ or $\limsup_{n\to\infty} a_n/b_n$ are positive reals, we employ the notion $a_n \preceq b_n$, iff there is a constant $C > 0$ with $a_n \leq C \cdot b_n$ for all $n \in \mathbb{N}$. Conversely, $a_n \succeq b_n$ stands for $b_n \preceq a_n$ and we denote $a_n \preceq b_n \preceq a_n$ by $a_n \asymp b_n$.

Generally, we will denote constants by the letter c and C and additional indices may state a dependence on the indicated variables. Moreover, the constants may change from line to line.

In addition, we define the ceiling function as $\lceil x \rceil := \min\{z \in \mathbb{Z} : z \geq x\}$.

As already introduced, we denote by $C([a,b])$ the set of continuous functions $f : [a,b] \to \mathbb{R}$, which is a Banach space equipped with the Sup-Norm

$$\|f\|_\infty := \sup_{x\in[a,b]} |f(x)|.$$

Furthermore, we write for some $r \in \mathbb{N}_0$

$$C^r([a,b]) := \left\{f : [a,b] \to \mathbb{R} :\ f^{(r)} \in C([a,b])\right\}$$

for the set of r-times continuously differentiable functions and $C^{r,r}([a,b]^2)$ refers to the two dimensional r-times continuously differentiable functions on $[a,b]^2$.

The classical Lebesgue-space for $1 \leq q \leq \infty$ on $[a,b]$ are denoted by $L^q := L^q([a,b],dt)$ and consists of all measurable functions $f : [a,b] \to \mathbb{R}$ with finite norm

$$\|f\|_{L^q} := \begin{cases} \left(\int\limits_a^b |f|^q d\lambda \right)^{1/q}, & 1 \leq q < \infty \\ \lambda\text{-ess-sup}|f|, & q = \infty. \end{cases}$$

Regarding sequences spaces on $\mathbb{R}$, we define for $1 \leq q \leq \infty$

$$l_q := \left\{ \xi \in \mathbb{R}^{\mathbb{N}} : \|\xi\|_{l_q} < \infty \right\}$$

with

$$\|\xi\|_{l_q} := \begin{cases} \left(\sum_{n \geq 1} |\xi_n|^q \right)^{1/q}, & 1 \leq q < \infty \\ \sup_{n \in \mathbb{N}} |\xi_n|, & q = \infty, \end{cases}$$

whereas c_0 refers to the set of all null sequences, i.e.

$$c_0 := \left\{ \xi \in \mathbb{R}^{\mathbb{N}} : \lim_{n \to \infty} |\xi_n| = 0 \right\}.$$

Finally, we write

$$c_{00} := \left\{ \xi \in \mathbb{R}^{\mathbb{N}} : \xi_n \neq 0 \text{ for only finite many } n \right\}$$

for all finite sequences.

3 Optimal Quantization

Having introduced some notations and basic facts, we now formulate the fundamental approximation problem, with which we deal in this work.

3.1 Definitions and problem description

Definition 3.1. Let $1 \leq p < \infty$ and $X \in L^p(E)$ be a Radon random vector with values in a general Banach space E.

A set $\alpha \subset E$ with $|\alpha| \leq N$ for $N \in \mathbb{N}$ is called *N-quantizer* and induces for the random variable X the *N-th quantization error*

$$e(X;\alpha)_p := \Big(\mathbb{E}\min_{a\in\alpha}\|X-a\|^p\Big)^{1/p}.$$

If we minimize for fixed N over all quantizers, we arrive at the *minimal quantization error* at level N

$$e_N(X)_p := e_N(X,E)_p := \inf\left\{\Big(\mathbb{E}\min_{a\in\alpha}\|X-a\|^p\Big)^{1/p} \,:\, \alpha\subset E,\, |\alpha|\leq N\right\}. \tag{3.1}$$

Moreover, each N-quantizer with

$$e(X;\alpha)_p = e_N(X)_p$$

is called *optimal N-quantizer.*

In some situations, i.e. the additivity of Proposition 3.2 (ii), it may be useful to consider the quantization error in a different scale, hence we define *dyadic quantization error* for $n \in \mathbb{N}_0$ as

$$r_n(X)_p := e_{2^n}(X)_p.$$

The quantization problem (3.1) may actually be stated in several alternative ways:

Proposition 3.1. *Let $1 \leq p < \infty$, E be a Banach space and $X \in L^p(E)$. Then*

$$\begin{aligned} e_N(X)_p &= \inf\left\{\Big(\mathbb{E}\|X-\hat{X}\|^p\Big)^{1/p} \,:\, \hat{X}\in L^p(E),\, |\hat{X}(\Omega)|\leq N\right\} \\ &= \inf\left\{(\mathbb{E}\|X-f(X)\|^p)^{1/p} \,:\, f:(E,\mathfrak{B})\to(E,\mathfrak{B}),\, |f(E)|\leq N\right\}. \end{aligned}$$

Proof. For $\hat{X} \in L^p(E)$ with $|\hat{X}(\Omega)| \leq N$, define a N-quantizer by $\alpha := \hat{X}(\Omega)$. Since

$$e_N^p(X)_p \leq \mathbb{E}\min_{a\in\alpha}\|X-a\|^p \leq \mathbb{E}\|X-\hat{X}\|^p$$

we get the first inequality by taking the infimum over all possible $\hat{X}$. Analogously, we choose for $f : (E, \mathfrak{B}) \to (E, \mathfrak{B})$ with $|f(E)| \leq N$ the quantization of X as $\hat{X} := f(X)$ and get the second inequality, so it remains to show only

$$\inf\left\{(\mathbb{E}\|X - f(X)\|^p)^{1/p} :\ f : (E, \mathfrak{B}) \to (E, \mathfrak{B}),\ |f(E)| \leq N\right\} \leq e_N(X)_p.$$

For $\alpha \subset E$ with $|\alpha| \leq N$ let $C_a(\alpha)$ be a *Voronoi*-partition of E, that is $C_a(\alpha)$ is a Borel-partition of E satisfying

$$C_a(\alpha) \subset \left\{x \in E : \|x - a\| = \min_{b \in \alpha}\|x - b\|\right\}.$$

Then $f := \sum_{a\in\alpha} a\mathbb{1}_{C_a(\alpha)}$ is Borel-measurable, $|f(E)| \leq N$ and

$$\begin{aligned}
\mathbb{E}\min_{a\in\alpha}\|X - a\|^p &= \sum_{a'\in\alpha} \mathbb{E}\min_{a\in\alpha} \mathbb{1}_{C_{a'}(\alpha)}\|X - a\|^p = \sum_{a'\in\alpha} \mathbb{E}\ \mathbb{1}_{C_{a'}(\alpha)}\|X - a'\|^p \\
&= \mathbb{E}\left(\sum_{a'\in\alpha} \mathbb{1}_{C_{a'}(\alpha)}\|X - a'\|^p\right) = \mathbb{E}\|X - f(X)\|^p \\
&\geq \inf\left\{\mathbb{E}\|X - f(X)\|^p :\ f : (E, \mathfrak{B}) \to (E, \mathfrak{B}),\ |f(E)| \leq N\right\}.
\end{aligned}$$

Taking again the infimum over all N-quantizers yields the assertion. □

The random vector $\hat{X} := f(X)$ is called *(Voronoi)-quantization* of X. Using this equivalence of the problem formulation (3.1), we may switch between these three problem descriptions in order to find optimal quantizers.

Furthermore, the quantization error obeys the following algebraic properties:

Proposition 3.2. *Let $1 \leq p < \infty$ with Radon random variables $X, Y \in L^p(E)$ on the Banach space E. Then*

(i) $e_1(X)_p \leq \|X\|_p$, $e_N(X)_p \to 0$ *as* $N \to \infty$ *and* $e_N(X)_p$ *is non-increasing,*

(ii) $e(X + Y; \alpha_1 + \alpha_2)_p \leq e(X; \alpha_1)_p + e(Y; \alpha_2)_p$
and in particular
$e_{N_1\cdot N_2}(X + Y)_p \leq e_{N_1}(X)_p + e_{N_2}(Y)_p$ *and*
$r_{n+m}(X + Y)_p \leq r_n(X)_p + r_m(Y)_p$, *(Additivity)*

(iii) $e_N(X) = 0$, *if* $|\operatorname{supp}\mathbb{P}^X| \leq N$.

Proof. (i) The monotonicity and the assertion about $e_1(X)_p$ follow directly from the definition of $e_N(X)_p$.

Moreover, since X is Radon, we may assume that $\mathrm{supp}(\mathbb{P}^X)$ is separable, which implies the existence of a countable dense subset $\overline{\{a_i, i \in \mathbb{N}\}} = \mathrm{supp}(\mathbb{P}^X)$. Hence we get

$$0 \le e_N^p(X)_p \le \mathbb{E} \min_{1 \le i \le N} \|X - a_i\|^p \to 0, \qquad \text{as } N \to \infty$$

by the Theorem of dominated convergence.

(ii) Let $\alpha_1, \alpha_2 \subset E$ be quantizers with $|\alpha_1| \le N_1$ and $|\alpha_2| \le N_2$. Then, their *Minkowski sum*

$$\alpha := \alpha_1 + \alpha_2 := \{a_1 + a_2 : a_i \in \alpha, i = 1, 2\}$$

defines a quantizer of level $|\alpha| \le N_1 \cdot N_2$, and we conclude

$$\begin{aligned} e_{N_1 \cdot N_2}(X + Y)_p &\le e(X + Y; \alpha)_p \\ &= \left(\mathbb{E} \min_{a_1 \in \alpha_1} \min_{a_2 \in \alpha_2} \|X + Y - a_1 - a_2\|^p \right)^{1/p} \\ &\le \left(\mathbb{E} \min_{a_1 \in \alpha_1} \|X - a_1\|^p \right)^{1/p} + \left(\mathbb{E} \min_{a_2 \in \alpha_2} \|Y - a_2\|^p \right)^{1/p} \\ &\le e(X; \alpha_1)_p + e(Y; \alpha_2)_p. \end{aligned}$$

Taking the infimum over all quantizers of level N_1 and N_2 resp. 2^n and 2^m yields the assertion.

(iii) If we have $|\mathrm{supp}(\mathbb{P}^X)| \le N$, we set $\alpha := \mathrm{supp}(\mathbb{P}^X)$ and clearly get $\mathbb{E} \min_{a \in \alpha} \|X - a\|^p = 0$. □

Another useful tool is the transition of quantizers by a linear operator.

Proposition 3.3. *Let $1 \le p < \infty$, E and F Banach spaces and $X \in L^p(E)$. Moreover assume $T : E \to F$ to be a bounded operator. Then, for every $N \in \mathbb{N}$ and quantizer $\alpha \subset E$*

(i) $e(TX; T\alpha)_p \le \|T\| \cdot e(X; \alpha)_p$.

(ii) If T is even an isometric isomorphism, then

$$e(TX; T\alpha)_p = e(X; \alpha)_p$$

and in particular

$$e_N(TX)_p = e_N(X)_p.$$

Proof. Part (i) is obvious from the definition of the quantization error and (ii) follows by applying (i) successively for T and T^{-1} with $\|T\| = \|T^{-1}\| = 1$. □

Note moreover, that if we regard the quantization problem of X on some larger space E', the quantization error may be reduced at most by the factor 2.

Proposition 3.4. *Let $\tau_{E'} : E \to E'$ for $E' \supset E$ denote an isometric embedding. Then, it holds*

$$e_N(\tau_{E'}(X))_p \le e_N(X)_p \le 2e_N(\tau_{E'}(X))_p.$$

Proof. The first inequality is obvious. For $\{a_1, \dots, a_N\} \subset E'$ and $\varepsilon > 0$, choose $\{b_1, \dots, b_N\} \subset E$ with

$$\|a_i - \tau_{E'}(b_i)\| \le (1+\varepsilon)\operatorname{dist}(a_i, \tau_{E'}(E)).$$

This implies

$$\|a_i - \tau_{E'}(b_i)\| \le (1+\varepsilon)\|a_i - \tau_{E'}(X)\|,$$

and in addition

$$\|X - b_i\| = \|\tau_{E'}(X) - \tau_{E'}(b_i)\| \le (2+\varepsilon)\|\tau_{E'}(X) - a_i\|.$$

Hence, we arrive at

$$e_N(X)_p \le \big(\mathbb{E} \min_{1\le i\le N} \|X - b_i\|^p\big)^{1/p} \le (2+\varepsilon)\big(\mathbb{E} \min_{1\le i\le N} \|\tau_{E'}(X) - a_i\|^p\big)^{1/p},$$

which yields the assertion, since $\{a_1, \dots, a_N\}$ and $\varepsilon > 0$ were chosen arbitrarily. □

3.2 Existence

In this general setting of Radon random vector with values in a Banach space, it is not clear anymore that there exists actually a quantizer α, which yields the minimal quantization error, that is the infimum in (3.1) actually stands as a minimum and we have

$$e(X;\alpha)_p = e_N(X)_p.$$

We summarize the most important results for the existence of optimal quantizers in the following theorem.

Theorem 3.5. *([GLP07], Thm 1, Prop 2, Thm 2)*
Let $1 \le p < \infty$ and X be a Radon random variable with values in a Banach space E. Then

(i) If E is 1-complemented in its bidual E^{**}, i.e. there exists a projection $P : E^{**} \to E$ with $\|P\| \le 1$, then for every $N \in \mathbb{N}$ there is an N-quantizer α, such that

$$e(X;\alpha)_p = e_N(X)_p.$$

(ii) Denote by $\tau_{E^{**}} : E \to E^{**}$ the canonical embedding of E into its bidual, then we have for every $N \in \mathbb{N}$

$$e_N(X)_p = e_N(\tau_{E^{**}}(X))_p.$$

Note that there always exists a projection P of norm one from E^{***} into E^*, hence E is 1-complemented, once it is a dual space of another Banach space. Consequently, optimal quantizers exist always in the bidual E^{**} and attain there the same quantization error as in E, as far as they belong to the original space E.

Furthermore, we may conclude that in the case of quantization on the Banach space $(C[0,1], \|\cdot\|_\infty)$, with which we will especially deal later on, any $X \in L^p(C[0,1])$ has at least one optimal quantizer in the space $L^\infty([0,1], dt)$ with $\|\cdot\|_{L^\infty} = \lambda$-ess-sup.

Theorem 3.5 moreover ensures the existence of optimal quantizers in Hilbert spaces and in the finite dimensional case of $(E, \|\cdot\|) = (\mathbb{R}^d, \|\cdot\|_{l^p}),\ 1 \le p \le \infty$.

3.3 Optimal Quantization rates and schemes

Unfortunately, there are only very few cases where it is possible to derive explicit solutions to the quantization problem of a random variable X (and then mostly for very small N only).

Hence, most results for optimal quantization are of asymptotic type, i.e. with a decreasing sequence (c_N) we have

$$e_N(X)_p \sim c_N \quad \text{as } N \to \infty$$

for the sharp asymptotics, or only

$$e_N(X)_p \asymp c_N \quad \text{as } N \to \infty$$

in the case of the weak asymptotics.

In addition, we call a sequence of N-quantizers $(\alpha^N)_{n\in\mathbb{N}}$ *(sharp) asymptotically optimal*, iff $e(X;\alpha^N)_p \asymp e_N(X)_p$ or $e(X;\alpha^N) \sim e_N(X)_p$ as $N \to \infty$ in the sharp case.

3.3.1 Finite dimensional results

In the finite dimensional setting, when regarding X as random vector on $(\mathbb{R}^d, \|\cdot\|)$, the sharp asymptotics of the quantization error are completely described by the Zador Theorem, which is in its final version due to Graf and Luschgy.

Theorem 3.6. *([GL00], Thm 6.2)*

Assume $1 \le p < p' < \infty$, $(E, \|\cdot\|) = (\mathbb{R}^d, \|\cdot\|)$ *and* $X \in L^{p'}(\mathbb{R}^d)$. *If we denote by g the Lebesgue density of the absolute continuous part of* $\mathbb{P}^X$, *then*

$$e_N(X)_p \sim C_{p,d,\|\cdot\|} \cdot \left(\int g^{d/(d+p)} d\lambda\right)^{1/p+1/d} \cdot N^{-1/d} \qquad \text{as } N \to \infty,$$

where the constant $C_{p,d,\|\cdot\|} > 0$ *is the limit of* $N^{1/d} \cdot e_N(U([0,1]^d))_p$ *for* $N \to \infty$ *and* $U([0,1]^d)$ *refers to the uniform distribution on* $[0,1]^d$.

A non-asymptotical result in dimension one, which gives a different view on the involving constants, is given by the Pierce-Lemma and plays a fundamental role in the upper for quantization error of mean regular stochastic processes on $L^p([0,1], dt), 1 \leq p < \infty$ (c.f. [LP08]]). It states in an extended version as follows

Lemma 3.7. *Let* $1 \leq p < p' < \infty$*. Then there exists a constant* $C_{p,p'} > 0$*, such that for every random variable* $X \in L^{p'}(\mathbb{R})$

$$e_N(X)_p \leq C_{p,p'} \cdot \|X\|_{p'} \cdot N^{-1} \quad \forall N \in \mathbb{N}.$$

There exists a direct counterpart for the multidimensional setting ([GL00], Cor 6.7), but in that version the constants are not independent of the dimension d, so it is not really applicable for block quantizers with increasing block-length, with which we will deal later on.

More appropriate for our purposes is a version due to J. Creutzig, which is in the first sight not very sharp, since it fails to have the proper order of Theorem 3.6, but this lack will, in fact, not be of high significance in our applications.

Lemma 3.8. *([Cre01], Prop* 4.6.4*)*
Let $1 \leq p < p' < \infty$ *and* $X \in L^{p'}(E)$ *with* $\operatorname{rk} X < d$*. Then for every* $N \in \mathbb{N}$

$$e_N(X)_p \leq C_{p,p'} \cdot \|X\|_{p'} \cdot N^{-(1-p/p')/d},$$

where the constant $C_{p,p'} > 0$ *depends on* p, p' *solely.*

3.3.2 Infinite dimensional results

Turning now to the infinite dimensional case, we will state only those results in detail, which allow the construction of optimal quantization schemes. Moreover, we illustrate the theorems by the example of the Brownian Motion, since, at first glance, the differences and similarities in the resulting rates of these theorems are not very striking only from the formal appearance of their assumptions.

Abstract Quantization Scheme. Since each N-quantizer α lies necessarily in the finite dimensional subspace $\operatorname{span}(\alpha) \subset E$, it is possible to project the random variable X onto an appropriate finite dimensional subspace of E and solve the optimal quantization problem there, which allows to apply the results of the former section. Nevertheless, it is at this point in no way clear how to choose these finite dimensional subspaces and how to "project" X onto them in the general Banach space setting.

Therefore, we start by introducing an abstract quantization scheme, which allows to describe the invertible transformation of the quantization problem on E to finite dimensional problems on some l_q^d-spaces, and consequently renders a construction of optimal quantizers by numerical methods possible.

To be more precise, this quantization scheme consists of

(i) a sequence of finite dimensional random variables in E approximating X,

(ii) a sequence of isomorphisms, which map the random variables from (i) a.s. into some finite dimensional l_q-space,

(iii) a sequence of quantizers in l_q.

Using these three objects, we will be able to describe every optimal and asymptotically optimal quantizer on some Banach space, with which we will deal in this work, in terms of finite dimensional quantizers on l_q.

In addition, we refer to the *rank* of a random variable Y as

$$\operatorname{rk} Y := \dim \operatorname{span}(\mathbb{P}^Y).$$

Definition 3.2. For $X \in L^p(E)$, $N \in \mathbb{N}$ and $q \in [1, \infty]$, let

(i) $(X_k)_{k\geq 1}$ be a sequence of finite dimensional random variables such that

$$\operatorname{rk} X_k \leq d_k \qquad \text{and} \qquad (d_k)_{k\geq 1} \in c_{00},$$

(ii) $I_k : E_k \to l_q^{d_k}$ be linear isomorphisms for subspaces $E_k \subset E$ with $\mathbb{P}^{X_k}(E_k) = 1$,

(iii) $\beta_k \subset l_q^{d_k}$ with $|\beta_k| \leq N_k$, such that $\prod_{k\geq 1} N_k \leq N$.

Then

$$(X_k, I_k, \beta_k)_{k\geq 1}$$

is called *Abstract Quantization Scheme* for X at level N. Moreover, a sequence of Abstract Quantization Schemes at level N

$$\left(X_k^N, I_k^N, \beta_k^N\right)_{k\geq 1}$$

for $N \to \infty$, where the isomorphisms are uniformly bounded by a common constant $C > 0$, i.e.

$$\|I_k^N\|, \|(I_k^N)^{-1}\| \leq C \quad \forall\, k, N \in \mathbb{N}, \tag{3.2}$$

should be denoted *Asymptotical Quantization Scheme*.

W.l.o.g we may assume $(d_k)_{k\geq 1}$ to be ordered non-increasingly. Moreover we will refer in the case $(d_k)_{k\geq 1} = (d_1, 0, \ldots)$ to a *Single-Block Design*, in the case $(d_k)_{k\geq 1} = (1, \ldots, 1, 0, \ldots)$ to a *Scalar-Product Design* and otherwise only to a *Product Design*.

Note that due to the condition $(d_k)_{k\geq 1} \in c_{00}$, there are only finite many random variables with $\operatorname{rk} X_k > 0$. All the other random variables vanish and therefore we have in fact to deal only with a finite number of random variables in the definition of the Abstract Quantization Scheme. Moreover, the same is true for the product $\prod_{k\geq 1} N_k \leq N$, where only finite many N_k may be greater than one.

In addition, we will not demand an explicit specification of the sequences $(d_k)_{k\geq 1}$ and $(N_k)_{k\geq 1}$, although it is in general a non-trivial task to derive these sequences in an optimal way, and the asymptotically optimal choices for $(d_k)_{k\geq 1}$ and $(N_k)_{k\geq 1}$ exhibit mostly a rather complicated form. Nevertheless, regarding the numerical construction of (asymptotically) optimal quantizers, we get better results for finite $N \in \mathbb{N}$ by solving numerically a so-called Block-Allocation-Problem specially tailored to the available quantizers β_k in some $l_q^{d_k}$-space (cf. [PP05] or [LPW08]).

Moreover, the smallest constant $C > 0$ which can be achieved by an Asymptotical Quantization Scheme is $C = 1$. Indeed, we always have

$$1 = \|\mathrm{id}\| \leq \|I_k^{-1}\|\|I_k\| \leq C^2,$$

hence the case $C = 1$ corresponds to the fact that all the I_k's are isometric isomorphisms. In that special case it is, due to Proposition 3.3, completely equivalent if we consider the quantization problem of X_k on E_k or $I_k X_k$ on $l_q^{d_k}$. Otherwise, we only get a weak equivalence, that is up to a constant.

For a given Abstract Quantization Scheme of level N, we may construct in a canonical way an N-quantizer for X by means of the Minkowski sum:

Proposition 3.9. *For $X \in L^p(E)$ and $N \in \mathbb{N}$ let $(X_k, I_k, \beta_k)_{k\in\mathbb{N}}$ be an Abstract Quantization Scheme of level N. Then*

$$\alpha := \sum_{k\geq 1} I_k^{-1}\beta_k := \left\{ \sum_{k\geq 1} I_k^{-1} b_k : b_k \in \beta_k \right\}$$

defines an N-quantizer for X with quantization error upper bound

$$e(X;\alpha)_p \leq \sum_{k\geq 1} \left\| I_k^{-1} \right\| e(I_k X_k;\beta_k)_p + \left\| X - \sum_{k\geq 1} X_k \right\|_p.$$

Proof. From the construction of α and the β_k we clearly have $\alpha \subset E$ and $|\alpha| \leq \prod_{k\geq 1}|\beta_k| \leq \prod_{k\geq 1} N_k \leq N$.

Concerning the quantization error we conclude with Proposition 3.2

$$\begin{aligned} e(X;\alpha)_p &\leq e\left(\sum_{k\geq 1} X_k;\alpha\right)_p + \left\| X - \sum_{k\geq 1} X_k \right\|_p \\ &\leq \sum_{k\geq 1} e(X_k; I_k^{-1}\beta_k)_p + \left\| X - \sum_{k\geq 1} X_k \right\|_p \\ &\leq \sum_{k\geq 1} \left\| I_k^{-1} \right\| e(I_k X_k;\beta_k)_p + \left\| X - \sum_{k\geq 1} X_k \right\|_p . \end{aligned}$$

□

The Hilbert space setting. Regarding the quantization problem on a separable Hilbert space H for $p = 2$ and $E = H$, we get in a canonical way an isomorphism onto l_2 by

$$I : H \to l_2, \qquad h \mapsto (\langle h, u_n \rangle)_{n\geq 1} ,$$

where the $(u_n)_{n\geq 1}$ denote an arbitrary orthonormal basis of H.

This isomorphism is even an isometric one, hence each restriction of I and I^{-1} to some closed subspace E_k is also of norm one and thus the uniform boundedness condition (3.2) for an Asymptotical Quantization Scheme is satisfied with constant $C = 1$.

Moreover, it was shown in [LP02] that in case of a centered Gaussian random variable X, an optimal quantizer α always lies in a subspace

$$U := \operatorname{span}(\alpha) \subset H,$$

which is spanned by the eigenvectors corresponding to the largest eigenvalues of the covariance operator.

Denote by $(\lambda_n)_{n\in\mathbb{N}}$ the decreasingly ordered eigenvalues of C_X and by $(e_n)_{n\in\mathbb{N}}$ the corresponding eigenvectors, which form an orthonormal basis of $\operatorname{supp}(\mathbb{P}^X)$. Then, an expansion of X in the basis $(e_n)_{n\in\mathbb{N}}$ yields

$$X = \sum_{n\geq 1} \langle X, e_n \rangle \, e_n \qquad \text{a.s..} \tag{3.3}$$

Moreover, the coefficients $\langle X, e_n \rangle$ are again centered Gaussians with covariance

$$\mathbb{E}\langle X, e_i \rangle \langle X, e_j \rangle = \langle e_i, C_X e_j \rangle = \langle e_i, \lambda_j e_j \rangle = \lambda_j \delta_{ij},$$

i.e.

$$(\langle X, e_n \rangle)_{n\geq 1} \stackrel{d}{=} \bigotimes_{n=1}^{\infty} \mathcal{N}(0, \lambda_n).$$

Using the notion $\xi_n := \langle X, e_n \rangle / \sqrt{\lambda_n}$, this expansion writes

$$X = \sum_{n \geq 1} \sqrt{\lambda_n} \xi_n e_n, \qquad \text{a.s.}$$

with $(\xi_n)_{n \in \mathbb{N}}$ i.i.d $\mathcal{N}(0,1)$-distributed and is also known as *Karhunen-Loève Expansion* of X.

Furthermore, we have for any closed subspace $V \subset H$ with $\alpha \subset V$ the orthogonal decomposition of the squared quantization error

$$e^2(X;\alpha)_2 = \mathbb{E} \min_{a \in \alpha} \|X - a\|^2 = \mathbb{E} \min_{a \in \alpha} \|\Pi_V X - a\|^2 + \mathbb{E}\|X - \Pi_V X\|^2, \tag{3.4}$$

where we denote by Π_V the orthogonal projection from V on H.

This yields

Proposition 3.10. *([LP02], Thm 3.2, et seqq)*

Let X be a centered Gaussian with values in a separable Hilbert space H. If we denote by $\lambda_1 \geq \lambda_2 \geq \ldots > 0$ the ordered sequence of eigenvalues of the positive semidefinite operator C_X, it holds

$$e_N^2(X)_2 = e_N^2\left(\bigotimes_{n=1}^{d_N} \mathcal{N}(0,\lambda_n)\right)_2 + \sum_{n > d_N} \lambda_n$$

with $d_N := \min\{\dim \operatorname{span}(\alpha) : \ \alpha$ is an optimal N-quantizer for $X\}$.

In terms of an Abstract Quantization Scheme, Proposition 3.10 defines a isometric Single-Block Design with

$$X_1 = \sum_{n=1}^{d_N} \langle X, e_n \rangle \, e_n,$$

$$I_1 : \operatorname{supp}(\mathbb{P}^{X_1}) \to l_2^{d_N}, \quad x \mapsto (\langle x, e_n \rangle)_{1 \leq n \leq d_N}$$

and

$$\beta_1 \subset l_2^{d_N}, \quad |\beta_1| \leq N \quad \text{such that} \quad e\left(\bigotimes_{n=1}^{d_N} \mathcal{N}(0,\lambda_n);\beta_1\right)_2 \leq e_N\left(\bigotimes_{n=1}^{d_N} \mathcal{N}(0,\lambda_n)\right)_2.$$

Thus, by Proposition 3.10

$$\alpha := I_1^{-1}\beta_1 := \sum_{n=1}^{d_N} (\beta_1)_n \, e_n := \left\{ \sum_{n=1}^{d_N} b_n \, e_n \, : \, (b_n)_{1 \leq n \leq d_N} \in \beta_1 \right\}$$

is an optimal N-quantizer for X on H.

In order to establish a sharp upper bound for the quantization error of a centered Gaussian on H, H. Luschgy and G. Pagès constructed in [LP04] a product design based on the expansion (3.3), i.e. for some sequence $(d_k)_{k\geq 1} \in c_{00}$ and $n_k := \sum_{j=1}^{k-1} d_j$ set

$$X_k := \sum_{n=1}^{d_k} \langle X, e_{n_k+n}\rangle \, e_{n_k+n}$$

$$I_k : \operatorname{supp}(\mathbb{P}^{X_k}) \to l_2^{d_k}, \quad x \mapsto (\langle x, e_{n_k+n}\rangle)_{1\leq n\leq d_k}$$

and

$$\beta_k \subset l_2^{d_k}, \quad |\beta_k| \leq N_k \quad \text{with} \quad e\left(\bigotimes_{n=1}^{d_k} \mathcal{N}(0, \lambda_{n_k+n}); \beta_k\right)_2 \leq \lambda_{n_k+1} \cdot e_{N_k}(\mathcal{N}(0, I_{d_k}))_2$$

for a certain sequence $(N_k)_{k\geq 1}$ with $\prod_{k\geq 1} N_k \leq N$.

Setting $\alpha := \sum_{k\geq 1} I_k^{-1}\beta_k$ and using a Hilbert space analogon of Proposition 3.9, which takes into account the orthogonality of $(e_n)_{n\in\mathbb{N}}$, we arrive at

$$e^2(X;\alpha)_2 = \sum_{k\geq 1} e^2(I_k X_k; \beta_k)_2 + \left\| X - \sum_{k\geq 1} X_k \right\|_2^2,$$

which yields with $n := \sum_{k\geq 1} d_k$

$$e^2(X;\alpha)_2 \leq \sum_{k=1}^{n} \lambda_{n_k+1}\, e_{N_k}(\mathcal{N}(0, I_{d_k}))_2 + \sum_{k>n} \lambda_k,$$

H. Luschgy and G. Pagès derived by means of a precise asymptotics for $C(d) := \sup_{N\geq 1} N^{2/d} \cdot e_N^2(\mathcal{N}(0, I_d))_2$ a sharp upper bound under mild regularity assumptions on the decay of the eigenvalues λ_j of the C_X.

Recall that a function $\varphi : (s, \infty) \to (0, \infty)$ for some $s \geq 0$ is said to be *regularly varying at infinity with index* $b \in \mathbb{R}$, iff for every $c > 0$

$$\varphi(cx) \sim c^b \varphi(x) \qquad \text{as } x \to \infty.$$

The sharp asymptotic formula then reads in combination with a corresponding lower bound as follows

Theorem 3.11. *([LP04], Thm 2.2)*

Let X be a centered Gaussian on $L^2([0,1], dt)$ and assume $\lambda_n \sim \varphi(n)$ as $n \to \infty$ for a decreasing and regularly varying function $\varphi : (s, \infty) \to (0, \infty)$ of index $-b < -1$ and some $s \geq 0$. Then

$$e_N(X)_2 \sim \left(\left(\frac{b}{2}\right)^{b-1} \frac{b}{b-1}\right)^{1/2} \psi(\log N)^{-1/2} \qquad \textit{as } N \to \infty$$

with $\psi(x) := \frac{1}{x\varphi(x)}$ for $x > s$.

If we consider a scalar product design for the expansion (3.3), i.e. for some $(d_k)_{k\geq 1} = (1,\dots,1,0,\dots) \in c_{00}$ we have

$$X_k := \begin{cases} \langle X, e_k\rangle\, e_k, & d_k = 1 \\ 0, & d_k = 0, \end{cases}$$

$$I_k : \operatorname{supp}(\mathbb{P}^{X_k}) \to (\mathbb{R}, |\cdot|), \quad x \mapsto \langle x, e_k\rangle$$

and

$$\beta_k \subset \mathbb{R}, |\beta_k| \leq N_k, \quad e(\mathcal{N}(0,\lambda_k);\beta_k)_2 \leq e_{N_k}(\mathcal{N}(0,\lambda_k))_2$$

with $\prod_{k\geq 1} N_k \leq N$, then applying the same techniques as in the non-scalar case does not yield the sharp constant anymore, but is only rate optimal (cf. [LP04], Thm 2.2(b)).

We illustrate the above theorem by the example of the Brownian Motion:

Example 3.1. Let W denote the Wiener measure on $L^2([0,1],dt)$. Then it is well known that the covariance operator C_W has eigenvalues

$$\lambda_n = \left(\frac{1}{\pi(n-1/2)}\right)^2.$$

Thus $\lambda_n \sim (\pi n)^{-2}$ and Theorem 3.11 yields with $b = 2$

$$e_N(W)_2 \sim \frac{\sqrt{2}}{\pi}(\log N)^{-1/2} \quad \text{as } N \to \infty.$$

Note that in the Hilbert space setting we either have an implicit formula for the minimal quantization error for finite $N \in \mathbb{N}$ by Proposition 3.10 and the sharp asymptotics due to Theorem 3.11. This is in fact much more as it will be possible when turning to the non-Hilbert space setting, where in general we only get a weak asymptotic of the quantization error.

Case $E = L^q([0,1],dt), 1 \leq q < \infty$. For $X \in L^p(E)$ with values in the Banach spaces $L^q([0,1],dt)$ with $1 \leq q < \infty$, there is an upper bound for general stochastic processes, which is based on scalar quantization of an expansion in the Haar basis. This upper bound relates the quantization error of the process to the Hölder-smoothness of its paths.

To be more precise, denote by $(e_n)_{n\geq 0}$ the *Haar basis* in $L^q([0,1],dt)$, i.e.

$$e_0 := \mathbb{1}_{[0,1]}, \quad e_1 := \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}$$

$$e_{2^k+j} := 2^{k/2}\, e_1(2^k \cdot - j), \quad k \geq 1, j \in \{0,\dots,2^k-1\}.$$

Then $(e_n)_{n\geq 0}$ is a Schauder basis of $L^q([0,1],dt)$ and we have

$$X = (X, e_0)\, e_0 + \sum_{k\geq 0}\sum_{j=0}^{2^k-1} (X, e_{2^k+j})\, e_{2^k+j} \quad \text{a.s.}$$

with bilinear form $(f,g) := \int_0^1 f(t)g(t)dt$.

Due to the monotonicity of the L^p-Norms, it is sufficient to consider the case $p = q$ only. Moreover, it is straightforward to verify

$$\left\| \sum_{j=0}^{2^k-1} (f, e_{2^k+j}) e_{2^k+j} \right\|_{L^q} = 2^{k/2-k/q} \left\| (f, e_{2^k+j})_{0\le j\le 2^k-1} \right\|_{l_q^{2^k}}.$$

Thus for some $(d_k)_{k\ge -1}$

$$X_{-1} := (X, e_0)\, e_0, \qquad X_k := \sum_{j=0}^{2^k-1} (X, e_{2^k+j}) e_{2^k+j}, \quad k \ge 0$$

$$\begin{aligned} I_{-1} &: \mathrm{supp}(\mathbb{P}^{X_{-1}}) \to (\mathbb{R}, |\cdot|), \quad && x \mapsto (x, e_0) \\ I_k &: \mathrm{supp}(\mathbb{P}^{X_k}) \to l_q^{2^k}, && x \mapsto 2^{k/2-k/q} \big((x, e_{2^k+j})\big)_{0\le j\le 2^k-1} \end{aligned}$$

and

$$\beta_{-1} \subset \mathbb{R}, \quad |\beta_{-1}| \le N_{-1}, \qquad e((X, e_0); \beta_{-1})_q \le e_{N_{-1}}((X, e_0))_q$$

$$\beta_k \subset l_q^{2^k}, \quad |\beta_k| \le \prod_{j=0}^{2^k-1} N_{2^k+j},$$

$$e\left(2^{k/2-k/q}\big((X, e_{2^k+j})\big)_{0\le j\le 2^k-1}; \beta_k\right)_q \le 2^{k/2} \max_{0\le j\le 2^k-1} e_{N_{2^k+j}}((X, e_{2^k+j}))_q$$

defines a product design, where the last inequality is justified by the estimate

$$\|Y\|_q^q = \mathbb{E} \sum_{j=0}^{2^k-1} |Y_j|^q \le 2^k \max_{0\le j\le 2^k-1} \mathbb{E}|Y_j|^q$$

with $Y \in L^q(l_q^{2^k})$. Therefore $\beta_k \subset l_q^{2^k}$ also admits a representation as scalar product quantizer by means of the cartesian product

$$\beta_k := 2^{k/2-k/q} \prod_{j=0}^{2^k-1} \gamma_{2^k+j}$$

for $\gamma_{2^k+j} \subset \mathbb{R}$, $|\gamma_{2^k+j}| \le N_{2^k+j}$ and $e\big((X, e_{2^k+j}); \gamma_{2^k+j}\big)_q \le 2^{k/q} e_{N_{2^k+j}}\big((X, e_{2^k+j})\big)_q$.

Setting

$$\alpha := I_{-1}^{-1}\beta_{-1} + \sum_{k\ge 0} I_k^{-1}\beta_k,$$

which implies

$$e(X, \alpha)_q \le e_{N_{-1}}\big((X, e_0)\big)_q + \sum_{k\ge 0} 2^{k/2} \max_{0\le j\le 2^k-1} e_{N_{2^k+j}}\big((X, e_{2^k+j})\big),$$

Luschgy and Pagès derived in [LP08] by means of Lemma 3.7 the following upper bound:

Theorem 3.12. *([LP08], Thm 1)*
Let $1 \leq p, q < \rho$ and $X \in L^\rho\left(L^q([0,1], dt)\right)$, such that

$$\mathbb{E}|X_t - X_s|^\rho \leq (\varphi(|t-s|))^\rho \quad \forall s, t \in [0,1]$$

with $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ non-decreasing and regularly varying with an index $b > 0$ at 0. Then

$$e_N(X)_p \leq C_{p,q}\, \varphi\left((\log N)^{-1}\right) \quad \forall N \in \mathbb{N}$$

with a constant $C_{p,q} > 0$.

Again we elucidate the situation in case of the Brownian Motion.

Example 3.2. Regarding the Brownian Motion W as random variable in $L^q([0,1], dt)$ for $1 \leq q < \infty$, we know that $W_t - W_s / \sqrt{|t-s|}$ is standard normally distributed for every $s, t \in [0,1]$. Thus

$$\left\| \frac{W_t - W_s}{\sqrt{|t-s|}} \right\|_2 = 1 \quad \forall s, t \in [0,1]$$

and Proposition 2.1 implies for $1 \leq \rho < \infty$ and a constant $C_\rho > 0$

$$\left\| \frac{W_t - W_s}{\sqrt{|t-s|}} \right\|_\rho \leq C_\rho \quad \forall s, t \in [0,1]$$

so that the assumptions of Theorem 3.12 are satisfied with $\varphi(x) = C_\rho\, x^{1/2}$ and the upper bound of the quantization error reads for any $1 \leq p < \infty$

$$e_N(W)_p \preceq (\log N)^{-1/2} \quad \text{as } N \to \infty.$$

Unfortunately even in the case of Gaussian X, this upper bound is not able to reproduce "smooth rates" of processes e.g. with paths in $C^k([0,1]), k \geq 1$, since the Hölder condition does not take into account a faster decay than a linear one.

Case $(E, \|\cdot\|) = (C[0,1], \|\cdot\|_\infty)$. A similar approach using scalar quantization for an appropriate series expansion was investigated in [LP07] for Gaussian X with values in the Banach space $(C[0,1], \|\cdot\|_\infty)$.

Here we start with a series representation

$$X = \sum_{n \geq 1} \xi_n f_n, \qquad \text{a.s.}$$

for ξ_n i.i.d $\mathcal{N}(0,1)$-distributed and $f_n \in C[0,1]$. For $N \in \mathbb{N}$ and some $d := (1, \ldots, 1, 0, \ldots) \in c_{00}$ with $m := \sum_{k \geq 1} d_k$ set

$$X_k := \begin{cases} \xi_k f_k & k \leq m \\ 0 & k > m \end{cases}$$

$$I_k : \mathrm{supp}(\mathbb{P}^{X_k}) \to (\mathbb{R}, |\cdot|), \quad \vartheta f_k \mapsto \|f_k\|_\infty \vartheta$$

and

$$\beta_k \subset \mathbb{R}, |\beta_k| \leq N_k \text{ such that } e(\|f_k\|_\infty \xi_k; \beta_k)_p \leq \|f_k\|_\infty e_{N_k}(\mathcal{N}(0,1))_p$$

for $\prod_{k\geq 1} N_k \leq N$.

Hence the quantizer

$$\alpha := \sum_{k\geq 1} I_k^{-1}\beta_k := \sum_{k=1}^{m} \beta_k f_k / \|f_k\|_\infty$$

yields

$$e(X;\alpha)_p \leq \sum_{k=1}^{m} \|f_k\|_\infty e_{N_k}(\mathcal{N}(0,1))_p + \Big\| \sum_{k>m} \xi_k f_k \Big\|_p$$

from which the following upper bound was derived:

Theorem 3.13. *([LP07], Thm 3) Let $1 \leq p < \infty$ and X be a centered Gaussian with values in the Banach space $(C[0,1], \|\cdot\|_\infty)$, such that there exists a representation*

$$X = \sum_{n\geq 1} \xi_n f_n \qquad a.s.,$$

where the ξ_n's are i.i.d standard normals and the $f_n \in C[0,1]$ satisfy

(i) $\|f_n\|_\infty \preceq n^\vartheta \log(1+n)^\gamma$ as $n \to \infty$, with $\vartheta > 1/2, \gamma \geq 0$

(ii) f_n is α-Hölder-continuous with Hölder constant $[f_n]_\alpha \preceq n^\beta$ as $n \to \infty$, and $\alpha \in (0,1], \beta \in \mathbb{R}$.

Then

$$e_N(X)_p \preceq \frac{(\log\log N)^{\vartheta+\gamma}}{(\log N)^{\vartheta-1/2}} \qquad as\ N \to \infty.$$

Hence, in the case $\gamma = 0$ this upper bound produces an additional $\log\log N$-term in contrast to the rates from Theorem 3.11 and 3.12, which the following example reveals.

Example 3.3. We consider for the Brownian Motion W as random variable on $(C[0,1], \|\cdot\|_\infty)$ the same series expansion as of example 3.1, i.e. for $\lambda_n = (\pi(n-1/2))^{-2}$, $e_n(t) = \sin(t/\sqrt{\lambda_n})$ and ξ_n a sequence of i.i.d standard normals we get

$$W = \sum_{n\geq 1} \sqrt{\lambda_n}\xi_n e_n,$$

where the convergence occurs almost surely in $(C[0,1], \|\cdot\|_\infty)$.

Setting $f_n := \sqrt{\lambda_n} e_n$, the assumptions of Theorem 3.13 are satisfied with $\vartheta = 1, \gamma = 0, \alpha = 1$ and $\beta = 0$. Hence, we conclude for $1 \leq p < \infty$

$$e_N(W)_p \preceq (\log \log N)^{1/2} \cdot (\log N)^{-1/2} \quad \text{as } N \to \infty,$$

which differs from the rates of examples 3.1 and 3.2 by a $\log \log N$-term. Moreover, there is no series expansion, which would yield a better rate.

In fact, it can be derived using a non-constructive relationship between the optimal quantization error and the small-ball probability of W (see [GLP03] or [DFMS03]), that the true rate of W on the Banach space $(C[0,1], \|\cdot\|_\infty)$ is

$$e_N(W)_p \asymp (\log N)^{-1/2} \quad \text{as } N \to \infty.$$

Moreover, this gap of the additional $\log \log N$-term between this upper bound for scalar quantizers and the true rate occurs for all known examples of one dimensional Gaussian processes (see, [LP07]).

In section 4, as the main result of this work, we will present a new constructive upper bound, which matches the true rate in case of the Banach space $(C[0,1], \|\cdot\|_\infty)$, where the so far developed approaches fail.

4 A new upper Bound

4.1 Constructive upper bound for the quantization error on Banach spaces

For the quantization of a Radon random variable X in the general Banach space setting, we first derive a general upper bound for the dyadic quantization error, which is based on the approximation power of finite dimensional random variables X_k to X.

Therefore, suppose $X_k \in L^p(E), k \in \mathbb{N}$ to be a sequence of random variables with

$$\operatorname{rk} X_k \leq k \quad \text{ and } \quad \|X - X_k\|_p \to 0 \quad \text{as } k \to \infty.$$

Note, that such a sequence always exists since $\operatorname{supp}(\mathbb{P}^X)$ is separable for X Radon.

As a matter of fact, the rate of convergence to zero of $\|X - X_k\|_p$ dominates the dyadic quantization error in the following way:

Theorem 4.1. *Let $1 \leq p < p' < \infty$ and $X \in L^{p'}(E)$ for a Banach space E. Assume furthermore, that there is a sequence of random variables $X_k \in L^{p'}(E)$ with*

$$\operatorname{rk} X_k \leq c2^k \quad \text{ and } \quad \|X - X_k\|_{p'} \preceq 2^{-\alpha k} \quad \text{as } k \to \infty, \tag{4.1}$$

for some $\alpha > 0$ and $c \in \mathbb{N}$.

Then, there is a sequence $Z_k \in L^{p'}(E)$ with $Z_{c'2^k} = X_k$,

$$\operatorname{rk} Z_k \leq k \quad \text{ and } \quad \|X - Z_k\|_{p'} \preceq k^{-\alpha} \quad \text{as } k \to \infty,$$

for some $c' \in \mathbb{N}$, such that

$$r_n(Z_n)_p \preceq n^{-\alpha} \quad \text{ as } n \to \infty,$$

and in particular we may state for the dyadic quantization error of X

$$r_n(X)_p \preceq n^{-\alpha} \quad \text{ as } n \to \infty.$$

Proof. We first consider only natural numbers

$$n = c_{p,p',\alpha} \cdot 2^M, \quad M = 1, 2, \ldots$$

for a constant $c_{p,p',\alpha} \in \mathbb{N}$ specified later on.

Suppose to have $X_k \in L^p(E)$ with

$$\operatorname{rk} X_k \leq c2^k \quad \text{ and } \quad \|X - X_k\|_{p'} \leq C \cdot 2^{-\alpha k},$$

for $k \in \mathbb{N}$ and constants $C, c > 0$.

If we set

$$X_0 := 0 \quad \text{ and } \quad Y_k := X_k - X_{k-1}, \quad k \in \mathbb{N},$$

then it holds for Y_k, that

$$\operatorname{rk} Y_k \le \operatorname{rk} X_k + \operatorname{rk} X_{k-1} \le c2^k + c2^{k-1} < c2^{k+1}$$

and

$$\begin{aligned}\|Y_k\|_{p'} &\le \|X - X_k\|_{p'} + \|X - X_{k-1}\|_{p'} \\ &\le C\left(2^{-\alpha k} + 2^{-\alpha(k-1)}\right) \\ &\le C_\alpha \cdot 2^{-\alpha k}.\end{aligned} \tag{4.2}$$

By this, we have constructed a series of random variables Y_k, for which we can control both rank and size, hence they become suitable for the use of Lemma 3.8.

Clearly, we have due to the construction of Y_k for $M \in \mathbb{N}$

$$X_M = \sum_{k=1}^{M} Y_k,$$

which induces by the additivity of r_n in Proposition 3.2

$$r_{\sum_{k=1}^{M} n_k}(X_M)_p \le \sum_{k=1}^{M} r_{n_k}(Y_k)_p \tag{4.3}$$

for some n_k to be specified later on.

Applying the Creutzig Lemma 3.8 for Y_k with $\operatorname{rk} Y_k < c2^{k+1}$, we conclude from (4.2)

$$\begin{aligned} r_{n_k}(Y_k)_p &\le C_{p,p'} \, \|Y_k\|_{p'} \, 2^{-n_k(1-p/p')/c2^{k+1}} \\ &\le C_{p,p',\alpha} \cdot 2^{-\alpha k} \cdot 2^{-n_k(1-p/p')/c2^{k+1}}. \end{aligned}$$

If we fix now n_k as

$$n_k := \left\lceil \frac{c2^{k+1}(1+\alpha)(M-k)}{1-p/p'} \right\rceil$$

we arrive at

$$r_{n_k}(Y_k)_p \le C_{p,p',\alpha} \cdot 2^{-\alpha k} \cdot 2^{-(1+\alpha)(M-k)} = C_{p,p',\alpha} \cdot 2^{-\alpha M} \cdot 2^{k-M},$$

which yields

$$\begin{aligned} r_{\sum_{k=1}^{M} n_k}(X_M)_p &\le \sum_{k=0}^{M} r_{n_k}(Y_k)_p \le C_{p,p',\alpha} \cdot 2^{-\alpha M} \sum_{k=1}^{M} 2^{k-M} \\ &\le C_{p,p',\alpha} \cdot 2^{-\alpha M}. \end{aligned} \tag{4.4}$$

It remains now to show that

$$\sum_{k=1}^{M} n_k \leq c_{p,p',\alpha}\, 2^M.$$

With

$$\begin{aligned} n_k &\leq 1 + \frac{c2^{k+1}(1+\alpha)(M-k)}{1-p/p'} \\ &\leq c_{p,p',\alpha} \cdot 2^{k+1}(M-k) \end{aligned}$$

we arrive at

$$\begin{aligned} \sum_{k=1}^{M} n_k &\leq c_{p,p',\alpha} \sum_{k=1}^{M} 2^{k+1}(M-k) \\ &= c_{p,p',\alpha} \cdot 4\,(2^M + 2 - M) \\ &\leq c_{p,p',\alpha}\; 2^M, \end{aligned}$$

so using the monotonicity of r_n, we have proved so far

$$r_{c_{p,p',\alpha}\,2^M}(X_M)_p \leq C_{p,p',\alpha}\, 2^{-\alpha M}. \tag{4.5}$$

For the case of a general $n \in \mathbb{N}$, we first consider the case $n \geq 2\,c'$ with

$$c' := \max\{c, c_{p,p',\alpha}\} \tag{4.6}$$

. Here we choose $M \in \mathbb{N}$, such that

$$c'\,2^M \leq n < c'\,2^{M+1}, \tag{4.7}$$

which implies

$$2^{-\alpha M} < (2c')^\alpha n^{-\alpha}. \tag{4.8}$$

Thus, we may set $Z_n := X_M$, which yields

$$\operatorname{rk} Z_n = \operatorname{rk} X_M \leq c\,2^M \leq n \quad \text{and} \quad \|X - Z_n\|_{p'} \leq C\; 2^{-\alpha M} < C\,(2c')^\alpha\, n^{-\alpha},$$

and we arrive due to the monotonicity of r_n and (4.5) at

$$\begin{aligned} r_n(Z_n)_p = r_n(X_M)_p &\leq r_{c_{p,p',\alpha}2^M}(X_M)_p \\ &\leq C_{p,p',\alpha}\, 2^{-\alpha M} < C_{p,p',\alpha}\,(2c')^\alpha\, n^{-\alpha}. \end{aligned}$$

In the case $n < 2\,c'$, we clearly have

$$1 < (2c')^{\alpha}\, n^{-\alpha}, \tag{4.9}$$

so we get for $Z_n := 0$

$$\operatorname{rk} Z_n = 0 \le n \qquad \text{and} \qquad \|X - Z_n\|_{p'} = \|X\|_{p'} < \|X\|_{p'}\,(2c')^{\alpha}\, n^{-\alpha},$$

which implies

$$r_n(Z_n)_p \le \|X\|_p < \|X\|_{p'}\,(2c')^{\alpha}\, n^{-\alpha}.$$

Regarding the dyadic quantization error of X, we decompose X into

$$X = Z_n + (X - Z_n),$$

hence Proposition 3.2 yields

$$r_n(X)_p \le r_n(Z_n)_p + \|X - Z_n\|_p,$$

which implies the assertion. □

With exactly the same arguments as in the last step of the proof, we may carry over the assertion to the ordinary quantization error $e_N(X)_p$

Corollary 4.2. *In the situation of Theorem 4.1, we get the existence of a sequence $Z_k \in L^{p'}(E)$ with*

$$\operatorname{rk} Z_k \le \log k \quad \textit{and} \quad \|X - Z_k\|_{p'} \preceq (\log k)^{-\alpha} \quad \textit{as } k \to \infty,$$

such that

$$e_N(Z_N)_p \preceq (\log N)^{-\alpha} \qquad \textit{as } N \to \infty,$$

and in particular it holds for the optimal quantization error of X

$$e_N(X)_p \preceq (\log N)^{-\alpha} \qquad \textit{as } N \to \infty.$$

Due to the preceding theorem, the approximating sequence X_k, which is a-priori appropriate for constructive quantization, since the X_k are finite dimensional, has to fulfill a certain approximation rate to achieve a corresponding lower bound for X.

This approximation problem is better known as average Kolmogorov n-width, which we introduce in the following.

4.2 n-width

Definition 4.1. Let $1 \le p < \infty$, $X \in L^p(E)$ for a Banach space E and $n \in \mathbb{N}_0$. Then we call

$$d_n(X)_p := \inf\{\|X - Y\|_p : Y \in L^p(E), \operatorname{rk} Y \le n\}$$

the *n-th (average) Kolmogorov width.*

This quantity enjoys, similar to the quantization error, the following equivalent descriptions:

Proposition 4.3. *(cf. [Cre01], Prop. 2.7.4)*
For $1 \le p < \infty$, $X \in L^p(E)$ and $n \in \mathbb{N}$ it holds

$$\begin{aligned} d_n(X) &= \inf\{\|Q_U X\|_p : U \subset E \text{ is a subspace with } \dim U \le n\} \\ &= \inf\{\|X - f(X)\|_p : f : (E, \mathfrak{B}) \to (E, \mathfrak{B}), \dim\operatorname{span} f(E) \le n\}, \end{aligned} \tag{4.10}$$

where $Q_U : E \to E/U$ denotes the canonical quotient mapping from E onto the quotient space E/U, i.e. we have $\|Q_U X\| = \inf_{u \in U} \|X - u\|$.

Note that if X is Gaussian, then $Q_U X$ is either Gaussian as image of a continuous linear mapping, so that we have by Proposition 2.1 for any $1 \le p, q < \infty$, a constant $C_{p,q} > 0$, such that for any Gaussian X and $n \in \mathbb{N}$

$$C_{p,q}^{-1} \, d_n(X)_p \le d_n(X)_q \le C_{p,q} \, d_n(X)_p. \tag{4.11}$$

Hence, the Kolmogorov n-width of a Gaussian random variable is up to a constant the same for any power $p \in [1, \infty)$, and we will formulate these results, where the constant does not play a prominent role, only for the case

$$d_n(X) := d_n(X)_2.$$

Furthermore, the Kolmogorov n-width exhibits the following algebraic properties:

Proposition 4.4. *Let $1 \le p < \infty$, and $X, Y \in L^p(E)$ on the Banach space E. Then*

(i) $d_0(X)_p = \|X\|_p$, $d_n(X)_p \to 0$ *as* $n \to \infty$ *and* $d_n(X)_p$ *is non-increasing,*

(ii) $d_{n+m}(X)_p \le d_n(X)_p + d_m(X)_p$,

(iii) $d_n(X) = 0$, *if* $\operatorname{rk} X \le n$.

Since the proof of Proposition 4.4 runs completely in analogue to those of the quantization error, we omit them here.

Note moreover, that we always have

$$d_n(X)_p \le e_n(X)_p,$$

since a quantization rule $f : (E, \mathfrak{B}) \to (E, \mathfrak{B})$ with $|f(E)| \le n$ always spans a subspace with dimension $\le n$.

Regarding the convergence rate of $d_n(X)_p \to 0$ for $n \to \infty$, we only state here an estimate for the finite dimensional Gaussian random variables on l_∞ by J. Creutzig, which will be an important tool in the following section and generalizes a result of V.E. Maiorov ([Mai93]) for the case $u = id_{2,\infty}$.

Lemma 4.5. *(cf. [Cre01], Thm* 4.5.1*)*
For some $C > 0$, the estimate

$$d_n(u(\gamma)) \le C \; \|u\|_{l_{2,\infty}} \left(\log \left(\frac{em}{n+1} \right) \right)^{1/2}$$

is valid for every $u \in \mathcal{L}(l_2^m, l_\infty^m)$ and $n \le m \in \mathbb{N}$, where γ denotes a standard normally distributed random variable on l_2^m.

Since we have employed in the Abstract Quantization Schemes so far only finite dimensional random variables derived from linear series expansions of X, i.e. by a linear transformation on X of finite rank, we also introduce the average linear n-width, where we replace the condition of f being only Borel-measurable by f linear and bounded in (4.10).

Definition 4.2. Let $1 \le p < \infty$, $X \in L^p(E)$ and $n \in \mathbb{N}$. Then

$$l_n(X)_p := \inf \{ \|X - f(X)\|_p : \; f \in \mathcal{L}(E, E), \, \dim f(E) \le n \}$$

is called the *n-th (average) linear width.*

Again we realize that $X - f(X)$ with $f \in \mathcal{L}(E, E)$ is Gaussian, hence the linear n-width is, due to the equivalence of Gaussian moments, up to a constant the same for any power $p \in [1, \infty)$. So we may restrict in some cases to

$$l_n(X) := l_n(X)_2.$$

We clearly always have

$$d_n(X)_p \le l_n(X)_p \tag{4.12}$$

and in case of a Gaussian random variable X with values in a Hilbert space, the estimate even stands as equality with

$$d_n^2(X) = l_n^2(X) = \sum_{j>n} \lambda_j, \tag{4.13}$$

where λ_j denotes the decreasingly-ordered eigenvalues of the covariance operator C_X (see e.g. [Rit00], Ch III, Prop 24).

At this point, the question is raised if there exists a reverse estimate of (4.12), e.g. there is a constant $C_E > 0$, such that

$$l_n(X) \leq C_E \cdot d_n(X)$$

for any Gaussian random variable X with values in the Banach space E and $n \in \mathbb{N}$.

In fact, the above assertion follows from a much stronger result of G. Pisier (cf. [Cre01], Cor 3.4.2), if the Banach space E is B-convex, i.e. it does **not** contain $(l_1^n)_{n\in\mathbb{N}}$ uniformly (see, e.g. [Pis89], Ch 2 for more details on B-Convexity).

As a matter of fact, $L^q([0,1], dt)$ for $q \in (1, \infty)$ is B-convex, which explains why it was possible to derive asymptotical optimal quantizers in section 3.3.2, which were solely based on linear transformations to reduce X into finite dimensional random variables.

In case of an arbitrary Banach space E, it was again J. Creutzig who established for a general constant $C > 0$ the estimate

$$l_n(X) \leq C(1 + \log n)\, d_n(X) \tag{4.14}$$

with X being any Gaussian random variable on a Banach space E and $n \in \mathbb{N}$ (cf. [Cre01], Thm 4.4.1).

Hence, we see that $l_n(X)$ and $d_n(X)$ may differ by a log-term if E is not B-convex, which corresponds to the additional $\log\log$-term in the quantization error of the scheme for the case $q = \infty$ in section 3.3.2.

Consequently, we will, with regard to Theorem 4.1, focus on a nonlinear transformation of the random variable X into a finite-dimensional random vector by means of a constructive approach, which will be subject of the following section.

4.3 Constructive upper bound for the Kolmogorov n-width

We here state the main result about an explicit upper bound for the n-width of Gaussian random variables on E.

Theorem 4.6. *Let $1 \leq p < \infty$ and X be a centered Gaussian on E. Suppose that there is an representation of X as*

$$X = \sum_{k=0}^{\infty} X_k \qquad a.s.$$

satisfying

$$\operatorname{rk} X_k \leq 2^k + r$$

with some $r \in \mathbb{N}_0$.

Assume furthermore, that for some $\alpha > 0$ and some constant $C_r > 0$, there are linear isomorphisms $I_k : E_k \to l_\infty^{2^k+r}$, where $E_k \subset E$ denotes a subspace with $\mathbb{P}^{X_k}(E_k) = 1$, such that

(i) $\|I_k^{-1}\| \le C_r$

(ii) $I_k X_k \overset{d}{=} u_k(\gamma_k)$,

(iii) $\|u_k\|_{l_{2,\infty}} \preceq 2^{-\alpha k}$ *as* $k \to \infty$,

with $u_k \in \mathcal{L}(l_2^{2^k+r}, l_\infty^{2^k+r})$ and γ_k denoting a standard normally distributed random variable on $l_2^{2^k+r}$.

Then, there exists a sequence of random variables $Y_n \in L^p(E)$ with $\operatorname{rk} Y_n \le n$, such that

$$\|X - Y_n\|_p \preceq n^{-\alpha} \quad \text{as } n \to \infty,$$

which particularly yields

$$d_n(X)_p \preceq n^{-\alpha} \quad \text{as } n \to \infty.$$

Proof. Again we first only consider natural numbers

$$n = c_r \cdot 2^N, \quad N = 1, 2, \dots$$

for some $c_r \in \mathbb{N}$ specified later on.

Set $m_k := 2^k + r$ and denote the image of X_k under I_k by ξ_k, i.e.

$$\xi_k := I_k X_k.$$

Thus, ξ_k is a normally distributed random variable on $l_\infty^{m_k}$.

Now choose $\eta_k \in L^p(\mathbb{R}^{m_k})$, such that

$$\operatorname{rk} \eta_k \le n_k \quad \text{and} \quad \|\xi_k - \eta_k\|_p \le 2\, d_{n_k}(\xi_k)_p, \quad k \ge 0, \tag{4.15}$$

where we may define n_k as

$$n_k := \begin{cases} 2^k + r & 0 \le k \le N \\ 2^{2N-k} & N < k \le 2N \\ 0 & 2N < k. \end{cases}$$

This way, we obtain

$$\begin{aligned} \sum_{k\ge 0} n_k &= \sum_{k=0}^{N} (2^k + r) + \sum_{k=N+1}^{2N} 2^{2N-k} \\ &= (N+1)r + \sum_{k=0}^{N} 2^k + \sum_{k=0}^{N-1} 2^k \qquad (4.16) \\ &= (N+1)r + 2^{N+1} - 1 + 2^N - 1 \\ &\le c_r\, 2^N \end{aligned}$$

with constant $c_r := (2r+3) \in \mathbb{N}$.

If we then define

$$Y_n := \sum_{k\geq 0} I_k^{-1}\eta_k, \tag{4.17}$$

which is actually a finite sum, we obviously have

$$\operatorname{rk} Y_n \leq \sum_{k\geq 0} n_k \leq c_r\ 2^N.$$

Regarding the approximation error for X, it then holds by the construction of the η_k and the assumption (i) for the isomorphism I_k

$$\begin{aligned}\|X - Y_n\|_p &= \Big\|\sum_{k\geq 0} X_k - I_k^{-1}\eta_k\Big\|_p \leq \sum_{k\geq 0}\|X_k - I_k^{-1}\eta_k\|_p \\ &\leq C_r \sum_{k\geq 0}\|\xi_k - \eta_k\|_p \leq C_r \sum_{k\geq 0} d_{n_k}(\xi_k)_p.\end{aligned} \tag{4.18}$$

Note, that $d_{n_k}(\xi_k)_p = 0$ for $k \leq N$ by the choice of the n_k's, so we may conclude from (4.18), Lemma 4.5 and assumption (iii)

$$\|X - Y_n\|_p \leq C_r \sum_{k>N} d_{n_k}(\xi_k)_p \leq C_{r,p} \sum_{k>N} 2^{-\alpha k}\left(\log\left(\frac{em_k}{n_k+1}\right)\right)^{1/2}. \tag{4.19}$$

Furthermore, we have for $N < k \leq 2N$

$$\begin{aligned}\log\left(\frac{em_k}{n_k+1}\right) &\leq \log(em_k/n_k) = \log(e2^{2k-2N} + er2^{k-2N}) \\ &\leq C_r(k-N)\end{aligned} \tag{4.20}$$

and in the case $k > 2N$

$$\begin{aligned}\log\left(\frac{em_k}{n_k+1}\right) &= \log(e2^k + er) \leq C_r k \leq C_r k + C_r(k-2N) \\ &\leq C_r(k-N).\end{aligned} \tag{4.21}$$

Thus, (4.19) together with (4.20) and (4.21) now implies

$$\begin{aligned}\|X - Y_n\|_p &\leq C_{r,p} \sum_{k>N} 2^{-\alpha k}\sqrt{k-N} \\ &= C_{r,p}\cdot 2^{-\alpha N} \sum_{k\geq 1} 2^{-\alpha k}\sqrt{k} \\ &\leq C_{r,p,\alpha}\cdot (2^N)^{-\alpha},\end{aligned} \tag{4.22}$$

since $\sum_{k\geq 1} (2^\alpha)^{-k}\sqrt{k}$ is bounded for $\alpha > 0$.

To cover the case of a general $n \in \mathbb{N}$, we first restrict to $n \geq 2\,c_r$. Here we choose an $N \in \mathbb{N}$, such that

$$c_r\, 2^N \leq n < c_r\, 2^{N+1},$$

which implies

$$(2^N)^{-\alpha} < (2c_r)^\alpha\, n^{-\alpha}. \tag{4.23}$$

Setting $Y_n := Y_{c_r 2^N}$, and thus

$$\operatorname{rk} Y_n = \operatorname{rk} Y_{c_r 2^N} \leq c_r\, 2^N \leq n,$$

we arrive by (4.22) and (4.23) at

$$\|X - Y_n\|_p = \|X - Y_{c_r 2^N}\|_p \leq C_{r,p,\alpha}\,(2c_r)^\alpha\, n^{-\alpha}.$$

Conversely, for $n < 2\,c_r$ we may set $Y_n := 0$ and verify

$$\|X - Y_n\|_p = \|X\|_p \leq \|X\|_p (2c_r)^\alpha\, n^{-\alpha}.$$

Since $\operatorname{rk} Y_n \leq n$ for every $n \in \mathbb{N}$, we clearly have the inequality

$$d_n(X)_p \leq \|X - Y_n\|_p.$$

□

From the above proof we get an abstract scheme, which allows to construct approximating random variables in sense of the Kolmogorov n-width of order $n^{-\alpha}$ for Gaussian random variables on a Banach space: We compute an order optimal approximating sequence of random variables η_k for the normally distributed random variables ξ_k on $l_\infty^{m_k}$ and define Y_n as in (4.17) by means of the isomorphisms I_k^{-1} from $l_\infty^{m_k}$ into E_k.

4.4 Approximation of X on $C[0,1]$ by Spline-functions

It is now left to fill the assumptions of Theorem 4.6 with some constructive ingredients for a centered Gaussian X on $(C[0,1], \|\cdot\|_\infty)$.

Therefore, denote by $\mathcal{S}_k^r$ the Schoenberg space on $[0,1]$ of order r generated by 2^k subintervals of length 2^{-k} i.e., we have for $k \in \mathbb{N}_0$ the sequence of simple knots (including auxiliary ones)

$$t_j = j\,2^{-k}, \qquad j = -r+1, \ldots, 2^k + r - 1$$

and write Λ_k for the set of indices $\{-r+1 \leq j \leq 2^k - 1\}$.

Hence, $\mathcal{S}_k^r$ is due to the unique spline representation for B-splines in (2.9) of dimension $m_k := |\Lambda_k| = 2^k + r - 1$.

Note that, in this case, we can inductively derive from (2.6) that the normalized B-Splines $N_j^k, j \in \Lambda_k$ in $\mathcal{S}_k^r$ are translate dilates of the single B-spline

$$N^r(x) := N(x; 0, 1, \ldots, r), \tag{4.24}$$

that is we have

$$N_j^k(x) = N^r(2^k x - j), \quad j \in \Lambda_k. \tag{4.25}$$

Since N^r is continuous for $r > 1$, we get $\mathcal{S}_k^r \subset C[0,1]$ for any $k \in \mathbb{N}, r > 1$. As a matter of fact, $\mathcal{S}_k^1$ is contained only in the larger space $\mathbb{D}[0,1]$, which will lead to some special treatment later on. Nevertheless, the Schoenberg spaces $\mathcal{S}_k^r$ are, as finite dimensional subspaces of $\mathbb{D}[0,1]$, again Banach spaces equipped with the $\|\cdot\|_\infty$-Norm.

Denote by Q_k the restriction of the Quasi-Interpolant (2.12) to $C[0,1]$, i.e.

$$Q_k : C[0,1] \to \mathcal{S}_k^r, \quad f \mapsto \sum_{j \in \Lambda_k} (\gamma_j^k, f)\, N_j^k$$

for some $\gamma_j^k \in (C[0,1])^*$ and normalized B-spline functions N_j^k.

If we set

$$T_0 := Q_0 \quad \text{and} \quad T_k := Q_k - Q_{k-1}, \quad k = 1, 2, \ldots, \tag{4.26}$$

then T_k also maps $C[0,1]$ into $\mathcal{S}_k^r$ and we arrive at the representation

$$\begin{aligned} T_k f &= \sum_{j \in \Lambda_k} (\gamma_j^k, f)\, N_j^k - \sum_{j \in \Lambda_{k-1}} (\gamma_j^{k-1}, f)\, N_j^{k-1} \\ &= \sum_{j \in \Lambda_k} (c_j^k, T_k f)\, N_j^k \\ &= \sum_{j \in \Lambda_k} (T_k^* c_j^k, f)\, N_j^k, \end{aligned} \tag{4.27}$$

where $T_k^* : (\mathcal{S}_k^r)^* \to (C[0,1])^*$ refers to the adjoint of the operator T_k and the $c_j^k \in (\mathcal{S}_k^r)^*$ denote the de Boor-Fix functionals from (2.8). Thus, $T_k^* c_j^k$ is a continuous, linear functional on $C[0,1]$.

In addition, we have by Theorem 2.6

$$\sum_{k=0}^{n} T_k f = Q_n f \to f \quad \text{as } n \to \infty,$$

and there is a constant $C_r > 0$ independent of f and k, such that

$$\begin{aligned} \|T_k f\|_\infty &\leq \|f - Q_k f\|_\infty + \|f - Q_{k-1} f\|_\infty \\ &\leq C_r \left(w_r(f, 2^{-k})_\infty + w_r(f, 2^{-(k-1)})_\infty \right) \\ &\leq C_r w_r(f, 2^{-k})_\infty, \end{aligned}$$

where the last inequality follows from Proposition 2.7.

This, in turn, implies by the virtue of Proposition 2.4 and (4.27)

$$\|(T_k^* c_\cdot^k, f)\|_{l_\infty^{m_k}} \leq D_r^{-1} \|T_k f\|_\infty \leq C_r w_r(f, 2^{-k})_\infty \quad \forall f \in C[0,1]. \tag{4.28}$$

If we now turn to the situation of Theorem 4.6, we may set

$$X_k := T_k X \tag{4.29}$$

and arrive at

$$X = \sum_{k=0}^{\infty} X_k \qquad \text{a.s.}$$

in $(C[0,1], \|\cdot\|_\infty)$ resp. $(\mathbb{D}[0,1], \|\cdot\|_\infty)$. Moreover, for the bijective coordinate mapping

$$I_k : \mathcal{S}_k^r \to l_\infty^{m_k}, \quad S = \sum_{j \in \Lambda_k} (c_j^k, S)\, N_j^k \mapsto \big((c_j^k, S)\big)_{j \in \Lambda_k} \tag{4.30}$$

it holds with some constant $D_r > 0$ from Proposition 2.4

$$\|I_k\| \leq D_r^{-1} \quad \text{ and } \quad \|I_k^{-1}\| \leq 1, \tag{4.31}$$

and we conclude furthermore that

$$I_k X_k = \big((T_k^* c_j^k, X)\big)_{j \in \Lambda_k} \tag{4.32}$$

is a normally distributed random variable on $l_\infty^{m_k}$.

Hence, it remains for the application of Theorem 4.6 to determine the norm of an operator $u_k \in \mathcal{L}(l_2^{m_k}, l_\infty^{m_k})$, which satisfies

$$u_k \gamma_k \stackrel{d}{=} I_k X_k$$

where γ_k refers to a standard normal Gaussian on the Hilbert space $l_2^{m_k}$.

If we consider the covariance operator of the random variable $I_k X_k$, that is $C_{I_k X_k} : (l_\infty^{m_k})^* \to l_\infty^{m_k}$ and recall that $l_1^{m_k}$ is the topological dual space of $l_\infty^{m_k}$, then there is a linear operator u with adjoint u^* and

$$l_1^{m_k} \xrightarrow{u_k^*} l_2^{m_k} \xrightarrow{u_k} l_\infty^{m_k}$$

which provides a factorization of $C_{I_k X_k}$, i.e.

$$C_{I_k X_k} = u_k u_k^*.$$

We then clearly have

$$u_k \gamma_k \stackrel{d}{=} I_k X_k$$

and get from (2.3) for the operator norm

$$\|u_k\|_{l_{2,\infty}^{m_k}} = \left(\|C_{I_kX_k}\|_{l_{1,\infty}^{m_k}}\right)^{1/2}. \tag{4.33}$$

But for the latter expression, we immediately can verify

$$\begin{aligned}
\|C_{I_kX_k}\|_{l_{1,\infty}^{m_k}} &= \sup\left\{\|C_{I_kX_k}u\|_{l_\infty^{m_k}} : u \in l_1^{m_k}, \|u\|_{l_1^{m_k}} \le 1\right\} \\
&= \max_{i,j\in\Lambda_k} |(e_j, C_{I_kX_k}e_i)| \\
&= \max_{i,j\in\Lambda_k} |\mathbb{E}(e_j, I_kX_k)(e_i, I_kX_k)| \\
&= \max_{i,j\in\Lambda_k} |\mathbb{E}(T_k^*c_j^k, X)(T_k^*c_i^k, X)| \\
&= \max_{i,j\in\Lambda_k} |(T_k^*c_j^k, C_XT_k^*c_i^k)|
\end{aligned} \tag{4.34}$$

by (4.32) and the notion of e_i as the evaluation functional on the i-th component in $l_1^{m_k}$ and $l_\infty^{m_k}$.

The final estimate for the last expression is now given by the following Lemma. Here we will identify C_X with a tensor in $C[0,1] \otimes_\lambda C[0,1]$ as well as with a function in $C([0,1]^2)$, which is justified by the isometrically isomorphic identities

$$C[0,1] \otimes_\lambda C[0,1] \cong C([0,1], C([0,1])) \cong C([0,1]^2)$$

from Proposition 2.2.

Lemma 4.7. *There is a constant $C_r > 0$, such that for any covariance operator $C_X : (C[0,1])^* \to C[0,1]$ with associated covariance function $\Gamma(s,t) := (\delta_s, C_X\delta_t) \in C([0,1]^2)$, it holds true that*

$$\max_{i,j\in\Lambda_k} \left|(T_k^*c_j^k, C_XT_k^*c_i^k)\right| \le C_r w_{r,r}(\Gamma, 2^{-k})_\infty.$$

Proof. We will show only

$$\max_{i,j\in\Lambda_k} \left|\left(T_k^*c_j^k \otimes T_k^*c_i^k\right)\left(\sum_{l=1}^n x_l \otimes y_l\right)\right| \le C_r \sup_{0<h_1,h_2\le 2^{-k}} \left\|\left(\Delta_{h_1}^r \otimes \Delta_{h_2}^r\right)\left(\sum_{l=1}^n x_l \otimes y_l\right)\right\|_\lambda$$

for arbitrary $\sum_{l=1}^n x_l \otimes y_l \in C[0,1] \otimes C[0,1]$, which is due to the continuity of the involved operators sufficient for the assertion.

Thus, by repeated use of (4.28) and the linearity of Δ_h^r, we verify

$$\begin{aligned}
\max_{i,j\in\Lambda_k}\Big|\left(T_k^* c_j^k \otimes T_k^* c_i^k\right)\Big(\sum_{l=1}^n x_l \otimes y_l\Big)\Big| &= \max_i \max_j \Big|\sum_{l=1}^n T_k^* c_j^k(x_l) T_k^* c_i^k(y_l)\Big| \\
&= \max_i \max_j \Big| T_k^* c_j^k \Big(\sum_{l=1}^n T_k^* c_i^k(y_l) x_l\Big)\Big| \\
&\le C_r \max_i \sup_{0<h_1\le 2^{-k}} \sup_{s\in[0,1-h_1]} \Big|\Delta_{h_1}^r \Big(\sum_{l=1}^n T_k^* c_i^k(y_l) x_l\Big)(s)\Big| \\
&= C_r \sup_{h_1} \sup_s \max_i \Big| T_k^* c_i^k \Big(\sum_{l=1}^n \Delta_{h_1}^r x_l(s) y_l\Big)\Big| \\
&\le C_r^2 \sup_{h_1} \sup_s \sup_{0<h_2\le 2^{-k}} \sup_{t\in[0,1-h_2]} \Big|\Delta_{h_2}^r \Big(\sum_{l=1}^n \Delta_{h_1}^r x_l(s) y_l\Big)(t)\Big| \\
&= C_r^2 \sup_{h_1,h_2} \sup_{s,t} \Big|\sum_{l=1}^n \Delta_{h_1}^r x_l(s) \Delta_{h_2}^r y_l(t)\Big| \\
&= C_r^2 \sup_{h_1,h_2} \Big\|\left(\Delta_{h_1}^r \otimes \Delta_{h_2}^r\right)\Big(\sum_{l=1}^n x_l \otimes y_l\Big)\Big\|_\lambda .
\end{aligned}$$

□

Hence, the assumptions of Theorem 4.6 are satisfied and we arrive at the following new upper bound, which relates the quantization error of X to the smoothness of its covariance functions.

Theorem 4.8. *Let $1 \le p < \infty$ and X be a centered Gaussian on $(C[0,1], \|\cdot\|_\infty)$ with covariance function $\Gamma(s,t) := \mathbb{E}X_s X_t$, which satisfies for some $r \in \mathbb{N}$ and some $\alpha > 0$*

$$w_{r,r}(\Gamma, 2^{-k})_\infty \preceq 2^{-\alpha k} \qquad \text{as } k \to \infty.$$

Then, it holds

$$d_n(X)_p \preceq n^{-\alpha/2} \qquad \text{as } n \to \infty,$$

and

$$e_N(X)_p \preceq (\log N)^{-\alpha/2} \qquad \text{as } N \to \infty.$$

Proof. We start with the case $r \ge 2$, since in this setting we have $\mathcal{S}_k^r \subset C[0,1]$ and may apply Theorem 4.6 on the Banach space $(E, \|\cdot\|) = (C[0,1], \|\cdot\|_\infty)$. As above, we set

$$X_k := T_k X,$$

which implies

$$X = \sum_{k=0}^\infty X_k \quad \text{a.s.}, \qquad \operatorname{rk} X_k = 2^k + r - 1$$

and $\mathbb{P}^{X_k}(E_k) = 1$ for $E_k := \mathcal{S}_k^r$.

Moreover, the isometries $I_k : \mathcal{S}_k^r \to l_\infty^{m_k}$ are uniformly bounded due to Proposition 2.4.

In addition, the factorization $C_{I_k X_k} = u_k u_k^*$ yields as above mentioned

$$u_k \gamma_k \stackrel{d}{=} I_k X_k$$

and the assumptions on the smoothness of Γ in combination with Lemma 4.7, (4.33) and (4.34) imply

$$\|u_k\|_{l_{2,\infty}} \preceq 2^{-\alpha k/2} \qquad \text{as } k \to \infty.$$

Thus, the requirements of Theorem 4.6 are fulfilled and we get the existence of a sequence of random variables $Y_n \in L^p(E)$ with $\operatorname{rk} Y_n \leq n$ and

$$\|X - Y_n\|_p \preceq n^{-\alpha/2} \qquad \text{as } n \to \infty,$$

which yields the assertion about the Kolmogorov n-width.

Since X has Gaussian moments of any order, Theorem 4.6 also applies for $p' > p$, so that the assumptions of Theorem 4.1 are satisfied and we may conclude from Corollary 4.2

$$e_N(X)_p \preceq (\log N)^{-\alpha/2} \qquad \text{as } N \to \infty.$$

Concerning the case $r = 1$, we have to embed X into the Banach space $(\mathbb{D}[0,1], \|\cdot\|_\infty)$. Since the embedding $\tau_{\mathbb{D}[0,1]} : C[0,1] \to \mathbb{D}[0,1]$ is continuous, $\tau_{\mathbb{D}[0,1]}(X)$ is again a Radon random variable on $\mathbb{D}[0,1]$.

Furthermore, this embedding does not alter the optimal quantization error nor the Kolmogorov n-width of X. This can be seen as follows.

Let Q_k^2 denote the Quasi-Interpolant of order 2, i.e. we have from (2.10)

$$Q_k^2 f = \sum_{j \in \Delta_k} (\delta_{(j+1)/2^k}, f) N^2(2^k \cdot -j),$$

which is a continuous operator from $\mathbb{D}[0,1]$ into $\mathcal{S}_k^2$ with $\|Q_k\| \leq 1$, since

$$\|Q_k^2 f\|_\infty \leq \max_{j \in \Delta_k} |f((j+1)/2^k)| \leq \|f\|_\infty.$$

Thus, we may map any random variable Y on $\mathbb{D}[0,1]$ into $\mathcal{S}_k^2 \subset C[0,1]$ by means of Q_k^2 and conclude for the approximation error of $Q_k^2 Y$

$$\begin{aligned} \|X - Q_k^2 Y\|_\infty &\leq \|Q_k^2 X - Q_k^2 Y\|_\infty + \|X - Q_k^2 X\|_\infty \\ &\leq \|\tau_{\mathbb{D}[0,1]}(X) - Y\|_\infty + \|X - Q_k^2 X\|_\infty, \end{aligned}$$

where $\|X - Q_k^2 X\|_\infty$ becomes arbitrarily small for $k \to \infty$, due to Theorem 2.6 and $X \in C[0,1]$.

Hence, the assertion follows again from Theorem 4.6 and Corollary 4.2 for $\tau_{\mathbb{D}[0,1]}(X)$ on $(E, \|\cdot\|) := (\mathbb{D}[0,1], \|\cdot\|_\infty)$. □

Note that it is even possible to sharpen the above results for the choices $r = 1, 2$ in the following way:

From (2.6) we have for the "mother" B-splines from (4.24)

$$N^1(x) = \mathbb{1}_{[0,1)}(x) \quad \text{and} \quad N^2(x) = x\,\mathbb{1}_{[0,1)}(x) + (2-x)\mathbb{1}_{[1,2)}(x) \tag{4.35}$$

and may set according to (2.10)

$$c_j^k := \begin{cases} \delta_{j/2^k} & r=1 \\ \delta_{(j+1)/2^k} & r=2 \end{cases}, \quad j \in \Lambda_k, \tag{4.36}$$

since in the case $r = 1$, each spline $S \in \mathcal{S}_k^1$ is constant on $[j/2^k, (j+1)/2^k)$. Moreover, the c_j^k are already bounded linear functionals on $C[0,1]$ resp $\mathbb{D}[0,1]$, hence they coincide with the γ_j^k from the Quasi-Interpolant Q_k.

Presenting here only the further proceeding for $r = 2$, we conclude for $k \geq 1$:

$$\begin{aligned} T_k f(x) &= \sum_{j\in\Lambda_k} (\delta_{(j+1)/2^k}, f)\, N^2(2^k x - j) - \sum_{j\in\Lambda_{k-1}} (\delta_{(j+1)/2^{k-1}}, f)\, N^2(2^{k-1}x - j) \\ &= \sum_{l\in\Lambda_k} \Big[\sum_{j\in\Lambda_k} \big(\delta_{(j+1)/2^k}, f\big)\big(\delta_{(l+1)/2^k}, N^2(2^k \cdot - j)\big) - \\ &\quad - \sum_{j\in\Lambda_{k-1}} \big(\delta_{(j+1)/2^{k-1}}, f\big)\big(\delta_{(l+1)/2^k}, N^2(2^{k-1} \cdot - j)\big)\Big] N^2(2^k x - l). \end{aligned} \tag{4.37}$$

Using

$$\big(\delta_{(l+1)/2^k}, N^2(2^k \cdot - j)\big) = N^2\big(l+1-j\big) = \begin{cases} 0 & l<j \\ 1 & l=j \\ 0 & l>j \end{cases}$$

and

$$\big(\delta_{(l+1)/2^k}, N^2(2^{k-1} \cdot - j)\big) = N^2\big((l+1)/2-j\big) = \begin{cases} 0 & l<2j \\ 1/2 & l=2j \\ 1 & l=2j+1 \\ 1/2 & l=2j+2 \\ 0 & l>2j+2 \end{cases}$$

we recognize that all the terms in (4.37) with odd l vanish and conclude

$$T_k f(x) = \sum_{\substack{l\in\Lambda_k \\ l \text{ even}}} \Big(-\frac{1}{2}\delta_{l/2^k} + \delta_{(l+1)/2^k} - \frac{1}{2}\delta_{(l+2)/2^k}, f\Big) N^2(2^k x - l)$$

and

$$T_k^* c_l^k = \begin{cases} -\frac{1}{2}\delta_{l/2^k} + \delta_{(l+1)/2^k} - \frac{1}{2}\delta_{(l+2)/2^k} & l \text{ even} \\ 0 & l \text{ odd.} \end{cases}$$

In addition, we have for $k = 0$

$$T_0 f(x) = Q_0 f(x) = (\delta_0, f) N^2(x+1) + (\delta_1, f) N^2(x),$$

which implies

$$T_0^* c_l^0 = \delta_{l+1}.$$

Hence, we arrive at

$$\begin{aligned} f &= \sum_{k\geq 0} T_k f = \sum_{k\geq 0} \sum_{j\in\Delta_k} (T_k^* c_j^k, f)\, N^2(2^k \cdot -j) \\ &= (\delta_0, f) N^2(\cdot + 1) + (\delta_1, f) N^2 \\ &\qquad + \sum_{k\geq 1} \sum_{j=0}^{2^{k-1}-1} (-\frac{1}{2}\delta_{2j/2^k} + \delta_{(2j+1)/2^k} - \frac{1}{2}\delta_{(2j+2)/2^k}, f)\, N^2(2^k \cdot -2j), \end{aligned} \tag{4.38}$$

which is in fact nothing else than an expansion of f in the famous *Faber Schauder-Basis* of $C[0,1]$ (c.f. [Fab10]).

Since $N^2(2^k x - 2j), j \in \{0, \ldots, 2^{k-1} - 1\}$ even consists of hat functions with disjoint support $[j/2^{k-1}, (j+1)/2^{k-1}]$ (see figure 4.1) and

$$\|N^2(2^k \cdot -2j)\|_\infty = 1,$$

we have

$$\begin{aligned} \|T_k f\|_\infty &= \max_{j\in\Lambda_k} |(T_k^* c_j^k, f)| \\ &= 2 \max_{0\leq j\leq 2^{k-1}-1} \big|(\delta_{2j/2^k} - 2\delta_{(2j+1)/2^k} + \delta_{(2j+2)/2^k}, f)\big|. \end{aligned} \tag{4.39}$$

The last expression can also be written in terms of the finite difference operator, i.e.

$$(\delta_{2j/2^k} - 2\delta_{(2j+1)/2^k} + \delta_{(2j+2)/2^k}, f) = \Delta^2_{2^{-k}} f(j/2^{k-1}),$$

such that the analogon of (4.28) now reads

$$\|(T_k^* c_\cdot^k, f)\|_{l_\infty^{m_k}} = 2 \max_{0\leq j\leq 2^{k-1}-1} \big|\Delta^2_{2^{-k}} f(j/2^{k-1})\big| \quad \forall f \in C[0,1].$$

Note moreover, that due to (4.39) the inequalities (4.31) now hold with $D_r = 1$, i.e. I_k defines an isometric isomorphism. Furthermore the analogon of Lemma

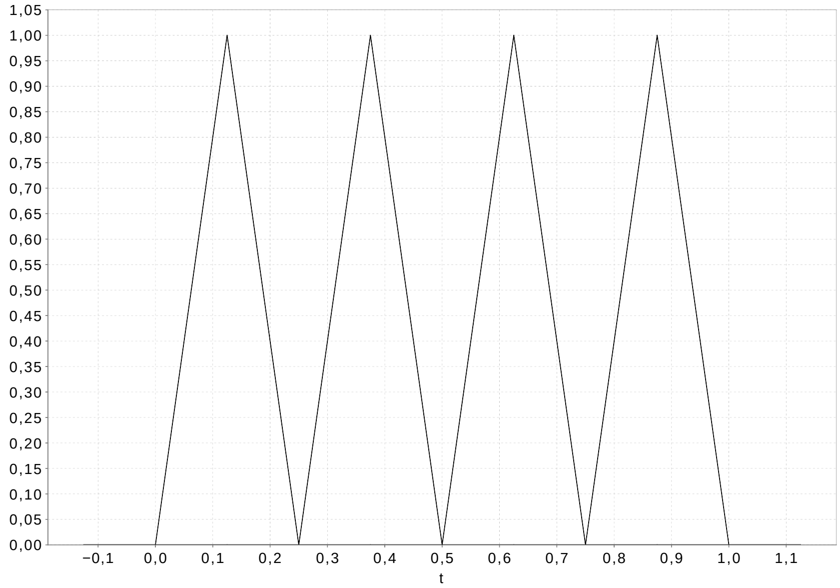

Figure 4.1: B-Splines of order 2 as Faber Schauder-Basis in $C[0,1]$, i.e. $N^2(2^k x - 2j)$ for $j \in \{0, \dots, 2^{k-1}-1\}$ and $k = 3$.

4.7 may be developed as

$$\begin{aligned}
\max_{i,j\in\Lambda_k}\left|(T_k^* c_j^k, C_X T_k^* c_i^k)\right| = 4 &\max_{0\le i,j\le 2^{k-1}-1} \\
&\left|\mathbb{E}(\delta_{2i/2^k} - 2\delta_{(2i+1)/2^k} + \delta_{(2i+2)/2^k}, X)(\delta_{2j/2^k} - 2\delta_{(2j+1)/2^k} + \delta_{(2j+2)/2^k}, X)\right| \\
&= 4\max_{i,j}\left|\Delta^2_{(2^{-k},2^{-k})}\Gamma(i/2^{k-1}, j/2^{k-1})\right| \\
&\le 4 \sup_{0<h\le 2^{-k}} \sup_{s,t\in[0,1-2h]} \left|\Delta^2_{(h,h)}\Gamma(s,t)\right|.
\end{aligned} \tag{4.40}$$

Regarding the case $r = 1$, we arrive in the same way at the representation

$$f = \sum_{k\ge 0} T_k f = (\delta_0, f)N^1 + \sum_{k\ge 1}\sum_{j\in\Delta_{k-1}} (\delta_{(2j+1)/2^k} - \delta_{2j/2^k}, f)N^1(2^k \cdot -2j),$$

which implies the estimate

$$\max_{i,j\in\Lambda_k}\left|(T_k^* c_j^k, C_X T_k^* c_i^k)\right| \le \sup_{0<h\le 2^{-k}} \sup_{s,t\in[0,1-2h]} \left|\Delta^1_{(h,h)}\Gamma(s,t)\right|. \tag{4.41}$$

If we now assume that there is a function $g : [-1,1] \to \mathbb{R}$, such that

$$\left|\Delta^r_{(h,h)}\Gamma(s,t)\right| = \left|\left(\Delta_h^2 T_{-h}\right)^r g(t-s)\right| \quad \forall s,t \in [0, 1-rh], \tag{4.42}$$

we clearly get

$$\begin{aligned}\sup_{0<h\le 2^{-k}}\sup_{s,t\in[0,1-rh]}\left|\Delta^r_{(h,h)}\Gamma(s,t)\right| &= \sup_{0<h\le 2^{-k}}\sup_{z\in[-1,1-2rh]}\left|\Delta^{2r}_h g(z)\right| \\ &\le w_{2r}(g,2^{-k}).\end{aligned} \tag{4.43}$$

This condition is especially fulfilled for stationary X, since then we have

$$\Gamma(s,t)=\Gamma(0,|t-s|)$$

so that we may set

$$g(t):=\Gamma(0,|t|)$$

and arrive at

$$\Delta^0_{(h,h)}\Gamma(s,t)=\Gamma(s,t)=g(t-s)=\left(\Delta^2_h T_{-h}\right)^0 g(t-s).$$

Furthermore, we may conclude inductively for $l<r$, since $\Delta_{(h,h)}$ and T_{-h} commute,

$$\begin{aligned}\Delta^l_{(h,h)}\Gamma(s,t) &= \Delta^{l-1}_{(h,h)}\Delta_{(h,h)}\Gamma(s,t)\\ &= \Delta^{l-1}_{(h,h)}\Gamma(s+h,t+h)-\Delta^{l-1}_{(h,h)}\Gamma(s+h,t)\\ &\quad -\Delta^{l-1}_{(h,h)}\Gamma(s,t+h)+\Delta^{l-1}_{(h,h)}\Gamma(s,t)\\ &= \left(\Delta^2_h T_{-h}\right)^{l-1}g(t-s)-\left(\Delta^2_h T_{-h}\right)^{l-1}g(t-s+h)\\ &\quad -\left(\Delta^2_h T_{-h}\right)^{l-1}g(t-s-h)+\left(\Delta^2_h T_{-h}\right)^{l-1}g(t-s)\\ &= \left(\Delta^2_h T_{-h}\right)^{l-1}\left(-\Delta^2_h T_{-h}\right)g(t-s)\\ &= -\left(\Delta^2_h T_{-h}\right)^{l}g(t-s),\end{aligned}$$

which yields condition (4.42).

Thus, we may conclude again from Theorem 4.6 and Corollary 4.2 in conjunction with (4.40), (4.41) and (4.43):

Theorem 4.9. *For $1\le p<\infty$ and $r\in\{1,2\}$ let X be a centered Gaussian on $(C[0,1],\|\cdot\|_\infty)$, whose covariance function $\Gamma(s,t):=\mathbb{E}X_sX_t$ satisfies*

$$\sup_{0<h\le 2^{-k}}\sup_{s,t\in[0,1-rh]}\left|\Delta^r_{(h,h)}\Gamma(s,t)\right|\preceq 2^{-\alpha k}\qquad \text{as } k\to\infty,$$

or assume for $l\in\{0,\dots,r\}$ the existence of a function $g:[-1,1]\to\mathbb{R}$ with

$$\left|\Delta^l_{(h,h)}\Gamma(s,t)\right|=\left|\left(\Delta^2_h T_{-h}\right)^l g(t-s)\right|\quad \forall s,t\in[0,1-lh]$$

and

$$w_{2l}(g,2^{-k})_\infty\preceq 2^{-\alpha k}\qquad \text{as } k\to\infty$$

for some $\alpha > 0$.

Then, it holds

$$d_n(X)_p \preceq n^{-\alpha/2} \qquad \text{as } n \to \infty,$$

and

$$e_N(X)_p \preceq (\log N)^{-\alpha/2} \qquad \text{as } N \to \infty.$$

Note that we clearly have for $r = 1, 2$ and $\Gamma \in C[0,1]^2$

$$\sup_{0<h\leq 2^{-k}} \sup_{s,t\in[0,1-2h]} \left|\Delta^2_{(h,h)}\Gamma(s,t)\right| \leq w_{r,r}(\Gamma, 2^{-k}).$$

Thus, it holds with $\alpha > 0$

$$\left\{\Gamma : w_{r,r}(\Gamma, 2^{-k}) \preceq 2^{-\alpha k}, k \to \infty\right\} \subset \left\{\Gamma : \sup_h \sup_{s,t} \left|\Delta^2_{(h,h)}\Gamma(s,t)\right| \preceq 2^{-\alpha k}, k \to \infty\right\}$$

and Theorem 4.9 indeed sharpens the results of Theorem 4.8.

Another useful tool to derive an upper bound for the bivariate modulus of smoothness is the following proposition.

Proposition 4.10. *For $\Gamma : [0,1]^2 \to \mathbb{R}$ denote the section of Γ by $\Gamma_s := \Gamma(s,\cdot)$. Then, for $r \in \mathbb{N}_0$ and $\delta > 0$*

$$w_{r,r}(\Gamma, \delta)_\infty \leq 2^r \sup_{s\in[0,1]} w_r(\Gamma_s, \delta)_\infty.$$

Proof. Since the linear operators $\Delta_{(h_1,h_2)}$ and T_h commute, we may conclude using the binomial theorem

$$\begin{aligned}
\left|\Delta^r_{(h_1,h_2)}\Gamma(s,t)\right| &= \left|\left(T_{(h_1,0)} - I\right)^r \left(T_{(0,h_2)} - I\right)^r \Gamma(s,t)\right| \\
&= \left|\left(\sum_{l=0}^{r} \binom{r}{l} (-1)^{r-l}\, T^l_{(h_1,0)}\right) \left(T_{(0,h_2)} - I\right)^r \Gamma(s,t)\right| \\
&\leq \sum_{l=0}^{r} \binom{r}{l} \left|\left(T_{(0,h_2)} - I\right)^r \Gamma(s+lh,t)\right| \\
&= \sum_{l=0}^{r} \binom{r}{l} \left|\Delta^r_{h_2}\Gamma_{s+lh}(t)\right|
\end{aligned}$$

and it follows

$$\begin{aligned}
w_{r,r}(\Gamma,\delta)_\infty &= \sup_{0<h_1,h_2\leq\delta} \sup_{s,t\in[0,1-rh]} \left|\Delta^r_{(h_1,h_2)}\Gamma(s,t)\right| \\
&\leq \sum_{l=0}^{r} \binom{r}{l} \sup_{0<h_1\leq\delta} \sup_{s\in[0,1-rh]} \sup_{0<h_2\leq\delta} \sup_{t\in[0,1-rh]} \left|\Delta^r_{h_2}\Gamma_{s+lh_1}(t)\right| \\
&\leq \sum_{l=0}^{r} \binom{r}{l} \sup_{s\in[0,1]} \sup_{0<h_2\leq\delta} \sup_{t\in[0,1-rh]} \left|\Delta^r_{h_2}\Gamma_s(t)\right| \\
&= 2^r \sup_{s\in[0,1]} w_r(\Gamma_s,\delta)_\infty.
\end{aligned}$$

$\square$

Having proceeded so far in the analysis, we are able to apply the results to concrete Gaussian processes, which reveals that the upper bound in all theses cases attains the true rate.

4.5 Examples

We start with the most prominent example of a Gaussian process having its paths in $C[0,1]$, the Brownian Motion.

Brownian Motion. The Brownian Motion is the centered Gaussian process W with paths in $C[0,1]$ and covariance function

$$\Gamma^W(s,t) = \min\{s,t\} = \frac{1}{2}\left(|t| + |s| - |t-s|\right).$$

For the finite difference of the section Γ_s^W, we verify

$$\begin{aligned} \Delta_h \Gamma_s^W(t) &= \frac{1}{2}\big(|t+h| + |s| - |t-s+h| \\ &\qquad - |t| - |s| + |t-s|\big) \\ &= \frac{1}{2}\left(|t+h| - |t| - (|t-s+h| + |t-s|)\right) \\ &= \Delta_h g(t) - \Delta_h g(t-s) \end{aligned} \tag{4.44}$$

with

$$g : [-1,1] \to \mathbb{R}, \quad t \mapsto \frac{1}{2}\big|t\big|.$$

Hence, it suffices to regard only

$$\Delta_h g(t) = \frac{1}{2}\big(|t+h| - |t|\big),$$

for which we derive with $h > 0$

$$\big|\Delta_h g(t)\big| = \frac{1}{2}h \qquad \forall t \in [0, 1-h] \cup [-1, -h]$$

and

$$\big|\Delta_h g(t)\big| = \frac{1}{2}\big|t+h+t\big| \le \frac{1}{2}\big|t+h\big| + \frac{1}{2}\big|t\big| \le h \qquad \forall t \in (-h, 0).$$

This yields

$$w_1(\Gamma_s^W, 2^{-k})_\infty \le 2 \sup_{0<h\le 2^{-k}} \sup_{t\in[-1,1-h]} \big|\Delta_h g(t)\big| \le 2 \cdot 2^{-k}$$

and from Proposition 4.10, we may conclude

$$w_{1,1}(\Gamma^W, 2^{-k})_\infty \leq 2 \sup_{s\in[0,1]} w_1(\Gamma_s^W, 2^{-k})_\infty \leq 4 \cdot 2^{-k},$$

so that Theorem 4.8 implies

$$e_N(W)_p \preceq (\log N)^{-1/2}.$$

This rate is sharp, since in [LP02] e.g., it was shown that

$$e_N(W)_2 \asymp (\log N)^{-1/2}$$

for W in $L^2([0,1], dt)$, which gives obviously a lower bound for the quantization on $(C[0,1], \|\cdot\|_\infty)$.

Brownian Bridge. If we impose the additional condition $W(1) = 0$ a.s. to the paths of the Brownian Motion, we arrive at the Brownian Bridge, which we will denote by B.

This process is characterized by the covariance function

$$\Gamma^B(s,t) = \min\{s,t\} - st,$$

which clearly decomposes into

$$\Gamma^B(s,t) = \Gamma^W(s,t) - f(s,t)$$

for $f(s,t) := s \cdot t$. Using

$$\sup_{s\in[0,1]} |\Delta_h f_s(t)| = \sup_{s\in[0,1]} |s(t+h) - st| = h,$$

we arrive at

$$w_1(\Gamma_s^B, 2^{-k})_\infty \leq w_1(\Gamma_s^W, 2^{-k})_\infty + w_1(f_s, 2^{-k})_\infty \leq C\, 2^{-k}$$

and conclude from Theorem 4.8 in combination with Proposition 4.10

$$e_N(B)_p \preceq (\log N)^{-1/2}.$$

According to [LP02], this is again the sharp rate.

Fractional Brownian Motion. Another modification of the Brownian motion is achieved by fractional integration with parameter $2H \in (0,2)$. The resulting process is called Fractional Brownian motion and will be denoted by W_H. It is uniquely determined as centered Gaussian process with paths in $C[0,1]$ and covariance function

$$\Gamma^{W_H}(s,t) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

Note that for $H = 1/2$, we arrive at the ordinary Brownian Motion.

Using the section $\Gamma_s^{W_H}$, we derive

$$\begin{aligned}
\Delta_h^2 \Gamma_s^{W_H} &= \frac{1}{2}\big(|t+2h|^{2H} + |s|^{2H} - |t-s+2h|^{2H} \\
&\quad - 2|t+h|^{2H} - 2|s|^{2H} + 2|t-s+h|^{2H} \\
&\quad + |t|^{2H} + |s|^{2H} - |t-s|^{2H}\big) \\
&= \frac{1}{2}\Big(|t+2h|^{2H} - 2|t+h|^{2H} + |t|^{2H} \\
&\quad - \left(|t-s+2h|^{2H} - 2|t-s+h|^{2H} + |t-s|^{2H}\right)\Big) \\
&= \Delta_h^2 g(t) - \Delta_h^2 g(t-s)
\end{aligned}$$

with

$$g(t) := \frac{1}{2}\big|t\big|^{2H}.$$

If we first focus on $H \in (0, 1/2]$, we conclude from the elementary inequality

$$(a+b)^\alpha \le a^\alpha + b^\alpha, \quad \alpha \in (0,1], a, b \ge 0,$$

for $h > 0$ and $t \in [0, 1-h]$

$$\big|\Delta_h g(t)\big| = \frac{1}{2}\left((t+h)^{2H} - t^{2H}\right) \le \frac{1}{2}h^{2H}.$$

Analogously, for $t \in [-1, -h]$

$$\begin{aligned}
\big|\Delta_h g(t)\big| &= \frac{1}{2}\big|(-t-h)^{2H} - (-t)^{2H}\big| \\
&= \frac{1}{2}\left((-t-h+h)^{2H} - (-t)^{2H}\right) \le \frac{1}{2}h^{2H},
\end{aligned}$$

whereas in the case $t \in (-h, 0)$ we verify

$$\begin{aligned}
\big|\Delta_h g(t)\big| &= \frac{1}{2}\big|(t+h)^{2H} - (-t)^{2H}\big| \\
&\le \frac{1}{2}\big|(-t-h+h)^{2H}\big| + \frac{1}{2}\big|(-t)^{2H}\big| \le h^{2H}.
\end{aligned}$$

This yields for $H \in (0, 1/2]$

$$w_1(g, 2^{-k})_\infty \leq 2^{-2Hk}. \tag{4.45}$$

In the case $H \in (1/2, 1)$, we use the fact that g is differentiable with derivative

$$g'(t) = \begin{cases} H|t|^{2H-1} & t \geq 0 \\ -H|t|^{2H-1} & t < 0 \end{cases}$$

Using exact the same arguments as above, we may conclude, since $2H - 1 \leq 1$,

$$w_1(g', 2^{-k})_\infty \leq 2H \cdot 2^{-(2H-1)k}$$

and Proposition 2.7 implies

$$w_2(g, 2^{-k})_\infty \leq 2^{-k} w_1(g', 2^{-k})_\infty \leq 2H \cdot 2^{-2Hk}. \tag{4.46}$$

So, we finally get from (4.45) and (4.46) for general Hurst index $H \in (0, 1)$

$$w_{2,2}(\Gamma^{W_H})_\infty \leq 4 \sup_{s\in[0,1]} w_2(\Gamma_s^{W_H}, 2^{-k})_\infty \leq C_H 2^{-2Hk}$$

with $C_H > 0$ and Theorem 4.8 yields

$$e_N(W_H)_p \preceq (\log N)^{-H}.$$

This rate is again, by the lower bound in $L^2([0,1], dt)$ from [LP04], optimal.

Remark. The above example of the Fractional Brownian Motion could also be tackled by Theorem 4.9. Indeed, we may verify

$$\begin{aligned} \Delta^1_{h,h} \Gamma^{W_H}(s,t) &= \frac{1}{2}\Big(|t+h|^{2H} + |s+h|^{2H} - |t-s|^{2H} \\ &\quad - |t|^{2H} - |s+h|^{2H} + |t-s-h|^{2H} \\ &\quad - |t+h|^{2H} - |s|^{2H} + |t-s+h|^{2H} \\ &\quad + |t|^{2H} + |s|^{2H} - |t-s|^{2H}\Big) \\ &= \frac{1}{2}\Big(|t-s-h|^{2H} - 2|t-s|^{2H} + |t-s+h|^{2H}\Big) \\ &= \big(\Delta_h^2 T_{-h}\big) g(t-s) \end{aligned}$$

for $g(t) = \frac{1}{2}\big|t\big|^{2H}$, so that the assertion follows even with $r = 1$ from (4.46) and Theorem 4.9.

Fractional Ornstein Uhlenbeck-Process. Another class of centered Gaussian processes is given by the Ornstein-Uhlenbeck processes, which are solution of Stochastic Differential equations. We immediately discuss the fractional integrated case, which we denote by X_ρ. Its covariance function writes

$$\Gamma^{X_\rho}(s,t) = \exp(-\alpha|t-s|^\rho)$$

with $\alpha > 0$ and $\rho \in (0,2)$.

This process is stationary, thus, in view of Theorem 4.9, we may focus on the function

$$f : [-1,1] \to \mathbb{R}, \quad t \mapsto \exp(-\alpha|t|^\rho).$$

Since $\tilde{f}(z) := \exp(-\alpha z)$ is Lipschitz-continuous on $[-1,1]$, we have for $\rho \in (0,1]$ and $h > 0$

$$\left|\Delta_h f\right| = |f(t+h) - f(t)| \le [\tilde{f}]_{\mathrm{Lip}}\left||t+h|^\rho - |t|^\rho\right| \le [\tilde{f}]_{\mathrm{Lip}} h^\rho$$

using the same estimates as in the case of the Fractional Brownian Motion.

For $\rho \in (1,2)$, we regard the function

$$g(t) := |t|^\rho,$$

which is continuously differentiable, thus has a bounded derivative on $[-1,1]$ and we get

$$f'(t) = -\alpha g'(t) f(t).$$

This yields

$$\begin{aligned}\left|\Delta_h f'(t)\right| &= \alpha\left|g'(t+h)f(t+h) - g'(t)f(t)\right| \\ &\le \alpha\left|g'(t+h)f(t+h) - g'(t)f(t+h)\right| + \alpha\left|g'(t)f(t+h) - g'(t)f(t)\right|.\end{aligned}$$

As in the case of the fractional Brownian Motion and since f is bounded on $[-1,1]$, we conclude

$$\left|g'(t+h)f(t+h) - g'(t)f(t+h)\right| \le \|f\|_\infty \left|\Delta_h g'(t)\right| \le \|f\|_\infty\, h^{\rho-1}$$

and

$$\left|g'(t)f(t+h) - g'(t)f(t)\right| \le \|g'\|_\infty [f]_{\mathrm{Lip}}\, h.$$

Since we are only interested in small values of h, we finally get for $h \in (0,1]$

$$\left|\Delta_h f'(t)\right| \le C h^{\rho-1}$$

with a constant $C > 0$. Hence, by Proposition 2.7 we arrive at

$$w_2(f, 2^k)_\infty \le C 2^{-\rho k}$$

for any $\rho \in (0,2)$ and Theorem 4.9 now implies

$$e_N(X_\rho) \preceq (\log N)^{-\rho/2},$$

which is by [LP04] the true rate.

Note that in all above examples we were able to attain the optimal rate by means of splines of order $r = 1$.

r-fold integrated processes. The situation looks different when we consider more smoother processes, e.g. r-fold integrate ones. Suppose that X is a centered Gaussian process on $C[0,1]$ whose covariance function satisfies

$$w_{l,l}(\Gamma^X, 2^{-k})_\infty \leq C 2^{-\alpha k}$$

for some $\alpha > 0$, $l \in \mathbb{N}$ and a constant $C > 0$. Then

$$X_t^1 := \int_0^t X_s ds$$

defines the once-integrated version of X and recursively we set

$$X_t^r := \int_0^t X_s^{r-1} ds$$

for the r-fold integrated versions of X with $r \in \mathbb{N}$. Thus, $X^r \in C^r[0,1]$ and we clearly have

$$D^r X_t^r = X_t,$$

where D^r denotes the r-th differential operator, so that we arrive at

$$D^{r,r}\Gamma^{X^r}(s,t) = D^{r,r}\mathbb{E}X_s^r X_t^r = \mathbb{E}D^{r,r}X_s^r X_t^r = \Gamma^X(s,t).$$

Using Proposition 2.8, we may derive

$$\begin{aligned} w_{l+r,l+r}(\Gamma^{X^r}, 2^{-k})_\infty &\leq (2^{-k})^{2r} w_{l,l}(D^{r,r}\Gamma^{X^r}, 2^{-k})_\infty \\ &= C 2^{-2rk} w_{l,l}(\Gamma^X, 2^{-k})_\infty \\ &\leq C 2^{-(2r+\alpha)k} \end{aligned}$$

and Theorem 4.8 implies

$$e_N(X^r)_p \preceq (\log N)^{-(r+\alpha/2)}.$$

4.6 Notes

An important tool in the development of the preceding results is the use of a "discretization" technique, which is based on differences of finite dimensional blocks, for which we can control both their dimension and their size.

This discretization technique appeared first in [Mai75] and [Höl79] and then became popular in many approximation theoretic settings c.f. the books [Pie86], [CS90], [LGM96] and [Pin85].

The proof of Theorem 4.6 has his origin in a series of papers by V.E. Maiorov, where average n-widths are computed for some special processes (e.g. [Mai96] and [Mai93]). For the proof of Theorem 4.1, we followed an idea of J. Creutzig ([Cre01]) for a Carl-type inequality. Whereas the idea to put all these pieces together and to derive a Jackson-type Theorem for the Quantization and average n-widths numbers with respect to the modulus of smoothness of the covariance function, seems to be new.

5 New Optimal Schemes

5.1 New schemes

To describe an Abstract Quantization Scheme, which achieves asymptotically the optimal quantization rate as presented in section 4, let X be a centered Gaussian on the Banach space $(C[0,1], \|\cdot\|_\infty)$.

For some $r \in \mathbb{N}$, we denote again by $\mathcal{S}_k^r$ the Schoenberg space of order r with 2^k simple knots in $[0,1]$, i.e.

$$t_j = j\, 2^{-k}, \qquad j = -r+1, \ldots, 2^k + r - 1$$

with index set $\Lambda_k := \{-r+1 \le j \le 2^k - 1\}$ and $m_k := 2^k + r - 1$.

Moreover, $T_k : C[0,1] \to \mathcal{S}_k^r$, denotes again the projection on $\mathcal{S}_k^r$ from (4.26), and

$$I_k : \mathcal{S}_k^r \to l_\infty^{m_k}, \quad S = \sum_{j \in \Lambda_k} (c_j^k, S)\, N_j^k \mapsto \big((c_j^k, S)\big)_{j \in \Lambda_k},$$

the isometries from (4.30) with de Boor-Fix functionals $c_j^k \in (\mathcal{S}_k^r)^*$, which satisfy due to Proposition 2.4

$$\|I_k\| \cdot \|I_k^{-1}\| \le D_r^{-1} \qquad \forall k \in \mathbb{N}_0.$$

In addition, we set again

$$X_k := T_k X = \sum_{j \in \Lambda_k} (T_k^* c_j^k, X)\, N_j^k.$$

In this setting the proofs of Theorems 4.6 and 4.1 contain a product quantizer design, which yields an upper bound for the quantization error depending on the smoothness of the covariance function Γ^X of X.

This scheme relies on the explicit construction of some rate optimal solutions Y_M to the Kolmogorov n-width problem, for which the proof of Theorem 4.6 and the proceeding of section 4.4 also provide a general principle to reduce this infinite dimensional problem to some approximation problem on $l_\infty^{m_k}$, which may be solved by numerical methods.

Nevertheless, we will in the sequel introduce a further Asymptotical Quantization Scheme, which does no longer rely on the explicit construction of the Y_M. Instead, the existence of such random variables will be only of theoretical importance, and the practical part of this problem can be, due to the uniform boundedness of the isomorphisms I_k, I_k^{-1}, carried over to the optimal quantization problem on some $l_\infty^{m_k}$-space.

Therefore, we define for some $c' \in \mathbb{N}$ and $n := c'\, 2^M$

$$Z^n := \sum_{k=0}^{2M} X_k = \sum_{k=0}^{2M} \sum_{j \in \Lambda_k} (T_k^* c_j^k, X)\, N_j^k,$$

denote by

$$J^n : \mathcal{S}^r_{2M} \to l_\infty^{m_{2M}}, \quad S = \sum_{j \in \Lambda_{2M}} (c_j^k, S)\, N_j^k \mapsto \big((c_j^k, S)\big)_{j \in \Lambda_{2M}},$$

the uniformly bounded isomorphisms from Proposition 2.4, and assume

$$\beta^n \subset l_\infty^{m_{2M}},\ |\beta^n| \le 2^n \quad \text{such that} \quad \Big(\mathbb{E} \min_{b \in \beta^n} \|J^n Z^n - b\|^p_{l_\infty}\Big)^{1/p} \le e_{2^n}(J^n Z^n)_p.$$

Thus, (Z^n, J^n, β^n) defines a dyadic Abstract Quantization Design with quantizer

$$\alpha^n := \big(J^n\big)^{-1} \beta^n,$$

which attains the following quantization rate

Theorem 5.1. *Let* $1 \le p < \infty$, $r \in \mathbb{N}$ *and* X *a centered Gaussian on* $(C[0,1], \|\cdot\|_\infty)$ *with covariance function* Γ^X *satisfying*

$$w_{r,r}(\Gamma^X, 2^{-k})_\infty \preceq 2^{-\alpha k} \qquad as\ k \to \infty.$$

Then there exists an asymptotical Abstract Quantization Scheme $(\tilde{Z}^N, \tilde{J}^N, \tilde{\beta}^N)$ *with* $(\tilde{Z}^{2^n}, \tilde{J}^{2^n}, \tilde{\beta}^{2^n}) = (Z^n, J^n, \beta^n)$ *such that for*

$$\tilde{\alpha}^N := \big(\tilde{J}^N\big)^{-1} \tilde{\beta}^N$$

it holds

$$\Big(\mathbb{E} \min_{a \in \tilde{\alpha}^N} \|X - a\|^p_\infty\Big)^{1/p} \preceq \big(\log N\big)^{-\alpha/2} \qquad as\ N \to \infty.$$

Proof. Using exactly the same arguments as in Theorem 4.1, it is sufficient to prove the assertion for the quantization error of α^n only for

$$n := n(M) := c' \cdot 2^M,$$

i.e. we show

$$\Big(\mathbb{E} \min_{a \in \alpha^n} \|X - a\|^p_\infty\Big)^{1/p} \preceq 2^{-\alpha M/2} \qquad \text{as } M \to \infty.$$

As in the situation of Theorem 4.8, the assumptions of Theorem 4.6 on X_k and I_k are, due to the condition on the smoothness of Γ_X, satisfied for $\alpha > 0$.

Hence, we may set for some $p' > p$

$$Y_M := \sum_{k \ge 0} I_k^{-1} \eta_k = \sum_{k=0}^{2M} I_k^{-1} \eta_k$$

with η_k from (4.15). This yields for the constant $c_r := 2r + 1$ from (4.16)

$$\operatorname{rk} Y_M \le c_r 2^M \qquad \text{and} \qquad \|X - Y_M\|_p \preceq 2^{-\alpha M/2}, \qquad \text{as } M \to \infty.$$

Thus, Y_M satisfies the assumptions of Theorem 4.1.

Moreover, we get

$$\|Z^n - Y_M\|_{p'} = \left\| \sum_{k=0}^{2M} X_k - I_k^{-1}\eta_k \right\|_{p'} \leq \sum_{k\geq 0} \|X_k - I_k^{-1}\eta_k\|_{p'},$$

so that we arrive in the situation of (4.18) and therefore may conclude in the same way

$$\|Z^{n(M)} - Y_M\|_{p'} \preceq 2^{-\alpha M/2}, \qquad \text{as } M \to \infty.$$

We then derive from the construction of the α^n, β^n and Propositions 3.2 and 3.4

$$\begin{aligned}
\left(\mathbb{E} \min_{a\in\alpha^n} \|X - a\|_\infty^p\right)^{1/p} &\leq \left(\mathbb{E} \min_{a\in\alpha^n} \|Z^n - a\|_\infty^p\right)^{1/p} + \|X - Z^n\|_p \\
&\leq \left\| \left(J^n\right)^{-1} \right\| \cdot \left(\mathbb{E} \min_{b\in\beta^n} \|J^n Z^n - b\|_{l_\infty}^p\right)^{1/p} + \|X - Z^n\|_p \\
&\leq \left\| \left(J^n\right)^{-1} \right\| \cdot e_{2^n}(J^n Z^n)_p + \|X - Z^n\|_p \\
&\leq \left\| \left(J^n\right)^{-1} \right\| \cdot \left\| J^n \right\| \cdot e_{2^n}(Z^n; \mathcal{S}_{2M}^r)_p + \|X - Z^n\|_p \\
&\leq 2 \cdot \left\| \left(J^n\right)^{-1} \right\| \cdot \left\| J^n \right\| \cdot e_{2^n}(Z^n)_p + \|X - Z^n\|_p,
\end{aligned}$$

since $\mathcal{S}_{2M}^r \subset C[0,1]$ (resp. $\mathbb{D}[0,1]$ in the case $r = 1$) and $\mathbb{P}^{Z^n}(\mathcal{S}_{2M}^r) = 1$.

In addition, the J^n are uniformly bounded, so we arrive with some constant $C > 0$ at

$$\left(\mathbb{E} \min_{a\in\alpha^n} \|X - a\|_\infty^p\right)^{1/p} \leq C e_{2^n}(Z^n)_p + \|X - Z^n\|_p.$$

To apply the result of Theorem 4.1, we have to pass from the quantization error of Z^n to the quantization of the "smaller" random variable Y_M.

Indeed, this is accomplished by the estimate

$$e_{2^n}(Z^n)_p \leq e_{2^n}(Y_M)_p + \|Z^n - Y_M\|_p$$

and, from Theorem 4.1, we then get for $n = c'\,2^M$ with c' from (4.6)

$$e_{2^n}(Y_M)_p = r_n(Y_M)_p \leq C 2^{-\alpha M/2}.$$

Thus, we may conclude

$$\begin{aligned}
\left(\mathbb{E} \min_{a\in\alpha^n} \|X - a\|_\infty^p\right)^{1/p} &\leq C e_{2^n}(Y_M)_p + \|Z^n - Y_M\|_p + \|X - Z^n\|_p \\
&\leq C 2^{-\alpha M/2} + 2\|Z^n - Y_M\|_p + \|X - Y_M\|_p \\
&\leq C 2^{-\alpha M/2},
\end{aligned}$$

which yields the assertion. $\square$

The price we have to pay, if we overcome the explicit construction of the random variable Y_M and instead define quantizers directly for the simpler random variable Z^n, is an increase in the dimension of the quantization problem.

In fact, provided that it holds $w_{r,r}(\Gamma^X, 2^{-k})_\infty \preceq 2^{-\alpha k}$, Corollary 4.2 states that an asymptotical quantizer with rate $(\log N)^{-\alpha/2}$ can be constructed by quantizing a random variable $\tilde{Y}^N$ with dimension $\log N$.

In contrast, Theorem 5.1 yields that $\tilde{Z}^N$, which also attains rate $(\log N)^{-\alpha/2}$, is up to a constant of dimension $(\log N)^2$, that is asymptotically the dimension of $\tilde{Y}^N$ to the square.

An explicit numerical construction of quantizers for the Brownian Motion on the Banach space $C[0,1]$ using the above Abstract Quantization Scheme will be presented in section 6.

5.2 Comparisons with the known schemes

To compare our new schemes with the already known ones for $L^q([0,1], dt)$ with $1 \leq q < \infty$, first of all we return to the product design from the proof of Theorem 4.1.

Recall that the random variables Y_k in the proof of that theorem are constructed from differences of the finite dimensional random variables X_k, which are assumed to attain a given approximation rate. In addition, the X_k are optimally chosen as solution to the Kolmogorov n-width problem. In the Hilbert space setting, we know from (4.13) that

$$X_n := \sum_{j=1}^{n} \langle X, e_j \rangle \, e_j,$$

with e_j the first eigenvectors of C_X, is a solution to this n-width problem. Thus,

$$Y_k := X_k - X_{k-1} = \sum_{j=1}^{n_k - n_{k-1}} \langle X, e_{n_{k-1}+j} \rangle \, e_{n_{k-1}+j}$$

for $n_k := \operatorname{rk} X_k$, coincides with the finite dimensional approximations X_k from the Abstract Quantization Scheme of Theorem 3.11, which yields the sharp asymptotics for the quantization problem on Hilbert spaces.

Hence, the product quantizer design based on the Y_k as differences of solutions to the n-width problem, which leads in Theorem 4.1 to a weak upper bound for the general Banach space setting, induces in the Hilbert space setting an asymptotical Abstract Quantization Scheme, which reaches even the sharp constant.

On the other hand, in the case $E = L^q([0,1], dt)$, $1 \leq q < \infty$, note that the Haar Basis $(e_n)_{n \geq 0}$ from section 3.3.2 is obviously made up by splines of order $r = 1$.

If we proceed as in section 4.4, the de Boor-Fix functionals $c_j^k = \delta_{j/2^k}$ for $r = 1$ are no longer well-defined as linear functionals for $f \in L^q([0,1], dt)$, so that we may set, using the notion $(f, g) := \int_0^1 f(t)g(t)dt$,

$$\gamma_j^k : L^q([0,1], dt) \to \mathbb{R}, \quad f \mapsto 2^k (N_j^k, f) = 2^k \int_{j/2^k}^{(j+1)/2^k} f(t)\, dt,$$

which defines an isometric extension of c_j^k from $\mathcal{S}_k^1$ to $L^q([0,1], dt)$ and therefore is suited as Hahn-Banach extension for the Quasi-Interpolant $Q_k : L^q([0,1], dt) \to \mathcal{S}_k^1$, $f \mapsto \sum_{j \in \Lambda_k} (\gamma_j^k, f) N_j^k$.

In addition, the operator $T_k = Q_k - Q_{k-1}$ may be developed in this case as

$$T_k f = \sum_{j \in \Lambda_{k-1}} (\gamma_{2j}^k - \gamma_j^{k-1}, f)\, N_{2j}^k + (\gamma_{2j+1}^k - \gamma_j^{k-1}, f)\, N_{2j+1}^k. \tag{5.1}$$

Since $N_j^{k-1} = N_{2j}^k + N_{2j+1}^k$, it is straightforward to derive

$$\gamma_{2j}^k - \gamma_j^{k-1} = 2^{k-1}(N_{2j}^k - N_{2j+1}^k) = \gamma_j^{k-1} - \gamma_{2j+1}^k$$

and (5.1) now reads

$$T_k f = \sum_{j \in \Lambda_{k-1}} 2^{k-1}(N_{2j}^k - N_{2j+1}^k, f)(N_{2j}^k - N_{2j+1}^k).$$

Returning to the Haar basis $(e_n)_{n \geq 0}$ from section 3.3.2, we clearly have

$$e_1 = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)} = N_0^1 - N_1^1,$$

which yields for $k \geq 1$, $j \in \{0, \ldots, 2^k - 1\}$

$$e_{2^k + j} = 2^{k/2}(N_{2j}^{k+1} - N_{2j+1}^{k+1}).$$

This, in turn, implies

$$T_k f = \sum_{j=0}^{2^{k-1}-1} (e_{2^{k-1}+j}, f)\, e_{2^{k-1}+j}$$

and reveals that the proceeding of section 4.4 is an extension of the Abstract Quantization Scheme from [LP08] to $(E, \|\cdot\|) = (\mathbb{D}[0,1], \|\cdot\|_\infty)$. Furthermore, this approach is also capable to take into account a higher smoothness by means of splines of order $r > 1$.

6 Numerical results

We finally present some numerical results for the Abstract Quantization Scheme from section 5.1 in the case of the Brownian Motion W on $(C[0,1], \|\cdot\|_\infty)$ and $p = 2$.

Thus, we know from section 4.5 that the covariance function $\Gamma^W(s,t) = \min\{s,t\}$ satisfies

$$w_{1,1}(\Gamma^W, 2^{-k})_\infty \preceq 2^{-k} \quad \text{as } k \to \infty,$$

and we may choose $r \in \mathbb{N}$ arbitrarily to achieve the optimal rate $(\log N)^{-1/2}$ for W.

Nevertheless, we decide for $r = 2$, since in that case the isomorphisms J^N are isometric ones, as will be seen later on.

Hence, we set for $N \in \mathbb{N}$ and some $M \in \mathbb{N}$ with $N \leq 2^{c' 2^{M/2}}$, c' from (4.6),

$$Z^N := \sum_{k=0}^{M} \sum_{j \in \Lambda_k} (T_k^* c_j^k, W) N_j^2$$

$$J^N : \mathcal{S}_M^2 \to l_\infty^{m_M}, \quad S \mapsto \big((c_j^k, S)\big)_{j \in \Lambda_M}$$

$$\beta^N \subset l_\infty^{m_M},\ |\beta^N| \leq N, \text{ such that } \Big(\mathbb{E} \min_{b \in \beta^N} \|J^N Z^N - b\|_{l_\infty}^p\Big)^{1/p} \leq e_N(J^N Z^N)_p.$$

The corresponding quantizer α^N for W then reads

$$\alpha^N := \big(J^N\big)^{-1} \beta^N = \sum_{j \in \Lambda_M} (\beta^N)_j\, N_j^M. \tag{6.1}$$

Concerning the isomorphisms $J^N : \mathcal{S}_M^2 \to l_\infty^{m_M}$, we recognize that S is piecewise linear on each interval $\big(j/2^M, (j+1)/2^M\big)$, hence

$$\begin{aligned} \|S\|_\infty &= \Big\| \sum_{i \in \Lambda_M} (c_i^M, S) N_i^M \Big\|_\infty = \max_{j \in \Lambda_M} \Big| \sum_{i \in \Lambda_M} (c_i^M, S) N^2(j - i) \Big| \\ &= \max_{j \in \Lambda_M} \big|(c_j^M, S)\big| = \|J^N S\|_{l_\infty}, \end{aligned}$$

with B-spline function N^2 from (4.25) and (4.35) so that $N^2(j-i) = \delta_{ij}$ for $i, j \in \Lambda_M$, which reveals that the isomorphisms J^N are indeed isometrical for $r = 2$.

Examine the proceeding of section 5.1 in detail, we realize that the essential estimate for the quantization error of the above scheme (Z^N, J^N, β^N) reads

$$\Big(\mathbb{E} \min_{a \in \alpha^N} \|W - a\|_\infty^p\Big)^{1/p} \leq \bigg(\mathbb{E} \min_{a \in \alpha^N} \Big\| \sum_{k=0}^{M} T_k W - a \Big\|_\infty^p \bigg)^{1/p} + \sum_{k > M} \|T_k W\|_p.$$

Hence, we may conclude that an increase in the block size M for fixed N may only lower this bound, since for $M' > M$ we get

$$\begin{aligned}\Big(\mathbb{E}\min_{a\in\alpha^N_{M'}}\Big\|\sum_{k=0}^{M'}T_kW-a\Big\|^p_\infty\Big)^{1/p} &\le e_N\Big(\sum_{k=0}^{M'}T_kW;\,\mathcal{S}^2_{M'}\Big)_p\\ &\le e_N\Big(\sum_{k=0}^{M'}T_kW;\,\mathcal{S}^2_{M}\Big)_p\\ &\le e_N\Big(\sum_{k=0}^{M}T_kW;\,\mathcal{S}^2_{M}\Big)_p+\sum_{k=M+1}^{M'}\|T_kW\|_p\end{aligned}$$

using the fact that $\mathcal{S}^2_M\subset\mathcal{S}^2_{M'}$ and J^N being isometric isomorphisms.

This actually justifies that we will later on, in the numerical applications, may choose M as large as possible within the capabilities of the numerical algorithms.

Having now a closer look at the distribution of the random variables J^NZ^N, note that we may choose from (4.36)

$$c^M_j=\delta_{(j+1)/2^M},\quad j\in\Lambda_M.$$

Following (4.38), Z^N now reads

$$\begin{aligned}Z^N &= (\delta_0,W)\,N^0_{-1}+(\delta_1,W)\,N^0_0\\ &\quad+\sum_{k=1}^{M}\sum_{j=0}^{2^{k-1}-1}\Big(-\frac{1}{2}\delta_{2j/2^k}+\delta_{(2j+1)/2^k}-\frac{1}{2}\delta_{(2j+2)/2^k},W\Big)N^M_{2j}.\end{aligned}$$

Setting

$$\xi_{-1}:=(\delta_0,W),\qquad \xi_0:=(\delta_1,W)$$
$$\xi_l:=2^{(k+1)/2}\Big(-\frac{1}{2}\delta_{2j/2^k}+\delta_{(2j+1)/2^k}-\frac{1}{2}\delta_{(2j+2)/2^k},W\Big),\quad l=2^{k-1}+j$$

for $1\le k\le M$ and $0\le j\le 2^{k-1}-1$, we immediately see $\xi_{-1}\equiv 0$, and some additional calculations reveal

$$\mathbb{E}\xi_i\xi_l=\delta_{il},\quad 0\le i,l\le 2^M-1,$$

which means that $(\xi_l)_{l\ge0}$ is an i.i.d sequence of standard normals. Using the canonical expansion of Z^N in the B-spline basis, we arrive at

$$\begin{aligned}Z^N &= \xi_0\,N^0_0+\sum_{k=1}^{M}2^{-(k+1)/2}\sum_{j=0}^{2^{k-1}-1}\xi_{2^{k-1}+j}\,N^k_{2j}\\ &= \sum_{i\in\Lambda_M}\Big((\delta_{(i+1)/2^M},N^0_0)\,\xi_0+\sum_{k=1}^{M}2^{-(k+1)/2}\sum_{j=0}^{2^{k-1}-1}(\delta_{(i+1)/2^M},N^k_{2j})\,\xi_{2^{k-1}+j}\Big)N^M_i,\end{aligned}\tag{6.2}$$

where the first identity discloses the fact that Z^N for $N \to \infty$ leads to the well-known *Lévy-Ciesielski* expansion of the Brownian Motion (cf. [Cie61]).

To emphasize the linear transformation of the ξ_l within the big brackets of the above identity, we define $\Sigma := (\gamma_{il})_{0 \le i,l \le 2^M-1}$ using (4.25) and (4.35) as

$$\gamma_{il} := \begin{cases} N^2\big((i+1)/2^M\big), & l = 0 \\ 2^{-(k+1)/2} N^2\big((i+1)/2^{M-k} - 2j\big), & l = 2^{k-1} + j \end{cases}$$

for $1 \le k \le M$, $0 \le j \le 2^{k-1} - 1$ and $0 \le i \le 2^M - 1$. Moreover, we set

$$\zeta := \Sigma\, \xi$$

and arrive at

$$Z^N = \sum_{i=0}^{2^M-1} \sum_{l=0}^{2^M-1} \gamma_{il}\, \xi_l\, N_i^M = \sum_{i=0}^{2^M-1} (\Sigma\xi)_i\, N_i^M = \sum_{i=0}^{2^M-1} \zeta_i\, N_i^M \tag{6.3}$$

with

$$\zeta \stackrel{d}{=} \mathcal{N}(0, \Sigma\Sigma^T).$$

Hence,

$$\big((J^N Z^N)_i\big)_{0 \le i \le 2^M-1} \stackrel{d}{=} \mathcal{N}(0, \Sigma\Sigma^T),$$

and $(\beta_i^N)_{0 \le i \le 2^M-1}$ has to be chosen as optimal N-quantizer for the normal distributed random variable ζ on $l_\infty^{2^M}$, which is a standard problem in finite dimensional quantization by numerics.

Therefore, we consider for $X \stackrel{d}{=} \mathcal{N}(0, \Sigma\Sigma^T)$ the *Distortion function*

$$D_N^X : (\mathbb{R}^d)^N \to \mathbb{R}, \quad y = (y_j^i)_{\substack{1 \le j \le d \\ 1 \le i \le N}} \mapsto \mathbb{E} \min_{1 \le i \le N} \|X - y^i\|^2_{l_\infty^d},$$

as objective function, which we want to minimize.

Note that a careful rereading of the proof of Lemma 4.10 in [GL00] reveals that D_N^X is also in case of the l_∞-norm differentiable, since the derivative $D(\|\cdot\|_{l_\infty})$ is only nonexistent on the set $\{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_i = x_j \text{ for some } i \neq j\}$, which consists of hyperplanes in $\mathbb{R}^d$ and therefore is of $\mathbb{P}^X$-measure zero.

Hence, we may apply a Large-Scale Optimization method based on gradient information to find critical points of D_N^X. To be more precise, we employed the Trust-Region method from [Gay83], which uses a BFGS-Update formula for second order gradient information, to solve the optimization problem

$$D_N^X(y) \to \min_{y \in (\mathbb{R}^d)^N}.$$

This choice has proven in our setting to converge very quickly, which is important in our case, since each evaluation of D_N^X is very time-consuming.

Concerning the latter problem, we have to deal with numerical integration in dimension $d = 4, 8$ or 16. Thus, we cannot anymore employ deterministic integrations methods and therefore have decided for a Quasi Monte-Carlo method, i.e. we implement a Monte-Carlo method based on the Sobol-numbers from [BF88]. As initialization for the Trust-Region method, we have chosen a scalar product quantizer, based on the Lèvy-Ciesielski expansion (6.2).

To be more precise, this scalar product design states for $N \in \mathbb{N}$ as follows. For some $m := 2^M \in \mathbb{N}$ and $l := 2^{k-1} + j$ set

$$X_0 := \xi_0 N_0^0$$
$$X_l := \begin{cases} 2^{-(k+1)/2} \xi_l N_{2j}^k & l \leq m \\ 0 & l > m. \end{cases}$$

Moreover, we define isomorphisms by

$$I_0 : \mathrm{supp}(\mathbb{P}^{X_0}) \to (\mathbb{R}, |\cdot|), \quad \vartheta N_0^0 \mapsto \vartheta$$
$$I_l : \mathrm{supp}(\mathbb{P}^{X_l}) \to (\mathbb{R}, |\cdot|), \quad \vartheta N_{2j}^k \mapsto \vartheta$$

and quantizers in $(\mathbb{R}, |\cdot|)$ by

$$\beta_0 \subset \mathbb{R}, |\beta_0| \leq N_0, \quad e(\xi_0; \beta_0)_2 \leq e_{N_0}(\mathcal{N}(0,1))_2$$
$$\beta_l \subset \mathbb{R}, |\beta_l| \leq N_l, \quad e(2^{-(k+1)/2}\xi_l; \beta_l)_2 \leq 2^{-(k+1)/2} e_{N_l}(\mathcal{N}(0,1))_2.$$

In addition, the quantization grid sizes N_l are chosen as solution to the allocation problem

$$\left\{ e_{N_0}(\mathcal{N}(0,1))_2 + \sum_{k=0}^{M} 2^{-(k+1/2)} \sum_{j=0}^{2^{k-1}-1} e_{N_l}(\mathcal{N}(0,1))_2 \right\} \to \min_{\prod_{l=0}^m N_l \leq N}.$$

The scalar product quantizer for W then reads

$$\alpha_{\mathrm{sc}} := \sum_{l=0}^{m} I_l^{-1} \beta_l.$$

Note that m is naturally bounded by $\log_2 N$, since the quantizer grid sizes N_l have to satisfy $\prod_{l \geq 0} N_l \leq N$ and the optimal quantizer for $\mathcal{N}(0,1)$ of size 1 is $\{0\}$.

A product quantizer α_{sc} for $N = 12$ is shown in Figure 6.1. As already mentioned, such a quantizer served as initialization for the Trust-Region method to minimize D_N^X for $X \stackrel{d}{=} \mathcal{N}(0, \Sigma\Sigma^T)$.

Moreover, using the Single-Block Design (Z^N, J^N, β^N) from the beginning of this section, we introduce for $M = 2, 3, 4$ the notions $\alpha_{4\mathrm{d}}, \alpha_{8\mathrm{d}}, \alpha_{16\mathrm{d}}$ to refer to

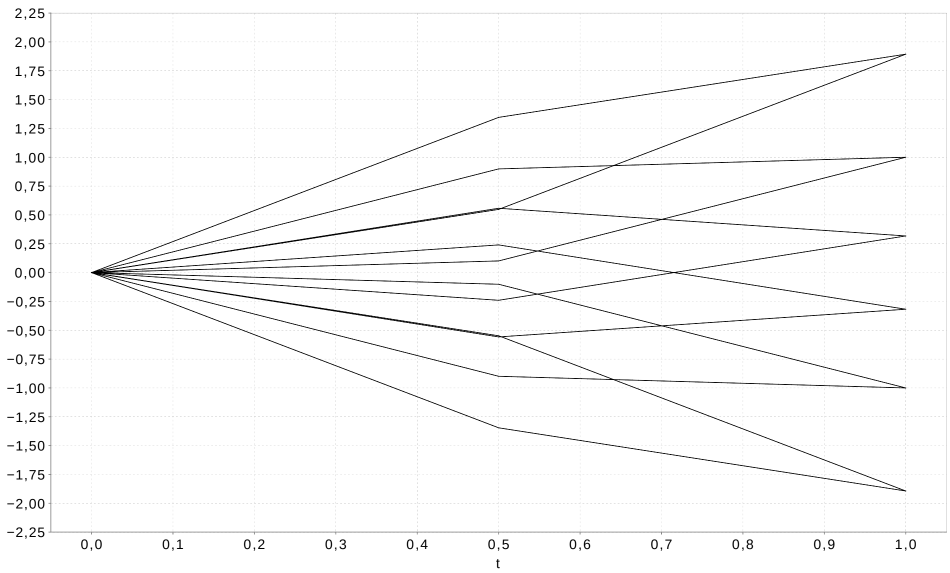

Figure 6.1: Quantizer α_{sc} for $N = 12$, which serves as initialization for the optimization method to produce the quantizers from Figure 6.2.

the quantizer α from (6.1) with corresponding dimension 2^M, $M \in \{2, 3, 4\}$. The same convention will hold for the random variables Z^N.

Again, we have drawn some plots for the quantizers $\alpha_{4\mathrm{d}}, \alpha_{8\mathrm{d}}, \alpha_{16\mathrm{d}}$ and $N = 12$ in Figure 6.2. In addition, we illustrate in Table 6.1 the approximation power of these quantizers by means of their quantization error with respect to the random variables Z^N in the spline spaces $\mathcal{S}^2_M$ for $M \in \{2, 3, 4\}$.

α	$\mathbb{E}\min\limits_{a\in\alpha}\|Z^N_{4\mathrm{d}} - a\|^2_\infty$	$\mathbb{E}\min\limits_{a\in\alpha}\|Z^N_{8\mathrm{d}} - a\|^2_\infty$	$\mathbb{E}\min\limits_{a\in\alpha}\|Z^N_{16\mathrm{d}} - a\|^2_\infty$
α_{sc}	0.2255	0.3010	0.3607
$\alpha_{4\mathrm{d}}$	**0.2147**	0.2902	0.3492
$\alpha_{8\mathrm{d}}$	0.2153	**0.2890**	0.3480
$\alpha_{16\mathrm{d}}$	0.2154	0.2891	**0.3479**

Table 6.1: Distortions for the quantizers from Figures 6.1 and 6.2, i.e. $N = 12$.

Of most interest is of course the last column of Table 6.1, which allows an objective comparison of the quantization power of these designs and the influence of an increasing quantization dimension.

In fact, we observe that for $N = 12$ an increase of the block size from 8 to 16 does not further reduce the quantization error significantly.

Moreover, we present a quantizer for $N = 128$ in Figure 6.3 and additional quantization errors in Table 6.2.

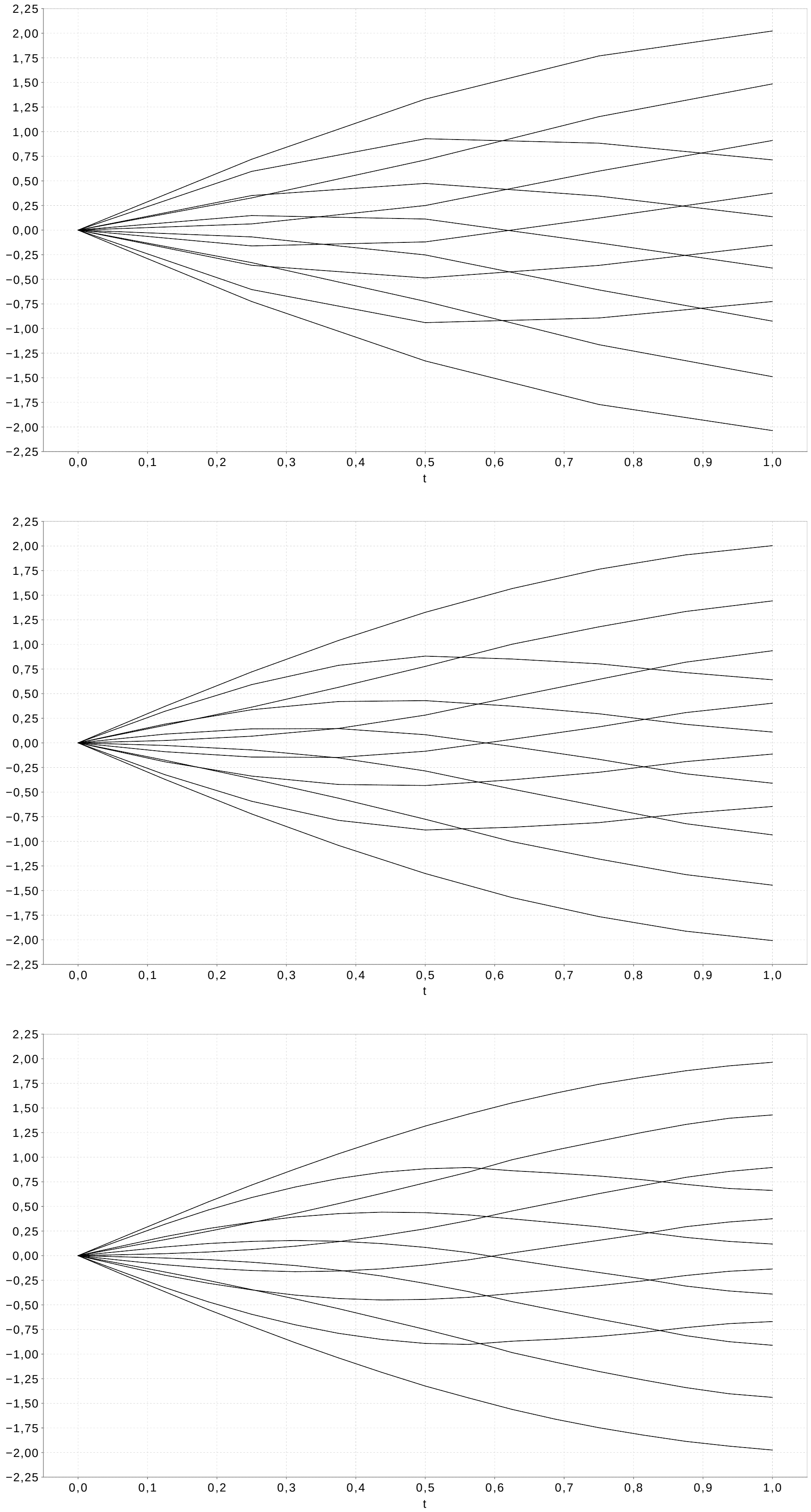

Figure 6.2: Quantizers $\alpha_{4\mathrm{d}}, \alpha_{8\mathrm{d}}, \alpha_{16\mathrm{d}}$ of size $N = 12$ for W on $(C[0,1], \|\cdot\|_\infty)$.

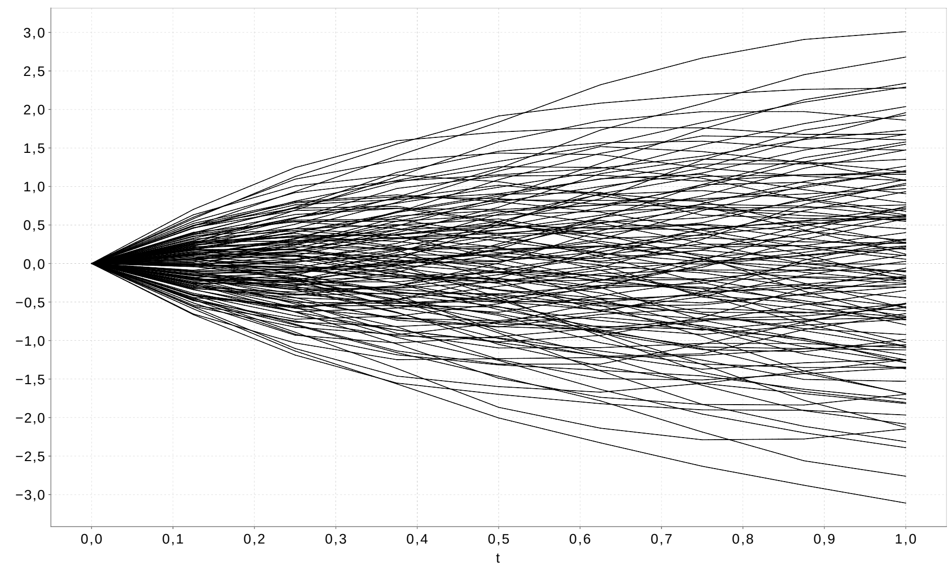

Figure 6.3: Quantizer $\alpha_{8\mathrm{d}}$ of size $N = 128$ for W on $(C[0,1], \|\cdot\|_\infty)$.

N	$\mathbb{E} \min_{a \in \alpha_{8\mathrm{d}}} \|Z_{8\mathrm{d}}^N - a\|_\infty^2$
12	0.2890
48	0.1804
96	0.1469
480	0.0960
1080	0.0773

Table 6.2: Distortions of the quantizer $\alpha_{8\mathrm{d}}$.

All the computations were performed using 10^6 Sobol numbers for the Quasi-Monte Carlo Integration in the evaluation of D_N^X and $\varepsilon = 10^{-6}$ as stopping criterion for $\|\nabla D_N^X\|$.

In fact, it is well known that Trust-Region methods are only converging to a local minimum of D_N^X. Therefore, we have no theoretical evidence that these quantizers reflect the global minimum of D_N^X. But, due to comparisons with different initializations, which also included random initialization, we are convinced to be very close to the global optimum.

Additionally, it would be sufficient to have only rate optimal quantizers for the construction of an asymptotically optimal quantization to the Brownian Motion. In the Hilbert space setting, e.g., it is possible to examine the asymptotical behaviour of the mapping

$$N \mapsto (\log N) \cdot \mathbb{E} \min_{a \in \alpha^N} \|W - a\|_{L^2}^2,$$

to give evidence that a computed sequence of quantizers α^N achieves the optimal infinite dimensional rate as in [LPW08].

Unfortunately, it is not possible to compute $\mathbb{E}\min_{a\in\alpha}\|W-a\|_\infty^2$ in our setting, since we have, in contrast to the Hilbert space setting, no exact formula for the approximation error term $\mathbb{E}\|W-Z^N\|_\infty^2$.

Due to the fact that the appearance of the quantizers from Figure 6.2 is similarly smooth as the one from the quantizers for W on the Hilbert space $L^2([0,1],dt)$ of e.g. [LPW08], it is plausible to compare these two quantizers.

In Figure 6.4, we have drawn the quantizers $\alpha_{16\mathrm{d}}$ from Figure 6.2 (the dark paths) and an optimal quantizer α_{L^2} for W on $L^2([0,1],dt)$ (light-coloured paths). The latter one was computed using the scheme from Proposition 3.10, i.e. a single block of dimension 4 from the expansion of W in the eigenbasis of the covariance operator.

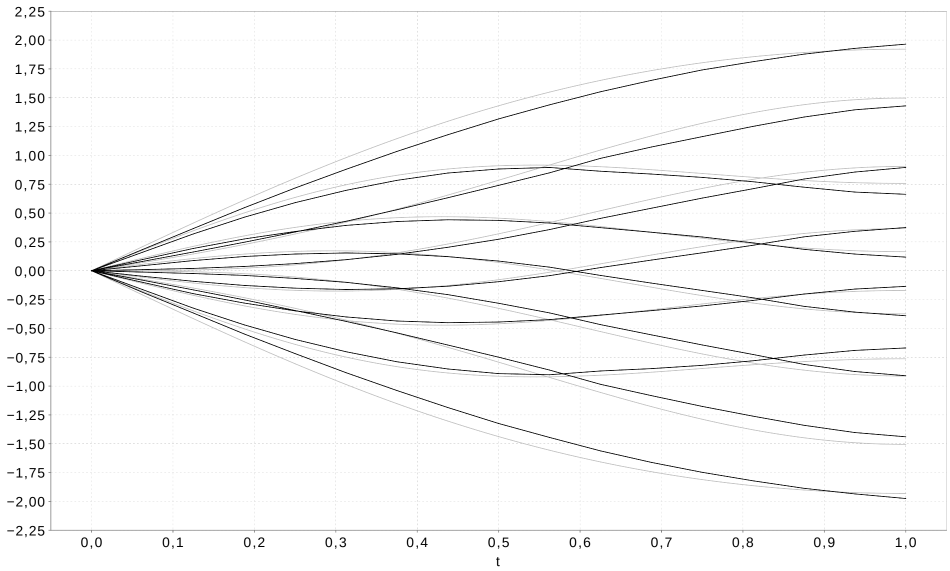

Figure 6.4: Quantizer $\alpha_{16\mathrm{d}}$ of size $N=12$ for W on $(C[0,1],\|\cdot\|_\infty)$ (dark paths) and the quantizer α_{L^2} for W on $L^2([0,1],dt)$ (light-coloured paths).

It is obvious that these two quantizers resemble each other very much, whereas the $\|\cdot\|_\infty$-quantizer has a bit less curved and more linear paths.

In fact, the differences becomes more visible if we regard the quantization error of these two quantizers at a fixed timepoint t. E.g., for $t=1$ we have $W(1)\stackrel{d}{=}\mathcal{N}(0,1)$, i.e. we regard the quantization error of $\alpha(1):=\{a(1):a\in\alpha\}$ for $\alpha=\alpha_{16\mathrm{d}},\alpha_{L^2}$ to $\mathcal{N}(0,1)$.

As a matter of fact, this yields $\mathbb{E}\min_{a\in\alpha_{16\mathrm{d}}}|W(1)-a(1)|^2\approx 0.0237$ and $\mathbb{E}\min_{a\in\alpha_{L^2}}|W(1)-a(1)|^2\approx 0.0275$. To interpret this difference, note that the optimal error which could be achieved is $e_{12}^2(\mathcal{N}(0,1))\approx 0.0163$.

Hence, compared to the optimal quantization error for a one dimensional normal distribution, the functional quantizer for W on $(C[0,1], \|\cdot\|_\infty)$ at $t = 1$ yields a relative difference of 44.8%, whereas the quantizer for $L^2([0,1], dt)$ causes a relative difference of 68.4% compared with the optimal one for dimension one.

7 Open Problems / Future prospects

Besides the successful construction of an asymptotically optimal Quantization Scheme on the Banach space $(C[0,1], \|\cdot\|_\infty)$ and a new upper bound for the optimal quantization error and the (average) Kolmogorov n-width based on the smoothness of the covariance function of the underlying Gaussian process, this work clearly raises new questions and problems.

The most striking task is of course the question for a corresponding lower bound to Theorem 4.8, i.e. whether $w_{r,r}(\Gamma, 2^{-k})_\infty \succeq 2^{-\alpha k}$ implies $e_N(X)_p \succeq (\log N)^{-\alpha/2}$, which would provide evidence that Theorem 4.8, for a proper choice of the order r, always yields the true rate.

In addition, one may ask for an inverse result of Theorem 4.8, which would be of Bernstein type, i.e. $e_N(X)_p \preceq (\log N)^{-\alpha/2}$ implies $w_{r,r}(\Gamma, 2^{-k})_\infty \preceq 2^{-\alpha k}$.

Note that these two statements are not equivalent, unless we can ensure the existence of the limits $e_N(X)_p/(\log N)^{-\alpha_1/2}$ and $w_{r,r}(\Gamma, 2^{-k})_\infty/2^{-\alpha_2 k}$ for some $\alpha_1, \alpha_2 \in (0, \infty)$.

Otherwise, one could try to establish a connection between the smoothness of the covariance function and the rate of decay for the eigenvalues of the covariance operator in the Hilbert space setting.

This immediately leads to the question if there also holds a constructive upper bound for Gaussian X on $L^q([0,1], dt)$, $1 \leq q < \infty$, which is based on the modulus of smoothness in these L^q-spaces, i.e. $w_{r,r}(\Gamma, 2^{-k})_q$.

Furthermore, motivated by some related problems in deterministic approximation theory (cf. [Pin85], Ch IV.5), one may wonder if a non-equidistant spline approximation may also yield the best possible approximation error for finite $n \in \mathbb{N}$ in sense of the Kolmogorov n-width.

Finally, it would be interesting to explore the performance of the new quantizers in numerical applications like the ones discussed in [PP05].

Especially those cases would be worth further investigations, where quantization is used as cubature formula on the Wiener space for a functional $F : C[0,1] \to \mathbb{R}$, which is only continuous with respect to the $\|\cdot\|_\infty$-Norm and not any longer for the $\|\cdot\|_{L^2}$-Norm. This happens for example in the case of the payoff-functional of a lookback option.

Additionally, the new possibilities which arise from the Spline expansion of W for hybrid methods, i.e. quantization as variance reduction in Monte Carlo-methods, seem promising challenges.

Notation Index

$\sim$	sharp asymptotics	p. 16
$\preceq$	weak asymptotic domination	p. 16
$\asymp$	weak asymptotics	p. 16
$\lceil x \rceil$	smallest integer greater equal x	p. 16
$\|f\|_{L^q}$	L^q-norm of the function f	p. 17
$\|\xi\|_{l_q}$	l^q-norm of the sequence ξ	p. 17
$\|f\|_\infty$	supremum norm of the function f	p. 16
$\|X\|_p$	p-norm of the random variable X	p. 5
$x \otimes y$	tensor product of x and y	p. 8
α	quantizer in E	p. 18
β	quantizer in some l_q-space	p. 24
$\mathfrak{B}(E)$	σ-field generated by the Borel sets in E	p. 5
c_0	null sequences	p. 17
c_{00}	finite sequences	p. 17
$C_a(\alpha)$	Voronoi cell of a induced by α	p. 19
c_j	de Boor-Fix functional	p. 13
c_j^k	de Boor-Fix functional on $\mathcal{S}_k^r$	p. 44
C_X	covariance operator of X	p. 6
$C([a,b])$	continuous functions on $[a,b]$	p. 16
$\mathbb{D}([a,b])$	càdlàg functions on $[a,b]$	p. 14
Δ_h^r	finite difference operator of order r	p. 15
$\Delta_{(h_1,h_2)}^{r_1,r_2}$	bivariate finite difference operator of order r_1, r_2	p. 16
$d_n(X)_p$	Kolmogorov n-width of X	p. 38
D_N^X	Distortion rate function of X	p. 67
$(E, \|\cdot\|)$	a Banach space	p. 5

E^*	topological dual space of E	p. 5
$e_N(X)_p$	optimal quantization error of X at level N	p. 18
$e(X;\alpha)_p$	quantization error of the quantizer α for X	p. 18
Γ^X	covariance function of X	p. 46
γ_j	Hahn-Banach extension of c_j	p. 14
γ_j^k	Hahn-Banach extension of c_j^k	p. 44
Λ_k	index set of $\mathcal{S}_k^r$	p. 43
$\mathcal{L}(E,F)$	linear and continuous operators from E to F	p. 6
$l_n(X)_p$	linear n-width of X	p. 39
$L^p(E)$	Bochner-integrable random variables on E with finite $\|X\|_p$	p. 6
$L^q([a,b],dt)$	Lebesgue-integrable functions on $[a,b]$ with finite $\|f\|_{L^q}$	p. 17
l_q	sequences with finite $\|\xi\|_{l_q}$	p. 17
N_j	B-spline function on $\mathcal{S}_T^r$	p. 11
N_j^k	B-spline function on $\mathcal{S}_k^r$	p. 44
N^r	mother B-spline of order r	p. 44
$(\Omega,\mathfrak{A},\mathbb{P})$	an abstract Probability Space	p. 5
Q_k	Quasi-Interpolant for $\mathcal{S}_k^r$	p. 44
Q_T	Quasi-Interpolant for $\mathcal{S}_T^r$	p. 14
$\operatorname{rk} X$	rank of the random variable X	p. 24
$r_n(X)_p$	dyadic quantization error of X	p. 18
$\mathcal{S}_T^r$	Schoenberg space of order r for a knot sequence T	p. 11
$\mathcal{S}_k^r$	Schoenberg space of order r with 2^k equidistant intervals	p. 43
T_h	translation operator	p. 15
T_k	difference operator of the Quasi-Interpolants Q_k	p. 44
$w_r(f,t)_q$	r-th modulus of smoothness of f	p. 15
$w_{r_1,r_2}(\Gamma,t)_q$	bivariate modulus of smoothness of order r_1,r_2 for Γ	p. 16
$(X_k,I_k,\beta_k)_{k\geq 1}$	abstract Quantization Scheme	p. 24

References

[BF88] P. Bratley and B. L. Fox. Algorithm 659: implementing sobol's quasirandom sequence generator. *ACM Trans. Math. Softw.*, 14(1):88–100, 1988.

[Bog98] V.I. Bogachev. *Gaussian Measures.* AMS, 1998.

[CDMGR08] J. Creutzig, S. Dereich, T. Müller-Gronbach, and K. Ritter. Infinite-dimensional quadrature and approximation of distributions, 2008. To appear in Found. Comput. Math.

[Cie61] Z. Ciesielski. Hölder conditions for realizations of Gaussian processes. *Trans. Amer. Math. Soc.*, 99:403–413, 1961.

[Cre01] J. Creutzig. *Approximation of Gaussian Random Vectors in Banach Spaces.* PhD thesis, FSU Jena, 2001.

[CS66] H. B. Curry and I. J. Schoenberg. On Pólya frequency functions. IV. The fundamental spline functions and their limits. *J. Analyse Math.*, 17:71–107, 1966.

[CS90] B. Carl and I. Stephani. *Entropy, compactness and the approximation of operators.* Cambridge University Press, 1990.

[dBF73] C. de Boor and G. J. Fix. Spline approximation by quasiinterpolants. *J. Approximation Theory*, 8:19–45, 1973.

[DF93] A. Defant and K. Floret. *Tensor norms and operator ideals*, volume 176 of *North-Holland Mathematics Studies.* North-Holland Publishing Co., Amsterdam, 1993.

[DFMS03] S. Dereich, F. Fehringer, A. Matoussi, and M. Scheutzow. On the link between small ball probabilities and the quantization problem for Gaussian measures on Banach spaces. *J. Theoret. Probab.*, 16(1):249–265, 2003.

[DL93] R. A. DeVore and G. G. Lorentz. *Constructive approximation*, volume 303. Springer-Verlag, Berlin, 1993.

[DS06] S. Dereich and M. Scheutzow. High-resolution quantization and entropy coding for fractional Brownian motion. *Electron. J. Probab.*, 11(28):700–722, 2006.

[Fab10] G. Faber. Über die Orthogonalfunktionen des Herrn *Haar. Deutsche Math.-Ver.*, 19:104–112, 1910.

[Gay83] D. M. Gay. Algorithm 611: Subroutines for unconstrained minimization using a model/trust-region approach. *ACM Trans. Math. Softw.*, 9(4):503–524, 1983.

[GL00] S. Graf and H. Luschgy. *Foundations of Quantization for Probability Distributions*. Lecture Notes in Mathematics $n^0$1730. Springer, Berlin, 2000.

[GLP03] S. Graf, H. Luschgy, and G. Pagès. Functional quantization and small ball probabilities for Gaussian processes. *J. Theoret. Probab.*, 16(4):1047–1062, 2003.

[GLP07] S. Graf, H. Luschgy, and G. Pagès. Optimal quantizers for Radon random vectors in a Banach space. *J. Approx. Theory*, 144(1):27–53, 2007.

[GN98] R.M. Gray and D.L. Neuhoff. Quantization. *IEEE Trans. Inform.*, 44:2325–2383, 1998.

[Höl79] K. Höllig. Approximationszahlen von Sobolev-Einbettungen. *Math. Ann.*, 242(3):273–281, 1979.

[LC85] W. A. Light and E. W. Cheney. *Approximation theory in tensor product spaces*, volume 1169 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1985.

[LGM96] G. G. Lorentz, M. v. Golitschek, and Y. Makovoz. *Constructive approximation*, volume 304. Springer-Verlag, Berlin, 1996.

[LP02] H. Luschgy and G. Pagès. Functional quantization of stochastic processes. *J. Funct. Anal.*, 196:486–531, 2002.

[LP04] H. Luschgy and G. Pagès. Sharp asymptotics of the functional quantization problem for gaussian processes. *Ann.Probab.*, 32:1574–1599, 2004.

[LP07] H. Luschgy and G. Pagès. High-resolution product quantization for Gaussian processes under sup-norm distortion. *Bernoulli*, 13(3):653–671, 2007.

[LP08] H. Luschgy and G. Pagès. Functional quantization rate and mean regularity of processes with an application to Lévy processes. *Ann. Appl. Probab.*, 18(2):427–469, 2008.

[LPW08] H. Luschgy, G. Pagès, and B. Wilbertz. Asymptotically optimal quantization schemes for gaussian processes on hilbert spaces, 2008. To appear in ESAIMPS.

[LT91] M. Ledoux and M. Talagrand. *Probability in Banach spaces*, volume 23. Springer-Verlag, Berlin, 1991.

[Mai75] V.E. Maiorov. Discretization of the problem of diameters. *Uspekhi Mat. Nauk*, 30(6):179–180, 1975.

[Mai93] V.E. Maiorov. Average n-widths of the Wiener space in the L_∞-norm. *J. Complexity*, 9:222–230, 1993.

[Mai96] V.E. Maiorov. About widths of Wiener space in the L_q-norm. *J. Complexity*, 12(1):47–57, 1996.

[Pag08] G Pagès. Quadratic optimal functional quantization of stochastic processes and numerical applications. In A. Keller, S. Heinrich, and H. Niederreiter, editors, *Monte Carlo and Quasi-Monte Carlo Methods 2006*, pages 101–142. Springer-Verlag, Berlin, 2008.

[Pie86] A. Pietsch. *Eigenvalues and s-numbers.* Cambridge University Press, New York, NY, USA, 1986.

[Pin85] A. Pinkus. *n-widths in Approximation Theory*, volume 7. Springer-Verlag, Berlin, 1985.

[Pis89] G. Pisier. *The volume of convex bodies and Banach space geometry*, volume 94 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1989.

[PP03] G. Pagès and J. Printems. Optimal quadratic quantization for numerics: the gaussian case. *Monte Carlo Methods and Applications*, 9(2):135–166, 2003.

[PP05] G. Pagès and J. Printems. Functional quantization for numerics with an application to option pricing. *Monte Carlo Methods and Applications*, 11(4):407–446, 2005.

[PPP04] G. Pagès, H. Pham, and J. Printems. Optimal quantization methods and applications to numerical methods and applications in finance. In S. Rachev, editor, *Handbook of Computational and Numerical Methods in Finance*, pages 253–298. Birkhäuser, 2004.

[Rit00] K. Ritter. *Average-case analysis of numerical problems.* Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2000.

[Sch81] L. L. Schumaker. *Spline functions: basic theory.* John Wiley & Sons Inc., New York, 1981.

[VTC87] N. N. Vakhania, V. I. Tarieladze, and S. A. Chobanyan. *Probability distributions on Banach spaces*, volume 14. D. Reidel Publishing Co., Dordrecht, 1987.