Time–Optimal Control: Algorithms based upon Moment Equations and Parametric Programming

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Preface

The optimal control of hyperbolic systems is an important problem in engineering. In this book, a self–contained exposition of the problem is given, a numerical method for the time–optimal control of hyperbolic systems is presented and a convergence analysis is made. Numerical examples are given for the problem of the rotating Euler–Bernoulli beam.

The method is based on the theory of moment problems. In the algorithm, the numerical solution of certain Volterra equations plays an important role. The ideas of the method are also related to parametric programming. A special feature of the convergence analysis is the detailed investigation of the properties of the sequence of optimal value functions of the discretized parametric auxiliary problems.

Moreover, the optimal value function of the original problem is investigated thoroughly.

Vorwort

Die optimale Steuerung hyperbolischer Systeme ist ein wichtiges Problem in den Ingenieurwissenschaften. Wir stellen das Problem dar und schlagen ein numerisches Verfahren für die zeitoptimale Steuerung hyperbolischer Systeme vor. Für dieses Verfahren entwickeln wir eine Konvergenzanalyse. Für das Problem des rotierenden Euler–Bernoulli Balkens geben wir numerische Beispiele.

Der Algorithmus basiert auf der Theorie der Momentenprobleme. In dem Verfahren spielt die Lösung gewisser Volterra–Gleichungen zweiter Art eine wichtige Rolle. Die Ideen des Verfahrens hängen auch mit der parametrischen Optimierung zusammen. Die ausführliche Untersuchung der Eigenschaften der Folge der Optimalwertfunktionen der diskretisierten Probleme ist eine Besonderheit der Analyse. Die Optimalwertfunktion des Ausgangsproblems wird ebenfalls gründlich untersucht.

Contents

Chapter 1

Introduction

1.1 The Problem

In this book, the control of a system is studied whose evolution in time is governed by a linear partial differential equation.

For given initial conditions and control functions, the solution can be expressed as a series, where the eigenfunctions and the eigenvalues of an operator which appears in the partial differential equation occur.

Due to the series representation of the solution, the control functions that steer the system from the given intial state to a desired terminal state can be characterized as the solution set of an infinite system of moment equations. This approach via moment problems has originally been given by Russel (see [39]). It is also considered in [3]. We consider exact control, that is the terminal state that we want to reach is prescribed exactly.

We consider the problem of time–optimal control subject to an upper bound on the L^2–norm of the image of an affine linear operator applied to the control function.

1.2 The Method

We present a numerical method for the solution of the problem of time–optimal control stated above, that is based on the properties of the value function of a certain parametric auxiliary problem.

In this auxiliary problem, the controlling time is the parameter. The moment equations are taken as equality constraints, and the L^2–norm combined with the affine linear operator that occurs in the definition of the inequality constraint of the problem of time–optimal control is taken as the objective function.

Since for the optimal controlling time, the inequality constraint in the problem of time–optimal control is active, the optimal value of the auxiliary problem with the optimal controlling time as fixed parameter is known. This fact is the foundation of the basic idea of the method, which is due to Krabs: It is to find the optimal controlling time as the point, where the optimal value function of the parametric auxiliary problem attains a certain known value, so basically, the optimal controlling time is determined as the root of a certain function.

To analyse this approach, it is interesting to consider the regularity of the optimal value function. The corresponding results presented here are original. The investigations are related to the results about the marginal function in parametric programming, see for example [51], [29], [13].

We show the continuity of the value function. For the computations, the system of moment equations is truncated to obtain a finite number of equality constraints. In this way, a sequence of optimal value functions corresponding to the discretized problems is generated. We show that this sequence converges uniformly to the optimal value function fo the original problem. The proof is based on Dini's Theorem.

We show that under weak assumptions, the optimal value functions of the discretized problems are three times differentiable. The proof is based on the implicit function theorem. The first and the second derivative are easy to compute, once the solution of the discretized problem is known. Hence we can use Newton's method to compute approximations for the optimal controlling time. A combination with a bisection approach yields a method that converges globally quadratically. Since the evaluation of the second derivative is also cheap, it makes sense to apply higher order methods too, such as Halley's method. A combination with a bisection approach yields global cubic convergence.

We present a sufficient condition for the differentiability of the optimal value function of the original problem. If the optimal controls are pointwise uniformly bounded, the optimal value function of the original problem is Lipschitz.

1.3 The Example of the Euler–Bernoulli Beam

For our computational examples, we consider the problem of the control of a rotating beam, which is a classical problem of control theory that has been studied in numerous papers. An Euler–Bernoulli beam is considered whose movement is controlled by a torque at the axis of the rotation.

The control function is the angular acceleration. It is assumed to be a square integrable function of time, that is we work in the Hilbert space $L^2[0, T]$.

The problem is to control the beam from a given intial state to a prescribed terminal state, for example to rotate it by a given angle in such a way that at the end of the controlling time, no vibrations occur and the beam is completely at rest.

The torque is computed from the control function by applying a linear Volterra–operator of the second kind. For the inequality constraint we require that the L^2–norm of the torque function is less than or equal to a given bound. This has a physical interpretation as a constraint on the energy that is used to steer the beam.

For the solution of each of the corresponding discretized auxiliary problems, a number of Volterra–equations has to be solved which is equal to the number of moment equations. This is the most time–consuming part of the computation.

The kernel in the Volterra–equations is given as a series. For the solution of the Volterra–equations, we use the fact that the truncated kernel has finite rank, hence we can obtain an approximate solution through the solution of a linear initial value problem. For the numerical solution, we can use an implicit scheme, where due to the special structure of the matrix the corresponding systems of linear equations can be solved analytically.

We analyse the convergence of the computed approximations of the optimal controlling time as the number of moment equations and the number of terms in the truncated kernel increase and the stepsize in the numerical integration decreases.

Now we mention some of the papers where the rotating beam has been studied:

In [41], optimal control on a fixed time interval under L^∞–constraints is considered, and an algorithm is given, that is based on Galerkin approximations and ordinary differential equations.

In [25], the problem of time–optimal control of a rotating beam as described above is considered. In [24], the exact controllability of the system is shown, using a trigonometric inequality by Ingham. For the numerical solution, Krabs proposes a secant method. No numerical examples are given.

In [42], a beam with interior damping of Voigt–type is considered and a method for feedback stabilization is given. This approach is given for a more general case in [40].

In [8], a chain of serially connected Euler–Bernoulli beams is studied and it is shown that uniform exponential stabilization can be achieved by stabilizing at one end point of the composite beam.

In [27], [28] networks of beams are considered.

Chapter 2

Hyperbolic Systems

In this chapter, we consider systems that are governed by partial differential equations of the hyperbolic type. We prove a result about the existence of a unique solution that depends continuously on the initial data.

As an example, we consider the partial differential equation of the Euler–Bernoulli beam. We give a representation of the solution as a series.

2.1 The Problem

Consider a system that is described by an evolution equation of the second order in time of the form

$$\frac{d^2y(t)}{dt^2} + Ay(t) = R(t),\ t \geq 0 \tag{2.1}$$

with initial conditions

$$y(0) = y_0,\ dy(0)/dt = y_1. \tag{2.2}$$

Now we introduce the formal setting. Let H be a Hilbert space over the field of real numbers. Let $(\phi_j)_{j\in\mathbb{N}}$ be an orthonormal Schauder basis of H. Let $(\lambda_j)_{j\in\mathbb{N}}$ be a sequence of numbers that are greater than zero.

Let $c = (c_j)_{j\in\mathbb{N}}$ be a sequence of real numbers and $r \in \mathbb{R}$. Define

$$\|c\|_r = \left(\sum_{j=1}^{\infty} |c_j|^2 (\lambda_j)^r\right)^{1/2}. \tag{2.3}$$

We define the space of sequences

$$l_r^2 = \{(c_j)_{j\in\mathbb{N}} : \|c\|_r < \infty\}. \tag{2.4}$$

We also use the notation $\|c\|_{l_r^2} = \|c\|_r$.

Define the subspace of H

$$W_r = \left\{ f \in H : f = \sum_{j=1}^{\infty} c_j\phi_j,\ \|f\|_{W_r} := \|(c_j)_{j\in\mathbb{N}}\|_r < \infty \right\}. \tag{2.5}$$

It is evident that $W_0 = H$.

We consider a linear operator A from W_r to W_{r-2} that is given by

$$A(\sum_{j=1}^{\infty} c_j\phi_j) = \sum_{j=1}^{\infty} \lambda_j c_j \phi_j. \tag{2.6}$$

Obviously, for all $h \in W_r$ we have

$$\|Ah\|_{W_{r-2}} = \|h\|_{W_r} \tag{2.7}$$

The operator A has a pure point spectrum and the eigenfunctions $(\phi_j)_{j\in\mathbb{N}}$ with the corresponding strictly positive eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$.

In the applications, the operator A will usually be given as a differential operator and the spectrum has to be computed somehow. In this section, we started with the eigenfunctions for the sake of ease of exposition. The approach in this section is close to the presentation in [3].

For $T > 0$, define the space $L^2(0,T;W_r)$ of measurable functions $g : (0,T) \mapsto W_r$ such that

$$\|g\|_{L^2(0,T;W_r)} := \left(\int_0^T \|g(t)\|_{W_r}^2\, dt\right)^{1/2} < \infty \tag{2.8}$$

and the space $C(0,T;W_r)$ of continuous functions $d : [0,T] \mapsto W_r$ with the norm $\|d\|_{C(0,T;W_r)} := \max_{t\in[0,T]} \|d(t)\|_{W_r}$.

Functions $g \in L^2(0,T;W_r)$ can be represented in the form

$$g(t) = \sum_{j=1}^{\infty} g_j(t)\phi_j,$$

with $g_j \in L^2(0,T)$ and $\sum_{j=1}^{\infty} \|g_j\|_{L^2(0,T)}^2 \lambda_j^r < \infty$.

Let $\mathcal{D}(0,T)$ be the set of infinitely differentiable real–valued functions on $[0,T]$ whose support is a compact subset of $(0,T)$. For $j \in I\!N$, let dg_j/dt be the generalized derivative (in the sense of distributions) of g_j.

Then for $\psi \in \mathcal{D}(0,T)$ we have

$$\frac{dg_j}{dt}(\psi) = -\int_0^T g_j(t)\psi'(t)\,dt.$$

This implies the inequality

$$\left|\frac{dg_j}{dt}(\psi)\right|^2 \le \|g_j\|^2_{L^2(0,T)}\|\psi'\|^2_{L^2(0,T)}.$$

Define

$$\frac{dg}{dt} = \sum_{j=1}^{\infty}\frac{dg_j}{dt}\phi_j,\ \ \frac{dg}{dt}(\psi) = \sum_{j=1}^{\infty}\frac{dg_j}{dt}(\psi)\phi_j,\ \text{for } \psi \in \mathcal{D}(0,T).$$

Hence we see that dg/dt is a well–defined continuous map from $\mathcal{D}(0,T)$ into W_r satisfying the equality

$$\frac{dg}{dt}(\psi) = -\int_0^T g(t)\psi'(t)\,dt.$$

Let $R \in L^2(0,T;W_{r-1})$, $y_0 \in W_r$, $y_1 \in W_{r-1}$. We are looking for a solution of our evolution equation with values in the space W_r that is continuous with respect to t.

2.2 Existence of a Unique Solution

A function $y \in C(0,T;W_r)$ is said to be a solution of the problem (2.1), (2.2) if the sum $d^2y/(dt^2) + Ay$ is an element of the space $L^2(0,T;W_{r-1})$ and equal to R and the initial conditions hold as equalities in the spaces W_r and W_{r-1} respectively. Let me remind the reader of the fact that we have assumed (2.6).

Theorem 2.2.1 (see Theorem III.2.1.,[3], p.154)
Let $R \in L^2(0,T;W_{r-1})$, $y_0 \in W_r$, and $y_1 \in W_{r-1}$.

Then there exists a unique solution of problem (2.1), (2.2) and the map

$$(R, y_0, y_1) \mapsto (y, dy/dt)$$

of space $L^2(0,T;W_{r-1})\times W_r \times W_{r-1}$ to the space $C(0,T;W_r)\times C(0,T;W_{r-1})$ is continuous.

Proof The proof is based on Fourier expansions. The functions R, y_0 and y_1 can be represented as

$$R(t) = \sum_{j=1}^{\infty} R_j(t)\phi_j, \quad y_0 = \textstyle\sum_{j=1}^{\infty} y_j^0 \phi_j, \quad y_1 = \displaystyle\sum_{j=1}^{\infty} y_j^1 \phi_j$$

with

$$\sum_{j=1}^{\infty} \|R_j\|^2_{L^2(0,T)} \lambda_j^{r-1} < \infty, \quad \textstyle\sum_{j=1}^{\infty} |$$

which implies that

$$|y_j(t)|^2\lambda_j^r \;\; \leq \;\; 3|y_j^0|^2\lambda_j^r + 3|y_j^1|\lambda_j^{r-1} + 3\|R_j\|^2_{L^2(0,T)}T\lambda_j^{r-1}. \quad (2.13)$$

Hence $y(t) \in W_r$ for all $t \in [0,T]$.

Equation (2.12) implies that

$$|dy_j(t)/dt| \;\; \leq \;\; \sqrt{\lambda_j}|y_j^0| + |y_j^1| + \|R_j\|_{L^2(0,T)}\sqrt{T},$$

thus we have

$$|dy_j(t)/dt|^2 \;\; \leq \;\; 3\lambda_j|y_j^0|^2 + 3|y_j^1|^2 + 3T\|R_j\|^2_{L^2(0,T)}$$

which implies that

$$|dy_j(t)/dt|^2\lambda_j^{r-1} \leq 3|y_j^0|^2\lambda_j^r + 3|y_j^1|^2\lambda_j^{r-1} + 3T\lambda_j^{r-1}\|R_j\|^2_{L^2(0,T)}. \quad (2.14)$$

Hence $dy(t)/dt \in W_{r-1}$ for all $t \in [0,T]$.

Moreover, the series (2.9) converges in W_r uniformly in t, $t \in [0,T]$. Therefore, inequality (2.13) and the Weierstrass theorem imply that the function y is continuous in t in the norm of W_r. Analogously, we see that dy/dt is a continuous function from $[0,T]$ into W_{r-1}. Equality (2.13) as an equality in the space $L^2(0,T;W_{r-1})$ and equalities (2.2) as equalities in the spaces W_r, W_{r-1} respectively follow directly from (2.10).

To see that the solution is uniquely determined, note that every solution can be represented in the form (2.9), and the coefficients have to solve the initial value problems (2.10).

Since the unique solution of (2.10) is given by (2.11), the uniqueness of the solution of (2.1), (2.2) follows.

Inequality (2.13) implies

$$\|y\|^2_{L^2(0,T;W_r)} \leq 3\|y_0\|^2_{W_r} + 3\|y_1\|^2_{W_{r-1}} + 3T\|R\|^2_{L^2(0,T;W_{r-1})} \quad (2.15)$$

and inequality (2.14) implies

$$\|dy/dt\|^2_{L^2(0,T;W_{r-1})} \leq 3\|y_0\|^2_{W_r} + 3\|y_1\|^2_{W_{r-1}} + 3T\|R\|^2_{L^2(0,T;W_{r-1})} \quad (2.16)$$

and the asserted continuity follows. □

2.3 Example: The Euler–Bernoulli Beam

In this section we want to apply Theorem 2.2.1 to the partial differential equation describing the Euler–Bernoulli beam. In particular, we want to give a representation of its solution as a series.

The treatment given in the previous section depends on the expansion in the eigenfunctions. To use this approach numerically for a particular operator the eigenfunctions have to be computed somehow. Then it has to be shown that the eigenfunctions form a complete orthonormal system. For some operators (e.g. of Sturm–Liouville type), this is guaranteed by general results.

In this section, we want to work with the Hilbert space $L^2(0,l)$. So we have to show that the operator A corresponding to our example has a pure point spectrum and that the eigenfunctions form a basis of $L^2(0,l)$. Fortunately, for this particular example, it is possible to compute the eigenfunctions and the eigenvalues of A analytically. The asymptotic behaviour of the eigenvalues can be described very accurately.

2.3.1 $Ay = y_{xxxx}$

Let $l > 0$ be given. In this section we consider the operator

$$Ay(x) = d^4 y(x)/dx^4,$$

and we want to compute the eigenfunctions contained in the set

$$D(A) = \{z \in C^{(4)}([0,l]) \text{ with } z(0) = z'(0) = 0 = z''(l) = z'''(l)\}.$$

Lemma 2.3.1 *Eigenfunctions z_1, z_2 corresponding to different eigenvalues $\lambda_1 \neq \lambda_2$ are orthogonal in $L^2(0,l)$, i.e.*

$$\int_0^l z_1(x) z_2(x)\, dx = 0.$$

All eigenvalues are strictly positive.

The first assertion of Lemma 2.3.1 follows from the equation

$$\begin{aligned}
\lambda_2 \int_0^l z_1(x) z_2(x)\, dx &= \int_0^l z_1(x) z_2^{(4)}(x)\, dx \\
&= z_1(x) z_2^{(3)}(x)|_0^l - \int_0^l z_1^{(1)}(x) z_2^{(3)}(x)\, dx \\
&= -z_1^{(1)}(x) z_2^{(2)}(x)|_0^l + \int_0^l z_1^{(2)}(x) z_2^{(2)}(x)\, dx \\
&= \int_0^l z_1^{(2)}(x) z_2^{(2)}(x)\, dx \\
&= \lambda_1 \int_0^l z_1(x) z_2(x)\, dx.
\end{aligned}$$

By partial integration, we can also see that

$$\lambda_1 \int_0^l z_1(x)^2\,dx = \int_0^l (z_1^{(2)}(x))^2\,dx.$$

Since $z_1 \neq 0$, due to the definition of $D(A)$ this implies $\lambda_1 > 0$. □

For $\lambda > 0$, we examine the differential equation $z^{(4)}(x) = \lambda z(x)$. The general solutions of this differential equation is

$$z(x) = A\sin(\lambda^{1/4}x) + B\cos(\lambda^{1/4}x) + C\sinh(\lambda^{1/4}x) + D\cosh(\lambda^{1/4}x),$$

with coefficients A, B, C, D. (Using the Wronskian, it can be shown that sin, cos, sinh, cosh are linearly independent functions and hence we have a basis of the space of solutions.)

The conditions $z(0) = z'(0) = 0$ yield the equations $B = -D$ and $A = -C$. Hence the eigenfunctions have to be of the form

$$z(x) = A(\sin(\lambda^{1/4}x) - \sinh(\lambda^{1/4}x)) + B(\cos(\lambda^{1/4}x) - \cosh(\lambda^{1/4}x)),$$

which implies that for the derivatives we have

$$z''(l) = A\lambda^{1/2}(-\sin(\lambda^{1/4}l) - \sinh(\lambda^{1/4}l)) - B\lambda^{1/2}(\cos(\lambda^{1/4}l) + \cosh(\lambda^{1/4}l)),$$

$$z'''(l) = A\lambda^{3/4}(-\cos(\lambda^{1/4}l) - \cosh(\lambda^{1/4}l)) + B\lambda^{3/4}(\sin(\lambda^{1/4}l) - \sinh(\lambda^{1/4}l)).$$

The conditions $z''(l) = 0 = z'''(l)$ yield a linear system for the coefficients A, B. A solution $z \neq 0$ can only exist, if the determinant of this system equals zero, that is if

$$(-\sin(\lambda^{1/4}l) - \sinh(\lambda^{1/4}l))(\sin(\lambda^{1/4}l) - \sinh(\lambda^{1/4}l))$$
$$-(-\cos(\lambda^{1/4}l) - \cosh(\lambda^{1/4}l))^2 = 0,$$

which is equivalent to the equation

$$\cos(\lambda^{(1/4)}l)\cosh(\lambda^{(1/4)}l) = -1$$

(on account of $\cos^2 + \sin^2 = 1$, $\cosh^2 - \sinh^2 = 1$).

Hence the eigenvalues of A are the positive solutions of the equation

$$\cos(\lambda^{1/4}l) + 1/\cosh(\lambda^{1/4}l) = 0.$$

Lemma 2.3.2 *The function*

$$F(x) = \cos(x) + 1/\cosh(x)$$

has a countable number of positive roots.

Let $(x_j)_{j\in I\!N}$ *denote the strictly increasing sequence of roots.* *Let* $y_j = j\pi - \pi/2$ *and* $\delta_1 = \pi/2$, $\delta_j = \arcsin(1/\cosh(y_{j-1}))$ *for* $j \geq 2$.

If j *is uneven, we have* $x_j \in (y_j, y_j + \delta_j)$.

If j *is even, we have* $x_j \in (y_j - \delta_j, y_j)$.

Proof If j is uneven, for $x \in [y_j + \delta_j, y_{j+1} - \delta_{j+1}]$ we have

$$\begin{aligned} F(x) &< \max\{-\sin(\delta_j), -\sin(\delta_{j+1})\} + 1/\cosh(y_j) \\ &= \max\{-1/\cosh(y_{j-1}), -1/\cosh(y_j)\} + 1/\cosh(y_j) \\ &= -1/\cosh(y_j) + 1/\cosh(y_j) \\ &= 0. \end{aligned}$$

Hence F has no root on the interval $[y_j + \delta_j, y_{j+1} - \delta_{j+1}]$.

If j is even, for $x \in [y_j, y_{j+1}]$ we have

$$F(x) \geq 0 + 1/\cosh(x) > 0.$$

Hence F has no root on the interval $[y_j, y_{j+1}]$.

For all $j \in I\!N$, we have

$$F(y_j) = \cos(j\pi - \pi/2) + 1/\cosh(y_j) = 1/\cosh(y_j) > 0.$$

If j is uneven, we have

$$\begin{aligned} F(y_j + \delta_j) &= \cos(y_j + \delta_j) + 1/\cosh(y_j + \delta_j) \\ &= -\sin(\delta_j) + 1/\cosh(y_j + \delta_j) \\ &< -\sin(\delta_j) + 1/\cosh(y_{j-1}) \\ &\leq 0. \end{aligned}$$

If j is even, we have

$$\begin{aligned} F(y_j - \delta_j) &= \cos(y_j - \delta_j) + 1/\cosh(y_j - \delta_j) \\ &= \sin(-\delta_j) + 1/\cosh(y_j - \delta_j) \\ &< -\sin(\delta_j) + 1/\cosh(y_{j-1}) \\ &\leq 0. \end{aligned}$$

Since the function F is continuous, the existence of zeros of F in the intervals $(y_j, y_j + \delta_j)$, $(y_j - \delta_j, y_j)$ respectively follows from the intermediate value theorem.

We have the derivative $F'(x) = -\sin(x) - \tanh(x)/\cosh(x)$.

If j is uneven, for $x \in (y_j, y_j + \delta_j)$ we have

$$\begin{aligned} F'(x) &\leq -\sin(y_j + \delta_j) - \tanh(y_j)/\cosh(y_j + \delta_j) \\ &= -\cos(\delta_j) - \tanh(y_j)/\cosh(y_j + \delta_j) \\ &< 0. \end{aligned}$$

Hence F is strictly decreasing on $(y_j, y_j + \delta_j)$ and has at most one root on this interval.

If j is even, for $x \in (y_j - \delta_j, y_j)$ we have

$$\begin{aligned} F'(x) &\geq -\sin(y_j - \delta_j) - 1/\cosh(y_j - \delta_j) \\ &\geq \cos(\delta_j) - 1/\cosh(\pi) \\ &\geq \sqrt{1 - 1/\cosh^2(y_{j-1})} - 1/\cosh(\pi) \\ &\geq \sqrt{1 - 1/\cosh^2(\pi/2)} - 1/\cosh(\pi) \\ &> 0. \end{aligned}$$

Hence F is strictly increasing on $(y_j - \delta_j, y_j)$ and hence F has at most one root on this interval. □

For the eigenvalues λ_j of A we have $\lambda_j^{1/4} l = x_j$, hence $\lambda_j = (x_j/l)^4$.

Remark 2.3.1 The numbers x_j can be computed numerically by using Newton's method applied to the function $F(y_j + \cdot)$ with the zero as the starting point. Note that for $j \geq 12$ we have $\delta_j \leq 10^{-14}$, hence for $j > 12$ the numbers y_j are good approximations of the x_j.

To obtain the eigenfunctions, we choose $B = -1$. The equation $z''(l) = 0$ yields

$$A = (\cos(\lambda_j^{1/4} l) + \cosh(\lambda_j^{1/4} l))/(\sin(\lambda_j^{1/4} l) + \sinh(\lambda_j^{1/4} l)) =: \gamma_j.$$

Thus we have the eigenfunctions

$$\varphi_j(x) = \gamma_j(\sin(\lambda_j^{(1/4)} x) - \sinh(\lambda_j^{(1/4)} x)) - \cos(\lambda_j^{(1/4)} x) + \cosh(\lambda_j^{(1/4)} x). \tag{2.17}$$

Since $\cos x_j = -1/\cosh x_j$, we have

$$\begin{aligned} \cos(x_j) + \cosh(x_j) &= \cosh(x_j) - 1/\cosh(x_j) \\ &= (\cosh^2(x_j) - 1)/\cosh(x_j) \\ &= \sinh^2(x_j)/\cosh(x_j). \end{aligned}$$

Moreover, we have

$$\begin{aligned}\sin(x_j) &= (-1)^{j+1}\sqrt{1-\cos^2(x_j)}\\ &= (-1)^{j+1}\sqrt{1-1/\cosh^2(x_j)}\\ &= (-1)^{j+1}\left(\sqrt{\cosh^2(x_j)-1}\right)/\cosh(x_j)\\ &= (-1)^{j+1}\sinh(x_j)/\cosh(x_j).\end{aligned}$$

Hence we can conclude that

$$\begin{aligned}\gamma_j &= (\sinh^2(x_j)/\cosh(x_j))/((-1)^{j+1}\sinh x_j/\cosh x_j+\sinh x_j)\\ &= \sinh(x_j)/(\cosh(x_j)((-1)^{j+1}/\cosh x_j+1))\\ &= \sinh(x_j)/(\cosh(x_j)+(-1)^{j+1}).\end{aligned}$$

For j uneven, this yields

$$\begin{aligned}\gamma_j &= \sinh(x_j)/(\cosh(x_j)+1)\\ &= \tanh(x_j/2)\end{aligned}$$

and for j even, we have

$$\begin{aligned}\gamma_j &= \sinh(x_j)/(\cosh(x_j)-1)\\ &= \coth(x_j/2).\end{aligned}$$

Hence Lemma 2.3.2 implies that if j is uneven, we have

$$\gamma_j \in [\tanh(j\pi/2-\pi/4),1]$$

and if j is even, we have

$$\gamma_j \in [1,\coth((j-1)\pi/2)].$$

Thus we see that the sequence $(\gamma_j)_{j\in\mathbb{N}}$ converges to 1 very fast.

Lemma 2.3.3 *For the functions φ_j defined in (2.17) we have*

$$\int_0^l \varphi_j(x)^2\,dx = l.$$

Proof We have $\int_0^l \varphi_j(x)^2\,dx = l\int_0^1 \varphi_j(tl)^2\,dt$.

If j is uneven, we have

$$\varphi_j(tl) = \tanh(x_j/2)(\sin(x_jt)-\sinh(x_jt))-\cos(x_jt)+\cosh(x_jt).$$

Using the program Mathematica, we obtain the formula

$$\begin{aligned}
&\int_0^1 (\tanh(w/2)(\sin(wt) - \sinh(wt)) - \cos(wt) + \cosh(wt))^2\,dt \\
&= \frac{2w + 2w\cosh(w) - 4\sin(w) + \sin(2w) + \sinh(w) - 4\cos(w)\sinh(w) + \cos(2w)\sinh(w)}{4w\cosh^2(w/2)} \\
&= \frac{2\left(2w(\cosh(w)+1) + \sin(w)(-4+2\cos(w)) + \sinh(w)(2\cos^2(w) - 4\cos(w))\right)}{4w(\cosh(w)+1)}
\end{aligned}$$

$$\begin{aligned}
&= 1 + \frac{2\left(\sinh(w)(2\cos^2(w) - 4\cos(w)) + \sin(w)(-4+2\cos(w))\right)}{4w(\cosh(w)+1)} \\
&= 1 + \frac{\left(\sinh(w)(\cos^2(w) - 2\cos(w)) + \sin(w)(-2+\cos(w))\right)}{w(\cosh(w)+1)} \\
&= 1 + \frac{1}{w}\tanh(w/2)\left(\cos^2(w) - 2\cos(w) + \frac{\sin(w)}{\sinh(w)}(-2+\cos(w))\right).
\end{aligned}$$

For j uneven, we have

$$\sin(x_j) = \sqrt{1-\cos^2(x_j)} = \sqrt{1 - 1/\cosh^2(x_j)} = \tanh(x_j).$$

For $w = x_j$, this implies the equation

$$\begin{aligned}
&\cos^2(w) - 2\cos(w) + \frac{\sin(w)}{\sinh(w)}(-2+\cos(w)) \\
&= \cos^2(w) - 2\cos(w) + \frac{1}{\cosh(w)}(-2+\cos(w)) \\
&= \cos^2(w) - 2\cos(w) - \cos(w)(-2+\cos(w)) \\
&= 0.
\end{aligned}$$

Thus for j uneven, we see that $\int_0^1 \varphi_j(tl)^2\,dt = 1$.

If j is even, we have

$$\varphi_j(tl) = \coth(x_j/2)(\sin(x_jt) - \sinh(x_jt)) - \cos(x_jt) + \cosh(x_jt).$$

Again using Mathematica, we obtain the formula

$$\begin{aligned}
&\int_0^1 (\coth(w/2)(\sin(wt) - \sinh(wt)) - \cos(wt) + \cosh(wt))^2\,dt \\
&= \frac{-2w + 2w\cosh(w) - 4\sin(w) - \sin(2w) + \sinh(w) + 4\cos(w)\sinh(w) + \cos(2w)\sinh(w)}{4w\sinh^2(w/2)} \\
&= \frac{2w(\cosh(w)-1) + \sin(w)(-4-2\cos(w)) + \sinh(w)(4\cos(w) + 2\cos^2(w))}{2w(\cosh(w)-1)}
\end{aligned}$$

$$= 1 + \frac{1}{w} \coth(w/2) \left(\cos^2(w) + 2\cos(w) + \frac{\sin(w)}{} \right.$$

yields the number 1.00000000, so Formula (2.18) is more suitable for numerical purposes than (2.17).

Let

$$\phi_j(x) = \varphi_j(x)/\sqrt{l}. \tag{2.19}$$

Then Lemma 2.3.3 implies that the functions $(\phi_j)_{j\in\mathbb{N}}$ form an orthonormal system.

The question remains: Is the sequence $(\phi_j)_{j\in\mathbb{N}}$ a Schauder basis of $L^2(0,l)$? A positive answer follows from a result in Chapter 1, 16 in [31]: Using an approach that is based on finite differences, it is shown that Parseval's equation holds, that is

$$\text{for all } f \in L^2(0,l) \text{ we have } \int_0^l f^2(x)\,dx = \sum_{j=1}^{\infty}\left(\int_0^l f(x)\phi_j(x)\,dx\right)^2. \tag{2.20}$$

Since the sequence $(\phi_j)_{j\in\mathbb{N}}$ is an orthonormal system, (2.20) is equivalent to its completeness.

Thus the operator A satisfies the assumptions of Section 2.1 for the Hilbert space $H = L^2(0,l)$.

Remark 2.3.2 The eigenfunctions ϕ_j appear in many papers (see for example [41], [24]). However, the intervals containing the eigenvalues λ_j in Lemma 2.3.2 and the simple representations of the γ_j that are very useful for numerical purposes seem to be new.

2.3.2 The Series Representation of the Solution

Let $D > 0$. The number D measures the stiffness of the beam. The Euler–Bernoulli beam is defined by the initial boundary value problem

$$\frac{d^2y(x,t)}{(dt)^2} + D\frac{d^4y(x,t)}{(dx)^4} = R(x,t),\ x \in [0,l], t \geq 0. \tag{2.21}$$

with initial conditions

$$y(x,0) = y_0(x),\ dy(x,0)/dt = y_1(x) \tag{2.22}$$

and boundary conditions

$$y(0,t) = y'(0,t) = 0 = y''(l,t) = y'''(l,t). \tag{2.23}$$

The proof of Theorem 2.2.1 implies that for the solution of this problem, we have a representation as a series.

In the physical formulation, we have $D = EI/m$, where E is Young's modulus of the material, I is the moment of inertia of the cross section and m is the mass per unit length.

Let $H = L^2(0, l)$. Assume that $R \in L^2(0, T; H)$, $y_0 \in W_1$, $y_1 \in H$. Let

$$\begin{aligned} R_j(t) &= \int_0^l R(x,t)\phi_j(x)\,dx, \\ y_j^0 &= \int_0^l y_0(x)\phi_j(x)\,dx, \\ y_j^1 &= \int_0^l y_1(x)\phi_j(x)\,dx. \end{aligned}$$

Let λ_j, ϕ_j and A be as in 2.3.1. The operator DA has the eigenvalues $D\lambda_j$. We have

$$\{f \in H : f = \sum_{j=1}^{\infty} c_j\phi_j,\ \|f\|_{W_1}^2 = D\sum_{j=1}^{\infty} |c_j|^2\lambda_j < \infty\} = W_1.$$

Obviously, $W_2 \subset W_1$.

Lemma 2.3.4 *Let $f \in C^{(4)}(0, l)$ with $f(0) = f'(0) = 0 = f''(l) = f'''(l)$. Then $f \in W_2$.*

Proof Similar as in the proof of Lemma 2.3.1, we can show that

$$\begin{aligned} \hat{f}(j) := \int_0^l f(x)\phi_j(x)\,dx &= \frac{1}{\lambda_j}\int_0^l f(x)\phi_j^{(4)}(x)\,dx \\ &= \frac{1}{\lambda_j}\int_0^l f^{(4)}(x)\phi_j(x)\,dx \\ &=: \frac{1}{\lambda_j}\hat{(f^{(4)})}(j). \end{aligned}$$

Since $f^{(4)} \in L^2(0, l)$, we have $\left(\hat{(f^{(4)})}(j)\right)_{j\in\mathbb{N}} \in l_0^2$.

Hence $(\lambda_j\hat{f}(j))_{j\in\mathbb{N}} = \left(\hat{(f^{(4)})}(j)\right)_{j\in\mathbb{N}} \in l_0^2$. Thus $f \in W_2$. $\Box$

Let $y_j(t) = y_j^0\cos(\sqrt{D\lambda_j}t) + y_j^1\sin(\sqrt{D\lambda_j}t)/\sqrt{D\lambda_j}$

$$+\int_0^t R_j(\tau)\sin(\sqrt{D\lambda_j}(t-\tau))/\sqrt{D\lambda_j}\,d\tau.$$

Then

$$y(x,t) = \sum_{j=1}^{\infty} y_j(t)\phi_j(x) \tag{2.24}$$

is the unique solution of our problem in $C(0,T;H)$ in the sense of Theorem 2.2.1.

2.4 Reachable Set and Moment Problems

In this section, we define the reachable set and characterize it in terms of a moment problem. From another point of view, this means that we characterize the set of successful controls as the solution set of a moment problem. Again we follow the treatment given in [3], Chapter III.

2.4.1 Characterization of the Reachable Set

Let H, A and W_r be as in Section 2.1. Let U be a Hilbert space, B a bounded linear operator from U to W_{r-1} and $\mathcal{U} = L^2(0,T;U)$. We consider the control system

$$\frac{d^2y(t)}{dt^2} + Ay(t) = Bu(t),\ t \geq 0,\ u \in \mathcal{U} \tag{2.25}$$

with initial conditions

$$y(0) = y_0,\ dy(0)/dt = y_1. \tag{2.26}$$

Definition 2.4.1 *The* **reachable set** $\mathcal{R}(T, y_0, y_1)$ *from* y_0, y_1 *at time* T *is the set of all points of the form* $(y(T), dy(T)/dt) \in W_r \times W_{r-1}$, *where* y *is the solution of (2.25) (2.26) for some* $u \in \mathcal{U}$.

Each element of the reachable set $\mathcal{R}(T, y_0, y_1)$ can be uniquely represented in the form

$$y(T) = \sum_{j=1}^{\infty} y_j(T)\phi_j,\ dy(T)/dt = \sum_{j=1}^{\infty} \left(\frac{dy_j(T)}{dt}\right)\phi_j. \tag{2.27}$$

We have

$\|y(T)\|_{W_r} = \|(y_j(T))_{j\in\mathbb{N}}\|_r$ and $\|dy(T)/dt\|_{W_{r-1}} = \|(dy_j(T)/dt)_{j\in\mathbb{N}}\|_{r-1}$.

Therefore, the set $\mathcal{R}(T, y_0, y_1)$ is isometric to the set $\hat{\mathcal{R}}(T, (y_0^j)_{j\in\mathbb{N}}, (y_1^j)_{j\in\mathbb{N}}) \subset l_r^2 \times l_{r-1}^2$ consisting of the pairs of sequences $((y_j(T))_{j\in\mathbb{N}}, (dy_j(T)/dt)_{j\in\mathbb{N}})$ corresponding to $(y(T), dy(T)/dt) \in \mathcal{R}(T, y_0, y_1)$.

Let $y(t, u, y_0, y_1)$ denote the solution of system (2.25), (2.26). Then we have

$$\begin{aligned} y(t, u, y_0, y_1) &= y(t, 0, y_0, y_1) + y(t, u, 0, 0) \\ &=: S_0(t)(y_0, y_1) + K_0(t)u. \end{aligned}$$

In this way, we have defined bounded linear operators $S_0(t) : W_r \times W_{r-1} \to W_r$ and $K_0(t) : \mathcal{U} \to W_r$ where

$$\begin{aligned} S_0(t)(y_0, y_1) &= \sum_{j=1}^{\infty} \left(y_j^0 \cos(\sqrt{\lambda_j} t) + y_j^1 \sin(\sqrt{\lambda_j} t)/\sqrt{\lambda_j} \right) \phi_j \text{ and} \\ K_0(t)u &= \sum_{j=1}^{\infty} \left(\int_0^t (Bu(\tau))_j \sin(\sqrt{\lambda_j}(t-\tau))/\sqrt{\lambda_j}\, d\tau \right) \phi_j. \end{aligned}$$

Here, the functions $(Bu(\tau))_j$ are defined by the relation

$$Bu(\tau) = \sum_{j=1}^{\infty} (Bu(\tau))_j \phi_j.$$

For the time–derivative, we have

$$\begin{aligned} dy(t, u, y_0, y_1)/dt &= dy(t, 0, y_0, y_1)/dt + dy(t, u, 0, 0)/dt \\ &=: S_1(t)(y_0, y_1) + K_1(t)u, \end{aligned}$$

with $S_1(t) : W_r \times W_{r-1} \to W_{r-1}$ and $K_1(t) : \mathcal{U} \to W_{r-1}$ where

$$\begin{aligned} S_1(t)(y_0, y_1) &= \sum_{j=1}^{\infty} \left(-y_j^0 \sqrt{\lambda_j} \sin(\sqrt{\lambda_j} t) + y_j^1 \cos(\sqrt{\lambda_j} t) \right) \phi_j \text{ and} \\ K_1(t)u &= \sum_{j=1}^{\infty} \left(\int_0^t (Bu(\tau))_j \cos(\sqrt{\lambda_j}(t-\tau))\, d\tau \right) \phi_j. \end{aligned}$$

Thus we have the reachable set

$$\mathcal{R}(T, y_0, y_1) = \{(S_0(T)(y_0, y_1) + K_0(T)u,\ S_1(T)(y_0, y_1) + K_1(T)u) :\ u \in \mathcal{U}\}.$$

We are mainly interested in the set

$$\mathcal{R}(T) := \mathcal{R}(T, 0, 0) = \{(K_0(T)u, K_1(T)u) :\ u \in \mathcal{U}\}.$$

Define the corresponding set of coordinate sequences

$$\begin{aligned}\hat{\mathcal{R}}(T) &= \hat{\mathcal{R}}(T,(0)_{j\in I\!N},(0)_{j\in I\!N})\\ &= \{(((K_0(T)u)_j)_{j\in I\!N},\ ((K_1(T)u)_j)_{j\in I\!N}):\ u\in\mathcal{U}\}.\end{aligned}$$

At this point, we need the fact that the topological dual space of W_r is W_{-r}. We define the linear operator $B^*: W_{-r+1}\to U$ by the relation

$$\psi(Bv)=\langle v,B^*\psi\rangle_U,\ v\in U,\psi\in W_{-r+1}.$$

In particular, $(Bv)_j=\langle v,B^*\phi_j\rangle_U,\ v\in U,\ j\in I\!N$.

For $((y_j(T))_{j\in I\!N},(dy_j(T)/dt)_{j\in I\!N})\in\hat{\mathcal{R}}(T)$ we have

$$\begin{aligned}y_j(T) &= \int_0^T (Bu(\tau))_j\sin(\sqrt{\lambda_j}(T-\tau))/\sqrt{\lambda_j}\,d\tau\\ &= \int_0^T \sin(\sqrt{\lambda_j}(T-\tau))/\sqrt{\lambda_j}\langle u(\tau),B^*\phi_j\rangle_U\,d\tau \text{ and} && (2.28)\\ dy_j(T)/dt &= \int_0^T \cos(\sqrt{\lambda_j}(T-\tau))\langle u(\tau),B^*\phi_j\rangle_U\,d\tau. && (2.29)\end{aligned}$$

Trigonometric identities imply that the system of equations (2.28) and (2.29) is equivalent to the equations

$$\begin{aligned}\sin(\sqrt{\lambda_j}T)\sqrt{\lambda_j}y_j(T)+\cos(\sqrt{\lambda_j}T)\frac{dy_j(T)}{dt} &= \int_0^T\cos(\sqrt{\lambda_j}\tau)\langle u(\tau),B^*\phi_j\rangle_U\,d\tau\\ -\cos(\sqrt{\lambda_j}T)\sqrt{\lambda_j}y_j(T)+\sin(\sqrt{\lambda_j}T)\frac{dy_j(T)}{dt} &= \int_0^T\sin(\sqrt{\lambda_j}\tau)\langle u(\tau),B^*\phi_j\rangle_U\,d\tau.\end{aligned}$$

Hence we can state the following theorem.

Theorem 2.4.1 *The reachable set $\mathcal{R}(T)$ is isometric to the set $\hat{\mathcal{R}}(T)$, which is equal to the set of all pairs of sequences $((\alpha_j)_{j\in I\!N},(\beta_j)_{j\in I\!N})$ for which the moment problem*

$$\begin{aligned}\sin(\sqrt{\lambda_j}T)\sqrt{\lambda_j}\alpha_j+\cos(\sqrt{\lambda_j}T)\beta_j &= \int_0^T\cos(\sqrt{\lambda_j}\tau)\langle u(\tau),B^*\phi_j\rangle_U\,d\tau\\ -\cos(\sqrt{\lambda_j}T)\sqrt{\lambda_j}\alpha_j+\sin(\sqrt{\lambda_j}T)\beta_j &= \int_0^T\sin(\sqrt{\lambda_j}\tau)\langle u(\tau),B^*\phi_j\rangle_U\,d\tau.\end{aligned}$$

is solvable with some $u\in\mathcal{U}$.

If we want to control our system to the zero position, we ask whether the relation

$$0\in\mathcal{R}(T,y_0,y_1)=(S_0(T),S_1(T))+\mathcal{R}(T)$$

holds. This is equivalent to the relation

$$-(S_0(T), S_1(T)) \in \mathcal{R}(T).$$

Due to Theorem 2.4.1 and the definition of S_0 and S_1, this is in turn equivalent to the statement that the moment problem

$$\sqrt{\lambda_j} y_j^0 = \int_0^T \sin(\sqrt{\lambda_j}\tau)\langle u(\tau), B^*\phi_j\rangle_U \, d\tau \tag{2.30}$$

$$-y_j^1 = \int_0^T \cos(\sqrt{\lambda_j}\tau)\langle u(\tau), B^*\phi_j\rangle_U \, d\tau \tag{2.31}$$

is solvable with some $u \in \mathcal{U}$.

2.4.2 Example: The Euler–Bernoulli Beam

Let A be as in Section 2.3.1. For the Euler–Bernoulli beam, we have $H = L^2(0,l)$, $U = I\!R$, $\mathcal{U} = L^2(0,T)$. For $v \in I\!R$, $Bv = -xv \in L^2(0,l)$.

Hence we consider the initial boundary value problem

$$\frac{d^2 y(x,t)}{dt^2} + D\frac{d^4 y(x,t)}{dx^4} = -xu(t),\ x \in [0,l], t \geq 0. \tag{2.32}$$

with initial conditions (2.22) and boundary conditions (2.23).

We have $(Bv)_j = -v\int_0^l x\phi_j(x)\,dx$, with ϕ_j as defined in (2.19).

Lemma 2.4.1 *(cf. [24], p.451) Let ϕ_j be defined as in (2.19). Then*

$$\int_0^l x\phi_j(x)\,dx = \frac{2}{\sqrt{l\lambda_j}}. \tag{2.33}$$

Proof With φ_j as defined in (2.17) we have

$$\begin{aligned}
\int_0^l x\phi_j(x)\,dx &= (1/\lambda_j)\int_0^l x\phi_j^{(4)}(x)\,dx \\
&= (1/\lambda_j)\left(x\phi_j^{(3)}(x)|_0^l - \int_0^l \phi_j^{(3)}(x)\,dx\right) \\
&= (1/\lambda_j)\left(-\phi_j^{(2)}(x)|_0^l\right) \\
&= (1/\lambda_j)\phi_j^{(2)}(0) \\
&= (1/(\lambda_j\sqrt{l}))\varphi_j^{(2)}(0) \\
&= (1/(\lambda_j\sqrt{l}))2\sqrt{\lambda_j} \\
&= \frac{2}{\sqrt{l\lambda_j}} \qquad \square
\end{aligned}$$

Hence we have

$$(Bv)_j = -2v/(\sqrt{l\lambda_j}). \tag{2.34}$$

We are looking for a control function u for which at time T, the system satisfies the end conditions

$$y(x,T) = 0,\ dy(x,T)/dt = 0,\ x \in [0,l].$$

Due to (2.30), (2.31) this requirement to the control function is equivalent to the conditions

$$\int_0^T \sin(\sqrt{D\lambda_j}\tau)u(\tau)\,d\tau = -\frac{\lambda_j\sqrt{l}\sqrt{D}}{2}\int_0^l y_0(x)\phi_j(x)\,dx, \tag{2.35}$$

$$\int_0^T \cos(\sqrt{D\lambda_j}\tau)u(\tau)\,d\tau = \frac{\sqrt{\lambda_j}\sqrt{l}}{2}\int_0^l y_1(x)\phi_j(x)\,dx,\ j \in I\!N. \tag{2.36}$$

Thus the feasible controls must solve the above trigonometric moment problem. The above equations can also be considered as a countable number of equality constraints for the control functions.

For the sake of completeness, we state the series–representation of the solution for the Euler–Bernoulli beam:

$$y(x,t) = \tag{2.37}$$

$$\sum_{j=1}^{\infty}\left(y_j^0\cos(\sqrt{D\lambda_j}t) + y_j^1\sin(\sqrt{D\lambda_j}t)/\sqrt{D\lambda_j}\right)\phi_j$$

$$-\sum_{j=1}^{\infty}\left(\frac{2}{\lambda_j\sqrt{lD}}\int_0^t u(\tau)\sin(\sqrt{D\lambda_j}(t-\tau))\,d\tau\right)\phi_j.$$

Chapter 3

Moment Problems in Hilbert Space

As we have seen, the controls that steer a hyperbolic system of the type considered in Chapter 2 from a given initial state to a given target state can be described as the elements of the solution set of a certain trigonometric moment problem. To develop and analyse an algorithm that uses this fact numerically, we need some results about general moment problems in Hilbert space, that are presented in this chapter.

Usually in the literature solutions of moment problems with minimal norm are considered. For our application, we need the more general case of an objective function that is given by the norm of the image of the control under an affine linear transformation.

Let H be a Hilbert space. Let $(f_j)_{j\in\mathbb{N}}$ be a sequence of elements of H and $(c_j)_{j\in\mathbb{N}}$ a sequence of scalars. The corresponding moment problem is to find a point $f \in H$ such that

$$\langle f, f_j\rangle_H = c_j \text{ for all } j \in \mathbb{N}. \tag{3.1}$$

Definition 3.0.2 (see [50]) *A sequence $(f_j)_{j\in\mathbb{N}}$ is called a* **Bessel sequence***, if for all $f \in H$ we have*

$$\sum_{j=1}^{\infty} |\langle f, f_j\rangle_H|^2 < \infty.$$

The sequence is called a **Riesz–Fischer sequence***, if the moment problem (3.1) has a solution for all $(c_j)_{j\in\mathbb{N}} \in l_0^2$.*

The **moment space** *of $(f_j)_{j\in\mathbb{N}}$ is the space*

$$\{(\langle f, f_j\rangle_H)_{j\in\mathbb{N}} : f \in H\}.$$

The moment space of $(f_j)_{j\in I\!N}$ is $l_0^2 = l^2$, if and only if $(f_j)_{j\in I\!N}$ is both a Bessel and a Riesz–Fischer sequence.

Lemma 3.0.2 (see [50], Proposition 2, p. 154) *If $(f_j)_{j\in I\!N}$ is a Bessel sequence, then there exists a constant M such that for every $f \in H$*

$$\sum_{j=1}^{\infty} |\langle f, f_j\rangle_H|^2 \le M\|f\|_H^2.$$

If $(f_j)_{j\in I\!N}$ is a Riesz–Fischer sequence, then there exists a constant m such that for all $(c_j)_{j\in I\!N} \in l_0^2$ the moment problem (3.1) has at least one solution f satisfying

$$\|f\|_H^2 \le \frac{1}{m}\sum_{j=1}^{\infty} |c_j|^2 .$$

The numbers m and M are called bounds for the sequence $(f_j)_{j\in I\!N}$. The lemma can be proved with the help of the uniform boundedness principle.

Theorem 3.0.2 (see [50], Theorem 3, p. 155)

(i) The sequence $(f_j)_{j\in I\!N}$ is a Bessel sequence with bound M if and only if the inequality

$$\|\sum_{j=1}^{N} c_j f_j\|_H^2 \le M\sum_{j=1}^{N} |c_j|^2$$

holds for all $N \in I\!N$, $(c_j)_{j\in I\!N} \in l_0^2$.

(ii) The sequence $(f_j)_{j\in I\!N}$ is a Riesz–Fischer sequence with bound $m > 0$ if and only if the inequality

$$m\sum^{N}$$

For the proof of (ii) given in [50], the following Theorem is used.

Theorem 3.0.3 (see Theorem 2, p. 151 in [50]) *Let $(f_j)_{j\in I\!N}$ be a sequence of elements of the Hilbert space H, $(c_j)_{j\in I\!N}$ be a sequence of scalars and $m > 0$. The following statements are equivalent:*

(i) There exists $w \in H$ with $\|w\|_H \le m$ and $\langle w, f_j\rangle_H = c_j$, $(j \in I\!N)$.

(ii) For all $N \in I\!N$, $(a_1, ..., a_N)^T \in I\!R^N$ we have the inequality

$$|\sum_{i=1}^{N} a_i c_i| \le m\|\sum_{i=1}^{N} a_i f_i\|_H.$$

Proof Statement (i) implies for all $N \in I\!N$, $(a_1, ..., a_N)^T \in I\!R^N$ the inequality

$$\begin{aligned} |\sum_{i=1}^{N} a_i c_i| &= |\sum_{i=1}^{N} a_i \langle w, f_i\rangle_H| \le |\langle w, \sum_{i=1}^{N} a_i f_i\rangle_H| \\ &\le \|w\|_H \, \|\sum_{i=1}^{N} a_i f_i\|_H \le m\|\sum_{i=1}^{N} a_i f_i\|_H. \end{aligned}$$

Assume now that statement (ii) holds.
Let Y denote the closure of span$\{f_j, j \in I\!N\}$. Define a linear functional μ of H by $\mu(\sum_{j=1}^{\infty} a_j f_j) = \sum_{j=1}^{\infty} a_j c_j$, and $\mu(f) = 0$ for $f \in Y^{\perp}$. Condition (ii) implies that μ is bounded with $\|\mu\| \le m$. The Riesz representation theorem implies the existence of $w \in H$ such that $\mu(f) = \langle w, f\rangle_H$ for all $f \in H$. Since $\|w\|_H \le m$, w is a point as required in (i). $\Box$

In the sequel we use the following easy statements.

Statement 3.0.1 *For u,$w \in H$ with $\langle u-w, w\rangle_H = 0$, we have $\|w\|_H \le \|u\|_H$. If $u \ne w$, we have $\|w\|_H < \|u\|_H$.*

This is easily seen from

$$0 \le \|u - w\|_H^2 = \|u\|_H^2 - 2\langle u, w\rangle_H + \|w\|_H^2 = \|u\|_H^2 - 2\|w\|_H^2 + \|w\|_H^2 = \|u\|_H^2 - \|w\|_H^2. \Box$$

Statement 3.0.2 *Let $(x_j)_{j\in I\!N}$ be a sequence of elements of H.*

Then $\lim_{j\to\infty} \|x_j - x\|_H = 0$ if and only if $\lim_{j\to\infty} \|x_j\|_H = \|x\|_H$ and $(x_j)_{j\in I\!N}$ converges weakly to x.

The assertion follows from the fact that for all $j \in I\!N$, we have

$$\|x_j - x\|_H^2 = \|x_j\|_H^2 - 2\langle x, x_j\rangle_H + \|x\|_H^2. \Box$$

For the numerical solution of problems with moment equations as equality constraints we replace the infinite system

$$\langle f, f_j \rangle_H = c_j \ (j \in I\!N)$$

by the sequence of finite systems

$$\langle f, f_j \rangle_H = c_j \ (j \in \{1, ..., N\}),$$

with $N \in I\!N$.

Consider now the optimization problem

$$P_\infty : \ \min \|Su - b\|_H^2 \text{ s.t. } \ \langle u, f_j \rangle_H = c_j \ (j \in I\!N),$$

with $b \in H$ and a continuous bijective linear map $S : H \to H$. For the numerical solution, this problem is replaced by a sequence of problems with a finite number of equality constraints. The discretized problems can be solved by solving a finite system of linear equations.

Theorem 3.0.4 (see [25], p. 153) *Let $S : H \to H$ be a continuous bijective linear map, $b \in H$, $(c_1, c_2, ..., c_N)^T \in I\!R^N$ and $f_1, ..., f_N \in H$ be linearly independent. The problem*

$$P_N : \ \min \|Su - b\|_H^2 \ \text{ s.t. } \langle u, f_j \rangle_H = c_j, \ j \in \{1, ..., N\}$$

has a unique solution u_N that satisfies the equality

$$Su_N - b = \sum_{j=1}^{N} \eta_j H_j, \ \textit{with } H_j = (S^*)^{-1} f_j$$

and the coefficients η_j that solve the linear system

$$\sum_{j=1}^{N} \langle H_i, H_j \rangle_H \eta_j = c_i - \langle b, H_i \rangle_H, \ i \in \{1, ..., N\}. \tag{3.2}$$

Here S^* denotes the adjoint of the operator S.

Proof Since the functions $H_1, ..., H_N$ are linearly independent, the Gram matrix $(\langle H_i, H_j \rangle_H)_{i,j=1}^N$ is positive definite. Hence the linear system (3.2) has a unique solution. Therefore

$$\begin{aligned}\langle Su^N - b, H_i \rangle_H &= \sum_{i=1}^{N} \eta_j \langle H_i, H_j \rangle_H \\ &= c_i - \langle b, H_i \rangle_H, \ i \in \{1, ..., N\}\end{aligned}$$

hence $\langle u^N, f_j\rangle_H = c_j$, $j \in \{1, ..., N\}$.

Let $w^N = Su^N - b$. Let $u \in H$ be given with $u \neq u^N$ and $\langle u, f_j\rangle_H = c_j$, $j \in \{1, ..., N\}$. Then $Su - b \neq w^N$ and $\langle Su - b, H_j\rangle_H = \langle w^N, H_j\rangle_H$, hence $\langle Su - b - w^N, w^N\rangle_H = 0$. Statement 3.0.1 implies $\|w^N\|_H^2 < \|Su - b\|^2$. Thus u^N is the unique solution of Problem P_N. □

For $N \in I\!N$, we define

$$\lambda_N = \min\{\|Su - b\|_H^2 \text{ s.t. } \langle u, f_j\rangle_H = c_j,\ j \in \{1, ..., N\}\}.$$

Then the sequence $(\lambda_j)_{j\in I\!N}$ is increasing.

Theorem 3.0.5 (see [22], p. 71) *The following statements are equivalent.*

(i) $\lim_{N\to\infty} \lambda_N < \infty$

(ii) Problem P_∞ has a unique solution u^*.

If (ii) is valid, then u^* is the unique solution of the system of moment equations

$$\langle u, f_j\rangle_H = c_j,\ j \in I\!N$$

with $Su - b$ in the closure of span$\{(S^*)^{-1} f_j,\ j \in I\!N\}$. Moreover,

$$\lim_{N\to\infty} \|u^* - u^N\|_H = 0,\ \lim_{N\to\infty} \lambda_N = \|Su^* - b\|_H^2$$

with u^N as defined in Theorem 3.0.4.

Proof If (ii) is valid, for all $N \in I\!N$ the inequality $\lambda_N \leq \|Su^* - b\|_H^2$

and Statement 3.0.1 implies $\|Su^* - b\|_H < \|Su - b\|_H$. We have

$$\|Su^* - b\|_H^2 \leq \liminf_{N\to\infty} \|Su^N - b\|_H^2 = \lim_{N\to\infty} \lambda_N \leq \|Su^* - b\|_H^2,$$

hence $\lim_{N\to\infty} \|Su^N - b\|_H = \|Su^* - b\|_H$, and $\lim_{N\to\infty}\langle Su^N - Su^*, H_j\rangle_H = 0$ for all $j \in I\!N$, and Statement 3.0.2 implies the assertion. □

Remark It is not necessary to assume that the space H is separable.

3.1 Example: The Euler–Bernoulli Beam

In Section 2.4.2 we have seen that the controls for which the end conditions

$$y(x,T) = 0,\ y_t(x,T) = 0,\ x \in [0,l] \tag{3.3}$$

hold, satisfy the trigonometric moment equations (2.35), (2.36).

We want to consider the problem of a rotating Euler–Bernoulli beam, where the beam rotates about an axis through its fixed end. Problems of the control of rotating Euler–Bernoulli beams have been considered in numerous papers, see for example [25], [26], [41], [21], [5], [42], [30].

Let $\psi(t)$ be the angle of rotation at time t. Then we have the additional initial conditions

$$\psi(0) = \psi_0,\ \psi'(0) = \psi_1 \tag{3.4}$$

for the initial angle ψ_0 and the initial angular velocity ψ_1. The control is the angular acceleration $u = \psi''$. The transverse vibrations of the beam are described by equation (2.32).

We want to steer the beam to a position of rest, so we have the additional end condition that the angular velocity at time T be zero,

$$\psi'(T) = 0. \tag{3.5}$$

If we prescribe the angle where the beam comes to rest, we have also the end condition

$$\psi(T) = \psi_2. \tag{3.6}$$

The end condition (3.5) is equivalent to the moment equation

$$\int_0^T u(t)\,dt = -\psi_1 \tag{3.7}$$

and the end condition (3.6) is equivalent to the moment equation

$$\int_0^T t u(t)\,dt = \psi_0 - \psi_2. \tag{3.8}$$

So the controls steering the system to the desired end state are the solutions of the moment problem consisting of the equations

$$\int_0^T \sin(\sqrt{D\lambda_j}\tau)u(\tau)\,d\tau = -\frac{\lambda_j\sqrt{l}\sqrt{D}}{2}\int_0^l y_0(x)\phi_j(x)\,dx, \tag{3.9}$$

$$\int_0^T \cos(\sqrt{D\lambda_j}\tau)u(\tau)\,d\tau = \frac{\sqrt{\lambda_j}\sqrt{l}}{2}\int_0^l y_1(x)\phi_j(x)\,dx,\ j \in \mathbb{N} \tag{3.10}$$

$$\int_0^T u(t)\,dt = -\psi_1 \text{ and if (3.6) is prescribed} \tag{3.11}$$

$$\int_0^T tu(t)\,dt = \psi_0 - \psi_2. \tag{3.12}$$

The controllability of the system (2.22), (2.23), (3.4), (2.32) (3.3), (3.5), (3.6) is equivalent to the solvability of the moment problem (3.9), (3.10), (3.11), (3.12).

This approach to controllability via moment problems has been studied by Krabs in [23]. Earlier, Russel has considered this approach (see [39]).

The basic idea of the proofs is to show that an inequality of the type that appears in Theorem 3.0.2 (iii) holds. For trigonometric moment problems, this can sometimes be done using the results of Ingham given in [16]. Before these results can be applied, the problem has to be transformed to a complex trigonometric moment problem. This approach is given in detail in [23], Chapter 1.

Theorem 3.1.1 *For $T > 0$, the functions*

$$\{1, t, \sin(\sqrt{D\lambda_j}t), \cos(\sqrt{D\lambda_j}t),\ j \in \mathbb{N}\}$$

form a Riesz–Fischer sequence in $L^2(0,T)$.

Proof See Section 1.2.3 in [23].□

Theorem 3.1.2 *For $T > 0$, the functions*

$$\{1, t, \sin(\sqrt{D\lambda_j}t), \cos(\sqrt{D\lambda_j}t),\ j \in \mathbb{N}\}$$

form a Bessel sequence in $L^2(0,T)$.

Proof See [22], Theorem II.2.8.□

Note that for all functions y_0, $y_1 \in C^{(4)}(0,l)$, Lemma 2.3.4 implies that the numbers on the right hand side of the moment equations (3.9), (3.10) form a sequence in l_0^2. Hence Theorem 3.1.1 implies the following result.

Theorem 3.1.3 *For* $T > 0$, y_0, $y_1 \in C^4(0,l)$, *the moment problem (3.9), (3.10), (3.11), (3.12) has a solution* u *in* $L^2(0,T)$.

Chapter 4

The Computation of $H_j = (S^*)^{-1} f_j$

As the title indicates, in this chapter we consider a computational aspect. The reader who is more interested in theoretical considerations can skip this and the next chapter and continue with Chapter 6. However, the numerical examples presented here are important since they show that the theory developed up to now can be used for computations.

As we have seen in Theorem 3.0.4, the computation of the solution of the discretized problem P_N requires the knowledge of the functions $H_j = (S^*)^{-1} f_j$, $j \in \{1, ..., N\}$.

Numerically, this means that we have to solve the equations

$$S^* H_j = f_j, \; j \in \{1, ..., N\}.$$

In the applications, the Hilbert space H will often be $L^2(0, T)$. In this chapter, we show how for this space we can solve the equations if S is a Volterra–type operator with a finite rank kernel.

This is motivated by our example, the Euler–Bernoulli beam where the torque at the axis is given by a Volterra–type operator with a kernel that is given as an infinite series. The truncated series is a finite rank kernel.

4.1 Volterra Operators with Finite Rank Kernel

Let a number $\ > 0$ be given. Let K and f be continuous functions on the interval $[0,T]$. We consider an operator of the form

$$Su(t) = \ u(t) - \int_0^t K(t-s)u(s)\,ds,\ t \in [0,T]. \tag{4.1}$$

Then we have the adjoint

$$S^*u(t) = \ u(t) - \int_t^T K(s-t)u(s)\,ds.$$

This can easily be seen using Fubini's theorem on the square $[0,T]\times[0,T]$ (see [24], (3.9)). We want to find a function u such that

$$S^*u(t) = f(t),\ t \in [0,T]. \tag{4.2}$$

For this purpose, we start with a transformation to the standard form of Volterra–equations of the second kind.

Lemma 4.1.1 *If $v \in C(0,T)$ is the continuous solution of the linear Volterra equation of the second kind*

$$v(x) - \int_0^x \frac{K(x-y)}{\ }v(y)\,dy = \frac{f(T-x)}{\ },\ x \in [0,T] \tag{4.3}$$

then $u(x) := v(T-x)$ satisfies the equation

$$u(x) - \int_x^T K(y-x)u(y)\,dy = f(x),\ x \in [0,T].$$

Proof Since K and f are continuous, equation (4.3) has a unique continuous solution (see Theorem 3.1 in [35]). Using the substitution formula we get the equation

$$\begin{aligned} & v(T-x) - \int_0^{T-x} (K(T-x-y)/\ )v(y)\,dy \\ = \ & u(x) - \int_T^x (K(s-x)/\ )v(T-s)\,(-ds) \\ = \ & u(x) - \int_x^T (K(s-x)/\ )u(s)\,ds. \end{aligned}$$

Since $f(T-(T-x)) = f(x)$, the assertion follows. □

Now assume that the Kernel K has the representation

$$K(x-y)/ \quad = -\sum_{i=1}^{n} P_i(x)Q_i(y), \tag{4.4}$$

with continuous functions P_i and Q_i, i.e. that K is a finite rank kernel. In this case, the solution of the equation (4.3) can be computed by means of the solution of an initial value problem in $I\!R^n$.

Theorem 4.1.1 (see [35], Theorem 1.1, p.9) *Let*

$$k(t,s) = -\sum_{i=1}^{n} P_i(t)Q_i(s). \tag{4.5}$$

Assume that P_i, Q_i and a given function g are continuous in $[0,T]$.

Then the linear equation

$$v(t) - \int_0^t k(t,s)v(s)\,ds = g(t)$$

has a solution

$$v(t) = g(t) - \sum_{i=1}^{n} P_i(t)y_i(t),$$

where the y_i are the solutions of the system

$$\begin{aligned} y_i'(t) &= Q_i(t)\{g(t) - \sum_{j=1}^{n} P_j(t)y_j(t)\},\ i \in \{1,...,n\} \\ y(0) &= 0. \end{aligned}$$

Thus equation (4.2) has the solution

$$u(x) = v(T-x) = f(x)/ \quad - \sum_{i=1}^{n} P_i(T-x)y_i(T-x), \tag{4.6}$$

where the y_i are the solutions of the initial value problem

$$\begin{aligned} y_i'(t) &= Q_i(t)\{f(T-t)/ \quad - \sum_{j=1}^{n} P_j(t)y_j(t)\},\ i \in \{1,...,n\} \\ y(0) &= 0. \end{aligned}$$

4.1.1 Example: The Torque at the Axis of the Beam

The torque $M(t)$ at the axis of the beam at time t for a control function u is given by the equation

$$M(t) = (\quad + l^3/3)u(t) + \int_0^l x y_{tt}(x,t)\,dx$$

(see [24], (1.2)). The control of a rotating beam with torque of minimal norm has also been studied in [4].

The differential equation (2.32) and the boundary condition (2.23) imply

$$\begin{aligned}
\int_0^l x y_{tt}(x,t)\,dx &= -u(t)\int_0^l x^2\,dx - D\int_0^l x y_{xxxx}(x,t)\,dx \\
&= -u(t)l^3/3 - D\left[x y_{xxx}(x,t)|_0^l - \int_0^l y_{xxx}(x,t)\,dx \right] \\
&= -u(t)l^3/3 - D y_{xx}(0,t)
\end{aligned}$$

Hence $M(t) = \quad u(t) - D y_{xx}(0,t)$.

The series representation (2.37) implies

$$\begin{aligned}
y_{xx}(0,t) &= \sum_{j=1}^{\infty} (y_j^0 \cos(\sqrt{D\lambda_j} t) + y_j^1 \sin(\sqrt{D\lambda_j} t)/\sqrt{D\lambda_j} \\
&\quad + \int_0^t \frac{-2u(\tau)}{\sqrt{l\lambda_j}} \sin(\sqrt{D\lambda_j}(t-\tau))/\sqrt{D\lambda_j}\,d\tau)\phi_{jxx}(0)
\end{aligned}$$

with ϕ_j as in (2.19). According to (2.17), we have $\varphi_{jxx}(0) = 2\sqrt{\lambda_j}$, hence $\phi_{jxx}(0) = 2\sqrt{\lambda_j}/\sqrt{l}$. Thus we have

$$\begin{aligned}
M(t) &= \quad u(t) - D\sum_{j=1}^{\infty} [\frac{2\sqrt{\lambda_j}}{\sqrt{l}} y_j^0 \cos(\sqrt{D\lambda_j} t) + \frac{2}{\sqrt{D}\sqrt{l}} y_j^1 \sin(\sqrt{D\lambda_j} t) \\
&\quad - \frac{4}{l\sqrt{D\lambda_j}} \int_0^t u(\tau)\sin(\sqrt{D\lambda_j}(t-\tau)\,d\tau].
\end{aligned}$$

We define the function b by

$$b(t) = D\sum_{j=1}^{\infty} \left[\frac{2\sqrt{\lambda_j}}{\sqrt{l}} y_j^0 \cos(\sqrt{D\lambda_j} t) + \frac{2}{\sqrt{D}\sqrt{l}} y_j^1 \sin(\sqrt{D\lambda_j} t) \right]$$

and the function K by

$$K(t) = D \sum_{j=1}^{\infty} \frac{-4}{l\sqrt{D\lambda_j}} \sin(\sqrt{D\lambda_j}t). \tag{4.7}$$

Then we see that $M(t)+b(t)$ is given by an operator S of the form (4.1),

$$M(t) + b(t) = Su(t) = \quad u(t) - \int_0^t K(t-s)u(s)\,ds.$$

The series in the definition of K converges absolutely and uniformly. Thus K is continuous. For numerical computations, we approximate the kernel by a finite sum. For this purpose we define

$$\begin{aligned} K_N(t) &= \sqrt{D} \sum_{j=1}^{N} \frac{-4}{l\sqrt{\lambda_j}} \sin(\sqrt{D\lambda_j}t), \qquad (4.8) \\ S_N u(t) &= \quad u(t) - \int_0^t K_N(t-s)u(s)\,ds. \end{aligned}$$

We want to compute the solution u of the equation

$$S_N^* u = f. \tag{4.9}$$

For a continuous right hand side f, this equation has a continuous solution. Using the resolvent kernel (see [35], p.36), it is easy to show that for N large enough, this solution is arbitrarily close in the sense of the maximum norm on $[0,T]$ to the continuous solution of the Volterra equation with the kernel K.

A trigonometric identity implies that $K_N(x-y)/$

$$= \frac{-\sqrt{D}}{l} \sum_{j=1}^{N} \frac{4}{\sqrt{\lambda_j}} \left(\sin(\sqrt{D\lambda_j}x)\cos(\sqrt{D\lambda_j}y) - \cos(\sqrt{D\lambda_j}x)\sin(\sqrt{D\lambda_j}y) \right).$$

We define

$$\begin{aligned} P_{2i-1}(t) &= \tfrac{1}{\kappa}\sin(\sqrt{D\lambda_j}t), Q_{2i-1}(t) &= \frac{4\sqrt{D}}{l\sqrt{\lambda_j}}\cos(\sqrt{D\lambda_j}t) \\ P_{2i}(t) &= \tfrac{-1}{\kappa}\cos(\sqrt{D\lambda_j}t), Q_{2i}(t) &= \frac{4\sqrt{D}}{l\sqrt{\lambda_j}}\sin(\sqrt{D\lambda_j}t). \end{aligned}$$

Then (4.4) holds, and we can apply Theorem 4.1.1. Let $(y_1, ..., y_{2N})^T$ be the solution of the initial value problem

$$y'_{2i-1}(t) = \frac{4\sqrt{D}}{l\sqrt{\lambda_j}} \cos(\sqrt{D\lambda_j}t) \cdot \tag{4.10}$$

$$\left(\frac{f(T-t)}{} - \frac{1}{} \sum_{j=1}^{N} \Big(\right.$$

We can write the differential equations (4.10), (4.11) in the form

$$y' = r - \frac{1}{\ }Ay.$$

Let

$$\begin{aligned} q_{2i-1} &= \tfrac{4\sqrt{D}}{\sqrt{\lambda_j}}\cos(\sqrt{D\lambda_j}t), & q_{2i} &= \tfrac{4\sqrt{D}}{\sqrt{\lambda_j}}\sin(\sqrt{D\lambda_j}t), \\ p_{2i-1} &= \sin(\sqrt{D\lambda_j}t), & p_{2i} &= -\cos(\sqrt{D\lambda_j}t). \end{aligned}$$

Then $A = qp^T$ and $r = (f(T-t)/\ \ )q$.

The trapezoidal method for the solution of the initial value problem yields

$$y_{k+1} = y_k + \frac{h}{2}(r_k - \frac{1}{\ }A_k y_k + r_{k+1} - \frac{1}{\ }A_{k+1}y_{k+1}).$$

Hence

$$\begin{aligned} \left(I + \frac{h}{2\ }A_{k+1}\right) y_{k+1} &= y_k + \frac{h}{2}(r_k + r_{k+1} - \frac{1}{\ }A_k y_k) \\ &= y_k + \frac{h}{2}(r_k + r_{k+1}) - \frac{h}{2\ }(p_k^T y_k)q_k \\ &=: b_{k+1}, \end{aligned}$$

which implies the equation

$$y_{k+1} = b_{k+1} + \frac{h}{2\ }\delta_{k+1}q_{k+1}, \tag{4.15}$$

with

$$\delta_{k+1} = \frac{p_{k+1}^T b_{k+1}}{-1 - h(p_{k+1}^T q_{k+1})/(2\ \ )}.$$

The following figures show the truncated kernel K_{100} for $D \in \{1, 10, 100, 1000\}$. The values where computed on an equidistant grid consisting of 1000 points.

Figure 4.1.1

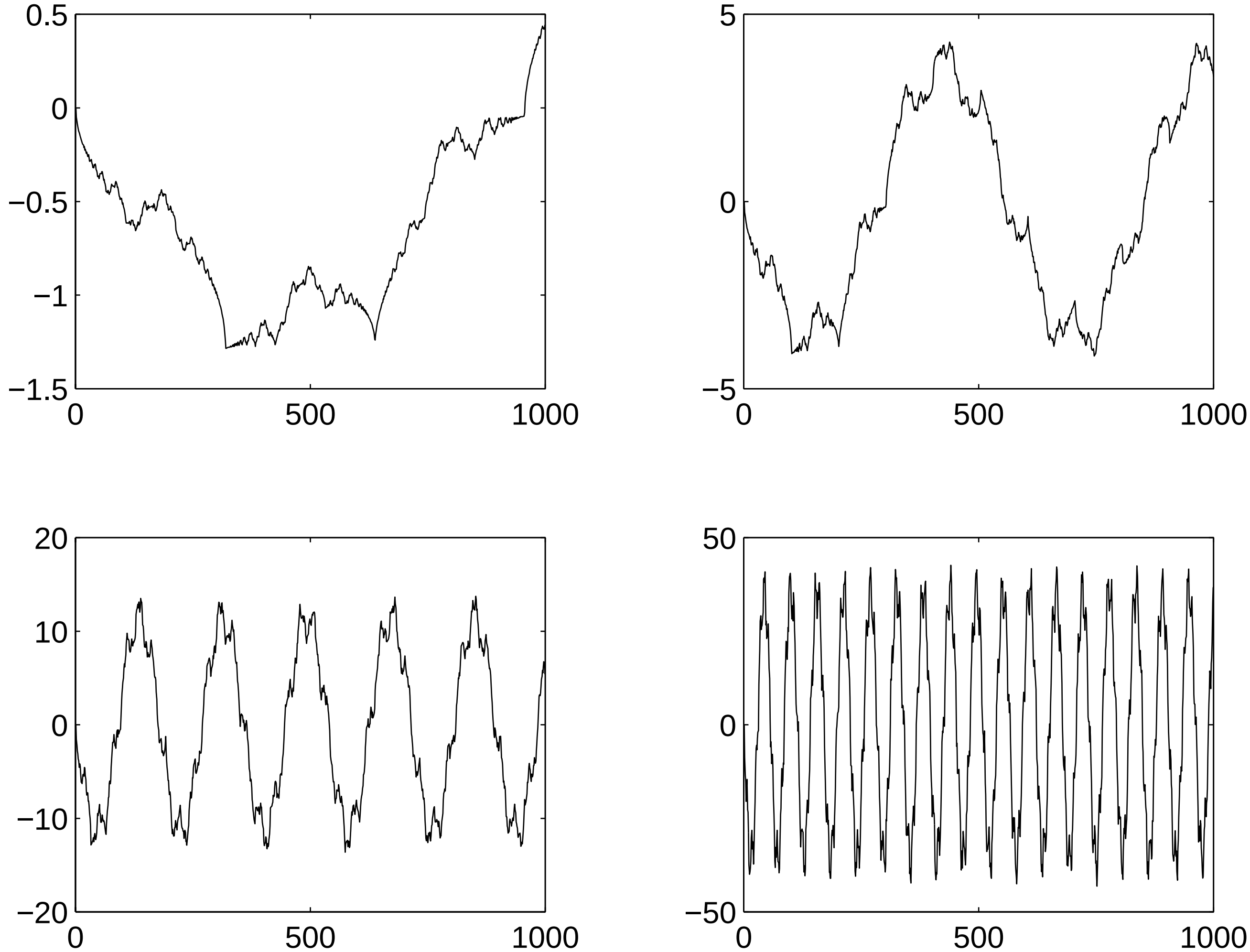


Figure 4.1.1. The Graph of K_{100} on the interval $[0,1]$ for $l = 1$, $D = 1$, 10, 100, 1000.

To obtain the angular acceleration u from the torque function $M = Su - b$, we have to solve the equation

$$Su = M + b.$$

For numerical purposes, we consider the equation

$$S_N u = M + b, \tag{4.16}$$

where the kernel K in S is replaced by the finite sum K_N. Let $g(t) = (M(t) + b(t))/\ $. We want to solve the Volterra equation of the second kind

$$u(t) - \int_0^t \frac{K_N(t-s)}{\ } u(s)\,ds = g(t).$$

Since equation (4.4) holds, we can apply Theorem 4.1.1. Let y be the solution of the initial value problem

$$\begin{aligned}
&y'_{2i-1}(t) = \frac{4\sqrt{D}}{l\sqrt{\lambda_j}} \cos(\sqrt{D\lambda_j}t) \cdot \\
&\left(g(t) - \frac{1}{\ } \sum_{j=1}^{N} \left(\sin(\sqrt{D\lambda_j}t) y_{2j-1}(t) - \cos(\sqrt{D\lambda_j}t) y_{2j}(t) \right) \right) \\
&y'_{2i}(t) = \frac{4\sqrt{D}}{l\sqrt{\lambda_j}} \sin(\sqrt{D\lambda_j}t) \cdot \\
&\left(g(t) - \frac{1}{\ } \sum_{j=1}^{N} \left(\sin(\sqrt{D\lambda_j}t) y_{2j-1}(t) - \cos(\sqrt{D\lambda_j}t) y_{2j}(t) \right) \right) \\
&y(0) = 0.
\end{aligned}$$

Then equation (4.16) has the solution

$$u(t) = g(t) - \frac{1}{\ } \sum_{i=1}^{N} \left(\sin(\sqrt{D\lambda_j}(t)) y_{2j-1}(t) - \cos(\sqrt{D\lambda_j}(t)) y_{2j}(t) \right).$$

4.1.2 Numerical Examples

For our numerical examples, we consider the functions $f_0(t) = z$

The following tables give the values $H_i(t) = (S^*)^{-1}f_i$ and the values $S^*H_i(t) - z_i(t)$, where $S^*H_i(t)$ was computed using numerical integration with a Newton–Cotes formula. For the numerical integration, the same grid points as for the solution of the initial value problem were used.

For Table 4.1.1, we used the parameter values $= 10$, $T = 1$, $N = 100$ and $h = 2^{-12}$.

Table 4.1.1

$D = 1$		
	$H_0(t)$	$S^*H_0(t) - z_0(t)$
$t = 0.00$	0.093854	$0.813E - 07$
$t = 0.25$	0.093910	$0.621E - 07$
$t = 0.50$	0.096116	$0.362E - 07$
$t = 0.75$	0.098787	$0.120E - 06$
$t = 1.00$	0.100000	$0.000E + 00$
	$H_8(t)$	$S^*H_8(t) - z_8(t)$
$t = 0.00$	0.100050	$-0.122E - 08$
$t = 0.25$	0.036975	$0.475E - 07$
$t = 0.50$	-0.072568	$0.368E - 07$
$t = 0.75$	-0.090958	$0.295E - 07$
$t = 1.00$	0.004938	$0.000E + 00$

$D = 10$		
	$H_0(t)$	$S^*H_0(t) - z_0(t)$
$t = 0.00$	0.097782	$-0.480E - 06$
$t = 0.25$	0.094919	$-0.164E - 06$
$t = 0.50$	0.099202	$0.419E - 06$
$t = 0.75$	0.093816	$0.439E - 06$
$t = 1.00$	0.100000	$0.000E + 00$
	$H_8(t)$	$S^*H_8(t) - z_8(t)$
$t = 0.00$	0.099942	$0.156E - 05$
$t = 0.25$	0.023960	$-0.542E - 06$
$t = 0.50$	-0.088891	$-0.746E - 06$
$t = 0.75$	-0.065404	$-0.118E - 05$
$t = 1.00$	0.058260	$0.000E + 00$

$D = 100$		
	$H_0(t)$	$S^* H_0(t) - z_0(t)$
$t = 0.00$	0.095617	$-0.768E-05$
$t = 0.25$	0.096455	$0.477E-05$
$t = 0.50$	0.098427	$-0.571E-05$
$t = 0.75$	0.094023	$-0.419E-05$
$t = 1.00$	0.100000	$0.000E+00$
	$H_8(t)$	$S^* H_8(t) - z_8(t)$
$t = 0.00$	0.100028	$0.902E-05$
$t = 0.25$	0.079595	$0.959E-05$
$t = 0.50$	0.025289	$-0.451E-05$
$t = 0.75$	-0.039950	$-0.330E-05$
$t = 1.00$	-0.088044	$0.000E+00$

$D = 1000$		
	$H_0(t)$	$S^* H_0(t) - z_0(t)$
$t = 0.00$	0.099944	$-0.314E-04$
$t = 0.25$	0.093691	$0.104E-04$
$t = 0.50$	0.099816	$-0.433E-04$
$t = 0.75$	0.093637	$-0.347E-04$
$t = 1.00$	0.100000	$0.000E+00$
	$H_8(t)$	$S^* H_8(t) - z_8(t)$
$t = 0.00$	0.099876	$-0.132E-05$
$t = 0.25$	0.074517	$-0.922E-05$
$t = 0.50$	0.005809	$0.719E-06$
$t = 0.75$	-0.066274	$0.144E-04$
$t = 1.00$	-0.099795	$0.000E+00$

For Table 4.1.2, we change the value of N to $N = 50$ and leave the other parameters as before, $\ = 10$, $T = 1$ and $h = 2^{-12}$.

Table 4.1.2

$D = 1$		
	$H_0(t)$	$S^*H_0(t) - z_0(t)$
$t = 0.00$	0.093854	$0.722E-07$
$t = 0.25$	0.093910	$0.127E-07$
$t = 0.50$	0.096116	$0.243E-07$
$t = 0.75$	0.098787	$0.711E-07$
$t = 1.00$	0.100000	$0.000E+00$
	$H_8(t)$	$S^*H_8(t) - z_8(t)$
$t = 0.00$	0.100050	$0.201E-08$
$t = 0.25$	0.036975	$0.353E-07$
$t = 0.50$	-0.072568	$0.198E-07$
$t = 0.75$	-0.090958	$0.109E-07$
$t = 1.00$	0.004938	$0.000E+00$

$D = 10$		
	$H_0(t)$	$S^*H_0(t) - z_0(t)$
$t = 0.00$	0.097782	$-0.520E-06$
$t = 0.25$	0.094919	$-0.235E-06$
$t = 0.50$	0.099202	$0.146E-06$
$t = 0.75$	0.093816	$0.239E-06$
$t = 1.00$	0.100000	$0.000E+00$
	$H_8(t)$	$S^*H_8(t) - z_8(t)$
$t = 0.00$	0.099942	$0.153E-05$
$t = 0.25$	0.023960	$-0.502E-06$
$t = 0.50$	-0.088891	$-0.748E-06$
$t = 0.75$	-0.065404	$-0.121E-05$
$t = 1.00$	0.058260	$0.000E+00$

$D = 100$		
	$H_0(t)$	$S^*H_0(t) - z_0(t)$
$t = 0.00$	0.095617	$-0.722E - 05$
$t = 0.25$	0.096455	$0.532E - 05$
$t = 0.50$	0.098427	$-0.537E - 05$
$t = 0.75$	0.094023	$-0.411E - 05$
$t = 1.00$	0.100000	$0.000E + 00$
	$H_8(t)$	$S^*H_8(t) - z_8(t)$
$t = 0.00$	0.100028	$-0.111E - 05$
$t = 0.25$	0.079595	$0.520E - 05$
$t = 0.50$	0.025289	$-0.660E - 06$
$t = 0.75$	-0.039950	$0.495E - 06$
$t = 1.00$	-0.088044	$0.000E + 00$

$D = 1000$		
	$H_0(t)$	$S^*H_0(t) - z_0(t)$
$t = 0.00$	0.099944	$-0.289E - 04$
$t = 0.25$	0.093690	$0.110E - 04$
$t = 0.50$	0.099815	$-0.396E - 04$
$t = 0.75$	0.093637	$-0.312E - 04$
$t = 1.00$	0.100000	$0.000E + 00$
	$H_8(t)$	$S^*H_8(t) - z_8(t)$
$t = 0.00$	0.099876	$0.166E - 06$
$t = 0.25$	0.074517	$-0.939E - 05$
$t = 0.50$	0.005809	$0.308E - 07$
$t = 0.75$	-0.066275	$0.145E - 04$
$t = 1.00$	-0.099795	$0.000E + 00$

Now we change the value of . Note that with decreasing values of , the approximation errors increase. Table 4.1.3 contains the values for $N = 50$, $= 1$, $T = 1$ and $h = 2^{-12}$. For $D = 1000$, the errrors become quite large so that the parameter value $= 1$ appears to be a lower bound for computations with the stepsize $h = 2^{-12}$ for $D = 1000$.

Table 4.1.3

$D = 1$		
	$H_0(t)$	$S^*H_0(t) - z_0(t)$
$t = 0.00$	0.592661	$0.522E - 06$
$t = 0.25$	0.505152	$-0.123E - 06$
$t = 0.50$	0.647049	$0.615E - 07$
$t = 0.75$	0.882986	$0.651E - 06$
$t = 1.00$	1.000000	$0.000E + 00$
	$H_8(t)$	$S^*H_8(t) - z_8(t)$
$t = 0.00$	1.007443	$0.241E - 07$
$t = 0.25$	0.359598	$0.347E - 06$
$t = 0.50$	-0.735773	$0.192E - 06$
$t = 0.75$	-0.913761	$0.106E - 06$
$t = 1.00$	0.049381	$0.000E + 00$

$D = 10$		
	$H_0(t)$	$S^*H_0(t) - z_0(t)$
$t = 0.00$	0.991446	$-0.523E - 05$
$t = 0.25$	0.507202	$-0.218E - 05$
$t = 0.50$	0.980079	$0.147E - 05$
$t = 0.75$	0.512083	$0.256E - 05$
$t = 1.00$	1.000000	$0.000E + 00$
	$H_8(t)$	$S^*H_8(t) - z_8(t)$
$t = 0.00$	1.001801	$0.153E - 04$
$t = 0.25$	0.273718	$-0.475E - 05$
$t = 0.50$	-0.878003	$-0.740E - 05$
$t = 0.75$	-0.665288	$-0.121E - 04$
$t = 1.00$	0.582597	$0.000E + 00$

$D = 100$		
	$H_0(t)$	$S^* H_0(t) - z_0(t)$
$t = 0.00$	0.534325	$-0.639E - 04$
$t = 0.25$	0.848073	$0.531E - 04$
$t = 0.50$	0.796741	$-0.511E - 04$
$t = 0.75$	0.564493	$-0.335E - 04$
$t = 1.00$	1.000000	$0.000E + 00$
	$H_8(t)$	$S^* H_8(t) - z_8(t)$
$t = 0.00$	0.984125	$-0.112E - 04$
$t = 0.25$	0.862619	$0.526E - 04$
$t = 0.50$	0.320507	$-0.561E - 05$
$t = 0.75$	-0.362338	$0.538E - 05$
$t = 1.00$	-0.880438	$0.000E + 00$

$D = 1000$		
	$H_0(t)$	$S^* H_0(t) - z_0(t)$
$t = 0.00$	0.623301	$-0.201E - 03$
$t = 0.25$	0.741808	$0.160E - 03$
$t = 0.50$	0.865530	$-0.379E - 03$
$t = 0.75$	0.962083	$-0.335E - 03$
$t = 1.00$	1.000000	$0.000E + 00$
	$H_8(t)$	$S^* H_8(t) - z_8(t)$
$t = 0.00$	0.866093	$0.429E - 06$
$t = 0.25$	0.930710	$-0.914E - 04$
$t = 0.50$	0.288906	$0.266E - 05$
$t = 0.75$	-0.570488	$0.146E - 03$
$t = 1.00$	-0.997954	$0.000E + 00$

Chapter 5

Numerical Solution of Problem P_N

In this chapter, we consider the numerical solution of problem P_N based on Theorem 3.0.4 for the example of the rotating Euler–Bernoulli beam.

The eigenvalues $\lambda_j = (x_j/l)^4$ are computed by Newton's method as described in Remark 2.3.1. Table 5.0.4 contains the numbers $x_1^2, ..., x_{10}^2$ that appear as factors in the frequencies in the trigonometric functions. Due to Lemma 2.3.2, we have the approximation $((2j-1)\pi/2)^2$ for x_j^2.

Table 5.0.4

x_1^2	3.51601526850015
x_2^2	22.0344915646668
x_3^2	61.6972144135491
x_4^2	120.901916052306
x_5^2	199.859530116803
x_6^2	298.555530967730
x_7^2	416.990786056605
x_8^2	555.165247555763
x_9^2	713.078917978976
x_{10}^2	890.731797198301

The fact that these numbers grow rather fast causes numerical difficulties, since for large values of D (i.e. for rather stiff beams), functions occur that oscillate wildly.

The eigenfunctions $\phi_j = \varphi_j/\sqrt{l}$ are computed as in equation (2.18).

The Fourier coefficients

$$a_j := \int_0^l y_0(x)\phi_j(x)\,dx,\ b_j := \int_0^l y_1(x)\phi_j(x)\,dx$$

are computed with a Newton–Cotes rule. For our implementation, we used the following notation.

$$\begin{aligned}
c_{-1} &= \psi_0, \\
c_0 &= -\psi_1, \\
c_j &= -0.5\lambda_{(j+1)/2}\,\sqrt{l}\,\sqrt{D}\,a_{(j+1)/2} \text{ for } j \text{ uneven}, \\
c_j &= 0.5\sqrt{\lambda_{j/2}}\,\sqrt{l}\,b_{j/2} \text{ for } j \text{ even}.
\end{aligned}$$

$$\begin{aligned}
z_{-1}(t) &= t, \\
z_0(t) &= 1, \\
z_j(t) &= \sin\left(\sqrt{\lambda_{(j+1)/2}D}\ t\right) \text{ for } j \text{ uneven}, \\
z_j(t) &= \cos\left(\sqrt{\lambda_{j/2}D}\ t\right) \text{ for } j \text{ even}.
\end{aligned}$$

With this notation, our moment equations have the form

$$\int_0^T u(t)z_j(t)\,dt = c_j. \tag{5.1}$$

We have to approximate the Gram matrix

$$\left(\int_0^T H_i(t)H_j(t)\,dt\right)_{ij}.$$

For this purpose, we start with the computation of the values

$$H_i(x_j),\ j \in \{0, ..., m\} \tag{5.2}$$

on an equidistant grid $x_j = j/m$, $j \in \{0, ..., m\}$ by the method described in Section 4.1.1.

For our numerical experiments, we have first solved the corresponding initial value problem (4.10), (4.11) with subroutine D02BBF from the NAG library which is an implementation of a Runge–Kutta method. For a stiffness check of the system we used subroutine D02BDF, which indicated that the system is not stiff.

Then and in our computations for time–optimal control presented in Chapter 7, we have used the trapezoidal method for the solution of the initial value problems as described at the end of Section 4.1.1 with a constant stepsize h. Unless stated otherwise, we have used the truncated kernel K_{100} and the stepsize $h = 2^{-12}$.

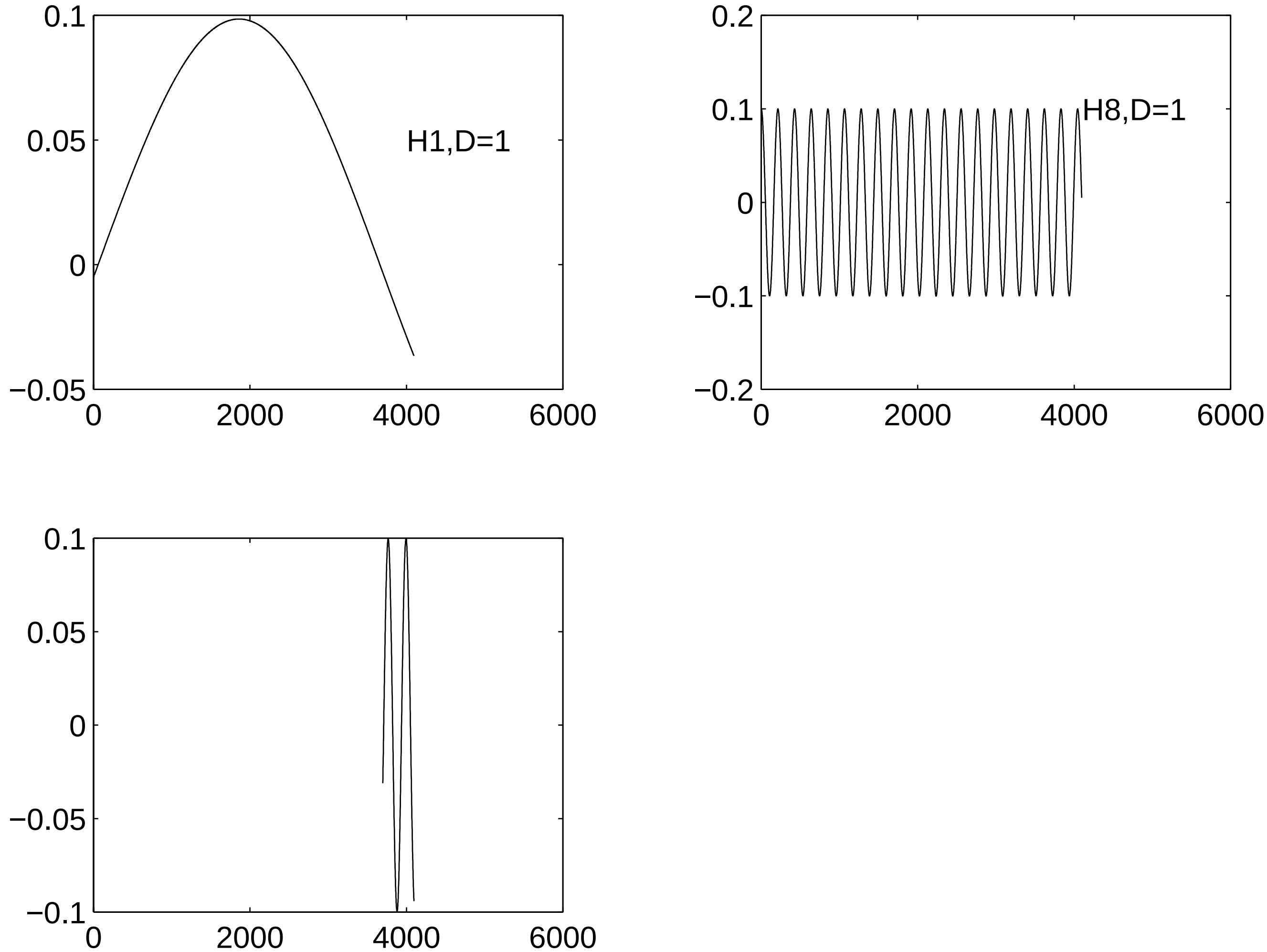


Figure 5.0.2 shows the graphs of the functions H_1 and H_8 for $l = 1$, $= 10$, $D = 1$ and $D = 1000$. The values where computed on an equidistant grid of 4096 points on the interval $[0, 1]$.

Figure 5.0.3

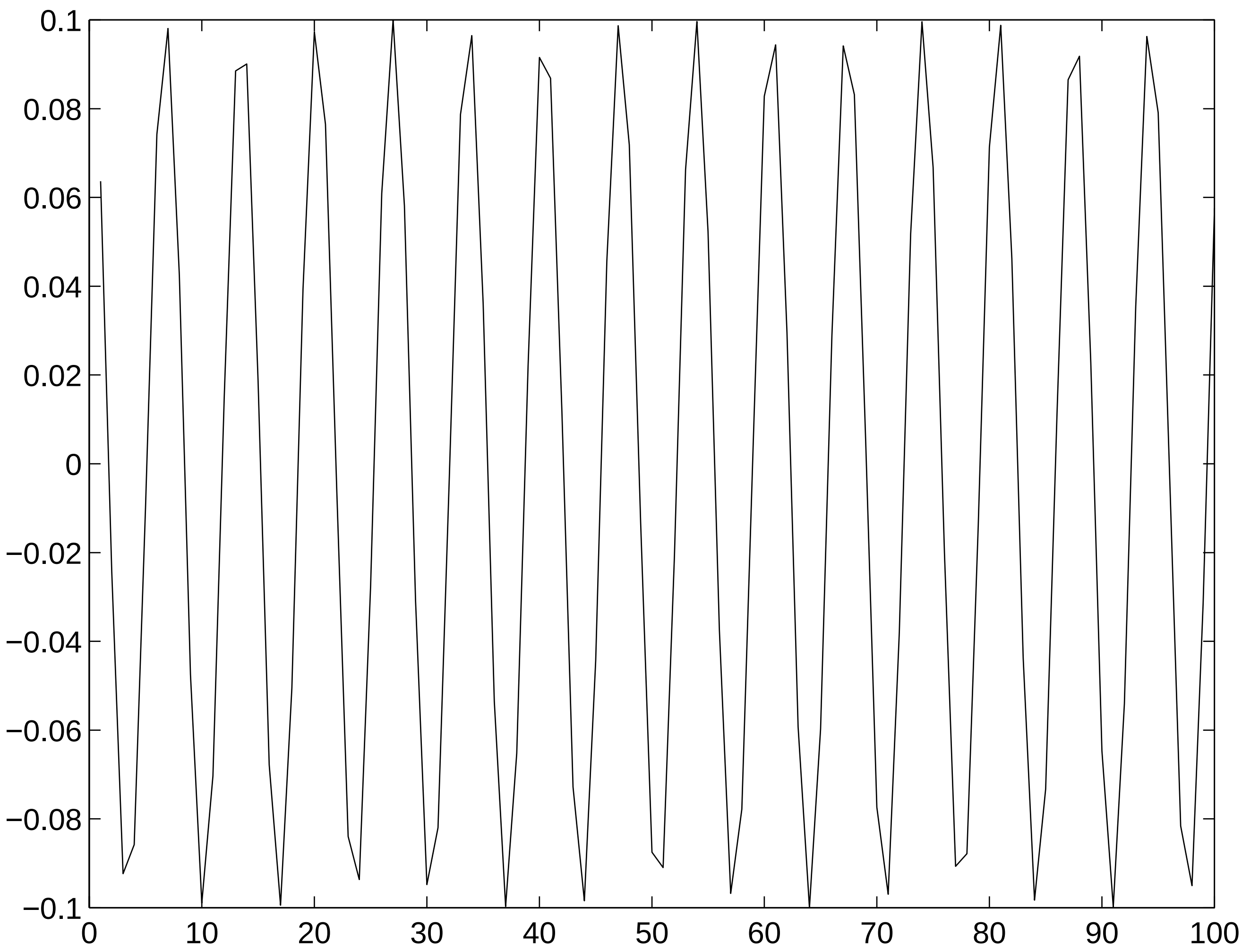


Figure 5.0.3. The first 100 grid points of H_8 for $D = 1000$.

The graph of H_8 for $D = 1000$ shown in Figure 5.0.2 illustrates that it is reasonable to use a very fine grid. It appears to deserve a closer inspection. Figure 5.0.3 shows the corresponding values for the first 100 grid points. The computed points of the graph are connected with straight lines.

With the computed approximations of the values (5.2), we computed the scalar products $\int_0^T H_i(t)H_j(t)\,dt$ with a Newton–Cotes formula. In our implementation, we have used a generalized Simpson rule. On account of the oscillations of the functions H_i we used a large number of grid points (e.g. $m = 4096$).

The linear System (3.2) for the coefficients η_j was solved with subroutine F04AMF from the NAG library, that uses a QR–decomposition. Our implementation is based on the program given in [48].

We used the initial data $\psi_0 = -2$, $\psi_1 = 0$, $y_0 = y_1 = 0$. So initially the beam is at rest at the angle -2 and we have $c_j = 0$ for $j > 0$. The end conditions were $\psi(T) = \psi'(T) = 0$. So we steer the beam to a position of rest at angle 0.

We used the data $T = 1$, $l = 1$ and $\quad = 10$.

We worked with the truncated kernel K_{100}. The values $H_i(x_j)$, $i \in \{-1, 0, 1, ..., 8\}$ were computed on an equidistant grid of 4096 points. We solved the problem

$$P_M : \quad \min \int_0^T (Su(t) - b(t))^2\,dt$$

$$s.t. \int_0^T u(t)z_j(t)\,dt = c_j,\ j \in \{-1, 0, 1, ..., M\}$$

for the values $M = 8$ and $D \in \{1, 10, 100, 1000\}$. The optimal value of P_M is denoted by $\omega(M)$.

Figure 5.0.4 shows the torque functions $\sum_{i=-1}^{M} \eta_i H_i(t)$ for the computed coefficients η_i. The functions were evaluated on an equidistant grid consisting of 4096 points.

Comparable computational results based on the method of moments have apparently not been given in the literature.

As the physical intuition suggests, the torque function looks much nicer if the stiffness of the beam is higher.

Figure 5.0.4

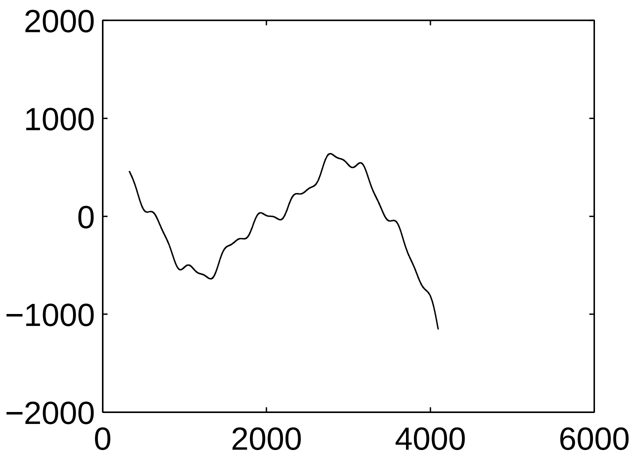


Table 5.0.5 contains the computed values $\omega(M)$ and the coefficients η of the computed solutions shown in Figure 5.0.4.

Table 5.0.5

	$D=1$	$D=10$
$\omega(8)$	207332	6009.18
η_{-1}	-103666.175179571	-3004.58945797609
η_0	51833.0875930414	1502.29472911265
η_1	-7387.45765146898	-341.906396775370
η_2	-38992.5646651368	-304.056798382331
η_3	-23.4998299900517	-61.5554967928653
η_4	-1085.98318319345	17.8441219573877
η_5	-577.027488395252	-24.2019121243438
η_6	-367.678732746883	3.96767012563677
η_7	-170.881646232623	-10.3312672492900
η_8	162.642036446255	-5.30577791657202

	$D=100$	$D=1000$
$\omega(8)$	5161.4080	5135.2056
η_{-1}	-2580.70401126334	-2567.60280778742
η_0	1290.35200572575	1283.80140389871
η_1	-25.6857050234166	-25.9077843943739
η_2	-113.613699310472	-55.7137423930246
η_3	-19.8970650795290	-6.86843851751824
η_4	4.39266402485629	-2.31573968024444
η_5	-5.00963386143109	$-1.553989245281026E-2$
η_6	3.50273269829957	-0.409739976397877
η_7	-0.211644877802561	$1.834163567310364E-3$
η_8	0.840416830934177	$-2.960031395761198E-2$

We continue our parameter studies by choosing a smaller value of , namely $= 1$. The other parameters are $l = 1$, $T = 1$, $M = 8$. Figure 5.0.5 shows the value of the torque functions for $D = 1$, 10, 100, 1000 computed on a grid of 4096 equidistant points.

Figure 5.0.5

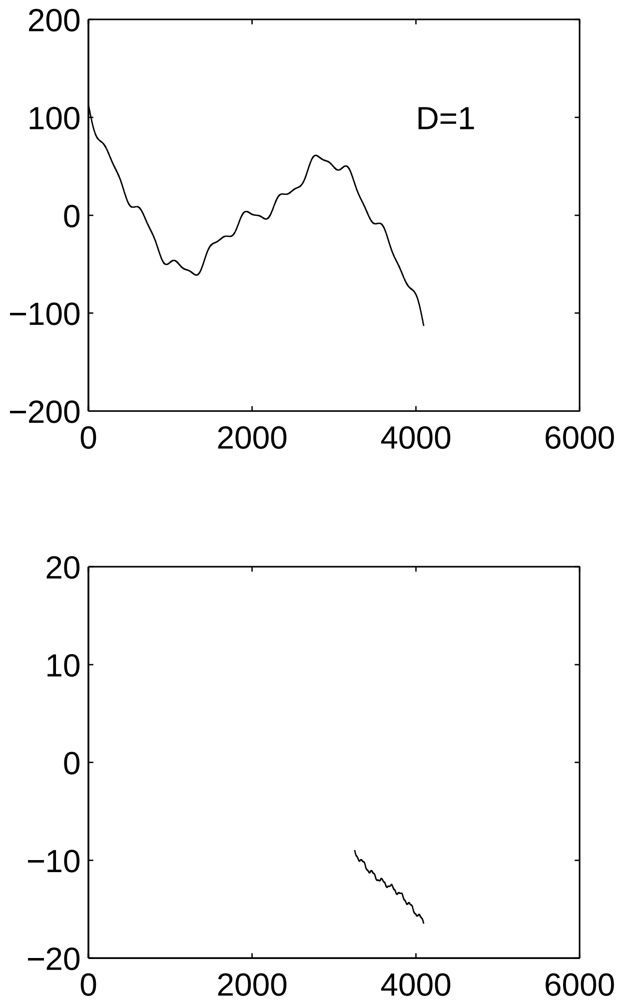


Figure 5.0.6

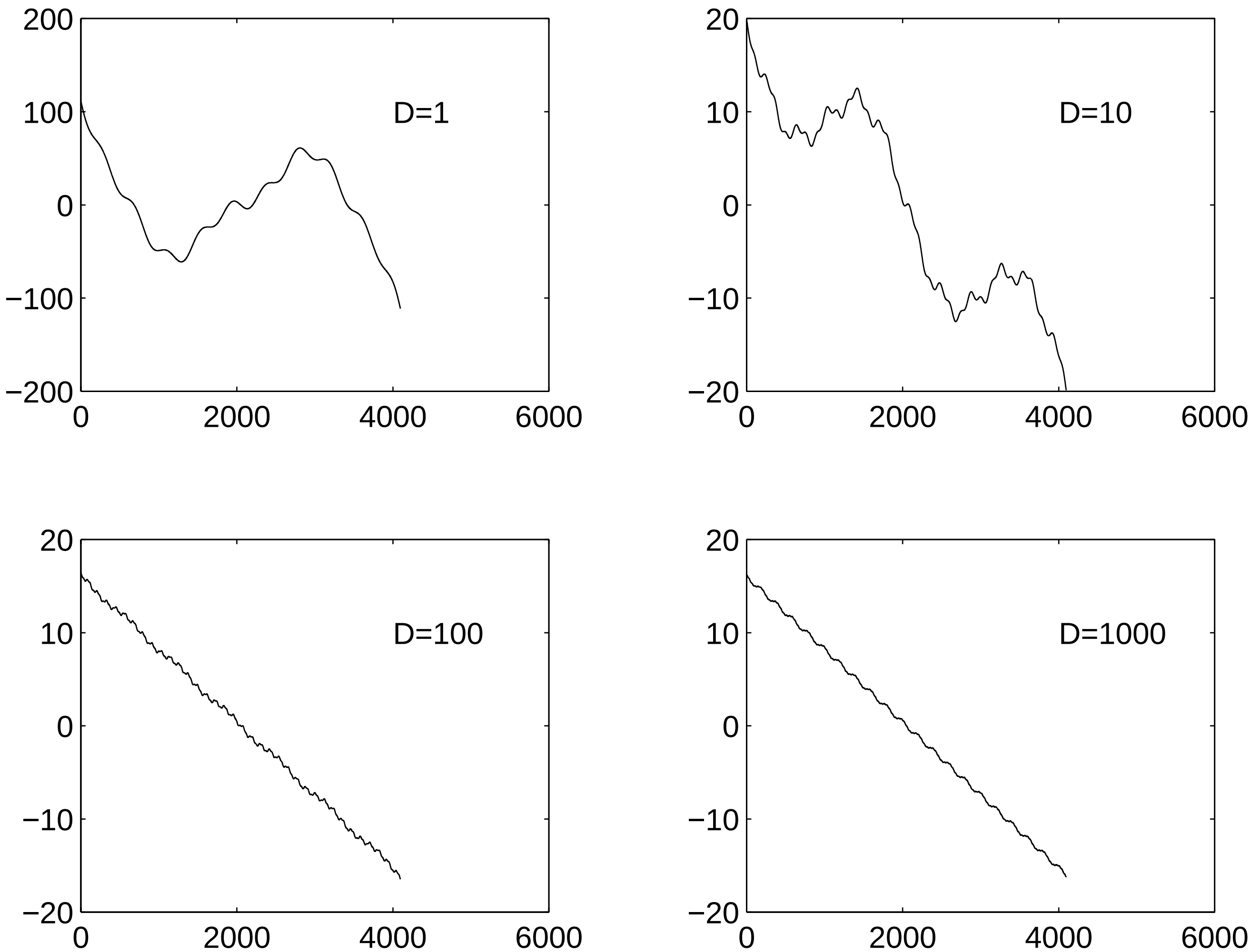


Figure 5.0.6. The Graph of the torque functions on $[0, 1]$ for $\quad = 1$ and $M = 6$

Table 5.0.6 contains the computed values $\omega(M)$ and the coefficients η of the computed torque functions $\sum_{i=-1}^{M} \eta_i H_i$ shown in Figures 5.0.5 and 5.0.6.

Table 5.0.6

	$D = 1$	
	$M = 6$	$M = 8$
$\omega(M)$	1953.70223	1956.26899
η_{-1}	-976.851116	-978.134493
η_0	488.425558	489.067227
η_1	-69.5594830	-69.6424391
η_2	-361.318889	-361.757994
η_3	-0.216388267	-0.215504078
η_4	-11.7394866	-11.7031530
η_5	-5.73562417	-5.69064187
η_6	-3.65546229	-3.62682958
η_7		-1.64946679
η_8		1.57022874

	$D = 10$	
	$M = 6$	$M = 8$
$\omega(M)$	100.124008	100.142664
η_{-1}	-50.0620038	-50.0713322
η_0	25.0310037	25.0356670
η_1	-6.11269044	-6.11406356
η_2	-4.99746687	-5.00032853
η_3	-0.876674650	-0.877990826
η_4	0.264965246	0.265310299
η_5	-0.383831551	-0.385099796
η_6	$6.319181767E-2$	$6.339439690E-2$
η_7		-0.171560367
η_8		$-8.816897422E-2$

	$D = 100$	
	$M = 6$	$M = 8$
$\omega(M)$	85.3989837	85.3989925
η_{-1}	-42.6994919	-42.6994963
η_0	21.3497455	21.3497469
η_1	-2.95876032	-2.95874899
η_2	-3.92199904	-3.92200512
η_3	-0.165040869	-0.165041769
η_4	$6.751015501E - 2$	$6.750941322E - 002$
η_5	$-5.201929151E - 2$	$-5.201631680E - 002$
η_6	$3.862737539E - 2$	$3.862424872E - 002$
η_7		$-6.697841442E - 004$
η_8		$4.146153767E - 003$

	$D = 1000$	
	$M = 6$	$M = 8$
$\omega(M)$	85.3640599	85.3640944
η_{-1}	-42.6820300	-42.6820472
η_0	21.3410155	21.3410233
η_1	-4.66253217	-4.66252954
η_2	-1.70433837	-1.70434809
η_3	$-7.206619944E - 2$	$-7.206734396E - 2$
η_4	$6.832755733E - 2$	$6.832426489E - 2$
η_5	$6.074123826E - 3$	$6.074410399E - 3$
η_6	$-3.305413956E - 2$	$-3.305523040E - 2$
η_7		$1.408419035E - 3$
η_8		$-8.173360917E - 3$

If the initial data are changed to $\psi_0 = -4$, $\psi_1 = 0$, and the other parameters are chosen as before as $= 1$, $l = 1$, $T = 1$, $M = 6$, the coefficients η_i of the torque functions with minimal norm are twice the coefficients of the solutions corresponding to the initial data $\psi_0 = -2$, and the optimal value is four times the optimal value corresponding to the initial data $\psi_0 = -2$.

More generally, with $y_0 = y_1 = 0$ if the initial data are changed from ψ_0, ψ_1 to $\lambda\psi_0$, $\lambda\psi_1$, the coefficients of the optimal torque functions are also multiplied with λ.

It is interesting to compare the torque functions shown in Figure 5.0.5 with the torque functions $Su - b$ generated by control functions u that solve the problem

$$\min \int_0^1 u(t)^2 \, dt$$
$$s.t. \int_0^T u(t) z_j(t) \, dt = c_j, \; j \in \{-1, 0, 1, ..., M\}.$$

Let $\nu(M)$ denote the value of this problem. This problem has an objective function of a simpler structure than problem P_M and can be solved much easier numerically, since the costly computation of the functions H_j is not necessary.

Figure 5.0.7

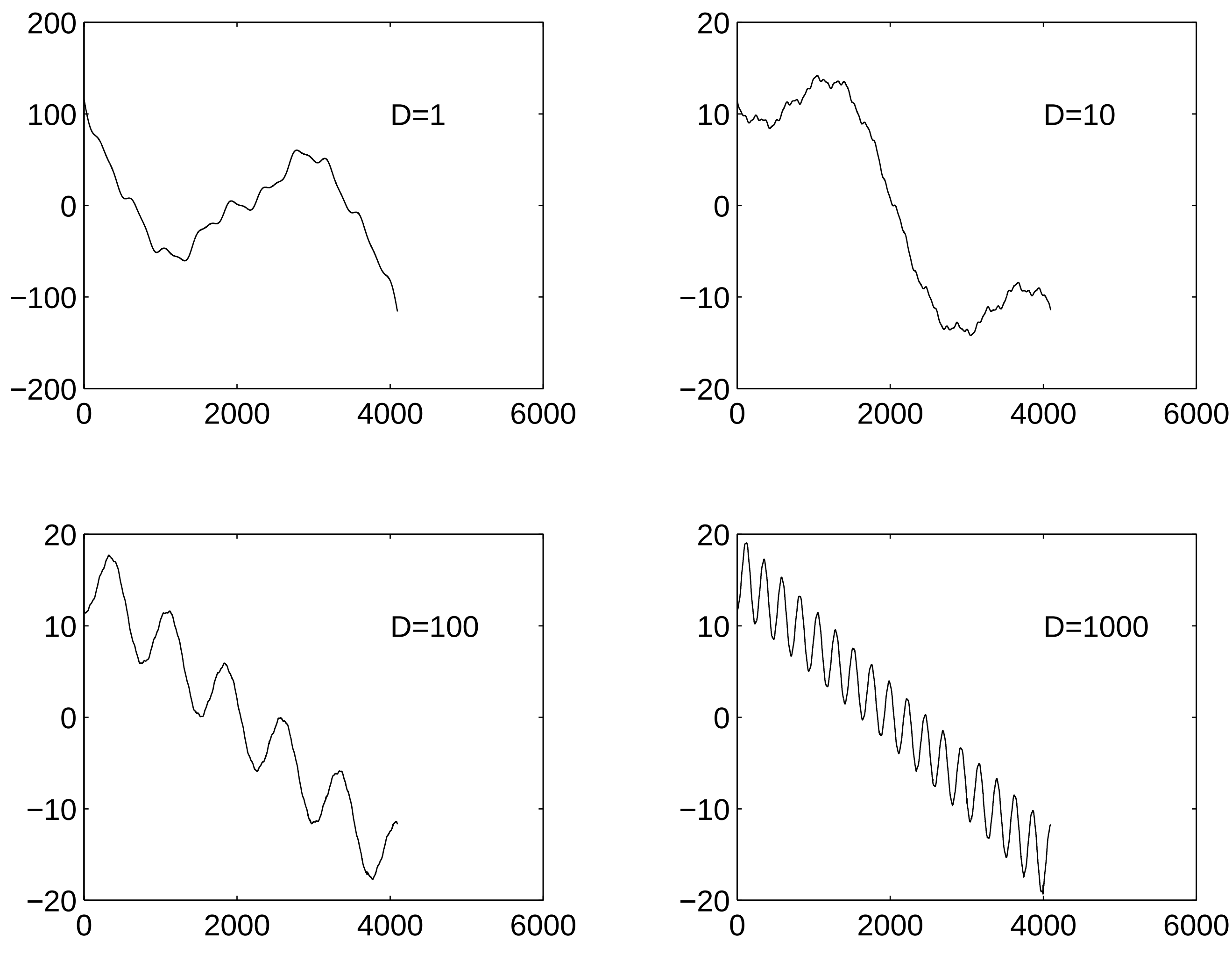


Figure 5.0.7. The torque functions for angular acceleration with minimal norm on $[0, 1]$.

Figure 5.0.7 shows the torque functions for the same data that were used for Figure 5.0.5. It illustrates that in a way the complicated objective function $\|Su - b\|^2$ causes a regularization with respect to the torque: If we compute a control which yields a torque with minimal norm, we obtain torque functions that have small amplitudes as in Figure 5.0.5, in particular for large values of D.

In contrast to this situation, the amplitudes of the torque functions in Figure 5.0.7 increase with D. In particular the oscillations of the torque function for $D = 1000$ are quite large, the amplitudes are much larger than the amplitudes of the oscillations the torque function with minimal norm.

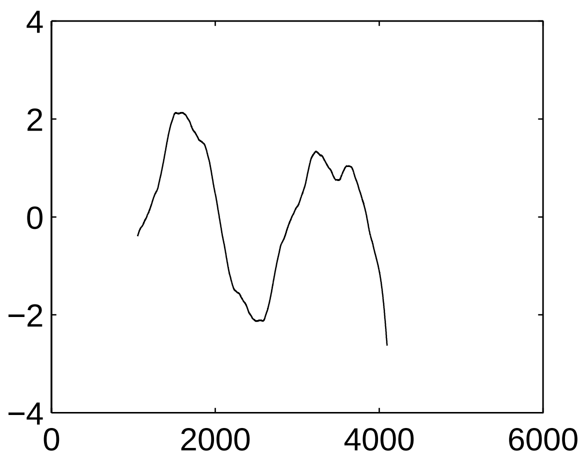
Table 5.0.7 contains the computed values $\nu(M)$ and the coefficients η of the computed angular accelerations $u(t) = \sum_{i=-1}^{M} \eta_i z_i(t)$, whose torque functions are shown in Figure 5.0.7.

Table 5.0.7

	$D = 1$	$D = 10$
$\nu(8)$	2087.16098	55.6682732
η_{-1}	-1043.58049	-27.8341366
η_0	521.790244	13.9170683
η_1	-74.4871941	-2.97140438
η_2	-393.218466	-2.62657523
η_3	-0.233322869	-0.586711805
η_4	-10.7646631	0.170027871
η_5	-5.77927821	-0.228272220
η_6	-3.68251580	$3.742168640E-2$
η_7	-1.71553359	$-9.693928866E-2$
η_8	1.63281080	$-4.978427570E-2$
	$D = 100$	$D = 1000$
$\nu(8)$	48.1362100	48.0344568
η_{-1}	-24.0681050	-24.0172284
η_0	12.0340525	12.0086142
η_1	-0.139243278	-0.146157153
η_2	-0.448045127	-0.206815186
η_3	-0.200262182	$-6.103085798E-2$
η_4	$4.409224883E-2$	$-2.028476554E-2$
η_5	$-5.035384095E-2$	$-7.565543740E-5$
η_6	$3.519265082E-2$	$-1.397195681E-3$
η_7	$-2.235204329E-3$	$-1.193747558E-5$
η_8	$8.864417848E-3$	$3.730507036E-4$

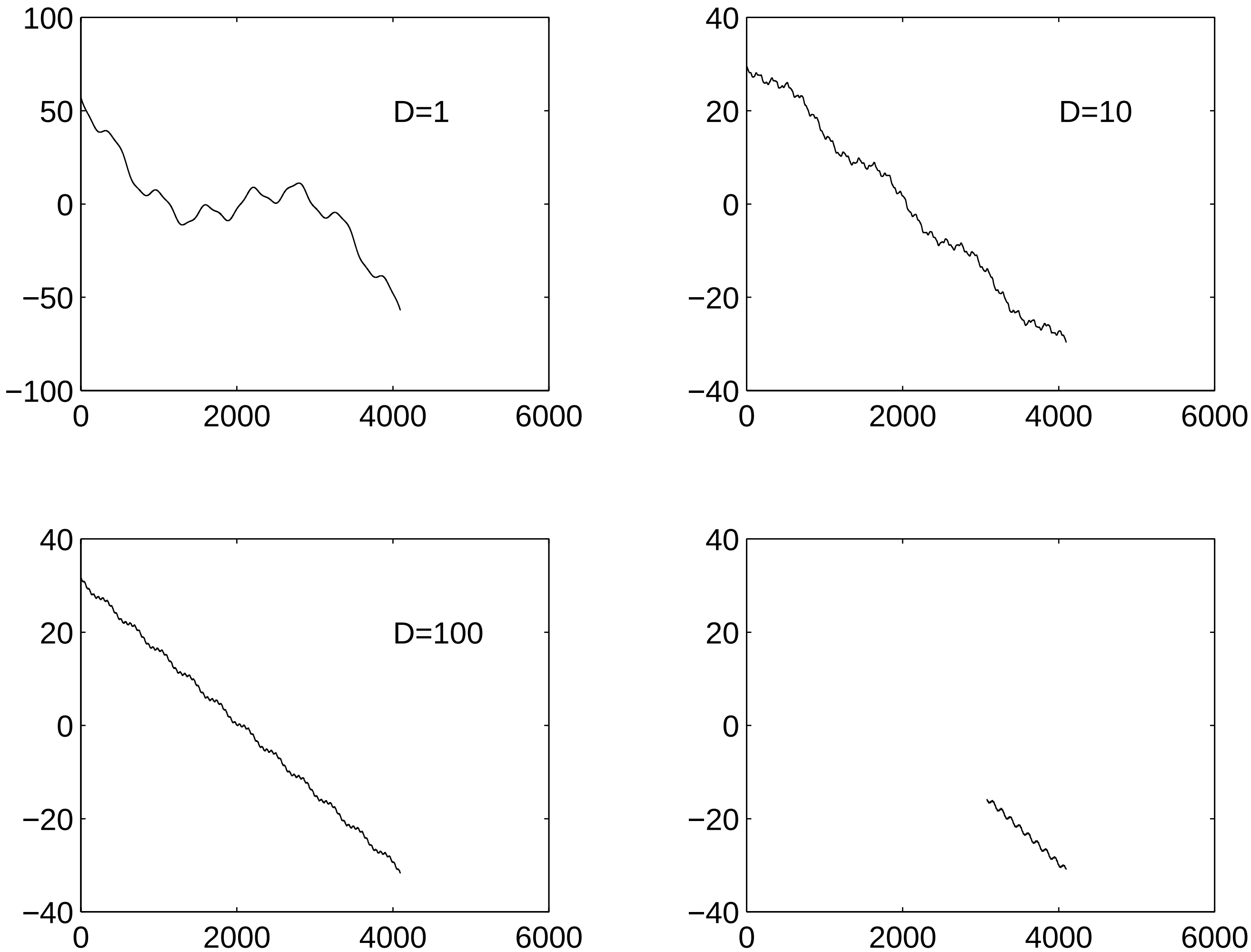
Figure 5.0.8 contains the graphs of the differences of the torque functions with minimal norm shown in Figure 5.0.5 and the torque functions shown in Figure 5.0.7. The number of oscillations of the difference functions increases with D. The maximal absolute value of the differences for $D = 100$ and $D = 1000$ is very similar.

Figure 5.0.8



rameters are chosen as $l = 1$, $\quad = 10$, $M = 8$. Figure 5.0.9 shows the value of the torque functions for $D = 1, 10, 100, 1000$ computed on a grid of 4096 equidistant points.

Figure 5.0.9



Except for the value of , we use the same data as for Figure 5.0.4. Figure 5.0.10 shows the graphs of the torque functions for $= 100$, $l = 1$, $T = 1$, $M = 8$, $N = 100$, $h = 2^{-12}$.

Figure 5.0.10

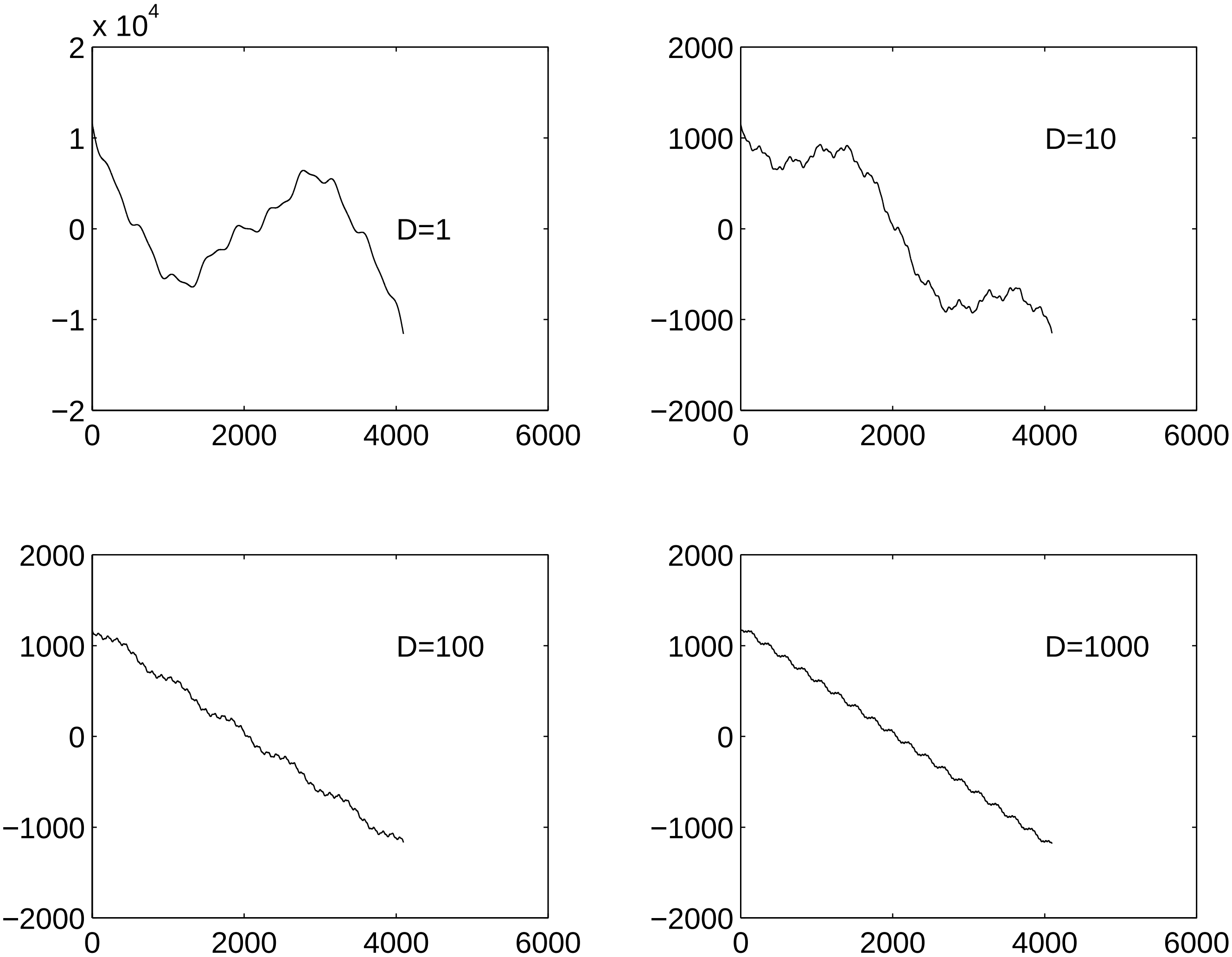


Figure 5.0.10. The Graph of the torque functions on $[0, 1]$ for $= 100$.

Now we investigate how the optimal value of Problem P_M changes, if the number N of terms in the truncated series for the kernel K is changed. We use the data $l = 1$, $T = 1$, $M = 8$, $h = 2^{-14}$.

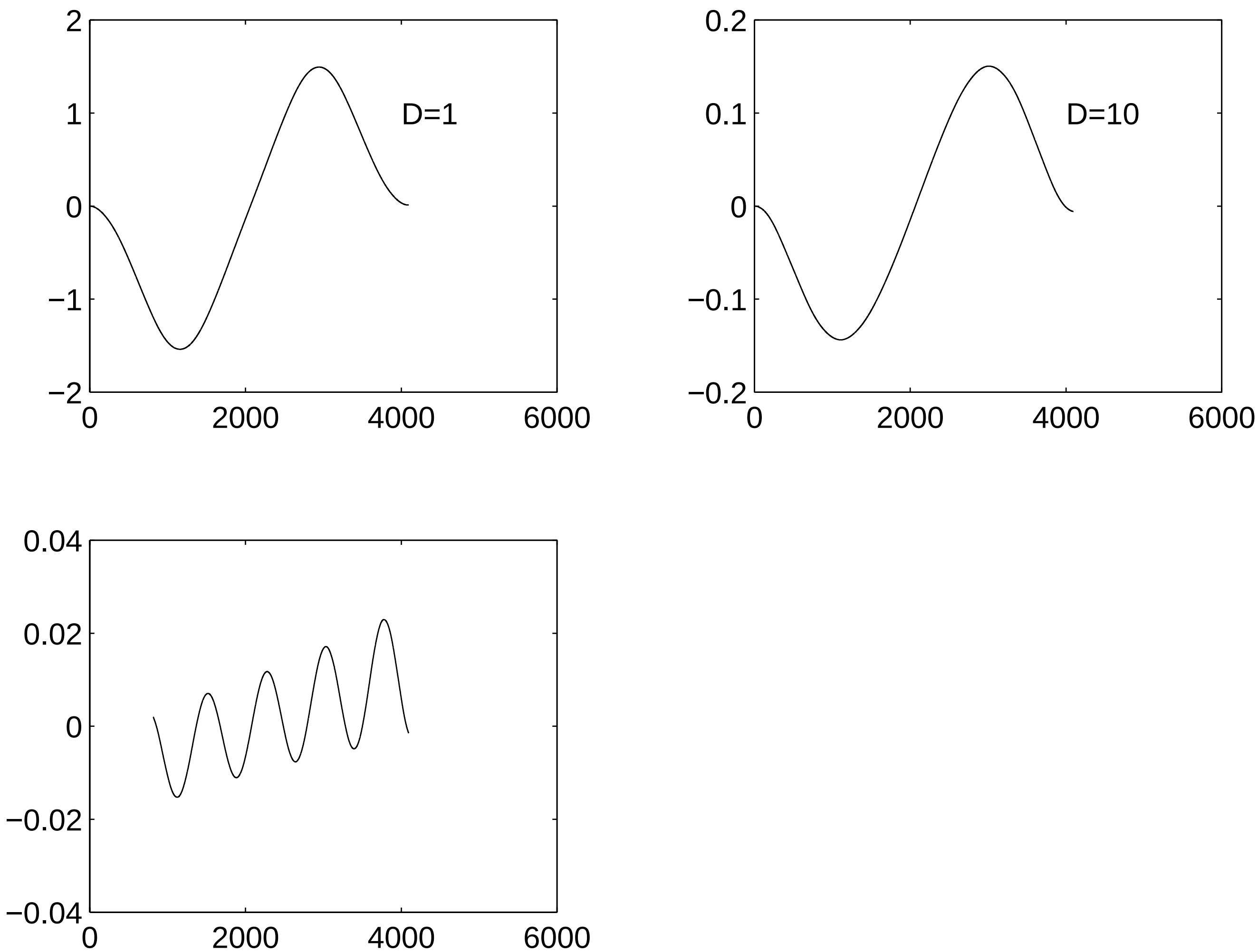
Table 5.0.8

$= 1$				
	$D = 1$	$D = 10$	$D = 100$	$D = 1000$
$N = 1$	1908	98.67	84.16	84.11
$N = 2$	1950	99.90	85.20	85.17
$N = 3$	1954	100.07	85.34	85.30
$N = 4$	1955	100.11	85.37	85.34
$N = 5$	1956	100.13	85.39	85.35
$N = 100$	1956	100.14	85.40	85.36

$= 10$				
	$D = 1$	$D = 10$	$D = 100$	$D = 1000$
$N = 1$	206832	5998	5152	5125
$N = 2$	207266	6007	5160	5133
$N = 3$	207313	6009	5161	5135
$N = 100$	207332	6009	5164	5135

Note that the numbers in Table 5.0.8 increase with N.

The following figure shows the graph of the function $y(1,t)$, $t \in [0,1]$ for $\quad = 10$, $l = 1$, $T = 1$, $M = 8$, $h = 2^{-12}$ and $N = 100$ for the angular acceleration corresponding to torque function with minimal norm shown in Figure 5.0.4. This graph describes the deflection of the tip of the beam.



For $D = 1$ and $D = 10$, the movement has a simple structure. The beam moves once to the right–hand side and once to the left–hand side. The first movement is due to the acceleration and the second part of the movement is due to the negative acceleration in the second half of the time–interval.

For $D = 100$ and $D = 1000$, the movement is much more complicated, since the tip of the beam oscillates more often. Note that the maximal amplitudes occur at the beginning and at the end of the movement.

Chapter 6

Optimal Value Functions

In Chapter 5 we have considered the numerical solution of Problem P_N for a fixed steering time. For the example of the Euler–Bernoulli beam, Problem P_N is the problem of the computation of a control function for which the corresponding torque function has minimal L^2–norm and which satisfies a truncated system of moment equations. The torque function is given by the image of the control function under a Volterra operator.

Problem P_N appeared as a discretization of Problem P_∞. Theorem 3.0.5 guarantees the strong convergence of the solutions of P_N to the solution of P_∞ for $N \to \infty$. For all these problems, one fixed Hilbert space is considered, which corresponds to the fact that the steering time T is fixed.

To check the stability of the model, it is useful to examine the behaviour of the optimal value as a function of T. This sensitivity analysis is particularly important since it is related to problems of time–optimal control.

In this chapter, we examine the behaviour of the optimal value functions of P_N and P_∞. We also analyse how the sequence of optimal value functions of the Problems P_N converges to the optimal value function of P_∞.

For Problem P_∞ with the complete countable set of moment equations as equality constraints, we show that the optimal value is a continuous function of the steering time T. So we consider P_∞ as a time–parametric program. A particular difficulty in the analysis of such time–parametric programs is due to the fact that in a natural way not only one fixed Hilbert space but a whole parametric family of spaces depending on the controlling time occurs.

To prove the continuity of the value function of P_∞, we use the following scheme which is used in parametric optimization (see [13]). First we show that the solutions of P_∞ are uniformly bounded with respect to T. We use this fact to prove that the value function is lower semicontinuous. Then we introduce a dual problem and show that the dual solutions are also uniformly bounded. This fact allows us to prove that the value function is upper semicontinuous.

In a similar way, we can show that the value function of Problem P_N is continuous.

Using Dini's Theorem, we show that on a given time–interval, the sequence of optimal value functions corresponding to the problems P_N converges uniformly to the optimal value function of the original problem. This result is important for the stability of the numerical approach via the moment equations. It guarantees that for a given accuracy, a truncation level N exists that allows to approximate the optimal value function of problem P_∞ by the optimal value function of problem P_N on a whole time–interval with that accuracy.

Our problem differs from the standard minimum norm problem because instead of the L^2–norm of the control function we allow for a more general objective function which is given by the L^2–norm of the image of the control function under an affine linear map.

The numerical examples in Chapter 5 have illustrated that for the rotating Euler–Bernoulli beam, this approach makes sense since minimizing the L^2–norm of the torque instead of the L^2–norm of the angular acceleration yields controls for which the corresponding torque functions look much nicer (see Figures 5.0.4 and 5.0.7).

Due to the generality of the objective function, the analysis is more complicated than for the standard minimum norm problem. A transformation of our objective function to the norm as the standard objective function yields a problem that differs from the standard minimum norm problem because the right—hand side of the moment equations depends on the time–parameter T. This means that the results that are given in [23] for the standard minimum norm problem are not applicable. For example, for the objective function considered here in general the optimal value function is not necessarily decreasing. Hence the results presented here are original and cannot be obtained as conclusions from the results about the standard minimum norm problem.

If the objective function of Problem P_N is given by a Volterra–

operator, we can show that the corresponding optimal value function is continuously differentiable. Moreover, we give a formula that shows how the derivative can be computed. This formula is important for numerical purposes, since it allows to apply Newton's method to compute a time where the optimal value function attains a given prescribed value. In this way, certain problems of time–optimal control can be solved (see Chapter 7).

The tool that we use to prove the differentiability is the implicit function theorem. It also yields the result that the coefficients in the linear combination that represents the optimal control function are continuously differentiable functions of the steering time.

For the example of the Euler–Bernoulli beam with the L^2–norm of the torque function as objective function of Problem P_N and P_∞, we can show that the corresponding optimal value functions are decreasing. This result also holds for the standard minimum norm problem. Krabs uses the monotonicity in his proof of con

For $u \in Z(0,\overline{T})$, instead of $\|u|_{[\min\{T_1,T_2\},\max\{T_1,T_2\}]}\|_{(T_1,T_2)}$ we write $\|u\|_{(T_1,T_2)}$; analogously, for $u, v \in Z(0,\overline{T})$ we use the notation $\langle u, v\rangle_{(T_1,T_2)}$.

For our analysis it is essential, that we do not work in only one space, but use a whole (time–)parametric family of spaces.

For all $T \in (0,\overline{T}]$, let $S_T : Z(0,T) \to Z(0,T)$ be a continuous linear map that is bijective and for which the following equality holds for all $u \in Z(0,\overline{T})$:

$$S_T(u|_{[0,T]}) = (S_{\overline{T}}u)|_{[0,T]}. \tag{6.1}$$

As an example for S_T consider the Volterra operator with a constant > 0 and kernel $K \in C(0,\overline{T})$:

$$(S_T u)(t) = \quad u(t) - \int_0^t K(t-s)u(s)\,ds. \tag{6.2}$$

The adjoint operators of S_T and S_T^{-1} are denoted by S_T^*, $(S_T^{-1})^*$ respectively. In example (6.2) we have

$$(S_T^* u)(t) = \quad u(t) - \int_t^T K(s-t)u(s)\,ds.$$

Lemma 6.1.1 *For all $T \in [0,\overline{T}]$ for all $y \in Z(0,\overline{T})$ we have*

$$S_T^{-1}(y|_{[0,T]}) = (S_{\overline{T}}^{-1}y)|_{[0,T]}.$$

Moreover $\|S_T^*\| = \|S_T\| \leq \|S_{\overline{T}}\|$ and $\|(S_T^{-1})^*\| = \|S_T^{-1}\| \leq \|S_{\overline{T}}^{-1}\|$.

Proof. Let $y \in Z(0,\overline{T})$, $T \in [0,\overline{T}]$. Let $u_1 = (S_{\overline{T}}^{-1}y)|_{[0,T]}$ and $u_2 = S_T^{-1}(y|_{[0,T]})$.

Then $S_T u_2 = y|_{[0,T]}$ and (6.1) implies

$$S_T u_1 = S_T((S_{\overline{T}}^{-1}y)|_{[0,T]}) = (S_{\overline{T}}(S_{\overline{T}}^{-1}y))|_{[0,T]} = y|_{[0,T]} = S_T u_2.$$

Hence $u_1 = u_2$.

For $u \in Z(0,T)$, define $\tilde{u} \in Z(0,\overline{T})$ by $\tilde{u}|_{[0,T]} := u$, $\tilde{u}|_{(T,\overline{T}]} = 0$. Then $\|u\|_{(0,T)} = \|\tilde{u}\|_{[0,\overline{T})}$. Hence (6.1) implies $\|S_T u\|_{(0,T)} = \|S_{\overline{T}}\tilde{u}\|_{(0,T)} \leq \|S_{\overline{T}}\tilde{u}\|_{(0,\overline{T})} \leq \|S_{\overline{T}}\|\|\tilde{u}\|_{(0,\overline{T})} = \|S_{\overline{T}}\|\,\|u\|_{(0,T)}$. Thus $\|S_T\| \leq \|S_{\overline{T}}\|$. The inequality $\|S_T^{-1}\| \leq \|S_{\overline{T}}^{-1}\|$ follows analogously. The equality $\|S_T\| = \|S_T^*\|$ is always valid (see [37] p. 90). □

We assume that for all $u \in Z(0,\overline{T})$ and $T_j \in (0,\overline{T}]$ ($j \in I\!N$) with $\lim_{j\to\infty} T_j = T$ we have

$$\lim_{j\to\infty} \left\| \left(S^*_{T_j}(u(\cdot\,\overline{T}/T_j))\right)(\cdot\,T_j/\overline{T}) - \left(S^*_T(u(\cdot\,\overline{T}/T))\right)(\cdot\,T/\overline{T})\right\|_{(0,\overline{T})} = 0. \tag{6.3}$$

For the example of the Volterra operator in (6.2) we have for all $u \in Z(0,\overline{T})$ and $t \in [0,T]$

$$\begin{aligned}\left(S^*_T(u(\cdot\,\overline{T}/T))\right)(t) = &\ u(t\overline{T}/T) - \int_t^T K(s-t)u(s\overline{T}/T)\,ds \\ = &\ u(t\overline{T}/T) - \frac{T}{\overline{T}}\int_{t\overline{T}/T}^{\overline{T}} K(xT/\overline{T}-t)u(x)\,dx\end{aligned}$$

hence we conclude that for all $y \in [0,\overline{T}]$ we have

$$\left(S^*_T(u(\cdot\,\overline{T}/T))\right)(yT/\overline{T}) = \ u(y) - \frac{T}{\overline{T}}\int_y^{\overline{T}} K((x-y)T/\overline{T})u(x)\,dx.$$

Let

$$\begin{aligned}D_j(y) &= \left(S^*_{T_j}(u(\cdot\,\overline{T}/T_j))\right)(\cdot\,T_j/\overline{T}) - \left(S^*_T(u(\cdot\,\overline{T}/T))\right)(\cdot\,T/\overline{T}) \\ &= \int_y^{\overline{T}} \left[\frac{T}{\overline{T}}K((x-y)\frac{T}{\overline{T}}) - \frac{T_j}{\overline{T}}K((x-y)\frac{T_j}{\overline{T}})\right] u(x)\,dx.\end{aligned}$$

Then Hölder's inequality implies that

$$D_j(y)^2 \le \int_y^{\overline{T}} (u(x))^2\,dx \int_y^{\overline{T}} \left[\frac{T}{\overline{T}}K((x-y)\frac{T}{\overline{T}}) - \frac{T_j}{\overline{T}}K((x-y)\frac{T_j}{\overline{T}})\right]^2 dx.$$

Since the function K is uniformly continuous on $[0,\overline{T}]$, $\lim_{j\to\infty} T_j = T$ implies that

$$\max_{t\in[0,\overline{T}]} \left|\frac{T}{\overline{T}}K(t\frac{T}{\overline{T}}) - \frac{T_j}{\overline{T}}K(t\frac{T_j}{\overline{T}})\right| \to 0 \ \ (j\to\infty)$$

hence

$$\lim_{j\to\infty}\int_0^{\overline{T}} D_j(y)^2\,dy = 0,$$

thus (6.3) is valid.

Let $(z_j)_{j\in I\!N} \in (Z(0,\overline{T}))^{I\!N}$ be a sequence of functions, $b \in Z(0,\overline{T})$ and $c \in l^2$.

For $\alpha \in I\!R \cup \{\infty\}$ and $e \in l^2$ we define the set

$$U(T,\alpha,e)$$

$$= \{u \in Z(0,T) : \|S_T u - b\|^2_{(0,T)} \le \alpha^2 \text{ and } \langle u, z_j\rangle_{(0,T)} = e_j \text{ for all } j \in I\!N\}.$$

We make the following assumptions:

A0 A number $\beta \in I\!R$ is given such that the set $U(\overline{T},\beta,c)$ is nonempty.

A1 There exist constants $\underline{T}$, M, $P > 0$ such that for all $N \in I\!N$, $(a_1, \ldots, a_N) \in I\!R^N$ we have

$$\begin{aligned}(1/M)(\sum_{i=1}^N a_i^2)^{1/2} &\le \|\sum_{i=1}^N a_i z_i\|_{(0,\underline{T})}\\ &\le \|\sum_{i=1}^N a_i z_i\|_{(0,\overline{T})}\\ &\le P(\sum_{i=1}^N a_i^2)^{1/2}.\end{aligned}$$

For trigonometric moment problems, the validity of the inequality in A1 can sometimes be verified with the help of a result of Ingham (see [16]). Usually (e.g. in [47], Lemma 4.1, p.120 and [22] (II.2.11)) in the theory of moment problems a similar inequality for one fixed space is considered; in contrast to the present work the parametric aspect is not taken into account. Condition A1 is equivalent to the statement that for all $T \in [\underline{T}, \overline{T}]$, the functions z_i form a Riesz–basis of the closure of their linear span. This means that the family $(z_i)_{i\in I\!N}$ is isomorphic to an orthonormal basis of the closure of its linear span (see [3], p.26 or [50], p.30). Condition A1 is also equivalent to the statement that for all $T \in [\underline{T}, \overline{T}]$, the Gram–matrix

$$\left(\langle z_i, z_j\rangle_{(0,T)}\right)_{i,j=1}^{\infty}$$

generates a linear bounded invertible operator on l^2. Riesz–bases can also be characterized in terms of biorthogonal sequences (see [50], Theorem 9, p. 32).

Using Lemma 6.1.1, it is easy to prove that Assumption A1 implies the following Lemma.

Lemma 6.1.2 *Let* $\hat{M} = M\,\|S_{\overline{T}}\|$, $\hat{P} = P\|S_{\overline{T}}^{-1}\|$. *For* $T \in [\underline{T}, \overline{T}]$, $j \in \mathbb{N}$ *define* $H_j(T) = (S_T^*)^{-1} z_j$. *Then for all* $N \in \mathbb{N}$, $(a_1, ..., a_N) \in \mathbb{R}^N$, $T \in [\underline{T}, \overline{T}]$ *we have*

$$(1/\hat{M})(\sum_{i=1}^{N} a_i^2)^{1/2} \le \| \sum_{i=1}^{N} a_i H_i(T) \|_{(0,T)} \le \hat{P}(\sum_{i=1}^{N} a_i^2)^{1/2}.$$

The assertion of Lemma 6.1.2 means that the sequence $(H_j(T))_{j\in\mathbb{N}}$ is a Bessel sequence with bound $\hat{P}^2$ and a Riesz–Fischer sequence with bound $1/\hat{M}^2$ (see Theorem 3.0.2).

6.1.2 The Problem

For $T \in [\underline{T}, \overline{T}]$ define the parametric optimization problem $P_\infty(T)$:

$$\min \|S_T u - b\|_{(0,T)}^2 \;\text{ s.t. }\; \langle u, z_j \rangle_{(0,T)} = c_j \;\text{ for all } j \in \mathbb{N}.$$

Let $\omega(T)$ denote the value of $P_\infty(T)$.

Note that in the theory of moment problems (e.g. in [47]), usually instead of $\|S_T u - b\|_{(0,T)}^2$ the objective function $\|u\|_{(0,T)}^2$ is considered that yields so called normal solutions.

6.1.3 The Discretized Problem

Since $P_\infty(T)$ has an infinite number of equality constraints, for numerical purposes it is necessary to examine a discretized problem $P_N(T)$, where only the first N equality constraints of problem $P_\infty(T)$ are considered.

For $T \in [\underline{T}, \overline{T}]$, $N \in \mathbb{N}$ define the parametric optimization problem $P_N(T)$:

$$\min \|S_T u - b\|_{(0,T)}^2 \;\text{ s.t. }\; \langle u, z_j \rangle_{(0,T)} = c_j \;\text{ for all } j \in \{1, ..., N\}.$$

Let $\omega_N(T)$ denote the value of $P_N(T)$. Then for all $T \in [\underline{T}, \overline{T}]$, we have $\omega_{N+1}(T) \ge \omega_N(T)$.

The solution of problem $P_N(T)$ is characterized in Theorem 3.0.4 which is restated here with the notation of the parametric case.

Lemma 6.1.3 *Let* $T \in [\underline{T}, \overline{T}]$, $N \in \mathbb{N}$. *For* $j \in \{1, ..., N\}$, *define* $H_j(T) = (S_T^*)^{-1} z_j$. *Define* $\eta_N(T) = (\eta_i^N(T))_{i=1}^N \in \mathbb{R}^N$ *as the solution* *of the linear system*

$$\Big(\langle H_i(T), H_j(T)\rangle_{(0,T)}\Big)_{i,j=1}^{N} \eta_N(T) = \Big(c_i - \langle b, H_i(T)\rangle_{(0,T)}\Big)_{i=1}^{N}.$$

Then $u_N(T) = S_T^{-1}(\sum_{i=1}^N \eta_i^N(T)H_i(T) + b)$ *is the unique solution of* $P_N(T)$.

6.1.4 Solvability of problem $P_\infty(T)$

To analyse the solvability of problem $P_\infty(T)$, we need an additional assumption.

A2 For all $N \in I\!N$, $S \in [0, \overline{T}]$, $T \in [\underline{T}, \overline{T}]$, $S < T$ the functions $z_1|_{[S,T]}, ..., z_N|_{[S,T]}$ are linearly independent.

Assume that in the sequel, A2 is valid.

Lemma 6.1.4 *For all* $S \in [0, \overline{T}]$, $T \in [\underline{T}, \overline{T}]$, $S \leq T$, $u \in Z(S,T)$ *the following inequality holds:*

$$\sum_{i=1}^{\infty} \Big(\langle u, H_i(T)\rangle_{(S}$$

Lemma 6.1.2 implies that for all $N \in I\!N$, $(a_1, ..., a_N) \in I\!R^N$, we have the inequality

$$\begin{aligned} \|\sum_{i=1}^{N} a_i H_i(T)\|_{(S,T)} &\leq \|\sum_{i=1}^{N} a_i H_i(T)\|_{(0,T)} \\ &\leq \hat{P} \left(\sum_{i=1}^{N} a_i^2 \right)^{1/2}. \end{aligned}$$

This implies that for all $y \in I\!R^N$, we have

$$y^T y \leq \hat{P}^2 y^T \left(G_N(S,T)\right)^{-1} y.$$

Thus we have

$$\begin{aligned} \sum_{i=1}^{N} \left(\langle u, H_i(T) \rangle_{(S,T)} \right)^2 &= U_N^T U_N \\ &\leq \hat{P}^2 U_N^T \left(G_N(S,T)\right)^{-1} U_N \\ &= \hat{P}^2 \alpha_N^T G_N(S,T) \alpha_N \\ &= \hat{P}^2 \|u_N\|_{(S,T)}^2 \\ &\leq \hat{P}^2 \|u\|_{(S,T)}^2. \end{aligned}$$

Since this inequality holds for all $N \in I\!N$, the assertion follows. □

Lemma 6.1.5 *For all $T \in [\underline{T}, \overline{T}]$ there exists an element $v_*(T)$ of the closure of span$\{H_i(T) : i \in I\!N\}$ such that for all $i \in I\!N$ the equality*

$$\langle v_*(T), H_i(T) \rangle_{(0,T)} = c_i - \langle b, H_i(T) \rangle_{(0,T)} \tag{6.4}$$

is valid. Moreover, $u_*(T) = S_T^{-1}(v_*(T) + b)$ is the unique solution of $P_\infty(T)$.

Proof. Let $T \in [\underline{T}, \overline{T}]$, $N \in I\!N$ be given and let
$G_N(T) = (\langle H_i(T), H_j(T) \rangle_{(0,T)})_{ij=1}^N$. Define

$$V_N = (c_i - \langle b, H_i(T) \rangle_{(0,T)})_{i=1}^N \in I\!R^N.$$

As in Lemma 6.1.3, let

$$\eta_N(T) = (G_N(T))^{-1} V_N, \; v_N(T) = \sum_{i=1}^{N} \eta_i^N(T) H_i(T).$$

On account of Lemma 6.1.3 and Lemma 6.1.2 we have the inequality

$$\begin{aligned}
\|v_N(T)\|^2_{(0,T)} &= \eta_N(T)^T G_N(T)\eta_N(T) \\
&= V_N^T (G_N(T))^{-1} V_N \\
&\leq \hat{M}^2 V_N^T V_N \\
&\leq \hat{M}^2 \gamma(T),
\end{aligned}$$

$$\text{with } \gamma(T) = \sum_{i=1}^{\infty} (c_i - \langle b, H_i(T)\rangle_{(0,T)})^2.$$

Due to Lemma 6.1.4, $\gamma(T)$ is finite. Hence the sequence $(v_N(T))_{N\in \mathbb{N}}$ is bounded, and thus contains a weakly convergent subsequence. Let $v_*(T)$ denote a weak cluster point of $(v_N(T))_{N\in\mathbb{N}}$. For all i, $N \in \mathbb{N}$ with $i \leq N$ we have

$$\langle v_N(T), H_i(T)\rangle_{(0,T)} = c_i - \langle b, H_i(T)\rangle_{(0,T)}.$$

Thus due to the definition of weak convergence we obtain for all $i \in \mathbb{N}$

$$\langle v_*(T), H_i(T)\rangle_{(0,T)} = c_i - \langle b, H_i(T)\rangle_{(0,T)}.$$

For all $N \in \mathbb{N}$ we have v

We introduce a dual problem for $P_\infty(T)$ and show that the corresponding dual solutions are also uniformly bounded on $[\underline{T}, \overline{T}]$. We use this fact to show that ω is upper semicontinuous.

Lemma 6.1.6 (Uniform boundedness of the primal solutions) *The solutions of $P_\infty(T)$ are uniformly bounded on $[\underline{T}, \overline{T}]$, that is there exists $r \in I\!R$, such that for all $T \in [\underline{T}, \overline{T}]$ we have*

$$\|u_*(T)\|_{(0,T)} \leq r.$$

Proof. Let $T \in [\underline{T}, \overline{T}]$. Let $\gamma(T)$ be defined as in the proof of Lemma 6.1.5. Then due to Lemma 6.1.4 we have

$$\begin{aligned}
\sqrt{\gamma(T)} &= \left(\sum_{i=1}^{\infty}(c_i - \langle b, H_i(T)\rangle_{(0,T)})^2\right)^{1/2} \\
&\leq \|c\|_{l^2} + \left(\sum_{i=1}^{\infty}\langle b, H_i(T)\rangle_{(0,T)}^2\right)^{1/2} \\
&\leq \|c\|_{l^2} + \hat{P}\|b\|_{(0,T)} \\
&\leq \|c\|_{l^2} + \hat{P}\|b\|_{(0,\overline{T})} =: R.
\end{aligned}$$

Since $v_*(T)$ is a weak cluster point of the sequence $(v_N(T))_{N\in I\!N}$ due to the lower semi–continuity of $\|\cdot\|_{(0,T)}^2$ we have

$$\|v_*(T)\|_{(0,T)}^2 \leq \hat{M}^2\gamma(T) \leq \hat{M}^2 R^2.$$

According to Lemma 6.1.5, we have

$$u_*(T) = S_T^{-1}(v_*(T) + b).$$

By Lemma 6.1.1, this yields the inequality

$$\begin{aligned}
\|u_*(T)\|_{(0,T)} &\leq \|S_T^{-1}\| \left(\|v_*(T)\|_{(0,T)} + \|b\|_{(0,T)}\right) \\
&\leq \|S_{\overline{T}}^{-1}\| \left(\hat{M}R + \|b\|_{(0,\overline{T})}\right) =: r,
\end{aligned}$$

and the assertion follows. □

Lemma 6.1.7 (Lower semicontinuity) *The function ω is lower semicontinuous on $[\underline{T}, \overline{T}]$.*

Proof. Let $T \in [\underline{T}, \overline{T}]$ and a sequence $(T_l)_{l\in I\!N} \in [\underline{T}, \overline{T}]^{I\!N}$ converging to T be given. For $k \in I\!N$, let $u_k = u_*(T_k) \in Z(0, T_k)$. Due to Lemma 6.1.6 there is $r \in I\!R$ such that for all k we have: $\|u_k\|_{(0,T_k)} \le r$.

Define $\tilde{u}_k(\cdot) = u_k(\cdot\, T_k/\overline{T}) \in Z(0, \overline{T})$. Then

$$\|\tilde{u}_k\|_{(0,\overline{T})} = (\overline{T}/T_k)^{1/2} \|u_k\|_{(0,T_k)} \le (\overline{T}/T_k)^{1/2} r.$$

Hence the sequence $(\tilde{u}_k)_{k\in I\!N}$ is bounded. Thus there exists a subsequence that converges weakly to a point $\tilde{u}_* \in Z(0, \overline{T})$. Assume without restriction that the whole sequence $(\tilde{u}_k)_{k\in I\!N}$ is weakly convergent.

We have $u_j(\cdot) = \tilde{u}_j(\cdot \overline{T}/T_j)$. Define $w_*(\cdot) = \tilde{u}_*(\cdot \overline{T}/T)$. Let $\tilde{z}_l^j = z_l(\cdot T_j/\overline{T}) \in Z(0, \overline{T})$. For all $l \in I\!N$ we have $c_l = \langle u_j, z_l\rangle_{(0,T_j)} = (T_j/\overline{T}) \langle \tilde{u}_j, \tilde{z}_l^j\rangle_{(0,\overline{T})}$.

Let $\tilde{z}_l^*(\cdot) = z_l(\cdot T/\overline{T}) \in Z(0, \overline{T})$. Then

$$\lim_{j\to\infty} \|\tilde{z}_l^j - \tilde{z}_l^*\|_{(0,\overline{T})} = 0.$$

For fixed $l \in I\!N$, this can be seen as follows. Let $\epsilon > 0$ be given. Then there exists $g \in C(0, \overline{T})$ such that

$$\int_0^{\overline{T}} (g(x) - z_l(x))^2 \, dx < \epsilon^2$$

(see (9.7) in [49]). Since g is uniformly continuous on $[0, \overline{T}]$, there exists a number $n_1(\epsilon)$ such that for all $j > n_1(\epsilon)$

$$\int_0^{\overline{T}} (g(xT_j/\overline{T}) - g(xT/\overline{T}))^2 \, dx < \epsilon^2.$$

Moreover, we have

$$\int_0^{\overline{T}} (z_l(xT_j/\overline{T}) - g(xT_j/\overline{T}))^2 \, dx = (\overline{T}/T_j) \int_0^{T_j} (z_l(t) - g(t))^2 \, dt \le \epsilon^2 \overline{T}/\underline{T}$$

and

$$\int_0^{\overline{T}} (z_l(xT/\overline{T}) - g(xT/\overline{T}))^2 \, dx \le \epsilon^2 \overline{T}/\underline{T}.$$

Hence Minkowski's inequality implies

$$(\int_0^{\overline{T}} (z_l(xT_j/\overline{T}) - z_l(xT/\overline{T}))^2 \, dx)^{1/2} \le 3\epsilon \sqrt{\overline{T}/\underline{T}}.$$

Hence for all $l \in I\!N$

$$\langle \tilde{u}_*, \tilde{z}_l^* \rangle_{(0,\overline{T})} = \lim_{j\to\infty} \langle \tilde{u}_j, \tilde{z}_l^* \rangle_{(0,\overline{T})} = \lim_{j\to\infty} \langle \tilde{u}_j, \tilde{z}_l^j \rangle_{(0,\overline{T})} = \lim_{j\to\infty} (\overline{T}/T_j) c_l = (\overline{T}/T) c_l.$$

Hence we have

$$\langle w_*, z_l \rangle_{(0,T)} = (T/\overline{T}) \langle \tilde{u}_*, \tilde{z}_l^* \rangle_{(0,\overline{T})} = (T/\overline{T})(\overline{T}/T) c_l = c_l.$$

Thus we have $w_* \in U(T, \infty, c)$ and hence $\omega(T) \le \|S_T w_* - b\|^2_{(0,T)}$.

The function $u \mapsto \|u\|_{(0,T)}$, $Z(0,T) \to I\!R$ is sequentially weakly lower semicontinuous (as the supremum of sequentially weakly continuous functions, see [37], Prop. 1.5.12).

Let $\tilde{b}(\cdot) = b(\cdot T/\overline{T})$. Let $v_j = S_{T_j} u_j - b \in Z(0, T_j)$ and $\tilde{v}_j(\cdot) = v_j(\cdot\, T_j/\overline{T}) \in Z(0, \overline{T})$. Let $v_* = S_T w_* - b$ and $\tilde{v_*}(\cdot) = v_*(\cdot\, T/\overline{T}) \in Z(0, \overline{T})$. For $f \in Z(0, \overline{T})$, let $\hat{f}_j(\cdot) = f(\cdot\, \overline{T}/T_j) \in Z(0, T_j)$ and $\hat{f}(\cdot) = f(\cdot\, \overline{T}/T) \in Z(0,T)$. Then

$$\langle f, \tilde{v}_j \rangle_{(0,\overline{T})} = \langle f, (S_{T_j} u_j)(\cdot\, T_j/\overline{T}) \rangle_{(0,\overline{T})} - \langle f, b(\cdot\, T_j/\overline{T}) \rangle_{(0,\overline{T})}.$$

We have

$$\begin{aligned} \langle f, (S_{T_j} u_j)(\cdot\, T_j/\overline{T}) \rangle_{(0,\overline{T})} &= \langle \hat{f}_j, (S_{T_j} u_j) \rangle_{(0,T_j)} (\overline{T}/T_j) \\ &= \langle S_{T_j}^* \hat{f}_j, u_j \rangle_{(0,T_j)} (\overline{T}/T_j) \\ &= \langle (S_{T_j}^* \hat{f}_j)(\cdot\, T_j/\overline{T}), \tilde{u}_j \rangle_{(0,\overline{T})}. \end{aligned}$$

Thus assumption (6.3) and the weak convergence of the sequence $(\tilde{u}_j)_{j\in I\!N}$ imply

$$\begin{aligned} \lim_{j\to\infty} \langle f, (S_{T_j} u_j)(\cdot\, T_j/\overline{T}) \rangle_{(0,\overline{T})} &= \langle (S_T^* \hat{f})(\cdot\, T/\overline{T}), \tilde{u}_* \rangle_{(0,\overline{T})} \\ &= \langle S_T^* \hat{f}, w_* \rangle_{(0,T)} (\overline{T}/T) \\ &= \langle \hat{f}, S_T w_* \rangle_{(0,T)} (\overline{T}/T) \\ &= \langle f, (S_T w_*)(\cdot\, T/\overline{T}) \rangle_{(0,\overline{T})}. \end{aligned}$$

Moreover, since $\|b(\cdot T_j/\overline{T}) - \tilde{b}(\cdot)\|_{(0,\overline{T})} \to 0$ $(j \to \infty)$ we have

$$\lim_{j\to\infty} \langle f, b(\cdot\, T_j/T) \rangle_{(0,\overline{T})} = \langle f, \tilde{b} \rangle_{(0,\overline{T})}.$$

Thus we can conclude that

$$\lim_{j\to\infty}\langle f,\tilde{v}_j\rangle_{(0,\overline{T})} = \langle f,(S_T w_* - b)(\cdot\, T/\overline{T})\rangle_{(0,\overline{T})} = \langle f,\tilde{v}_*\rangle_{(0,\overline{T})},$$

so the sequence $(\tilde{v}_j)_{j\in I\!N}$ converges weakly to $\tilde{v}_*$.

Thus we have

$$\begin{aligned}
\omega(T) &\leq \|v_*\|^2_{(0,T)}\\
&= (T/\overline{T})\|\tilde{v}_*\|^2_{(0,\overline{T})}\\
&\leq (T/\overline{T})\liminf_{j\to\infty}\|\tilde{v}_j\|^2_{(0,\overline{T})}\\
&= \liminf_{j\to\infty}(T_j/\overline{T})\|\tilde{v}_j\|^2_{(0,\overline{T})}\\
&= \liminf_{j\to\infty}\|v_j\|^2_{(0,T_j)}\\
&= \liminf_{j\to\infty}\omega(T_j).
\end{aligned}$$

Hence $\omega(T)\leq \liminf_{k\to\infty}\omega(T_k)$, that is ω is lower semicontinuous in T. $\square$

To show the upper semicontinuity of ω, we use the coefficients of $v_*(T)$ written as a linear combination of the functions $H_i(T)$.

These coefficients form a sequence in l^2 and can be used to express the optimal value $\omega(T)$.

Lemma 6.1.8 *Let $T\in[\underline{T},\overline{T}]$. Then there exist $(\alpha_i(T))_{i\in I\!N}\in l^2$ such that*

$$\begin{aligned}
v_*(T) &= \sum_{i=1}^{\infty}\alpha_i(T)H_i(T) \text{ and}\\
\omega(T) &= \sum_{i=1}^{\infty}\alpha_i(T)(c_i - \langle b, H_i(T)\rangle_{(0,T)}).
\end{aligned}$$

Moreover, for all $i\in I\!N$ the following equality is valid:

$$\sum_{j=1}^{\infty}\alpha_j(T)\langle H_i(T),H_j(T)\rangle_{(0,T)} = c_i - \langle b,H_i(T)\rangle_{(0,T)}. \tag{6.5}$$

Proof. According to Lemma 6.1.5, we have

$$v_*(T)\in span\{H_i(T), i\in I\!N\}.$$

Hence there exists $(\alpha_i(T))_{i\in\mathbb{N}}$ such that $v_*(T) = \sum_{i=1}^{\infty}\alpha_i(T)H_i(T)$. Lemma 6.1.2 implies $(\alpha_i(T))_{i\in\mathbb{N}} \in l^2$. Since $v_*(T) = S_T u_*(T) - b$, we have

$$\begin{aligned}
\omega(T) &= \|v_*(T)\|^2_{(0,T)} \\
&= \langle \sum_{i=1}^{\infty} \alpha_i(T) H_i(T), v_*(T)\rangle_{(0,T)} \\
&= \sum_{i=1}^{\infty} \alpha_i(T) \langle H_i(T), v_*(T)\rangle_{(0,T)} \\
&= \sum_{i=1}^{\infty} \alpha_i(T)(c_i - \langle b, H_i(T)\rangle_{(0,T)},
\end{aligned}$$

where the last equality follows from equation (6.4), which also implies (6.5). □

In the next Lemma, we introduce a maximization problem with value $\omega(T)$, i.e. a dual problem for $P_\infty(T)$.

Lemma 6.1.9 (Dual Problem) *Let $T \in [\underline{T}, \overline{T}]$. Then we have*

$$\omega(T)$$

$$= \sup_{\alpha\in l^2} - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_i\alpha_j \langle H_i(T), H_j(T)\rangle_{(0,T)} + 2\sum_{j=1}^{\infty}\alpha_j\left(c_j - \langle b, H_j(T)\rangle_{(0,T)}\right).$$

Proof. For $T \in [\underline{T}, \overline{T}]$, $\alpha \in l^2$, define

$$h(T,\alpha) = - \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_i\alpha_j \langle H_i(T), H_j(T)\rangle_{(0,T)} + 2\sum_{j=1}^{\infty}\alpha_j\left(c_j - \langle b, H_j(T)\rangle_{(0,T)}\right).$$

Let $\alpha(T) = (\alpha_i(T))_{i\in\mathbb{N}}$ be as in Lemma 6.1.8. Then Lemma 6.1.8 implies

$$\begin{aligned}
h(T,\alpha(T)) &= -\|v_*(T)\|^2 + 2\sum_{j=1}^{\infty}\alpha_j(T)\left(c_j - \langle b, H_j(T)\rangle_{(0,T)}\right) \\
&= -\omega(T) + 2\omega(T) \\
&= \omega(T). \qquad (6.6)
\end{aligned}$$

Hence we have the inequality

$$\omega(T) \le \sup_{\alpha\in l^2} h(T,\alpha).$$

For $\alpha \in l^2$, $v \in Z(0,T)$ define

$$\phi(T,v,\alpha) = \|v\|_{(0,T)}^2 + 2\sum_{j=1}^{\infty} \alpha_j \left(c_j - \langle b, H_j(T)\rangle_{(0,T)} - \langle v, H_j(T)\rangle_{(0,T)} \right) \tag{6.7}$$

Lemma 6.1.4 implies, that $\phi(T,v,\alpha)$ is well–defined.

According to Lemma 6.1.5, we have

$$\|v_*(T)\|_{(0,T)}^2 = \omega(T)$$

and hence (6.5) implies that for all $\alpha \in l^2$

$$\phi(T, v_*(T), \alpha) = \|v_*(T)\|_{(0,T)}^2 = \omega(T).$$

For all $\alpha \in l^2$, the map $\phi(T,\cdot,\alpha)$ is coercive and strictly convex, hence the set

$$M_{\min}(T) = \{v \in Z(0,T) : \phi(T,v,\alpha) = \inf_{w \in Z(0,T)} \phi(T,w,\alpha)\}$$

is nonempty and consists of a single element.

Let $\alpha \in l^2$ be fixed and $M_{\min}(T) = \{w_*\}$. Since the map $\phi(T,\cdot,\alpha) : Z(0,T) \to I\!R$ is Fréchet–differentiable, we can derive the equation

$$w_* = \sum_{j=1}^{\infty} \alpha_j H_j(T).$$

Thus we have

$$\begin{aligned} &\phi(T, w_*, \alpha) \\ &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_i\alpha_j \langle H_i(T), H_j(T)\rangle_{(0,T)} + 2\sum_{j=1}^{\infty} \alpha_j \left(c_j - \langle b, H_j(T)\rangle_{(0,T)} \right) \\ &-2\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_i\alpha_j \langle H_i(T), H_j(T)\rangle_{(0,T)} \\ &= h(T,\alpha). \end{aligned}$$

Hence for all $\alpha \in l^2$ we have

$$\begin{aligned} h(T,\alpha) &= \inf_{v \in Z(0,T)} \phi(T,v,\alpha) \\ &\leq \phi(T, v_*(T), \alpha) \\ &= \omega(T). \end{aligned} \tag{6.8}$$

This implies

$$\sup_{\alpha \in l^2} h(T,\alpha) \le \omega(T),$$

and the assertion follows. $\Box$

Lemma 6.1.10 (Uniqueness of the dual solutions) *For all $T \in [\underline{T}, \overline{T}]$, the point $(\alpha_i(T))_{i\in\mathbb{N}} \in l^2$ as defined in Lemma 6.1.8 is uniquely determined and the unique solution of the dual problem stated in Lemma 6.1.9.*

Proof. Let $\alpha(T) = (\alpha_i(T))_{i\in\mathbb{N}}$ be as in Lemma 6.1.8. Equation (6.6) implies that $\alpha(T)$ solves the dual problem.

Lemma 6.1.2 implies that the function $h(T,\cdot) : l^2 \to \mathbb{R}$ is strictly concave, hence the dual solution is unique.

Hence $\alpha(T)$ is uniquely determined. $\Box$

Note that for all $T \in [\underline{T}, \overline{T}]$, the dual solution is an element of the space l^2 that is a space which is independent of T. This fact is very convenient for our analysis.

Lemma 6.1.11 (Uniform boundedness of the dual solutions) *Let $T \in [\underline{T}, \overline{T}]$ and $(\alpha_i(T))_{i\in\mathbb{N}}$ be as in Lemma 6.1.8. There exists $r \in \mathbb{R}$, such that for all $T \in [\underline{T}, \overline{T}]$ we have*

$$\sum_{i=1}^{\infty} (\alpha_i(T))^2 \le r.$$

Proof. According to Lemma 6.1.2, for all $T \in [\underline{T}, \overline{T}]$ we have

$$\begin{aligned}
\left(\sum_{i=1}^{\infty} \alpha_i(T)^2\right)^{1/2} &\le \hat{M} \| \sum_{i=1}^{\infty} \alpha_i(T) H_i(T) \|_{(0,T)} \\
&= \hat{M} \| v_*(T) \|_{(0,T)} \\
&\le \hat{M} R
\end{aligned}$$

with R as defined in the proof of Lemma 6.1.6. The assertion follows with $r = \hat{M}R$. $\Box$

Lemma 6.1.12 *Let $u \in Z(0, \overline{T})$. For $T \in [\underline{T}, \overline{T}]$, $i \in \mathbb{N}$ define*

$$d_i(T) = \langle u, H_i(T) \rangle_{(0,T)}.$$

Then for all $T \in [\underline{T}, \overline{T}]$, we have

$$\lim_{t \to T, t \in [\underline{T}, \overline{T}]} \sum_{i=1}^{\infty} (d_i(t) - d_i(T))^2 = 0.$$

Proof. According to Lemma 6.1.4, for all $t \in [\underline{T}, \overline{T}]$, we have $(d_i(t))_{i \in \mathbb{N}} \in l^2$. From the definition, we have

$$d_i(T) = \langle u, (S_T^*)^{-1} z_i \rangle_{(0,T)} = \langle S_T^{-1} u, z_i \rangle_{(0,T)}.$$

Let T_1, $T_2 \in [\underline{T}, \overline{T}]$, $T_1 < T_2$. Then Lemma 6.1.1 implies

$$\begin{aligned} d_i(T_2) - d_i(T_1) &= \langle S_{T_2}^{-1} u, z_i \rangle_{(0,T_2)} - \langle S_{T_2}^{-1} u, z_i \rangle_{(0,T_1)} \\ &= \langle S_{T_2}^{-1} u, z_i \rangle_{(T_1,T_2)}. \end{aligned}$$

Analogously to Lemma 6.1.4 we can prove (by replacing $H_i(T)$ by z_i) that for all $v \in Z(T_1, T_2)$ we have

$$\sum_{i=1}^{\infty} \langle v, z_i \rangle_{(T_1,T_2)}^2 \leq P^2 \|v\|_{(T_1,T_2)}.$$

This implies

$$\begin{aligned} \sum_{i=1}^{\infty} (d_i(T_2) - d_i(T_1))^2 &= \sum_{i=1}^{\infty} \langle S_{T_2}^{-1} u, z_i \rangle_{(T_1,T_2)}^2 \\ &\leq P^2 \|S_{T_2}^{-1} u\|_{(T_1,T_2)}^2. \end{aligned}$$

On account of

$$\lim_{t \to T, t \in [\underline{T}, \overline{T}]} \|S_{\overline{T}}^{-1} u\|_{(t,T)} = 0,$$

the assertion follows. □

Lemma 6.1.13 (Upper semicontinuity) *The function ω is upper semicontinuous on $[\underline{T}, \overline{T}]$.*

Proof. Let $T \in [\underline{T}, \overline{T}]$ and a sequence $(T_j)_{j \in \mathbb{N}} \in [\underline{T}, \overline{T}]^{\mathbb{N}}$ converging to T be given. Then for all $u \in Z(0, \overline{T})$, we have

$$\lim_{j \to \infty} \|u\|_{(0,T_j)} = \|u\|_{(0,T)}.$$

Moreover, Lemma 6.1.12 implies

$$\begin{aligned} \lim_{k \to \infty} \sum_{j=1}^{\infty} \left(\langle b, H_j(T_k) \rangle_{(0,T_k)} - \langle b, H_j(T) \rangle_{(0,T)} \right)^2 &= 0 \text{ and} \\ \lim_{k \to \infty} \sum_{j=1}^{\infty} \left(\langle u, H_j(T_k) \rangle_{(0,T_k)} - \langle u, H_j(T) \rangle_{(0,T)} \right)^2 &= 0. \end{aligned}$$

Let $(\nu^j)_{j\in\mathbb{N}} \in (l^2)^{\mathbb{N}}$ be a weakly convergent sequence converging to ν^*.

Then for ϕ as defined in (6.7) we have

$$\begin{aligned}
&\lim_{k\to\infty} \phi(T_k, u|_{[0,T_k]}, \nu^k) \\
&= \lim_{k\to\infty} \|u\|^2_{(0,T_k)} + 2\sum_{j=1}^{\infty} \nu_j^k \left(c_j - \langle b, H_j(T_k)\rangle_{(0,T_k)} - \langle u, H_j(T_k)\rangle_{(0,T_k)} \right) \\
&= \phi(T, u|_{(0,T)}, \nu^*),
\end{aligned}$$

i.e. the map

$$(T,\nu) \mapsto \phi(T, u|_{(0,T)}, \nu),\ [\underline{T}, \overline{T}] \times l^2 \to \mathbb{R}$$

is sequentially weakly continuous. From (6.8), we have

$$h(T,\nu) = \inf_{u\in Z(0,\overline{T})} \phi(T, u|_{(0,T)}, \nu).$$

Hence h is the infimum of sequentially weakly continuous maps. Thus Proposition 1.5.12 in [37] implies that h is sequentially weakly upper semicontinuous, i.e.

$$\limsup_{j\to\infty} h(T_j, \nu^j) \le h(T, \nu^*).$$

For $t \in [\underline{T}, \overline{T}]$, let $\alpha(t) = (\alpha_i(t))_{i\in\mathbb{N}}$. According to Lemma 6.1.11 there exists $r \in \mathbb{R}$ such that for all k we have

$$\sum_{i=1}^{\infty} (\alpha_i(T_k))^2 \le r.$$

Hence there exists a subsequence $(t_j)_{j\in\mathbb{N}}$ of $(T_j)_{j\in\mathbb{N}}$ for which we have

$$\limsup_{k\to\infty} h(T_k, \alpha(T_k)) = \lim_{k\to\infty} h(t_k, \alpha(t_k))$$

and such that the sequence $(\alpha(t_k))_{k\in\mathbb{N}} \in (l^2)^{\mathbb{N}}$ converges weakly to a point $\alpha^* \in l^2$. Then due to Lemma 6.1.9 we have

$$\begin{aligned}
\limsup_{k\to\infty} \omega(T_k) &= \limsup_{k\to\infty} h(T_k, \alpha(T_k)) \\
&= \lim_{k\to\infty} h(t_k, \alpha(t_k)) \\
&\le h(T, \alpha^*) \\
&\le \omega(T).
\end{aligned}$$

Hence $\limsup_{k\to\infty} \omega(T_k) \leq \omega(T)$, i.e. ω is upper semicontinuous on $[\underline{T}, \overline{T}]$. □

Now we state the main result of this section.

Theorem 6.1.1 (Continuity) *The function ω is continuous on $[\underline{T}, \overline{T}]$.*

Proof. Lemma 6.1.7 and Lemma 6.1.13 together yield the assertion. □

6.1.6 Continuity of the Value Function for the Discretized Problem

Lemma 6.1.14 *For all $N \in I\!N$, the function ω_N is continuous on $[\underline{T}, \overline{T}]$.*

Proof. The assertion follows analogously to Theorem 6.1.1, by replacing the infinite series by finite sums and the infinite systems of moment equations by the corresponding finite systems. The dual solutions of problem $P_N(T)$ are elements of $I\!R^N$. □

6.1.7 Uniform Convergence of the Value Functions for the Discretized Problems

In this section we present the result that is announced in the title, a theorem about uniform convergence of the optimal value functions for the discretized problems. This theorem shows that if the discretization level is large enough, the discretized problem yields an arbitrarily good approximation for the optimal value function ω, uniformly on the whole interval $[\underline{T}, \overline{T}]$.

Theorem 6.1.2 (Uniform Convergence) *The sequence $(\omega_N)_{N\in I\!N}$ converges uniformly and monotone to ω on $[\underline{T}, \overline{T}]$.*

Proof. The definitions of $P_\infty(T)$ and $P_N(T)$ imply that for all $N \in I\!N$ we have

$$\omega_N(T) \leq \omega_{N+1}(T) \leq \omega(T).$$

Hence for all $T \in [\underline{T}, \overline{T}]$, the sequence $(\omega_N(T))_{N\in I\!N}$ is convergent and

$$\lim_{N\to\infty} \omega_N(T) \leq \omega(T).$$

The proof of Lemma 6.1.5 implies that

$$\begin{aligned} \lim_{N\to\infty} \omega_N(T) &= \liminf_{N\to\infty} \|v_N(T)\|^2_{(0,T)} \\ &\geq \|v_*(T)\|^2_{(0,T)} \\ &= \omega(T), \end{aligned}$$

where we have used the fact that the function $\|\cdot\|_{(0,T)}$ is sequentially weakly lower semicontinuous. Hence for all $T \in [\underline{T}, \overline{T}]$, we have

$$\lim_{N\to\infty} \omega_N(T) = \omega(T).$$

Thus the sequence of functions $(\omega_N)_{N\in\mathbb{N}}$ converges pointwise to the function ω. By Lemma 6.1.14, for all $N \in \mathbb{N}$ the functions ω_N are continuous. By Theorem 6.1.1, the limit function ω is also continuous. Hence Dini's Theorem (see [37] implies the uniform convergence. □

In the next theorem, we summarize our results.

Theorem 6.1.3 *For all $N \in \mathbb{N}$, the optimal value functions ω_N of the discretized problems are continuous. The value function ω of the original problem is also continuous.*

The sequence $(\omega_N)_{N\in\mathbb{N}}$ converges uniformly and monotone to ω on $[\underline{T}, \overline{T}]$.

If the functions ω and ω_M are strictly decreasing, the inverse functions ω_M^{-1} exist on the intervals $[\omega_M^{-1}(\overline{T}), \omega_M^{-1}(\underline{T})]$ and ω^{-1} exists on the interval $[\omega^{-1}(\overline{T}), \omega^{-1}(\underline{T})]$ and these functions are continuous and strictly decreasing.

Let $y_M = \omega_M^{-1}(\overline{T})$. Then $\omega_M(y_M) = \overline{T}$, hence $\omega(y_M) \geq \omega_M(y_M) = \overline{T}$, thus $\omega^{-1}(\overline{T}) \geq y_M = \omega_M^{-1}(\underline{T})$. Hence for all $M \geq M_0$, the functions ω_M^{-1} are all defined on the intervals $[\omega^{-1}(\overline{T}), \omega_{M_0}^{-1}(\underline{T})]$.

Theorem 6.1.4 *If the functions ω and ω_M are strictly decreasing, the sequence of inverse functions $\{\omega_M^{-1}\}_{M=M_0}^{\infty}$ converges uniformly to the function ω^{-1} on the interval $[\omega^{-1}(\overline{T}), \omega_{M_0}^{-1}(\underline{T})]$.*

Proof. With similar arguments as above we can show that for all $M \geq M_0$, $y \in [\omega^{-1}(\overline{T}), \omega_{M_0}^{-1}(\underline{T})]$ the following inequality holds: $\omega_M^{-1}(y) \leq \omega_{M+1}^{-1}(y) \leq \omega^{-1}(y)$.

Let $L = \lim_{M\to\infty} \omega_M^{-1}(y)$. Then Theorem 6.1.3 implies that

$$|\omega(L) - y| = \lim_{M\to\infty} |\omega(L) - \omega_M(\omega_M^{-1}(y))|$$

$$\begin{aligned} &\leq \lim_{M\to\infty} |\omega(L) - \omega(\omega_M^{-1}(y))| + |\omega(\omega_M^{-1}(y)) - \omega_M(\omega_M^{-1}(y))| \\ &= 0, \end{aligned}$$

hence $L = \omega^{-1}(y)$.

Since the functions ω_M^{-1} and ω^{-1} are continuous and the sequence ω_M^{-1} converges pointwise monotone to the function ω^{-1} on the interval $[\omega^{-1}(\overline{T}), \omega_{M_0}^{-1}(\underline{T})]$ we can again apply Dini's Theorem which yields the assertion. $\square$

6.2 Differentiability of the Value Function for the Discretized Problem

In this section we assume that the operator S_T that appears in the objective function of Problem $P_N(T)$ is a Volterra operator with a constant > 0 and a continuous kernel $K \in C(0, \overline{T})$:

$$(S_T u)(t) = \quad u(t) - \int_0^t K(t-s)u(s)\, ds.$$

We also assume that the functions z_j ($j \in I\!N$) and b are in $C(0, \overline{T})$.

We show that these assumptions imply that the optimal value function ω_N is continuously differentiable. Moreover, we point out how the derivative can be computed.

As in the previous section, let $H_j(T) = (S_T^*)^{-1} z_j$. We prove that for all $t \in (0, T)$, the derivative $\partial_T H_j(T)(t)$ exists. This parameter derivative $\partial_T H_j(T)$ can be computed as the solution of a Volterra equation, which differs from the Volterra–equation for $H_j(T)$ only in the right–hand side.

In our formula for the derivative of the optimal value function, the functions $H_j(T)$ and $\partial_T H_j(T)$ appear. For the proof of the differentiability of the optimal value function, we use the fact that the coefficient vector $\eta_N(T)$ is the solution of a system of linear equations which depend differentiably on the parameter T. In the proof, we apply the implicit function theorem to this system.

6.2.1 The Parameter Derivative of $H_j(T)$

Theorem 6.2.1 *Let $z \in C(0, \overline{T})$ be given. For $T \in [0, \overline{T}]$, let $G(T,t)$ be the solution of the Volterra equation*

$$G(T,t) - \int_t^T K(s-t)G(T,s)\,ds = z(T)K(T-t)/\ \ ,\ t \in [0,T]$$

(i.e. $G(T,\cdot) = (z(T)/\ \ )(S_T^*)^{-1}K(T-\cdot)$) and $H(T,t)$ be the solution of the Volterra equation

$$H(T,t) - \int_t^T K(s-t)H(T,s)\,ds = z(t),\ t \in [0,T]$$

(i.e. $H(T,\cdot) = (S_T^*)^{-1}z$). Then for all $T \in [0,\overline{T}]$, for all $t \in [0,T]$ the function $H(\cdot,t)$ is continuously differentiable and

$$\partial_T H(T,t) = G(T,t).$$

Proof. Since the functions K and z are continuous, the functions $H(T,\cdot)$ and $G(T,\cdot)$ are well–defined as the unique continuous solutions of the corresponding Volterra equations (see Theorem 3.1 in [35] and Lemma 4.1.1).

First we show that the function $G(\cdot,t)$ is continuous on $[t,\overline{T}]$.

According to Lemma 4.1.1, the function $v(T,t) = G(T,T-t)$ is the solution of the equation

$$v(T,t) - \int_0^t \frac{K(t-s)}{\ }v(T,s)\,ds = \frac{z(T)}{2}K(t).$$

This equation defines the function $v(T,\cdot)$ on the interval $[0,\overline{T}]$. According to Theorem 3.5 in [35], the solution of this problem can be expressed using the resolvent kernel R for the difference kernel $K/\ \ $, namely as

$$v(T,t) = \frac{z(T)}{2}K(t) + \int_0^t R(t-s)\frac{z(T)}{2}K(s)\,ds$$

where R is the solution of

$$R(t) - \int_0^t \frac{K(t-s)}{\ }R(s)\,ds = \frac{K(t)}{\ }.$$

Thus we have $v(T,t) = R(t)z(T)/\ \ $. Hence for all $T \in [0,\overline{T}]$ we have

$$\lim_{h\to 0}\max_{t\in[0,\overline{T}]} |v(T+h,t) - v(T,t)| = 0,$$

where for $T \in \{0, \overline{T}\}$ the corresponding one–sided limits are used.

Since the function $v(T, \cdot)$ is continuous, this implies that the function $G(\cdot, t)$ is continuous on $[t, \overline{T}]$, because we have

$$\begin{aligned} |G(T+h,t) - G(T,t)| &= |v(T+h, T+h-t) - v(T, T-t)| \\ &\leq \max_{s \in [0,\overline{T}]} |v(T+h,s) - v(T,s)| \\ &+ |v(T, T+h-t) - v(T, T-t)|. \end{aligned}$$

Define

$$I(T,t) = \int_t^T G(s,t)\, ds + \frac{z(t)}{}.$$

Then we have

$$I(T,t) = \int_t^T G(s,t)\, ds + z(t).$$

Moreover, the Volterra equation defining G implies

$$\begin{aligned} \int_t^T G(s,t)\, ds &= \int_t^T \frac{z(s)}{} K(s-t)\, ds + \int_t^T \int_t^s K(u-t) G(s,u)\, du\, ds \\ &= \int_t^T \frac{z(s)}{} K(s-t)\, ds + \int_t^T \int_u^T K(u-t) G(s,u)\, ds\, du \\ &= \int_t^T \frac{z(s)}{} K(s-t)\, ds + \int_t^T K(u-t) (\int_u^T G(s,u)\, ds)\, du \\ &= \int_t^T K(s-t) I(T,s)\, ds. \end{aligned}$$

Hence we have

$$I(T,t) = \int_t^T K(s-t) I(T,s)\, ds + z(t),$$

which implies $I(T,t) = H(T,t)$. Hence $\partial_T H$ exists and we have

$$\partial_T H(T,t) = G(T,t).$$

So the assertion follows.

$$= \sum_{k=1}^{N}\sum_{l=1}^{N} \eta_k^N(T)\eta_l^N(T)\langle H_k(T), H_l(T)\rangle_{(0,T)}.$$

Let $A_N(T)$ be the positive definite matrix

$$A_N(T) = \Big(\langle H_k(T), H_l(T)\rangle_{(0,T)}\Big)_{k,l=1}^{N}.$$

Define

$$r_N(T) = \Big(c_j - \langle b, H_j(T)\rangle_{(0,T)}\Big)_{j=1}^{N}.$$

Then by Lemma 6.1.3 we have

$$\omega_N(T) = (\eta_N(T))^T A_N(T)\eta_N(T) = (\eta_N(T))^T r_N(T). \tag{6.9}$$

With this representation of the function ω_N we can prove the following Lemma.

Lemma 6.2.1 *The function ω_N is continuously differentiable on $[0,\overline{T}]$ and*

$$\omega_N'(T) = 2(\eta_N(T))^T r_N'(T) - (\eta_N(T))^T A_N'(T)\eta_N(T). \tag{6.10}$$

Let $G_j(T) = \partial_T H_j(T)$. The map r_N is continuously differentiable on $[0,\overline{T}]$ and

$$r_N'(T) = \Big(-b(T)z_j(T)/\quad - \langle b, G_j(T)\rangle_{(0,T)}\Big)_{j=1}^{N}.$$

The map A_N is continuously differentiable on $[0,\overline{T}]$ and

$$A_N'(T) = \Big(z_i(T)z_j(T)/\quad^2 + \langle H_i(T), G_j(T)\rangle_{(0,T)} + \langle G_i(T), H_j(T)\rangle_{(0,T)}\Big)_{i,j=1}^{N}.$$

Proof According to Theorem 6.2.1, $G_j(T)$ is well–defined and continuous. Let

$$a_{kl}(T) = \langle H_k(T), H_l(T)\rangle_{(0,T)}.$$

Then the function a_{kl} is differentiable and

$$a_{kl}'(T) = H_k(T)(T)H_l(T)(T) + \langle H_k(T), G_l(T)\rangle_{(0,T)} + \langle G_k(T), H_l(T)\rangle_{(0,T)}.$$

The definition of $H_j(T)$ implies the equation

$$H_j(T)(T) = z_j(T)/\quad.$$

Hence $A_N'(T)$ exists and the representation that is given in the statement of the Lemma follows. Analogously, the assertion for $r_N'(T)$ follows.

According to Lemma 6.1.3 we have the equation

$$A_N(T)\eta_N(T) - r_N(T) = 0.$$

The matrix $A_N(T)$ is regular. Thus the implicit function theorem implies that the map η_N is continuously differentiable and

$$\eta_N'(T) = A_N^{-1}(T)\left(r_N'(T) - A_N'(T)\eta_N(T)\right).$$

The representation (6.9) of ω_N yields

$$\begin{aligned}
\omega_N'(T) &= (\eta_N'(T))^T r_N(T) + \eta_N(T)^T r_N'(T) \\
&= \left(r_N'(T) - A_N'(T)\eta_N(T)\right)^T A_N^{-1}(T) r_N(t) + \eta_N(T)^T r_N'(T) \\
&= \left(r_N'(T) - A_N'(T)\eta_N(T)\right)^T \eta_N(T) + \eta_N(T)^T r_N'(T) \\
&= 2\eta_N(T)^T r_N'(T) - (\eta_N(T))^T A_N'(T)\eta_N(T). \ \Box
\end{aligned}$$

Hence from the vectors $\eta_N(T)$, $r_N'(T)$ and the matrix $A_N'(T)$ the derivative $\omega_N'(T)$ can be computed without solving a system of linear equations. To compute the derivative $\eta_N'(T)$, the solution of a system of linear equations is necessary.

Remark 6.2.1 If $A_N'(T)$ is regular, $r_N'(T) = 0$ and $\eta_N(T) \neq 0$, the formula for ω_N' implies that $\omega_N'(T) < 0$, hence the function ω_N is strictly decreasing in a neighbourhood of T.

A sufficient condition for $r_N'(T) = 0$ is $b = 0$. If $c \neq 0$ we have $\eta_N(T) \neq 0$. Hence if $A_N'(T)$ is regular, $b = 0$ and $c \neq 0$, the function ω_N is strictly decreasing.

In our example of the Euler–Bernoulli beam, we have $b = 0$ if $y_j^0 = y_j^1 = 0$ $(j \in \mathbb{N})$. So if we have $y_0 = y_1 = 0$ in the initial conditions, the equation $b = 0$ is valid. This implies that $\omega_N'(T) \leq 0$ and the function ω_N is decreasing. If in addition $A_N'(T)$ is regular, $(\psi_1, \psi_0 - \psi_2) \neq (0, 0)$, we have $c \neq 0$ and the function ω_N is strictly decreasing.

Lemma 6.2.2 *If the kernel K satisfies the equation $K(0) = 0$, then $G_j(T) = 0$. Hence in that case*

$$\begin{aligned}
r_N'(T) &= (-b(T)z_j(T)/\ \ )_{j=1}^N, \\
A_N'(T) &= (z_i(T)z_j(T)/\ \ ^2)_{i,j=1}^N.
\end{aligned}$$

Proof Theorem 6.2.1 implies that the functions $G_j(T)$ are continuously differentiable with respect to T and that the derivative $\partial_T G_j(T)$ is the solution of the Volterra equation

$$\partial_T G_j(T)(t) - \int_t^T K(s-t)\partial_T G_j(T)(s)\, ds = z(T)K(0)K(T-t)/\ ^2.$$

If $K(0) = 0$, this yields $\partial_T G_j(T) = 0$. This implies that for all $T \in (0, \overline{T})$ $t \in (0, T)$ we have

$$G_j(T)(t) = G_j(t)(t) = z(t)K(0)/\ ^2 = 0.$$

Now the remaining assertions follow from Lemma 6.2.1. □

For the kernel in the example of the Euler–Bernoulli beam we have $K(0) = 0$. Lemma 6.2.2 implies that in this case the computation of the derivative $\omega_M'(T)$ is very cheap. Once the solution $\eta_N(T)$ is computed, we get the derivative $\omega_N'(T)$ almost for free!

Lemma 6.2.3 *If the functions b and z_j are (continuously) differentiable, the function ω_N is twice (continuously) differentiable on $[0, \overline{T}]$.*

The maps r_N and A_N are also twice (continuously) differentiable on $[0, \overline{T}]$.

Proof In the proof of Lemma 6.2.2, we have seen that the derivative $\partial_T G_j(T)$ exists.

Lemma 6.2.1 implies that the map r_N is twice differentiable on $[0, \overline{T}]$ and $r_N''(T) =$

$$\left(-b'(T)z_j(T)/\ \ - b(T)z_j'(T)/\ \ - b(T)G_j(T)(T) - \langle b, \partial_T G_j(T)\rangle_{(0,T)}\right)_{j=1}^N.$$

The map A_N is also twice differentiable on $[0, \overline{T}]$ and

$$\begin{aligned} A_N''(T) = \Big(& z_i'(T)z_j(T)/\ ^2 + z_i(T)z_j'(T)/\ ^2 \\ &+ H_i(T)(T)G_j(T)(T) + \langle G_i(T), G_j(T)\rangle_{(0,T)} + \langle H_i(T), \partial_T G_j(T)\rangle_{(0,T)} \\ &+ G_i(T)(T)H_j(T)(T) + \langle \partial_T G_i(T) H_j(T)\rangle_{(0,T)} + \langle G_i(T), G_j(T)\rangle_{(0,T)}\Big)_{i,j=1}^N \end{aligned}$$

The representation of $\omega_N'(T)$ given in Lemma 6.2.1 implies

$$\begin{aligned} \omega_N''(T) &= 2(\eta_N'(T))^T r_N'(T) + 2(\eta_N(T))^T r_N''(T) \\ &-2(\eta_N'(T))^T A_N'(T)\eta_N(T) - (\eta_N(T))^T A_N''(T)\eta_N(T). \end{aligned}$$

□

Remark 6.2.2 If $K(0) = 0$ we have

$$r_N''(T) = \Big(-b'(T)z_j(T)/ \quad - b(T)z_j'(T)/ \quad \Big)_{j=1}^N,$$

$$A_N''(T) = \Big(z_i'(T)z_j(T)/ \quad ^2 + z_i(T)z_j'(T)/ \quad ^2\Big)_{i,j=1}^N.$$

So in this case the derivative $\omega_N''(T)$ is easy to compute: Only to obtain $\eta_N'(T)$ a system of linear equations has to be solved. The matrix and the right–hand side of this system is not costly to compute.

The existence of ω_N'' is used in the proof of quadratic convergence of Newton's method applied to the function $\omega_N - \beta^2$ ($\beta \in I\!R$) at the end of Section 7.5.2 in Chapter 7 about time–optimal control.

In the example of the Euler–Bernoulli beam, the functions z_j are C^∞. If $b = 0$, it is obviously also C^∞. In general, the function b is not differentiable.

If b is sufficiently regular, the following Lemma is interesting.

Lemma 6.2.4 *Let $n \in I\!N$. If the functions b and z_j are n–times (continuously) differentiable, the function ω_N and the maps r_N, η_N and A_N are $(n+1)$–times (continuously) differentiable on $[0, \overline{T}]$.*

Proof By induction, Theorem 6.2.1 implies that the functions $G_j(T)$ are infinitely often differentiable and that the derivative $\partial_T^{(n)} G_j(T)$ is the solution of the Volterra equation

$$\partial_T^{(n)} G_j(T)(t) - \int_t^T K(s-t)\partial_T^{(n)} G_j(T)(s)\, ds) = z(T)K(0)^n K(T-t)/ \quad ^{n+1}.$$

Hence the definition of $r_N(T)$ implies that r_N is $(n+1)$–times (continuously) differentiable, and this is also true for $A_N(T)$.

The equation $A_N(T)\eta_N(T) - r_N(T) = 0$ implies that $\eta_N(T)$ is $(n+1)$–times (continuously) differentiable on $[0, \overline{T}]$. Now the assertion follows from the equation $\omega_N(T) = (\eta_N(T))^T r_N(T)$. $\square$

The Lemma suggests that for the computation of a root of the function $\omega_N - \beta^2$, it makes sense to use methods that use both the first and the second derivative, like Euler's or Halley's method (see e.g. [36]).

Figure 6.2.1 shows the values of the functions ω_8 for the example of the Euler–Bernoulli beam on an equidistant grid consisting of 100 points on the interval $[1, 2]$ for $= 1$, $\psi_0 = -2$, $l = 1$ and $D \in \{1, 10, 100, 1000\}$.

Figure 6.2.2 shows the values of the functions $(\omega_8 - \omega_6)/\omega_8$ for the example of the Euler–Bernoulli beam on an equidistant grid consisting of 100 points on the interval $[1, 2]$ for $= 1$, $\psi_0 = -2$, $l = 1$ and

$D \in \{1, 10, 100, 1000\}$.

Figure 6.2.1

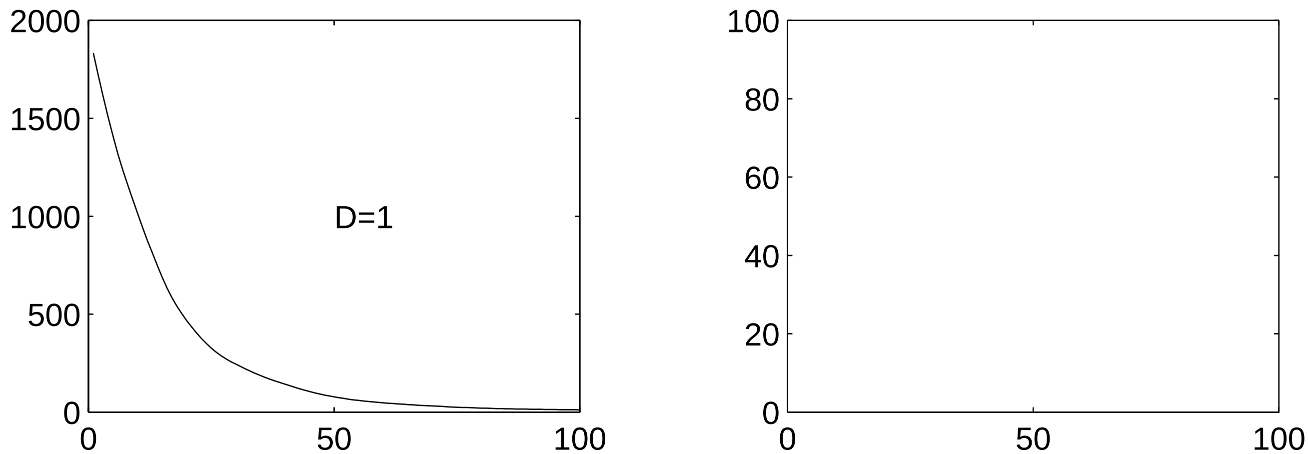


Figure 6.2.2

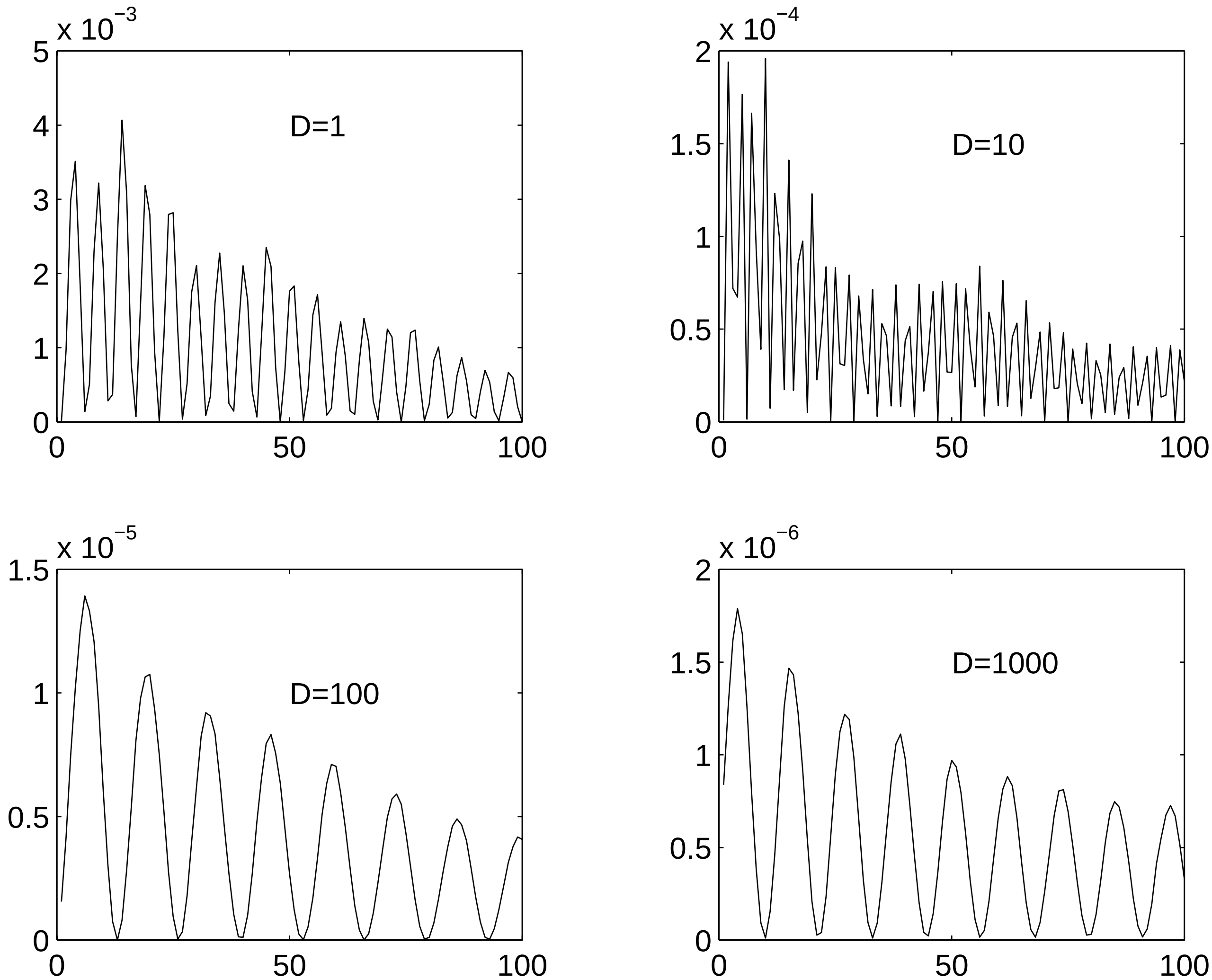


Figure 6.2.2. The Graph of the functions $(\omega_8 - \omega_6)/\omega_8$ on $[1, 2]$ for $\quad = 1$

Figure 6.2.3 shows the values of the derivatives ω_8' for the example of the Euler–Bernoulli beam on an equidistant grid consisting of 100 points on the interval $[1, 2]$ for $\quad = 1$, $\psi_0 = -2$, $l = 1$ and $D \in \{1, 10, 100, 1000\}$.

Figure 6.2.3

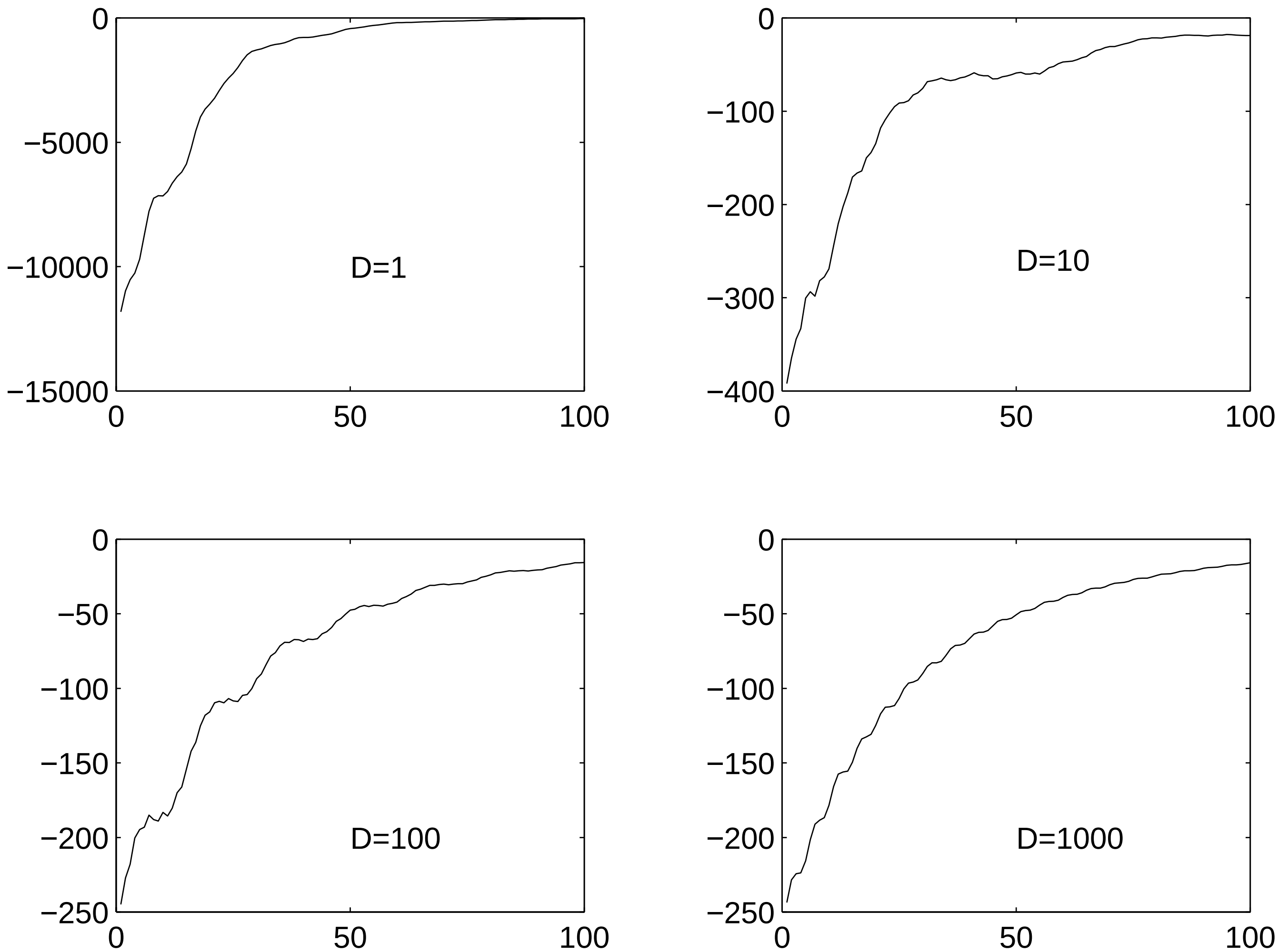


Figure 6.2.3. The Graph of the derivative ω_8' on $[1, 2]$ for $\quad = 1$

6.2.3 Euler–Bernoulli beam: ω_∞ is decreasing

In this subsection, we consider the problem of finding a control for the Euler–Bernoulli beam for which the corresponding torque function has minimal norm in $L^2[0,T]$. We show that the corresponding optimal value function is decreasing. Our problem is

$$P_\infty(T): \quad \min \|M(t)\|^2_{(0,T)} \; s.t. \int_0^T u(t) z_j(t)\, dt = c_j, \; j \in \{i, i+1, ..., M\},$$

where $i = -1$ or $i = 0$, $M(t) = (\quad + l^3/3)u(t) + \int_0^1 x y_{tt}(x,t)\, dx$ is as in Subsection 4.1.1 and the moment equations are as in (3.9), (3.10), (3.11) and possibly (3.12). For the value $\omega_\infty(T)$ of Problem $P_\infty(T)$ we have the following result:

Theorem 6.2.2 *For the problem of finding a control for the Euler–Bernoulli beam for which the torque function has minimal norm in $L^2[0,T]$, the corresponding optimal value function ω_∞ is decreasing on $[\underline{T}, \overline{T}]$.*

Proof Let $\underline{T} \le T_1 < T_2 \le \overline{T}$. Let u_1 be the solution of $P_\infty(T_1)$. We define $u(t) := u_1(t)$ if $t \le T_1$ and $u(t) := 0$ if $t > T_1$.

We consider the initial boundary value problem with initial conditions $y(x,T_1) = y_t(x,T_1) = 0$, boundary conditions $y(0,t) = y'(0,t) = 0 = y''(l,t) = y'''(l,t)$ and the beam equation

$$y_{tt}(x,t) + D y_{xxxx}(x,t) = -x u(t), \quad x \in [0,l], t \ge T_1.$$

Then the unique solution is $y(x,t) = 0$, $t \ge T_1$. Hence $y_{tt}(x,t) = 0$, $t \ge T_1$. Thus for $t \ge T_1$, we have

$$(Su - b)(t) = M(t) = (\quad + l^3/3)u(t) + \int_0^1 x y_{tt}(x,t)\, dx = 0.$$

Hence $(Su-b)(t) = 0$, if $t \ge T_1$. Since u satisfies the moment equations in $P_\infty(T_2)$, we conclude that

$$\omega_\infty(T_2) \le \|Su - b\|^2_{(0,T_2)} = \|Su_1 - b\|^2_{(0,T_1)} = \omega_\infty(T_1),$$

and the assertion follows. $\square$

6.3 Is the Optimal Value Function of P_∞ Differentiable?

In the area of parametric optimization, results about the one–sided differentiability of optimal value functions are well–known (see for example [10], [13], [14], [46], [7], [44], [2], [29], [32], [33], [1], [34], [11], [6]).

Points where the right–hand side derivative and the left–hand side derivative are not equal can occur if the primal or the dual solution is not uniquely determined.

For problem $P_\infty(T)$, both the primal and the dual solution are uniquely determined. However, due to the special structure of the problem the general results are not applicable here.

To illustrate the difficulties connected with the question of differentiability of the optimal value function, we give an example. We present conditions on the solutions of $P_\infty(T)$ and on the sequences $\eta(T)$ that give the solutions of the Problems $P_\infty(T)$ that guarantee the differentiability of the optimal value function. To my knowledge, results of this type have not appeared in the literature before.

6.3.1 An Example

Let x_j and δ_j be as in Lemma 2.3.2. Let $\lambda_j = x_j^4$ and $T_c = 4/\pi$. For $j \in I\!N$, define

$$\begin{aligned} z_{2j}(t) &= \cos(\sqrt{\lambda_j} t), \\ z_{2j-1}(t) &= \sin(\sqrt{\lambda_j} t). \end{aligned}$$

According to Lemma 2.3.2 we have $\sqrt{\lambda_j} = (j\pi - \pi/2 + \epsilon_j)^2$, with $|\epsilon_j| \le |\delta_j|$, so $\lim_{j\to\infty} \epsilon_j = 0$.

Hence we have

$$\lim_{j\to\infty} \cos(\sqrt{\lambda_j} T_c) = \lim_{j\to\infty} \cos(\pi(4j(j-1)+1)) = \cos(\pi) = -1$$

and

$$\lim_{j\to\infty} \sin(\sqrt{\lambda_j} T_c) = \sin(\pi) = 0.$$

Hence there exists $K_0 \in I\!N$ such that for all $j \ge K_0$,

$$\cos(\sqrt{\lambda_j} T_c) + \sin(\sqrt{\lambda_j} T_c) \le -1/2. \tag{6.11}$$

Let $\beta_{2j} = 1/j$ and $\beta_{2j-1} = 1/j$. Then $(\beta_j)_{j\in I\!N} \in l^2$. Since the $(z_j)_{j\in I\!N}$ are a Bessel sequence (see Theorem 3.1.2), the function

$$v(t) := \sum_{j=1}^{\infty} \beta_j z_j(t)$$

is in $L^2(0, T_c)$.

Let $c_j = \langle v, z_j\rangle_{(0,T_c)}$ $\quad (j \in I\!N)$. Then $(c_j)_{j\in I\!N} \in l^2$. Consider the problem

$$P_\infty(T): \quad \min \|u\|^2_{(0,T)} \text{ s.t.}$$
$$\langle u, z_j\rangle_{(0,T)} = c_j \quad (j \in I\!N).$$

Then Lemma 6.1.5 implies that v is the unique solution of Problem $P_\infty(T_c)$.

According to Lemma 6.1.9, we have

$$\omega(T) = \sup_{\alpha\in l^2} - \sum_{i,j=1}^{\infty} \alpha_i\alpha_j\langle z_i, z_j\rangle_{(0,T)} + 2\sum_{j=1}^{\infty} \alpha_j c_j.$$

Hence the following inequality is valid for all $h > 0$:

$$\begin{aligned}
\omega(T_c - h) - \omega(T_c) &= \sup_{\alpha\in l^2} - \sum_{i,j=1}^{\infty} \alpha_i\alpha_j\langle z_i, z_j\rangle_{(0,T_c-h)} + 2\sum_{j=1}^{\infty} \alpha_j c_j \\
&\quad + \sum_{i,j=1}^{\infty} \beta_i\beta_j\langle z_i, z_j\rangle_{(0,T_c)} - 2\sum_{j=1}^{\infty} \beta_j c_j \\
&\geq \sum_{i,j=1}^{\infty} \beta_i\beta_j\langle z_i, z_j\rangle_{(T_c-h,T_c)} \\
&= \int_{T_c-h}^{T_c} v^2(t)\,dt.
\end{aligned}$$

Hence for all $h > 0$ we have

$$\frac{\omega(T_c - h) - \omega(T_c)}{-h} \leq \sum_{i,j=1}^{\infty} \beta_i\beta_j \frac{(-1)}{h} \int_{T_c-h}^{T_c} z_i(t)z_j(t)\,dt.$$

Hence we have

$$\limsup_{h\to 0+} \frac{\omega(T_c - h) - \omega(T_c)}{-h} \quad \leq \tag{6.12}$$
$$\limsup_{h\to 0+} \frac{(-1)}{h} \int_{T_c-h}^{T_c} (v(t))^2 \, dt.$$

But (6.11) implies the inequality

$$\sum_{i=2K_0-1}^{\infty} \beta_i z_i(T_c) = \sum_{i=K_0}^{\infty} \frac{1}{i}\left(\cos(\sqrt{\lambda_j}T_c) + \sin(\sqrt{\lambda_j}T_c)\right) \leq \sum_{i=K_0}^{\infty} -\frac{1}{2i} = -\infty.$$

Thus we have $v(T_c) = -\infty$. If T_c is a regular point of v^2 in the sense that

$$\lim_{h\to 0+} \frac{1}{h}\int_{T_c-h}^{T_c} v(t)^2\,dt = v(T_c)^2,$$

then (6.12) implies

$$\lim_{h\to 0+} \frac{\omega(T_c-h)-\omega(T_c)}{-h} = -\infty,$$

and ω is not differentiable at T_c. If $\limsup_{h\to 0+} -\int_{T_c-h}^{T_c} v(t)^2\,dt/h$ is a real number, then (6.12) implies an upper bound for the left–hand side upper Dini–derivative of ω.

6.3.2 Sufficient Conditions for Differentiability and Lipschitz Continuity of ω

For $T \in [\underline{T}, \overline{T}]$, consider the standard minimum norm problem

$$P_\infty(T): \quad \min \|u\|^2_{(0,T)} \ \text{ s.t.}$$

$$\langle u, z_j\rangle_{(0,T)} = c_j \quad (j \in I\!N)$$

with optimal value $\omega(T)$. Let Assumptions A1 and A2 be valid.

We start with results about the regularity of ω with assumptions that are stated as conditions on the solution u of $P_\infty(T)$.

Lemma 6.3.1 *Let $T \in (\underline{T}, \overline{T}]$ be given.*

a) For all $h \in (0, T-\underline{T}]$, the following inequality holds:

$$\omega(T-h) \geq \omega(T) + \langle u_*, u_*\rangle_{(T-h,T)},$$

where u_* is the solution of Problem $P_\infty(T)$.

b) If T is a regular point in the sense that

$$\liminf_{h\to 0+} \frac{1}{h}\langle u_*, u_*\rangle_{(T-h,T)} \geq u_*(T)^2, \tag{6.13}$$

then

$$\limsup_{h\to 0+} \frac{\omega(T-h)-\omega(T)}{-h} \leq u_*(T)^2.$$

Remark 6.3.1 Lemma 6.3.1 states that if the end–point of the time–interval is a regular point for the solution of Problem $P_\infty(T)$ in the sense of (6.13), then we have an upper bound for the left–hand side upper Dini–derivative of the optimal value function in terms of the value of the optimal solution at the end–point of the time–interval.

Proof

a) Let $\beta \in l^2$ be such that $u_* = \sum_{j=1}^\infty \beta_j z_j$. According to Lemma 6.1.9 we have

$$\omega(T) = \sup_{\alpha \in l^2} - \sum_{i,j=1}^\infty \alpha_i \alpha_j \langle z_i, z_j \rangle_{(0,T)} + 2 \sum_{j=1}^\infty \alpha_j c_j.$$

Hence for all $h \in (0, T - \underline{T})$, the following inequality is valid:

$$\begin{aligned}
\omega(T-h) - \omega(T) &= \sup_{\alpha \in l^2} - \sum_{i,j=1}^\infty \alpha_i \alpha_j \langle z_i, z_j \rangle_{(0,T-h)} + 2 \sum_{j=1}^\infty \alpha_j c_j \\
&\quad + \sum_{i,j=1}^\infty \beta_i \beta_j \langle z_i, z_j \rangle_{(0,T)} - 2 \sum_{j=1}^\infty \beta_j c_j \\
&\geq \sum_{i,j=1}^\infty \beta_i \beta_j \langle z_i, z_j \rangle_{(T-h,T)} \\
&= \langle u_*, u_* \rangle_{(T-h,T)}.
\end{aligned}$$

b) Assertion a) implies that for all $h \in (0, T - \underline{T})$, we have

$$\frac{\omega(T-h) - \omega(T)}{-h} \leq -\frac{1}{h} \langle u_*, u_* \rangle_{(T-h,T)}.$$

Now the assumption that T is a regular point yields the assertion. □

Now we give a sufficient condition for differentiability of ω in T that depends on the behaviour of the solutions in a neighbourhood of T.

Theorem 6.3.1 *Let $T \in [\underline{T}, \overline{T}]$ be given. For $t \in [\underline{T}, \overline{T}]$, let $u_*(t)$ be the solution of Problem $P_\infty(t)$. Assume that T is a regular point in the sense that*

$$\lim_{h \to 0} \frac{1}{h} \langle u_*(T+h), u_*(T) \rangle_{(T,T+h)} = (u_*(T)(T))^2, \tag{6.14}$$

where $u_*(T) = \sum_{i=1}^\infty \eta_i z_i$ is defined on the interval $[0, \overline{T}]$ since $z_i \in Z(0, \overline{T})$.

Then ω is differentiable in T and

$$\omega'(T) = -(u_*(T)(T))^2.$$

Proof For $T \in [\underline{T}, \overline{T}]$, define the linear operator $A(T) : l^2 \to l^2$, $A(T)\alpha = (\sum_{j=1}^{\infty} \alpha_j \langle z_i, z_j \rangle_{(0,T)})_{i=1}^{\infty}$. Assumption $A1$ implies that $A(T)$ is a bounded invertible operator, see Theorem 9 in [50], p.32.

Let $\eta(t)$ be such that $u_*(t) = \sum_{i=1}^{\infty} \eta_i(t) z_i$. Then we have $\eta(t) = A(t)^{-1}c$.

For $h \neq 0$ such that $T + h \in [\underline{T}, \overline{T}]$, we have $c = A(T)\eta(T) = A(T+h)\eta(T+h)$, hence $\eta(T) = A(T)^{-1}[A(T+h)\eta(T+h)]$, thus

$$\eta(T+h) - \eta(T) = A(T)^{-1}[A(T) - A(T+h)]\eta(T+h).$$

Since $\omega(T) = \sum_{i=1}^{\infty} \eta_i(T) c_i =: c^T \eta(T)$, we obtain

$$\begin{aligned}
\frac{\omega(T+h) - \omega(T)}{h} &= c^T \left(\frac{\eta(T+h) - \eta(T)}{h} \right) \\
&= -c^T A(T)^{-1} \frac{A(T+h) - A(T)}{h} \eta(T+h) \\
&= -\eta(T)^T \frac{A(T+h) - A(T)}{h} \eta(T+h) \\
&= -\frac{1}{h} \langle u_*(T+h), u_*(T) \rangle_{(T,T+h)},
\end{aligned}$$

and due to (6.14), for $h \to 0$ we obtain the assertion. $\square$

Now we give a sufficient condition for the Lipschitz continuity of ω that depends on the uniform boundedness of the solutions in L^∞.

Theorem 6.3.2 *If the solutions are uniformly bounded in the sense that there exists $M \in \mathbb{R}$ such that for all $t \in [\underline{T}, \overline{T}]$ we have*

$$|u_*(t)| \leq M \text{ almost everywhere in } [0, \overline{T}], \tag{6.15}$$

then ω is Lipschitz continuous on $[\underline{T}, \overline{T}]$, in the sense that

$$|\omega(t_1) - \omega(t_2)| \leq M^2 |t_1 - t_2|$$

for all t_1, $t_2 \in [\underline{T}, \overline{T}]$.

Proof From the proof of Theorem 6.3.1 we have

$$\begin{aligned}
|\omega(T+h) - \omega(T)| &\leq |\langle u_*(T+h), u_*(T) \rangle_{(T,T+h)}| \\
&\leq hM^2. \square
\end{aligned}$$

Now we give results about the regularity of ω with assumptions on the regularity of the sequence $\eta(t)$ that gives the solution of $P_\infty(t)$.

Theorem 6.3.3 *Assume that A1 holds and that the functions z_i are continuous and*

$$\max_{t\in[0,\overline{T}]} |z_i(t)| \le 1, \;\; i \in \mathbb{N} \tag{6.16}$$

and that they are continuously differentiable with

$$\max_{t\in[0,\overline{T}]} |z_i'(t)| \le \sqrt{\lambda_i}, \;\; i \in \mathbb{N} \tag{6.17}$$

where $(\lambda_i)_{i\in\mathbb{N}}$ is a sequence of positive numbers. Moreover, assume that

$$\text{for all } \; i, \; \lambda_i \ge 1 \text{ and that there is } \; s > 0 \text{ such that } \sum_{i=1}^{\infty} 1/\lambda_i^s < \infty. \tag{6.18}$$

Let $t \in [\underline{T}, \overline{T}]$ be such that $\eta(t) = A^{-1}(t)c \in l_r^2$, where $r > s+1$. Then

$$\liminf_{h\to 0+} \frac{\omega(t+h)-\omega(t)}{h} \ge -\eta(t)D(t)\eta(t),$$

$$\limsup_{h\to 0-} \frac{\omega(t+h)-\omega(t)}{h} \le -\eta(t)D(t)\eta(t),$$

Remark 6.3.2 For the trigonometric moment problem (2.35), (2.36) conditions (6.16) and (6.17) hold and (6.18) is valid for $s > 1/4$.

For the proof of Theorem 6.3.3, we need a number of Lemmas.

It is clear that Condition A1 is equivalent to the following statement:

There exist constants $M, P > 0$ such that for all $N \in \mathbb{N}$, $(a_1, ..., a_N)^T \in \mathbb{R}^N$ and $t \in [\underline{T}, \overline{T}]$ we have

$$\left(\frac{1}{M^2}\right)\left(\sum_{i=1}^N a_i^2\right) \le a^T A(t) a = \sum_{i=1}^N \sum_{j=1}^N a_i \langle z_i, z_j\rangle_{(0,t)} a_j \le P^2 \left(\sum_{i=1}^N a_i^2\right). \tag{6.19}$$

The following Lemma contains another equivalent formulation, which is well–known.

Lemma 6.3.2 *Condition A1 is is equivalent to the statement*

$$\text{for all } a \in l^2 : \frac{1}{M^2}\|a\|_{l^2} \le \|A(t)a\|_{l^2} \le P^2\|a\|_{l^2}. \tag{6.20}$$

It is also equivalent to the statements

$$\text{for all } a \in l^2 : \|A(t)a\|_{l^2} \le P^2\|a\|_{l^2} \text{ and} \tag{6.21}$$

$$A(t) \; \text{ is invertible and for all } a \in l^2 : \|A(t)^{-1}a\|_{l^2} \le M^2\|a\|_{l^2}. \tag{6.22}$$

Proof Assume that (6.22) and (6.21) hold. Then (6.21) implies that

$$a^T A(t)a \leq \|A(t)a\|_{l^2}\|a\|_{l^2} \leq P^2\|a\|_{l^2}$$

and (6.22) implies that

$$a^T A^{-1}(t)a \leq \|a\|_{l^2} M^2\|a\|_{l^2} = M^2\|a\|_{l^2}^2.$$

Since $A(t)$ is positive and symmetric, the generalized Cauchy–Schwarz inequality (see [15], p. 195) implies

$$\|a\|_{l^2}^4 = \left(a^T A(t)A^{-1}(t)a\right)^2 \leq$$

$$(a^T A(t)a)(a^T A^{-1}(t)a) \leq (a^T A(t)a)M^2\|a\|_{l^2}^2,$$

thus $a^T A(t)a \geq \|a\|_{l^2}^2/M^2$. Hence (6.19) holds.

Assume now that (6.19) is valid. Then we have

$$\frac{1}{M^2}\|a\|_{l^2}^2 \leq a^T A(t)a \leq \|a\|_{l^2}\|A(t)a\|_{l^2},$$

hence

$$\|A(t)a\|_{l^2} \geq \frac{1}{M^2}\|a\|_{l^2},$$

which implies (6.22), since $A(t)$ is surjective due to Theorem 3.0.2 (ii) (see [12], I.3.7). The generalized Cauchy–Schwarz inequality implies

$$\|a\|_{l^2}^4 \leq (a^T A(t)^{-1}a)(a^T A(t)a) \leq \|A(t)^{-1}a\|_{l^2}\|a\|_{l^2} P^2\|a\|_{l^2}^2,$$

hence $\|A(t)^{-1}a\|_{l^2} \geq \frac{1}{P^2}\|a\|_{l^2}$, which implies (6.21). $\Box$

Note that the dual space of l_r^2 is l_{-r}^2.

Lemma 6.3.3 *For $t \in [\underline{T}, \overline{T}]$, $\alpha \in l_r^2$, let*

$$(D(t)\alpha)_i = \sum_{i=1}^{\infty} z_i(t)z_j(t)\alpha_j.$$

Assume that (6.16) and (6.18) holds. Let $r > s$. Then $D(t)$ is a continuous linear map from l_r^2 into l_{-r}^2 and for all $\alpha \in l_r^2$ we have

$$\|D(t)\alpha\|_{l_{-r}^2} \leq \|\alpha\|_{l_r^2}\left(\sum_{i=1}^{\infty}\lambda_i^{-r}\right). \tag{6.23}$$

Proof Let $\beta \in l^2_r$. Then

$$
\begin{aligned}
|\beta^T D(t)\alpha| &= |\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\beta_i z_i(t) z_j(t)\alpha_j| \\
&\le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}|\beta_i\alpha_j| \\
&= \left(\sum_{i=1}|\beta_i|\right)\left(\sum_{j=1}^{\infty}|\alpha_j|\right) \\
&= \left(\sum_{i=1}^{\infty}|\beta_i|\lambda_i^{r/2}\lambda_i^{-r/2}\right)\left(\sum_{j=1}^{\infty}|\alpha_j|\lambda_j^{r/2}\lambda_j^{-r/2}\right) \\
&\le \|\beta\|_{l^2_r}\left(\sum_{i=1}^{\infty}\lambda_i^{-r}\right)\|\alpha\|_{l^2_r},
\end{aligned}
$$

where for the last line we have applied the Cauchy–Schwarz inequality twice. Hence the inequality (6.23) follows. □

Lemma 6.3.4 *Assume that A1, (6.16), (6.17) and (6.18) hold. Let* $r > s+1$. *For* $t \in [\underline{T}, \overline{T}]$, $\alpha \in l^2_r$, *let*

$$\left(\bar{A}(t)\alpha\right)_i = \sum_{j=1}^{\infty}\langle z_i, z_j\rangle_{(0,t)}\alpha_j.$$

Then $\bar{A}(t)$ *is a bounded linear operator from* l^2_r *into* l^2_{-r}. $\bar{A}(t)$ *is Fréchet–differentiable with respect to* t, *and*

$$\left(\bar{A}'(t)\alpha\right)_i = \sum_{i=1}^{\infty} z_i(t) z_j(t)\alpha_j = (D(t)\alpha)_i.$$

Proof Due to A1, for $\alpha \in l^2_r$ we have

$$\|\bar{A}(t)\alpha\|_{l^2_{-r}} \le \|\bar{A}(t)\alpha\|_{l^2} \le P^2\|\alpha\|_{l^2} \le P^2\|\alpha\|_{l^2_r}.$$

Let $h \neq 0$ be such that $t+h \in [\underline{T}, \overline{T}]$. The Taylor–expansion implies the existence of numbers $\xi_{ij} \in (0, \overline{T})$ such that

$$\left|\frac{1}{h}\langle z_i, z_j\rangle_{(t,t+h)} - z_i(t) z_j(t)\right|$$

$$= \frac{|h|}{2}\left|z_i(\xi_{ij}) z_j'(\xi_{ij}) + z_i'(\xi_{ij}) z_j(\xi_{ij})\right| \le \frac{|h|}{2}(\sqrt{\lambda_i} + \sqrt{\lambda_j}).$$

Let $\alpha \in l_r^2$. Define $A_i = \sum_{j=1}^{\infty}(\sqrt{\lambda_i} + \sqrt{\lambda_j})\alpha_j$. Then for all $\beta \in l_r^2$, we have

$$\begin{aligned}\left|\sum_{i=1}^{\infty} A_i\beta_i\right| &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |\beta_i|(\sqrt{\lambda_i} + \sqrt{\lambda_j})|\alpha_j| \\ &\leq \left(\sum_{i=1}^{\infty}|\beta_i|\sqrt{\lambda_i}\right)\left(\sum_{j=1}^{\infty}|\alpha_j|\right) + \left(\sum_{i=1}^{\infty}|\beta_i|\right)\left(\sum_{j=1}^{\infty}|\alpha_j|\sqrt{\lambda_j}\right).\end{aligned}$$

For $q \geq s$, we have

$$C_q := \left(\sum_{i=1}^{\infty}\frac{1}{\lambda_i^q}\right)^{1/2} < \infty. \tag{6.24}$$

For $\gamma \in l_r^2$, we have

$$\begin{aligned}\sum_{i=1}^{\infty}|\gamma_i|\sqrt{\lambda_i} &= \sum_{i=1}^{\infty}|\gamma_i|\lambda_i^{r/2}\lambda_i^{(1-r)/2} \\ &\leq \|\gamma\|_{l_r^2}\left(\sum_{i=1}^{\infty}\frac{1}{\lambda_i^{r-1}}\right) = \|\gamma\|_{l_r^2}C_{r-1}.\end{aligned}$$

Moreover,

$$\sum_{i=1}^{\infty}|\gamma_i| \leq \|\gamma\|_{l_r^2}C_r.$$

Hence for all $\beta \in l_r^2$ we have

$$\left|\sum_{i=1}^{\infty}A_i\beta_i\right| \leq \|\beta\|_{l_r^2}\|\alpha\|_{l_r^2}2C_{r-1}C_r.$$

Thus we conclude that

$$\|(A_i)_i\|_{l_{-r}^2} \leq 2\|\alpha\|_{l_r^2}C_{r-1}C_r.$$

Hence

$$\begin{aligned}&\left\|\left[\frac{\bar{A}(t+h)-\bar{A}(t)}{h} - D(t)\right]\alpha\right\|_{l_{-r}^2} \\ =\; &\left\|\left(\sum_{j=1}^{\infty}[\frac{1}{h}\langle z_i, z_j\rangle_{(t,t+h)} - z_i(t)z_j(t)]\alpha_j\right)_i\right\|_{l_{-r}^2} \\ \leq\; &\left\|\left(\frac{h}{2}\sum_{j=1}^{\infty}(\sqrt{\lambda_i}+\sqrt{\lambda_j})\alpha_j\right)_i\right\|_{l_{-r}^2}\end{aligned}$$

$$\begin{aligned} &= \frac{|h|}{2}\|(A_i)_i\|_{l^2_{-r}} \\ &\leq |h| \, \|\alpha\|_{l^2_r} C_{r-1} C_r. \end{aligned} \tag{6.25}$$

Hence for $h \to 0$ the assertion that $\bar{A}$ is Fréchet–differentiable in t follows. □

Proof of Theorem 6.3.3 Let $h \neq 0$ be such that $t + h \in [\underline{T}, \overline{T}]$. Then we have

$$\begin{aligned} \omega(t+h) - \omega(t) &= \sup_{\alpha \in l^2} - \sum_{i,j=1}^{\infty} \alpha_i \alpha_j \langle z_i, z_j \rangle_{(0,t+h)} + 2 \sum_{j=1}^{\infty} \alpha_j c_j \\ &\quad + \sum_{i,j=1}^{\infty} \eta_i(t)\eta_j(t)\langle z_i, z_j\rangle_{(0,t)} - 2\sum_{j=1}^{\infty} \eta_j(t) c_j \\ &\geq - \sum_{i,j=1}^{\infty} \eta_i(t)\eta_j(t)\langle z_i, z_j\rangle_{(t,t+h)} \\ &= -\eta(t)^T (A(t+h) - A(t))\eta(t). \end{aligned}$$

Since $\eta(t) \in l_r^2$, inequality (6.25) inplies

$$\begin{aligned} &|\eta(t)^T (A(t+h) - A(t))\eta(t)/h - \eta(t) D(t) \eta(t)| \\ &\leq \|\eta(t)\|_{l_r^2} \|(\frac{A(t+h) - A(t)}{h} - D(t))\eta(t)\|_{l^2_{-r}} \\ &\qquad \leq \|\eta(t)\|_{l_r^2} |h| \, \|\eta(t)\|_{l_r^2} C_{r-1} C_r. \end{aligned}$$

Thus we have

$$\lim_{h \to 0} \eta(t)^T \frac{A(t+h) - A(t)}{h} \eta(t) = \eta(t)^T D(t) \eta(t).$$

Hence we conclude that

$$\begin{aligned} \liminf_{h \to 0+} \frac{\omega(t+h) - \omega(t)}{h} &\geq - \lim_{h \to 0+} \eta(t)^T \frac{A(t+h) - A(t)}{h} \eta(t) \\ &= -\eta(t)^T D(t) \eta(t). \end{aligned}$$

The last assertion follows analogously. □

To prove a result about the differentiability of ω, we need the following Lemma.

Lemma 6.3.5 *Let a sequence $(t_k)_{k \in \mathbb{N}}$ with $t_k \in [\underline{T}, \overline{T}]$, ($k \in \mathbb{N}$) be given. Assume that $\lim_{k \to \infty} t_k = t$. For $T \in [\underline{T}, \overline{T}]$, let $\eta(T) = A(T)^{-1} c$ and $u(T) = \sum_{i=1}^{\infty} \eta_i(T) z_i$.*

Assume that A1, (6.16) and (6.18) hold and that

$$\sup_{k\in \mathbb{N}} \|\eta(t_k)\|_{l_s^2} < \infty.$$

Then

$$\lim_{k\to\infty} u(t_k)(t) = u(t)(t).$$

Proof Let

$$L_1 = \limsup_{k\to\infty} u(t_k)(t).$$

There exists a subsequence $(t_{k_j})_{j\in \mathbb{N}}$ such that

$$L_1 = \lim_{j\to\infty} u(t_{k_j})(t).$$

Since it is bounded in l_s^2, the sequence $(\eta(t_{k_j}))_{j\in \mathbb{N}}$ possesses a weakly convergent subsequence. Let $(\tau_k)_{k\in \mathbb{N}}$ denote such a subsequence with weak limit $\tau_* \in l_s^2$. Assumptions (6.16) and (6.18) imply the inequality

$$\sum_{i=1}^{\infty} |z_i(t)|^2 \frac{1}{\lambda_i^s} \le \sum_{i=1}^{\infty} \frac{1}{\lambda_i^s} < \infty,$$

hence $(z_i(t))_{i\in \mathbb{N}} \in l_{-s}^2$. Hence the weak convergence implies that

$$\lim_{k\to\infty} \sum_{i=1}^{\infty} z_i(t)(\tau_k)_i = \sum_{i=1}^{\infty} z_i(t)(\tau_*)_i.$$

Thus we have

$$\begin{aligned} L_1 &= \lim_{j\to\infty} u(t_{k_j})(t) \\ &= \lim_{j\to\infty} \sum_{i=1}^{\infty} z_i(t)\eta_i(t_{k_j}) \\ &= \lim_{k\to\infty} \sum_{i=1}^{\infty} z_i(t)(\tau_k)_i \\ &= \sum_{i=1}^{\infty} z_i(t)(\tau_*)_i. \end{aligned}$$

Let $(l_k)_{k\in \mathbb{N}}$ be a sequence of natural numbers such that $\tau_k = \eta(t_{l_k})$. For all i, $k \in \mathbb{N}$, we have

$$\langle z_i, \sum_{j=1}^{\infty} (\tau_k)_j z_j \rangle_{(0,t_{l_k})} = c_i.$$

Hence for all $i \in I\!N$

$$\begin{aligned}
&c_i - \sum_{j=1}^{\infty} \langle z_i, z_j \rangle_{(0,t)} (\tau_*)_j \\
&= \lim_{k\to\infty} \sum_{j=1}^{\infty} \langle z_i, z_j \rangle_{(0,t_{l_k})} (\tau_k)_j - \sum_{j=1}^{\infty} \langle z_i, z_j \rangle_{(0,t)} (\tau_*)_j \\
&= \lim_{k\to\infty} \sum_{j=1}^{\infty} \left(\langle z_i, z_j \rangle_{(0,t_{l_k})} - \langle z_i, z_j \rangle_{(0,t)} \right) (\tau_k)_j + \sum_{j=1}^{\infty} \langle z_i, z_j \rangle_{(0,t)} (\tau_k - \tau_*)_j \\
&=: F_1 + F_2.
\end{aligned}$$

We have

$$|F_1| \le \lim_{k\to\infty} \sum_{j=1}^{\infty} |\langle z_i, z_j \rangle_{(t_{l_k},t)} (\tau_k)_j| \le \lim_{k\to\infty} |t_{l_k} - t| C_s \|\tau_k\|_{l_s^2} = 0,$$

where C_s is defined in (6.24). Since $(\langle z_i, z_j \rangle_{(0,t)})_{j\in I\!N} \in l^2$, we have $(\langle z_i, z_j \rangle_{(0,t)})_{j\in I\!N} \in l^2_{-s}$, hence the weak convergence implies $F_2 = 0$. Thus for all $i \in I\!N$, we have

$$\sum_{j=1}^{\infty} \langle z_i, z_j \rangle_{(0,t)} (\tau_*)_j = c_i,$$

hence $A(t)\tau_* = c$, and therefore $\tau_* = \eta(t)$. Hence

$$\limsup_{k\to\infty} u(t_k)(t) = L_1 = \sum_{i=1}^{\infty} z_i(t)(\tau_*)_i = \sum_{i=1}^{\infty} z_i(t)\eta_i(t) = u(t)(t).$$

Analogously, we can show that

$$\liminf_{k\to\infty} u(t_k)(t) = u(t)(t),$$

and the assertion follows. □

Theorem 6.3.4 *Assume that A1 holds and that the functions z_i are continuously differentiable and that (6.16), (6.17) and (6.18) hold.*

Let $t \in [\underline{T}, \overline{T}]$ be such that there exists a neighbourhood $U \subset [\underline{T}, \overline{T}]$ of t such that for all $t \in U$, $\eta(t) \in l_r^2$ where $r > s+1$ and

$$\sup_{s\in U} \|\eta(s)\|_{l_r^2} < \infty.$$

Then ω is differentiable in t and

$$\omega'(t) = -\eta(t) D(t) \eta(t).$$

Proof Let $h \neq 0$ be such that $t+h \in U$. Then we have

$$\begin{aligned}
\omega(t+h)-\omega(t) &= -\sum_{i,j=1}^{\infty} \eta_i(t+h)\eta_j(t+h)\langle z_i, z_j\rangle_{(0,t+h)} + 2\sum_{j=1}^{\infty} \eta_j(t+h)c_j \\
&\quad - \sup_{\alpha\in l^2}\left[-\sum_{i,j=1}^{\infty} \alpha_i\alpha_j\langle z_i, z_j\rangle_{(0,t)} + 2\sum_{j=1}^{\infty}\alpha_j c_j\right] \\
&\leq -\sum_{i,j=1}^{\infty} \eta_i(t+h)\eta_j(t+h)\langle z_i, z_j\rangle_{(t,t+h)} \\
&= -\eta(t+h)^T(A(t+h)-A(t))\eta(t+h).
\end{aligned}$$

Since $\eta(t+h) \in l^2_r$, inequality (6.25) inplies

$$\begin{aligned}
&|\eta(t+h)^T(A(t+h)-A(t))\eta(t+h)/h - \eta(t+h)D(t)\eta(t+h)| \\
&\quad \leq \|\eta(t+h)\|_{l^2_r} \|(\frac{A(t+h)-A(t)}{h} - D(t))\eta(t+h)\|_{l^2_{-r}} \\
&\quad \leq \|\eta(t+h)\|_{l^2_r}|h|\ \|\eta(t+h)\|
\end{aligned}$$

Remark 6.3.3 It would be desirable to have sufficient conditions that guarantee that the assumption that there exists a neighbourhood $U \subset [\underline{T}, \overline{T}]$ of t such that for all $s \in U$, $\eta(s) \in l^2_r$ and the inequality

$$\sup_{s\in U} \|\eta(s)\|_{l^2_r} < \infty$$

is valid. Unfortunately we did not find a condition that can be verified for trigonometric moment problems.

The following Lemma gives a sufficient condition for $\eta(t) \in l^2_r$. However, in general it is not applicable to trigonometric moment problems.

Lemma 6.3.6 *Assume that A1 is valid. Let $(\lambda_j)_{j\in I\!N}$ be a sequence of real numbers such that for all i, $\lambda_i \geq 1$. Let $r \in [0,\infty)$ and $t \in [\underline{T}, \overline{T}]$.*

If $c \in l^2_{2r}$ and $\eta(t) = A^{-1}(t)c \geq 0$ and for all i, j: $a_{ij}(t) \geq 0$, then $\eta(t) \in l^2_r$. Moreover,

$$\|\eta(t)\|_{l^2_r} \leq M^2 \|c\|_{l^2_{2r}}.$$

Proof Let $c \in l^2_{2r} \subset l^2$. Then A1 implies that $\eta(t) \in l^2$.

Define $\tilde{\eta} = (\eta_i(t)\lambda_i^r)_{i\in I\!N} \in l^2_{-2r}$ and $\hat{\eta} = (\eta_i(t)\lambda_i^{r/2}))_{i\in I\!N} \in l^2_{-r}$. Then

$$|\tilde{\eta}^T A(t)\eta(t)| = |\tilde{\eta}^T c| \leq \|\tilde{\eta}\|_{l^2_{-2r}} \|c\|_{l^2_{2r}} < \infty.$$

We have

$$\tilde{\eta}^T A(t)\eta(t) = \sum_{i,j=1}^{\infty} \eta_i(t)\lambda_i^r a_{ij}(t)\eta_j(t) = \eta^T A(t)\tilde{\eta}(t) = \sum_{i,j=1}^{\infty} \eta_i(t) a_{ij}(t)\lambda_j^r \eta_j(t).$$

For all i, $j \in I\!N$, we have

$$\begin{aligned} & \eta_i(t)\lambda_i^r a_{ij}(t)\eta_j(t) + \eta_i(t)a_{ij}(t)\lambda_j^r\eta_j(t) \\ = \; & \eta_i(t)a_{ij}(t)(\lambda_i^r + \lambda_j^r)\eta_j(t) \\ \geq \; & 2\eta_i(t)a_{ij}(t)\lambda_i^{r/2}\lambda_j^{r/2}\eta_j(t). \end{aligned}$$

Hence we conclude that

$$2|\tilde{\eta}^T A(t)\eta(t)| \geq 2\hat{\eta}^T A(t)\hat{\eta} \geq \frac{2}{M^2}\|\hat{\eta}\|^2_{l^2} = \frac{2}{M^2}\|\eta(t)\|^2_{l^2_r}.$$

Thus

$$\begin{aligned} \frac{1}{M^2}\|\eta(t)\|^2_{l^2_r} & \leq \|\tilde{\eta}\|_{l^2_{-2r}} \|c\|_{l^2_{2r}} \\ & = \|\eta\|_{l^2} \|c\|_{l^2_{2r}} \\ & \leq M^2 \|c\|_{l^2} \|c\|_{l^2_{2r}} \\ & \leq M^2 \|c\|^2_{l^2_{2r}}. \quad \square \end{aligned}$$

Theorem 6.3.5 *Assume that A1, (6.16), (6.17) and (6.18) hold. Let* $r > s + 1$. *Let* $t \in [\underline{T}, \overline{T})$ *be such that* $\eta(t) \in l_r^2$. *Then there exists a constant* $L(t) > 0$ *such that for all* $t_2 \in (t, \overline{T}]$, *the following inequality is valid:*

$$\omega(t) \geq \omega(t_2) \geq \omega(t) - L(t)\ (t_2 - t).$$

Proof Let $t_2 \in (t, \overline{T}]$ and $h = t_2 - t > 0$. Let u_* be the solution of $P_\infty(t)$. Define $\hat{u}(s) := u_*(s)$, if $s \in [0, t]$, $\hat{u}(s) := 0$ if $s \in (t, t_2]$. Then for all $i \in I\!N$ we have

$$\langle \hat{u}, z_i \rangle_{(0,t_2)} = \langle u_*, z_i \rangle_{(0,t)} = c_i,$$

hence

$$\omega(t_2) \leq \|\hat{u}\|^2_{(0,t+h)} = \|u_*\|^2_{(0,t)} = \omega(t).$$

Moreover, we have

$$\begin{aligned} \frac{\omega(t+h) - \omega(t)}{h} &\geq -\eta(t)^T \frac{A(t+h) - A(t)}{h} \eta(t) \\ &= -\eta(t)^T \left(\frac{A(t+h) - A(t)}{h} - D(t) \right) \eta(t) - \eta(t)^T D(t) \eta(t) \\ &\geq -\eta(t)^T D(t) \eta(t) - \|\eta(t)\|^2_{l_r^2} C_{r-1} C_r |h|, \end{aligned}$$

where the last line follows from (6.25).

Let $L(t) = \eta(t)^T D(t) \eta(t) + \|\eta(t)\|^2_{l_r^2} C_{r-1} C_r [\overline{T} - \underline{T}] > 0$. Then

$$\omega(t+h) \geq \omega(t) - L(t)\ h,$$

and the assertion follows. □

Remark 6.3.4 The fact that ω is decreasing is well–known, but the lower bound for $\omega(t_2)$ appears to be new.

Up to now we have mainly studied the optimal value function. The following lemma contains a result about the sensitivity of the optimal solutions with respect to the parameter t. The question of stability of the optimal solutions is an important topic in parametric optimization, see for example [17], [18], [20].

Lemma 6.3.7 *Let* $c \in l^2$. *For* $t \in [\underline{T}, \overline{T}]$, *let* $\eta(t) = A(t)^{-1} c$. *Assume that A1 and A2 are valid. Then for all* t_1, $t_2 \in [\underline{T}, \overline{T}]$, *the following inequality is valid:*

$$\|\eta(t_1) - \eta(t_2)\|_{l^2} \leq M^2 P \| \sum_{j=1}^{\infty} \eta_j(t_1) z_j \|_{(t_1,t_2)}.$$

In particular, we have

$$\lim_{t_2 \to t_1} \|\eta(t_1) - \eta(t_2)\|_{l^2} = 0.$$

Moreover, if (6.16) holds, r is such that

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i^r} < \infty$$

and $\eta(t_1) \in l_r^2$, we have

$$\|\eta(t_1) - \eta(t_2)\|_{l^2} \leq \sqrt{|t_1 - t_2|} M^2 P \|\eta(t_1)\|_{l_r^2} \left(\sum_{i=1}^{\infty} \frac{1}{\lambda_i^r} \right)^{1/2} \tag{6.26}$$

and

$$|\omega(t_1) - \omega(t_2)| \leq \sqrt{|t_1 - t_2|} M^2 P \|c\|_{l^2} \|\eta(t_1)\|_{l_r^2} \left(\sum_{i=1}^{\infty} \frac{1}{\lambda_i^r} \right)^{1/2}. \tag{6.27}$$

Proof We have

$$\begin{aligned} & A(t_2)(\eta(t_1) - \eta(t_2)) \\ = \;& A(t_2)\eta(t_1) - c \\ = \;& (A(t_2) - A(t_1))\eta(t_1). \end{aligned}$$

Let $u = \sum_{j=1}^{\infty} \eta_j(t_1) z_j$

$$
\begin{aligned}
&\leq \left(\sum_{i=1}^{\infty} |\eta_i(t_1)| \lambda_i^{r/2} \lambda_i^{-r/2} \right)^2 |t_1 - t_2| \\
&\leq \|\eta(t_1)\|_{l_r^2}^2 \left(\sum_{i=1}^{\infty} \frac{1}{\lambda_i^r} \right) |t_1 - t_2|,
\end{aligned}
$$

hence if $\eta(t_1) \in l_r^2$, (6.26) follows. Now (6.27) is a consequence of the equation

$$\omega(t_1) - \omega(t_2) = c^T(\eta(t_1) - \eta(t_2))$$

and the Cauchy–Schwarz inequality. □

Chapter 7

Time–Optimal Control

7.1 Introduction

In this chapter we consider the problem of time–optimal control with a constraint that is related to the L^2–norm. Again we use the fact that the set of feasible controls can be described as the solution set of certain moment–problems.

After an analysis of the problem, we propose a fast algorithm for its numerical solution.

We prove the existence of a unique time–optimal control function. We characterize the optimal controlling time as the smallest root of the optimal value function of a time–parametric optimization problem, which has a sequence of moment equations as equality constraints. We consider the discretized problems where this sequence is truncated and give analogous results about the existence of a unique solution.

We prove the montone convergence of the smallest roots of the value functions of the discretized problems to the optimal controlling time. These roots are the optimal controlling times corresponding to certain discretized problems.

We also study, how the optimal controlling time depends on the value of the upper bound in the constraint. We show that if the value function is strictly decreasing, the dependence is continuous.

We give an algorithm for the computation of the smallest root of the value function of the discretized problem. The algorithm is a Newton–bisection method that is based on the fact that the optimal value function is differentiable and that the cost to compute the derivative is very small compared with the cost to compute the function value. Since the second derivative can also be computed very cheaply, we consider methods that use the second derivative as well, namely Euler's and Halley's method.

For the example of the Euler–Bernoulli beam, the optimal value function appears to be strictly decreasing. We apply Newton's method without modifications and obtain fast convergence. For Halley's method that uses the second derivative as well, convergence is even faster.

We think that it makes sense to use the derivative to reduce the number of iterations since the evaluation of the optimal value function is very expensive. The main difficulty consists in the computation of the functions H_j. To compute the values, we have implemented the approach based on rank–one matrices described at the end of Chapter 4.

So the usage of the derivative saves a lot of computing time, compared for example with the secant method.

An alternative approach to problems of time–optimal control would be to transform the time intervals to a fixed interval, for example $[0,1]$ by introducing an additional variable. The drawback of this approach is that the resulting problem is not convex with respect to the new variable, which appears in a rather complicated way.

7.2 The Problem

In this chapter, we consider the situation as defined in 6.1.1. In particular, we assume that $A0$ and $A1$ hold. Our problem is to find the shortest time T for which we can find a control function that is contained in the set $U(T,\beta,C)$. Define

$$T^* = \inf\{T \in [\underline{T}, \overline{T}] : U(T,\beta,c) \neq \emptyset\},$$

that is the infimum of all points $T \in [\underline{T}, \overline{T}]$ for which there exists a control $u \in Z(0,T)$ for which $\|S_T u - b\|^2_{(0,T)} \leq \beta^2$ and which satisfies the moment equations, i.e. such that $\langle u, z_j\rangle_{(0,T)} = c_j$ for all $j \in I\!N$.

The lower bound $\underline{T}$ is introduced since only for $T \geq \underline{T}$, $A1$ guarantees that the set $U(T,\infty,c)$ is nonempty (see Theorem 3.0.3). For the example of the Euler–Bernoulli beam, $\underline{T}$ can be chosen arbitrarily small. Assumption $A0$ implies $T^* \leq \overline{T}$.

The following Lemma guarantees the existence of a unique time–optimal control. In addition, the minimal controlling time T^* is characterized as the smallest root of the function $\omega(\cdot) - \beta^2$.

Lemma 7.2.1 *We have $\omega(T^*) \leq \beta^2$.*

If $T^* > \underline{T}$, then $\omega(T^*) = \beta^2$ and $U(T^*, \beta, c) \neq \emptyset$. Moreover, the time–optimal control is uniquely determined.

Proof Lemma 6.1.5 implies that the set $U(T, \beta, c)$ is nonempty if and only if $\omega(T) \leq \beta^2$. Hence the definition of T^* implies

$$T^* = \inf\{T \in [\underline{T}, \overline{T}] : \omega(T) \leq \beta^2\}. \tag{7.1}$$

Thus if $T^* > \underline{T}$, the continuity of ω (see Theorem 6.1.1) implies $\omega(T^*) = \beta^2$. Now Lemma 6.1.5 implies the existence of a unique time–optimal control. $\square$

7.3 The Discretized Problem

If we truncate the system of moment equations, we obtain an approximation for T^* that can be computed numerically. For $N \in \mathbb{N}$, let

$$\begin{array}{rcl} T_N^* = \inf\{T \in [\underline{T}, \overline{T}] : & & \text{there exists } u \in Z(0,T) : \\ \langle u, z_j \rangle_{(0,T)} & = & c_j,\ j \in \{1, ..., N\} \ \text{ and} \\ \|S_T u - b\|^2_{(0,T)} & \leq & \beta^2\} \end{array}$$

Analogously to Lemma 7.2.1, the number T_N^* can be characterized as the smallest root of the function $\omega_N(\cdot) - \beta^2$, where ω_N is the optimal value function of the discretized problem P_N.

Lemma 7.3.1 *We have $\omega_N(T_N^*) \leq \beta^2$.*

If $T_N^* > \underline{T}$, then $\omega(T_N^*) = \beta^2$ and there exists a uniquely determined function $\tilde{u}_N = u_N(T^*)$ with $\langle \tilde{u}_N, z_j \rangle_{(0,T_N^*)} = c_j, \quad j \in \{1, ..., N\}$ and $\|S_T \tilde{u}_N - b\|^2_{(0,T_N^*)} = \beta^2$.

Proof Lemma 6.1.3 implies that

$$T_N^* = \inf\{T \in [\underline{T}, \overline{T}] :\ \omega_N(T) \leq \beta^2\}.$$

Thus if $T_N^* > \underline{T}$, the continuity of ω_N (see Lemma 6.1.14) implies $\omega_N(T_N^*) = \beta^2$. Now Lemma 6.1.3 implies the assertion. $\square$

Next we consider the sequence $(T_N^*)_{N \in \mathbb{N}}$. Since $\omega_N(T) \leq \omega_{N+1}(T) \leq \omega(T)$ for all $N \in \mathbb{N}$, $T \in [\underline{T}, \overline{T}]$ we have $T_N^* \leq T_{N+1}^* \leq T^*$. Hence the sequence $(T_N^*)_{N \in \mathbb{N}}$ is convergent and $\lim_{N \to \infty} T_N^* \leq T^*$.

Lemma 7.3.2 *The sequence $(T_N^*)_{N\in I\!N}$ converges monotonically to T^*. If $T^* > \underline{T}$, for N large enough, we have $\omega_N(T_N^*) = \beta^2$.*

Proof If $T^* = \underline{T}$, for all $N \in I\!N$ we have $\omega_N(\underline{T}) \leq \omega(\underline{T}) \leq \beta^2$, hence $T_N^* = \underline{T}$ for all $N \in I\!N$. Now assume that $T^* > \underline{T}$. By Lemma 7.2.1, we have $\omega(T^*) = \beta^2$. Let $L = \lim_{N\to\infty} T_N^*$. Suppose that $L = \underline{T}$. For all $N \in I\!N$, we have $\omega_N(T_N^*) \leq \beta^2$. Hence by Theorem 6.1.2

$$\omega(L) = \lim_{N\to\infty} \omega_N(T_N^*) \leq \beta^2.$$

Hence (7.1) implies that $T^* \leq L = \underline{T}$, a contradiction.

Hence for N large enough, we have $T_N^* > \overline{T}$ and Lemma 7.3.1 implies that $\omega_N(T_N^*) = \beta^2$. Hence Theorem 6.1.2 implies that $\omega(L) = \beta^2$. Since $L \leq T^*$, by (7.1) this yields $L = T^*$. $\square$

Remark 7.3.1 The assertion $\lim T_N^* = T^*$ has been proved in [25] in another way for the example of the rotating beam. The relationship between time–minimal controllability with norm–bounded controls and minimum–norm controls on fixed time intervals can be found in several works of Krabs, see e.g. [22].

7.4 Sensitivity with respect to β

Obviously the numbers T^* and T_N^* depend on the choice of the bound β. If the function ω is strictly decreasing, the value of T^* depends continuously on β. If ω is only decreasing, this cannot be guaranteed.

Lemma 7.4.1 *If the function ω is strictly decreasing, the value of T^* depends continuously on β for β such that $\omega(\overline{T}) < \beta^2$.*

Proof Let

$$T^*(\beta) = \inf\{T \in [\underline{T}, \overline{T}] : U(T, \beta, c) \neq \emptyset\}.$$

Let $\beta_1 < \beta_2$ be such that $\omega(\overline{T}) < \beta_1^2$ Then $\omega(T^*(\beta_1)) = \beta_1^2 < \omega(T^*(\beta_2)) = \beta_2^2$. Thus $T^*(\beta_1) > T^*(\beta_2)$. Hence the function T^* is strictly decreasing.

Let β be such that $\omega(\overline{T}) < \beta^2$. Then $\omega(T^*(\beta)) = \beta^2$. Let $\beta_k \to \beta-$. Then $L = \lim_{k\to\infty} T^*(\beta_k) \geq T^*(\beta)$, and $\omega(L) = \lim_{k\to\infty} \omega(T^*(\beta_k)) = \lim_{k\to\infty} \beta_k^2 = \beta^2$, hence $L = T^*(\beta)$.

Let $\beta_k \to \beta+$. Then $L = \lim_{k\to\infty} T^*(\beta_k) \leq T^*(\beta)$, and $\omega(L) = \beta^2$, hence $L = T^*(\beta)$. Hence the function T^* is continuous. $\square$

7.5 A Newton–Bisection Algorithm

7.5.1 A Regularization Method

Let a continuous function $v : \mathbb{R} \to \mathbb{R}$ and T_0 with $v(T_0) > 0$ and $\hat{T} > T_0$ with $v(\hat{T}) < 0$.

We consider the following problem: Find the smallest root of v in $[T_0, \hat{T}]$:

$$T := \min\{t \in [T_0, \hat{T}] : v(t) = 0\}.$$

If v is not strictly decreasing, several roots can exist on the interval $[T_0, \hat{T}]$. This is the motivation to use a regularization approach. The function v is regularized in such a way that a parametric family of strictly decreasing functions is generated, whoose roots converge to the smallest root of v. This approach is similar to the method of prox–regularization (vgl. [19]).

For $\alpha > 0$ we define the function

$$h(t, s) = v(t) - \alpha v(s)(t - s).$$

This means geometrically that we add a strictly monotone linear function to the function v, which yields a strictly decreasing function if the linear function decreases sufficiently steep.

The function h has the following properties:

1. $h(t, t) = v(t)$.
2. For $s \in [T_0, T)$, $v(s) > 0$, hence if $T_0 \le t < s < T$ then $h(t, s) > v(t) > 0$.
3. If $T_0 \le s < T$ and $s < T$ then $h(t, s) < v(t)$.
4. Let $s \in [T_0, T)$ be given. Then $h(T, s) = -\alpha v(s)(T - s) < 0$ and $h(s, s) = v(s) > 0$. Hence the function $h(\cdot, s)$ has a root in the interval (s, T).

Now we give an algorithm that generates a sequence $(s_k)_{k \in \mathbb{N}}$ that is increasing and converges to T.

Algorithm 7.1 (Regularization)

Step 0

Choose $s_0 < T$ (e.g. $s_0 = T_0$). Let $\bar{s}_1$ denote the smallest root of $h(\cdot, s_0)$ in $[T_0, \hat{T}]$. Choose $\epsilon_1 \in (0, 1]$. Compute s_1 with $\bar{s}_1 - \epsilon_1 \le s_1 \le \bar{s}_1$.

Step k

Let s_k be given. Let $\bar{s}_{k+1}$ denote the smallest root of $h(\cdot, s_k)$ in $[T_0, \hat{T}]$. Choose $\epsilon_{k+1} \in (0, 2^{-k}]$. Compute s_{k+1} with $\bar{s}_{k+1} - \epsilon_{k+1} \leq s_{k+1} \leq \bar{s}_{k+1}$.

Theorem 7.5.1 *The sequences $(\bar{s}_k)_{k\in\mathbb{N}}$ and $(s_k)_{k\in\mathbb{N}}$ generated by Algorithm 7.1 have the following properties:*

a) The sequences $(\bar{s}_k)_{k\in\mathbb{N}}$ and $(s_k)_{k\in\mathbb{N}}$ converge to T and for all $k \in \mathbb{N}$ we have $s_k < T$ and $\bar{s}_k < T$.

b) If v is differentiable and $v'(T) < 0$ and $\epsilon_i/|T - s_i| \to 0$ $(i \to \infty)$ then the sequence $(s_i)_{i\in\mathbb{N}}$ converges superlinearly to T in the sense that

$$\lim_{i\to\infty} \frac{s_{i+1} - T}{s_i - T} = 0.$$

If in addition $\epsilon_i/|T - s_i|^2 \to 0$ $(i \to \infty)$, then the sequence $(s_i)_{i\in\mathbb{N}}$ converges quadratically to T in the sense that

$$\lim_{i\to\infty} \frac{s_{i+1} - T}{(s_i - T)^2} \leq \alpha,$$

c) If in addition to the previous assumptions, v' is continuous in T, let $d_k := \min\{d \in \{s_k + 2^{-k}, \hat{T}\} : v(d) < 0\}$. Then $d_k > T$ and the sequence $(d_k)_{k\in\mathbb{N}}$ converges to T.

If in addition v' is bounded on the interval $[T_0, \hat{T}]$, the number α can be chosen such that we have

$$\sup_{i\in\mathbb{N}} \sup_{t\in[s_i-\epsilon_{i+1}, d_i]} \partial_t h(t, s_i) < 0.$$

Proof a) First we show by induction, that for all $k \in \mathbb{N}$ we have $\bar{s}_{k+1} \in (s_k, T)$ and $s_{k+1} \in (s_k - \epsilon_{k+1}, T)$.

$k = 0$: We have $h(s_0, s_0) = v(s_0) > 0$. For all $t < s_0$: $h(t, s_0) > v(t) > 0$. Hence $\bar{s}_1 > s_0$. On the other hand we have $h(T, s_0) < v(T) = 0$. Since $\bar{s}_1$ is the smallest root of $h(\cdot, s_0)$, this implies $\bar{s}_1 \in (s_0, T)$ and the definition of s_1 yields $s_1 \in (s_0 - \epsilon_1, T)$.

$k \mapsto k + 1$: Assume that $s_k \in (s_{k-1} - \epsilon_k, T)$ and thus $h(s_k, s_k) = v(s_k) > 0$. For all $t \in [T_0, s_k)$ we have $h(t, s_k) > v(t) > 0$. Hence $\bar{s}_{k+1} > s_k$. On the other hand we have $h(T, s_k) < v(T) = 0$. Hence $\bar{s}_{k+1} \in (s_k, T)$ and the definition of s_{k+1} yields $s_{k+1} \in (s_k - \epsilon_{k+1}, T)$.

Now we show $\lim_{k\to\infty} \bar{s}_k = T = \lim_{k\to\infty} s_k$. Let $\bar{s} = \liminf_{k\to\infty} \bar{s}_k$. Then $\bar{s} \leq T$. Since $\bar{s} \geq T_0$ this yields $v(\bar{s}) \geq 0$. Let $(k_j)_{j\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ be suc

Hence

$$0 \le \limsup_{i\to\infty} \frac{T - s_{i+1}}{(T-s_i)^2} \;\le\; \limsup_{i\to\infty} \frac{T - \bar{s}_{i+1} + \epsilon_{i+1}}{(T-s_i)^2}$$
$$\le\; \alpha \frac{v'(T)}{v'(T)} + 0 = \alpha.$$

c) The sequence $(s_k)_{k\in\mathbb{N}}$ converges superlinearly to T. Hence for all k that are sufficiently large we have $s_k + 2^{-k} > T$.

The inequality $v'(T) < 0$ and the continuity of v' imply the existence of a neighbourhood U of T with $v'(t) < 0$ for all $t \in U$. If k is large enough we have on account of $s_k \to T$: $s_k + 2^{-k} \in U$. Since v is strictly decreasing on U, $s_k + 2^{-k} > T$ yields the inequality $v(s_k + 1/k) < 0$ and thus $d_k = s_k + 2^{-k}$ for all except a finite number of k. Hence $d_k \to T$. Since $v(d_k) < 0$ we have $d_k > T$.

Let $S_1 = \sup_{t\in U} v'(t) < 0$ and $S_2 = \sup_{t\in[T_0,\hat{T}]} v'(t)$. For all except a finite number of i we have $[s_i - \epsilon_{i+1}, d_i] \subset U$; therefore on account of $s_i < T$ we have

$$S_3 = \inf_{i\in\mathbb{N}:[s_i-\epsilon_{i+1},d_i]\not\subset U} v(s_i) > 0.$$

Choose $\alpha \ge (1 + S_2)/S_3$. Then for all i with $[s_i - \epsilon_{i+1}, d_i] \not\subset U$ and for all $t \in [T_0, \hat{T}]$ we have the chain of inequalities

$$\begin{aligned} \partial_t h(t,s) &\le v'(t) - \alpha S_3 \\ &\le v'(t) - S_2 - 1 \\ &\le -1. \end{aligned}$$

Now $[s_i - \epsilon_{i+1}, d_i] \subset U$ and $t \in [s_i - \epsilon_{i+1}, d_i] \subset U$ imply

$$\partial_t h(t,s) \le S_1 < 0. \qquad \square$$

Remark 7.5.1 *If it is known a priori that v is decreasing (not necessarily strictly decreasing) it is also possible to use the regularization Algorithm 7.1 with $h(t,s) = v(t) - \alpha(t-s)$. Then $\partial_t h \le -\alpha$ and*

$$0 \le \frac{T - s_{i+1}}{T - s_i} \le \frac{T - \bar{s}_{i+1} + \epsilon_{i+1}}{T - s_i} \to \frac{\alpha}{\alpha - v'(T)}.$$

Hence if $v'(T) < 0$ the sequence $(s_i)_{i\in\mathbb{N}}$ converges linearly to T. If $v'(T) = 0$ the convergence is only sublinear.

7.5.2 Newton–Bisection Method

Now we give an algorithm for the computation of the smallest root of the functions $h(\cdot, s_k)$ that is required in Algorithm 7.1.

According to Theorem 7.5.1 c) we can assume that

$$h'(\cdot, s_k) \leq -\epsilon < 0 \text{ on } E_0 := [s_k - \epsilon_{k+1}, d_k].$$

So we give an algorithm for the computation of the smallest root in E_0 of a function that is strictly decreasing. The fact that the function is strictly decreasing allows us to show global superlinear convergence. This is the reason for our regularization effort in Algorithm 7.1.

We consider the following problem: Let functions H, G: $I\!R \to I\!R$ be given. Assume that H is continuous and $G(t) \neq 0$ for all $t \in I\!R$. (G is used as an approximation of the derivative of H).

We want to compute an approximation for the smallest root $\bar{s}$ of H, whose existence we also assume. Our algorithm combines a fast algorithm that determines the numbers n_k with a bisection method.

Algorithm 7.2 (Newton–Bisection)

Step 0 *Find an interval* $E_0 = [a_0, b_0]$ *containing* $\bar{s}$. *Choose* $\lambda_0 \in \{a_0, b_0\}$.

Step k *Given an interval* $E_k = [a_k, b_k]$ *containing* $\bar{s}$ *and* $\lambda_k \in \{a_k, b_k\}$.

Compute $n_k = \lambda_k - H(\lambda_k)/G(\lambda_k)$.

If $n_k \in E_k$ *compute* $H(n_k)$.

Take the left interval $[\alpha_k, \beta_k] \in \{[a_k, n_k], [n_k, b_k]\}$ *with* $\bar{s} \in [\alpha_k, \beta_k]$, *which is determined by the sign of* $H(n_k)$.

If $n_k \notin E_k$, *let* $[\alpha_k, \beta_k] = [a_k, b_k]$.

Compute the midpoint $c_k = (\beta_k + \alpha_k)/2$ *and* $H(c_k)$.

Take the left interval $E_{k+1} = [a_{k+1}, b_{k+1}] \in \{[\alpha_k, c_k], [c_k, \beta_k]\}$ *with* $\bar{s} \in E_{k+1}$.

If $n_k \leq a_{k+1}$, *let* $\lambda_{k+1} = a_{k+1}$. *If* $n_k \geq b_{k+1}$, *let* $\lambda_{k+1} = b_{k+1}$.

Lemma 7.5.1 *If for all* k, $\lambda_k \neq \bar{s}$, $\inf_{t \in E_0} |G(t)| > 0$ *and* $t_k \to \bar{s}$ *implies* $G(t_k) - H(t_k)/(t_k - \bar{s}) \to 0$, *then*

$$\lim_{k \to \infty} \frac{\lambda_{k+1} - \bar{s}}{\lambda_k - \bar{s}} = 0,$$

that is the sequence $(\lambda_k)_{k \in I\!N}$ *converges superlinearly to* $\bar{s}$.

Proof For all k we have $\lambda_k \in E_k$. Since $E_{k+1} \subset E_k$ and $b_{k+1} - a_{k+1} \leq (b_k - a_k)/2$ the sequence $(\lambda_k)_{k\in\mathbb{N}}$ is convergent. For all k we have $\bar{s} \in E_k$, hence $\lim_{k\to\infty} \lambda_k = \bar{s}$.

For all k we have $|\lambda_{k+1} - \bar{s}| \leq |n_k - \bar{s}|$. The definition of n_k implies

$$\begin{aligned} \frac{n_k - \bar{s}}{\lambda_k - \bar{s}} &= \frac{\lambda_k - H(\lambda_k)/G(\lambda_k) - \bar{s}}{\lambda_k - \bar{s}} \\ &= \frac{1}{G(\lambda_k)} \left(G(\lambda_k) - \frac{H(\lambda_k)}{\lambda_k - \bar{s}} \right). \end{aligned}$$

With the last assumption of the Lemma this yields

$$\lim_{k\to\infty} \frac{n_k - \bar{s}}{\lambda_k - \bar{s}} = 0.$$

On account of

$$\left| \frac{\lambda_{k+1} - \bar{s}}{\lambda_k - \bar{s}} \right| \leq \left| \frac{n_k - \bar{s}}{\lambda_k - \bar{s}} \right|$$

the assertion follows. □

Lemma 7.5.2 *If for all k, $\lambda_k \neq \bar{s}$, $\inf_{t\in E_0} |G(t)| > 0$ and $t_k \to \bar{s}$ implies $[G(t_k) - H(t_k)/(t_k - \bar{s})]/(t_k - \bar{s}) \to 0$, then*

$$\lim_{k\to\infty} \frac{\lambda_{k+1} - \bar{s}}{(\lambda_k - \bar{s})^2} < \infty,$$

that is the sequence $(\lambda_k)_{k\in\mathbb{N}}$ converges quadratically to $\bar{s}$.

Proof: As in Lemma 7.5.1.□

For N

and Lemma 7.5.1 implies superlinear convergence.

If the functions b, z_j ($j \in I\!N$) are continuously differentiable, Lemma 6.2.3 implies that H is twice continuously differentiable, hence

$$H'(t) - \frac{H(t)}{t - \bar{s}} = O(t - \bar{s})$$

and Lemma 7.5.2 implies quadratic convergence.

If ω_N is twice continuously differentiable, another possible choice of G is

$$G(t) = H'(t) - \frac{H(t)H''(t)}{2H'(t)}. \tag{7.3}$$

For this choice of G, the definition of n_k is as in Halley's method (see [36]).

If $H'(\bar{s}) \neq 0$, (7.2) holds and Lemma 7.5.1 implies superlinear convergence. Moreover, we have

$$G(t) - \frac{H(t)}{t - \bar{s}} = O(t - \bar{s})$$

and Lemma 7.5.2 guarantees quadratic convergence. If H is three times continuously differentiable we can say even more: Analogously to Lemma 7.5.2 we can show that

$$G(t) - \frac{H(t)}{t - \bar{s}} = O(t - \bar{s})^2 \tag{7.4}$$

and $\inf_{t \in E_0} |G(t)| > 0$ implies cubic convergence.

If H is three times continuously differentiable, (7.4) is valid for G as in (7.3). Hence we have cubic convergence.

7.6 Numerical Examples

For the example of the Euler–Bernoulli beam, it is possible to work with Newton's, Halley's and Euler's method without regularization and without a combination with a bisection method. In Figure 6.2.1, the graph of ω_8 looks like the graph of a convex function. Therefore, we started with points that were smaller than the root that was to be computed, since in this situation, for a convex function, Newton's method generates a sequence that converges monotonically to the root.

The methods that we used were defined in the following way:

$$
\begin{array}{rrl}
\text{Newton} & n_{k+1} = & n_k - \omega_8(n_k)/\omega_8'(n_k) \\
\text{Halley} & h_{k+1} = & h_k - \omega_8(h_k)/(\omega_8'(h_k) - \omega_8(h_k)\omega_8''(h_k)/(2\omega_8'(h_k))) \\
& \text{if} & 3|\omega_8(h_k)\omega_8''(h_k)| \leq 2\omega_8'(h_k)^2, \\
& h_{k+1} = & h_k - \omega_8(h_k)/\omega_8'(h_k) \text{ else} \\
\text{Euler} & e_{k+1} = & e_k - (\omega_8'(e_k) + \sqrt{\omega_8'(e_k)^2 - 2\omega_8(e_k)\omega_8''(e_k)})/\omega_8''(e_k) \\
& \text{if} & \omega_8'(e_k)^2 - 2\omega_8(e_k)\omega_8''(e_k) \geq 0 \text{ and}
\end{array}
$$

of iterations needed by the three methods is approximately the same.

Figure 7.6.1

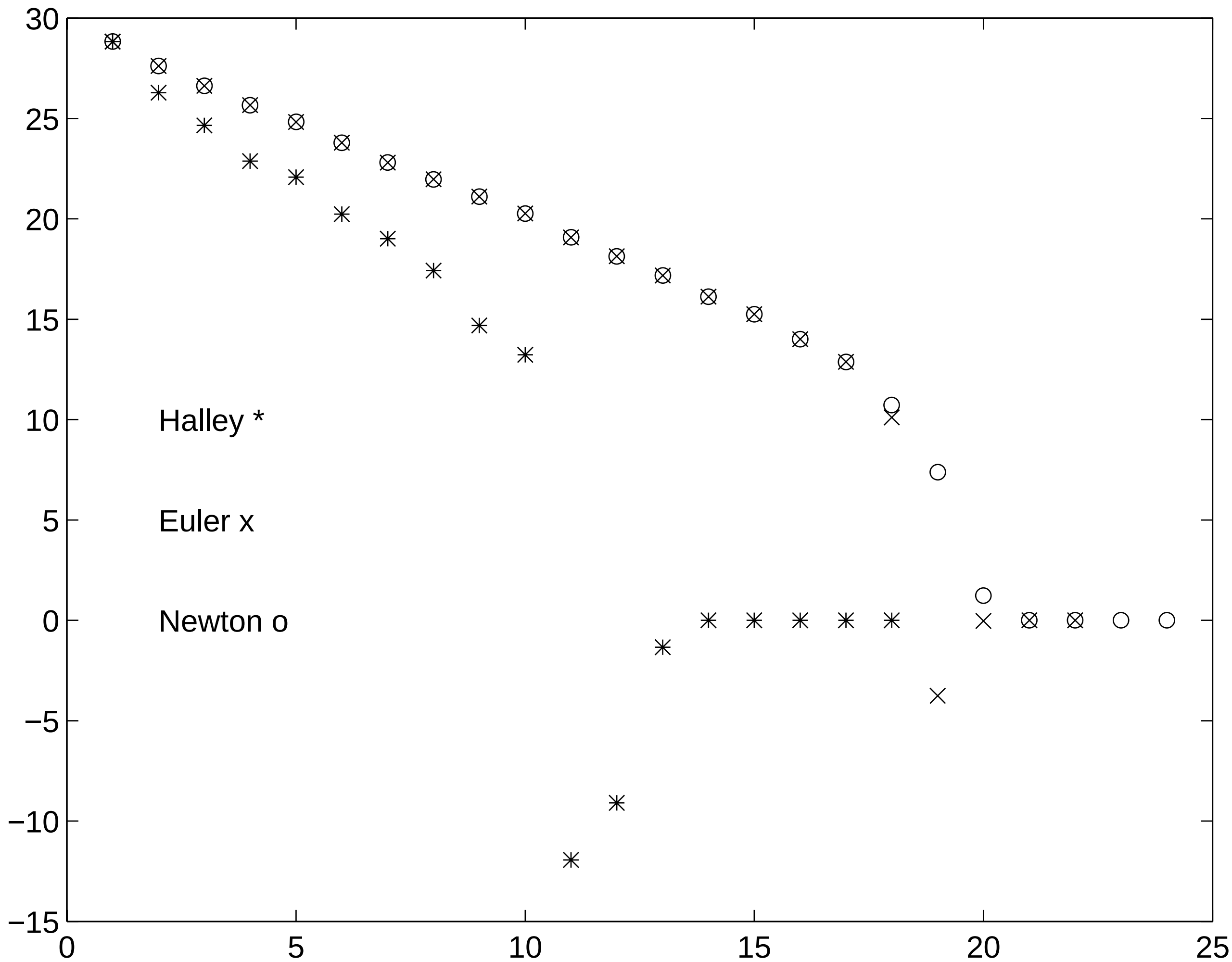


Figure 7.6.1:

The points $\mathrm{sign}(\omega_8(x_i))\log(1+|\omega_8(x_i)|)$ for the methods of Halley, Euler, Newton.

Example 7.6.2 For our second example, we choose $\beta = 100$ and start with the point $x_0 = 0.2$ for $D = 1$ and $x_0 = 0.1$ for $D \in \{10, 100, 1000\}$.

Table 7.6 contains the number of the iteration i where (7.5) was satisfied and the iteration stopped.

	Halley	Euler	Newton
$D = 1$	19	25	24
$D = 10$	13	19	20
$D = 100$	7	11	12
$D = 1000$	9	8	8

(7.6)

$$\begin{aligned} D = 1 : T_8^* &= 0.826303 \\ D = 10 : T_8^* &= 0.396888 \\ D = 100 : T_8^* &= 0.207121 \\ D = 1000 : T_8^* &= 0.205991 \end{aligned}$$

Example 7.6.3 For our third example, we choose β

	Halley	Euler	Newton
$D = 1$	10	13	13
$D = 10$	9	11	10
$D = 100$	9	11	11
$D = 1000$	11	12	11

(7.8)

$$\begin{aligned} D = 1 : T_8^* &= 4.51032 \\ D = 10 : T_8^* &= 4.41398 \\ D = 100 : T_8^* &= 4.39814 \\ D = 1000 : T_8^* &= 4.40270 \end{aligned}$$

The numerical results indicate that Halley's method is often faster than Newton's method if the iteration starts far away from the solution and large changes in the first derivative occur during the iteration.

If the iteration starts close to the solution, there is almost no difference between the iterates generated by Newton's and Halley's method. Euler's method does not work so well since often, far away from the solution, it has to use a Newton step.

7.7 Discretization Refinement

7.7.1 Introduction

The main work in the computation of a solution of Problem $P_N(T)$ with the method described in Chapter 5 consists in the computation of the functions $H_j(T)$.

Hence the amount of work is proportional to the number of the functions $H_j(T)$.

We assume that the assumptions A1 and A2 given in Chapter 6 hold.

Assume that S_T is a Volterra operator with a kernel that is given as a series where the truncated series yields a finite rank kernel. Then for each truncated kernel, the functions H_j can be computed using the solution of an initial value problem.

The dimension of the differential equation in the initial value problems is equal to the number k of terms in the truncated series of the kernel K, hence the computing time is also approximately proportional to k.

Moreover, the computing time depends on the stepsize that is used in the numerical solution of the initial value problem and for the numerical integration, which is used to compute approximations of the scalar products which are the numbers in the Gram matrix.

In our algorithm for the solution of the problem of time–optimal control, the computation of solutions of $P_N(T_j)$ is required for a number of points T_j. To reduce the number of iterations in this algorithm, a good starting point is essential. As such a starting point, we can use the root of a cheaper approximation of the function whoose root we want to compute.

For each smaller number k, each smaller number N and each larger stepsize h we obtain such an approximation.

Theoretically, we can compute an infinite number of roots $T^*_{N,k,h}$ corresponding to these functions.

In this section, we consider the convergence of these numbers $T^*_{N,k,h}$ for $N \to \infty$, $k \to \infty$ and $h \to 0$.

Again we can prove uniform convergence of the corresponding optimal value functions, provided that k is increasing fast enough with N and h is increasing fast enough with N and k.

We start here with the parameter N since it determines the size of the Gram matrix. The number k determines the size of the initial value problem, hence we consider h as the last parameter.

7.7.2 Sensitivity with respect to N and k

We start our analysis with a Lemma about the sensitivity to perturbations of the solution of linear Volterra equations of the second kind, that is based on Theorem 3.10 in [35].

Lemma 7.7.1 *Let $\;> 0$ and K, K_k and f be continuous functions on the interval $[0,\overline{T}]$.*

For $T \in (0,\overline{T}]$, let $v(T)$ denote the solution of the equation

$$v(T)(x) - \int_0^x \frac{K(x-y)}{\;} v(T)(y)\,dy = \frac{f(T-x)}{\;}$$

and $v_k(T)$ denote the solution of the equation

$$v_k(T)(x) - \int_0^x \frac{K_k(x-y)}{\;} v_k(T)(y)\,dy = \frac{f(T-x)}{\;}.$$

$$\begin{aligned} \textit{Let } \Delta K(k) &= \max_{x\in[0,\overline{T}]} \frac{|K(x)-K_k(x)|}{} \\ \textit{and } \Delta f(s) &= \max_{x,x+s\in[0,\overline{T}]} \frac{|f(x+s)-f(x)|}{}. \end{aligned}$$

Let $\bar{f} = \max_{x\in[0,\overline{T}]} \frac{|f(x)|}{\kappa}$ *and* $\bar{K} = \max_{x\in[0,\overline{T}]} \frac{|K(x)|}{\kappa}$.

Then for all $x \in [0, \min\{T, T+s\}]$ *the following inequality holds:*

$$|v(T)(x) - v_k(T+s)(x)| \leq$$

$$\Big\{\Delta f(s) + \Delta K(k)x(\bar{f} + \Delta f(s))\exp((\bar{K} + \Delta K(k))x)\Big\}\exp(\bar{K}x). \quad (7.9)$$

Proof The result follows directly from Theorem 3.10 in [35]. □

In the sequel, for $\eta \in I\!R^N$

$$\|\eta\| = (\sum_{i=1}^{N} |\eta_i|^2)^{1/2}$$

denotes the Euclidean norm and for $A \in I\!R^{N\times N}$, $\|A\|$ denotes the corresponding matrix norm.

Let $H_{j,k}(T)$ denote the function that solves the equation

$$H_{j,k}(T)(x) - \int_x^T K_k(y-x)H_{j,k}(T)(y)\,dy = z_j(x). \quad (7.10)$$

Lemma 4.1.1 implies the equation

$$H_{j,k}(T)(x) = v_k(T)(T-x),$$

with $v_k(T)$ as defined in Lemma 7.7.1 with $f(x) = z_j(x)$.

For the rest of this chapter, we assume that

$$M_0 = \sup_j \sup_{x\in[0,\overline{T}]} \frac{|z_j(x)|}{} < \infty.$$

Then according to Example 3.3 in [35] we have for all $x \in [0, T]$

$$|H_j(T)(x)| \leq M_0 \exp(\bar{K}\overline{T}).$$

We consider a sequence of kernels $(K_k)_{k\in I\!N}$ with $\lim_{k\to\infty} \Delta K(k) = 0$.

Lemma 7.7.1 implies that for all $x \in [0, \overline{T}]$, j, $k \in I\!N$ we have

$$|H_{j,k}(T)(x) - H_j(T)(x)| \leq \Delta K(k)\overline{T}M_0 \exp(2(\bar{K} + \Delta K(k))\overline{T}) =: B_k.$$

For the scalar products this yields for all $i,\, j \in I\!N$

$$\begin{aligned} \left|\langle H_{i,k}(T), H_{j,k}(T)\rangle_{(0,T)} - \langle H_i(T), H_j(T)\rangle_{(0,T)}\right| &\leq \\ \left|\langle H_{i,k}(T) - H_i(T), H_{j,k}(T)\rangle_{(0,T)}\right| &+ \\ \left|\langle H_i(T), H_{j,k}(T) - H_j(T)\rangle_{(0,T)}\right| &\leq \\ B_k\overline{T}(\max_{x\in[0,T]} |H_i(T)(x)| + \max_{x\in[0,T]} |H_j(T)(x)| + B_k) &\leq \\ B_k\overline{T}(2M_0\exp(\bar{K}\overline{T}) + B_k) &=: \quad \bar{B}_k. \end{aligned}$$

Let $A_{N,k}(T)$ be the $N \times N$ matrix

$$A_{N,k}(T) = \left(\langle H_{i,k}(T), H_{j,k}(T)\rangle_{(0,T)}\right)_{i,j=1}^{N}.$$

For $A \in I\!R^{N\times N}$, let $\|A\|_1$ denote the column–sum norm of the matrix,

$$\|A\|_1 = \max_{j\in\{1,\ldots,N\}} \sum_{i=1}^{N} |a_{ij}|.$$

Then we have $\|A\| \leq \sqrt{N}\|A\|_1$. Define $\Delta_{N,k}A(T) = A_{N,k}(T) - A_N(T)$. Then

$$\|\Delta_{N,k}A(T)\| \leq \sqrt{N}\|\Delta_{N,k}A(T)\|_1 \leq N^{3/2}\bar{B}_k. \tag{7.11}$$

With the notation as in Lemma 6.1.3, we have

$$A_N(T)\eta_N(T) = r_N(T) = (c_i - \langle b, H_i(T)\rangle_{(0,T)})_{i=1}^{N}.$$

Let $(b_k)_{k\in I\!N}$ be a sequence of approximations of b such that

$$\lim_{k\to\infty} \|b - b_k\|_{(0,\overline{T})} = 0.$$

Let $\eta_{N,k}(T)$ be the solution of the equation

$$A_{N,k}(T)\eta_{N,k}(T) = r_{N,k}(T) =: (c_i - \langle b_k, H_i(T)\rangle_{(0,T)})_{i=1}^{N}.$$

Let $\Delta_{N,k}r(T) = r_{N,k}(T) - r_N(T) = (\langle b - b_k, H_i(T)\rangle_{(0,T)})_{i=1}^{N}$.
Due to A2, Lemma 6.1.4 implies that for all k we have

$$\sum_{i=1}^{\infty}(\langle b - b_k, H_i(T)\rangle_{(0,T)})^2 \leq \hat{P}^2\|b - b_k\|_{(0,T)}^2.$$

Hence for all N, k we have

$$\|\Delta_{N,k}r(T)\| \leq \hat{P}\|b - b_k\|_{(0,T)}.$$

For k sufficiently large we have the well–known inequality (see [43], (1.86), p.37)

$$\|\eta_N(T) - \eta_{N,k}(T)\| \leq$$
$$\frac{\|A_N(T)^{-1}\|}{1 - \|A_N(T)^{-1}\| \, \|\Delta_{N,k}A(T)\|} \left\{\|\Delta_{N,k}r(T)\| + \|\Delta_{N,k}A(T)\| \, \|\eta_N(T)\|\right\}. \tag{7.12}$$

Lemma 6.1.2 implies that $\|A_N(T)^{-1}\|$ is uniformly bounded with respect to $T \in [\underline{T}, \overline{T}]$ and N, since the smallest eigenvalue of the symmetric and positive definite matrix $A_N(T)$ is greater than or equal to $1/\hat{M}^2$:

$$\|A_N(T)^{-1}\| \leq \hat{M}^2.$$

Lemma 6.1.4 implies that $\|r_N(T)\|$ is also uniformly bounded with respect to $T \in [\underline{T}, \overline{T}]$ and N, since

$$\|r_N(T)\| \leq \|c\|_{l^2} + \hat{P}\|b\|_{(0,T)}.$$

Since $\|\eta_N(T)\| \leq \|A_N(T)^{-1}\|\|r_N(T)\|$, this implies that we also have

$$\sup_N \sup_{T \in [\underline{T}, \overline{T}]} \|\eta_N(T)\| \leq M^2(\|c\|_{l_2} + \hat{P}\|b\|_{(0,\overline{T})}) < \infty.$$

Let $C_0 > 0$ be such that for all $k \in I\!N$, $\bar{B}_k \leq C_0 \Delta K(k)$.

Then for all k, N we have

$$\|\Delta_{N,k}A(T)\| \leq C_0 N^{3/2} \Delta K(k).$$

In this case, (7.12) implies that there is $C_2 > 0$ such that for all N, k sufficiently large we have

$$\|\eta_N(T) - \eta_{N,k}(T)\| \leq C_2 \left(N^{3/2}\Delta K(k) + \|b - b_k\|_{(0,T)}\right).$$

By (6.9), the value of ω_N can be represented in the form

$$\omega_N(T) = (\eta_N(T))^T r_N(T).$$

Define

$$\omega_{N,k}(T) = (\eta_{N,k}(T))^T r_{N,k}(T).$$

It is easy to see that $\omega_{N,k}(T)$ can be interpreted as the optimal value of a certain optimization problem $P_{N,k}(T)$.

For the difference of $\omega_N(T)$ and $\omega_{N,k}(T)$, we have the following inequality:

$$|\omega_N(T)-\omega_{N,k}(T)| \leq \|\eta_N(T)\| \, \|\Delta_{N,k} r(T)\| + \|r_{N,k}(T)\| \, \|\eta_N(T)-\eta_{N,k}(T)\|.$$

Hence there exists a constant $C_3 > 0$ such that for all N, k and $T \in [\underline{T}, \overline{T}]$ we have

$$|\omega_N(T) - \omega_{N,k}(T)| \leq C_3 \left(N^{3/2} \Delta K(k) + \|b - b_k\|_{(0,T)} \right).$$

Thus we have the following Lemma

Lemma 7.7.2 *There exists a constant $C_3 > 0$, $k_0 \in I\!N$ such that for all $\epsilon > 0$, for all N, $k \geq k_0$ with*

$$N^{3/2} \Delta K(k) + \|b - b_k\|_{(0,\overline{T})} \leq \epsilon$$

the following inequality holds:

$$\max_{T \in [\underline{T}, \overline{T}]} |\omega_N(T) - \omega_{N,k}(T)| \leq \epsilon C_3.$$

Hence we can prove the following Theorem:

Theorem 7.7.1 *For all $\epsilon > 0$ there is $k_0 > 0$, $N_0 > 0$ such that for all $N > N_0$, for all $k > k_0$ with*

$$\Delta K(k) + \|b - b_k\|_{(0,\overline{T})} / N^{3/2} \leq \epsilon / (2 N^{3/2} C_3), \tag{7.13}$$

we have

$$\max_{T \in [\underline{T}, \overline{T}]} |\omega(T) - \omega_{N,k}(T)| \leq \epsilon. \tag{7.14}$$

Proof Choose $\epsilon > 0$. Theorem 6.1.2 implies the existence of $N_0 > 0$ such that for all $N \geq N_0$ we have

$$\max_{T \in [\underline{T}, \overline{T}]} |\omega(T) - \omega_N(T)| \leq \epsilon/2.$$

According to Lemma 7.7.2, for all N, k for which (7.13) holds, we have

$$|\omega_N(T) - \omega_{N,k}(T)| \leq \epsilon/2,$$

hence if $N \geq N_0$ and (7.13) is valid, the inequality

$$|\omega(T) - \omega_{N,k}(T)| \leq |\omega(T) - \omega_N(T)| + |\omega_N(T) - \omega_{N,k}(T)| \leq \epsilon$$

holds, and the assertion follows. □

Remark 7.7.1 If we define $r_{N,k}(T)$ as $(c_i - \langle b_k, H_{i,k}(T) \rangle_{(0,T)})_{i=1}^N$ then Lemma 7.7.2 and Theorem 7.7.1 hold with N^2 instead of $N^{3/2}$. If $b = b_k = 0$ as in our numerical examples, the result with $N^{3/2}$ is applicable.

7.7.3 Sensitivity with respect to N, k and h

In this section, we assume that for all k, K_k is a finite rank kernel as defined in (4.8). For general finite rank kernels as in (4.5), an analogous discussion is possible.

Let ϵ_h denote the global discretization error corresponding to the stepsize h for the numerical solution of the initial value problem (4.10), (4.11), (4.12), so that we have the inequality

$$|y(x_k) - y_k| \leq \epsilon_h,$$

with $x_k = kh$, $k \in \{1, ..., T/h\}$, for the exact solution y and the values y_k of the approximate solution. Let $m = T/h \in I\!N$.

Let $H_{j,k,h}(T)$ be defined as

$$H_{j,k,h}(T)(x_k) = z_j(x_k)/ \; -$$

$$\frac{1}{-}\sum_{i=1}^{k} \left(\sin(\sqrt{D\lambda_j}x_{m-i})y_{2j-1}(x_{m-i}) - \cos(\sqrt{D\lambda_j}x_{m-i})y_{2j}(x_{m-i})\right).$$

Then due to (4.13), for the solution $H_{j,k}(T)$ of the Volterra equation (7.10), we have

$$|H_{j,k}(T)(x_k) - H_{j,k,h}(T)(x_k)| \leq \frac{1}{-}2k\epsilon_h. \tag{7.15}$$

This inequality shows that if k is increased, this only has a positive effect on the accuracy if at the same time, h is made smaller in such a way that ϵ_h decreases.

Let G_m^T denote a generalized Newton–Cotes rule for the approximation of integrals on the interval $[0, T]$ by using the values on an equidistant grid,

$$G_m^T(f) = \sum_{i=0}^{m} W_i^m f(x_i),\; W_i^m \geq 0,\; \sum_{i=0}^{m} W_i^m = T.$$

Assume that there exist $\sigma \in I\!N$, $Q_m > 0$ such that for the quadrature error, we have the inequality

$$\left|G_m^T(f) - \int_0^T f(t)\,dt\right| \leq Q_m \left|f^{(\sigma)}(\xi)\right| h^\sigma,$$

for some $\xi \in [0, T]$. Then

$$|\langle H_{i,k}(T)H_{j,k}(T)\rangle_{(0,T)} - G_m^T(H_{i,k}(T)H_{j,k}(T))| \leq$$

$$Q_m|(H_{i,k}(T)H_{j,k}(T))^{(\sigma)}(\xi)|h^\sigma.$$

Inequality (7.15) implies

$$|\langle H_{i,k}(T)H_{j,k}(T)\rangle_{(0,T)} - G_m^T(H_{i,k,h}(T)H_{j,k,h}(T))| \leq$$

$$Q_m|(H_{i,k}(T)H_{j,k}(T))^{(\sigma)}(\xi)|h^\sigma + \overline{T}2k\epsilon$$

Remark 7.7.2 In the case of the trapezoidal rule (4.15), we have $\epsilon_h = O(h^3)$ (see [9]).

In our implementation, for the quadrature we used the generalized Simpson rule, for which we have $\sigma = 4$.

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