# Dynamics of the Taylor Shift on Spaces of Holomorphic Functions

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Maike Rosa Thelen

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# Preface

In this work, we will consider discrete dynamical systems $(X, T)$ which consist of a state space $X$ and a linear operator $T$ acting on $X$. Given a state $x \in X$ at time zero, its state at time $n$ is determined by the $n$-th iteration

$$T^n x = (T \circ ... \circ T)x \quad (n \text{ times}).$$

We are interested in the long-term behaviour of this system, that means we want to know how the sequence $(T^n x)_{n\in\mathbb{N}}$ behaves for increasing $n$ and $x \in X$.

In the first chapter, we will sum up the relevant definitions and results of linear dynamics. In particular, in topological dynamics the notions of hypercyclic, frequently hypercyclic and mixing operators will be presented. In the setting of measurable dynamics, the most important definitions will be those of weakly and strongly mixing operators.

If $\Omega$ is an open set in the (extended) complex plane containing 0, we can define the Taylor shift operator on the space $H(\Omega)$ of functions $f$ holomorphic in $\Omega$ as

$$Tf(z) = \frac{f(z) - f(0)}{z} \quad (z \neq 0), \quad Tf(0) = f'(0).$$

In the second chapter, we will start examining the Taylor shift on $H(\Omega)$ endowed with the topology of locally uniform convergence. Depending on the choice of $\Omega$, we will study whether or not the Taylor shift is weakly or strongly mixing in the Gaussian sense.

Next, we will consider Banach spaces of functions holomorphic on the unit disc $\mathbb{D}$. The first section of this chapter will sum up the basic properties of Bergman and Hardy spaces in order to analyse the dynamical behaviour of the Taylor shift on these Banach spaces in the next part. In the third section, we study the space of Cauchy transforms of complex Borel measures on the unit circle first endowed with the quotient norm of the total variation and then with a weak-$*$ topology. While the Taylor shift is not even hypercyclic in the first case, we show that it is mixing for the latter case.

In Chapter 4, we will first introduce Bergman spaces $A^p(\Omega)$ for general open sets $\Omega$ and provide approximation results which will be needed in the next chapter where we examine the Taylor shift on these spaces on its dynamical properties. In particular, for $1 \leq p < 2$ we will find sufficient conditions for the Taylor shift to be weakly mixing or strongly mixing in the Gaussian sense. For $p \geq 2$, we consider specific Cauchy transforms in order to determine open sets $\Omega$ such that the Taylor shift is mixing on $A^p(\Omega)$. In both sections, we will illustrate the results with appropriate examples.

Finally, we apply our results to universal Taylor series. The results of Chapter 5 about the Taylor shift allow us to consider the behaviour of the partial sums of the Taylor expansion of functions in general Bergman spaces outside its disc of convergence.

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# Contents

# Chapter 1

# Preliminaries

## 1.1 Basic Definitions and Results from Linear Dynamics

In this work, we want to examine a specific operator on its dynamical properties, i.e. we want to see how the iterates of the operator behave. Note that if we speak of an operator, it will always be linear and continuous. For an operator $T\colon X \to X$ on a topological vector space $X$, we call the pair $(X, T)$ a dynamical system. Furthermore, for a topological vector space $X$ and $A \subset X$, we denote by $\mathrm{cl}_X(A) = \mathrm{cl}(A)$ the closure of $A$ in $X$ and by $\mathrm{int}_X(A) = \mathrm{int}(A)$ the interior of $A$. As usual, we say that $A$ is dense in $X$ if its closure equals $X$. In this chapter, we want to give an overview of the notions of linear dynamics which will be important to us in the course of this thesis. If not stated otherwise, the following definitions and results can be found in [3] and [22].

**Definition 1.1.1.** Let $T$ be an operator $T : X \to X$ on a separable topological vector space $X$.

1. $T$ is called *hypercyclic* if there exists some vector $x \in X$ such that the orbit $\mathrm{orb}(x, T) \coloneqq \{T^n x : n \in \mathbb{N}_0\}$ is dense in $X$.

2. $T$ is called *frequently hypercyclic* if there exists some $x \in X$ such that for every non-empty open set $U \subset X$

$$\underline{\mathrm{dens}}\{n \in \mathbb{N}_0 : T^n x \in U\} > 0$$

where the *lower density* of a subset $A \subset \mathbb{N}_0$ of the non-negative integers is defined by

$$\underline{\mathrm{dens}}(A) := \liminf_{N\to\infty} \frac{|\{0 \le n \le N : n \in A\}|}{N+1}.$$

3. A vector $x \in X$ is called a *periodic point* of $X$ if there exists some $n \in \mathbb{N}$ such that $T^n x = x$. Then $T$ is *chaotic* if it is hypercyclic and has a dense set of periodic points.

**Remark 1.1.2.** An equivalent formulation of hypercyclicity and frequent hypercyclicity shows the difference between the two concepts: A vector $x \in X$ is frequently hypercyclic for $T$ if and only if, for any non-empty open subset $U$ of $X$, there is a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that

$$T^{n_k} x \in U \text{ for all } k \in \mathbb{N}, \quad \text{and} \quad n_k = O(k).$$

In contrast, for a vector $x$ to be hypercyclic, the sequence does not necessarily have to be of order $O(k)$. This also shows that frequent hypercyclicity is stronger than mere hypercyclicity.

A subset of a complete metric space $X$ is called *nowhere dense* if the interior of its closure is empty. A set is of *first Baire category* if it is a countable union of nowhere dense sets and it is called *residual* if its complement is of first Baire category. Finally, we say that a property holds for *generically many elements* of $X$ if the property is satisfied on a residual set in the space.

**Remark 1.1.3.** In [30, Theorem 1] the author showed for a frequently hypercyclic operator $T$ on a complex separable Fréchet space $X$ that the set of frequently hypercyclic vectors is always of first Baire category in $X$. This also distinguishes frequent hypercyclicity from hypercyclicity because for a hypercyclic operator $T$ the set of hypercyclic vectors is residual.

**Definition 1.1.4.** Let $T$ be an operator on a topological vector space $X$.

1. $T$ is called *(topologically) transitive* if, for any pair $U$, $V$ of non-empty open subsets of $X$, there exists some $n \ge 0$ such that

$$T^n(U) \cap V \neq \emptyset.$$

2. $T$ is called *(topologically) mixing* if, for any pair $U$, $V$ of non-empty open subsets of $X$, there exists an integer $N \geq 0$ such that

$$T^n(U) \cap V \neq \emptyset \quad \text{for all } n \geq N.$$

$T$ is called *weakly mixing* if $T \times T$ is topologically transitive on $X \times X$.

**Remark 1.1.5.** The Birkhoff transitivity theorem (see e.g. [22, Theorem 2.19]) states that an operator $T$ on a Fréchet space $X$ is hypercyclic if and only if it is topologically transitive. This implies that topologically mixing is a stronger notion than hypercyclicity.

In Chapter 6 we want to apply the results of the prior chapters to partial sums of Taylor series. Therefore, we briefly introduce the concept of universality.

**Definition 1.1.6.** Let $X$ and $Y$ be metric spaces and let $T_n : X \to Y$, $n \in \mathbb{N}_0$ be a sequence of continuous maps.

1. An element $x \in X$ is called *universal for* $(T_n)_{n \in \mathbb{N}_0}$ if its orbit $\operatorname{orb}(x, (T_n)_{n \in \mathbb{N}_0}) := \{T_n x : n \in \mathbb{N}_0\}$ is dense in $Y$.

2. $(T_n)_{n \in \mathbb{N}_0}$ is called *(topologically) transitive* if, for any pair $U \subset X$ and $V \subset Y$ of non-empty open sets, there is some $n \geq 0$ such that

$$T_n(U) \cap V \neq \emptyset,$$

and along the same lines as before, it is called *(topologically) mixing* if the same holds for all $n$ sufficiently large. Furthermore, $(T_n)_{n \in \mathbb{N}_0}$ is called *weakly mixing* if $(T_n \times T_n)_{n \in \mathbb{N}}$ is topologically transitive on $X \times X$.

3. If $(T_n)_{n \in \mathbb{N}_0}$ has universal elements, then the set of those elements is called *algebraically generic* if it contains a dense vector subspace except 0.

In the case of universality we get a result similar to the Birkhoff transitivity theorem.

**Theorem 1.1.7** (Universality Criterion)**.** *Let $X$ be a complete metric space, $Y$ a separable metric space and $T_n : X \to Y$, $n \in \mathbb{N}_0$ continuous maps. Then the following assertions are equivalent:*

1. *$(T_n)_{n\in\mathbb{N}_0}$ is topologically transitive,*

2. *to every $x \in X$ and $y \in Y$, there exist sequences $(x_k)_{k\in\mathbb{N}}$ in $X$ and $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}_0$ such that*
$$x_k \to x \quad \textit{and} \quad T_{n_k} x_k \to y \quad (k \to \infty),$$

3. *there exists a dense set of points $x \in X$ such that $\operatorname{orb}(x, (T_n)_{n\in\mathbb{N}})$ is dense in $Y$.*

*If one of these conditions holds, then there are generically many elements in $X$ with dense orbit.*

**Remark 1.1.8.** Let $T\colon X \to X$ be an operator on a separable Fréchet space $X$. $T$ is called *hereditarily hypercyclic with respect to* $(n_k)_{k\in\mathbb{N}}$ if for each subsequence $(m_k)_{k\in\mathbb{N}}$ of $(n_k)_{k\in\mathbb{N}}$, there is some $x \in X$ such that $\{T^{m_k}x : k \in \mathbb{N}\}$ is dense in $X$. By the theorem of Bès-Peris (see e.g. [22, Theorem 3.15]), we know that $T$ is weakly mixing if and only if it is hereditarily hypercyclic with respect to some increasing sequence of positive integers. Along the same lines, $T$ is topologically mixing if and only if it is hereditarily hypercyclic with respect to the full sequence $(n)_{n\in\mathbb{N}}$ (see e.g. [22, Exercise 3.2.3]).

An easy way to show that an operator is frequently hypercyclic or mixing is by transforming it to another operator of which we already know that it has the desired property.

**Definition 1.1.9.** For two dynamical systems $T : X \to X$ and $S : Y \to Y$, $T$ is called *quasiconjugate* to $S$ (or $T$ *is a factor of* $S$) if there exists a continuous map $\phi : Y \to X$ with dense range such that $T \circ \phi = \phi \circ S$, i.e. the diagram

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ \phi\downarrow & & \downarrow\phi \\ X & \xrightarrow[T]{} & X \end{array}$$

commutes. If $\phi$ can be chosen to be a homeomorphism then $S$ and $T$ are called *conjugate.*

**Remark 1.1.10.** Hypercyclicity, frequent hypercyclicity and the mixing property are preserved under quasiconjugacy (see e.g. [22]). That means, if a dynamical system $S : Y \to Y$ has the property to be hypercyclic, frequently hypercyclic or mixing then every dynamical system $T : X \to X$ that is quasiconjugate to $S$ has the same property. In particular, this is the case if an operator is conjugate to another operator, that is, if $\phi$ can be chosen to be a homeomorphism.

## 1.2 Necessary and Sufficient Conditions

We start this section by giving some necessary conditions for an operator to be hypercyclic, frequently hypercyclic or chaotic.

**Definition 1.2.1.** Let $T$ be an operator on a Banach space $X$ and let $\|\cdot\|_{\text{op}}$ denote the operator norm.

1. $T$ is called a *contraction* if $\|T\|_{\text{op}} \leq 1$.
2. $T$ is called *power bounded* if

$$\sup_{n \geq 0} \|T^n\|_{\text{op}} < \infty.$$

For a power bounded operator each orbit is bounded which yields the following result.

**Proposition 1.2.2.** *No power bounded operator can be hypercyclic. In particular, no contraction is hypercyclic.*

**Remark and Definition 1.2.3.** Let $T\colon X \to X$ be an operator on a Fréchet space $X$. We denote by $\sigma(T)$ the spectrum of $T$. If $T$ is bijective, the open mapping theorem yields that its inverse is also continuous. In particular, $\lambda \in \sigma(T)$ if and only if at least one of the following statements is true

1. The range of $T - \lambda I$ is not all of $X$.
2. $T - \lambda I$ is not one-to-one (i.e. $\lambda$ is an eigenvalue of $T$). By $\sigma_0(T) \subset \sigma(T)$ we denote the *point spectrum of* $T$, i.e. the set of all eigenvalues of $T$.

With that, we get another necessary condition for hypercyclicity if $T$ is an operator acting on a Banach space. Note that the spectrum of a continuous operator on a Banach space is always compact.

**Theorem 1.2.4** (Kitai's Theorem)**.** *Let $X$ be a Banach space and $T\colon X \to X$ be a hypercyclic operator. Then every connected component of $\sigma(T)$ meets the unit circle.*

Finally, for frequent hypercyclicity (and chaos) of an operator $T$ on Banach spaces, the spectrum of $T$ has to fulfil an additional condition (see [3, Proposition 6.37]):

**Theorem 1.2.5.** *Let $X$ be a Banach space and $T$ be an operator on $X$. If $T$ is chaotic or frequently hypercyclic, then $\sigma(T)$ is a perfect set.*

We now want to list some sufficient conditions for the notions defined in the first section of this chapter. A useful tool to verify whether an operator is mixing is the variant of Kitai's Criterion which can be found e.g. in [22, Theorem 12.31 and Remark 3.5].

**Theorem 1.2.6** (Kitai Criterion)**.** *Let $T$ be an operator on a topological vector space $X$. If there are dense subsets $X_0$, $Y_0 \subset X$ such that*

1. *for any $x \in X_0$ it follows $T^n x \to 0$ $(n \to \infty)$,*
2. *for any $y \in Y_0$ there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $X$ such that $u_n \to 0$ and $T^n u_n \to y$ $(n \to \infty)$,*

*then $T$ is mixing.*

The following result will be concerned with notions of measurable dynamics whereas we only considered topological dynamics until now.

**Remark and Definition 1.2.7.** Let $(X, \Sigma, \mu)$ be a measure space where $\mu$ is a measure with full support, i.e. $\mu(A) > 0$ for every non-empty open set $A \subset X$. We say that a measurable function $T\colon (X, \Sigma, \mu) \to (X, \Sigma, \mu)$ is *measure-preserving* (or $\mu$ is $T$-*invariant*) if $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \Sigma$. In this setting, $T$ is called *ergodic* if it is measure-preserving and if for any two $A, B \in \Sigma$ with positive measure one can find an integer $n \geq 0$ such that

$$T^n(A) \cap B \neq \emptyset.$$

The notion of ergodicity can be viewed as a measure-theoretic analogue of topological transitivity. A consequence of Birkhoff's ergodic theorem is the following:

Let $T\colon (X, \Sigma, \mu) \to (X, \Sigma, \mu)$ be a measure-preserving transformation. Then the following are equivalent:

1. $T$ is ergodic,
2. for any $A, B \in \Sigma$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B).$$

It follows from the definition of ergodicity that it implies topological transitivity and hence also hypercyclicity. Actually, we get an even stronger result:

Let $T$ be an operator on $X$ such that there exists a $T$-invariant measure $\mu$ with full support. If $T$ is ergodic with respect to $\mu$, then $T$ is hypercyclic and the set of hypercyclic vectors has full measure. More precisely, almost every $x \in X$ has the following property: for every non-empty open set $A \subset X$ one has

$$\liminf_{N \to \infty} \frac{|\{0 \leq n \leq N : T^n x \in A\}|}{N+1} > 0,$$

i.e. $T$ is even frequently hypercyclic.

For the next result which can be found in [4], we will now define weakly and strongly mixing operators.

**Definition 1.2.8.** Let $(X, \Sigma, \mu)$ be a measure space where $X$ is a complex Fréchet space and $T : (X, \Sigma, \mu) \to (X, \Sigma, \mu)$ be a measure-preserving transformation.

1. $T$ is called *weakly mixing* (with respect to $\mu$) if

$$\frac{1}{N} \sum_{n=0}^{N-1} |\mu(A \cap T^{-n}(B)) - \mu(A)\mu(B)| \to 0 \quad (N \to \infty)$$

   for any measurable set $A, B \subset X$.

2. $T$ is called *strongly mixing* (with respect to $\mu$) if

$$\mu(A \cap T^{-n}(B)) \to \mu(A)\mu(B) \quad (n \to \infty)$$

   for any $A, B \in \Sigma$.

**Remark 1.2.9.** The notion of weakly mixing in a measure-theoretic sense is consistent with that in topological dynamics, since one can show that a measure-preserving transformation $T$ is weakly mixing if and only if $T \times T$ is ergodic on $(X \times X, \Sigma \otimes \Sigma, \mu \otimes \mu)$. Furthermore, if $T : X \to X$ is continuous and (weakly) mixing with respect to some measure $\mu$ with full support, then $T$ is also (weakly) mixing in the topological sense. In that case, $T$ is also frequently hypercyclic according to Remark and Definition 1.2.7.

**Definition 1.2.10.** Let $X$ be a complex Fréchet space, $\mathcal{B}$ the Borel-$\sigma$-algebra on $X$ and $\mu$ a measure on $\mathcal{B}$. A measure-preserving transformation $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ is called weakly (resp. strongly) mixing *in the Gaussian sense* if there exists a Gaussian probability measure $\mu$ such that $T$ is weakly (resp. strongly) mixing with respect to $\mu$. For the definition of Gaussian probability measures, see e.g. [3].

In line with [4], we define perfectly spanning $\mathbb{T}$-eigenvectors. For that, we will need the notion of sets of uniqueness. A short overview of the most important properties will be given in Appendix A.

**Definition 1.2.11.** Let $X$ be a complex, separable Fréchet space and $T$ an operator on $X$. Then $T$ has *perfectly spanning* $\mathbb{T}$*-eigenvectors* if for any countable set $D \subset \mathbb{T}$

$$\operatorname{span} \bigcup_{\lambda \in \mathbb{T} \setminus D} \ker(T - \lambda I) \text{ is dense in } X.$$

Furthermore, we say that $T$ has $\mathcal{U}_0$*-perfectly spanning* $\mathbb{T}$*-eigenvectors* if for any Borel set of extended uniqueness $D \subset \mathbb{T}$, the linear span of $\bigcup_{\lambda \in \mathbb{T} \setminus D} \ker(T - \lambda I)$ is dense in $X$.

In [4, Theorem 1] the following important result was proved. For the notion of the cotype of a Banach space which will occur in the third part of the theorem see for example [2].

**Theorem 1.2.12.** *Let $X$ be a complex separable Fréchet space and $T$ be an operator on $X$.*

1. *If the $\mathbb{T}$-eigenvectors of $T$ are perfectly spanning, then $T$ is weakly mixing in the Gaussian sense.*

2. *If the $\mathbb{T}$-eigenvectors are $\mathcal{U}_0$-perfectly spanning, then $T$ is strongly mixing in the Gaussian sense.*

3. *In 1. and 2., the converse implications are true if $X$ is a Banach space with cotype 2.*

# Chapter 2

# The Taylor Shift on $H(\Omega)$

## 2.1 The Taylor shift operator

We start this section by introducing some general notations from plane and spherical topology that will be used throughout this thesis. The extended complex plane $\mathbb{C}\cup\{\infty\}$ will be denoted by $\mathbb{C}_\infty$ and shall be equipped with the chordal metric. As usual, we determine $z/\infty := 0$ for $z \in \mathbb{C}$ and $z/0 := \infty$ for $z \in \mathbb{C}_\infty \setminus \{0\}$. We denote by $D_r$ the open disc around 0 with radius $0 \le r \le \infty$, i.e. $\{z \in \mathbb{C} : |z| < r\}$ and the unit disc will be briefly written as $\mathbb{D}$. The closed disc around 0 with radius $r$ will be written as $\Delta_r$ and the closed unit disc respectively as $\Delta$. The unit circle $\{z \in \mathbb{C} : |z| = 1\}$ will be denoted by $\mathbb{T}$. For $M \subset \mathbb{C}_\infty$ we set $M^{-1} := \{1/z : z \in M\}$ and

$$M^* := (\mathbb{C}_\infty \setminus M)^{-1}.$$

For an open set $\Omega \subset \mathbb{C}_\infty$ with $0 \in \Omega$, we have that $\Omega^*$ is compact in $\mathbb{C}$.

**Definition 2.1.1.** Let $\Omega \subset \mathbb{C}_\infty$ be an open set. Then we denote by $H(\Omega)$ the space of holomorphic functions on $\Omega$ vanishing at $\infty$ if $\infty \in \Omega$ where we call a function $f$ holomorphic at infinity if $f(z^{-1})$ is holomorphic at 0. Following the lines of [39], we can find compact sets $K_n$, $n \in \mathbb{N}$, with $K_n \subset \operatorname{int}(K_{n+1})$ and $\Omega = \bigcup_{n\in\mathbb{N}} K_n$. Then $H(\Omega)$ is topologized by the increasing sequence of seminorms

$$\|f\|_{K_n,\infty} := \max_{z\in K_n} |f(z)| \quad \text{for } f \in H(\Omega).$$

The sets

$$V_n = \{f \in H(\Omega)\colon \|f\|_{K_n,\infty} < \frac{1}{n}\}$$

form a convex local 0-base for $H(\Omega)$. Furthermore, one can show that the topology of $H(\Omega)$ is compatible with the complete metric

$$d(f,g) = \max_{n\in\mathbb{N}} \frac{2^{-n}\|f-g\|_{K_n,\infty}}{1+\|f-g\|_{K_n,\infty}}.$$

Therefore, it follows that $H(\Omega)$ forms a Fréchet space endowed with the seminorms

$$\|f\|_{K,\infty} := \max_{z\in K} |f(z)| \quad (f \in H(\Omega))$$

for $K \subset \Omega$ compact.

We now want to define the operator which will be studied for its dynamical behaviour throughout this work.

**Remark and Definition 2.1.2.** Let $\Omega \subset \mathbb{C}_\infty$ be open. If $0 \in \Omega$, we define $T := T_{H(\Omega)} : H(\Omega) \to H(\Omega)$ by

$$Tf(z) := \begin{cases} \frac{1}{z}(f(z)-f(0)), & z \neq 0 \\ f'(0), & z = 0. \end{cases} \tag{2.1}$$

One can easily see that for $n \in \mathbb{N}_0$

$$T^n f(z) = \begin{cases} \frac{1}{z^n}(f(z)-s_{n-1}f(z)), & z \neq 0 \\ a_n, & z = 0, \end{cases} \tag{2.2}$$

where $s_n f(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ denotes the $n$th partial sum of the Taylor expansion of $f$ around 0. The iterates of $T$ under $f$ have the Taylor series representation

$$T^n f(z) = \sum_{\nu=0}^{\infty} a_{\nu+n} z^\nu \tag{2.3}$$

for $|z| < \operatorname{dist}(0,\partial\Omega)$. Because of this property, $T$ is called the Taylor (backward) shift on $H(\Omega)$.

Furthermore, we can define the Taylor forward shift for open sets $\Omega$ with $\infty \notin \Omega$ as

$$S\colon H(\Omega) \to H(\Omega),\ Sf(z) = zf(z). \tag{2.4}$$

The name is justified because for $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$, we have again for $|z| < \operatorname{dist}(0, \partial\Omega)$ that

$$S^n f(z) = \sum_{\nu=0}^{\infty} a_{\nu-n} z^\nu$$

with $a_{-k} = 0$ for $k \in \mathbb{N}$. The forward shift is a right inverse to the Taylor shift since for $f \in H(\Omega)$, we have

$$TSf(z) = \frac{zf(z) - 0f(0)}{z} = f(z).$$

A first simple result regarding frequent hypercyclicity on $H(\mathbb{D})$ is the following proposition (for the proof see [41]).

**Proposition 2.1.3.** *The Taylor shift operator $T$ on $H(\mathbb{D})$ is frequently hypercyclic.*

In particular, one knows with Remark 1.1.3 that the set of the frequently hypercyclic vectors is of first Baire category.

**Remark 2.1.4.**

1. Let $\Omega \subset \mathbb{C}_\infty$ be open with $0 \in \Omega$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Then $\lambda T\colon H(\Omega) \to H(\Omega)$ is quasiconjugate to $T\colon H(\lambda^{-1}\Omega) \to H(\lambda^{-1}\Omega)$.
   For that, define the operator

   $$R\colon H(\lambda^{-1}\Omega) \to H(\Omega),\ Rf(z) = f(\lambda z).$$

   Then one can easily see that $R$ is continuous, has dense range and it holds that the diagram

   $$\begin{array}{ccc} H(\lambda^{-1}\Omega) & \xrightarrow{T} & H(\lambda^{-1}\Omega) \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R} \\ H(\Omega) & \xrightarrow[\lambda T]{} & H(\Omega) \end{array}$$

   commutes.

2. Let $\Omega$, $\Omega_0 \subset \mathbb{C}_\infty$ be open sets such that $\Omega \subset \Omega_0$ and $0 \in \Omega$. If $H(\Omega_0)$ is dense in $H(\Omega)$, then $T_{H(\Omega)}$ is quasiconjugate to $T_{H(\Omega_0)}$.

Let $j\colon H(\Omega_0) \to H(\Omega)$, $j(f) = f|_\Omega$ be the restriction of $f$ to $\Omega$. Then $j$ is continuous and $T_{H(\Omega)} \circ j = j \circ T_{H(\Omega_0)}$. Therefore, $T_{H(\Omega)}$ is quasiconjugate to $T_{H(\Omega_0)}$.

Note that in the setting above, $H(\Omega_0)$ is dense in $H(\Omega)$ if and only if $\Omega_0 \setminus \Omega$ has no compact component (Runge's Theorem, see e.g. [37, Theorem 13.5]).

**Remark 2.1.5.** By Proposition 2.1.3 and the second part of Remark 2.1.4, we have the following: For an open subset $\Omega$ of the unit disc $\mathbb{D}$ containing 0 such that $\mathbb{C} \setminus \Omega$ is connected, the Taylor shift $T\colon H(\Omega) \to H(\Omega)$ is frequently hypercyclic.

## 2.2 Eigenvalue Criteria

The Taylor shift operator on $H(\Omega)$ for general open sets $\Omega \subset \mathbb{C}_\infty$ was introduced in [7] and the following result was proved.

**Theorem 2.2.1** (Beise, Meyrath, Müller)**.** *Let $\Omega \subset \mathbb{C}_\infty$ be open with $0 \in \Omega$. Then the following are equivalent:*

1. *$T$ is mixing on $H(\Omega)$,*
2. *$T$ is hypercyclic on $H(\Omega)$,*
3. *each connected component of $\Omega^*$ intersects $\mathbb{T}$.*

It is our aim to study the weak and strong mixing property of the Taylor shift operator. As we have seen in the first chapter, a sufficient supply of unimodular eigenfunctions is useful in order to prove this. Therefore, we define for $\alpha \in \mathbb{C}$ the functions

$$\gamma(\alpha)\colon \{\alpha\}^* \to \mathbb{C},\ \gamma(\alpha)(z) \coloneqq \frac{1}{1-\alpha z} \quad (z \in \mathbb{C}_\infty \setminus \{1/\alpha\}). \tag{2.5}$$

**Proposition 2.2.2.** *Let $\Omega \subset \mathbb{C}_\infty$ be an open set with $0 \in \Omega$ and $T$ be the Taylor shift on $H(\Omega)$. Then*

$$\sigma(T) = \sigma_0(T) = \Omega^*$$

*where an eigenfunction to the eigenvalue $\alpha \in \Omega^*$ is given by $\gamma(\alpha)$. Furthermore, for $\alpha \in \Omega^*$ the kernel of $T - \alpha I$ is 1-dimensional and it fulfils*

$$\ker(T - \alpha I) = \operatorname{span} \gamma(\alpha).$$

*Proof.* One can easily see that, for $\alpha \in \Omega^*$, the multiples of $\gamma(\alpha)$ are eigenfunctions for the Taylor shift on $\Omega$ to the corresponding eigenvalue $\alpha$. In particular, the set $\Omega^*$ is contained in the point spectrum and therefore in the spectrum of $T$. On the other hand, for $\alpha \in 1/\Omega = \mathbb{C}_\infty \setminus \Omega^*$ it is easily seen that

$$S_\alpha : H(\Omega) \to H(\Omega),\ S_\alpha g(z) = \frac{zg(z) - g(1/\alpha)/\alpha}{1 - z\alpha} \tag{2.6}$$

(continuously extended at the point $1/\alpha$) defines the continuous inverse to $T - \alpha I$, so $\alpha \in 1/\Omega$ cannot be in the spectrum of $T$. Therefore, $\sigma(T) = \sigma_0(T) = \Omega^*$. Finally, suppose that $f \in H(\Omega)$ is an eigenfunction to the eigenvalue $\alpha \in \Omega^*$ with Taylor expansion $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ for $z$ sufficiently small. By (2.3) it follows

$$a_\nu = \alpha^\nu a_0 \quad \text{for all } \nu \in \mathbb{N}.$$

Therefore, $f(z) = a_0 \sum_{\nu=0}^{\infty} \alpha^\nu z^\nu = a_0 \gamma(\alpha)(z)$ so every eigenfunction of the Taylor shift has to be a multiple of $\gamma(\alpha)$. □

**Remark 2.2.3.** In the setting of Proposition 2.2.2, $T$ is invertible if and only if $\infty \in \Omega$ and for $g \in H(\Omega)$ with $g(z) = \sum_{\nu=0}^{\infty} b_\nu / z^{\nu+1}$ near $\infty$ the inverse is given by

$$T^{-1} g(z) = \sum_{\nu=0}^{\infty} \frac{b_{\nu+1}}{z^{\nu+1}}$$

near $\infty$. Hence, $T^{-1}$ is again a backward shift.

Approximation of holomorphic functions by rational functions with simple poles is of importance for the following results of this chapter. Therefore, we introduce a version of Runge's theorem which can be found in [28, Theorem 10.2].

**Theorem 2.2.4** (Runge)**.** *Let $\Omega \subset \mathbb{C}_\infty$ be open and $\Lambda \subset \Omega^*$ such that each connected component of $\Omega^*$ contains an accumulation point of $\Lambda$. Then* $\operatorname{span}\{\gamma(\alpha) : \alpha \in \Lambda\}$ *is dense in $H(\Omega)$.*

A first result regarding the frequent hypercyclicity of the Taylor shift was already shown in [41]. In the following, we call $A$ a non-trivial subarc of $\mathbb{T}$ if it is of the form $A = \{e^{it} : t \in (a, b)\}$ with $a < b$.

**Theorem 2.2.5.** *Let $0 \in \Omega \subset \mathbb{C}_\infty$ be an open set such that each component of $\Omega^*$ contains a non-trivial subarc of $\mathbb{T}$ and let $T$ be the Taylor shift on $H(\Omega)$. Then $T$ has a spanning $C^\infty$-eigenvector field and $T$ is mixing, chaotic and frequently hypercyclic on $H(\Omega)$.*

**Example 2.2.6.** Let $\Omega := \mathbb{C}_\infty \setminus B$ where $B$ is the closure of a non-trivial subarc of $\mathbb{T}$. With the previous theorem we obtain that the Taylor shift on $H(\Omega)$ is mixing, chaotic and frequently hypercyclic.

$A \subset \mathbb{C}$ is called a *$\mathcal{U}_0$-perfect set* if for every set $D \subset \mathbb{C}$ of extended uniqueness it follows that $A \setminus D$ is a perfect set. We apply Theorem 1.2.12 to obtain the following more general condition for the Taylor shift to be weakly or strongly mixing in the Gaussian sense and by the first chapter it is then also frequently hypercyclic.

**Theorem 2.2.7.** *Let $0 \in \Omega \subset \mathbb{C}_\infty$ be an open set such that each component of $\Omega^*$ meets $\mathbb{T}$ and $T$ be the Taylor shift on $H(\Omega)$.*

1. *If $\Omega^* \cap \mathbb{T}$ is a perfect set, then $T$ has perfectly spanning $\mathbb{T}$-eigenvectors.*

2. *If $\Omega^* \cap \mathbb{T}$ is a $\mathcal{U}_0$-perfect set, then $T$ has $\mathcal{U}_0$-perfectly spanning $\mathbb{T}$-eigenvectors.*

*Proof.* By definition, we need to show that for every countable subset $D \subset \mathbb{T}$ the linear span of $\bigcup_{\alpha \in \mathbb{T}\setminus D} \ker(T - \alpha I)$ is dense in $H(\Omega)$. For $\Lambda_D := (\Omega^* \cap \mathbb{T}) \setminus D$ we have

$$\operatorname{span} \bigcup_{\alpha \in \mathbb{T}\setminus D} \ker(T - \alpha I) = \operatorname{span}\{\gamma(\alpha) : \alpha \in \Lambda_D\}$$

and because every point of $\Omega^* \cap \mathbb{T}$ is an accumulation point of $\Lambda_D$, we obtain from Theorem 2.2.4 that $\operatorname{span}\{\gamma(\alpha) : \alpha \in \Lambda_D\}$ is dense in $H(\Omega)$. Therefore, the Taylor shift has perfectly spanning $\mathbb{T}$-eigenvectors. The second case follows along the same lines. □

**Theorem 2.2.8.** *Let $0 \in \Omega \subset \mathbb{C}_\infty$ be an open set such that each component of $\Omega^*$ meets $\mathbb{T}$ and $T$ be the Taylor shift on $H(\Omega)$.*

1. *If $\Omega^* \cap \mathbb{T}$ is a perfect set, then $T$ is weakly mixing in the Gaussian sense.*

2. *If $\Omega^* \cap \mathbb{T}$ is a $\mathcal{U}_0$-perfect set, then $T$ is strongly mixing in the Gaussian sense.*

*Proof.* The first and second statement follow from the fact that $T$ has perfectly spanning (respectively $\mathcal{U}_0$-perfectly spanning) $\mathbb{T}$-eigenvectors and Theorem 1.2.12. □

As already stated, the last theorem yields that under the given assumptions the Taylor shift is frequently hypercyclic. In contrast to that, we now want to use the Mellin transformation in order to find open sets $\Omega$ such that this is not the case for the Taylor shift on $H(\Omega)$.

Let $L \subset \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \pi\}$ be compact and convex and $\Omega_L := (e^L)^*$. Let $\mathrm{Exp}(L)$ denote the space of the entire functions $f$ of exponential type whose conjugate indicator diagram $K(f)$ is contained in $L$ (see Appendix B). Then the Mellin transformation $M$ is defined as

$$M : H(\Omega_L) \to \mathrm{Exp}(L),\ Mg(z) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{g(w)}{w^{z+1}} dw \quad (z \in \mathbb{C})$$

where $w^z := e^{z \log w}$ for $z \in \mathbb{C}$, $w \in \mathbb{C}_-$ with $\mathbb{C}_- := \mathbb{C} \setminus (-\infty, 0]$ and where $\gamma$ is a loop in $\mathbb{C}_- \setminus e^{-L}$ of index $-1$ with respect to the compact set $e^{-L}$. Furthermore, log denotes the principal branch of the logarithm. An introduction to the Mellin transformation can be found in Appendix B.

To obtain situations in which the Taylor shift is not frequently hypercyclic on $H(\Omega)$ for an open set $\Omega$, we now show that the Mellin transform conjugates it to the translation operator on the space of entire functions of exponential type.

**Theorem 2.2.9.** *Let $L \subset \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \pi\}$ be compact and convex. Then the Taylor shift operator on $H(\Omega_L)$ is conjugate to the translation operator*

$$\tau \colon \mathrm{Exp}(L) \to \mathrm{Exp}(L),\ \tau f(z) = f(z+1).$$

*Proof.* Let $\gamma$ be a loop in $e^{-L-\frac{1}{m}\Delta}$ of index $-1$ with respect to the points in $e^{-L}$. We already know that the Mellin transform $M : H(\Omega_L) \to \mathrm{Exp}(L)$ is a homeomorphism. Let $g \in H(\Omega_L)$, then one can easily see that for $n \in \mathbb{N}_0$

$$(M \circ T)(g)(n) = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{g(w) - g(0)}{w^{n+2}} dw = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{g(w)}{w^{(n+1)+1}} dw = (\tau \circ M)(g)(n)$$

Now $(M \circ T - \tau \circ M)(g) \in \mathrm{Exp}(L)$ and we have

$$\max_{z \in L} \mathrm{Im}\ z - \min_{z \in L} \mathrm{Im}\ z < 2\pi$$

by assumption, so with Carlson's theorem (see Appendix B.1.7) we get $M \circ T = \tau \circ M$. Thus, $T$ is conjugate to $\tau$ . □

For $v, w \in \mathbb{C}$, we denote by $[v, w] := \{z \in \mathbb{C} \colon z = \lambda v + (1 - \lambda)w,\ \lambda \in [0, 1]\}$ the line segment between $v$ and $w$.

**Remark 2.2.10.**
1. In [5, Theorem 1] it was proved that for $L := [v, w] \subset \{z \in \mathbb{C} : |\mathrm{Im}\ z| < \pi\}$ the translation operator $\tau$ on $\mathrm{Exp}(L)$ is frequently hypercyclic if and only if $v \neq w$ and $v, w \in i\mathbb{R}$.

2. With [6, Theorem 3] one obtains that for an entire function of exponential type $f$ such that $K(f) \subset \{z : \mathrm{Re}\ z \leq 0\}$ and $K(f) \cap \{z : \mathrm{Re}\ z = 0\}$ contains at most one element, then $f$ is not frequently hypercyclic for the translation operator $\tau$ with respect to the topology on $H(\mathbb{C})$.

Since the property of frequent hypercyclicity is preserved under conjugacy (see Remark 1.1.10), we obtain with Theorem 2.2.9 (see also [41]):

**Corollary 2.2.11.**
1. *Let $v, w$ be complex numbers and $L := [v, w] \subset \{z \in \mathbb{C} : |\mathrm{Im}\ z| < \pi\}$. Then the Taylor shift operator on $H(\Omega_L)$ is frequently hypercyclic if and only if $v \neq w$ and $v, w \in i\mathbb{R}$.*

2. *For a compact and convex set $L \subset \{z : |\mathrm{Im}\ z| < \pi\} \cap \{z : \mathrm{Re}\ z \leq 0\}$ such that there exists only one element $z_0 \in L$ with $\mathrm{Re}\ z_0 = 0$, the Taylor shift operator $T : H(\Omega_L) \to H(\Omega_L)$ is not frequently hypercyclic.*

From this corollary, we can easily deduce an open set $\Omega$ such that the Taylor shift operator is not frequently hypercyclic on $H(\Omega)$.

**Example 2.2.12.** Let $I = [x, y] \subset \mathbb{R}_+$ be an interval on the positive real axis with $1 \in I$. Then it follows for $L = [-\log y, -\log x]$ that the Taylor shift on $H(\Omega_L) = H(\mathbb{C}_\infty \setminus I)$ is not frequently hypercyclic. In particular, the special case $x = y = 1$ yields that the Taylor shift on $H(\mathbb{C}_\infty \setminus \{1\})$ is not frequently hypercyclic. Compared to that, let

$B := \{e^{it} \colon t \in [a, b]\}$ where $a, b \in \mathbb{R}$ with $a < b$. We have already seen in Example 2.2.6 that for $L := [ia, ib]$ the Taylor shift on $H(\Omega_L) = H(\mathbb{C}_\infty \setminus B)$ is frequently hypercyclic since $B$ is the closure of a non-trivial subarc.

# Chapter 3

# The Taylor Shift on Banach Spaces of Functions Holomorphic in the Unit Disc $\mathbb{D}$

## 3.1 Bergman and Hardy Spaces

In this section, we introduce the Hardy and Bergman spaces and review some basic properties. A function $f \in H(\mathbb{D})$ belongs to the Bergman space $A^p$, $0 < p < \infty$ if it fulfils

$$\|f\|_p = \left( \int_{\mathbb{D}} |f|^p d\lambda_2 \right)^{1/p} < \infty$$

where $\lambda_2$ is the 2-dimensional Lebesgue measure.
The following proposition can be found in [19, Theorem 1, p. 7] and shows that point-evaluation is a bounded linear functional on $A^p$ for $0 < p < \infty$.

**Proposition 3.1.1** (see [19]). *Each function $f \in A^p$ has the property*

$$|f(z)| \leq \pi^{-1/p} \delta(z)^{-2/p} \|f\|_p, \quad \textit{for all } z \in \mathbb{D}$$

*where $\delta(z) = \operatorname{dist}(z, \mathbb{T})$ is the distance from $z$ to the boundary of $\mathbb{D}$.*

This fact has several important implications which will be summarized in the following remark.

**Remark 3.1.2.**

1. If $f_n$ and $f$ are in $A^p$, $n \in \mathbb{N}$, with $\|f_n - f\|_p \to 0$ $(n \to \infty)$, then $f_n \to f$ uniformly on every compact subset of $\mathbb{D}$.

2. Because $A^p$ is a closed subspace of the complete space $L^p(\mathbb{D}, \lambda_2)$, the space $A^p$ is also complete. Thus, $A^p$ is a Banach space for $p \geq 1$ and for $p = 2$ it is a Hilbert space with inner product

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z)\overline{g(z)} d\lambda_2(z).$$

For $0 < p < 1$, the triangle inequality is no longer satisfied.

**Theorem 3.1.3** (see [19]). *For $1 < p < \infty$, the dual space of $A^p$ can be identified with $A^q$ where $1/p + 1/q = 1$. Each functional $\varphi \in (A^p)'$ has a unique representation*

$$\varphi(f) = \int_{\mathbb{D}} f\overline{g} d\lambda_2, \quad f \in A^p,$$

*for some $g \in A^q$.*

**Theorem 3.1.4** (see [19]). *Let $1 \leq p < \infty$. Then the polynomials are dense in $A^p$.*

Replacing the area integral by a line integral, we obtain Hardy spaces. For $f \in H(\mathbb{D})$, the function $f_r$ on $\mathbb{T}$ is defined by $f_r(e^{it}) = f(re^{it})$ for $0 \leq r < 1$. By $m$ we denote the normalized arc length measure on $\mathbb{T}$. Then the functions $f_r$ belong to the $L^p(\mathbb{T}, m)$-space for $0 \leq r < 1$ and $0 < p < \infty$ and the Hardy space $H^p$ is defined as the set of functions $f \in H(\mathbb{D})$ that fulfil

$$\|f\|_{H^p} = \sup_{0 \leq r < 1} \|f_r\|_{\mathbb{T},p} < \infty.$$

In [18], the following result regarding point-evaluation of functions in Hardy spaces can be found:

**Proposition 3.1.5** (see [18]). *If $0 < p < \infty$ and $f \in H^p$, then*

$$|f(z)| \leq 2^{1/p}\|f\|_p(1 - |z|)^{-1/p}.$$

**Remark 3.1.6.** As in Remark 3.1.2, if $f_n,\ f \in H^p$, $n \in \mathbb{N}$, with $\|f_n - f\|_{H^p} \to 0$ $(n \to \infty)$, then $f_n \to f$ locally uniformly on $\mathbb{D}$.

For an important property of functions in the Hardy spaces, we will need the notion of nontangential limits (see [38]): For $\zeta \in \mathbb{T}$ and $0 < r < 1$ we denote by $\Omega_{\zeta,r}$ the union of the disc $D_r$ and the line segments from $\zeta$ to the points of $D_r$. We say that a function $f$ holomorphic on $\mathbb{D}$ has a *nontangential limit* $f^*(\zeta)$ at $\zeta \in \mathbb{T}$ if for each $0 < r < 1$

$$f(z_n) \to f^*(\zeta) \quad (n \to \infty)$$

where $(z_n)_{n\in\mathbb{N}}$ is an arbitrary sequence in $\Omega_{\zeta,r}$ tending to $\zeta$. Then we have the following result for functions in the Hardy spaces (see e.g. [38, Theorem 17.11]):

**Theorem**: If $0 < p < \infty$ and $f \in H^p$, then the nontangential limits $f^*(\zeta)$ exist almost everywhere on $\mathbb{T}$ with nontangential limit $f^* \in L^p(\mathbb{T}, m)$ and $\|f^*\|_{\mathbb{T},p} = \|f\|_{H^p}$.

**Remark 3.1.7.** Using the theorem about nontangential limits, it can be shown that for $1 \leq p < \infty$, the space $H^p$ equipped with the norm $\|\cdot\|_{H^p}$ is a Banach space and $H^2$ is a Hilbert space. For $0 < p < 1$, $H^p$ is still a vector space but the triangle inequality is no longer satisfied for $\|\cdot\|_{H^p}$.

## 3.2 The Taylor shift on the Bergman and Hardy Spaces

We want to consider the Taylor shift $T = T_{A^p}$ on the Bergman space $A^p$ for $1 \leq p < \infty$. Note that $(A^p,\ T)$ is a dynamical system. In the previous chapter, we have seen that the eigenfunctions to the eigenvalue $\alpha \in \mathbb{D}^* = \Delta$ for the Taylor shift on $H(\mathbb{D})$ are given by multiples of $\gamma(\alpha)$ which was defined in (2.5). If $\alpha \in \mathbb{T}$, the functions $\gamma(\alpha)$ only lie in $A^p$ if $1 \leq p < 2$. To be more precise, the following is true:

1. $\sigma(T) = \Delta$ for all $1 \leq p < \infty$,
2. $\sigma_0(T) = \Delta$ for $1 \leq p < 2$ and $\sigma_0(T) = \mathbb{D}$ for $2 \leq p < \infty$.

**Remark 3.2.1.** Let $\ell^2 = \{x \in \mathbb{C}^{\mathbb{N}}\colon \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$ and $B_w$ be the weighted (backward) shift on $\ell^2$, that is

$$B_w(x_1, x_2, x_3, ...) = (w_2x_2, w_3x_3, w_4x_4, ...),$$

where $w = (w_n)_{n\in\mathbb{N}}$ is a sequence of nonzero scalars. Then $B_w$ is a self map if and only if the weights $w_n$, $n \geq 1$, are bounded. Furthermore, the following is true (see [22, Example 4.9(a) and Proposition 9.17]):

1. $B_w$ is hypercyclic on $\ell^2$ if and only if
$$\sup_{n\in\mathbb{N}} \prod_{\nu=1}^{n} |w_\nu| = \infty.$$
2. $B_w$ is mixing on $\ell^2$ if and only if
$$\lim_{n\to\infty} \prod_{\nu=1}^{n} |w_\nu| = \infty.$$
3. $B_w$ is chaotic on $\ell^2$ if and only if
$$\sum_{n=1}^{\infty} \frac{1}{\prod_{\nu=1}^{n} |w_\nu|^2} < \infty.$$
4. If $B_w$ is frequently hypercyclic, then there exists a subset $A \subset \mathbb{N}_0$ of positive lower density such that
$$\sum_{n\in A} \left(\prod_{\nu=0}^{n} w_\nu\right)^{-1} < \infty.$$

In [22, Example 4.9(b)], it was shown that for $w_n = (\frac{n+1}{n})^{1/2}$, $n \geq 1$ the weighted shift $B_w$ on $\ell^2$ is conjugate to the Taylor shift $T$ on $A^2$. Therefore, we obtain:

1. Since $\lim_{n\to\infty} \prod_{\nu=1}^{n} |\frac{\nu+1}{\nu}|^{1/2} = \lim_{n\to\infty}(n+1)^{1/2} = \infty$, the Taylor shift is mixing on $A^2$.
2. Since $\sum_{n=1}^{\infty} 1/(\prod_{\nu=1}^{\infty} |\frac{n+1}{n}|) = \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty$, the Taylor shift is not chaotic.

3. Since there does not exist a set $A = \{n_k : k \in \mathbb{N}\} \subset \mathbb{N}_0$ of positive lower density such that $\sum_{k=1}^{\infty} \frac{1}{(n_k+1)^{1/2}}$ converges, the Taylor shift is not frequently hypercyclic on $A^2$ (see [22, Example 9.18]).

Using the Kitai Criterion, it was shown in [8, Proposition 3.1] that the Taylor shift is mixing on $A^p$ for all $1 \leq p < \infty$. We now want to prove that $T$ is frequently hypercyclic on $A^p$ for $1 \leq p < 2$ as well as weakly and strongly mixing in the Gaussian sense.

**Remark 3.2.2.** For $1 \leq p < q < \infty$ it follows that $A^q \subset A^p$ and that the embedding

$$j\colon (A^q, \|\cdot\|_q) \to (A^p, \|\cdot\|_p)$$

is continuous. Furthermore, $j(A^q)$ is dense in $A^p$ since the polynomials are dense in $A^p$ for all $p \geq 1$ by Theorem 3.1.4. If $T_{A^p}$ denotes the Taylor shift on $A^p$ and $T_{A^q}$ the Taylor shift on $A^q$, then we obviously have that $T_{A^p} \circ j = j \circ T_{A^q}$, i.e. $T_{A^p}$ is quasiconjugate to $T_{A^q}$. We therefore obtain that whenever the Taylor shift is frequently hypercyclic on $A^q$, it is also frequently hypercyclic on $A^p$ for all $1 \leq p \leq q$. The same holds for all other dynamical properties that are preserved under quasiconjugacy.

In the following, vector-valued integration will play an important role. For a short introduction to the concept see [39, Chapter 3] for the case of Fréchet spaces or [17] for Banach spaces.

**Definition 3.2.3.** Let $E \in \mathcal{B}$ be bounded. By $M(E)$ we denote the set of complex measures concentrated on $E$ (see Appendix A) and we define the *Cauchy transform of* $\mu \in M(E)$ by

$$C\mu(z) := \int\limits_E \gamma(\zeta) d\overline{\mu}(\zeta) = \int\limits_E \frac{1}{1-\zeta\cdot} d\overline{\mu}(\zeta) \quad (z \in E^*).$$

Note that $C\mu$ is holomorphic in the interior of $E^*$ by [15, Proposition 5.2, Section 18.5], i.e.

$$C\colon M(E) \to H(\mathrm{int}(E^*))$$

and the mapping $C$ is called the *Cauchy transformation with respect to $E$.*

**Remark 3.2.4.** For a set $E \in \mathcal{B}$ with $0 \in \operatorname{int}(E^*)$ and $\mu \in M(E)$, the Cauchy transform is holomorphic for $|z| < \operatorname{dist}(0, \partial(E^*))$. For those $z$ we have

$$C\mu(z) = \int_E \gamma(\zeta)(z) d\overline{\mu}(\zeta) = \int_E \sum_{\nu=0}^{\infty} (\zeta z)^\nu d\overline{\mu}(\zeta) = \sum_{\nu=0}^{\infty} z^\nu \int_E \zeta^\nu d\overline{\mu}(\zeta).$$

Therefore, the coefficients $a_\nu$ of the Taylor expansion of the Cauchy transform $C\mu$ around 0 are given by

$$a_\nu = \int_E \zeta^\nu d\overline{\mu}(\zeta).$$

**Remark 3.2.5.** There is an immediate connection between the eigenfunctions of the Taylor shift and the Cauchy transformation: For an open set $\Omega$ with $0 \in \Omega$ and $\alpha \in \Omega^*$ we have

$$\gamma(\alpha) = \int_{\Omega^*} \gamma(\zeta) d\delta_\alpha(\zeta) = \int_{\Omega^*} \frac{1}{1-\zeta\cdot} d\delta_\alpha(\zeta) = C\delta_\alpha \tag{3.1}$$

where $\delta_\alpha \in M(\Omega^*)$ is the Dirac measure with respect to $\alpha$.

As one can see, the Cauchy transform can be viewed as a "mean" of the eigenfunctions of the Taylor shift. This gives an idea why it will be of importance later on in this work.

The following important result regarding Cauchy transforms can be found in [15, Section 18.5] and will be useful for the next theorem and also in several other settings of this work.

**Theorem 3.2.6** (see [15])**.** *Let $f$ be integrable with respect to the 2-dimensional Lebesgue measure $\lambda_2$ with compact support. Then the Cauchy transform of $fd\lambda_2$ is defined for all $z \in \mathbb{C}_\infty$ and continuous on $\mathbb{C}_\infty$.*

We can now show that the eigenfunctions of the Taylor shift with poles on the boundary of the unit circle are dense in $A^p$ for $1 < p < 2$.

**Theorem 3.2.7.** *Let $1 < p < 2$. If $M$ is a dense subset of $\mathbb{T}$, then*

$$\operatorname{span}\{\gamma(\alpha) \colon \alpha \in M\} \textit{ is dense in } A^p.$$

*Proof.* Let $1 < p < 2$ and $M$ be dense in $\mathbb{T}$. We choose $\varphi \in (A^p)'$ with $\varphi(\gamma(\alpha)) = 0$ for all $\alpha \in M$. By Theorem 3.1.3, there exists a function $g \in A^q$, where $q$ is conjugated to

$p$, such that

$$\varphi(f) = \int_{\mathbb{D}} f\overline{g}d\lambda_2.$$

Then we have for $d\mu = gd\lambda_2$

$$C\mu(\alpha) = \varphi(\gamma(\alpha)) = \int_{\mathbb{D}} \frac{\overline{g}(z)}{1 - z\alpha} d\lambda_2(z) = 0 \quad \text{for all } \alpha \in M.$$

$C\mu$ is holomorphic in $\text{int}(\mathbb{D}^*) = \mathbb{D}$ and by Theorem 3.2.6 continuous everywhere. Since $M$ is a dense subset of $\mathbb{T}$, we have that $C\mu \equiv 0$ on $\mathbb{T}$. But then the maximum principle yields that $C\mu \equiv 0$ on $\overline{\mathbb{D}}$. Using Remark 3.2.4, the coefficients $a_n$ of the Taylor expansion of $C\mu$ around 0 fulfil

$$0 = a_n = \int_{\mathbb{D}} z^n d\overline{\mu}(z) = \int_{\mathbb{D}} z^n \overline{g}(z) d\lambda_2(z) \quad \text{for all } n \in \mathbb{N}.$$

Thus, it follows for the monomials $p_n(z) = z^n$ that

$$\varphi(p_n) = \int_{\mathbb{D}} p_n \overline{g} d\lambda_2 = 0 \quad \text{for all } n \in \mathbb{N}$$

and since by Theorem 3.1.4 the polynomials are dense in $A^p$ the Hahn-Banach theorem yields that $\varphi(f) = 0$ for all $f \in A^p$. □

We can now show that we have a complete characterization of frequent hypercyclicity of the Taylor shift on $A^p$ for $1 \leq p < \infty$.

**Theorem 3.2.8.** *Let $1 \leq p < \infty$ and $T$ be the Taylor shift on $A^p$. Then the following are equivalent*

1. $1 \leq p < 2$,
2. *$T$ is strongly mixing in the Gaussian sense,*
3. *$T$ is weakly mixing in the Gaussian sense,*
4. *$T$ is frequently hypercyclic on $A^p$.*

*Proof.* Let $1 \leq p < 2$. By Remark 3.2.2 we can assume $1 < p < 2$ and we choose $D \subset \mathbb{T}$ to be an extended set of uniqueness. Since $D$ has arc length measure 0 (see Appendix A), it follows that $\mathbb{T} \setminus D$ is dense in $\mathbb{T}$. Theorem 3.2.7 yields that $\operatorname{span}\{\gamma(\zeta) \colon \zeta \in \mathbb{T} \setminus D\}$ is dense in $A^p$. According to Theorem 1.2.12 the Taylor shift $T$ is strongly mixing in the Gaussian sense. This implies that $T$ is also weakly mixing in the Gaussian sense which itself yields that $T$ is frequently hypercyclic on $A^p$.
For $p = 2$, Remark 3.2.1 yields that $T$ is not frequently hypercyclic and the same follows for $p > 2$: if there would exit some $p > 2$ such that $T$ is frequently hypercyclic on $A^p$, then Remark 3.2.2 would yield that $T$ is frequently hypercyclic on $A^2$ which is not the case. □

We will now briefly consider the Taylor shift $T = T_{H^p}$ on the Hardy space $H^p$ for $1 \leq p < \infty$. Note that $(H^p,\ T)$ is a dynamical system.

**Theorem 3.2.9.** *Let $T$ be the Taylor shift operator on $H^p$, $1 < p < \infty$. Then the following is true:*

1. *$T^n f \to 0$ in $H^p$ as $n \to \infty$ for every $f \in H^p$ and*
2. *$T$ is power bounded.*

*Proof.* Let $f \in H^p$, then $T^n f \in H^p$ has a nontangential limit $(T^n f)^*$ almost everywhere on $\mathbb{T}$ and

$$(T^n f)^*(\zeta) = \frac{f^*(\zeta) - s_{n-1}f(\zeta)}{\zeta^n} \quad m\text{-almost everywhere}$$

where $f^*$ is the nontangential limit of $f$. By Corollary 3 in [42], the Taylor series of $f$ converges for $1 < p < \infty$, so it follows

$$\|T^n f\|_{H^p} = \|(T^n f)^*\|_{\mathbb{T},p} = \|f^* - s_{n-1}f\|_{\mathbb{T},p} = \|f - s_{n-1}f\|_{H^p} \to 0 \quad (n \to \infty).$$

Since $f \in H^p$ was arbitrary, the uniform boundedness principle yields

$$\sup_{n \geq 0} \|T^n\|_{\text{op}} < \infty.$$

Thus, the Taylor shift is power bounded on $H^p$ for $1 < p < \infty$. □

In general, an easy implication of (2.3) for functions holomorphic in $\mathbb{D}$ is the following:

**Remark 3.2.10.** If $f \in H(\mathbb{D})$ with $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ and $a_\nu \to 0$ as $\nu \to \infty$ then $T^n f \to 0$ $(n \to \infty)$ locally uniformly on $\mathbb{D}$.
For that, let $K \subset \mathbb{D}$ be compact and $\varepsilon > 0$. Then there exists some $0 < r < 1$ such that $K \subset D_r$. Since $a_n \to 0$ $(n \to \infty)$, for $\delta = \varepsilon(1-r) > 0$ there exists some $N \in \mathbb{N}$ such that for every $n \geq N$ we have $|a_n| < \delta$. For $n \geq N$ it follows

$$\|T^n f\|_{K,\infty} = \max_{z \in K} |\sum_{\nu=0}^{\infty} a_{\nu+n} z^\nu| \leq \delta \sum_{\nu=0}^{\infty} r^\nu = \frac{\delta}{1-r} = \varepsilon$$

which yields the conclusion.

**Remark 3.2.11.** For $1 < p < \infty$, Remark 3.1.6 and Theorem 3.2.9 also show that $\|T^n f\|_{K,\infty} \to 0$ $(n \to \infty)$ for every compact $K \subset \mathbb{D}$. For $p = 1$, we still have locally uniform convergence of $T^n f$ on $\mathbb{D}$ by Remark 3.2.10 since for every $f \in H^1$ its Taylor coefficients $(a_n)_{n \in \mathbb{N}}$ fulfil $a_n \to 0$ as $n \to \infty$. However, Theorem 3.2.9 does not hold for $H^1$. In general, the Taylor series of functions $f \in H^1$ do not converge in $H^1$ by [42, Corollary 3] and therefore $T^n f$ does not converge to 0 in $H^1$ since otherwise this would contradict

$$\|T^n f\|_{H^1} = \|f - s_{n-1} f\|_{H^1}.$$

## 3.3 The Space of Cauchy Transforms

If not stated otherwise, the quoted results of this section can be found in [14]. In the previous section, we have already seen that there is a connection between the eigenfunctions of the Taylor shift and the Cauchy transforms of the Dirac measure with respect to the corresponding eigenvalue of the eigenfunction. Since eigenfunctions of unimodular eigenvalues play an important role for the dynamical behaviour of an operator, it suggests itself to consider the Taylor shift on the space of Cauchy transforms on the unit disc, that is

$$\mathcal{C} = \{C\mu \in H(\mathbb{D}) \colon \mu \in M(\mathbb{T})\}.$$

The space of complex measures $M := M(\mathbb{T})$ will be equipped with the total variation norm, i.e. $\|\mu\|_{TV} = |\mu|(\mathbb{T})$ for $\mu \in M$ (see Appendix A).

For a function $f \in \mathcal{C}$, there are a variety of measures $\mu \in M$ such that $f = C\mu$.

Therefore, we define

$$M(f) \coloneqq \{\mu \in M \colon f = C\mu\}$$

as the set of representing measures of $f$. Along the lines of [14], we want to equip $\mathcal{C}$ with a norm $\|\cdot\|$ such that $(\mathcal{C}, \|\cdot\|)$ becomes a Banach space.

**Remark 3.3.1.** Let $C(\mathbb{T})$ denote the Banach space of complex-valued continuous functions on $\mathbb{T}$ endowed with the uniform norm $\|\cdot\|_{\mathbb{T}}$. Using the Riesz representation theorem, one can show that the map $M \to C(\mathbb{T})',\ \mu \mapsto \varphi_\mu$ where

$$\varphi_\mu(f) = \int_{\mathbb{T}} f d\overline{\mu}$$

is an isometric isomorphism, for short we write $M \simeq C(\mathbb{T})'$. Furthermore, for the disc algebra $A \coloneqq A(\Delta) \coloneqq \{f \in C(\Delta) \colon f \text{ holomorphic in } \mathbb{D}\}$, its annihilator $A^\perp$ is a closed subspace of $M$ and given by

$$A^\perp = \{\mu \in M \colon \int_{\mathbb{T}} f d\overline{\mu} = 0 \text{ for all } f \in A\}.$$

By the theorem of F. and M. Riesz such annihilating measures $\mu$ take the form $d\mu = hdm$ where $h \in \overline{H_0^1} \coloneqq \{\overline{g} \colon g \in H^1,\ g(0) = 0\}$ so we can identify $A^\perp$ with $\overline{H_0^1}$. Since $A$ is a closed subspace of $C(\mathbb{T})$, Theorem 1.4.6 in [14] yields

$$A' \simeq M/\overline{H_0^1}.$$

One can show that $C\mu = 0$ if and only if $\mu \in A^\perp \simeq \overline{H_0^1}$ so the map $\mu + \overline{H_0^1} \mapsto C\mu$ from $M/\overline{H_0^1}$ to $\mathcal{C}$ is bijective. Therefore, it makes sense to endow $\mathcal{C}$ with the norm of $M/\overline{H_0^1}$ that is

$$\|C\mu\| = \inf\{\|d\mu + hdm\|_{TV} \colon h \in \overline{H_0^1}\}. \tag{3.2}$$

Hence $\mathcal{C} \simeq M/\overline{H_0^1} \simeq A'$ and $(\mathcal{C}, \|\cdot\|)$ is a Banach space. Furthermore, one can show that there exists a unique measure $\mu_f \in M(f)$ such that $\|f\| = \|\mu_f\|_{TV}$.

The first section of this chapter dealt with Hardy spaces. One can say the following regarding the relation of the Hardy spaces with the space of Cauchy transforms:

$$\bigcup_{p\geq 1} H^p \subset \mathcal{C} \subset \bigcap_{0<p<1} H^p$$

and the inclusions are strict (see [14]).

**Proposition 3.3.2.** *The spaces $H^p$, $1 \leq p < \infty$, are continuously embedded in $\mathcal{C}$.*

*Proof.* Without loss of generality, we can assume that $p = 1$. Let $f_n, f \in H^1$, $n \in \mathbb{N}$, with $\|f_n - f\|_{H^1} \to 0$ as $n$ tends to $\infty$. Then there exist nontangential limits $f_n^*, f^* \in L^1(\mathbb{T}, m)$ with $\|f_n^* - f^*\|_{\mathbb{T},1} \to 0$ $(n \to \infty)$ and for $\mu_n = f_n^* dm$, $\mu = f^* dm$ we have $C\mu_n = f_n$ and $C\mu = f$ (see e.g. [38, Theorem 17.11]). Using Theorem A.1.3 in the last equality, we obtain

$$\|f_n - f\| = \inf\{\|d\mu_n - d\mu + hdm\|_{TV} \colon h \in \overline{H_0^1}\} \leq \|\mu_n - \mu\|_{TV} = \int_{\mathbb{T}} |f_n^* - f^*| dm \to 0$$

as $n$ tends to $\infty$, so $f_n \to f$ also in $\mathcal{C}$ which yields the conclusion. □

**Remark 3.3.3.** In contrast to the case of the Hardy spaces, one can show that

$$A^p \not\subset \mathcal{C} \quad \text{for all } p \geq 1.$$

For that, let $f \in \mathcal{C}$ and $\mu \in M(f)$. The Taylor coefficients of $f$ are then given by $a_\nu = \int_{\mathbb{T}} \zeta^\nu d\overline{\mu}(\zeta)$ (see Remark 3.2.4) and therefore we have

$$|a_\nu| \leq \int_{\mathbb{T}} 1 d|\mu|(\zeta) = \|\mu\|_{TV},$$

i.e. $(a_\nu)_{\nu\in\mathbb{N}}$ is bounded. However, if $f \in A^p$, then

$$a_\nu = o(\nu^{1/p}) \tag{3.3}$$

and the exponent $1/p$ is best possible (see [19, Theorem 4, p. 85]). Thus, there exist $f \in A^p$ such that $f \notin \mathcal{C}$.

We recall the following definition (see e.g. [39]).

**Definition 3.3.4.** Suppose $M$ is a closed subspace of a topological vector space $X$. If there exists a closed subspace $N$ of $X$ such that

$$X = M + N \quad \text{and} \quad M \cap N = \{0\}$$

then $M$ is said to be *complemented in* $X$. In this case, $X$ is said to be the *direct sum* of $M$ and $N$ and we write

$$X = M \oplus N.$$

Using this, the Lebesgue decomposition theorem says that $M = M_a \oplus M_s$ where $M_a = \{\mu \in M \colon \mu \ll m\}$ is the set of absolutely continuous measures $\mu$ with respect to the normalized arc length measure $m$ and $M_s = \{\mu \in M \colon \mu \perp m\}$ is the set of singular measures with respect to $m$ (see Appendix A for definitions). Then we can also write the space of Cauchy transforms as the direct sum

$$\mathcal{C} = \mathcal{C}_a \oplus \mathcal{C}_s$$

where $\mathcal{C}_a = \{C\mu \colon \mu \in M_a\}$ and $\mathcal{C}_s = \{C\mu \colon \mu \in M_s\}$.

Unfortunately, $\mathcal{C}$ endowed with the norm given in (3.2) is not separable. To be more precise, the following holds

1. The polynomials are dense in $(\mathcal{C}_a, \|\cdot\|)$. Therefore, $\mathcal{C}_a$ equipped with the norm given in (3.2) is separable.

2. $\mathcal{C}_s$ equipped with the norm given in (3.2) is not separable.

Because of 1., it would make sense to ask for the dynamical properties of the Taylor shift on $\mathcal{C}_a$. We first want to consider the Taylor shift $T = T_{\mathcal{C}} \colon \mathcal{C} \to \mathcal{C}$. Then the operator is a self map (see e.g. [14]). Because of the connection of the eigenfunctions of the Taylor shift and the Cauchy transforms of the Dirac measure shown in (3.1) and the fact that the Dirac measure with respect to an isolated point is singular to the arc length measure $m$, we have $\gamma(\alpha) = C\delta_\alpha \in \mathcal{C}_s \subset \mathcal{C}$ for all $\alpha \in \mathbb{T}$.

**Theorem 3.3.5.** *For* $R : M \to M$, $R\mu = \overline{\mathrm{id}} d\mu$ *where* $\mathrm{id} \colon \mathbb{T} \to \mathbb{T}$, $\mathrm{id}(\zeta) = \zeta$ *is the* *identity mapping on the unit circle, the diagram*

$$\begin{array}{ccc} M & \xrightarrow{R} & M \\ {\scriptstyle C}\downarrow & & \downarrow{\scriptstyle C} \\ \mathcal{C} & \xrightarrow[T]{} & \mathcal{C} \end{array}$$

*commutes, i.e.* $C \circ R = T \circ C$.

*Proof.* Let $f \in \mathcal{C}$ and $\mu \in M(f)$. Since by [39, Exercise 3.24] we can change the order of integration and the Taylor shift, we obtain

$$Tf = \int_{\mathbb{T}} T\gamma(\zeta) d\overline{\mu}(\zeta) = \int_{\mathbb{T}} \zeta\gamma(\zeta) d\overline{\mu}(\zeta) = \int_{\mathbb{T}} \frac{\zeta}{1-\zeta\cdot} d\overline{\mu}(\zeta)$$

with the fact that $\gamma(\zeta) \in \mathcal{C}$ for $\zeta \in \mathbb{T}$. □

**Remark 3.3.6.** By induction, it follows for all $f \in \mathcal{C}$ and $n \in \mathbb{N}$ that

$$T^n f = T^n C\mu = \int_{\mathbb{T}} \zeta^n \gamma(\zeta) d\overline{\mu} = \int_{\mathbb{T}} \frac{\zeta^n}{1-\zeta\cdot} d\overline{\mu}(\zeta) \tag{3.4}$$

if $f = C\mu$.

We now want to restrict the Taylor shift to $\mathcal{C}_a$. First, note that it is still a self map because for $f \in \mathcal{C}_a$ there exists a measure $\mu$ with $f = C\mu$ and $\mu \ll m$ i.e. there is a function $g \in L^1(\mathbb{T}, m)$ with $d\mu = gdm$. Using Theorem 3.3.5, we get

$$Tf = \int_{\mathbb{T}} \frac{\zeta}{1-\zeta\cdot} d\overline{\mu}(\zeta) = \int_{\mathbb{T}} \frac{\zeta\overline{g}(\zeta)}{1-\zeta\cdot} dm(\zeta)$$

so $Tf = C\nu$ with $d\nu = \overline{\mathrm{id}} \cdot gdm$ and $\mathrm{id} \cdot \overline{g} \in L^1(\mathbb{T}, m)$. Thus, $\nu \in M_a$ and $Tf \in \mathcal{C}_a$.

Since $(\mathcal{C}_a, \|\cdot\|)$ is separable, one could suspect that the restriction of the Taylor shift to $\mathcal{C}_a$ is hypercyclic. The following theorem shows that this is not the case. An alternative proof for this statement can be found in [14].

**Theorem 3.3.7.** *The Taylor shift operator on* $(\mathcal{C}, \|\cdot\|)$ *has operator norm 1.*

*Proof.* Let $f \in \mathcal{C}$ and $\mu \in M(f)$. By Theorem 3.3.5, we have

$$\begin{aligned}\|Tf\| = \inf\{\|\overline{\mathrm{id}}d\mu + hdm\|_{TV} \colon h \in \overline{H_0^1}\} &= \inf\{\|\overline{\mathrm{id}}(d\mu + h \cdot \mathrm{id}dm)\|_{TV} \colon h \in \overline{H_0^1}\} \\ &= \inf\{\|d\mu + h \cdot \mathrm{id}dm\|_{TV} \colon h \in \overline{H_0^1}\}.\end{aligned}$$

The last equation is true because of Theorem A.1.3 in the Appendix. We now show that $\mathrm{id}\overline{H_0^1} = \overline{H^1}$. First, let $g \in \mathrm{id}\overline{H_0^1}$. Then there exists a function $h \in \overline{H_0^1}$ with $g = \mathrm{id} \cdot h$ and a corresponding $h^* \in L^1(\mathbb{T}, m)$ which has vanishing Fourier coefficients $a_n$ for $n \geq 0$. This yields $g^* = \mathrm{id} \cdot h^* \in L^1(\mathbb{T}, m)$ with Fourier coefficients $b_n = a_{n+1}$ which vanish for $n \geq -1$ and this implies $g \in \overline{H^1}$. For the other implication, let $g \in \overline{H^1}$. Then there exists an almost everywhere limit $g^* \in L^1(\mathbb{T}, m)$ with Fourier coefficients $a_n = 0$ for $n \geq 1$. If we consider $h^* \in L^1(\mathbb{T}, m)$ with Fourier coefficients $b_n = a_{n-1}$ vanishing for $n \geq 0$, then $g^* = \mathrm{id} \cdot h^*$ and $h^*$ corresponds to a function $h \in \overline{H_0^1}$ with $g = \mathrm{id} \cdot h$ so $g \in \mathrm{id}\overline{H_0^1}$. In particular, we have $\overline{H_0^1} \subset \mathrm{id}\overline{H_0^1}$ and therefore

$$\|Tf\| = \inf\{\|d\mu + gdm\|_{TV} \colon g \in \mathrm{id}\overline{H_0^1}\} \leq \inf\{\|d\mu + hdm\|_{TV} \colon h \in \overline{H_0^1}\} = \|f\|.$$

By definition of the operator norm this yields $\|T\|_{\mathrm{op}} \leq 1$. Finally, because for $\alpha \in \mathbb{T}$ we have that $\gamma(\alpha) \in \mathcal{C}$ and $\|T\gamma(\alpha)\| = \|\gamma(\alpha)\|$, it follows $\|T\|_{\mathrm{op}} = 1$. □

With this theorem and Proposition 1.2.2, we obtain that $T \colon \mathcal{C}_a \to \mathcal{C}_a$ equipped with the norm in (3.2) cannot be hypercyclic.

Since $\mathcal{C}$ is the norm dual of the disk algebra $A$ equipped with the uniform norm on $\Delta$, we can also consider the weak-$*$ topology on $\mathcal{C}$. Using this topology, we obtain a locally convex space. From the pointwise estimate

$$|f(z)| \leq \frac{1}{1 - |z|}\|f\| \quad \text{for all } z \in \mathbb{D},\ f \in \mathcal{C}$$

where $\|\cdot\|$ is the norm defined in (3.2) it can be shown that a sequence $(f_n)_{n\in\mathbb{N}}$ converges weak-$*$ in $\mathcal{C}$ if and only if it converges pointwise on $\mathbb{D}$ and $(f_n)_{n\in\mathbb{N}}$ is norm bounded in $\mathcal{C}$. Proposition 4.2.8 of [14] states that $\mathcal{C}$ is separable in the weak-$*$ topology. Furthermore, both $\mathcal{C}_a$ and $\mathcal{C}_s$ are weak-$*$ dense in $\mathcal{C}$.

**Remark 3.3.8.** We have already noted in Remark 3.3.1 that for every $f \in \mathcal{C}$ there exists a unique measure $\mu_f \in M(f)$ such that $\|f\| = \|\mu_f\|_{TV}$. We define

$$S_n f := \int_{\mathbb{T}} \frac{1}{\zeta^n} \gamma(\zeta) d\overline{\mu_f}(\zeta) = \int_{\mathbb{T}} \frac{1}{\zeta^n(1-\zeta\cdot)} d\overline{\mu_f}(\zeta) \tag{3.5}$$

for $n \in \mathbb{N}$ and $f \in \mathcal{C}$. We now want to show that for each $f \in \mathcal{C}$ the sequence $(S_n f)_{n\in\mathbb{N}}$ is bounded in norm: If we consider $g_n : \mathbb{C} \to \mathbb{C}$, $g_n(z) = z^n$, then $\overline{g_n} \in \overline{H_0^1}$ for all $n \in \mathbb{N}$ with $|g_n(z)| = 1$ for $z \in \mathbb{T}$. Since $0 \in \overline{H_0^1}$,

$$\begin{aligned} \|S_n f\| = \|C(g_n d\mu_f)\| &= \inf\{\|g_n d\mu_f + h dm\|_{TV} : h \in \overline{H_0^1}\} \\ &= \inf\{\|d\mu_f + h\overline{g_n} dm\|_{TV} : h \in \overline{H_0^1}\} \le \|\mu_f\|_{TV} = \|f\| \end{aligned}$$

for all $n \in \mathbb{N}$.

**Theorem 3.3.9.** *Let $T$ be the Taylor shift on $\mathcal{C}$ endowed with the weak-$*$ topology. Then $T$ is mixing.*

*Proof.* We consider $T$ on the weak-$*$ dense subset $\mathcal{C}_a$ of $\mathcal{C}$. As seen before, $T$ is then a self map. Along the same lines, we get that the restriction of $S_n$ to $\mathcal{C}_a$ is a self map for all $n \in \mathbb{N}$.
Now, let $f \in \mathcal{C}_a$ and $\mu \in M(f)$ such that $\mu \ll m$. Since $\gamma(z)$ is a continuous function on $\mathbb{T}$ for $z \in \mathbb{D}$, we have for $g \in L^1(\mathbb{T}, m)$ with $d\mu = g dm$ and a fixed $z \in \mathbb{D}$

$$\int_{\mathbb{T}} \zeta^n \frac{1}{1-\zeta z} d\mu(\zeta) = \int_{\mathbb{T}} \zeta^n \gamma(z)(\zeta) g(\zeta) dm(\zeta) \to 0 \quad (n \to \pm\infty)$$

by the Riemann-Lebesgue theorem. Therefore

$$T^n f(z) \to 0 \quad \text{and} \quad S_n f(z) \to 0$$

pointwise for $z \in \mathbb{D}$ as $n$ tends to $\infty$. To show the weak-$*$ convergence, we still need to prove that both $(T^n f)_{n\in\mathbb{N}}$ and $(S_n f)_{n\in\mathbb{N}}$ are norm bounded. For $(T^n f)_{n\in\mathbb{N}}$ this follows from Theorem 3.3.7 and for $(S_n f)_{n\in\mathbb{N}}$ from Remark 3.3.8. Since pointwise convergence on $\mathbb{D}$ and norm boundedness are equivalent to weak-$*$ convergence, we get that $T^n f \to 0$ and $S_n f \to 0$ in weak-$*$ topology as $n$ tends to $\infty$ for all $f \in \mathcal{C}_a$.
Finally, for $n \in \mathbb{N}$ and $f \in \mathcal{C}_a$ with $\mu_f \in M(f)$ such that $\|f\| = \|\mu_f\|_{TV}$ we have

$S_n f = \int_{\mathbb{T}} \overline{\zeta^n} \gamma(\zeta) d\overline{\mu_f}(\zeta) = C\nu$ where $d\nu = g_n d\mu_f$ and $g_n(\zeta) = \zeta^n$. Therefore

$$TS_n f = TC\nu = \int_{\mathbb{T}} \frac{\zeta}{1-\zeta\cdot} d\overline{\nu}(\zeta) = \int_{\mathbb{T}} \frac{1}{\zeta^{n-1}(1-\zeta\cdot)} d\overline{\mu_f}(\zeta) = S_{n-1} f.$$

Applying $T$ inductively $n$ times, we obtain $T^n S_n f = f$ for arbitrary $f \in \mathcal{C}_a$ and $n \in \mathbb{N}$. Thus, the Kitai Criterion yields that the Taylor shift is mixing on $\mathcal{C}$ equipped with the weak-$*$ topology. $\square$

# Chapter 4

# Bergman Spaces on General Open Sets

## 4.1 Properties of Bergman spaces on General Open Sets

At the beginning of the third chapter, we already considered Bergman spaces on the unit disc. In the following, Bergman spaces of functions defined on arbitrary open sets will play an important role. We therefore want to give a short overview of basic definitions and results.

Let $\Omega \subset \mathbb{C}$ be an open set in the complex plane. For $1 \leq p < \infty$ the Bergman space $A^p(\Omega)$ consists of all functions analytic in $\Omega$ that fulfil

$$\|f\|_{\Omega,p} := \|f\|_p := \left( \int\limits_{\Omega} |f|^p d\lambda_2 \right)^{1/p} < \infty.$$

Then $\|\cdot\|_p$ is a norm on $A^p(\Omega)$.

**Remark 4.1.1.** As in Proposition 3.1.1 it follows for all open sets $\Omega \subset \mathbb{C}$ that point-evaluation is a linear functional on $A^p(\Omega)$.

1. For $f_n,\ f \in A^p(\Omega)$, $n \in \mathbb{N}$, with $\|f_n - f\|_p \to 0$ $(n \to \infty)$ the sequence $(f_n)_{n\in\mathbb{N}}$ converges to $f$ locally uniformly on $\Omega$.

2. $A^p(\Omega)$ is a closed subspace of $L^p(\Omega)$, so $A^p(\Omega)$ is also complete. Therefore, the Bergman spaces are Banach spaces for $p \geq 1$ and $A^2(\Omega)$ is a Hilbert space.

In the following, we want to characterize situations in which the Bergman spaces are trivial, i.e. only contain the zero function. For the case $1 \leq p < 2$ a simple characterization can be found in [1, Proposition 11.1.1]:

**Remark 4.1.2.** Let $\Omega := \mathbb{C} \setminus K$ where $K$ is compact.

1. The Bergman space $A^1(\Omega)$ is trivial if and only if $K$ consists of at most two points.
2. For $1 < p < 2$ the Bergman space $A^p(\Omega)$ is trivial if and only if $K$ consists of at most one point.

For $p \geq 2$ and $K \subset \mathbb{C}$ compact, we can give a sufficient condition for the Bergman space $A^p(\mathbb{C} \setminus K)$ to be nontrivial:

**Remark 4.1.3.** One can show that if $K \subset \mathbb{C}$ is a compact set which is not totally disconnected and $\Omega = \mathbb{C} \setminus K$, then $A^p(\Omega)$ is not trivial for all $2 \leq p < \infty$. This is true since for such compact sets, we have that the analytic capacity $\gamma(K) > 0$ (for a definition and some basic results of analytic capacity see Chapter 8 of [20]). If the analytic capacity of $K$ is positive, then there exists a bounded function $f$ holomorphic in $\Omega$ vanishing at $\infty$ which is not equal to 0. Then $f^2 \not\equiv 0$ is bounded in $\mathbb{C} \setminus K$ as well and vanishes at least of order $O(1/z^2)$ as $z \to \infty$. This yields $f \in A^p(\Omega)$.

We now want to study the case $p = 2$: In [15, Theorem 9.5, p. 347] we can find a characterization of the space $A^2(\mathbb{C} \setminus K)$ where $K$ is a compact subset of $\mathbb{C}$. For that, we need the following definition in line with [36].

**Definition 4.1.4.** For a finite Borel measure $\mu$ on $\mathbb{C}$ concentrated on a compact set, its energy is given by

$$I(\mu) = \iint \log|z - w| d\mu(z) d\mu(w).$$

Then the *logarithmic capacity* of a subset $E$ of $\mathbb{C}$ is defined as

$$c(E) = \sup_{\mu} e^{I(\mu)}$$

where the supremum is taken over all Borel probability measures $\mu$ on $\mathbb{C}$ which are concentrated on a compact subset of $E$. It is understood that $e^{-\infty} = 0$ and $E$ is called a *polar set* if $c(E) = 0$.

**Theorem 4.1.5** (see [15])**.** *If $K$ is a compact subset of $\mathbb{C}$, then $A^2(\mathbb{C} \setminus K) = \{0\}$ if and only if $K$ is a polar set.*

We now want to know which sets are polar sets in order to see in which cases it makes sense to examine the Taylor shift operator for its dynamical behaviour. The results in the following remark can be found in [15].

**Remark 4.1.6.**

1. Countable sets are polar.
2. Polar sets are Lebesgue measurable zero sets.
3. If $K$ is compact and not totally disconnected, then $K$ is not polar.

From now on, for $1 < p < \infty$ we will denote by $q$ the conjugated exponent of $p$ if not stated otherwise. Similarly to the case $p = 2$, we obtain a characterization for the Bergman spaces $p > 2$ in terms of $q$-capacity. For a definition see [1]. Note that the authors define a more general $(\alpha, q)$-capacity $C_{\alpha,q}$ but since we only need the case $\alpha = 1$ we will denote by $C_q \coloneqq C_{1,q}$ the $(1, q)$-capacity and just refer to it as the $q$-capacity. Then [1, Proposition 11.1.1] yields the following:

**Theorem 4.1.7** (see [1])**.** *Let $K$ be a compact subset of $\mathbb{C}$ and $2 < p < \infty$. Then $A^p(\mathbb{C} \setminus K) = \{0\}$ if and only if $C_q(K) = 0$.*

It can be useful to consider Hausdorff measures in order to get information about the $q$-capacity of a set.

**Remark and Definition 4.1.8.** Let $h_\alpha \colon [0, \infty) \to [0, \infty)$, $h_\alpha(r) = r^\alpha$ for some $\alpha > 0$. For a set $E \subset \mathbb{C}$ there exists a countable number of (open or closed) balls $x_i + D_{r_i}$ with centre $x_i$ and radius $r_i \geq 0$, $i \in \mathbb{N}$ such that the union of these balls covers the set $E$. Then for any $\rho \in (0, \infty]$ we can define the set function

$$\Lambda_\alpha^{(\rho)}(E) \coloneqq \inf \sum_{i=1}^{\infty} h_\alpha(r_i)$$

where the infimum is taken over all coverings of $E$ of the kind described above with $\sup_{i\in\mathbb{N}} r_i \leq \rho$. Then $\Lambda_\alpha^{(\rho)}(E)$ is a decreasing function of $\rho$, so $\lim_{\rho\to 0}\Lambda_\alpha^{(\rho)}(E)$ exists (but may be infinite) and the *$\alpha$-dimensional Hausdorff measure of $E$* is defined as

$$\Lambda_\alpha(E) = \lim_{\rho\to 0}\Lambda_\alpha^{(\rho)}(E).$$

Note that $\Lambda_\alpha$ is a measure on the Borel sets of $\mathbb{C}$.

**Remark 4.1.9.** In line with [35] we call the 1-dimensional Hausdorff measure of $E$ the *linear measure of $E$*. For sets in $\mathbb{R}$ the definitions of linear measure and the Lebesgue measure coincide, hence the linear measure on $\mathbb{R}$ is identical with the Lebesgue measure. The same is true for the arc length measure. In particular, the linear measure of $E$ is positive if and only if the arc length measure is positive.

With this definition we can compare the $q$-capacity with the $(2-q)$-dimensional Hausdorff measure (see [1, Theorem 5.1.9]).

**Theorem 4.1.10** (see [1])**.** *Let $1 < q < 2$ and $E \subset \mathbb{C}$. Then there exists a constant $A \geq 0$ independent of the set $E$ such that*

$$C_q(E) \leq A\Lambda_{2-q}^{(1)}(E),$$

*and moreover $\Lambda_{2-q}(E) < \infty$ implies $C_q(E) = 0$.*

In particular, for $K \subset \mathbb{C}$ compact and $2 < p < \infty$ the Bergman space $A^p(\Omega)$ is trivial if the $(2-q)$-dimensional Hausdorff measure of $K$ is finite.

Later on, it will be our aim to examine the Taylor shift on general Bergman spaces on its dynamical behaviour. If the underlying set is of the form $\mathbb{C}\setminus K$ where $K$ is a compact set, the Taylor shift is not necessarily a self map on $A^p(\mathbb{C}\setminus K)$: Let $1 < p \leq 2$ and suppose that $1 \in \operatorname{int}(K)$. Then the function $f\colon \mathbb{C}\setminus K \to \mathbb{C}$, $f(z) = 1/(1-z)^2$ belongs to $A^p(\mathbb{C}\setminus K)$ but $Tf(z) = (2-z)/(1-z)^2$ does not decay rapidly enough at $\infty$ to be $p$-integrable on $\mathbb{C}\setminus K$ so $Tf \notin A^p(\mathbb{C}\setminus K)$. Nevertheless, we want to consider open sets which have bounded complement. Therefore, we define a modified version of the Bergman spaces:

**Definition 4.1.11.** Let $\Omega \subset \mathbb{C}_\infty$ be an open set containing $\infty$, i.e. $\Omega$ is of the form $\Omega = \mathbb{C}_\infty \setminus K$ where $K$ is a compact set in $\mathbb{C}$. For $\rho := 1 + \max_{z\in K}|z|$ the complement of $\Omega$ is contained in $D_\rho$. We define $A^p(\Omega)$ as the space of all functions $f$ holomorphic in $\Omega$ such that $f|_{\Omega\cap D_\rho}$ lies in $A^p(\Omega \cap D_\rho)$.

**Remark 4.1.12.** For a compact set $K$ in $\mathbb{C}$, we have the following relationship between the usual definition of the Bergman spaces and the modified one:

1. $A^1(\mathbb{C} \setminus K) = \{f \in A^1(\mathbb{C}_\infty \setminus K) \colon f(z) = O(1/z^3)\ (z \to \infty)\}$,
2. $A^p(\mathbb{C} \setminus K) = \{f \in A^p(\mathbb{C}_\infty \setminus K) \colon f(z) = O(1/z^2)\ (z \to \infty)\}$ for $1 < p \le 2$,
3. $A^p(\mathbb{C} \setminus K) = A^p(\mathbb{C}_\infty \setminus K)$ for $p > 2$.

**Remark 4.1.13.** Compared to Remark 4.1.2, for $1 \le p < 2$ and $z_0 \in \mathbb{C}$ we have for the alternative definition of the Bergman spaces that $A^p(\mathbb{C}_\infty \setminus \{z_0\})$ is not trivial. This is true since for example $\gamma(1/z_0) \in A^p(\mathbb{C}_\infty \setminus \{z_0\})$. Actually, $A^p(\mathbb{C}_\infty \setminus \{z_0\})$ is 1-dimensional and equal to the linear span of $\gamma(1/z_0)$.

In the following, we want to show that $A^p(\Omega)$ is complemented in $A^p(\Omega \cap D_\rho)$ for $\Omega \subset \mathbb{C}_\infty$ open with $\infty \in \Omega$.

**Theorem 4.1.14.** *Let $\Omega \subset \mathbb{C}_\infty$ be an open set with $\infty \in \Omega$. Then*

$$A^p(\Omega \cap D_\rho) = A^p(\Omega) \oplus A^p(D_\rho). \tag{4.1}$$

*Proof.* We first want to show that $A^p(\Omega)$ is closed in $A^p(\Omega \cap D_\rho)$. For that, let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $A^p(\Omega)$ and $f \in A^p(\Omega \cap D_\rho)$ with

$$f_n \to f \quad (n \to \infty).$$

We then need to show that $f$ is holomorphic in $\Omega$. By the first statement of Remark 4.1.1, $H(\Omega) \ni f_n \to f$ uniformly on every compact subset of $\Omega \cap D_\rho$. In particular, let $0 < r < \rho$ be so that the complement of $\Omega$ still is contained in $D_r$. Then $f_n \to f$ uniformly on $\partial D_r$ as $n$ tends to $\infty$. According to the maximum principle we have that $\max_{|z|=r} |f_n(z)| = \max_{|z|\ge r} |f_n(z)|$ for all $n \in \mathbb{N}$ so $f$ is the locally uniform limit on $\mathbb{C}_\infty \setminus D_r$ of a sequence of holomorphic functions in $\Omega$ which means that it has a holomorphic continuation to $\Omega$. Along the same lines, we get that $A^p(D_\rho)$ is closed in $A^p(\Omega \cap D_\rho)$.

If $f \in A^p(\Omega \cap D_\rho)$, then one can find $0 < r_1 < r_2 < \rho$ such that the complement of $\Omega$ is still contained in $D_{r_1}$. Since $f$ then is holomorphic in $V_{r_1,r_2} := \{z \in \mathbb{C} : r_1 < |z| < r_2\}$, we can write $f$ as the Laurent series

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu + \sum_{\nu=1}^{\infty} a_{-\nu} z^{-\nu} \quad \text{for all } z \in V_{r_1,r_2}$$

and we get that $f_1(z) := \sum_{\nu=0}^{\infty} a_\nu z^\nu$ is holomorphic in $D_{r_2}$ and $f_2(z) := \sum_{\nu=1}^{\infty} a_{-\nu} z^{-\nu}$ is holomorphic in $\mathbb{C}_\infty \setminus \Delta_{r_1}$. Furthermore, as $f_1 = f - f_2$ with $f \in A^p(\Omega \cap D_\rho)$ and $f_2 \in H(\mathbb{C}_\infty \setminus \Delta_{r_1}) \subset H(\mathbb{C}_\infty \setminus \Delta_{r_2})$, we obtain that $f_1$ has a holomorphic extension to $D_\rho$ and

$$\|f_1\|_{D_\rho,p} = \left( \int\limits_{D_\rho} |f_1|^p d\lambda_2 \right)^{1/p} = \left( \int\limits_{D_{r_2}} |f_1|^p d\lambda_2 + \int\limits_{V_{r_2,\rho}} |f - f_2|^p d\lambda_2 \right)^{1/p} < \infty.$$

This yields $f_1 \in A^p(D_\rho)$. As above, because $f_2 = f - f_1$, we obtain that $f_2$ has a holomorphic extension to $\Omega$ and we get

$$\|f_2\|_{\Omega \cap D_\rho,p} \leq \|f_2 + f_1\|_{\Omega \cap D_\rho,p} + \|f_1\|_{\Omega \cap D_\rho,p} = \|f\|_{\Omega \cap D_\rho,p} + \|f_1\|_{\Omega \cap D_\rho,p} < \infty.$$

Finally, we have that $A^p(\Omega) \cap A^p(D_\rho) = \{0\}$ because for $f \in A^p(\Omega) \cap A^p(D_\rho)$ it follows that $f$ is holomorphic in $\Omega$ and in $D_\rho$. Since $\mathbb{C}_\infty \setminus \Omega \subset D_\rho$, we get that $f \in H(\mathbb{C}_\infty)$. Thus, $f \equiv 0$ which completes the proof. □

**Remark 4.1.15.** An implication of the previous theorem is that $(A^p(\Omega), \|\cdot\|_{\Omega \cap D_\rho})$ is a Banach space.

**Remark 4.1.16.** In Theorem 1.2.12, statement 1. and 2. are equivalences if the underlying Banach space $X$ has cotype 2. Since the $L^p$-spaces have cotype 2 for $1 \leq p \leq 2$ (see e.g. [2]) and the cotype of a Banach space is inherited by subspaces, it follows that $A^p(\Omega)$ has the desired property for $p \in [1, 2]$ and $\Omega \subset \mathbb{C}_\infty$ open.

**Proposition 4.1.17.** *Let $K \subset \mathbb{C}$ be compact. If $A^2(\mathbb{C} \setminus K) = \{0\}$, then $A^2(\mathbb{C}_\infty \setminus K)$ is trivial as well.*

*Proof.* From [15, Theorem 9.5, p. 347], we can deduce for $\Omega = \mathbb{C}_\infty \setminus K$

$$A^2(\Omega \cap D_\rho) = A^2(D_\rho \setminus K) = A^2(D_\rho).$$

Hence, (4.1) yields that $A^2(\Omega) = \{0\}$. □

**Proposition 4.1.18.** *Let $X$ be a Fréchet space with $X = A \oplus B$. If for $M \subset A$ and $L \subset B$ the linear span of $M + L$ is dense in $X$, then* span $M$ *is dense in $A$ and* span $L$ *is dense in $B$.*

*Proof.* Let $x \in A$. By definition we have $x \in X$, so there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in the linear span of $M + L$ such that $x_n \to x$ in $X$ as $n$ tends to $\infty$. Then one can find linear combinations $a_n$ of vectors belonging to $M$ and $b_n$ of vectors belonging to $L$ such that $x_n = a_n + b_n$ for all $n \in \mathbb{N}$. Respectively, one obtains $a \in A$ and $b \in B$ with $x = a + b$. Since $x$ was a vector belonging to $A$ by assumption, it follows that $a = x$ and $b = 0$. Because of [39, Theorem 5.16], the projection $k\colon X \to A$, $k(a + b) = a$ is continuous and it follows

$$a_n = k(x_n) \to k(x) = x \quad (n \to \infty) \tag{4.2}$$

in $A$. Hence, the linear span of $M$ is dense in $A$. The second statement follows analogously. □

Let $\Omega \subset \mathbb{C}_\infty$ be open with $\infty \in \Omega$. Because of (4.1), we can apply Proposition 4.1.18 to obtain results on approximation by rational function in the modified Bergman space $A^p(\Omega)$ as we will do in the next section.

## 4.2 Approximation by Rational Functions

In order to obtain situations in which the Taylor shift has certain dynamical properties on $A^p(\Omega)$ we need to have results on approximation by rational functions. In [23, Theorem 1] one can find the following result regarding approximation in the Bergman spaces for $1 \leq p < 2$. For a measurable bounded set $E \subset \mathbb{C}$ and a set $A \subset E^*$ we will denote by $R^p_A(E)$ the closure in $L^p(E)$ of rational functions with simple poles in $1/A$.

**Theorem 4.2.1** (Hedberg). *If $1 \leq p < 2$ and $\Omega \subset \mathbb{C}$ is a bounded open set, then $A^p(\Omega) = R^p_{\Omega^*}(\Omega)$.*

For a compact set $K \subset \mathbb{C}$, the inner boundary of $K$ means the set of points in $\partial K$ which are not in the boundary of any component of the complement of $K$. Note that if $\Omega$ is a bounded domain, then the inner boundary of $\mathrm{cl}(\Omega)$ is empty. For the case $p \geq 2$, [23, Theorems 4 and 5] yield the next result:

**Theorem 4.2.2** (Hedberg). *Let $2 \leq p < \infty$ and $\Omega \subset \mathbb{C}$ be a bounded open set. If the inner boundary of $\mathrm{cl}(\Omega)$ is empty, then $R^p_{\mathrm{int}(\Omega^*)}(\Omega) = A^p(\Omega)$.*

**Remark 4.2.3.** The previous two theorems are statements on approximation by rational functions with simple poles whereas the related theorems in [23] are stated for rational functions with arbitrary poles. The proofs, however, show that simple poles suffice. Theorem 4.2.2 is actually stated for compact sets $E \subset \mathbb{C}$, i.e. that functions in $L^p_a(E) = \{f \in L^p(E) \colon f|_{\mathrm{int}(E)} \in H(\mathrm{int}(E))\}$ can be approximated by rational functions with simple poles in the complement of $E$. However, for an open bounded set $\Omega$ and $f \in A^p(\Omega)$, we can understand $f$ as a function in $L^p_a(\mathrm{cl}(\Omega))$ by defining $f \equiv 0$ on $\partial\Omega$. Then there exists a sequence of rational functions $(r_j)_{j \in \mathbb{N}}$ such that $r_j \to f$ in $L^p_a(\mathrm{cl}(\Omega))$. It follows

$$\|r_j - f\|_{p,\Omega} \leq \|r_j - f\|_{p,\mathrm{cl}(\Omega)} \to 0$$

which yields Theorem 4.2.2.

In order to apply Theorem 1.2.12 to the Taylor shift, we need to approximate functions in the Bergman spaces by rational functions having poles in predetermined sets $\Lambda$. Therefore, we introduce the notion of uniqueness sets according to [12].

**Definition 4.2.4.** Let $K \subset \mathbb{C}$ be compact in $\mathbb{C}$. Then $\Lambda \subset K$ is called a *uniqueness set for $K$* if every continuous function on $K$ that is holomorphic on the interior of $K$ and vanishes on $\Lambda$ vanishes identically.

For the case $1 \leq p < 2$, we can now prove that it is sufficient to have simple poles in a uniqueness set to approximate functions in $A^p(\Omega)$. We recall that $\gamma(\alpha)$ are the functions defined in (2.5). Note that for $1 \leq p < 2$ these functions are in $A^p(\Omega)$ for all $\alpha \in \Omega^*$.

**Theorem 4.2.5.** *Let $1 \le p < 2$ and $\Omega \subset \mathbb{C}_\infty$ be an open set which is either bounded in $\mathbb{C}$ or with $\infty \in \Omega$. If $\Lambda$ is a uniqueness set for $\Omega^*$, then*

$$\operatorname{span}\{\gamma(\alpha) \colon \alpha \in \Lambda\} \text{ is dense in } A^p(\Omega).$$

*Proof.* We first assume that $\Omega$ is bounded in $\mathbb{C}$. Let $\varphi \in (A^p(\Omega))'$ with $\varphi(\gamma(\alpha)) = 0$ for all $\alpha \in \Lambda$. There exists a function $g \in A^q(\Omega)$ with $1/p + 1/q = 1$ such that

$$\varphi(f) = \int_\Omega f\overline{g} d\lambda_2 \quad \text{for all } f \in A^p(\Omega).$$

For $\mu \in M(\Omega)$ with $d\mu = 1_\Omega g d\lambda_2$ the Cauchy transform

$$C\mu(\zeta) = \int_\Omega \frac{\overline{g(z)}}{1 - \zeta z} d\lambda_2(z)$$

fulfils $C\mu(\alpha) = \varphi(\gamma(\alpha)) = 0$ for all $\alpha \in \Lambda$. Since $C\mu$ is holomorphic in the interior of $\Omega^*$ and continuous everywhere by Theorem 3.2.6 (note that we are considering the case that $\Omega$ is bounded) and $\Lambda$ is a uniqueness set for $\Omega^*$ by assumption we have that $C\mu|_{\Omega^*} \equiv 0$ and thus

$$\varphi(\gamma(\alpha)) = 0 \quad \text{for all } \alpha \in \Omega^*.$$

So $\varphi$ is equal to 0 on the set of rational functions with simple poles in $\mathbb{C} \setminus \Omega$. According to Theorem 4.2.1 this yields that $\varphi$ is identically 0.

Now, let $\Omega$ be open with $\infty \in \Omega$. By the previous considerations, we have that $R^p_\Lambda(\Omega \cap D_\rho) = A^p(\Omega \cap D_\rho)$. By Proposition 4.1.18 it follows that $\operatorname{span}\{\gamma(\alpha) \colon \alpha \in \Lambda\}$ is dense in $A^p(\Omega)$. □

For the case $p \ge 2$, we want to obtain an approximation result using specific Cauchy transforms. These type of functions were already considered in [8].

**Remark 4.2.6.** For measurable $B \subset \mathbb{T}$ we define

$$f_B(z) \coloneqq Cm_B(z) = \int_B \frac{dm(\zeta)}{1 - \zeta z} \quad (z \in B^*) \tag{4.3}$$

where $dm_B = 1_B dm$. By [24, Theorem 1.7], we obtain that

$$\int_{\mathbb{T}} \frac{dm(\zeta)}{|1-\zeta z|} = O\left(\log \frac{1}{1-|z|}\right) \quad (|z| \to 1^-).$$

With the following estimation

$$\int_{\mathbb{D}} \left|\log \frac{1}{1-|z|}\right|^p d\lambda_2(z) = \frac{1}{\pi} \int_0^{2\pi} \int_0^1 \left|\log \frac{1}{1-|re^{it}|}\right|^p r dr dt \leq \frac{1}{\pi} \int_0^1 \left|\log \frac{1}{r}\right|^p dr$$

and the fact that $\log(1/\cdot) \in L^p([0,1])$ for all $p \geq 1$, we obtain that for an open set $\Omega$ with $B = \partial\Omega^* \cap \mathbb{T} \neq \emptyset$

$$\int_B |\gamma(\zeta)| dm(\zeta) \in L^p(\Omega)$$

for all $1 \leq p < \infty$.

The following proposition gives a sufficient condition for a set to be a uniqueness set (see e.g. [12]).

**Proposition 4.2.7.** *Let $K \subset \mathbb{C}$ be a compact set such that* $\operatorname{int}(K)$ *is a domain. Then $\Lambda$ is a uniqueness set for $K$ if $\partial K$ is a rectifiable Jordan curve and $\Lambda$ has positive linear measure.*

Using this, we can show that functions in $A^p(\Omega)$ for specific open sets $\Omega$ can be approximated by functions of the form as in (4.3).

**Theorem 4.2.8.** *Let $2 \leq p < \infty$ and $\Omega \subset \mathbb{C}_\infty$ be a domain which is either bounded in $\mathbb{C}$ or with $\infty \in \Omega$. If for every component $C$ of $\Omega^*$ the boundary of $C$ is a rectifiable Jordan curve and the intersection of $C$ with the unit circle $\mathbb{T}$ has positive m-measure, then*

$$\operatorname{span}\{f_B : B \subset \Omega^* \cap \mathbb{T} \textit{ with positive m-measure}\} \textit{ is dense in } A^p(\Omega).$$

*Proof.* We first assume that $\Omega$ is bounded in $\mathbb{C}$ and set

$$L := \operatorname{span}\{f_B : B \subset \Omega^* \cap \mathbb{T} \text{ with positive m-measure}\}.$$

We choose $\varphi \in (A^p(\Omega))'$ with $\varphi(f_B) = 0$ for all $B \subset \Omega^* \cap \mathbb{T}$ with positive $m$-measure. Since $A^p(\Omega)$ is a subspace of $L^p(\Omega)$ and $\varphi$ is a linear continuous functional, the Hahn-Banach theorem yields that $\varphi$ can be extended to $L^p(\Omega)$. Thus, there exists a function $g \in L^q(\Omega)$ where $q$ is the conjugated exponent to $p$ such that

$$\varphi(f) = \int\limits_{\Omega} f\overline{g}d\lambda_2$$

for all $f \in A^p(\Omega) \subset L^p(\Omega)$. Since $\int_B |\gamma(\zeta)|dm(\zeta) \in L^p(\Omega)$ for all $B \subset \Omega^* \cap \mathbb{T}$ with positive $m$-measure, Hölder's inequality yields

$$\int\limits_{\Omega} |\int\limits_{B} |\frac{g(z)}{1-\zeta z}|dm(\zeta)|d\lambda_2(z)$$
$$\leq \left(\int\limits_{\Omega} |g(z)|^q d\lambda_2(z)\right)^{1/q} \left(\int\limits_{\Omega} |\int\limits_{B} |\frac{1}{1-\zeta z}|dm(\zeta)|^p d\lambda_2(z)\right)^{1/p} < \infty.$$

This allows us to apply Fubini's theorem and for all $B \subset \Omega^* \cap \mathbb{T}$ with positive $m$-measure we get

$$\varphi(f_B) = \int\limits_{\Omega}\int\limits_{B} \frac{dm(\zeta)}{1-\zeta z}\overline{g(z)}d\lambda_2(z) = \int\limits_{B}\int\limits_{\Omega} \frac{\overline{g(z)}}{1-\zeta z}d\lambda_2(z)dm(\zeta) = 0.$$

For $\mu \in M(\Omega)$ with $d\mu = 1_\Omega g d\lambda_2$ the Cauchy transform of $\mu$ is continuous everywhere since we are considering the case that $\Omega$ is bounded and

$$\varphi(f_B) = \int\limits_{B} C\mu(\zeta)dm(\zeta) = 0$$

for all $B \subset \Omega^* \cap \mathbb{T}$ with positive $m$-measure, so by [38, Theorem 1.39] it follows that $C\mu = 0$ $m$-almost everywhere on $\Omega^* \cap \mathbb{T}$. Since each component $C$ of $\Omega^*$ is enclosed by a rectifiable Jordan curve and $C \cap \mathbb{T}$ has positive $m$-measure and therefore also positive linear measure, Proposition 4.2.7 yields for every component that $C \cap \mathbb{T}$ is a uniqueness set for $C$. We obtain that $C\mu|_{\Omega^*} \equiv 0$ because $C\mu$ is holomorphic in the interior of $\Omega^*$. Since $\Omega$ is bounded, we get that $\varphi(\gamma(\alpha)) = 0$ for all $\alpha$ in the interior of $\Omega^*$ which is, by assumption, not empty for every component of $\Omega^*$. Since $\Omega$ is a domain, it follows that

the inner boundary of $\mathrm{cl}(\Omega)$ is empty so we can apply Theorem 4.2.2 and obtain that $L$ is dense in $A^p(\Omega)$ which yields the assertion.
Along the same lines, we get in the unbounded case that

$$L := \mathrm{span}(\{f_B : B \subset \Omega^* \cap \mathbb{T} \text{ with positive m-measure}\} \cup \{\gamma(\alpha) : \alpha \in \mathbb{C} \setminus \Delta_\rho\})$$

is dense in $A^p(\Omega \cap D_\rho)$ and applying Proposition 4.1.18 we obtain that $\mathrm{span}\{f_B : B \subset \Omega^* \cap \mathbb{T}$ with positive m-measure$\}$ is dense in $A^p(\Omega)$ which concludes the proof. □

In [8], the authors already proved an approximation result using functions of the form (4.3). For their result, we first need the notion of Carathéodory domains. Note that for a compact set $K \subset \mathbb{C}$ the open set $\mathbb{C} \setminus K$ can have at most countably many components, exactly one of which is unbounded. The boundary of this unique component of $\mathbb{C} \setminus K$ is called the outer boundary of $K$. Then the outer boundary of an open bounded set is defined as the outer boundary of its closure. According to [15], one can define the following.

**Definition 4.2.9.** A Carathéodory domain is a bounded open connected subset of $\mathbb{C}$ whose boundary equals its outer boundary.

**Remark 4.2.10.** In [8, Lemma 3.3] it was shown that in case that $\Omega$ is a Carathéodory domain such that $\mathrm{cl}(\Omega)$ does not separate the plane and $\mathbb{T} \setminus \Omega$ contains a non-trivial subarc, then we can approximate functions in $A^p(\Omega)$ by functions as in (4.3). This result is a special case of the previous theorem.

Along the same lines as in Theorem 4.2.5 and using the identity theorem, we obtain the following result.

**Theorem 4.2.11.** *Let $2 \le p < \infty$ and $\Omega \subset \mathbb{C}_\infty$ be a domain which is either bounded in $\mathbb{C}$ or $\infty \in \Omega$. If every component $C$ of $\Omega^*$ has interior points and $\Lambda \subset \mathrm{int}(\Omega^*)$ such that $\Lambda$ has an accumulation point in every component $C$ of $\Omega^*$, then*

$$\mathrm{span}\{\gamma(\alpha) \colon \alpha \in \Lambda\}$$

*is dense in $A^p(\Omega)$.*

# Chapter 5

# The Taylor Shift on General Bergman Spaces

## 5.1 The Spectrum of the Taylor Shift on Bergman Spaces

As in Chapters 2 and 3, we want to examine the Taylor shift operator $T \coloneqq T_{\Omega,p}$ on the Bergman spaces for its dynamical properties. If $\Omega$ is an open set with $0 \in \Omega$ which is either bounded in $\mathbb{C}$ or $\infty \in \Omega$, then $(A^p(\Omega),\ T)$ is a dynamical system: For $f \in A^p(\Omega)$ there exists some $r > 0$ such that $f$ is holomorphic in $D_r$ which means that $Tf$ is integrable in a neighbourhood of 0 and thus is integrable on $\Omega$ itself.
If $\Omega$ is as required above, the Cauchy kernel provides a family of eigenfunctions for the operator $T$. The following theorem sums up the relations between the spectrum, the point spectrum and $\Omega^*$ for different values of $p$.

**Theorem 5.1.1.** *Let $\Omega$ be an open set with $0 \in \Omega$ which is either bounded in $\mathbb{C}$ or $\infty \in \Omega$. For the Taylor shift $T$ on $A^p(\Omega)$, the following holds:*

1. $\sigma_0(T) = \sigma(T) = \Omega^*$ *for* $1 \leq p < 2$,

2. $\operatorname{int}(\Omega^*) \subset \sigma_0(T) \subset \sigma(T) \subset \Omega^*$ *for* $p \geq 2$.

*Proof.* An elementary calculation shows that $S_\alpha$ on $A^p(\Omega)$ from (2.6) is the continuous inverse operator to $T - \alpha I$ for $\alpha \in 1/\Omega$. This yields that $\sigma(T) \subset \Omega^*$. For $\alpha \in \operatorname{int}(\Omega^*)$ the functions $\gamma(\alpha)$ belong to $A^p(\Omega)$ for all $p$ and it is clear that $\gamma(\alpha)$ is an eigenfunction to the eigenvalue $\alpha$. As stated above, for $p < 2$ the functions $\gamma(\alpha)$ belong to $A^p(\Omega)$ also for $\alpha \in \partial\Omega^*$ which yields $\sigma_0(T) = \Omega^*$. Since $\sigma_0(T) \subset \sigma(T)$, we obtain both statements. □

**Remark 5.1.2.** For $p \geq 2$ we have that $\mathrm{cl}(\mathrm{int}(\Omega^*)) \subset \mathrm{cl}(\sigma_0(T)) \subset \sigma(T) \subset \Omega^*$ with the fact that the spectrum of a continuous operator is always a closed subset of the complex plane. In particular, this means that if the closure of the interior of $\Omega^*$ equals $\Omega^*$, then the spectrum of $T$ also coincides with $\Omega^*$ for $p \geq 2$. For example, this is the case if $\Omega$ is a Carathéodory domain.

**Example 5.1.3.**

1. Let $\Omega := D_R \setminus \{1\}$ for some $R > 1$. For $f \in A^2(\Omega)$ the point 1 must be a removable singularity by [15, Proposition 9.3, page 345]. Therefore, $A^2(D_R \setminus \{1\}) = A^2(D_R)$ and it follows $1 \in \Omega^* \setminus \sigma(T)$.
2. Let $\Omega = \mathbb{D} \setminus (3/4 + D_{1/4})$. Since the area of $\Omega$ around 1 is sufficiently small, it follows $\gamma(1) \in A^2(\Omega)$ and therefore $1 \in \sigma_0(T)$ but $1 \notin \mathrm{int}(\Omega^*)$.

These examples show that, in general, the two inclusions in the second statement of Theorem 5.1.1 are not equalities.

**Remark 5.1.4.** We know that the spectrum of $T$ on $A^p(\Omega)$ equals $\Omega^*$ for $p < 2$ by Theorem 5.1.1 which yields that $T$, by Theorem 1.2.5, can only be frequently hypercyclic (or chaotic) if $\Omega^*$ is a perfect set. The same holds for the Taylor shift on $A^p(\Omega)$ for $p \geq 2$ if the closure of the interior of $\Omega^*$ equals $\Omega^*$.

The authors of [8] already considered the Taylor shift operator on general Bergman spaces and succeeded to prove the following result.

**Theorem 5.1.5** (Beise, Müller)**.** *Let $\Omega$ be a Carathéodory domain such that $\mathrm{cl}(\Omega)$ does not separate the plane. If $\mathbb{T} \setminus \Omega$ contains some arc, then $T$ is mixing on $A^p(\Omega)$ for all $1 \leq p < \infty$.*

## 5.2 Dynamics of the Taylor Shift for $1 \leq p < 2$

In this section, we want to study the dynamics of the Taylor shift operator on general Bergman spaces for $1 \leq p < 2$. In the following, for open $\Omega \subset \mathbb{C}_\infty$, we set $\Lambda_D := (\Omega^* \cap \mathbb{T}) \setminus D$ for subsets $D$ of the unit circle $\mathbb{T}$.

**Theorem 5.2.1.** *Let $1 \leq p < 2$ and $\Omega \subset \mathbb{C}_\infty$ be an open set with $0 \in \Omega$ which is either bounded in $\mathbb{C}$ or contains $\infty$. Furthermore, let $T$ be the Taylor shift on $A^p(\Omega)$.*

1. *If $\Lambda_D$ is a uniqueness set for $\Omega^*$ for all countable $D \subset \mathbb{T}$, then $T$ is weakly mixing in the Gaussian sense.*
2. *If $\Lambda_D$ is a uniqueness set for $\Omega^*$ for all sets $D \subset \mathbb{T}$ of extended uniqueness, then $T$ is strongly mixing in the Gaussian sense.*

*Proof.* Fix an arbitrary countable set $D \subset \mathbb{T}$ and $f \in A^p(\Omega)$. Since $\Lambda_D$ is a uniqueness set for $\Omega^*$ by assumption, Theorem 4.2.5 yields that

$$\bigcup_{\alpha \in \Lambda_D} \ker(T - \alpha I) = \operatorname{span}\{\gamma(\alpha)\colon \alpha \in \Lambda_D\} \text{ is dense in } A^p(\Omega)$$

which is sufficient for the Taylor shift to be weakly mixing in the Gaussian sense by Theorem 1.2.12. The second statement follows analogously for sets $D \subset \mathbb{T}$ of extended uniqueness. □

We now want to obtain examples of open sets $\Omega$ such that the Taylor shift on $A^p(\Omega)$ is weakly or strongly mixing in the Gaussian sense. For that, it is useful to have a verifiable condition like Proposition 4.2.7 to show whether a set is a uniqueness set. The following result gives more insight to that.

**Proposition 5.2.2** (see e.g. [12])**.** *Let $K \subset \mathbb{C}$ be a compact set. Then $\Lambda \subset K$ is a uniqueness set for $K$ if and only if $K \setminus \operatorname{cl}(\operatorname{int}(K))$ is a subset of the closure of $\Lambda$ and for every component $C$ of $\operatorname{int}(K)$ the set $\Lambda \cap \operatorname{cl}(C)$ is a uniqueness set for $\operatorname{cl}(C)$.*

**Example 5.2.3.** Let $B \subset \mathbb{T}$ be a perfect compact subset of the unit circle and $\Omega := \mathbb{C}_\infty \setminus B$. Then for all countable $D \subset \mathbb{T}$ we have that $\Lambda_D := B \setminus D$ is a uniqueness set for $B$. To verify that we apply Proposition 5.2.2: the first condition holds since $B$ is a perfect set by assumption and the interior of $B$ is empty since then $\Lambda_D$ is dense in $B$ for all countable $D \subset \mathbb{T}$. For the second condition there is nothing to show because $\operatorname{int}(B) = \emptyset$. Hence, if $1 \leq p < 2$, it follows from Theorem 5.2.1 that the Taylor shift operator on $A^p(\Omega)$ is weakly mixing in the Gaussian sense.

In the setting of this example, it follows that it is not only sufficient for $B \subset \mathbb{T}$ to be a perfect set but also necessary as the next theorem will show.

**Theorem 5.2.4.** *Let $1 \leq p < 2$ and $\Omega := \mathbb{C}_\infty \setminus B$ be open where $B \subset \mathbb{T}$. Furthermore, let $T$ be the Taylor shift on $A^p(\Omega)$. Then the following are equivalent:*

1. *$T$ is weakly mixing in the Gaussian sense,*

2. *$T$ is frequently hypercyclic,*

3. *$B$ is a perfect set.*

*Proof.* If $T$ is weakly mixing in the Gaussian sense, it is already frequently hypercyclic by Remark 1.2.9. Since the spectrum of $T$ equals $1/B$, it follows from the frequent hypercyclicity and Theorem 1.2.5 that $1/B$ is perfect and therefore also $B$ has to be a perfect set. Finally, with Example 5.2.3 we obtain that $T$ is weakly mixing in the Gaussian sense if $B$ is perfect. □

Compared to the prior theorem, we get the following result regarding the strongly mixing property. This follows from the fact that $A^p(\Omega)$ are Banach spaces with cotype 2 for $1 \leq p < 2$ and Theorem 1.2.12.

**Theorem 5.2.5.** *Let $1 \leq p < 2$ and $\Omega := \mathbb{C}_\infty \setminus B$ be open where $B \subset \mathbb{T}$. Furthermore, let $T$ be the Taylor shift on $A^p(\Omega)$. Then the following are equivalent:*

1. *$T$ is strongly mixing in the Gaussian sense,*

2. *$B$ is a $\mathcal{U}_0$-perfect set.*

These two theorems lead us to an example of an open set $\Omega$ such that the Taylor shift on $A^p(\Omega)$ is weakly but not strongly mixing in the Gaussian sense.

**Example 5.2.6.** Define $\Omega := \mathbb{C}_\infty \setminus e^{i\pi(2C-1)}$ where $C$ is the classical Cantor 1/3-set. Then $\Omega$ fulfils the conditions of Theorem 5.2.4, so the Taylor shift on $A^p(\Omega)$ is frequently hypercyclic and weakly mixing in the Gaussian sense for $1 \leq p < 2$. Because $e^{i\pi(2C-1)}$ is a set of extended uniqueness (see Appendix A), it follows that $T$ cannot be strongly mixing in the Gaussian sense.

Compared to that, we can construct a Cantor set $C$ which has locally positive arc length measure.

**Example 5.2.7.** Recursively, we want to define a descending sequence $(C_n)_{n\in\mathbb{N}}$ of sets with $C_{n+1} \subset C_n$ for all $n \in \mathbb{N}$. We start with $C_0 = [0,1]$ and obtain $C_{n+1}$ by taking out the open interval $((a+b)/2 - 1/(4\cdot 3^{n+1}), (a+b)/2 + 1/(4\cdot 3^{n+1}))$ with length $1/(2\cdot 3^{n+1})$ of each connected interval $[a,b]$ of $C_n$. Then we define $C := \bigcap_{n\in\mathbb{N}} C_n$. In every step, one removes $2^n$ intervals which are pairwise disjoint such that $\lambda(C_{n+1}) = \lambda(C_n) - 2^n/(2\cdot 3^{n+1})$. Recursively, we get

$$\lambda(C_n) = 1 - \frac{1}{4}\sum_{j=1}^{n}(\frac{2}{3})^j = \frac{1}{2} + \frac{1}{2}\cdot(\frac{2}{3})^n \to \frac{1}{2} \quad (n\to\infty)$$

which yields that $C$ has locally positive Lebesgue measure. On the other hand, for all $n \in \mathbb{N}$, the set $C_n$ consists of $2^n$ components $C_{n,j}$ with

$$\lambda(C_{n,j}) = \frac{1}{2^{n+1}} + \frac{1}{2}\cdot\frac{1}{3^n} \to 0 \quad (n\to\infty).$$

Thus, $C$ is totally disconnected. If we consider $\Omega = \mathbb{C}_\infty \setminus e^{i\pi(2C-1)}$, then the Taylor shift is strongly mixing in the Gaussian sense. This is true because all sets of extended uniqueness have arc length measure 0 by Appendix A, i.e. $e^{i\pi(2C-1)}$ is a $\mathcal{U}_0$-perfect set.

Another example for the Taylor shift to be strongly mixing in the Gaussian sense is the following.

**Example 5.2.8.** Let $1 \leq p < 2$ and $\Omega$ be a domain containing 0 whose boundary is a rectifiable Jordan curve and such that $\partial\Omega^* \cap \mathbb{T}$ has positive linear measure (and equivalently has positive arc length measure). Then for every set of extended uniqueness $D \subset \mathbb{T}$ the set $(\partial\Omega^* \cap \mathbb{T}) \setminus D$ still has positive linear measure since $D$ has arc length measure 0. So, with Proposition 4.2.7, we get that $(\partial\Omega^* \cap \mathbb{T}) \setminus D$ is a uniqueness set for $\Omega^*$ for every set of extended uniqueness $D \subset \mathbb{T}$. Thus, the Taylor shift on $A^p(\Omega)$ is strongly mixing in the Gaussian sense.
This also shows that for the case that $\Omega$ is a Carathéodory domain such that $\Omega^*$ contains a non-trivial subarc of $\mathbb{T}$, we have that the Taylor shift on $A^p(\Omega)$ is strongly mixing in the Gaussian sense.

To conclude this section, we want to give an example which clarifies the results for different values of $p$.

**Example 5.2.9.** Let $1 \leq p < \infty$ and $\Omega = \mathbb{C}_\infty \setminus B$ be open where $B \subset \mathbb{T}$ is a perfect set. Furthermore, let $T$ be the Taylor shift on $A^p(\Omega)$.

1. For $1 \leq p < 2$ we have seen in Theorem 5.2.4 that $T$ is weakly mixing in the Gaussian sense. $T$ is even strongly mixing in the Gaussian sense for the case that $B$ is a $\mathcal{U}_0$-perfect set.

2. $A^2(\Omega)$ is trivial if and only if $B$ is a polar set. So let $B$ be nonpolar. Since $A^2(\Omega)$ is a Banach space with cotype 2, the statements in Theorem 1.2.12 are equivalences. This yields that $T$ is not weakly mixing because for $\alpha \in 1/B$ the functions $\gamma(\alpha)$ are not in $A^2(\Omega)$, i.e.
$$\sigma_0(T) = \emptyset.$$
We do not know whether the Taylor shift is frequently hypercyclic on $A^2(\Omega)$.

3. Let $2 < p < \infty$ and $q$ be the conjugated exponent of $p$. If the $q$-capacity of $B$ is 0, then $A^p(\Omega)$ is trivial. Whenever this is not the case, we do not know whether the Taylor shift is frequently hypercyclic, weakly or strongly mixing in the Gaussian sense.

## 5.3 Dynamics of the Taylor Shift for $p \geq 2$

In the following, let $2 \leq p < \infty$. Contrary to the setting of the previous section, we are now in a situation where the functions $\gamma(\alpha)$ from (2.5) are not necessarily contained in $A^p(\Omega)$ whenever $\alpha \in \partial\Omega^*$. Therefore, we want to replace these functions by appropriate Cauchy transforms. If $\Omega$ is an open set containing 0, then the Cauchy transforms for measures $\mu \in M(\partial\Omega^*)$ are of interest for the Taylor shift on $A^p(\Omega)$ (see Definition 3.2.3). Note that $\Omega^*$ and therefore also $\partial\Omega^*$ are bounded in $\mathbb{C}$ since $0 \in \Omega$ by assumption. Kitai's Theorem shows that an operator can only be hypercyclic if every component of the spectrum intersects the unit circle. Therefore from now on, we will always assume that $\Omega$ is an open set containing 0 such that $\partial\Omega^* \cap \mathbb{T}$ is not empty.

**Definition 5.3.1.** For $1 \leq p < \infty$ and an open set $\Omega$, we define
$$\mathcal{M}_p(\partial\Omega^*) = \{\mu \in M(\partial\Omega^*) : \int\limits_{\partial\Omega^*} |\gamma(\zeta)| d|\mu|(\zeta) \in L^p(\Omega)\}.$$

Furthermore, let

$$\mathcal{C}_p(\partial\Omega^*) = \{C\mu \colon \mu \in \mathcal{M}_p(\partial\Omega^*)\}.$$

Following the lines of Chapter 3, for $f \in \mathcal{C}_p(\partial\Omega^*)$ we denote by $M(f) = \{\mu \in \mathcal{M}_p(\partial\Omega^*) : C\mu = f\}$ the set of representing measures for $f$.

Note that for $f \in \mathcal{C}_p(\partial\Omega^*)$ it follows that $f \in A^p(\Omega)$ since Cauchy transforms of measures $\mu \in \mathcal{M}_p(\partial\Omega^*)$ are holomorphic in $\Omega$ and

$$|\int\limits_{\partial\Omega^*} \gamma(\zeta) d\mu(\zeta)| \leq \int\limits_{\partial\Omega^*} |\gamma(\zeta)| d|\mu|(\zeta) \in L^p(\Omega).$$

**Theorem 5.3.2.** *Let $1 \leq p < \infty$ and $\Omega$ be open with $0 \in \Omega$. Furthermore, let $T$ be the Taylor shift on $\mathcal{C}_p(\partial\Omega^*)$. For $R \colon \mathcal{M}_p(\partial\Omega^*) \to \mathcal{M}_p(\partial\Omega^*)$, $R\mu = \overline{\mathrm{id}} d\mu$ the diagram*

$$\begin{array}{ccc} \mathcal{M}_p(\partial\Omega^*) & \xrightarrow{R} & \mathcal{M}_p(\partial\Omega^*) \\ {\scriptstyle C}\downarrow & & \downarrow{\scriptstyle C} \\ \mathcal{C}_p(\partial\Omega^*) & \xrightarrow[T]{} & \mathcal{C}_p(\partial\Omega^*) \end{array}$$

*commutes.*

*Proof.* Let $1 \leq p < \infty$ and $\mu \in \mathcal{M}_p(\partial\Omega^*)$. We first show that $R$ is a self map: Since $\partial\Omega^*$ is compact in $\mathbb{C}$, there exists a constant $c > 0$ such that $\sup_{z \in \partial\Omega^*} |z| \leq c$ and we obtain

$$\int\limits_{\partial\Omega^*} |\gamma(\zeta)| d|R\mu|(\zeta) = \int\limits_{\partial\Omega^*} |\gamma(\zeta)\zeta| d|\mu|(\zeta) \leq c \int\limits_{\partial\Omega^*} |\gamma(\zeta)| d|\mu|(\zeta).$$

It follows that $R\mu \in \mathcal{M}_p(\partial\Omega^*)$ because $\mu \in \mathcal{M}_p(\partial\Omega^*)$ by assumption.
Now, let $f \in \mathcal{C}_p(\partial\Omega^*)$ with $\mu \in M(f)$. As in Theorem 3.3.5, it follows

$$Tf = \int\limits_{\partial\Omega^*} \frac{\zeta}{1 - \zeta \cdot} d\overline{\mu}(\zeta),$$

i.e. $Tf = CR\mu$. Since $R$ is a self map on $\mathcal{M}_p(\partial\Omega^*)$, it follows that $Tf \in \mathcal{C}_p(\partial\Omega^*)$. $\square$

Inductively, we obtain for $f \in \mathcal{C}_p(\partial\Omega^*)$ and $\mu \in M(f)$

$$T^n f = \int\limits_{\partial\Omega^*} \zeta^n \gamma(\zeta) d\overline{\mu}(\zeta)$$

for all $n \in \mathbb{N}$.

We now want to guarantee that the iterates of the Taylor shift applied to appropriate Cauchy transforms converge to 0 in $A^p(\Omega)$. We can prove that this is the case if $\mu \in \mathcal{M}_p(\partial\Omega^*)$ with the additional property that $\mu$ is concentrated on $\partial\Omega^* \cap \mathbb{T}$ and a Rajchman measure (see Appendix A for a definition). Note that $\mu$ can be understood as a measure on the unit circle if it is concentrated on $\partial\Omega^* \cap \mathbb{T} \subset \mathbb{T}$ so it makes sense to speak of Rajchman measures.
Since we want to apply the Kitai Criterion in order to show that under specific assumptions the Taylor shift is mixing on $A^p(\Omega)$, we also need a right inverse of $T$. Therefore, as in Chapter 3, for $f \in \mathcal{C}_p(\partial\Omega^*)$ we fix an arbitrary representing measure $\mu_f \in M(f)$ and define

$$S_n f = \int\limits_{\partial\Omega^*} \frac{\gamma(\zeta)}{\zeta^n} d\overline{\mu_f}(\zeta) = \int\limits_{\partial\Omega^*} \frac{d\overline{\mu_f}(\zeta)}{\zeta^n(1-\zeta\cdot)} \tag{5.1}$$

for all $n \in \mathbb{N}$. Since $0 \notin \partial\Omega^*$, it follows, as in the proof of Theorem 5.3.2, that $S_n f \in \mathcal{C}_p(\partial\Omega^*)$.

**Theorem 5.3.3.** *Let $2 \leq p < \infty$ and $\Omega$ be an open set with $0 \in \Omega$. Furthermore, let $T$ be the Taylor shift operator on $A^p(\Omega)$ and $(S_n)_{n\in\mathbb{N}}$ be the sequence of mappings on $\mathcal{C}_p(\partial\Omega^*)$ defined in (5.1). If $f \in \mathcal{C}_p(\partial\Omega^*)$ such that $f$ is represented by a Rajchman measure $\mu_f \in M(f)$ concentrated on $B := \partial\Omega^* \cap \mathbb{T}$, then*

$$T^n f \to 0 \quad \text{and} \quad S_n f \to 0$$

*in $A^p(\Omega)$ as $n \to \infty$.*

*Proof.* Let $f \in \mathcal{C}_p(\partial\Omega^*)$ and $\mu_f \in M(f)$ such that $\mu_f$ is a Rajchman measure concentrated on $B$. With Theorem 5.3.2 we have

$$T^n f(z) = \int\limits_{\partial\Omega^*} \frac{\zeta^n}{1-\zeta z} d\overline{\mu_f}(\zeta) = \int\limits_{B} \frac{\zeta^n}{1-\zeta z} d\overline{\mu_f}(\zeta) \quad (z \in \Omega)$$

for all $n \in \mathbb{N}$. Because $\mu_f \in \mathcal{M}_p(\partial\Omega^*)$ is concentrated on $B \subset \mathbb{T}$, we have that the functions $\gamma(z)$ lie in $L^1(\mathbb{T}, |\mu|)$ for all $z \in \Omega$. Since $\mu_f$ is a Rajchman measure, Theorem A.2.6 yields that $\mu_{f,z}$ with $d\mu_{f,z} = \gamma(z)d\overline{\mu_f}$ is a Rajchman measure as well for all $z \in \Omega$. Thus,

$$T^n f(z) = \int\limits_{\mathbb{T}} \zeta^n \gamma(z)(\zeta) d\overline{\mu_f}(\zeta) = \int\limits_{\mathbb{T}} \zeta^n d\mu_{f,z}(\zeta) = \hat{\mu}_{f,z}(-n) \to 0$$

and

$$S_n f(z) = \int\limits_{\mathbb{T}} \zeta^{-n} \gamma(z)(\zeta) d\overline{\mu_f}(\zeta) = \int\limits_{\mathbb{T}} \zeta^{-n} d\mu_{f,z}(\zeta) = \hat{\mu}_{f,z}(n) \to 0$$

as $n$ tends to $\infty$. Furthermore,

$$|T^n f(z)| \leq \int\limits_B |\gamma(\zeta)| d|\mu_f|(\zeta) \quad \text{and} \quad |S_n f(z)| \leq \int\limits_B |\gamma(\zeta)| d|\mu_f|(\zeta) \quad \text{for all } n \in \mathbb{N}$$

where $\int_B |\gamma(\zeta)| d|\mu_f|(\zeta)$ is $p$-integrable on $\Omega$ by assumption. Lebesgue's theorem of dominated convergence then yields that $\|T^n f\|_p \to 0$ and $\|S_n f\|_p \to 0$ as $n$ tends to $\infty$. □

**Remark 5.3.4.** Let $\Omega$ be an open set and $B = \partial\Omega^* \cap \mathbb{T} \neq \emptyset$. We recall the definition of the functions $f_B$ from (4.3). We have seen in Remark 4.2.6 that for $dm_B = 1_B dm$ it follows

$$\int\limits_{\partial\Omega^*} |\gamma(\zeta)| dm_B(\zeta) \in L^p(\Omega)$$

for all $1 \leq p < \infty$ which yields that $m_B$ is a measure in $\mathcal{M}_p(\partial\Omega^*)$ concentrated on $B$ and hence $f_B \in \mathcal{C}_p(\partial\Omega^*)$. Since $1_B \in L^1(\mathbb{T})$ and the normalized arc length measure is a Rajchman measure, Theorem A.2.6 yields that $m_B$ is a Rajchman measure as well. Using the previous theorem, we obtain that the iterates of the Taylor shift applied to $f_B$ converge to 0.

Putting together the results of this section and Section 4.2, we now obtain that the Taylor shift is mixing on the Bergman space $A^p(\Omega)$ for the case $p \geq 2$ if $\Omega$ is as in the setting of Theorem 4.2.8.

**Theorem 5.3.5.** *Let $p \geq 2$ and $\Omega \subset \mathbb{C}_\infty$ be a domain set which is either bounded in $\mathbb{C}$ or $\infty \in \Omega$. If for every component $C$ of $\Omega^*$ the boundary of $C$ is a rectifiable Jordan* *curve and the intersection of $C$ with the unit circle $\mathbb{T}$ has positive $m$-measure, the Taylor shift on $A^p(\Omega)$ is mixing.*

*Proof.* By Theorem 4.2.8, we obtain that

$$L := \operatorname{span}\{f_B : B \subset \Omega^* \cap \mathbb{T} \text{ with positive } m\text{-measure}\}$$

is dense in $A^p(\Omega)$. Furthermore, let $(S_n)_{n\in\mathbb{N}}$ be the sequence of mappings defined in (5.1) restricted to $L$. Then Theorem 5.3.3 yields that $\|T^n f_B\|_p$ and $\|S_n f_B\|_p$ converge to 0 as $n \to \infty$ for all $f_B \in L$. As in Theorem 3.3.9 we get $T^n S_n f_B = f_B$ for all $n \in \mathbb{N}$ so the Kitai Criterion (see Theorem 1.2.6) yields the assertion. □

**Remark 5.3.6.** In the situation of the previous theorem, we have that the set of hypercyclic elements of the Taylor shift $T$ is algebraically generic, i.e. contains a dense vector subspace except 0. This follows directly from Theorem 2 in [11] and the fact that the assumptions of the Kitai Criterion are fulfilled as seen in the proof of the last theorem (see condition (A) in [11]).

**Remark 5.3.7.** Let $p \geq 2$ and $\Omega \subset \mathbb{C}_\infty$ be a domain such that it is either bounded in $\mathbb{C}$ or $\infty \in \Omega$. Furthermore, let the boundary of every component $C$ of $\Omega^*$ be a rectifiable Jordan curve and the intersection of $C$ with $\mathbb{T}$ be of positive $m$-measure. If $\mathbb{D} \subset \Omega$, then

$$\Phi\colon (A^p(\Omega), \|\cdot\|_{p,\Omega}) \to (A^p(\mathbb{D}), \|\cdot\|_{p,\mathbb{D}}),\ f \mapsto f$$

is a continuous map with dense image by Theorem 4.2.8, i.e. the Taylor shift on $A^p$ is quasiconjugate to the Taylor shift on $A^p(\Omega)$. As we have seen in Theorem 3.2.8, the Taylor shift is not frequently hypercyclic on $A^p$ whenever $p \geq 2$. This yields that for all $\Omega \supset \mathbb{D}$ which fulfil the requirements of Theorem 5.3.5, we cannot have frequent hypercyclicity of $T$ on $A^p(\Omega)$ for $p \geq 2$.

We now want to give some concrete examples of open sets $\Omega$ such that the Taylor shift is mixing on $A^p(\Omega)$.

**Example 5.3.8.** Let $\Omega := \mathbb{C}_\infty \setminus K$ where $K$ is a sector of the form

$$K := K_\rho := \{re^{it} : -\theta \leq t \leq \theta,\ 1 \leq r \leq \rho\} \tag{5.2}$$

for $\theta \in (0, \pi)$ and $1 < \rho < \infty$. Then $\Omega$ is a domain, $K$ is connected and the intersection of $\partial(1/K)$ with $\mathbb{T}$ is a non-trivial subarc. Theorem 5.3.5 yields that the Taylor shift is mixing on $A^p(\mathbb{C}_\infty \setminus K)$ for all $p \geq 2$. Furthermore, with Remark 5.3.7 we obtain that $T$ is not frequently hypercyclic on $A^p(\mathbb{C}_\infty \setminus K)$ for $p \geq 2$ and therefore it is also neither weakly nor strongly mixing in the Gaussian sense. By Theorem 5.2.1 the Taylor shift $T$ on $A^p(\Omega)$ is strongly mixing in the Gaussian sense if $1 \leq p < 2$.

The next example shows that for the Taylor shift to be mixing on an open set $\Omega \subset \mathbb{C}_\infty$ it is sufficient for the intersection of the complement of $\Omega$ with the unit circle $\mathbb{T}$ to be a Cantor set which has positive $m$-measure.

**Example 5.3.9.** We consider the Cantor set $C$ from Example 5.2.7. In every iteration, $[0,1] \setminus C_{n+1}$ consists of $2^{n+1} - 2^n = 2^n$ additional pairwise disjoint open intervals $I_k$, where $k \in \{2^n, ..., 2^{n+1} - 1\}$, compared to $[0,1] \setminus C_n$. For each of those components $I_k$, one can choose an infinitely differentiable function $r_k : [0,1] \to [0, \infty)$ with compact support $\operatorname{supp} r_k = \operatorname{cl}(I_k)$ and $a_k > 0$ such that

$$a_k \|r_k\|_{[0,1]} < \frac{1}{2^k} \quad \text{for } k \in \{2^n, ..., 2^{n+1} - 1\}$$

Then $r : [0,1] \to [0, \infty)$, $r(x) = \sum_{k=1}^{\infty} a_k r_k(x)$ is an infinitely differentiable function with $r(x) = 0$ for all $x \in C$.
For $\alpha \in (0, \pi)$ we define

$$\varphi : [-\alpha, \alpha] \to \mathbb{C},\ \varphi(x) = \left(1 + r\left(\frac{x - \alpha}{2\alpha}\right)\right) e^{ix}.$$

It follows that $\varphi \in C^\infty([-\alpha, \alpha], \mathbb{C})$ and is injective because of the uniqueness of the polar coordinate representation. Furthermore, it holds that $\varphi([-\alpha, \alpha]) \cap \mathbb{T} = e^{i\alpha(2C-1)}$ where $e^{i\alpha(2C-1)}$ has positive arc length measure. We can complete $\varphi([-\alpha, \alpha])$ to $J$ such that $J$ is a closed Jordan curve. Since $\varphi$ is infinitely often differentiable, we can assume that $J$ is rectifiable. We denote by $K$ the closure of the bounded component of the complement of this Jordan curve and set $\Omega = \mathbb{C}_\infty \setminus K$. Then $\Omega$ fulfils the requirements of Theorem 5.3.5, so the Taylor shift $T$ is mixing on $A^p(\Omega)$ for $p \geq 2$. As in the last example, we get with Remark 5.3.7 that $T$ is not frequently hypercyclic on $A^p(\Omega)$ for $p \geq 2$ whereas it is even strongly mixing in the Gaussian sense for $1 \leq p < 2$ which follows from Theorem 5.2.1.

Given a self map $T\colon X \to X$ on a topological vector space $X$, an interesting question to ask is how the arithmetic means of its iterates behave.

**Definition 5.3.10.** For an operator $T\colon X \to X$ on a topological vector space $X$, we define the *$n$-th Cesàro mean of $T$* by

$$C_n\colon X \to X,\ C_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

for $n \in \mathbb{N}$.

One easily sees that, for each $n \in \mathbb{N}$, the map $C_n$ is linear and continuous. In [27], the author introduced the notion of Cesàro hypercyclicity. Along those lines, we give the following definition.

**Definition 5.3.11.** Let $T\colon X \to X$ be an operator on a topological vector space $X$. We say that $T$ is *Cesàro mixing* if the sequence $(C_n)_{n\in\mathbb{N}}$ of its Cesàro means is mixing on $X$.

Now, we can consider the $n$-th Cesàro mean of the Taylor shift $T$ on $A^p(\Omega)$, $1 \leq p < \infty$ defined as

$$C_n\colon A^p(\Omega) \to A^p(\Omega),\ C_n f = \frac{1}{n}\sum_{k=0}^{n-1} T^k f$$

for open sets $\Omega$. As in the case of frequent hypercyclicity, the eigenfunctions of the Taylor shift will be a useful tool to verify that it is Cesàro mixing. Note that for all $\alpha \in \operatorname{int}(\operatorname{cl}(\Omega))$, we have that the eigenfunctions $\gamma(\alpha)$ of the Taylor shift are in the Bergman spaces for $1 \leq p < \infty$ and thus we have for those $\alpha$

$$C_n\gamma(\alpha) = \frac{1}{n}\sum_{k=0}^{n-1} T^k\gamma(\alpha) = \frac{1}{n}\left(\sum_{k=0}^{n-1}\alpha^k\right)\gamma(\alpha). \tag{5.3}$$

**Theorem 5.3.12.** *Let $\Omega \subset \mathbb{C}_\infty$ be an open set with $0 \in \Omega$ and such that it is either bounded in $\mathbb{C}$ or $\infty \in \Omega$. If each component $C$ of $\Omega^*$ fulfils $\operatorname{int}(C) \cap \mathbb{T} \neq \emptyset$, then the following is true:*

1. *The Taylor shift $T$ is Cesàro mixing on $A^p(\Omega)$ for $1 \leq p < 2$.*

2. *If in addition $\Omega$ is a domain, then the Taylor shift $T$ is Cesàro mixing on $A^p(\Omega)$ for $2 \leq p < \infty$.*

*Proof.* Let $\Omega \subset \mathbb{C}_\infty$ be as required. Using (5.3), we have that $C_n\gamma(\alpha) \to 0$ $(n \to \infty)$ in $A^p(\Omega)$ for $|\alpha| < 1$ and $C_n\gamma(\alpha) \to \infty$ $(n \to \infty)$ in $A^p(\Omega)$ for $|\alpha| > 1$. Define

$$L = \operatorname{span}\{\gamma(\alpha)\colon \alpha \in \operatorname{int}(\Omega^*),\ |\alpha| < 1\} \text{ and } M = \operatorname{span}\{\gamma(\alpha)\colon \alpha \in \operatorname{int}(\Omega^*),\ |\alpha| > 1\}.$$

Then both $L$ and $M$ are dense in $A^p(\Omega)$ by Theorem 4.2.5 and Theorem 4.2.11, respectively. Now, let $U$, $V$ be non-empty open sets in $A^p(\Omega)$ and take $f \in U \cap L$ and $g \in V \cap M$, so we can write $f = \sum_{i=1}^{k} a_i\gamma(\alpha_i)$ for $\alpha_i \in \operatorname{int}(\Omega^*)$ with $|\alpha_i| < 1$, $i \in \{1, ..., k\}$ and $g = \sum_{j=1}^{l} b_j\gamma(\beta_j)$ for $\beta_j \in \operatorname{int}(\Omega^*)$ with $|\beta_j| > 1$, $j \in \{1, ..., l\}$. We set

$$h_n = \sum_{j=1}^{l} b_j \frac{n}{\sum_{m=0}^{n-1} \beta_j^m} \gamma(\beta_j)$$

such that $h_n \to 0$ in $A^p(\Omega)$ as $n$ tends to $\infty$ with $C_n h_n = \sum_{j=1}^{m} b_j\gamma(\beta_j) = g$. Furthermore

$$C_n f = \sum_{i=1}^{k} a_i \frac{\sum_{m=0}^{n-1} \alpha_i^m}{n} \gamma(\alpha_i) \to 0 \quad (n \to \infty).$$

Thus, there exists $N \in \mathbb{N}$ such that for every $n \geq N$ we have $f + h_n \in U$ and $C_n(f + h_n) = C_n(f) + g \in V$. Therefore, $(C_n)_{n\in\mathbb{N}}$ is mixing. □

# Chapter 6

# Universal Taylor Series

## 6.1 Overview and First Results

After a short introduction to universality in the first chapter, we now want to consider a special case, namely universal Taylor series. We have already seen the definition of the disk algebra. For general compact sets $E \subset \mathbb{C}$, the function algebra $A(E)$ is defined as

$$A(E) := \{f \in C(E) : f \text{ holomorphic in } \operatorname{int}(E)\}.$$

Equipped with the uniform norm on $E$, $A(E)$ is a Banach space. Note that if the interior of $E$ is empty, it follows that $A(E)$ is equal to $C(E)$.

In the following, we will always consider functions $f \in H(\Omega)$ where $\Omega$ is an open set with $\mathbb{D} \subset \Omega$ and the complement of $\Omega$ intersects the unit circle. As usual, by $s_n f$ we denote the $n$-th partial sum of the Taylor expansion of $f$ around 0.

**Definition 6.1.1.** Let $E \subset \mathbb{C} \setminus \mathbb{D}$ be a set outside the unit disc, $\Lambda \subset \mathbb{N}_0$ be infinite and let $f \in H(\mathbb{D})$.

1. We say that $(s_n f)_{n\in\Lambda}$ is *pointwise universal with respect to* $E$ if for every function $g\colon E \to \mathbb{C}$ there exists a subsequence of $(s_n f)_{n\in\Lambda}$ converging pointwise to $g$ on $E$.

2. If $E$ is a compact set with connected complement, we say that $(s_n f)_{n\in\Lambda}$ is *uniformly universal with respect to* $E$ if for every function $g \in A(E)$ there exists a subsequence of $(s_n f)_{n\in\Lambda}$ converging uniformly to $g$ on $E$.

If $\Lambda = \mathbb{N}_0$, we also say that the function $f$ is uniformly universal (respectively pointwise universal) with respect to $E$.

In [33] the author was able to show that for a domain $\Omega \subset \mathbb{C}$ there exist generically many functions $f \in H(\Omega)$ such that $(s_n f)_{n\in\mathbb{N}}$ is uniformly universal with respect all $E \subset \mathbb{C} \setminus \Omega$ with connected complement. In particular, the case $\Omega = \mathbb{D}$ yields that there exist generically many functions $f \in H(\mathbb{D})$ such that for each non-trivial subarc of the unit circle $E \subset \mathbb{T}$ which is not equal to $\mathbb{T}$ and for every function $g$ continuous on $E$ a subsequence of $(s_n f)_{n\in\mathbb{N}}$ converges uniformly to $g$ on $E$.

One might think that an analogous result is true for functions in the Bergman spaces. However, in [40] the author proved that for $f \in A^p$ at most one continuous pointwise limit function can exist on each non-trivial subarc of $\mathbb{T}$. On the other hand, the authors of [8] were able to prove that there exist subsets of the unit circle with positive $m$-measure such that there are generically many functions in $A^p$ which are uniformly universal. First, we need the following definition:

Let $E$ be a proper subset of $\mathbb{T}$ with $m(E) > 0$. Then $E$ is said to satisfy *Carleson's condition* if

$$\ell(E) := \sum_k m(B_k)\log(1/m(B_k)) < \infty$$

where $\mathbb{T} \setminus E = \bigcup_k B_k$ is the finite or countable union of the pairwise disjoint open arcs $B_k$.

**Theorem 6.1.2** (Beise, Müller)**.** *Let $1 \leq p < \infty$ and $E \subset \mathbb{T}$ be closed with either $m(E) > 0$ and $E$ not containing a subset of positive $m$-measure satisfying Carleson's condition or else $m(E) = 0$. If $\Lambda \subset \mathbb{N}_0$ is infinite, then generically many $f \in A^p$ enjoy the property that $(s_n f)_{n\in\Lambda}$ is uniformly universal with respect to $E$.*

As a conclusion from this result, one obtains the existence of generically many $f \in A^p$ which are universal in the sense of Menshov, that is for each measurable function $g : \mathbb{T} \to \mathbb{C}$ there exists a subsequence of the partial sums $(s_n f)_{n\in\mathbb{N}}$ tending to $g$ almost everywhere on $\mathbb{T}$.

We now want to obtain a universality result for functions $f \in A^p(\Omega)$, where $\Omega \subset \mathbb{C}_\infty$ is a domain containing $\infty$, with respect to compact sets $E \subset \mathbb{C} \setminus \Omega$ lying in a bounded component of the complement of $\Omega$. We will apply two theorems from [29] in order to show the following results:

**Corollary 6.1.3.** *Let $\Omega = \mathbb{C}_\infty \setminus K$ be open where $K \subset \mathbb{C} \setminus \mathbb{D}$ is a connected compact set with nonempty interior meeting the unit circle $\mathbb{T}$ and such that* $\mathrm{int}(K)$ *has connected complement. Then there exist generically many functions $f \in A^p(\Omega)$ which are, for all $E \subset \mathrm{int}(K)$ compact with connected complement, uniformly universal with respect to $E$.*

*Proof.* Let $E \subset \mathrm{int}(K)$ be a compact set with connected complement. Then one can find an open connected set $\Omega_0$ with $\mathrm{cl}(\Omega) \subset \Omega_0$ such that its complement is also connected and $E \cap \Omega_0 = \emptyset$. Define $T_n : A^p(\Omega) \to A(E)$, $T_n f := s_n f|_E$. Since $H(\Omega_0) \subset A^p(\Omega)$ is dense and with [29, Theorem 1], we obtain that the set of universal functions is a dense $G_\delta$-set.
Now, let $(E_k)_{k\in\mathbb{N}}$ be the standard exhaustion of $\mathrm{int}(K)$ (see [39]). Note that $E_k$, $k \in \mathbb{N}$, has connected complement as well. For $k \in \mathbb{N}$ the first part of the proof yields the existence of generically many functions $f \in A^p(\Omega)$ which are uniformly universal with respect to $E_k$. Using Baire's theorem, we obtain that generically many $f \in A^p(\Omega)$ enjoy the property that for all $k \in \mathbb{N}$ the set of partial sums $\{s_n f|_{E_k} : n \in \mathbb{N}\}$ is dense in $A(E_k)$. For an arbitrary $E \subset \mathrm{int}(K)$ compact with connected complement, there exists some $k \in \mathbb{N}$ with $E \subset E_k$ and since the polynomials are dense in $A(E)$ and $A(E_k)$, $A(E_k)$ is dense in $A(E)$. Therefore, any $f \in A^p(\Omega)$ which is uniformly universal with respect to all $E_k$, $k \in \mathbb{N}$, is uniformly universal with respect to all $E \subset \mathrm{int}(K)$ compact with connected complement. □

Compared to the results we have seen until now, we want to analyse the behaviour of the Taylor expansions of functions $f \in A^p(\Omega)$ on compact sets $E \subset \Omega \setminus \mathbb{D}$.

**Corollary 6.1.4.** *Let $\Omega = \mathbb{C}_\infty \setminus K$ be an open set where $K \subset \mathbb{C} \setminus \mathbb{D}$ is a compact set meeting the unit circle $\mathbb{T}$ such that $K = \mathrm{cl}(\mathrm{int}(K))$. For any countable set $E \subset \mathbb{C} \setminus \Delta$ there exist generically many functions $f \in A^p(\Omega)$ which are pointwise universal with respect to $E$.*

*Proof.* First, let $E \subset \mathbb{C} \setminus \Delta$ be finite. Because $K$ equals the closure of its interior, we can choose $z_0 \in \mathrm{int}(K)$ such that $|z_0| < |z|$ for all $z \in E$ and define $\Omega_0 := \mathbb{C} \setminus \{z_0\}$. [29, Theorem 2] yields the existence of generically many functions $f \in H(\Omega_0)$ having the property that for every $h : E \to \mathbb{C}$ there is a subsequence of $(s_n f)_{n\in\mathbb{N}}$ converging to $h$ on

$E$. Since $H(\Omega_0) \subset A^p(\Omega)$ is dense and defining $T_n : A^p(\Omega) \to \mathbb{C}^E$ as $T_n f := s_n f|_E$, the Universality Criterion states that there is a dense $G_\delta$-set of functions $f \in A^p(\Omega)$ having the desired property.
If $E = \{z_j : j \in \mathbb{N}\}$ is countable, we define $E_k := \{z_1, ..., z_k\}$. Then using Baire's theorem we obtain that generically many $f \in A^p(\Omega)$ enjoy the property that for all $k \in \mathbb{N}$ the set of partial sums $\{s_n f|_{E_k} : n \in \mathbb{N}\}$ is dense in $\mathbb{C}^{E_k}$. For a function $f \in A^p(\Omega)$ of this kind and any $h : E \to \mathbb{C}$, there exists a strictly increasing subsequence $(n_j)_{j\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$|s_{n_j} f(z) - h(z)| < 1/j \quad (z \in E_j).$$

Then $s_{n_j} f$ tends to $h$ pointwise on $E$ as $j \to \infty$. □

**Example 6.1.5.** Let $K \subset \mathbb{C}_\infty \setminus \mathbb{D}$ be a sector as in (5.2) with $K \cap \mathbb{T} \neq \emptyset$ and $\Omega = \mathbb{C}_\infty \setminus K$.

1. By Corollary 6.1.3 if $E \subset \text{int}(K)$ is a compact set with connected complement, generically many $f \in A^p(\Omega)$ are uniformly universal on $E$.

2. Corollary 6.1.4 states that for countable sets $E \subset \mathbb{C} \setminus \Delta$ there are generically many functions $f \in A^p(\Omega)$ which are pointwise universal on $E$. As an example, we can choose $E := (\mathbb{Q} + i\mathbb{Q}) \setminus \Delta$. This differs from the first example since $E$ is also allowed to intersect $\Omega$, i.e. the set where the Taylor expansion of $f$ can be holomorphically extended. On the other hand, only countable sets $E$ are allowed.

## 6.2 Universality Results using the Taylor Shift

In the following, we want to apply Theorem 5.3.5 in order to obtain further information about the behaviour of the partial sums of the Taylor expansion of functions in Bergman spaces. For that, let $\Omega$ be a domain with $0 \in \Omega$ and $T$ the Taylor shift on $A^p(\Omega)$ for $1 \leq p < \infty$. As one can see from (2.2), the behaviour of the iterates $T^n f$ is closely related to that of the partial sums $s_n f$. This connection was already studied in [8]. The authors showed that in case that $\Omega$ is a Carathéodory domain containing a nontrivial subarc of $\mathbb{T}$ which does not separate the plane there exist generically many functions $f \in A^p(\Omega)$ having a subsequence of $(s_n f)_{n\in\mathbb{N}}$ tending to $f$ locally uniformly on $\Omega \cap \Delta$. Along those lines, we want to apply the results of the previous chapter in order to make statements concerning the boundary behaviour of the partial sums $s_n f$ for $f \in A^p(\Omega)$ for more general $\Omega$. Note that (2.2) implies

$$|T^{n+1}f| \geq |f - s_n f| \tag{6.1}$$

on $\Delta \cap \Omega$ for all $f \in A^p(\Omega)$. Using (6.1) and Theorem 5.3.5, we obtain the following result concerning the behaviour of the partial sums of the Taylor expansion on the boundary and outside its disc of convergence.

**Corollary 6.2.1.** *Let $\Omega \subset \mathbb{C}_\infty$ be a domain which is either bounded in $\mathbb{C}$ or $\infty \in \Omega$. Furthermore, let for every component $C$ of $\Omega^*$ the boundary of $C$ be a rectifiable Jordan curve, the intersection of $C$ with the unit circle $\mathbb{T}$ have positive $m$-measure and $\Lambda \subset \mathbb{N}_0$ be infinite.*

1. *For generically many $f \in A^p(\Omega)$, there is a subsequence of $(s_n f)_{n\in\Lambda}$ tending to $f$ in $A^p(\Omega \cap \mathbb{D})$ and locally uniformly on $\Omega \cap \Delta$.*

2. *If $\Omega \setminus \Delta \neq \emptyset$, then for generically many $f \in A^p(\Omega)$ there exists a subsequence of $(s_n f(z))_{n\in\Lambda}$ tending to $\infty$ locally uniformly on $\Omega \setminus \Delta$.*

*Proof.* Theorem 5.3.5 yields that the Taylor shift is mixing on $A^p(\Omega)$, so by Theorem 1.1.7 and Remark 1.1.8 there exist generically many universal functions $f \in A^p(\Omega)$ for $(T^{n+1})_{n\in\Lambda}$.
To prove the first statement, let $f$ be such a universal function. Then there exists a subsequence $(n_j)_{j\in\mathbb{N}}$ in $\Lambda$ with $T^{n_j+1}f \to 0$ in $A^p(\Omega)$ and (6.1) implies that $s_{n_j} f \to f$ in $A^p(\Omega \cap \mathbb{D})$ as $j \to \infty$. Using Remark 4.1.1 and again (6.1), we know that $s_{n_j} f \to f$ locally uniformly on $\Delta \cap \Omega$ as $j$ tends to $\infty$.
To prove the second statement, we know that for each of the universal functions $f$ and $z_0 \in \operatorname{int}(\Omega^*)$ there exists a subsequence $(n_j)_{j\in\mathbb{N}}$ in $\Lambda$ with $T^{n_j+1}f \to \gamma(z_0)$ in $A^p(\Omega)$. In particular, $\gamma(z_0) \in A^p(\Omega)$ is bounded below on compact subsets $K$ of $\Omega \setminus \Delta$ with $\gamma(z_0)(z) \neq 0$ for all $z \in \Omega$. Therefore, we have with (2.2) that $|s_{n_j} f - f| \to \infty$ as $j \to \infty$ uniformly on every compact $K \subset \Omega \setminus \Delta$ and thus also

$$|s_{n_j} f| \to \infty \quad (j \to \infty)$$

locally uniformly on $\Omega \setminus \Delta$. □

The first statement of Corollary 6.2.1 yields that for generically many $f \in A^p(\Omega)$ there is a subsequence of $(s_n f)_{n\in\mathbb{N}}$ tending to $f$ locally uniformly on $\mathbb{T}\cap\Omega$. A natural question to ask is whether there are finite limit functions different from $f$ on parts of $\mathbb{T}\cap\Omega$.
A known result by Fatou and M. Riesz states that for each function holomorphic in a domain $\Omega \supset \mathbb{D}$ such that the Taylor coefficients $(a_n)_{n\in\mathbb{N}}$ tend to 0 and each closed non-trivial subarc $\Gamma$ of $\Omega\cap\mathbb{T}$, the partial sums $(s_n f)_{n\in\mathbb{N}}$ converge uniformly to $f$ on $\Gamma$. For functions $f \in A^p$, we have seen in (3.3) that the Taylor coefficients $a_n$ satisfy

$$a_n = o(n^{1/p}) \quad (n \to \infty)$$

which is best possible so we cannot apply the result of Fatou and Riesz. Without needing a condition for the growth of the Taylor coefficients, Gardiner and Manolaki ([21, Theorem 1]) were able to show a similar result for functions holomorphic in the unit disc.

**Theorem 6.2.2** (Gardiner, Manolaki)**.** *Let $f$ be a function holomorphic in $\mathbb{D}$ and $(s_{n_j} f)_{j\in\mathbb{N}}$ be an arbitrary subsequence of $(s_n f)_{n\in\mathbb{N}}$ converging pointwise to a (finite) limit function $h$ on a subset $\Gamma$ of $\mathbb{T}$. If $f$ has nontangential limits $f^*(\zeta)$ for $\zeta \in \Gamma$, then $h = f^*$ on $\Gamma$ almost everywhere with respect to the normalized arc length measure $m$.*

In contrast to this result, we want to show that on small subsets of $\Omega\cap\mathbb{T}$ a maximal set of limit functions can exist for $f \in A^p(\Omega)$.

**Definition 6.2.3.** A closed subset $E$ of $\mathbb{T}$ is called a *Dirichlet set* if a subsequence of $(z^n)_{n\in\mathbb{N}}$ tends to 1 uniformly on $E$.

**Remark 6.2.4.** One can easily see that each finite set in $\mathbb{T}$ is a Dirichlet set. Furthermore, Dirichlet sets cannot have positive arc length measure but can have Hausdorff dimension 1 (see e.g. [25]).

The authors of [8] already showed for Carathéodory domains $\Omega$ and Dirichlet sets $E \subset \mathbb{T}\cap\Omega$ generically many $f \in A^p(\Omega)$ enjoy the property that for each $h \in C(E)$ there exists a subsequence of the partial sums of the Taylor expansion converging to $h$. Using Theorem 5.3.5 we obtain the following extension of that result.

**Theorem 6.2.5.** *Let $\Omega \subset \mathbb{C}_\infty$ be a domain which is either bounded in $\mathbb{C}$ or $\infty \in \Omega$. If for every component $C$ of $\Omega^*$ the boundary of $C$ is a rectifiable Jordan curve, the* *intersection of $C$ with the unit circle $\mathbb{T}$ has positive $m$-measure and $E \subset \mathbb{T} \cap \Omega$ is a Dirichlet set then generically many $f \in A^p(\Omega)$ are uniformly universal with respect to $E$.*

*Proof.* Since $E$ is a Dirichlet set, one can find $\Lambda \subset \mathbb{N}_0$ infinite with $z^{n+1} \to 1$ uniformly on $E$ as $n \to \infty$, $n \in \Lambda$. Because $E$ has connected complement, Mergelian's theorem yields that the polynomials are dense in $C(E)$, so we can assume that $h \in A^p(\Omega)$. Let $f$ be universal for $(T^{n+1})_{n\in\Lambda}$ which exists by the Universality Criterion and Remark 1.1.8. Since convergence in $A^p(\Omega)$ implies local uniform convergence, there is a subsequence $(n_j)_{j\in\mathbb{N}}$ in $\Lambda$ with $T^{n_j+1}f \to f - h$ locally uniformly on $\Omega$ as $j$ tends to $\infty$ and thus in particular uniformly on $E$. Then also

$$z^{n_j+1}T^{n_j+1}f(z) \to f(z) - h(z) \quad (j \to \infty)$$

uniformly on $E$ and therefore

$$s_{n_j}f(z) = f(z) - z^{n_j+1}T^{n_j+1}f(z) \to h(z) \quad (j \to \infty)$$

uniformly on $E$. □

**Corollary 6.2.6.** *Let $K = \{re^{it} \colon \theta_1 \leq t \leq \theta_2,\ \rho_1 \leq r \leq \rho_2\}$ for $\theta_1 < \theta_2$ and $0 < \rho_1 < \rho_2$ such that $K \cap \mathbb{T} \neq \emptyset$ and let $\Omega = \mathbb{C}_\infty \setminus K$. Then for any Dirichlet set $E \subset \mathbb{T} \cap \Omega$ and any $r \in [\rho_1, \rho_2]$ generically many $f \in A^p(\Omega)$ are uniformly universal with respect to $rE$.*

*Proof.* Let $E \subset \mathbb{T} \cap \Omega$ be a Dirichlet set and $r \in [\rho_1, \rho_2]$. By Theorem 6.2.5 there exist generically many functions $f \in A^p(\Omega/r)$ which are uniformly universal with respect to $E$. Since $R \colon A^p(\Omega/r) \to A^p(\Omega)$, $Rf(z) = f(rz)$ is continuous and bijective it directly follows that there exist generically many $f \in A^p(\Omega)$ such that for any $h \in C(rE)$ there exists a subsequence $(n_j)_{j\in\mathbb{N}}$ with $s_{n_j}f \to h$ uniformly on $rE$. □

In order to show our next result we first need some definitions. In comparison to the notion of nontangential limits which was introduced in Chapter 3, we say that a function $f$ on $\mathbb{D}$ has an unrestricted limit $f^*(\zeta)$ at a point $\zeta \in \mathbb{T}$ if

$$f(z_n) \to f^*(\zeta) \quad (n \to \infty)$$

for an arbitrary sequence $(z_n)_{n\in\mathbb{N}}$ tending to $\zeta$ in $\mathbb{D}$.

**Definition 6.2.7.** Let $f \in H(\mathbb{D})$ with $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ for $z \in \mathbb{D}$. If $(n_k)_{k\in\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$, we say that $f$ has *Hadamard-Ostrowski gaps relative to* $(n_k)_{k\in\mathbb{N}}$ if a sequence of integers $(p_k)_{k\in\mathbb{N}}$ with the following properties exists:

1. $n_{k-1} \leq p_k < n_k$ and $\limsup_{k\to\infty} p_k/n_k < 1$,
2. if $J$ is the set of integers $\nu$ such that $p_k < \nu \leq n_k$ for some $k \in \mathbb{N}$, then

$$\limsup_{J\ni\nu\to\infty} |a_\nu|^{1/\nu} < 1.$$

Note that for the proof of the following theorem, a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ is called *Cesàro summable* if the limit of its Cesàro means exists.

**Theorem 6.2.8.** *Let $E \subset \mathbb{C} \setminus \mathbb{D}$ be a set such that $E \cap \mathbb{T} \neq \emptyset$ and $E \setminus \Delta$ non-polar. If $f \in A^p$ is a function having an unrestricted limit for some $\zeta \in E \cap \mathbb{T}$ and such that there exists a subsequence of $(s_n f)_{n\in\mathbb{N}}$ which converges pointwise on $E$ to a function $h$, then $h(\zeta) = f(\zeta)$.*

*Proof.* By assumption, $f$ has radius of convergence 1. Since $E\setminus\Delta$ is non-polar, it follows that $\Delta \cup E$ has logarithmic capacity $> 1$. Let $(n_k)_{k\in\mathbb{N}}$ be a sequence of integers such that $(s_{n_k} f)_{k\in\mathbb{N}}$ converges to a function $h$ pointwise on $E$. In particular, the sequence $(s_{n_k} f(\zeta))_{k\in\mathbb{N}}$ is bounded for each $\zeta \in E$. Hence,

$$\limsup_{k\to\infty} |s_{n_k} f(z)|^{1/n_k} \leq 1 \quad \text{for all } z \in E.$$

Thus, Theorem 1 of [9] yields that $f$ has Hadamard-Ostrowski gaps relative to $(n_k)_{k\in\mathbb{N}}$. Using (3.3), the Taylor coefficients satisfy $a_n = o(n)$ as $n$ tends to $\infty$. Therefore, [34] yields that $(s_n f)_{n\in\mathbb{N}}$ is Cesàro summable at each point $\zeta \in \mathbb{T}$ at which $f$ has an unrestricted limit which is the case for some $\zeta \in E \cap \mathbb{T}$ by assumption. Using Lemma 3.2 in [16], we can conclude that $h(\zeta) = f(\zeta)$. □

**Remark 6.2.9.** Let $\Omega$ be an open set such that $\Omega^* \cap \mathbb{T} \neq \emptyset$, let $E \subset \Omega \setminus \mathbb{D}$ intersect the unit circle and $E \setminus \Delta$ be non-polar. Note that each $f \in A^p(\Omega)$ has unrestricted limits on $\Omega \cap \mathbb{T}$. Thus, if for $f \in A^p(\Omega)$ there exists a subsequence of $(s_n f)_{n\in\mathbb{N}}$ converging pointwise on $E$ to a function $h$, the assumptions of the previous theorem are fulfilled and we have $h(\zeta) = f(\zeta)$ for all $\zeta \in E \cap \mathbb{T}$.

On the other hand, if $E \subset \Omega \setminus \mathbb{D}$ is a set meeting the unit circle and $E \setminus \Delta$ is polar, we do not know whether there exist universal functions $f \in A^p(\Omega)$ with respect to $E$.

**Example 6.2.10.** Let $K = K_\rho \subset \mathbb{C} \setminus \mathbb{D}$ be a sector as in (5.2) with $K \cap \mathbb{T} \neq \emptyset$ and $\Omega = \mathbb{C}_\infty \setminus K$. Furthermore, let $E \subset \Omega \cap \mathbb{T}$ be a Dirichlet set. By Corollary 6.2.6 it follows that for any $r \in [1, \rho]$ there exist generically many $f \in A^p(\Omega)$ which are uniformly universal with respect to $rE$. However, if $E$ is a non polar Dirichlet set and we only add one point $\zeta \in \Omega \cap \mathbb{T}$ to $rE$ for $r \in (1, \rho]$, we obtain by Theorem 6.2.8 that there exists no function $f \in A^p(\Omega)$ which is universal on $rE \cup \{\zeta\}$.

Compared to Corollary 6.2.6 the following theorem shows that, for each given non polar set $E \subset \mathbb{T}$ and each $r > 1$, all sufficiently small sectors $K \subset \mathbb{C}$ as in (5.2) have the property that for any $f \in H(\mathbb{C}_\infty \setminus K)$ and any subsequence $(n_j)_{j\in\mathbb{N}}$ of $\mathbb{N}$ the partial sums $(s_{n_j} f)_{j\in\mathbb{N}}$ are not bounded on $rE$.

**Theorem 6.2.11.** *Let $E \subset \mathbb{C} \setminus \Delta$ be a non polar set. Then there is a sector $K$ whose size only depends on the capacity of $\Delta \cup E$ such that for all $f \in H(\mathbb{C}_\infty \setminus K)$ with $f(z) = \sum_{\nu=0}^\infty a_\nu z^\nu$ having radius of convergence 1 and any subsequence $(n_j)_{j\in\mathbb{N}}$ of $\mathbb{N}$ the partial sums $(s_{n_j} f)_{j\in\mathbb{N}}$ are not bounded on $E$.*

*Proof.* Let $E \subset \mathbb{C} \setminus \Delta$ be a non-polar set and $\varepsilon > 0$ sufficiently small such that $C_1 := (1+\varepsilon)/c(\Delta \cup E) < 1$. By [9, Theorem 1] there exists some $q \in (0, 1)$ such that

$$\limsup_{j\to\infty} \max_{qn_j \le \nu \le n_j} |a_\nu|^{1/\nu} \le C_1 \tag{6.2}$$

for all $f(z) = \sum_{\nu=0}^\infty a_\nu z^\nu$ with radius of convergence 1 fulfilling

$$\limsup_{j\to\infty} |s_{n_j} f(z)|^{1/n_j} \le 1 \quad \text{for all } z \in E \tag{6.3}$$

for some subsequence $(n_j)_{j\in\mathbb{N}}$ of $\mathbb{N}$. Let $C_2 \in (C_1, 1)$ and let $f$ be a function satisfying (6.3). By [32, Lemma 2] we obtain the existence of some sufficiently small $\rho > 1$ and $\theta \in (0, \pi)$ such that for all functions $\Phi$ of exponential type whose indicator diagram is contained in the rectangle

$$R = \{z \in \mathbb{C} : -\ln\rho \le \text{Re } z \le 0,\ -|\ln\theta| \le \text{Im } z \le |\ln\theta|\}$$

it holds

$$q < \underline{\text{dens}}\{\nu \in \mathbb{N}_0 : |\Phi(\nu)|^{1/\nu} \geq C_2\}. \tag{6.4}$$

We now show that $f$ cannot be holomorphically extended to $\mathbb{C}_\infty \setminus K$ where $K = K_\rho := e^{-R}$ is a sector as in (5.2): Suppose not. Then $f(z) = \sum_{\nu=0}^{\infty} \Phi(\nu) z^\nu$ for some $\Phi \in \text{Exp}(R)$ using Theorem B.2.2 and and the fact that the Mellin transformation is a bijection between $H(\mathbb{C}_\infty \setminus K)$ and $\text{Exp}(R)$ with Proposition B.2.3. By (6.2), this already yields for $\Lambda := \mathbb{N} \setminus \bigcup_{j \in \mathbb{N}} [qn_j, n_j]$

$$\begin{aligned} \underline{\text{dens}}\{\nu \in \mathbb{N}_0 : |\Phi(\nu)|^{1/\nu} \geq C_2\} &\leq \underline{\text{dens}}\{\nu \in \Lambda : |\Phi(\nu)|^{1/\nu} \geq C_2\} \\ &\leq \liminf_{j \to \infty} \frac{|\{\Lambda \ni \nu \leq n_j : |a_\nu|^{1/\nu} \geq C_2\}|}{n_j} \leq \liminf_{j \to \infty} \frac{qn_j}{n_j} = q \end{aligned}$$

which contradicts (6.4). □

# Appendix A

# Sets of Uniqueness and Rajchman Measures

The deep result of Theorem 1.2.12 is of importance when it comes to confirming whether an operator is weakly or strongly mixing in the Gaussian sense. For an operator $T : X \to X$ on a Fréchet space $X$ to be strongly mixing, it is sufficient that for any Borel set of extended uniqueness $D \subset \mathbb{T}$ the linear span of $\bigcup_{\lambda \in \mathbb{T} \setminus D} \ker(T - \lambda I)$ is dense in $X$. Furthermore, Rajchman measures will play a crucial role in Chapter 5. Therefore, we want to give a short introduction of these notions.

## A.1 Complex Measures

For a measurable space $(S, \Sigma)$, we say that $\mu : \Sigma \to \mathbb{C}$ is a *complex measure on* $S$ if it fulfils (i) $\nu(\emptyset) = 0$ and (ii) $\nu(\bigcup_{i \in I} A_i) = \sum_{i \in I} \nu(A_i)$ for all $A_i \in \Sigma$ pairwise disjoint where $I$ is either finite or countably infinite. Furthermore, for a complex measure $\mu$, we can write $\mu = \text{Re } \mu + i \cdot \text{Im } \mu$ and then we have the complex conjugate $\overline{\mu} = \text{Re } \mu - i \cdot \text{Im } \mu$. If not stated otherwise, the following results can be found in [38, Chapter 6].

**Definition A.1.1.** Let $\mu$ and $\nu$ be complex measures on $S$.

1. The *total variation* of $\mu$ is given by

$$|\mu|(E) = \sup_{P \in \mathcal{P}} \sum_{A \in P} |\mu(A)| \quad \text{for all measurable } E$$

where $\mathcal{P}$ is the set of all partitions $P$ of $E$ into a countable number of disjoint measurable subsets. The total variation of a measure $\mu$ is itself a positive finite measure.

2. For two complex measures $\mu$, $\nu$, we define

$$\mu \ll \nu \text{ if } \mu(B) = 0 \text{ for all Borel sets } B \text{ with } \nu(B) = 0.$$

   and say that $\mu$ is *absolutely continuous with respect to* $\nu$.

3. We call $\mu$ *concentrated on* $K \in \Sigma$ if $\nu(A) = \nu(A \cap K)$ for every $A \in \Sigma$. This is equivalent to the hypothesis that $\nu(A) = 0$ whenever $A \cap K = \emptyset$.

4. $\mu$, $\nu$ are called *mutually singular* and we write $\mu \perp \nu$ if there exists a pair of disjoint sets $A$ and $B$ such that $\mu$ is concentrated on $A$ and $\nu$ is concentrated on $B$.

**Remark A.1.2.** Let $\mu$ and $\nu$ be complex measures on $S$.

1. If $\mu$ is concentrated on a set $K \in \Sigma$, then so is $|\mu|$.

2. If $\mu \ll \nu$, then it also follows $|\mu| \ll \nu$. In particular, this yields $|\mu| \ll \mu$.

3. By the Radon-Nikodym Theorem we know that $\mu$ is absolutely continuous with respect to $\nu$ if and only if there exists some $f \in L^1(S, |\nu|)$ with $d\mu = f d\nu$.

**Theorem A.1.3.** *Let $\mu$ be a complex measure on $\Sigma$ and $d\nu = f d\mu$ for a function $f \in L^1(\mu)$. Then*

$$|\nu|(A) = \int_A |f| d|\mu| \quad (A \in \Sigma).$$

*Proof.* [38, Theorem 6.12] yields the existence of a measurable function $h$ on $S$ with $|h| = 1$ and such that $d\mu = h d|\mu|$. But then it directly follows from [38, Theorem 6.13] with $d\nu = f d\mu = f h d\mu$ that

$$|\nu|(A) = \int_A |fh| d|\mu| = \int_A |f| d|\mu|$$

for all $A \in \Sigma$ which yields the conclusion. □

**Remark A.1.4.** 1. Let $\mu$ be a positive and $\nu$ be a complex measure such that there exists a complex valued function $f$ with $d\nu = f d\mu$. Then $d\overline{\nu} = \overline{f} d\mu$.

2. For two complex measures $\mu, \nu$ and a function $f$ with $d\nu = f d\mu$ it holds $d\overline{\nu} = \overline{f} d\overline{\mu}$. This is true because for $\mu$ there exists a function $h$ with $|h| = 1$ and $d\mu = hd|\mu|$ by [38, Theorem 6.12]. Then $d\nu = fhd|\mu|$ and since $|\mu|$ is a positive measure it follows with 1. that

$$d\overline{\nu} = \overline{fh}d|\mu| = \overline{f}d\overline{\mu}.$$

**Remark A.1.5.** Let $\mathcal{B}$ be the Borel-$\sigma$-algebra on $\mathbb{C}$ and $E \in \mathcal{B}$. Then we denote by $M(E)$ the set of complex measures $\mu : \mathcal{B} \to \mathbb{C}$ concentrated on $E$. If $\mathcal{B}(E)$ is the Borel-$\sigma$-algebra on $E$, we can understand measures $\mu : \mathcal{B}(E) \to \mathbb{C}$ as measures in $M(E)$ by setting

$$\mu(A) = \mu(A \cap E) \quad \text{for all } A \in \mathcal{B}.$$

## A.2 Rajchman Measures and Sets of Uniqueness

If not stated otherwise, the following definitions and results can be found in [26].

**Definition A.2.1.** A set $D \subset \mathbb{T}$ is called a *set of uniqueness* if for every trigonometric series

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad (c_n \in \mathbb{C})$$

which is 0 for $e^{ix} \in \mathbb{T} \setminus D$ we have that the series is identically 0. Otherwise it is called a *set of multiplicity*. We denote by $\mathcal{U}$ the class of sets of uniqueness and by $\mathcal{M}$ the class of sets of multiplicity.

It follows directly from the definition that if $D \in \mathcal{U}$, we have that every $A \subset D$ is a set of uniqueness.

**Theorem A.2.2** (Cantor, Lebesgue)**.** *Every countable closed set $D \subset \mathbb{T}$ is a set of uniqueness.*

**Theorem A.2.3.** *Every Lebesgue measurable set of uniqueness $D \subset \mathbb{T}$ has arc length measure 0.*

**Remark A.2.4.** There exist perfect sets of uniqueness. For example, Rajchman was able to show that $e^{i\pi(2C-1)}$ where $C$ is the classical Cantor 1/3-set is a set of uniqueness. On the other hand, Cantor sets with positive arc length measure are sets of multiplicity.

**Definition A.2.5.** A measure $\mu \in M(\mathbb{T})$ is called a *Rajchman measure* if $\hat{\mu}(n) \to 0$ as $|n| \to \infty$ where $\hat{\mu}(n) = \int e^{-int} d\mu(t)$ is the $n$-th Fourier-Stieltjes coefficient of $\mu$.

**Theorem A.2.6.** *Let $\nu$ be a Rajchman measure and $\mu \ll \nu$. Then $\mu$ is also a Rajchman measure.*

**Definition A.2.7.** A set $D \subset \mathbb{T}$ is called an *extended set of uniqueness* if for every positive Rajchman measure $\mu$ it follows that $\mu(D) = 0$. Otherwise, $D$ is called a *set of restricted multiplicity*. We denote by $\mathcal{U}_0$ the class of extended sets of uniqueness and by $\mathcal{M}_0$ the class of sets of restricted multiplicity.

**Remark A.2.8.** As the normalized arc length measure is a positive Rajchman measure it follows that $m(D) = 0$ for all extended sets of uniqueness $D$.

We call a set $D \subset \mathbb{T}$ universally measurable if it is measurable for all positive Borel measures on $\mathbb{T}$. In particular, all Borel sets are universally measurable.

**Proposition A.2.9.** *Every universally measurable set of uniqueness is of extended uniqueness.*

**Example A.2.10.** Since $e^{i\pi(2C-1)}$ is a Borel set and a set of uniqueness, the previous proposition yields that it is also an extended set of uniqueness.

# Appendix B

# Entire Functions of Exponential Type

Since entire functions of exponential type play an important role in Chapter 2 of this work, we will give a short introduction to the topic according to [13].

## B.1 Basic Definitions and Results

**Definition B.1.1.** Let $\Phi$ be an entire function

(i) The *maximum modulus* of $\Phi$ is defined by $M_\Phi(r) = \max_{|z|=r} |\Phi(z)|$.

(ii) $\Phi$ is called *entire function of exponential type* if

$$\tau(\Phi) \coloneqq \limsup_{r\to\infty} \frac{\ln M_\Phi(r)}{r} < \infty$$

**Definition B.1.2.** Let $\Phi$ be an entire function of exponential type. Then

$$h_\Phi : [-\pi, \pi) \to [-\infty, \infty), \quad h_\Phi(t) = \limsup_{r\to\infty} \frac{\ln|\Phi(re^{it})|}{r}$$

is called *Phragmén-Lindelöf indicator function* of $\Phi$.

**Definition B.1.3.** Let $K \subset \mathbb{C}$ be a non-empty, compact and convex set. Then

$$H_K : \mathbb{C} \to \mathbb{C} \quad H_K(z) = \sup_{u \in K} \text{Re } (uz)$$

is called *support function* of the set $K$.

In the following, we list some useful properties of the support function due to [10] and [31].

**Proposition B.1.4.** *Let $K, L \subset \mathbb{C}$ be non-empty, compact and convex. Then the following assertions hold:*

1. $H_{K+L} = H_K + H_L$.
2. *$K$ is a subset of $L$ if and only if $H_K \leq H_L$.*

Let $\Phi$ be an entire function if exponential type and $\zeta \in \mathbb{T}$. Then we set

$$W(\zeta) := \{z \in \mathbb{C} : \text{Re } (z\zeta) > h_\Phi(\arg \zeta)\}$$

Given this set, we define the *Laplace-transform* as follows

$$\mathcal{B}\Phi(z, \zeta) := \zeta \int_0^\infty \Phi(t\zeta) e^{-zt\zeta} dt \quad (z \in W(\zeta))$$

One can show the following properties for the Laplace-transform

1. $z \mapsto \mathcal{B}\Phi(z, \zeta)$ is holomorphic on $W(\zeta)$.
2. For $\zeta, \zeta' \in \mathbb{T}$ we have

$$\mathcal{B}\Phi(z, \zeta) = \mathcal{B}\Phi(z, \zeta') \quad (z \in W(\zeta) \cap W(\zeta'))$$

With the second property, one can glue the Laplace-transforms together to a well-defined function $\mathcal{B}\Phi$ on $\bigcup_{\zeta \in \mathbb{T}} W(\zeta)$. This function is called *Borel-transform* of $\Phi$.

**Definition B.1.5.** Let $\Phi$ be an entire function of exponential type. Then $K(\Phi) := \mathbb{C} \setminus \bigcup_{\zeta \in \mathbb{T}} W(\zeta)$ is called *conjugate indicator diagram* of $\Phi$, where $K(\Phi)$ is compact and convex.

**Definition B.1.6.** For a compact and convex set $K \subset \mathbb{C}$, we denote by $\mathrm{Exp}(K)$ the set of entire functions $f$ of exponential type whose conjugate indicator diagram $K(f)$ is a subset of $K$.

Equipped with the family of seminorms

$$\|f\|_m = \sup_{z\in\mathbb{C}} |f(z)| e^{-\frac{1}{m}|z| - H_K(z)}$$

$\mathrm{Exp}(K)$ forms a Fréchet space (see [10]), where $m$ is a positive integer, $f \in \mathrm{Exp}(K)$ and $H_K : \mathbb{C} \to \mathbb{C}$ denotes the support function.

**Theorem B.1.7** (Carlson's theorem)**.** *Let $\Phi$ be an entire function of exponential type and $K(\Phi)$ its conjugate indicator diagram such that*

$$\max_{z\in K(\Phi)} \mathrm{Im}\ z - \min_{z\in K(\Phi)} \mathrm{Im}\ z < 2\pi$$

*If $\Phi(n) = 0$ for $n \in \mathbb{N}_0$, then $\Phi \equiv 0$.*

## B.2 The Mellin transformation

**Definition B.2.1.** For $L \subset \{z \in \mathbb{C} : |\mathrm{Im}\ z| < \pi\}$ compact and convex the *Mellin transformation* is given by

$$M : H((e^L)^*) \to \mathrm{Exp}(L), \quad Mg(z) := -\frac{1}{2\pi i} \int_\gamma \frac{g(w)}{w^{z+1}} dw \quad (z \in \mathbb{C})$$

where $w^z := e^{z \log w}$, $z \in \mathbb{C}$, $w \in \mathbb{C}_-$ with $\mathbb{C}_- := \mathbb{C} \setminus (-\infty, 0]$ and where $\gamma$ is a loop in $\mathbb{C}_- \setminus e^{-L}$ of index $-1$ with respect to the compact set $e^{-L}$, and log denotes the principal branch of the logarithm.

With [10, pages 266-270] we know that the Mellin transformation is a bijective and linear map.

**Theorem B.2.2.** *For a given function $\Phi \in \mathrm{Exp}(L)$ and $z$ with small modulus, it holds*

$$M^{-1}\Phi(z) = \sum_{\nu=0}^{\infty} \Phi(\nu) z^\nu$$

*and for $z$ with large modulus*

$$M^{-1}\Phi(z) = -\sum_{\nu=1}^{\infty} \Phi(-\nu) z^{-\nu}.$$

In the following, we prove that the Mellin transformation is also continuous:

**Proposition B.2.3.** *Let $L \subset \{z \in \mathbb{C} : |\mathrm{Im}\, z| < \pi\}$ be compact and convex. Then the Mellin transformation is a topological isomorphism.*

*Proof.* Let $L$ be a compact and convex subset of $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \pi\}$. We already know that the Mellin transformation is a linear, bijective map. Because of its linearity, it is sufficient to show that $M$ is continuous at 0. For that purpose, let $m$ be a positive integer and $g_n \in H((e^L)^*)$, $n \in \mathbb{N}$, with $g_n \to 0$ $(n \to \infty)$. Now, choose $\gamma$ as a loop in $e^{-L-\frac{1}{m}\Delta}$ of index $-1$ with respect to the points in $e^{-L}$ so that $\log(|\gamma|) \subset \frac{1}{m}\Delta + L$ where $|\gamma|$ denotes the trace of the $\gamma$. Since $\frac{1}{m}\Delta + L$ is convex, with Proposition B.1.4 we know that

$$H_{\mathrm{conv}(\log(|\gamma|))} \leq H_{\frac{1}{m}\Delta + L} = H_{\frac{1}{m}\Delta} + H_L.$$

So, with $c := \int_\gamma |dw|/|w|$ we obtain

$$\begin{aligned}
\|Mg_n\|_m &= \sup_{z\in\mathbb{C}} \Big|\frac{1}{2\pi i}\int_\gamma \frac{g_n(w)}{w^{z+1}} dw\Big| e^{-\frac{1}{m}|z| - H_K(z)} \\
&\leq \frac{c}{2\pi} \sup_{z\in\mathbb{C}} \max_{w\in|\gamma|} \Big|\frac{g_n(w)}{e^{z\log(w)}}\Big| e^{-\frac{1}{m}|z| - H_K(z)} \\
&\leq \frac{c}{2\pi} \max_{w\in|\gamma|} |g_n(w)| \sup_{z\in\mathbb{C}} e^{H_{\mathrm{conv}(\log(|\gamma|))}(z) - \frac{1}{m}|z| - H_K(z)} \\
&\leq \frac{c}{2\pi} \max_{w\in|\gamma|} |g_n(w)| \to 0 \quad (n \to \infty).
\end{aligned}$$

With that we obtain the continuity and, finally, the continuity of the inverse map follows from the open mapping theorem (see e.g. [39, Corollary 2.12]). □

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