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Uniqueness Results and Universality of Fourier Series in Dirichlet-type Spaces

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Summary

Analysis is primarily concerned with the examination of limiting processes. It is important to note that not all sequences will necessarily converge toward a specific value or object. Indeed, there are instances where these processes have been observed to diverge in the most extreme manner possible. This kind of extreme behavior is frequently associated with a concept known as universality which gained attention since the beginning of the 20th century. The phenomenon of universality is a fascinating topic that warrants further exploration and understanding. This work aims to delve into this intriguing subject, shedding light on its complexities and its significance in the study of limiting processes.

The notion of universality was first identified by Fekete [58] in 1914 who demonstrated the existence of a real power series

$$\sum_{k=1}^{n} a_k x^k$$

which diverges for any $x \in [-1,1] \setminus \{0\}$ and that in the most conceivable way, namely for any continuous function h on $[-1,1]$ that equals 0 at the origin, there exists a subsequence n_j, such that

$$\sum_{k=1}^{n_j} a_k x^k \to h \quad \text{uniformly}$$

as j tends to ∞. One might be tempted to think that the occurrence of universal elements is a rare phenomenon, but usually the opposite effect appears. Recall that investigations of Fekete revealed the existence of one universal element, however, he did not quantify the amount of universal elements. Subsequent studies pointed out that if a class contains a universal element, then it is often the case that there is a great amount of universal elements in a topological sense.

Although universality has fascinated over the last decades, there are still numerous open questions in this field that require further investigation. In this work, we will mainly focus on classes of functions whose Fourier series are universal in the sense that they allow us to approximate uniformly any continuous function defined on a suitable subset of the unit circle.

The structure of this thesis is as follows. In the first chapter, we will initially introduce the most important notation which is needed for our following discussion. Subsequently, after recalling the notion of universality in a general context, we will revisit significant

results concerning universality of Taylor series. The focus here is particularly on universality with respect to uniform convergence and convergence in measure. By a result of Menshov, we will transition to universality of Fourier series which is the central object of study in this work.

In the second chapter, we recall spaces of holomorphic functions which are characterized by the growth of their coefficients. In this context, we will derive a relationship to functions on the unit circle via an application of the Fourier transform. In the second part of the chapter, our attention is devoted to the $\mathcal{D}^p_{\text{harm}}$ spaces which can be viewed as the set of harmonic functions contained in the $W^{1,p}(\mathbb{D})$ Sobolev spaces. In this context, we will also recall the Bergman projection. Thanks to the intensive study of the latter in relation to Sobolev spaces, we can derive a decomposition of $\mathcal{D}^p_{\text{harm}}$ spaces which may be seen as analogous to the Riesz projection for L^p spaces. Owing to this result, we are able to provide a link between $\mathcal{D}^p_{\text{harm}}$ spaces and spaces of holomorphic functions on $\mathbb{C}_\infty \setminus \mathbb{S}$ which turns out to be a crucial step in determining the dual of $\mathcal{D}^p_{\text{harm}}$ spaces. The last section of this chapter deals with the Cauchy dual which has a close connection to the Fantappié transform. As an application, we will determine the Cauchy dual of the spaces D_α and D^p_{harm}, two results that will prove to be very helpful later on. Finally, we will provide a useful criterion that establishes a connection between the density of a set in the direct sum $X \oplus Y$ and the Cauchy dual of the intersection of the respective spaces.

The subsequent chapter will delve into the theory of capacities and, consequently, potential theory which will prove to be essential in formulating our universality results. In addition to introducing further necessary terminologies, we will define capacities in the first section following [16], however in the frame of separable metric spaces, and revisit the most important results about them. Simultaneously, we make preparations that allow us to define the Li_α-capacity which will turn out to be equivalent to the classical Riesz α-capacity. The Li_α-capacity proves to be more adapted to the D_α spaces. It becomes apparent in the course of our discussion that the Li_α-capacity is essential to prove uniqueness results for the class D_α. This leads to the centerpiece of this chapter which forms the energy formula for the Li_α-capacity on the unit circle. More precisely, this identity establishes a connection between the energy of a measure and its corresponding Fourier coefficients. We will briefly deal with the complement-equivalence of capacities before we revisit the concept of Bessel and Riesz capacities, this time, however, in a much more general context, where we will mainly rely on [1]. Since we defined capacities on separable metric spaces in the first section, we can draw a connection between Bessel capacities and Li_α-capacities. To conclude this chapter, we would like to take a closer look at the geometric meaning of capacities. Here, we will point out a connection between the Hausdorff dimension and the polarity of a set, and transfer it to the Li_α-capacity. Another aspect will be the comparison of Bessel capacities across different dimensions, in which the theory of Wolff potentials crystallizes as a crucial auxiliary tool.

In the fourth chapter of this thesis, we will turn our focus to the theory of sets of uniqueness, a subject within the broader field of harmonic analysis. This theory has a close relationship with sets of universality, a connection that will be further elucidated in

the upcoming chapter. The initial section of this chapter will be dedicated to the notion of sets of uniqueness that is specifically adapted to our current context. Building on this concept, we will recall some of the fundamental results of this theory. In the subsequent section, we will primarily rely on techniques from previous chapters to determine the closed sets of uniqueness for the class $\mathcal{D}_\alpha$. The proofs we will discuss are largely influenced by [16, p. 178] and [9, pp. 82]. One more time, it will become evident that the introduction of the Li_α-capacity in the third chapter and the closely associated energy formula on the unit circle, were the pivotal factors that enabled us to carry out these proofs.

In the final chapter of our discourse, we will present our results on universality. To begin, we will recall a version of the universality criterion which traces back to the work of Grosse-Erdmann (see [26]). Coupled with an outcome from the second chapter, we will prove a result that allows us to obtain the universality of a class using the technique of simultaneous approximation. This tool will play a key role in the proof of our universality results which will follow hereafter. Our attention will first be directed toward the class D_α with α in the interval $(0, 1]$. Here, we summarize that universality with respect to uniform convergence occurs on closed and α-polar sets $E \subset \mathbb{S}$. Thanks to results of Carleson and further considerations, which particularly rely on the favorable behavior of the Li_α-kernel, we also find that this result is sharp. In particular, it may be seen as a generalization of the universality result for the harmonic Dirichlet space. Following this, we will investigate the same class, however, this time for $\alpha \in [-1, 0)$. In this case, it turns out that universality with respect to uniform convergence occurs on closed and $(-\alpha)$-complement-polar sets $E \subset \mathbb{S}$. In particular, these sets of universality can have positive arc measure. In the final section, we will focus on the class D^p_{harm}. Here, we manage to prove that universality occurs on closed and $(1, p)$-polar sets $E \subset \mathbb{S}$. Through results of Twomey [68] combined with an observation by Girela and Pélaez [23], as well as the decomposition of D^p_{harm}, we can deduce that the closed sets of universality with respect to uniform convergence of the class D^p_{harm} are characterized by $(1, p)$-polarity. We conclude our work with an application of the latter result to the class D^p. We will show that the closed sets of divergence for the class D^p are given by the $(1, p)$-polar sets.

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Contents

Chapter 1

Preliminaries

The first section of this chapter is intended to briefly introduce some basic and indispensable notation and definitions that are frequently used throughout this work. In the subsequent section, we outline some main results concerning universality, where we first put our attention to analytic function spaces and provide additional definitions as the need arises. In the course of this section, we will move on to universality of Fourier series.

1.1 Basic notation

As usual, the set of integers is denoted by $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ will be written for the positive and nonnegative integers, respectively. Moreover, $\mathbb{C}$ stands for the set of all complex numbers. Throughout this thesis, we endow the set of complex numbers with the Euclidean norm denoted by $|\cdot|$. We write $\mathbb{C}^*$ for the set of all complex numbers excluding 0 and $\mathbb{C}_\infty$ for the extended plane of complex numbers. In this context, we set $1/\infty = 0$. In addition to that, $\mathbb{D}$ denotes the (open) unit disk, i.e. the set of all complex numbers $z \in \mathbb{D}$ that satisfy $|z| < 1$ and $\mathbb{S}$ is written for the unit circle in the complex plane. Furthermore, we write $\mathbb{D}_{\text{ext}}$ for the set $\{z \in \mathbb{C}_\infty : |z| > 1\}$. As usual, $\mathbb{R}$ and $\mathbb{Q}$, stand for the real and rational numbers, respectively.

If A is a subset of a topological space, we write $\text{int}(A)$, $\text{cl}(A)$, and ∂A for the interior, the closure, and the boundary of A, respectively.

For a measurable space (Ω, Σ), we use the notation $\mathcal{M}(\Omega, \Sigma)$ for the set of all measurable functions $f : (\Omega, \Sigma) \to (\mathbb{C}, \mathcal{B}(\mathbb{C}))$, where $\mathcal{B}(\mathbb{C})$ denotes the Borel σ-algebra on $\mathbb{C}$. If Ω is a topological space and if Σ is the Borel σ-algebra $\mathcal{B}(\Omega)$ on Ω, then we usually write $\mathcal{M}(\Omega)$ instead of $\mathcal{M}(\Omega, \Sigma)$. In case μ defines a measure (see Appendix A) on (Ω, Σ), then a set $A \in \Sigma$ is called a **μ-null set** if $\mu(A) = 0$. A property is said to hold **μ-almost everywhere** (μ-a.e.), if it holds for every $\omega \in \Omega$ except a μ-null set A. If the considered measure is clear from the context, we say for the sake of simplicity null set and almost everywhere (a.e.), respectively.

For an open set $\Omega \subset \mathbb{C}$ we recall that $H(\Omega)$ denotes the set of holomorphic functions in Ω. We always equip this space with the topology of locally uniform convergence which can

be described by a countable set of seminorms, and which furthermore makes it a Fréchet space [64, p. 34]. In case that $\infty \in \Omega \subset \mathbb{C}_\infty$ open, f is contained in $H(\Omega)$ if f additionally vanishes at ∞, i.e. $f(1/z) \to 0$ if $z \to 0$.

Finally, for a compact set $E \subset \mathbb{C}$, $C(E)$ denotes the set of all functions $f : E \to \mathbb{C}$ which are continuous on E. Moreover, $A(E)$ denotes the set of all $f \in C(E)$ with the property that f is holomorphic in $\text{int}(E)$. These spaces will always be equipped with the maximum norm $\|\cdot\|_\infty := \|\cdot\|_{E,\infty}$, unless clearly stated otherwise which makes them Banach spaces.

1.2 Known results

We start this section with the definition of universality motivated by [26] which is crucial and one of the central terms for this work.

Definition 1.2.1 Let X be a vector space and $(Y, \mathcal{T})$ be a topological vector space. Moreover, let $T_\iota : X \to Y$ be a set of linear operators for $\iota \in I$. We say that an element $x \in X$ is $(Y, \mathcal{T})$-**universal** or shorter $\mathcal{T}$-**universal** or Y-**universal** (for the family $\{T_\iota : \iota \in I\}$) if the set

$$\{T_\iota x : \iota \in I\}$$

is dense in $(Y, \mathcal{T})$.

In case the topology on Y is induced by a metric d or a norm $\|\cdot\|$ we occasionally write d-universal or $\|\cdot\|$-universal.

Throughout this work the set I in Definition 1.2.1 will always take the role of the positive integers. An immediate observation is that countable families of operators can only have universal elements in separable topological vector spaces, and hence only those are of our interest!

In the sequel, for $k \in \mathbb{N}_0$, we write $p_k : \mathbb{C} \to \mathbb{C}$ defined by

$$p_k(z) := z^k \tag{1.1}$$

for the k-th **monomial**.

Definition 1.2.2 Let X be the Fréchet space of holomorphic functions on the unit disk and let $(Y, \mathcal{T})$ be a topological vector space over the field $\mathbb{C}$ which encompasses the algebraic polynomials. For $n \in \mathbb{N}$, we define the family of linear operators $T_n : X \to Y$ via

$$T_n(f) := \sum_{k=0}^{n} a_k p_k \qquad (f \in X),$$

where a_k denotes the k-th **Taylor coefficient** of the function f evaluated at the point 0. The operator T_n is called the n-th partial **Taylor sum**.

A set A contained in a topological vector space $(Y, \mathcal{T})$ is called a G_δ-set if A can be written as a countable intersection of open sets. A set $M \subset Y$ which contains a *dense* G_δ-set is called **residual**. In case $(Y, \mathcal{T})$ is a locally compact Hausdorff space or if (Y, d) is a complete metric space, we know thanks to Baire's theorem (see e.g. [64, 2.2 Baire's theorem]) that every countable intersection of dense G_δ-sets is a dense G_δ-set. In particular, every countable intersection of residual sets is residual.

We proceed by presenting some significant results in research on universal Taylor series. We start with a result that traces back to Nestoridis.

Theorem 1.2.3 (Nestoridis, [56, Theorem 2.6])
There exists a residual set $M \subset H(\mathbb{D})$ such that any function $f \in M$ is $(A(E), \|\cdot\|_\infty)$-universal for the family $\{T_n : n \in \mathbb{N}\}$ for all compact $E \subset \mathbb{C}\backslash\mathbb{D}$ with connected complement.

The latter theorem does not assume any growth condition on the function $f \in H(\mathbb{D})$, so the question arises of whether the theorem holds if the growth of Taylor coefficients is restricted. According to Melas, Nestoridis, and Papadoperakis [50, Theorem 1.2], a necessary condition for universal elements in sense of Theorem 1.2.3 is that for any $p > 1$, the Taylor coefficients have the property

$$(a_k) \notin O\Big(\exp(k/\ln(k)^p)\Big).$$

In particular, (a_k) cannot have polynomial growth.

In order to formulate the next theorem, we shall introduce Hardy spaces on the unit disk which form one of the central objects of study in complex analysis since the beginning of the last century. Before, we briefly recall the arc measure on the unit circle.

Definition 1.2.4 Recall that $\mathcal{B}(\mathbb{S})$ is the collection of Borel sets on $\mathbb{S}$. The (normalized) **arc measure** m on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ is defined as the pushforward measure of the normalized Lebesgue measure on $(-\pi, \pi]$ under the mapping $\varphi : (-\pi, \pi] \to \mathbb{S}$, $\varphi(t) = e^{it}$.

Moreover, we write m_2 for the (normalized) **area measure** on the unit disk $\mathbb{D}$.

Remark 1.2.5 The arc measure has the property of rotational invariance which mainly follows from the translation invariance of the Lebesgue measure. Thus, the arc measure is the unique Haar measure on the compact abelian group $(\mathbb{S}, \cdot)$ with the property $m(\mathbb{S}) = 1$. The reader is referred to [63, Chapter 1] for a basic introduction to the theory of Haar measures.

Definition 1.2.6 Let $p \in [1, \infty)$. The space $\mathcal{H}^p$ is defined as the set of all holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{\mathcal{H}^p} := \sup_{0<r<1} \left(\int_{\mathbb{S}} |f_r|^p \, dm \right)^{1/p} < \infty,$$

where $f_r := f(r\,\cdot)$ for $r \in (0,1)$, and together with the norm $\|\cdot\|_{\mathcal{H}^p}$, $\mathcal{H}^p$ is called **Hardy space** with parameter p. In addition to this, the space $\mathcal{H}^\infty$ is defined as the set of all bounded holomorphic functions on $\mathbb{D}$ equipped with the supremum norm on $\mathbb{D}$.

One can show that the class $\mathcal{H}^p$ is complete for all $p \in [1,\infty]$ and that the polynomials form a dense set if $p \neq \infty$. It is well known that $\mathcal{H}^\infty$ is not separable and that the closure of polynomials in $\mathcal{H}^\infty$ is given by the disk algebra $A(\mathrm{cl}(\mathbb{D}))$.

At this point, we refrain from presenting more far-reaching results on Hardy spaces. Instead, we refer the reader for example to [12] or [42].

Theorem 1.2.7 (Beise and Müller, [5, Theorem 1.1])
Let $p \in [1,\infty)$ and suppose that $E \subset \mathbb{S}$ is a closed m-null set. Then there exists a residual set of holomorphic functions contained in $\mathcal{H}^p$ which are $(C(E), \|\cdot\|_\infty)$-universal for the family $\{T_n : n \in \mathbb{N}\}$.

The proof of this theorem is based on a technique that involves simultaneous approximation. More precisely, it uses the fact that for any closed set E of vanishing arc measure, for each given $f \in \mathcal{H}^p$, where $p \in [1,\infty)$, and each $h \in C(E)$ there exists a sequence of polynomials (q_n) such that (q_n) approximates f and h simultaneously, which means

$$\|f - q_n|_{\mathbb{D}}\,\|_{\mathcal{H}^p} \to 0 \quad \text{and} \quad \|h - q_n|_E\,\|_\infty \to 0 \quad (n \to \infty).$$

Theorem 1.2.7 has the characteristic that it is sharp in the sense that universality cannot occur on sets with positive arc measure which is an immediate consequence of the famous Carleson-Hunt theorem in case $p \in (1,\infty)$. The case $p = 1$ involves some extra arguments and will be discussed in the following remark.

Remark 1.2.8 A result of Körner [37, Theorem A.1] states that the space $\mathcal{H}^1$ exhibits elements f whose corresponding Taylor series $(T_n f)$ diverge everywhere on $\mathbb{S}$. Nevertheless, according to Fatou's theorem, it is known that nontangential limits of $f \in \mathcal{H}^1$ exist m-almost everywhere.

A result of Gardiner and Manolaki [21, Theorem 1] says in particular that if a subsequence of $(T_n f)$ converges to g pointwise on $E \subset \mathbb{S}$, then the nontangential limit function of $f \in \mathcal{H}^1$ and the function g coincide m-a.e. on E. Hence, a function $f \in \mathcal{H}^1$ can only be $(C(E), \|\cdot\|_\infty)$-universal if E is of vanishing arc measure.

Due to the result of Körner combined with the insights of Gardiner and Manolaki, here universality with respect to $\|\cdot\|_\infty$ and divergence of the Taylor series fall apart.

Next, we put our focus on the Dirichlet space $\mathcal{D}$ which is defined as the space of all holomorphic functions f on the unit disk such that the derivative of $p_1 f$ is square integrable with respect to the area measure m_2. It is known that $\mathcal{D}$ is contained in any $\mathcal{H}^p$ whenever $1 \leq p < \infty$. Indeed, the boundary behavior of functions in the Dirichlet space is much more favorable as it is the case for functions in Hardy spaces. We refer to Beurling's theorem (see for instance [16, Theorem 3.2.1]) which says that for any function in $\mathcal{D}$ the radial

limits exist up to a set of vanishing logarithmic capacity. By Abel's theorem and Fejér's Tauberian theorem, it follows that $(T_n f)(\zeta)$ converges if and only if the nontangential limit of f exists at ζ. Hence, $(T_n f)$ converges on $\mathbb{S}$ up to a set of vanishing logarithmic capacity. By [54, Theorem 4] it follows that for every closed set $E \subset \mathbb{S}$ of vanishing logarithmic capacity, there exists a residual set of holomorphic functions in $\mathcal{D}$ which are $(C(E), \|\cdot\|_\infty)$-universal. Again, the result was proved by an application of simultaneous approximation which is possible due to findings that go back to the work of Khrushchev and Peller [36].

In the sequel, we want to introduce another space of holomorphic functions.

Definition 1.2.9 Let $p \in [1, \infty)$. The space $\mathcal{A}^p$ is defined as the collection of all holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ which fulfill

$$\|f\|_{\mathcal{A}^p} := \left(\int_{\mathbb{D}} |f|^p \, dm_2 \right)^{1/p} < \infty.$$

Equipped with the norm $\|\cdot\|_{\mathcal{A}^p}$, $\mathcal{A}^p$ is called **Bergman space** with parameter p.

For a detailed introduction to the theory of Bergman spaces, the reader is referred to [13] or [33]. At this point, it should be noted that $\mathcal{A}^p$ is complete (see e.g. [13, p. 8]). Moreover, one can show that the polynomials form a dense set in $\mathcal{A}^p$ (see e.g. [13, §2.2 Theorem 3] and consequently, $\mathcal{A}^p$ is separable.

We shall introduce some further designations in order to formulate the next universality result. According to [7, Definition 2.2.2], a sequence of measurable functions $(f_n)_{n\in\mathbb{N}}$ defined on a measure space (Ω, Σ, μ) is said to **converge in measure** to a measurable function f, if for all $\varepsilon > 0$

$$\mu\Big(\big\{\omega \in \Omega : |f_n(\omega) - f(\omega)| > \varepsilon\big\}\Big) \to 0 \quad (n \to \infty).$$

In case (Ω, Σ, μ) is a *finite* measure space and if the sequence (f_n) converges in measure to a function f, then there exists a subsequence (f_{n_j}) such that

$$f_{n_j} \to f \quad \mu\text{-a.e.} \tag{1.2}$$

as j tends to infinity (see e.g. [7, 2.2.5 Theorem]).

For a measure space (Ω, Σ, μ) the space $L^0(\mu) := L^0(\Omega, \Sigma, \mu)$ is defined as the vector space $\mathcal{M}(\Omega, \Sigma)$ endowed with the topology induced by convergence in measure with respect to μ.

This prepares us to consider the next universality result, here for Bergman spaces. The reader is advised that the proof of this theorem relies on deep considerations that go back to [35, Theorem 4.1] which can be interpreted as a result of simultaneous approximation (compare [46, Theorem 1.1]).

Theorem 1.2.10 (Beise and Müller, [5, Theorem 1.3])
Let $p \in [1, \infty)$. Then there exists a residual set of holomorphic functions contained in $\mathcal{A}^p$ which are $L^0(m)$-universal for the family $\{T_n : n \in \mathbb{N}\}$.

In particular, by exploiting (1.2), there is an $f \in \mathcal{A}^p$ such that for each $h \in \mathcal{M}(\mathbb{S})$ there is a subsequence n_j with

$$T_{n_j}(f) \to h \quad m\text{-a.e.}$$

The concept of $L^0(\mu)$-universality is often referred to as Menshov universality and remains in the focus of current research as illustrated by a recent result of Limani [46] who proved a statement concerning Menshov universality for the little Bloch space. The latter space is defined as the collection of all holomorphic functions on $\mathbb{D}$ that meet the condition

$$\lim_{|z| \to 1^-} (1 - |z|^2) f'(z) = 0.$$

It is well-known that the little Bloch space is contained in $\mathcal{A}^p$ for any $p \in [1, \infty)$. In contrast to the Bergman space which exhibits functions whose sequence of Taylor coefficients is unbounded, the sequence of Taylor coefficients of any function in the little Bloch space approaches 0.

If Σ is a σ-algebra and $A \in \Sigma$, then $\Sigma_A := \{A \cap B : B \in \Sigma\}$ is the trace σ-algebra of A in Σ. If μ is a measure on Σ and if $A \in \Sigma$, then we write $\mu|_A$ for the measure μ restricted to Σ_A.

Theorem 1.2.11 (Limani, [46, Corollary 8.3])
For any $\delta \in (0, 1)$ there exists a set E with $m(E) > \delta$ such that there is a residual set of functions in the little Bloch space which are $L^0(\, m|_E\,)$-universal for the family $\{T_n : n \in \mathbb{N}\}$.

Reflecting once again on the latter three theorems, it can be said that they are all based on simultaneous approximation. In the upcoming chapters, we will deploy techniques that make it indispensable to move on to trigonometric series instead of Taylor series for obtaining results on simultaneous approximation which allow us in turn to deduce results on universality.

For this purpose, in alignment with (1.1), we need to extend the definition of monomials by defining $p_k : \mathbb{C}^* \to \mathbb{C}$,

$$p_k(z) := z^k,$$

for $k \in -\mathbb{N}$. Moreover, we write

$$\mathcal{P} := \left\{ \sum_{k=-n}^{n} a_k p_k : n \in \mathbb{N}_0, a_k \in \mathbb{C} \right\}$$

for the linear span of monomials. The elements of $\mathcal{P}$ are **trigonometric polynomials**. If the coefficients of the monomials with negative exponent are all 0, we recover **algebraic polynomials**.

We introduce another family of operators that act on the set $\mathbb{C}^{\mathbb{Z}}$. In the sequel, we refer to the elements in $\mathbb{C}^{\mathbb{Z}}$ as two-sided sequences.

Definition 1.2.12 For a given sequence $a = (a_k)_{k\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ the operators $S_n : \mathbb{C}^{\mathbb{Z}} \to \mathcal{P}$ are defined by

$$S_n(a) := \sum_{k=-n}^{n} a_k p_k \qquad (a \in \mathbb{C}^{\mathbb{Z}},\ n \in \mathbb{N}).$$

We also use the notation $a^{\smile}$ which will stand for the sequence $(S_n(a))_{n\in\mathbb{N}}$.

Furthermore, we write $S_n^+, S_n^- : \mathbb{C}^{\mathbb{Z}} \to \mathcal{P}$

$$S_n^+(a) := \sum_{k=0}^{n} a_k p_k \quad \text{and} \quad S_n^-(a) := \sum_{k=-n}^{-1} a_k p_k \qquad (a \in \mathbb{C}^{\mathbb{Z}},\ n \in \mathbb{N}),$$

respectively.

The operators in the definition above will guide us throughout this thesis and play a central role in the forthcoming theorem which goes back to Menshov and treats the question if there exists a two-sided zero tending sequence which is $L^0(m)$-universal for the family $\{S_n : n \in \mathbb{N}\}$. Menshov gave a positive answer to this question in 1945. Recall that $c_0(\mathbb{Z})$ stands for the set of all two-sided sequences which vanish as $|k| \to \infty$.

Theorem 1.2.13 (Menshov, [51])
There is a sequence $a \in c_0(\mathbb{Z})$ such that a is $L^0(m)$-universal for the family $\{S_n : n \in \mathbb{N}\}$.

To lay the groundwork for the forthcoming discussions in this work, we wish to introduce supplementary findings from [67]. These results draw attention to sequences that surpass polynomial growth. In particular, there exists a two-sided sequence $a \in \mathbb{C}^{\mathbb{Z}}$ which is $(C(\mathbb{S}), \mathcal{T}_{\text{pw}})$-universal for the family $\{S_n : n \in \mathbb{N}\}$, where $\mathcal{T}_{\text{pw}}$ denotes the topology of pointwise convergence on $C(\mathbb{S})$, even if the sequence a only has subexponential growth. The situation changes dramatically when the growth falls below $k \mapsto \exp(k^\alpha)$ for some $\alpha \in (0, 1)$. Nevertheless, our subsequent discussions will concentrate on classes of sequences that exhibit only polynomial growth.

A set $E \subset \mathbb{S}$ is called **arc** if E is connected and E is called a nontrivial arc if E contains more than one element.

Theorem 1.2.14 (Shkarin, [67, Theorem 1.6 and Theorem 1.7])
Let $(c_n)_{n\in\mathbb{N}_0}$ be a sequence in $[0, \infty)$ which satisfies

$$\lim_{n\to\infty} n(c_n - c_{n-1}) = \infty$$

and define X as the set of two-sided complex sequences $a \in \mathbb{C}^{\mathbb{Z}}$ such that $|a_k| \le \exp(c_{|k|})$ for any $k \in \mathbb{Z}$.

1) If

$$\sum_{n=0}^{\infty} \frac{c_n}{n^2+1} < \infty,$$

then for each nontrivial arc $E \subset \mathbb{S}$ the set

$$\Big\{h \in C(E) : \, S_{n_j}(a)|_E \to h \text{ pointwise on } E, \text{ for some sequence } (n_j)\Big\}$$

consists of at most one element. In particular, there exists no element $a \in X$ which is $(C(E), \mathcal{T}_{\mathrm{pw}})$-universal for the family $\{S_n : n \in \mathbb{N}\}$.

2) If

$$\sum_{n=0}^{\infty} \frac{c_n}{n^2+1} = \infty,$$

then there exists an $a \in X$ which is $(C(\mathbb{S}), \mathcal{T}_{\mathrm{pw}})$-universal for the family $\{S_n : n \in \mathbb{N}\}$.

Using the latter theorem, we aim to distinguish the difference between Menshov universality and universality with respect to uniform convergence. For $p \in [1, \infty)$, we define A^p as the collection of all complex sequences $(a_k)_{k\in\mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0}$ which satisfy

$$f = \sum_{k=0}^{\infty} a_k \, p_k|_{\mathbb{D}} \in \mathcal{A}^p.$$

We equip the set A^p with the inherited norm of $\mathcal{A}^p$, that means

$$\|a\|_{A^p} := \|f\|_{\mathcal{A}^p} \quad (a \in A^p).$$

Due to [13, §3.3 Theorem 4], any sequence (a_k) in A^p has the property

$$(a_k) \in o(k^{1/p}).$$

In particular, any sequence in A^p is of polynomial growth. Theorem 1.2.10 says that A^p has elements which are $L^0(m)$-universal for the family $\{S_n^+ : n \in \mathbb{N}\}$. However, by Theorem 1.2.14, for any nontrivial and closed arc E there is *no* $a \in A^p$ which is $(C(E), \|\cdot\|_\infty)$-universal for the family $\{S_n^+ : n \in \mathbb{N}\}$. This shows in particular that Menshov universality and universality with respect to uniform convergence fall apart for the class A^p. However, for any $p \in [1, \infty)$ there exists $E \subset \mathbb{S}$ closed with positive arc measure and an $a \in A^p$ which is $(C(E), \|\cdot\|_\infty)$-universal for the family $\{S_n^+ : n \in \mathbb{N}\}$ (see [5, Theorem 2.2]).

In the ensuing discourse, the space $(Y, \mathcal{T})$ in Definition 1.2.1 will transition into the role of $(C(E), \|\cdot\|_\infty)$, where $E \subset \mathbb{S}$ is a closed set. Note that the topology induced by uniform convergence is stronger than the topology of pointwise convergence. By invoking Theorem 1.2.14, we deduce that two-sided sequences with polynomial growth cannot be $(C(E), \|\cdot\|_\infty)$-universal if E is a nontrivial arc.

Chapter 2

Duality

2.1 Two-sided spaces of holomorphic functions

In this section, we will focus on introducing various spaces of holomorphic functions on $\mathbb{C}_\infty \setminus \mathbb{S}$. Their properties will be explored in depth in the course of this chapter. We will also recall the Fourier transform of distributions on the unit circle.

The (external) direct sum $U \oplus V$ of two vector spaces U and V over the field of real or complex numbers $\mathbb{K}$ is defined as the vector space $U \times V$ with addition

$$(u_1, v_1) + (u_2, v_2) := (u_1 + u_2, v_1 + v_2) \quad (u_1, u_2 \in U,\ v_1, v_2 \in V)$$

and scalar multiplication

$$\lambda(u, v) := (\lambda u, \lambda v) \quad (\lambda \in \mathbb{K},\ u \in U,\ v \in V).$$

In case U and V are even normed vector spaces, a norm on $U \oplus V$ is given by $\sqrt{\|\cdot\|_U^2 + \|\cdot\|_V^2}$ and if U and V are Banach spaces, then $U \oplus V$ is also a Banach space. If W is a subspace of U, U/W is the quotient space of U and W. Moreover, we write $\overline{H(\mathbb{D})}$ for the set of all antiholomorphic function on $\mathbb{D}$, that means $f \in \overline{H(\mathbb{D})}$ if and only if $\overline{f} \in H(\mathbb{D})$.

Definition 2.1.1 Let X be a subspace of $\mathbb{C}^{\mathbb{Z}}$ such that

$$\limsup_{|k| \to \infty} |a_k|^{1/|k|} \leq 1 \qquad \Big(a = (a_k)_{k \in \mathbb{Z}} \in X\Big). \tag{2.1}$$

We define the one-sided spaces

$$X_+ := \left\{ a_+ \in H(\mathbb{D}) : a_+ = \sum_{k=0}^{\infty} a_k \, p_k|_{\mathbb{D}}\,,\ a \in X \right\},$$

as well as

$$X_- := \left\{ a_- \in H(\mathbb{D}_{\text{ext}}) : a_- = \sum_{k=1}^{\infty} a_{-k} \, p_{-k}|_{\mathbb{D}_{\text{ext}}}\,,\ a \in X \right\}.$$

Moreover, we define

$$X_{\pm} := X_+ \oplus X_-. \tag{2.2}$$

It is also possible to identify the space X_- with a set of antiholomorphic functions on the unit disk $\mathbb{D}$ since, by observing the decomposition $\mathbb{D}_{\text{ext}} \ni z = r\zeta$ with $r > 1$ and $\zeta \in \mathbb{S}$, we have

$$\sum_{k=1}^{\infty} a_{-k} z^{-k} = \sum_{k=1}^{\infty} a_{-k} (r\zeta)^{-k} = \sum_{k=1}^{\infty} a_{-k} (\overline{\zeta}^k / r^k). \tag{2.3}$$

This concept is also stable concerning the radial limit. Recall that each harmonic function f on the unit disk can be uniquely decomposed into its holomorphic part $f_{\text{H}} \in H(\mathbb{D})$ and its antiholomorphic part $f_{\text{AH}} \in \overline{H(\mathbb{D})}/\mathbb{C}$ such that

$$f = f_{\text{H}} + f_{\text{AH}}. \tag{2.4}$$

In particular, we can consider $X_{\pm}$ as a space of harmonic functions on the unit disk.

Remark 2.1.2 Given the functions $a_+ \in X_+$ and $a_- \in X_-$, we can regain *uniquely* the initial sequence a with the help of the Cauchy integral formula, which means

$$a_k = -i \int_{\mathbb{S}} \frac{a_+(r\zeta)}{(r\zeta)^{k+1}} dm(\zeta) \quad (k \in \mathbb{N}_0),$$

for an arbitrary $r \in (0, 1)$, and

$$a_k = -i \int_{\mathbb{S}} \frac{a_-(r\zeta)}{(r\zeta)^{k+1}} dm(\zeta) \quad (k \in -\mathbb{N}),$$

for an arbitrary $r \in (1, \infty)$. As a result of the latter two identities, the vector spaces X and $X_{\pm}$ are isomorphic.

In the sequel, our attention is directed toward sequences of polynomial growth which is the collection of all $a \in \mathbb{C}^{\mathbb{Z}}$ such that there exists an $n \in \mathbb{N}$ with

$$\lim_{|k| \to \infty} \frac{a_k}{|k|^n} = 0.$$

Definition 2.1.3 Let $\alpha \in \mathbb{R}$. We define

$$D_\alpha := \left\{ a \in \mathbb{C}^{\mathbb{Z}} : \sum_{k=-\infty}^{\infty} (|k| + 1)^\alpha |a_k|^2 < \infty \right\}.$$

Moreover, we define $\mathcal{D}_\alpha := (D_\alpha)_{\pm}$ (cf. (2.2)). We equip the space D_α with the inner product $\langle \cdot, \cdot \rangle_{D_\alpha} : D_\alpha \times D_\alpha \to \mathbb{C}$,

$$\langle a, b \rangle_{D_\alpha} := \sum_{k=-\infty}^{\infty} (|k| + 1)^\alpha a_k \overline{b_k} \quad (a, b \in D_\alpha). \tag{2.5}$$

One can show that $(D_\alpha, \langle \cdot, \cdot \rangle_{D_\alpha})$ is a Hilbert space. In the sequel, we write $\| \cdot \|_{D_\alpha}$ for the norm induced by the inner product of D_α.

Due to Remark 2.1.2, each function in $\mathcal{D}_\alpha$ can be uniquely identified with an element contained in D_α. Thus, we can inherit the inner product and hence the corresponding norm to the space $\mathcal{D}_\alpha$ where we write $\langle \cdot, \cdot \rangle_{\mathcal{D}_\alpha}$ and $\| \cdot \|_{\mathcal{D}_\alpha}$, respectively. Consequently, the spaces D_α and $\mathcal{D}_\alpha$ are isometrically isomorphic.

We aim to revisit the Fourier transform on the unit circle. For this purpose, it is essential to introduce further notation. Let $p \in [1, \infty)$ and suppose that (X, Σ, μ) is a measure space. Then $\mathcal{L}^p(\mu)$ is the vector space of all functions $f \in \mathcal{M}(X, \Sigma)$ that satisfy

$$\|f\|_{\mathcal{L}^p(\mu)}^p := \int |f|^p d\mu < \infty.$$

It is well known that $\| \cdot \|_{\mathcal{L}^p(\mu)}$ defines a seminorm on $\mathcal{L}^p(\mu)$. The Fischer-Riesz theorem states that the quotient space of $\mathcal{L}^p(\mu)$ and the space of all functions that vanish μ-a.e. is a Banach space. We write for this space $L^p(\mu) = (L^p(\mu), \| \cdot \|_{L^p(\mu)})$ in the sequel.

Moreover, if $A \in \Sigma$ and if $f \mathbb{1}_A$ is μ-integrable we write

$$\int_A f d\mu := \int f \mathbb{1}_A d\mu,$$

where $\mathbb{1}_A$ is the indicator function of the set A.

In the further course of this thesis, certain L^p spaces will occur frequently. We shall introduce some special notation for those in the following.

Example 2.1.4 We assume $p \in [1, \infty)$ and $N \in \mathbb{N}$.

i) If the underlying measure space is given by $(\mathbb{R}^N, \mathcal{B}(\mathbb{R}^N), \lambda_N)$ where λ_N denotes the N-dimensional Lebesgue measure, then we write $L^p(\mathbb{R}^N)$ instead of $L^p(\lambda_N)$. Moreover, if Ω is an open set in $\mathbb{R}^N$, we write $L^p(\Omega)$ instead of $L^p(\, \lambda_N|_\Omega \,)$.

ii) We write $L^p(\mathbb{S})$ for the space $L^p(m)$ where we recall that m is the arc measure on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ (see Definition 1.2.4). In the space $L^p(\mathbb{S})$, the following orthogonality relation applies which will be a crucial tool in the following course of this work

$$\int p_k \, dm = \frac{1}{2\pi} \int_{(0,2\pi)} e^{ikt} d\lambda(t) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \qquad (k \in \mathbb{Z}). \tag{2.6}$$

Definition 2.1.5 1) The **Fourier transform** on the unit circle $\mathcal{F} : L^1(\mathbb{S}) \to \mathbb{C}^{\mathbb{Z}}$ is defined by

$$\mathcal{F}(h) := \left(\int h \overline{p_k} \, dm \right)_{k \in \mathbb{Z}} \qquad (h \in L^1(\mathbb{S})).$$

Furthermore, $\mathcal{F}(h)(k)$ is the k-th **Fourier coefficient**. For the sake of abbreviation, we commonly write $\widehat{h}$ instead of $\mathcal{F}(h)$ and $\widehat{h}(k)$ instead of $\mathcal{F}(h)(k)$, respectively.

2) For h in $L^1(\mathbb{S})$,

$$S_n h := S_n \widehat{h} = \sum_{k=-n}^{n} \widehat{h}(k) p_k$$

is the n-th **partial Fourier sum** of h. Following Definition 1.2.12, we write as well $\widehat{h}^{\vee}$ for $(S_n h)_{n\in\mathbb{N}}$ which is denoted as **Fourier series** of the function h.

Numerous authors have studied the Fourier transform on the unit circle and its associated properties. For an introductory overview of fundamental characteristics, we direct the reader to the works of Katznelson [34] and Koerner [38]. However, we intend to provide a concise compilation of properties of the Fourier transform which will prove necessary in subsequent discussions.

Remark 2.1.6 A fundamental property of the Fourier transform is its injectivity. Moreover, the Riemann-Lebesgue lemma asserts that the Fourier transform of an m-integrable function is a zero-tending sequence. However, the Fourier transform is not onto the set $c_0(\mathbb{Z})$. On the other hand, if the Fourier transform is considered on $L^2(\mathbb{S})$, Plancherel's theorem yields that $\mathcal{F}$ is an isometric isomorphism between the spaces $L^2(\mathbb{S})$ and D_0.

In the sequel, we aim to extend the Fourier transform to distributions. The space of **sequences of rapid decay**, denoted by $s(\mathbb{Z})$, equipped with the norms

$$\max_{0\le j\le n} \sup_{k\in\mathbb{Z}} |k|^j |a_k| \qquad \Big(a \in s(\mathbb{Z}), n \in \mathbb{N}_0\Big)$$

defines a Fréchet space.

Furthermore, $C^\infty(\mathbb{S})$ is the space of smooth functions on $\mathbb{S}$. We equip $C^\infty(\mathbb{S})$ with the norms

$$\max_{0\le j\le n} \|h^{(j)}\|_\infty \qquad \Big(h \in C^\infty(\mathbb{S}), n \in \mathbb{N}_0\Big), \tag{2.7}$$

where $h^{(j)}$ denotes the j-th derivative of the function h. In the sequel, the space $C^\infty(\mathbb{S})$ endowed with (2.7) will be denoted as **Schwartz space** (on $\mathbb{S}$).

By observing the completeness of $C^\infty(\mathbb{S})$, [34, 4.4 Theorem], and Banach's isomorphism theorem, one obtains that the Fourier transform establishes an isomorphism between $C^\infty(\mathbb{S})$ and $s(\mathbb{Z})$.

Suppose that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed vector spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The **operator norm** $\|\cdot\|_{\mathrm{op}}$ of a linear and continuous operator $T : X \to Y$ is the supremum of $\|T(x)\|_Y$ taken over all x contained in the unit ball of X. If $Y = \mathbb{K}$, then

$$X' := \Big\{\varphi : X \to \mathbb{K} : \varphi \text{ is linear and continuous}\Big\}$$

equipped with the operator norm is the (norm) **dual** of X and the elements $\varphi \in X'$ are functionals.

It is possible to extend the definition of the dual if X assumes the role of a Fréchet space. In this case, X' is the strong dual of X. We refrain from delving into further details and instead, refer the reader to [49, 25 Fréchet spaces and (DF)-spaces].

We extend the Fourier transform to the space $C^\infty(\mathbb{S})'$ whose elements are called **tempered distribution**. We occasionally write $\langle \cdot, \varphi \rangle$ instead of φ if $\varphi \in X'$.

Definition 2.1.7 Let $\varphi \in C^\infty(\mathbb{S})'$. Since the monomials are contained in $C^\infty(\mathbb{S})$ we can define the **Fourier transform** of φ by

$$\mathcal{F}(\varphi) = (\langle p_k, \varphi \rangle)_{k \in \mathbb{Z}}.$$

Again, $(\mathcal{F}(\varphi))(k)$ is the k-th **Fourier coefficient** of φ and according to Definition 2.1.5, we occasionally write $\widehat{\varphi}$ instead of $\mathcal{F}(\varphi)$ and $\widehat{\varphi}(k)$ for the k-th Fourier coefficient.

In line with Definition 2.1.5, the operator

$$S_n \varphi := \sum_{k=-n}^{n} \widehat{\varphi}(k) p_k$$

is the n-th **partial Fourier sum**, and correspondingly, $(\widehat{\varphi})^\vee$ will be denoted as the **Fourier series** of the tempered distribution φ.

We like to briefly discuss that Definition 2.1.5 is consistent with the latter definition since one can regard the elements of $L^1(\mathbb{S})$ as a subset of $C^\infty(\mathbb{S})'$ via the embedding

$$\iota : L^1(\mathbb{S}) \to C^\infty(\mathbb{S})', \qquad L^1(\mathbb{S}) \ni h \mapsto \int h \, \overline{(\cdot)} \, dm \in C^\infty(\mathbb{S})'. \tag{2.8}$$

We can ensure that the two definitions of the Fourier transform correspond since

$$(\mathcal{F}\iota(h))(k) = \langle p_k, \iota(h) \rangle = \int h \overline{p_k} \, dm = (\mathcal{F}h)(k) \quad (k \in \mathbb{Z}).$$

Thus, Definition 2.1.7 can be regarded as a generalization of Definition 2.1.5.

Next, we will show that one can characterize the range of the Fourier transform of tempered distributions. Details can be found in the following proposition.

Proposition 2.1.8 *A sequence $a = (a_k)_{k \in \mathbb{Z}} \in \mathbb{C}^\mathbb{Z}$ is of polynomial growth if and only if the sequence a is the Fourier transform of a tempered distribution.*

Proof: Let $a \in \mathbb{C}^\mathbb{Z}$ be of polynomial growth. Then we define $\varphi : C^\infty(\mathbb{S}) \to \mathbb{C}$

$$\langle h, \varphi \rangle := \sum_{k=-\infty}^{\infty} a_k \, \widehat{h}(k) \quad (h \in C^\infty(\mathbb{S})). \tag{2.9}$$

Clearly, we have that φ is linear. Since a is of polynomial growth we find an $n \in \mathbb{N}$ and a $C \in (0, \infty)$ such that

$$|\langle h, \varphi \rangle| \leq \sum_{k=-\infty}^{\infty} |a_k| \, |\widehat{h}(k)| \leq \sum_{k=-\infty}^{\infty} \frac{C}{(|k|+1)^2} \cdot (|k|+1)^{n+2} |\widehat{h}(k)|.$$

Moreover, [34, p. 25, (4.5)] yields

$$|k|^{n+2}|\widehat{h}(k)| \leq \|h^{(n+2)}\|_\infty \quad (k \in \mathbb{Z}).$$

In particular, with $M := C \cdot \sum_{k=-\infty}^{\infty} (|k|+1)^{-2} < \infty$, we have

$$|\langle h, \varphi \rangle| \leq M \cdot \sup_{k \in \mathbb{Z}} (|k|+1)^{n+2} |\widehat{h}(k)| \leq M \cdot 2^{n+2} \cdot \|h^{(n+2)}\|_\infty,$$

and by taking account of (2.7), φ is continuous and thus a tempered distribution. We obtain the "if" part by observing

$$\langle p_j, \varphi \rangle = \sum_{k=-\infty}^{\infty} a_k \widehat{p_j}(k) = \sum_{k=-\infty}^{\infty} a_k \delta_{k,j} = a_j \quad (j \in \mathbb{Z}).$$

Vice versa, if φ is a tempered distribution, that means φ is continuous on the Schwartz space $C^\infty(\mathbb{S})$, we conclude that there exists an $M \in (0, \infty)$ and an $n \in \mathbb{N}$ such that

$$|\langle p_k, \varphi \rangle| \leq M \max_{0 \leq j \leq n} \|p_k^{(j)}\|_\infty \leq M|k|^n \quad (k \in \mathbb{Z})$$

which implies in turn that $(\langle p_k, \varphi \rangle)_{k \in \mathbb{Z}}$ is of polynomial growth. □

Definition 2.1.9 Assume that $X \subset \mathbb{C}^{\mathbb{Z}}$ such that any element in X is at most of polynomial growth. We define

$$\widetilde{X} := \{\varphi \in C^\infty(\mathbb{S})' : \widehat{\varphi} \in X\}.$$

According to this definition, for $D \subset C^\infty(\mathbb{S})'$, we write $\widehat{D}$ for the range of D under the Fourier transform $\mathcal{F}$.

Due to the injectivity of the Fourier transform, we can uniquely identify each φ in $\widetilde{X}$ with its Fourier transform $\widehat{\varphi}$ in X.

Example 2.1.10 i) We write $c_{00}(\mathbb{Z})$ for the set of two-sided sequences with finite support. We have

$$(c_{00}(\mathbb{Z}))^\vee = \left\{\varphi \in C^\infty(\mathbb{S})' : \widehat{\varphi} \in c_{00}(\mathbb{Z})\right\} = \left\{ \sum_{k \in \mathrm{supp}(\widehat{\varphi})} \widehat{\varphi}(k) p_k : \widehat{\varphi} \in c_{00}(\mathbb{Z}) \right\}.$$

Due to the embedding in (2.8) and since $\widehat{\varphi} \in c_{00}(\mathbb{Z})$ has finite support, we can identify the space $(c_{00}(\mathbb{Z}))^\vee$ with the trigonometric polynomials $\mathcal{P}$ on $\mathbb{S}$.

ii) Due to Plancherel's theorem, any element of $\widetilde{D_0}$ corresponds to a function $h \in L^2(\mathbb{S})$. In particular, for each $\alpha \geq 0$, $\widetilde{D_\alpha}$ can be identified with a subset of $L^2(\mathbb{S})$.

One can show that for any $\alpha < 0$ there exists a sequence $a \in D_\alpha$ such that a is not the Fourier transform of an m-integrable function. In particular, we have $\widetilde{D_\alpha} \not\subset L^1(\mathbb{S})$.

Proposition 2.1.11 *For all $\alpha \in \mathbb{R}$, the set $c_{00}(\mathbb{Z})$ is dense in $(D_\alpha, \|\cdot\|_{D_\alpha})$. In particular, D_α is separable.*

Proof: Let $\alpha \in \mathbb{R}$ and $a \in D_\alpha$. For any $\varepsilon > 0$ there exists an $n \in \mathbb{N}$ such that

$$\sum_{|k|>n} (|k|+1)^\alpha |a_k|^2 < \varepsilon. \tag{2.10}$$

Thus, we only need to set $b := a \cdot \mathbb{1}_{\{-n,\dots,n\}}$ which clearly has finite support and due to (2.10), we have $\|a-b\|_{D_\alpha} < \varepsilon$ which yields the assertion. □

Note that Example 2.1.10 in combination with Proposition 2.1.11 leads to the conclusion that the trigonometric polynomials $\mathcal{P}$ are dense in $\widetilde{D_\alpha}$, and thus the latter is in particular separable.

We want to highlight a connection between the preimage of D_α under the Fourier transform and the spaces $\mathcal{D}_\alpha$. A major tool will play Fatou's jump theorem which can be regarded as the analog of the Plemelj formulas for half-planes, see for example [47].

Remark 2.1.12 Let $\alpha \geq 0$. As mentioned in Example 2.1.10, we can associate to each element contained in $\widetilde{D_\alpha}$ an $L^2(\mathbb{S})$ function. We consider the **Cauchy transform** $C : L^1(\mathbb{S}) \to H(\mathbb{C}_\infty \setminus \mathbb{S})$ which is defined by

$$(Ch)(z) = \int \frac{h(\zeta)}{1-z\overline{\zeta}}\, dm(\zeta) \quad (h \in L^1(\mathbb{S})).$$

The function Ch is called the Cauchy transform of h. If one applies the geometric series expansion of the Cauchy kernel $\zeta \mapsto (1-z\overline{\zeta})^{-1}$ it follows that C maps $\widetilde{D_\alpha}$ bijective to $\mathcal{D}_\alpha$. By recalling Definition 2.1.1, we obtain

$$\widehat{h}_+ = Ch|_{\mathbb{D}} \quad \text{and} \quad \widehat{h}_- = Ch|_{\mathbb{D}_{\text{ext}}}.$$

It is known that $\widehat{h}_+$ and $\widehat{h}_-$ have nontangential interior and exterior limits, respectively (see e.g. [10, p. 54]). Beyond this, Fatou's jump theorem (cf. [10, p. 55]) permits us to regain the initial function, namely, we have

$$h(\zeta) = \lim_{r \nearrow 1} \left(\widehat{h}_+(r\zeta) - \widehat{h}_-(\zeta/r) \right) \quad \text{for } m\text{-almost every } \zeta \in \mathbb{S}.$$

In the case $\widehat{h}(k) = 0$ for $k \in \mathbb{Z} \setminus \mathbb{N}_0$, we get the Fatou theorem which says that the radial limit of any function in $\mathcal{H}^p$ exists m-almost everywhere and the radial limit function is contained in $L^p(\mathbb{S})$.

Let us point out that the assertions in the remark above cannot hold for $\alpha < 0$ since here, D_α contains sequences that do not emanate of an m-integrable function.

In the following remark, we want to demonstrate a connection between weighted Bergman spaces and the spaces $(D_\alpha)_+$. We will mainly refer to [33] and also to [70] where a generalization of weighted Bergman spaces was discussed.

Remark 2.1.13 i) The definition of Bergman spaces (see Definition 1.2.9) on the unit disk can be extended as follows. Let $\beta > -1$. Then the weighted Bergman space $\mathcal{A}^p_\beta$ is the set of all holomorphic functions in $H(\mathbb{D})$ with

$$\|f\|^p_{\mathcal{A}^p_\beta} := \int_{\mathbb{D}} |f|^p (1 - |\cdot|^2)^\beta dm_2 < \infty, \tag{2.11}$$

where $\beta = 0$ gives us the classical Bergman space $\mathcal{A}^p$. These spaces were considered in [4], [33] and [70]. In the sequel, we focus on the case $p = 2$ which is the Hilbert space case. Recall that Γ denotes the gamma function (see Appendix C). We use [33, p. 4] to infer that

$$e_k(z) = \left(\frac{\Gamma(k + 2 + \beta)}{k!\Gamma(2 + \beta)} \right)^{1/2} z^k \quad (k \in \mathbb{N}_0)$$

defines an orthonormal basis in $\mathcal{A}^2_\beta$. Thanks to Parseval's identity, we can deduce

$$\|f\|^2_{\mathcal{A}^2_\beta} = \sum_{k=0}^{\infty} \frac{k!\Gamma(2 + \beta)}{\Gamma(k + 2 + \beta)} |a_k|^2,$$

where a_k denotes the k-th Taylor coefficient of the function f about the point 0. Due to Stirling's formula, we obtain an $M \in (1, \infty)$ only depending on β such that

$$\frac{1}{M}(k + 1)^{-\beta-1} \leq \frac{k!\Gamma(2 + \beta)}{\Gamma(k + 2 + \beta)} \leq M(k + 1)^{-\beta-1} \quad (k \in \mathbb{N}_0), \tag{2.12}$$

and therefore as well

$$\frac{1}{M} \sum_{k=0}^{\infty} (k + 1)^{-\beta-1} |a_k|^2 \leq \|f\|^2_{\mathcal{A}^2_\beta} \leq M \sum_{k=0}^{\infty} (k + 1)^{-\beta-1} |a_k|^2.$$

In particular, we have

$$(D_\alpha)_+ = \mathcal{A}^2_{-(\alpha+1)} \quad (\alpha > 0). \tag{2.13}$$

ii) It is not possible to extend the definition of weighted Bergman spaces (cf. 2.11) by considering $\beta \leq -1$ since this space only contains the zero function. However, Zhao and Zhu (see [70, p. 2] and [70, Corollary 11]) tackled this problem by observing that $f \in \mathcal{A}^p_\beta$ if and only if the k-th radial derivative of f weighted with $z \mapsto (1 - |z|)^{k+\beta}$ is m_2-integrable. This enables considering weighted Bergman spaces for $\beta \in \mathbb{R}$. By [70, Theorem 40] and an application of Stirling's formula as in (2.12), it follows that $f \in \mathcal{A}^p_\beta$ if and only if the Taylor coefficients of f satisfy

$$\sum_{k=0}^{\infty} (k + 1)^{-\beta-1} |a_k|^2 < \infty,$$

and therefore the identity in (2.13) holds for $\alpha \in \mathbb{R}$.

By the latter remark, it follows immediately that a function $g \in H(\mathbb{C}_\infty \setminus \mathbb{S})$ is contained in $\mathcal{D}_\alpha$ if and only if

$$g|_{\mathbb{D}} \in \mathcal{A}^2_{-(\alpha+1)} \qquad \text{and} \qquad g|_{\mathbb{D}_{\text{ext}}} \circ p_{-1} \in \mathcal{A}^2_{-(\alpha+1)}.$$

It is noteworthy that the class $(D_\alpha)_+$ encompasses other well-known holomorphic spaces. Specifically, for $\alpha = 0$, $(D_\alpha)_+$ is isometrically isomorphic to the Hardy space $\mathcal{H}^2$ (see also [12, p. 93]). Since $(D_1)_+ = \mathcal{A}^2_{-2}$ due to Remark 2.1.13, we get by integration with polar coordinates and Parseval's formula

$$(D_1)_+ = \mathcal{D}.$$

The reader is also referred to [16, Theorem 1.1.2] for a more detailed proof.

2.2 Harmonic Dirichlet spaces

In this section, we investigate another class of two-sided sequences $a \in \mathbb{C}^{\mathbb{Z}}$. In contrast to the approach we pursued in the foregoing section where the functions of the class $\mathcal{D}_\alpha$ were characterized by a growth condition of its coefficients, here, the coefficients will be given indirectly via an integral condition on harmonic functions on $\mathbb{D}$ which can also be considered as a subset of $H(\mathbb{C}_\infty \setminus \mathbb{S})$ via (2.3).

Suppose $p \in [1, \infty)$ and $\Omega \subset \mathbb{R}^N$ open. Then the **Sobolev space** $W^{1,p}(\Omega)$ is defined as the collection of all $f \in L^p(\Omega)$ whose (weak) partial derivatives are also contained in $L^p(\Omega)$ endowed with the Sobloev norm $\|\cdot\|_{W^{1,p}(\Omega)}$ given by

$$\|f\|_{W^{1,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)} \quad (f \in W^{1,p}(\Omega)),$$

where ∇f denotes the (weak) gradient of the function f. Let us mention that $W^{1,p}(\Omega)$ defines a Banach space (see e.g. [43, Theorem 11.12 i)]). Moreover, by [43, Theorem 11.35]), it follows in particular that $C(\mathrm{cl}(\Omega)) \cap C^\infty(\Omega)$ forms a dense set in $W^{1,p}(\Omega)$. We refrain from presenting further properties of Sobolev spaces and refer the reader to [43, Chapter 11] which encompasses a more detailed introduction. With this in mind, we will first recall a generalization of the Dirichlet space, and then we introduce harmonic Dirichlet spaces. Note that we identify $\mathbb{R}^2$ with the complex plane. Especially the case $\Omega = \mathbb{D}$ is of our interest.

Definition 2.2.1 Suppose $p \in [1, \infty)$.

1) The space $\mathcal{D}^p$ is defined as the set of all holomorphic functions on $\mathbb{D}$ whose derivative is contained in the Bergman space $\mathcal{A}^p$. Equipped with the norm

$$\|f\|_{\mathcal{D}^p} := \left(\int |(p_1 f)'|^p \, dm_2 \right)^{1/p},$$

$\mathcal{D}^p$ becomes a Banach space. In the sequel, we refer to $\mathcal{D}^p$ as **L^p-type Dirichlet space** or shorter Dirichlet space (with parameter p).

2) We define the **harmonic Dirichlet space** $\mathcal{D}^p_{\text{harm}}$ as the collection of all harmonic functions contained in the Sobolev space $W^{1,p}(\mathbb{D})$ equipped with the norm $\|\cdot\|_{W^{1,p}(\mathbb{D})}$. We will often write $\|\cdot\|_{\mathcal{D}^p_{\text{harm}}}$ instead of $\|\cdot\|_{W^{1,p}(\mathbb{D})}$.

At this place, we remark that $\mathcal{D}^p$ is also the space of all holomorphic functions in $W^{1,p}(\mathbb{D})$ which is a consequence of the Cauchy-Riemann equations. In case $p = 2$, we recover the classical Dirichlet space $\mathcal{D}$. Furthermore,

$$\mathcal{H}^2 = (D_0)_+ \supset (D_1)_+ = \mathcal{D}.$$

By Douglas' formula (see e.g. [16, Theorem 1.5.1]), it follows that a function $f \in \mathcal{H}^2$ is contained in the Dirichlet space $\mathcal{D}$ if its radial limit function f^* satisfies

$$\int\int \left|\frac{f^*(z) - f^*(w)}{z - w}\right|^2 dm(z)dm(w) < \infty.$$

One can even show that $\mathcal{D}^p \subset \mathcal{H}^p$ for any $p \in (1, \infty)$ (see e.g. [11, p. 88]). Thus, each function in $\mathcal{D}^p$ can be identified with its corresponding radial limit function on $\mathbb{S}$. This observation allows us to define D^p as the preimage of $\mathcal{D}^p$ under the Fourier transform. We always equip this space with the inherited norm of the space $\mathcal{D}^p$.

The Hilbert space case $\mathcal{D}^2_{\text{harm}}$ has received considerable attention in the literature (see for instance [66, Section 2]). The focus of these studies is on the boundary behavior and the Fourier series of functions within this space. Significant focus has been placed on the spaces $\mathcal{D}^p_{\text{harm}}$. In the subsequent discussion, we will provide a concise summary of their relevant properties.

First, we remark that $\mathcal{D}^p_{\text{harm}}$ is a Banach space as the kernel of the Laplacian, an elliptic operator on $W^{1,p}(\mathbb{D})$. An additional point of interest is the relationship between the functions in $\mathcal{D}^p_{\text{harm}}$ and their corresponding radial limit functions. According to [17, 5.5 Traces, Theorem 1], these radial limit functions are always contained in $L^p(\mathbb{S})$. However, by additional arguments of trace theory, it can be shown that the radial limit functions of elements in $\mathcal{D}^p_{\text{harm}}$ are contained in a specific type of Besov spaces. These spaces are often referred to as Sobolev-Slobodeckij spaces.

For $p \in [1, \infty)$, we define $\mathcal{B}^p_{1-1/p}(\mathbb{S})$ as the space of all $h \in L^p(\mathbb{S})$ which fulfill

$$\int\int \left|\frac{h(z) - h(w)}{z - w}\right|^p dm(z)dm(w) < \infty. \tag{2.14}$$

This space is equipped with the norm $\|\cdot\|_{\mathcal{B}^p_{1-1/p}(\mathbb{S})}$ defined as the sum of the $L^p(\mathbb{S})$ norm and the p-th root of the expression in (2.14).

Remark 2.2.2 Let $p \in (1, \infty)$. Since $\mathbb{S} \subset \mathbb{C}$ is a smooth curve we can use a result of Harjulehto [30, p. 2374] which implies that there exists a linear and continuous operator

$$\text{Tr} : W^{1,p}(\mathbb{D}) \to \mathcal{B}^p_{1-1/p}(\mathbb{S})$$

such that

$$\mathrm{Tr}(f) = f_{|\mathbb{S}} \quad \Big(f \in C(\mathrm{cl}(\mathbb{D})) \cap W^{1,p}(\mathbb{D})\Big).$$

In particular, Tr maps the subspace $\mathcal{D}^p_{\mathrm{harm}}$ continuously to $\mathcal{B}^p_{1-1/p}(\mathbb{S})$. Here, we can infer that Tr restricted to $\mathcal{D}^p_{\mathrm{harm}}$ is a one-to-one mapping which is a consequence of $\mathcal{B}^p_{1-1/p}(\mathbb{S}) \subset L^p(\mathbb{S})$ and the uniqueness result for the Dirichlet problem on the unit disk $\mathbb{D}$ (compare [64, 11.9 Theorem]). As previously discussed, in case $p = 2$, it follows by Douglas' formula that Tr is onto (see also [66, p. 38]), that means

$$\mathrm{Tr} : W^{1,2}(\mathbb{D}) \to \mathcal{B}^2_{1/2}(\mathbb{S})$$

is an isomorphism. Whether Tr is also an isomorphism in case $p \neq 2$ remains an open question to the best of our knowledge.

In the sequel, we write D^p_{harm} for the space of all two-sided complex sequences that emanate from the coefficients of a harmonic function contained in $\mathcal{D}^p_{\mathrm{harm}}$. Since a harmonic function is uniquely identified by its coefficients we are allowed to inherit the norm of $\mathcal{D}^p_{\mathrm{harm}}$ to D^p_{harm} which makes it a Banach space. As a consequence of Remark 2.2.2, we obtain the inclusion $D^p_{\mathrm{harm}} \subset (\mathcal{B}^p_{1-1/p}(\mathbb{S}))^\wedge$ for any $p \in (1, \infty)$.

In the sequel, a function f is said to be a **harmonic polynomial**, if there exists a sequence $(a_k)_{k\in\mathbb{Z}} \in c_{00}(\mathbb{Z})$, such that

$$f = \sum_{k\geq 0} a_k\, p_k + \sum_{k<0} a_k\, \overline{p_{-k}}.$$

Proposition 2.2.3 *For all $p \in [1, \infty)$, the harmonic polynomials are dense in $\mathcal{D}^p_{\mathrm{harm}}$.*

Proof: Let $p \in [1, \infty)$. We perform this proof in two steps. First, we show for all $f \in W^{1,p}(\mathbb{D})$ that

$$f_r \to f \quad (r \to 1) \tag{2.15}$$

in $W^{1,p}(\mathbb{D})$ where $f_r := f(r\,\cdot)$ for $r \in (0, 1)$. For this purpose, we consider the family of operators $T_r : W^{1,p}(\mathbb{D}) \to W^{1,p}(\mathbb{D})$ defined by

$$T_r(f) := f(r\cdot).$$

The substitution $z \mapsto z/r$ reveals for all $f \in W^{1,p}(\mathbb{D})$ and $r \in (0, 1)$ the following equations

$$\|T_r(f)\|^p_{L^p(\mathbb{D})} = r \int_{r\mathbb{D}} |f|^p dm_2 \qquad \text{and} \qquad \|\nabla T_r(f)\|^p_{L^p(\mathbb{D})} = r^{p+1} \int_{r\mathbb{D}} |\nabla f|^p dm_2.$$

Consequently,

$$\|T_r(f)\|^p_{W^{1,p}(\mathbb{D})} \leq \|f\|^p_{W^{1,p}(\mathbb{D})} \quad (f \in W^{1,p}(\mathbb{D})),$$

and therefore the family $\{T_r : r \in (0, 1)\}$ is uniformly bounded. Moreover, the limit in (2.15) holds for $C(\mathrm{cl}(\mathbb{D})) \cap C^\infty(\mathbb{D})$ which is dense in $W^{1,p}(\mathbb{D})$ due to [43, Theorem 11.35].

By an application of the Banach-Steinhaus theorem, we deduce that the limit in (2.15) holds for each $f \in W^{1,p}(\mathbb{D})$.

Now, it remains to show that for any harmonic function g on an open superset U of $\mathrm{cl}(\mathbb{D})$, there exists a sequence of harmonic polynomials (q_n) on $\mathbb{D}$ such that

$$\| g|_{\mathbb{D}} - q_n\|_{\mathcal{D}^p_{\mathrm{harm}}} \to 0 \quad (n \to \infty).$$

If g is harmonic on U, then $g|_{\mathbb{S}}$ is smooth and thus its corresponding Fourier series, denoted by $(b_k)_{k\in\mathbb{Z}}$, is of rapid decay. We define g_n as the partial sums of g, which means

$$g_n := \sum_{k=0}^{n} b_k p_k + \sum_{k=1}^{n} b_{-k}\overline{p_k} \quad (n \in \mathbb{N}).$$

According to

$$\|p_k\|_{\mathcal{D}^p_{\mathrm{harm}}} \leq 2k+1 \quad (k \in \mathbb{N}_0),$$

we derive

$$\|g - g_n\|_{\mathcal{D}^p_{\mathrm{harm}}} \leq \sum_{|k|>n} |b_k|(2k+1).$$

As the sequence $(b_k)_{k\in\mathbb{Z}}$ is of rapid decay, the right-hand side converges to 0 as n tends to ∞ which yields the desired result. □

We aim to introduce another key concept. More precisely, we revisit the Bergman projection on the unit disk $\mathbb{D}$. Recall that a linear function $T : X \to X$ on the normed vector space X is a projection if $T \circ T = T$, (see e.g. [64, 5.15 Projections]).

Moreover, we recall an orthogonality relation on $\mathbb{D}$, analogous to (2.6). Here, we only consider algebraic monomials and their complex conjugate.

Let $j, k \in \mathbb{N}_0$. By definition of the area measure m_2, we obtain by using polar coordinates the following identity

$$\int_{\mathbb{D}} p_k\overline{p_j}\, dm_2 = 2\int_0^1 r^{k+j} \int_{\mathbb{S}} \zeta^{k-j} dm(\zeta)\, rdr = \frac{1}{j+1}\delta_{k,j}. \tag{2.16}$$

Consequently, for all $j \in \mathbb{N}_0$, the following equation holds

$$\int_{\mathbb{D}} w^j \cdot \sum_{k=0}^{\infty}(k+1)z^k\overline{w^k}\, dm_2(w) = z^j \quad (z \in \mathbb{D}).$$

This motivates the definition below.

Definition 2.2.4 The **Bergman kernel** $B : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ is defined by

$$B(z,w) := \sum_{k=0}^{\infty}(k+1)(z\overline{w})^k = \frac{1}{(1-z\overline{w})^2}.$$

For $p \in (1,\infty)$ the corresponding **Bergman projection** $P : L^p(\mathbb{D}) \to L^p(\mathbb{D})$ is defined by

$$P(f)(z) := \int_{\mathbb{D}} f(w)B(z,w)dm_2(w).$$

The Bergman projection is a *bounded* operator which maps $L^p(\mathbb{D})$ onto $\mathcal{A}^p$. Moreover,

$$P|_{\mathcal{A}^p} = \mathrm{id}|_{\mathcal{A}^p}.$$

The latter result can be found in [13, Chapter 2]. For further properties of the Bergman projection, we refer to [13].

It will be convenient to clarify how the Bergman projection acts on the conjugated, nonconstant monomials. If we follow the definition of the Bergman projection, we obtain by similar reasoning as in (2.16) the following identity

$$P(\overline{p_j})(z) = \sum_{k=0}^{\infty} \left(\frac{1}{k+1} \int_{\mathbb{D}} \overline{p_k} \cdot \overline{p_j} dm_2 \right) z^k = \delta_{0,j} = 0 \quad (j \in \mathbb{N}).$$

We like to work out a decomposition of $\mathcal{D}^p_{\mathrm{harm}}$ which can be seen as an analog to the Riesz projection theorem for $L^p(\mathbb{S})$ established by M. Riesz [62] in case that $p \in (1, \infty)$. The reader is also referred to [10, Theorem 3.2.1]. We emphasize that this theorem holds *not* if $p = 1$ or $p = \infty$. An explicit example in the case of $p = 1$ is presented in [10, p. 68]. It comes as a surprise that the analog of the Riesz projection theorem holds for the space of harmonic functions in $L^p(\mathbb{D})$ even if $0 < p \leq 1$ which goes back to a result of Hardy and Littlewood (see for instance [13, p. 54]).

Fortunately, the Bergman projection in combination with the space $W^{1,p}(\mathbb{D})$ attracted much attention in recent years which allows us to fall back to known results. We previously recall the definition of the internal direct sum and the decomposition theorem for projectors.

Definition 2.2.5 Let $(X, \|\cdot\|)$ be a normed vector space and suppose that E is a closed subspace in X. The space E is said to be complemented in X if there exists a closed subspace $F \subset X$ with the properties

1) $X = E + F$,

2) $E \cap F = \{0\}$.

In this case X is denoted as the (internal) direct sum of E and F and we use the notation

$$X = E \oplus F.$$

In this context, we recall that a continuous projection T on a normed vector space X generates the following decomposition (see e.g. [64, 5.16 Theorem (a)])

$$X = \ker(T) \oplus \mathrm{im}(T). \tag{2.17}$$

If X is complete and if $X = E \oplus F$, one obtains by an application of the closed graph theorem that the projection T with kernel E and range F is continuous (see e.g. [64, 5.16 Theorem (b)]).

Remark 2.2.6 First, by [45, Theorem 5] (see as well [4, Corollary 6.3]), it follows that the Bergman projection is a continuous projection in $W^{1,p}(\mathbb{D})$ with

$$P|_{\mathcal{D}^p} \,=\, \mathrm{id}|_{\mathcal{D}^p}\,,$$

in case $p \in (1,\infty)$. In particular, since $\mathcal{D}^p = \mathcal{A}^p \cap \mathcal{D}^p_{\mathrm{harm}}$, $P|_{\mathcal{D}^p_{\mathrm{harm}}}$ is a continuous projection in $\mathcal{D}^p_{\mathrm{harm}}$ onto $\mathcal{D}^p$. We observe that this result is a nontrivial extension of [13, Chapter 2, Theorem 5] where we recall that the derivative is involved due to the definition of the $W^{1,p}(\mathbb{D})$ norm. Hence, by using the decomposition theorem for continuous projectors (see (2.17)), we infer that the space $\mathcal{D}^p_{\mathrm{harm}}$ can be represented in the following way

$$\mathcal{D}^p_{\mathrm{harm}} = \mathrm{im}\Big(\, P|_{\mathcal{D}^p_{\mathrm{harm}}}\,\Big) \oplus \ker\Big(\, P|_{\mathcal{D}^p_{\mathrm{harm}}}\,\Big).$$

Let us determine the kernel of the Bergman projection. P and the identity I are bounded operators in $\mathcal{D}^p_{\mathrm{harm}}$, so $I-P$ is also bounded. Due to the identity

$$\ker\Big(\, P|_{\mathcal{D}^p_{\mathrm{harm}}}\,\Big) = \mathrm{im}\Big(\, (I-P)|_{\mathcal{D}^p_{\mathrm{harm}}}\,\Big),$$

and the fact that each harmonic function on $\mathbb{D}$ can be uniquely decomposed in the sense of (2.4), we deduce by continuity of $I-P$ that

$$\ker\Big(\, P|_{\mathcal{D}^p_{\mathrm{harm}}}\,\Big) = \Big\{ f \in \mathcal{D}^p_{\mathrm{harm}} : f \in \overline{H(\mathbb{D})}/\mathbb{C} \Big\}.$$

In the sequel, we dedicate ourselves to determine the dual space of $\mathcal{D}^p_{\mathrm{harm}}$. We start with the following remark which draws attention to the dual space of $\mathcal{D}^p$. Henceforth, q denotes the conjugate exponent of p which means $1/p + 1/q = 1$ whenever $p \in (1,\infty)$.

Remark 2.2.7 Let $p \in (1,\infty)$. Due to [33, Theorem 1.16] the norm dual of $\mathcal{A}^p$ can be represented by $\mathcal{A}^q$ under the integral pairing

$$\langle f, g\rangle := \int_{\mathbb{D}} f\,\overline{g}\; dm_2 \quad (f \in \mathcal{A}^p,\ g \in \mathcal{A}^q). \tag{2.18}$$

Furthermore, we have that

$$T : \mathcal{D}^p \to \mathcal{A}^p, \quad T(f) := (p_1 f)' \tag{2.19}$$

is an isometric isomorphism, that means $\mathcal{D}^p$ and $\mathcal{A}^p$ are isometrically isomorphic. It follows that the norm dual of $\mathcal{D}^p$ can also be represented by $\mathcal{A}^q$ via the pairing

$$\langle f, g\rangle_{\mathcal{D}^p} := \int_{\mathbb{D}} (p_1 f)' \cdot \overline{g}\; dm_2 \quad (f \in \mathcal{D}^p,\ g \in \mathcal{A}^q). \tag{2.20}$$

We shall now concentrate on the kernel of the Bergman projection of $\mathcal{D}^p_{\text{harm}}$, namely the set of all antiholomorphic functions in $\mathcal{D}^p_{\text{harm}}$ that vanish at 0. We like to elaborate on a representation of its dual space with the help of the fact that we have already found a representation of the dual space of $\mathcal{D}^p$. In alignment with [66, p. 37], we utilize the following notation for the subsequent antiholomorphic function spaces

$$\overline{\mathcal{D}^p} = \{\overline{f} : f \in \mathcal{D}^p\} \qquad \text{as well as} \qquad \overline{\mathcal{A}^q} = \{\overline{g} : g \in \mathcal{A}^q\}$$

for $p \in (1, \infty)$ and q in the role of the conjugate exponent of p. These spaces will inherit the norm of $\mathcal{D}^p$ and $\mathcal{A}^q$, respectively, that means

$$\|f\|_{\overline{\mathcal{D}^p}} := \|\overline{f}\|_{\mathcal{D}^p} \qquad \text{and} \qquad \|g\|_{\overline{\mathcal{A}^q}} := \|\overline{g}\|_{\mathcal{A}^q} \qquad (f \in \overline{\mathcal{D}^p},\ g \in \overline{\mathcal{A}^q}).$$

Therefore, if $p \in (1, \infty)$, the spaces $\overline{\mathcal{D}^p}$ and $\overline{\mathcal{A}^q}$ are isometrically isomorphic to $\mathcal{D}^p$ and $\mathcal{A}^q$, respectively via the mapping $f \mapsto \overline{f}$. Consequently, the dual space of $\overline{\mathcal{D}^p}$ can be identified with $\overline{\mathcal{A}^q}$ via the pairing

$$\langle f, g\rangle_{\overline{\mathcal{D}^p}} := \left\langle \overline{f}, \overline{g} \right\rangle_{\mathcal{D}^p} \qquad (f \in \overline{\mathcal{D}^p},\ g \in \overline{\mathcal{A}^q}). \tag{2.21}$$

We want to determine the dual space of $\ker\Big(P|_{\mathcal{D}^p_{\text{harm}}} \Big)$ which is given by the quotient space $\overline{\mathcal{D}^p}/\mathbb{C}$. The dual space of a quotient space is given by the annihilator of the subspace. This applied to our case means that we need to determine the set of all $g \in \overline{\mathcal{A}^q}$ that fulfill for any $c \in \mathbb{C}$ the equation

$$\langle c, g\rangle_{\overline{\mathcal{D}^p}} = \langle \overline{c}, \overline{g}\rangle_{\mathcal{D}^p} = \overline{c} \int_{\mathbb{D}} g dm_2 = 0. \tag{2.22}$$

Due to (2.16) the latter equation holds if and only if $g(0) = 0$. In other words, we obtain

$$\ker\Big(P|_{\mathcal{D}^p_{\text{harm}}} \Big)' = \overline{\mathcal{A}^q}/\mathbb{C}. \tag{2.23}$$

To determine the dual space of $\mathcal{D}^p_{\text{harm}}$, we need the following remark which draws attention to the dual space of the direct sum of two normed vector spaces.

Remark 2.2.8 Suppose that X and Y are normed vector spaces. Then $T : (X \oplus Y)' \to X' \oplus Y'$ defined by

$$T(u) = (u \circ \iota_X, u \circ \iota_Y)$$

is an isomorphism. Here, ι_X and ι_Y are the canonical projections onto X and Y, respectively, that means

$$\iota_X((x, y)) = (x, 0) \quad \text{and} \quad \iota_Y((x, y)) = (0, y) \qquad ((x, y) \in X \oplus Y).$$

By combining (2.23) with (2.20), we obtain due to Remark 2.2.8 the following representation of the dual space of $\mathcal{D}^p_{\text{harm}}$. Note that we write $P_- := I - P$ in the sequel.

Proposition 2.2.9 *For all $p \in (1,\infty)$ the dual space of $\mathcal{D}^p_{\mathrm{harm}}$ is given by $\mathcal{A}^q \oplus \overline{\mathcal{A}^q}/\mathbb{C}$ under the pairing*

$$\langle f, g\rangle_{\mathcal{D}^p_{\mathrm{harm}}} := \langle Pf, Pg\rangle_{\mathcal{D}^p} + \langle P_- f, P_- g\rangle_{\overline{\mathcal{D}^p}} \qquad \Big(f \in \mathcal{D}^p_{\mathrm{harm}},\ g \in \mathcal{A}^q \oplus \overline{\mathcal{A}^q}/\mathbb{C}\Big). \tag{2.24}$$

So far, our focus has been solely on Bergman spaces on the unit disk. However, for the steps that follow, it becomes necessary to extend its definition to $\mathbb{D}_{\mathrm{ext}}$.

Definition 2.2.10 Let $p \in [1,\infty)$. The space $\mathcal{A}^p_{\mathrm{ext}}$ is defined as the collection of all holomorphic functions $f : \mathbb{D}_{\mathrm{ext}} \to \mathbb{C}$ which satisfy

$$\|f\|_{\mathcal{A}^p_{\mathrm{ext}}} := \left(\int_{\mathbb{D}_{\mathrm{ext}}} |f(z)|^p \, \frac{d\lambda_2(z)}{\pi|z|^4}\right)^{1/p} < \infty.$$

One can show that $\mathcal{A}^p_{\mathrm{ext}}$ together with $\|\cdot\|_{\mathcal{A}^p_{\mathrm{ext}}}$ is a Banach space. We call the space $\mathcal{A}^p_{\mathrm{ext}}$ **outer Bergman space** (with parameter p).

Recall that $A^p \subset \mathbb{C}^{\mathbb{N}_0}$ is the set of all sequences that emanate from the Taylor coefficients of a function contained in $\mathcal{A}^p$. In the sequel, we write A^p_{ext} for the set of all sequences in $\mathbb{C}^{-\mathbb{N}}$ such that $(a_{-k})_{k\in-\mathbb{N}} \in A^p/\mathbb{C}$. We like to obtain a relation between A^p_{ext} and $\mathcal{A}^p_{\mathrm{ext}}$. The following remark serves as preparation. It highlights a connection between the spaces $\mathcal{A}^p/\mathbb{C}$ and $\mathcal{A}^p_{\mathrm{ext}}$.

Remark 2.2.11 Let $p \in [1,\infty)$ and suppose that $f \in \mathcal{A}^p_{\mathrm{ext}}$. By the substitution $z \mapsto 1/z$, we obtain

$$\|f\|^p_{\mathcal{A}^p_{\mathrm{ext}}} = \int_{\mathbb{D}_{\mathrm{ext}}} |f(z)|^p \, \frac{d\lambda_2(z)}{\pi|z|^4} = \int_{\mathbb{D}} |f(z^{-1})|^p \, \frac{d\lambda_2}{\pi}(z) = \big\|f \circ p_{-1}\big\|^p_{\mathcal{A}^p}.$$

We observe that any function in $\mathcal{A}^p_{\mathrm{ext}}$ vanishes as z tends to ∞. Hence,

$$T : \mathcal{A}^p_{\mathrm{ext}} \to \mathcal{A}^p/\mathbb{C}, \quad f \mapsto f \circ p_{-1}$$

is an isometric isomorphism. In particular, $\mathcal{A}^p_{\mathrm{ext}}$ and $\overline{\mathcal{A}^p}/\mathbb{C}$ are isometrically isomorphic.

Due to the definition of A^p_{ext}, Definition 2.1.1, and the forgoing remark, we immediately obtain the following result.

Proposition 2.2.12 *For all $p \in [1,\infty)$, the spaces $\mathcal{A}^p_{\mathrm{ext}}$, $\overline{\mathcal{A}^p}/\mathbb{C}$, and A^p_{ext} are isometrically isomorphic. Moreover,*

$$(A^p_{\mathrm{ext}})_- = \mathcal{A}^p_{\mathrm{ext}}.$$

The latter proposition yields for any $q \in (1,\infty)$ that the spaces $A^q \oplus A^q_{\mathrm{ext}}$ and $\mathcal{A}^q \oplus \overline{\mathcal{A}^q}/\mathbb{C}$ are isometrically isomorphic. By Proposition 2.2.9, we deduce that $A^q \oplus A^q_{\mathrm{ext}}$ is the dual space of D^p_{harm} under the pairing

$$\langle a, b\rangle_{D^p_{\mathrm{harm}}} := \langle f, g\rangle_{\mathcal{D}^p_{\mathrm{harm}}}, \tag{2.25}$$

where $a \in D^p_{\mathrm{harm}}$ and $b \in A^q \oplus A^q_{\mathrm{ext}}$ are the corresponding elements of $f \in \mathcal{D}^p_{\mathrm{harm}}$ and $g \in \mathcal{A}^q \oplus \overline{\mathcal{A}^q}/\mathbb{C}$, respectively.

2.3 The Cauchy dual

In the following, the Cauchy dual will be of particular importance to us since it is one of the essential tools we need to prove our universality results later. Building on this, we will classify this concept and derive the Cauchy dual of the class D_α and $\mathcal{D}^p_{\mathrm{harm}}$. Subsequently, we will show a connection between the Cauchy dual and simultaneous approximation. We keep in mind that the dual space of a normed vector space X is denoted by X' and for $\varphi \in X'$ and $x \in X$ we will occasionally use the notation $\langle x, \varphi \rangle$ instead of $\varphi(x)$. Moreover, recall that $\mathbb{D}_{\mathrm{ext}} = \mathbb{C}_\infty \setminus \mathrm{cl}(\mathbb{D})$.

Definition 2.3.1 Let $(X, \|\cdot\|)$ be a normed vector space and let $(x_k)_{k\in\mathbb{Z}} \in X^{\mathbb{Z}}$ such that $M := \mathrm{span}(\{x_k : k \in \mathbb{Z}\})$ is dense in X. In case

$$\limsup_{|k|\to\infty} \|x_k\|^{1/|k|} \leq 1,$$

we define $C := C_M : X' \to H(\mathbb{C}_\infty \setminus \mathbb{S})$ by

$$(C\varphi)(z) := \begin{cases} \sum_{k=0}^{\infty} \langle x_k, \varphi \rangle z^k, & z \in \mathbb{D} \\ -\sum_{k=1}^{\infty} \langle x_{-k}, \varphi \rangle z^{-k}, & z \in \mathbb{D}_{\mathrm{ext}} \end{cases} \qquad (\varphi \in X').$$

In the sequel, we call the function C_M **Cauchy transform** with respect to X relative to M. Furthermore, the space

$$X^{*,M} := C_M(X')$$

is denoted as **Cauchy dual** of X relative to M. In case M is clear from the context, we write X^* instead of $X^{*,M}$.

We shall argue in which sense the Cauchy transform of an $L^1(\mathbb{S})$ function coincides with the one of a functional contained in X' in the setting of Definition 2.3.1. We have already encountered a problem of this kind defining the Fourier transform. There, we saw that the Fourier transform of tempered distributions is an extension of the Fourier transform of m-integrable functions since $L^1(\mathbb{S})$ is embedded in $C(\mathbb{S})'$ (compare (2.8)). The same phenomenon occurs here. Details are elaborated on in the second part of the following remark.

Remark 2.3.2 i) Assume that X, the sequence $(x_k)_{k\in\mathbb{Z}}$, and M are as in Definition 2.3.1. The Cauchy transform C_M is injective since if we have $\varphi \in X'$ and $C_M(\varphi) = 0$, we obtain likewise

$$\langle x_k, \varphi \rangle = 0 \quad (k \in \mathbb{Z}),$$

as C_M takes values in $H(\mathbb{C}_\infty \setminus \mathbb{S})$ combined with the identity theorem. Since the set $M = \mathrm{span}(\{x_k : k \in \mathbb{Z}\})$ is dense in X and since φ is continuous, φ must be zero.

ii) If $X = C(E)$ for $E \subset \mathbb{S}$ compact and if the sequence (x_k) in Definition 2.3.1 takes the role of (p_{-k}), we can represent the Cauchy transform C_M as well in another way. For this purpose, we recall that the norm dual of $C(E)$ is given by the space of complex Borel measures $\boldsymbol{\mathcal{M}}_{\mathbb{C}}(E)$ equipped with the total variation norm $\|\cdot\|_{\mathrm{TV}}$ (see as well Appendix A) via the pairing

$$\langle f, \mu \rangle = \int f \, d\mu \quad \Big(f \in C(E),\ \mu \in \boldsymbol{\mathcal{M}}_{\mathbb{C}}(E) \Big).$$

The latter result is the well-known Riesz representation theorem, see for example [64, 6.19 Theorem]. Hence, we can identify each element in $C(E)'$ with a complex measure $\mu \in \boldsymbol{\mathcal{M}}_{\mathbb{C}}(E)$. This enables us to obtain an equivalent representation of the Cauchy transform by applying the geometric series expansion of the Cauchy kernel. We get

$$\sum_{k=0}^{\infty} \langle p_{-k}, \mu \rangle z^k = \int \sum_{k=0}^{\infty} (z\overline{\zeta})^k d\mu(\zeta) = \int \frac{1}{1 - z\overline{\zeta}} d\mu(\zeta) \quad (z \in \mathbb{D}),$$

and analogously

$$-\sum_{k=1}^{\infty} \langle p_k, \mu \rangle z^{-k} = \int \frac{1}{1 - z\overline{\zeta}} d\mu(\zeta) \quad (z \in \mathbb{D}_{\mathrm{ext}}).$$

In the special case that μ defines a measure that can be represented by a density function $h \in L^1(\mathbb{S})$ with respect to m, i.e. $\mu = hdm$, we find the form exposed in Remark 2.1.12, that means $C\mu = Ch$.

In the following remark, we briefly recall a concept that can be seen as an extension of the Cauchy transform, where we mainly fall back to findings exposed in [6, Section 1.6]. Let

$$K_n := \Big\{ z \in \mathbb{C} : 1 - 1/n \leq |z| \leq 1 + 1/n \Big\} \quad (n \in \mathbb{N}).$$

Then $(K_n)_{n\in\mathbb{N}}$ is a sequence of compact sets in $\mathbb{C}$ such that $K_{n+1} \subset \mathrm{int}(K_n)$ and $\mathbb{S} = \bigcap_{n\in\mathbb{N}} K_n$. The germs of holomorphic functions on $\mathbb{S}$, denoted by $H(\mathbb{S})$, are given by the inductive limit of $A(K_n)$ equipped with the inductive limit topology. As a consequence of Arzela-Ascoli's theorem and Montel's theorem, one can show that $H(\mathbb{S})$ is a DFS space (see e.g. [53, Theorem 1.5.5]). The strong dual of $H(\mathbb{S})$, denoted by $H(\mathbb{S})'$, is the space of **analytic functionals**. More details are provided in [53, Appendix A] and [53, §5]. Since $H(\mathbb{S})$ is a dense subspace in $C^\infty(\mathbb{S})$ and has a stronger topology (see e.g. [6, p. 98]), we find that the tempered distributions can be considered as a subspace of the analytic functionals.

Remark 2.3.3 Due to [6, p. 99], for all $z \in \mathbb{C}_\infty \backslash \mathbb{S}$ the function $\zeta \mapsto \zeta/(\zeta - z) = 1/(1 - z/\zeta)$ is contained in $H(\mathbb{S})$. The **Fantappié transform** (compare [6, Definition 1.6.9]) is defined

by $\Phi : H(\mathbb{S})' \to H(\mathbb{C}_\infty \setminus \mathbb{S})$,

$$(\Phi\varphi)(z) := \left\langle \zeta \mapsto \frac{1}{1 - z/\zeta}, \varphi \right\rangle = \begin{cases} \sum_{k=0}^{\infty} \langle p_{-k}, \varphi \rangle z^k, & z \in \mathbb{D} \\ -\sum_{k=1}^{\infty} \langle p_k, \varphi \rangle z^{-k}, & z \in \mathbb{D}_{\text{ext}} \end{cases} \qquad \Big(\varphi \in H(\mathbb{S})',\ z \in \mathbb{C}_\infty \setminus \mathbb{S}\Big),$$

where the latter equality is due to [6, Proposition 1.6.10] together with [6, Definition 1.6.8]. Consequently, if X is a normed vector space such that $H(\mathbb{S}) \subset X$ and if the sequence $(x_k)_{k\in\mathbb{Z}}$ in Definition 2.3.1 takes the role of $(p_{-k})_{k\in\mathbb{Z}}$ for $k \in \mathbb{Z}$, then the Fantappié transform can be seen as an extension of the Cauchy transform. Finally, we remark that Φ is an isomorphism between the spaces $H(\mathbb{S})'$ and $H(\mathbb{C}_\infty \setminus \mathbb{S})$.

After having introduced the notion of the Cauchy dual, we now want to calculate the Cauchy dual of the spaces D_α and $\mathcal{D}^p_{\text{harm}}$. The approach is the following. First, we have to find a representation for the dual space of X, and hereafter, we have to determine the image of X' under the Cauchy transform for a given set whose span is dense in X.

We start with the Hilbert space D_α. We apply the Riesz representation theorem for Hilbert spaces which indicates in particular that $\iota : D_\alpha \to D'_\alpha$, $\iota(a) = \langle \cdot, a \rangle_{D_\alpha}$ is an (antilinear) isometric isomorphism. Thus, it remains to calculate the image of D_α under the Cauchy transform. It turns out to be advantageous if the sequence $(x_k)_{k\in\mathbb{Z}}$ in Definition 2.3.1 takes the role of the unit vectors $(e_k)_{k\in\mathbb{Z}}$, where we bear in mind that the span of the unit vectors is dense in D_α due to Proposition 2.1.11. Details are elaborated below.

Proposition 2.3.4 *Let $\alpha \in \mathbb{R}$. Then the Cauchy dual of D_α with respect to $M := \operatorname{span}\{e_k : k \in \mathbb{Z}\}$ is given by $\mathcal{D}_{-\alpha}$.*

Proof: We calculate the image of D_α under the Cauchy transform with respect to M. Suppose that $\varphi \in D_\alpha$. By the definition of the scalar product on D_α (see (2.5)), we have for all $k \in \mathbb{Z}$

$$\langle e_k, \varphi \rangle_{D_\alpha} = \sum_{j=-\infty}^{\infty} (|j|+1)^\alpha e_k(j)\overline{\varphi}(j) = (|k|+1)^\alpha \overline{\varphi}(k).$$

Moreover, by the definition of D_α, we deduce

$$\sum_{j=-\infty}^{\infty} (|j|+1)^{-\alpha} \Big|(|j|+1)^\alpha \overline{\varphi}(j)\Big|^2 = \sum_{j=-\infty}^{\infty} (|j|+1)^\alpha |\overline{\varphi}(j)|^2 < \infty,$$

and consequently

$$b := (|\cdot|+1)^\alpha \overline{\varphi} \in D_{-\alpha} \qquad \text{if and only if} \qquad \varphi \in D_\alpha.$$

We go ahead by establishing

$$(C\varphi)(z) := \begin{cases} \sum_{k=0}^{\infty} \langle e_k, \varphi \rangle_{D_\alpha} z^k = \sum_{k=0}^{\infty} b_k z^k, & z \in \mathbb{D} \\ -\sum_{k=1}^{\infty} \langle e_{-k}, \varphi \rangle_{D_\alpha} z^{-k} = -\sum_{k=1}^{\infty} b_{-k} z^{-k}, & z \in \mathbb{D}_{\text{ext}} \end{cases} \qquad (\varphi \in D_\alpha),$$

and therefore the Cauchy dual of D_α is given by $\mathcal{D}_{-\alpha}$. $\square$

Now, we turn to the calculation of the Cauchy dual of $\mathcal{D}^p_{\text{harm}}$. Since we already calculated the dual space of $\mathcal{D}^p_{\text{harm}}$ which is given by $\mathcal{A}^q \oplus \overline{\mathcal{A}^q}/\mathbb{C}$ due to Proposition 2.2.9, it only remains to determine the image of the latter one under the Cauchy transform.

Recall that the Cauchy dual depends on the underlying dense set, called M in Definition 2.3.1. Due to Proposition 2.2.3, we know that the harmonic polynomials are dense in $\mathcal{D}^p_{\text{harm}}$. Since the harmonic monomials on the unit disk are the analog to the trigonometric monomials on the unit circle, we let M take the role of the harmonic polynomials which means for $k \in \mathbb{Z}$ that x_k (compare again Definition 2.3.1) takes the role of

$$\begin{aligned} &\bullet\ p_k,\ k \in \mathbb{N}_0, \\ &\bullet\ \overline{p_{-k}},\ k \in -\mathbb{N}. \end{aligned} \tag{2.26}$$

This setting allows us to obtain the following representation for the Cauchy dual of $\mathcal{D}^p_{\text{harm}}$.

Proposition 2.3.5 *Let $p \in (1,\infty)$ and assume that the sequence $(x_k)_{k\in\mathbb{Z}}$ is specified as in (2.26). Then the Cauchy dual of $\mathcal{D}^p_{\text{harm}}$ relative to $\operatorname{span}\{x_k : k \in \mathbb{Z}\}$ is given by*

$$(\mathcal{D}^p_{\text{harm}})^* = \mathcal{A}^q \oplus \mathcal{A}^q_{\text{ext}}.$$

Moreover, if $(x_k)_{k\in\mathbb{Z}} = (e_k)_{k\in\mathbb{Z}}$ the Cauchy dual of D^p_{harm} relative to $\operatorname{span}\{x_k : k \in \mathbb{Z}\}$ is given by

$$(D^p_{\text{harm}})^* = \mathcal{A}^q \oplus \mathcal{A}^q_{\text{ext}}.$$

Proof: By the definition of the Cauchy dual, we have to evaluate $\langle x_k, \varphi\rangle$ for all $k \in \mathbb{Z}$ and $\varphi \in (\mathcal{D}^p_{\text{harm}})'$ for x_k specified as in (2.26). We use Proposition 2.2.9 which implies that for all $\varphi \in (\mathcal{D}^p_{\text{harm}})'$, we can find a $g \in \mathcal{A}^q \oplus \overline{\mathcal{A}^q}/\mathbb{C}$ such that

$$\langle x_k, \varphi\rangle = \langle x_k, g\rangle_{\mathcal{D}^p_{\text{harm}}},$$

where the pairing was specified in (2.24). Let us evaluate this term in two steps, namely for $k \in \mathbb{N}_0$ and for $k \in -\mathbb{N}$. By definition of the Bergman projection P, we get for any $k \in \mathbb{N}_0$

$$P(p_k) = p_k \quad \text{and} \quad P_-(p_k) = 0.$$

The function g can be decomposed in the sense of (2.4) with $g_{\text{H}} \in \mathcal{A}^q$ and $g_{\text{AH}} \in \overline{\mathcal{A}^q}/\mathbb{C}$. Moreover, since $A^q \oplus A^q_{\text{ext}}$ and $\mathcal{A}^q \oplus \overline{\mathcal{A}^q}/\mathbb{C}$ are isometrically isomorphic, there exists a sequence $(b_j)_{j\in\mathbb{Z}} \in A^q \oplus A^q_{\text{ext}}$ such that

$$g = g_{\text{H}} + g_{\text{AH}} = \sum_{j=0}^{\infty} b_j p_j + \sum_{j=1}^{\infty} b_{-j}\overline{p_j}. \tag{2.27}$$

We obtain by (2.24)

$$\langle p_k, g\rangle_{\mathcal{D}^p_{\mathrm{harm}}} = (k+1)\int_{\mathbb{D}} p_k\,\overline{g_{\mathrm{H}}}\,dm_2 \quad (k\in\mathbb{N}_0). \tag{2.28}$$

Due to [13, p. 31, Theorem 4], the partial Taylor sums $T_n g_{\mathrm{H}}$ converge to g_{H} with respect to the norm $\|\cdot\|_{\mathcal{A}^q}$. Consequently, for any $k\in\mathbb{N}_0$,

$$\left|\sum_{j=0}^{n}\overline{b_j}\int_{\mathbb{D}} p_k\,\overline{p_j}dm_2 - \int_{\mathbb{D}} p_k\,\overline{g_{\mathrm{H}}}dm_2\right| \le \left\|T_n g_{\mathrm{H}} - g_{\mathrm{H}}\right\|_{\mathcal{A}^q} \to 0 \quad (n\to\infty).$$

Together with the orthogonality relation in (2.16), we obtain

$$\langle p_k, g\rangle_{\mathcal{D}^p_{\mathrm{harm}}} = (k+1)\sum_{j=0}^{\infty}\overline{b_j}\int_{\mathbb{D}} p_k\,\overline{p_j}dm_2 = (k+1)\sum_{j=0}^{\infty}\overline{b_j}\,\frac{1}{j+1}\delta_{k,j} = \overline{b_k} \quad (k\in\mathbb{N}_0). \tag{2.29}$$

Now, we argue analogously for $k\in-\mathbb{N}$ where $x_k = \overline{p_{-k}}$. We deduce that

$$P(\overline{p_{-k}}) = 0 \quad \text{and} \quad P_-(\overline{p_{-k}}) = \overline{p_{-k}} \quad (k\in-\mathbb{N}).$$

Hence, the first summand of the pairing in (2.24) drops out and we obtain

$$\langle\overline{p_{-k}}, g\rangle_{\mathcal{D}^p_{\mathrm{harm}}} = (k+1)\int_{\mathbb{D}} p_{-k} g_{\mathrm{AH}}dm_2 \quad (k\in-\mathbb{N}).$$

The application of [13, p. 31, Theorem 4] to the function $\overline{g_{\mathrm{AH}}}\in\mathcal{A}^q$ and the orthogonality relation in (2.16) leads to

$$\langle\overline{p_{-k}}, g\rangle_{\mathcal{D}^p_{\mathrm{harm}}} = (k+1)\sum_{j=1}^{\infty} b_{-j}\int_{\mathbb{D}} p_{-k}\,\overline{p_j}dm_2 = b_k \quad (k\in-\mathbb{N}). \tag{2.30}$$

By the definition of x_k, we summarize that

$$\langle x_k, g\rangle_{\mathcal{D}^p_{\mathrm{harm}}} = \begin{cases}\overline{b_k}, & k\in\mathbb{N}_0\\ b_k & k\in-\mathbb{N}.\end{cases} \tag{2.31}$$

Since $\overline{\mathcal{A}^q}$ is isometrically isomorphic to $\mathcal{A}^q$ and since $(A^q_{\mathrm{ext}})_- = \mathcal{A}^q_{\mathrm{ext}}$ due to Proposition 2.2.12, we finally obtain by the definition of the Cauchy dual

$$(\mathcal{D}^p_{\mathrm{harm}})^* = (A^q)_+ \oplus (A^q_{\mathrm{ext}})_- = \mathcal{A}^q \oplus \mathcal{A}^q_{\mathrm{ext}}.$$

To determine the Cauchy dual of D^p_{harm}, we first observe that $p_k\in\mathcal{D}^p_{\mathrm{harm}}$ corresponds to $e_k\in D^p_{\mathrm{harm}}$ for $k\in\mathbb{N}_0$, and similarly $\overline{p_k}\in\mathcal{D}^p_{\mathrm{harm}}$ corresponds to $e_{-k}\in D^p_{\mathrm{harm}}$ for $k\in\mathbb{N}$. Furthermore, the dual space of D^p_{harm} is given by $A^q\oplus A^q_{\mathrm{ext}}$. Let $b\in A^q\oplus A^q_{\mathrm{ext}}$ and let g be the corresponding function in $\mathcal{A}^q\oplus\overline{\mathcal{A}^q}/\mathbb{C}$. Applying (2.25) yields

$$\langle e_k, b\rangle_{D^p_{\mathrm{harm}}} = \langle x_k, g\rangle_{\mathcal{D}^p_{\mathrm{harm}}} = \begin{cases}\overline{b_k}, & k\in\mathbb{N}_0\\ b_k, & k\in-\mathbb{N}\end{cases}.$$

Thus, the Cauchy dual of D^p_{harm} is equal to the Cauchy dual of $\mathcal{D}^p_{\text{harm}}$. $\square$

To finish this chapter, we underscore a connection between the intersection of the Cauchy duals of two normed vector spaces and the concept of simultaneous approximation.

Proposition 2.3.6 *Let X and Y be two normed vector spaces and let $(x_k)_{k\in\mathbb{Z}} \in X^{\mathbb{Z}}$ as well as $(y_k)_{k\in\mathbb{Z}} \in Y^{\mathbb{Z}}$ such that $M_X := \text{span}(\{x_k : k \in \mathbb{Z}\})$ and $M_Y := \text{span}(\{y_k : k \in \mathbb{Z}\})$ are dense sets in X and Y, respectively. Then the following two statements are equivalent*

i) $X^* \cap Y^* = \{0\}$.

ii) $M := \text{span}\Big\{(x_k, y_k) : k \in \mathbb{Z}\Big\}$ *is dense in* $X \oplus Y$.

Proof: Firstly, suppose that $(\varphi, \psi) \in (X \oplus Y)' = X' \oplus Y'$ where the latter equality is due to Remark 2.2.8. According to the definition of the Cauchy dual, we have

$$0 = (\varphi, -\psi)(x_k, y_k) = \varphi(x_k) - \psi(y_k) \quad (k \in \mathbb{Z})$$

if and only if $C_{M_X}(\varphi) = C_{M_Y}(\psi)$.

Let us assume i). As $X^* \cap Y^* = \{0\}$, it follows that $C_{M_X}(\varphi) = C_{M_Y}(\psi) = 0$ and by the injectivity of the Cauchy transform (see Remark 2.3.2), we obtain $\varphi = \psi = 0$. Applying Hahn-Banach's theorem yields the denseness of M in $X \oplus Y$.

Conversely, assume that ii) is valid. If $C_{M_X}(\varphi) = C_{M_Y}(\psi)$, then $(\varphi, -\psi) = 0$ on the dense set M and by continuity, we infer

$$(\varphi, -\psi) = 0.$$

Thus, $C_{M_X}(\varphi) = C_{M_Y}(\psi) = 0$. $\square$

Chapter 3

Capacities

3.1 Kernels and capacities

In the sequel, we recall some notions from potential theory. We mainly follow [16] where the theory of potentials and capacities has been discussed in the frame of compact metric spaces. However, in the further course of this chapter, it will prove useful to define certain terms within the context of separable metric spaces.

Before, we fix some notation concerning measure theory (see also Appendix A). If X is a topological space, we write $\boldsymbol{\mathcal{M}}(X)$ for the set of all σ-finite Borel measures on $(X, \mathcal{B}(X))$ and $\mathrm{Prob}(X)$ for the set of probability measures in $\boldsymbol{\mathcal{M}}(X)$. Moreover, $\mathrm{supp}(\mu)$ is the support of a measure μ. If μ_1 and μ_2 are two σ-finite Borel measures on $(X, \mathcal{B}(X))$, then we write $\mu_1 \otimes \mu_2$ for the product measure of μ_1 and μ_2.

Finally, in case (X, d) is a metric space, the **diameter** $\mathrm{diam}(A)$ of $A \subset X$ is given by the supremum of $d(x, y)$ taken over all x, y contained in A.

Definition 3.1.1 Let (X, d) be a separable metric space. We call a decreasing and continuous function $K : (0, \mathrm{diam}(X)] \to [0, \infty)$ with

$$K(t) \to +\infty =: K(0) \quad (t \to 0)$$

a **kernel function**. Moreover, we refer to the pair (K, d) as **kernel**. If the considered metric d is clear from the context, we simply write K instead of (K, d). For a kernel (K, d) and $\mu \in \boldsymbol{\mathcal{M}}(X)$, the **potential function** $(K, d)\mu : X \to [0, \infty]$ of μ with respect to K and d is defined by

$$K\mu(x) := (K, d)\mu(x) := \int K\Big(d(x, y)\Big) d\mu(y) \in [0, \infty].$$

Furthermore, the (K, d)**-energy** of μ is defined by

$$\begin{aligned} I_K(\mu) := I_{(K,d)}(\mu) &:= \int \Big((K, d)\mu\Big)(x)\, d\mu(x) = \int\int K\Big(d(x, y)\Big)\, d\mu(y) d\mu(x) \\ &= \int K\Big(d(x, y)\Big)\, d(\mu \otimes \mu)(y, x) \in [0, \infty]. \end{aligned}$$

When the metric in use is clear, we drop the symbol d from the notation $(K,d)\mu$ and $I_{(K,d)}(\mu)$ as done for kernels. However, bear in mind that the potential function depends on the metric d.

We will recall an important concept motivated by [16]. In the sequel, we assign the infimum of the empty set to $+\infty$ as well as the supremum of the empty set to 0. Additionally, we set $1/+\infty = 0$ and $1/0 = +\infty$.

Definition 3.1.2 Suppose that E is a *compact* subset of the separable metric space (X,d) and let $K = (K,d)$ be a kernel. We define

$$I_K(E) := I_{(K,d)}(E) := \inf\left\{I_{(K,d)}(\mu) : \mu \in \mathrm{Prob}(X),\ \mathrm{supp}(\mu) \subset E\right\} \tag{3.1}$$

In the sequel, we refer to $I_{(K,d)}(E)$ as the (K,d)**-energy** or shorter K-energy of the set E. Moreover,

$$c_K(E) := c_{(K,d)}(E) := (I_{(K,d)}(E))^{-1} = \sup\left\{\frac{1}{I_{(K,d)}(\mu)} : \mu \in \mathrm{Prob}(X),\ \mathrm{supp}(\mu) \subset E\right\} \tag{3.2}$$

is called (K,d)**-capacity** or shorter K-capacity of the set E.

If the kernel is obvious from the context, we drop the prefix (K,d) or K in the latter notation which means that we only say energy or capacity of the set E.

Remark 3.1.3 By [16, Corollary 2.1.5], we have that the capacity of a compact set E is always finite. Moreover, if $E_n \subset X$ is a decreasing sequence of compact sets and if $E = \bigcap\limits_{n\in\mathbb{N}} E_n$, we have (cf. [16, Theorem 2.1.6])

$$c_K(E) = \lim_{n\to\infty} c_K(E_n). \tag{3.3}$$

Definition 3.1.2 does not specify if there is a probability measure such that the infimum in (3.1) is attained. We are particularly interested in measures that possess this characteristic. This interest prompts the subsequent definition.

Definition 3.1.4 Let (X,d) be a separable metric space, $E \subset X$ be a compact set, and $K = (K,d)$ be a kernel. A measure $\nu \in \mathrm{Prob}(X)$ with $\mathrm{supp}(\nu) \subset E$ is called a (K,d)**-equilibrium measure** or K-equilibrium measure of E if $I_K(\nu) < \infty$ and

$$I_K(\nu) \le I_K(\mu) \quad \Big(\mu \in \mathrm{Prob}(X),\ \mathrm{supp}(\mu) \subset E\Big).$$

Remark 3.1.5 In the setting of the latter definition, one can show that each compact subset of X has a K-equilibrium measure. This result is mainly a consequence of the Banach-Alaoglu theorem and the Stone-Weierstrass theorem (see e.g. [16, Theorem 2.2.2]).

So far, we only considered the capacity of compact sets. However, it appears beneficial to extend this concept to arbitrary sets using the notion of inner and outer capacities. For this purpose, we assume that (X, d) is a *compact* metric space. The following definition is based on [16, Definition 2.1.8].

Definition 3.1.6 Let (X, d) be a compact metric space and let $A \subset X$. Moreover, suppose that $K = (K, d)$ is a kernel. Then the **inner K-capacity** of A is defined by

$$c_K(A) := c_{(K,d)}(A) := \sup\left\{c_{(K,d)}(E) : E \subset A,\ E \text{ compact}\right\}.$$

Moreover, the **outer K-capacity** of A is defined by

$$c_K^*(A) := c_{(K,d)}^*(A) := \inf\left\{c_{(K,d)}(U) : A \subset U,\ U \text{ open in } X\right\}.$$

A set A that satisfies the condition $c_K(A) = c_K^*(A)$ is called **capacitable**.

Open sets and compact sets possess the property of capacitability, a conclusion that is derived from the previous definition and (3.3). If the kernel K is *consistent* (see e.g. [20, Definition, p. 167]) every Borel set in a compact metric space is capacitable. This is a result of [20, Theorem 4.5] which uses Choquet's capacitability theorem and [20, p. 188].

We extend the definition of the energy for arbitrary sets $A \subset X$ in the following way

$$I_K(A) := I_{(K,d)}(A) := \begin{cases} c_K^*(A)^{-1}, & c_K^*(A) > 0 \\ \infty, & c_K^*(A) = 0 \end{cases}.$$

In the upcoming remark, we aim to highlight some fundamental characteristics of outer capacities. In particular, the third point turns out to be essential. A proof can for instance be found in [16, Theorem 2.1.9].

Remark 3.1.7 Let (X, d) be a compact metric space and let K be a kernel. Outer capacities fulfill the following three properties.

i) $c_K^*(\emptyset) = 0$.

ii) For all sets $A_1, A_2 \subset X$ with $A_1 \subset A_2$ we have $c_K^*(A_1) \leq c_K^*(A_2)$.

iii) Outer capacities are σ-subadditive that means for any sequence $(A_j)_{j\in\mathbb{N}} \in X^{\mathbb{N}}$, we have

$$c_K^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} c_K^*(A_j).$$

Since $c_K(A) \leq c_K^*(A)$ for any $A \subset X$ (see [16, p. 18]) and since the K-energy of a singleton is always ∞, we obtain that any countable set is of vanishing outer K-capacity.

It is worth noting that certain authors have proposed concepts of capacities that do not adhere to all three properties previously mentioned, particularly the third one. This difference arises because these authors do not require a kernel function to take nonnegative values. For example, consider [61] where K is the logarithm equipped with the Euclidean distance $d_{|\cdot|}$ on $\mathbb{C}$.

Outer capacities bear some similarities to measures. Therefore, we recall in the subsequent definition a concept that can be considered analogous to null sets. This concept is also utilized by other authors, see for example [1, Definition 2.2.5] or [61, p. 160].

Definition 3.1.8 Let (X, d) be a compact metric space and let K be a kernel. We say that a property holds c_K^*-**quasieverywhere** if the property holds for all $x \in X$ except for those which belong to a set $A \subset X$ of vanishing outer K-capacity. Moreover, if $c_K^*(A) = 0$ we call the set A c_K-**polar**. Once more, if the kernel K is clear from the context, we simply say that A is polar or that the property holds quasieverywhere.

The forthcoming theorem, a pivotal result in potential theory, goes back to Otto Frostman. It is one of the essential findings in potential theory. The original theorem can be found in [19]. We will present a version that has been slightly adjusted to better align with our ongoing discussion. A proof can for instance be found in [16, Theorem 2.2.3].

Theorem 3.1.9 (Frostman's Theorem)
Let E be a compact and non-polar subset of the compact metric space (X, d) and let K be a kernel. Denote ν as a corresponding K-equilibrium measure of E. Then the following two estimates hold

1) $K\nu \leq I_K(E)$ *on* $\operatorname{supp}(\nu)$,

2) $K\nu \geq I_K(E)$ *quasieverywhere on* E.

In particular, the equality $K\nu = I_K(E)$ *holds quasieverywhere on* $\operatorname{supp}(\nu)$.

To conclude this section, we recall the notion of equivalent capacities. Our approach is based on [1, p. 20] (see also [2]) which requires for the equivalence of two capacities that the ratio is bounded above and below.

In the sequel, we assume that (X, d_X) and (Y, d_Y) are two separable metric spaces. In case $X \cap Y \neq \emptyset$, then

$$(X \cap Y, d_X + d_Y) \tag{3.4}$$

is also a metric space where the metric $d_X + d_Y$ is restricted to the set $(X \cap Y)^2$. For the sake of brevity, we occasionally write $X \cap Y$ instead of (3.4).

Definition 3.1.10 Suppose that (X, d_X) and (Y, d_Y) are two separable metric spaces such that $X \cap Y$ is a compact metric space. Furthermore, let $K = (K, d_X)$ and $L = (L, d_Y)$ be two kernels.

1) We say that the two capacities c_K and c_L are equivalent on $X \cap Y$ if there is a constant $A \in [1, \infty)$ such that for any $F \subset X \cap Y$

$$\frac{1}{A} \cdot c_K^*(F) \leq c_L^*(F) \leq A \cdot c_K^*(F).$$

2) We say that the kernels K and L are equivalent on $X \cap Y$ if there exist constants $B \in [1, \infty)$ and $M \in [0, \infty)$ such that

$$\frac{1}{B} \cdot K\Big(d_X(x,y)\Big) - M \leq L\Big(d_Y(x,y)\Big) \leq B \cdot K\Big(d_X(x,y)\Big) + M \quad (x, y \in X \cap Y).$$

In the subsequent proposition, we derive a sufficient criterion for the equivalence of two capacities.

Proposition 3.1.11 *Assume that (X, d_X) and (Y, d_Y) are two separable metric spaces such that $X \cap Y$ is a compact metric space. Moreover, suppose that $K = (K, d_X)$ and $L = (L, d_Y)$ are equivalent on $X \cap Y$. Then the corresponding capacities are equivalent on $X \cap Y$.*

Proof: First, suppose that $E \subset X \cap Y$ is a compact set. The equivalence of the kernels K and L yields

$$\begin{aligned} \frac{1}{B} \cdot I_K(\mu) - M &= \int\int \frac{1}{B} \cdot K\Big(d_X(x,y)\Big) - M \; d\mu(y) d\mu(x) \leq \int\int L\Big(d_Y(x,y)\Big) d\mu(y) d\mu(x) \\ &= I_L(\mu) \end{aligned}$$

for any measure $\mu \in \mathrm{Prob}(X \cap Y)$ with $\mathrm{supp}(\mu) \subset E$. In particular,

$$\frac{1}{B} \cdot I_K(E) \leq I_L(E) + M$$

and since $c_L(E) \leq c_L(X \cap Y) < \infty$ due to Remark 3.1.3, we get for all $E \subset X \cap Y$ compact

$$c_L(E) \leq B(1 + M c_L(E)) \cdot c_K(E) \leq B(1 + M c_L(X \cap Y)) \cdot c_K(E). \tag{3.5}$$

Define the constant $A := B(1 + M c_L(X \cap Y))$. By the definition of the inner capacity and (3.5), we obtain for any $F \subset X \cap Y$

$$\begin{aligned} c_L(F) &= \sup\Big\{c_L(E) : E \subset F, \; E \text{ compact}\Big\} \leq \sup\Big\{A \cdot c_K(E) : E \subset F, \; E \text{ compact}\Big\} \\ &= A \cdot c_K(F). \end{aligned}$$

In particular, the latter inequality holds for any $U \subset X \cap Y$ open. By the definition of outer capacities and the same argument as in the forgoing estimate, we obtain

$$c_L^*(F) \leq A \cdot c_K^*(F) \quad \Big(F \subset X \cap Y\Big).$$

The "only if" part follows by observing

$$\frac{1}{B} \cdot L\Big(d_Y(x,y)\Big) - M \leq K\Big(d_X(x,y)\Big)$$

which is valid due to the equivalence of the kernels K and L and analogous argumentation as in the "if" part. □

As an immediate consequence of Proposition 3.1.11, it follows that the c_K-polar sets and the c_L-polar sets correspond.

3.2 Riesz α-capacity and Li_α-capacity

Up to this point, our discussion has revolved around arbitrary kernel functions. Henceforth, we will shift our attention to specific families of kernels which possess additional properties such as convexity that are important for our subsequent discussions. Let us recall a well-known family of kernel functions in the ensuing definition.

Definition 3.2.1 Let $\alpha \in (0,1]$. The function $R_\alpha : (0,\infty) \to [0,\infty)$

$$R_\alpha(t) := \begin{cases} 1/t^{1-\alpha}, & \alpha \in (0,1), \\ \max\{\ln(2/t), 0\}, & \alpha = 1 \end{cases} \qquad (t \in (0,\infty))$$

and $R_\alpha(0) := \lim\limits_{t\to 0} R_\alpha(t) = +\infty$ is denoted as **Riesz kernel function** of degree α for $\alpha \in (0,1)$ and **logarithmic kernel function** in event of $\alpha = 1$.

We explore the fundamental properties of the Riesz and the logarithmic kernel function. First, for $\alpha \in (0,1]$, the function R_α satisfies the conditions specified in Definition 3.1.1 and R_α is also a convex function. The kernel functions R_α have more favorable properties if they are considered on suitable metric spaces, for example, it can be shown that the arc measure m is the *unique* $(R_\alpha, d_{|\cdot|})$-equilibrium measure of the set $\mathbb{S}$ for any $\alpha \in (0,1]$ which is a consequence of [24, Corollary 6].

Definition 3.2.2 Let (X,d) be a separable metric space. If $\alpha \in (0,1)$, we call (R_α, d) **Riesz kernel** of degree α with respect to the metric d. In case $\alpha = 1$, we refer to (R_1, d) as **logarithmic kernel** with respect to the metric d. If d is the Euclidean metric $d_{|\cdot|}$ on $\mathbb{C}$, we define for $\alpha \in (0,1]$

$$C_\alpha := c_{(R_\alpha, d_{|\cdot|})}.$$

In case $\alpha \in (0,1)$, the capacity C_α is denoted as **Riesz α-capacity** or simply **α-capacity**. For $\alpha = 1$, the capacity C_1 is called **logarithmic capacity** or sometimes 1-capacity.

In the sequel, we investigate properties of capacities defined on $\mathbb{S}$ equipped with the spherical metric which we recall in the upcoming definition. Note that ln is the inverse

function of $\exp : \mathbb{R} \to (0, \infty)$. For any $z \in \mathbb{C}^*$ there is a unique $t \in (-\pi, \pi]$ such that $z = |z|e^{it}$. The principle branch of the logarithm $\log : \mathbb{C}^* \to \mathbb{C}$ is defined by

$$\log(z) := \ln(|z|) + it \quad (z \in \mathbb{C}^*) \tag{3.6}$$

and the argument arg of a complex number $z \in \mathbb{C}^*$ is given by t. The function $\mathrm{Re}(\log) : \mathbb{C}^* \to \mathbb{C}$ is harmonic and log restricted to the set $\mathbb{C}_- := \mathbb{C} \setminus (-\infty, 0]$ is a holomorphic function.

For $z, w \in \mathbb{S}$ with $\arg(z) \leq \arg(w)$, we define the arc

$$A_z^w := \Big\{ \zeta \in \mathbb{S} : \arg(\zeta) \in [\arg(z), \arg(w)] \Big\} \tag{3.7}$$

as well as $A_w^z := \mathbb{S} \setminus A_z^w$ in case $\arg(z) < \arg(w)$.

Definition 3.2.3 The function $\sigma : \mathbb{S} \times \mathbb{S} \to [0, \pi]$

$$\sigma(z, w) := 2\pi \min \Big\{ m(A_z^w),\ m(A_w^z) \Big\}$$

is the **spherical metric** on $\mathbb{S}$.

Since the spherical metric will play a fundamental role throughout this thesis, let us collect three important facts. First, for any $u \in \mathbb{S}$

$$\sigma(1, u) = |\arg(u)|. \tag{3.8}$$

Since the spherical metric is invariant under rotation, we get for any $z, w \in \mathbb{S}$

$$\sigma(z, w) = \sigma(1, w\overline{z}) = |\arg(w\overline{z})|. \tag{3.9}$$

Furthermore, the following estimate holds for all $z, w \in \mathbb{S}$ with $z \neq w$

$$1 \leq \frac{\sigma(z, w)}{d_{|\cdot|}(z, w)} \leq \frac{\pi}{2}. \tag{3.10}$$

The compact metric space $(\mathbb{S}, \sigma)$ will be of particular interest in this thesis. One of our central concerns is the derivation of an identity that relates the energy of a measure μ to the corresponding Fourier coefficients of μ. For the logarithmic kernel and a finite Borel measure μ on $\mathbb{S}$, the following identity holds (see e.g. [16, Theorem 2.4.4])

$$I_{(R_1, d_{|\cdot|})}(\mu) = \sum_{k=1}^{\infty} \frac{|\widehat{\mu}(k)|^2}{k} + \ln(2)\mu(\mathbb{S})^2.$$

However, it is known that the latter identity cannot hold if $(R_1, d_{|\cdot|})$ is replaced by $(R_\alpha, d_{|\cdot|})$ and $1/k$ by $1/k^\alpha$, respectively. Thus, we pursue the goal of deriving an alternative

family of kernel functions K_α which allow us to extend the latter identity to $\alpha \in (0,1)$ and whose corresponding capacities are equivalent to Riesz α-capacities.

Our approach is based on the observation that the principal branch of the logarithm has the following well-known representation

$$\log\left(\frac{1}{1-z}\right) = \sum_{k=1}^{\infty} \frac{z^k}{k} \quad (z \in \mathbb{D}).$$

The Dirichlet test together with Abel's convergence theorem implies that the latter representation is still valid for $z \in \mathbb{S} \setminus \{1\}$. Now, the idea arises of replacing the factor $1/k$ by $1/k^\alpha$ for $\alpha \in (0,1)$. This brings into play the polylogarithm which is defined by $\mathrm{Li}_\alpha : \mathbb{D} \to \mathbb{C}$

$$\mathrm{Li}_\alpha(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^\alpha} \quad (z \in \mathbb{D}),$$

for $\alpha \in \mathbb{R}$. In case $\alpha > 0$, the latter series converges on $\mathbb{S} \setminus \{1\}$ by Dirichlet's test. Moreover, we can extend the polylogarithm analytically to the set $\mathbb{C} \setminus [1, \infty)$ with the help of the subsequent integral representation (see (C.2))

$$\mathrm{Li}_\alpha(z) = \frac{z}{\Gamma(\alpha)} \int_0^\infty \frac{s^{\alpha-1}}{e^s - z} ds \quad z \in \mathbb{C} \setminus [1, \infty). \tag{3.11}$$

This implies that the polylogarithm is a holomorphic function on $\mathbb{C} \setminus [1, \infty)$. We show that the real part of the polylogarithm concatenated with the function $t \mapsto e^{it}$ is decreasing on $(0, \pi]$.

At this point, we advise the reader that we have summarized some known results, estimates, and identities about the polylogarithm in Appendix C. We will refer to these results whenever the need arises.

Proposition 3.2.4 *For all $\alpha \in (0,1]$ the function*

$$(0, \pi] \ni t \mapsto \mathrm{Re}(\mathrm{Li}_\alpha(e^{it})) \tag{3.12}$$

is continuous and decreasing with $\lim_{t \to 0} \mathrm{Re}(\mathrm{Li}_\alpha(e^{it})) = +\infty$.

Proof: First, the assertion is clear for $\alpha = 1$ since (C.4) yields for any $t \in (0, \pi]$

$$\mathrm{Re}(\mathrm{Li}_1(e^{it})) = -\mathrm{Re}(\log(1 - e^{it})) = -\ln(|1 - e^{it}|). \tag{3.13}$$

Let us turn to the case $\alpha \in (0,1)$. The continuity of (3.12) is a consequence of the continuity of Li_α on $\mathbb{S} \setminus \{1\}$.

We show that (3.12) is decreasing. By the integral representation of the polylogarithm in (3.11), we get for $t \in (0, \pi]$

$$\Gamma(\alpha)\mathrm{Re}(\mathrm{Li}_\alpha(e^{it})) = \mathrm{Re}\left(e^{it} \int_0^\infty \frac{s^{\alpha-1}}{e^s - e^{it}}\, ds\right) = \int_0^\infty s^{\alpha-1} \mathrm{Re}\Big(\frac{1}{e^{s-it} - 1}\Big)\, ds.$$

Let us have a closer look at the term with the real part. We use the identity $\mathrm{Re}(1/z) = \mathrm{Re}(\overline{z})/|z|^2$, $z \neq 0$. Hereafter, we deploy Euler's formula, and observe that cosine is an even function. Together with the definition of the hyperbolic cosine, we deduce

$$\mathrm{Re}\Big(\frac{1}{e^{s-it}-1}\Big) = \frac{e^s\cos(t)-1}{|e^{s-it}-1|^2} = \frac{e^s\cos(t)-1}{e^{2s}-2e^s\cos(t)+1} = \frac{1}{2}\cdot\frac{\cos(t)-e^{-s}}{\cosh(s)-\cos(t)}. \tag{3.14}$$

We take the derivative on the right-hand side in (3.14) and obtain

$$\frac{\partial}{\partial t}\,\frac{\cos(t)-e^{-s}}{\cosh(s)-\cos(t)} = -\,\frac{(\cosh(s)-e^{-s})\sin(t)}{(\cosh(s)-\cos(t))^2}. \tag{3.15}$$

Let us analyze the right-hand side by keeping in mind that $s \in (0,\infty)$ and $t \in (0,\pi]$. It is clear that the numerator is nonnegative by observing $\cosh(s) - e^{-s} = \sinh(s)$ and that the sine function is nonnegative on $(0,\pi]$. Thus, (3.15) is nonpositive for all $t \in (0,\pi]$ and $s \in (0,\infty)$. Consequently, the derivative of (3.12) is nonpositive and therefore, (3.12) is decreasing.

Moreover, by (C.5) and (C.9), we obtain for any $\alpha \in (0,1)$

$$\sigma(1,e^{it})^{\alpha-1} = |t|^{\alpha-1} \in O\Big(\mathrm{Re}(\mathrm{Li}_\alpha(e^{it}))\Big)$$

as $t \to 0$ which yields the assertion. □

Since the function in (3.12) is continuous, decreasing, and unbounded at 0, we are permitted to define a kernel function by adding a constant to the real part of the polylogarithm. Thus, we are prepared to introduce the all-important kernel which will henceforth serve as the default kernel on the compact metric space $(\mathbb{S},\sigma)$.

Definition 3.2.5 Let $\alpha \in (0,1]$. With $L_\alpha := \mathrm{Li}_\alpha(-1) \in \mathbb{R}$ we define the **Li_α-kernel function** $K_\alpha : (0,\pi] \to [0,\infty)$ via

$$K_\alpha(t) := \mathrm{Re}(\mathrm{Li}_\alpha(e^{it})) - L_\alpha$$

and $K_\alpha(0) := \lim_{t\to 0} K_\alpha(t) = +\infty$. In this situation, we set $\mathrm{Li}_\alpha(1) := +\infty$ for the sake of consistency. With the spherical metric we call

$$K_\alpha = (K_\alpha, \sigma)$$

Li_α-kernel. Moreover, we denote the corresponding capacity

$$c_\alpha := c_{K_\alpha}$$

as **Li_α-capacity**.

As discussed above, R_α is convex for all $\alpha \in (0,1]$. We show that this is also true for the Li_α-kernel function K_α by considering the second derivative. In addition, we will need some results contained in Appendix C.

Proposition 3.2.6 *The Li_α-kernel function K_α is convex for all $\alpha \in (0,1]$.*

Proof: The case $\alpha = 1$ is an immediate consequence of (3.13). Therefore, let us focus on the case $\alpha \in (0,1)$. We first make use of (C.7) which yields

$$\frac{d^2}{dt^2}\mathrm{Li}_\alpha(e^{it}) = -\mathrm{Li}_{\alpha-2}(e^{it}) \quad (t \in (0,\pi]). \tag{3.16}$$

Thus, it is sufficient to show that $\mathrm{Re}\Big(\mathrm{Li}_{\alpha-2}(e^{it})\Big) \leq 0$ for all $t \in (0,\pi)$. Since $\alpha \in (0,1)$ that means $\beta := \alpha - 2 \in (-2,-1)$, we can invoke the integral representation stated in (C.3). By observing $\log(-e^{it}) = i(\pi - t)$, $t \in (0,\pi)$ we deduce

$$\mathrm{Li}_\beta(e^{it}) = \int_0^\infty \frac{s^{-\beta}\sin(\beta\pi/2 - i\cdot s(\pi - t))}{\sinh(\pi s)} ds \quad (t \in (0,\pi)). \tag{3.17}$$

It is enough to consider the real part of the sine term since $s \mapsto s^{-\beta}$ and the hyperbolic sine are functions that only take positive values on $(0,\infty)$. By Euler's formula, it follows

$$\mathrm{Re}\Big(\sin\Big(\beta\pi/2 - i\cdot s(\pi - t)\Big)\Big) = \sin(\beta\pi/2)\cosh(s(\pi - t)).$$

The hyperbolic cosine is bounded below by 1 on $\mathbb{R}$. Moreover, since $-2 < \beta < -1$, we obtain

$$\sin(\beta\pi/2) \in (-1,0).$$

Hence, the integrand in (3.17) is negative for all $s \in (0,\infty)$ which allows us finally to conclude in combination with (3.16) that K_α is convex on $(0,\pi]$. □

In the rest of this section, our goal is to forge a link between the α-capacity and the Li_α-capacity. This is inspired by the observation that for all $z, w \in \mathbb{S}$ with $z \neq w$, the subsequent equations are valid, primarily utilizing (C.4) and (3.9)

$$K_1(\sigma(z,w)) = \mathrm{Re}(\mathrm{Li}_1(w\overline{z})) - L_1 = -\ln(|1 - w\overline{z}|) + \ln(2) = R_1(d_{|\cdot|}(z,w)). \tag{3.18}$$

We immediately obtain that K_1 and $(R_1, d_{|\cdot|})$ induce the same capacities on $\mathbb{S}$. The latter equation does not hold for K_α and $(R_\alpha, d_{|\cdot|})$ if $\alpha \in (0,1)$. However, we will show that the α-capacity and the Li_α-capacity are equivalent for $\alpha \in (0,1)$. To achieve this, we need to perform some computations, a portion of which is delegated to Appendix C. Nevertheless, we do not want to forego presenting the main considerations in the following remark where we mainly rely on (C.6). Be advised that ζ and Γ are the zeta function and the gamma function, respectively. Their definitions can be found in Appendix C. Moreover, for $\beta \in \mathbb{R}$ and $w \in \mathbb{C}_-$ we have

$$w^\beta = \exp\Big(\beta\log(w)\Big),$$

where log is the principal branch of the logarithm (see (3.6)).

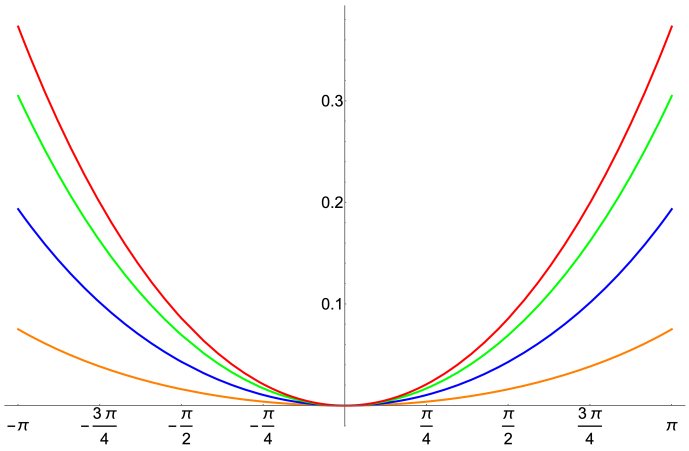

Figure 3.1: Plot of h_α (see (3.21)) for $\alpha = 0.3$ (orange), $\alpha = 0.6$ (blue), $\alpha = 0.8$ (green), and $\alpha = 0.9$ (red).

Remark 3.2.7 Let $\alpha \in (0,1)$. Due to (C.6) we have for all $t \in (-\pi, \pi)$

$$\mathrm{Li}_\alpha(e^{it}) - \Gamma(\alpha-1)(-it)^{\alpha-1} - \zeta(\alpha) = \sum_{k=1}^{\infty} \frac{\zeta(\alpha-k)}{k!}\,(it)^k. \tag{3.19}$$

Moreover, by the definition of the principle branch of the logarithm, we obtain for any $t \in (-\pi, \pi) \setminus \{0\}$

$$\mathrm{Re}\Big((-it)^{\alpha-1}\Big) = \mathrm{Re}\Big(|t|^{\alpha-1} \cdot \exp\Big(-i\pi\frac{\alpha-1}{2}\Big)\Big) = \cos\Big(\pi\frac{\alpha-1}{2}\Big) \cdot |t|^{\alpha-1}.$$

Define

$$\gamma_\alpha := \Gamma(1-\alpha)\cos(\pi(\alpha-1)/2). \tag{3.20}$$

We observe that $|t| = \sigma(1, e^{it})$ for $t \in (-\pi, \pi)$. Taking the real part in (3.19) yields

$$h_\alpha(t) := \mathrm{Re}\Big(\mathrm{Li}_\alpha(e^{it})\Big) - \gamma_\alpha \sigma(1, e^{it})^{\alpha-1} - \zeta(\alpha) = \sum_{k=1}^{\infty} (-1)^k \frac{\zeta(\alpha-2k)}{(2k)!} \cdot t^{2k} \quad (t \in (-\pi, \pi)).$$

We show that h_α is convex on $(-\pi, \pi)$. It is known that $\zeta(-2k) = 0$ for $k \in \mathbb{N}$. Moreover, the zeta function changes the sign at any root which is a consequence of [44, Theorem 9]. Since $\zeta < 0$ on $(-2\pi, 0)$, we deduce

$$(-1)^k \zeta(\alpha - 2k) > 0 \quad (k \in \mathbb{N}).$$

Thus, h_α is convex as a series of convex functions. Moreover, h_α is an even, nonnegative and bounded function on $(-\pi, \pi)$. Consequently, h_α can be continuously extended to the

boundary points $\{\pm\pi\}$. Since convex functions on compact intervals attain their maximum at the boundary, we deduce

$$0 \leq h_\alpha(t) \leq L_\alpha - \gamma_\alpha \pi^{\alpha-1} - \zeta(\alpha) \quad (t \in [-\pi, \pi]). \tag{3.21}$$

By the definition of the Riesz- and the Li_α-kernel function, the latter estimate yields

$$\gamma_\alpha \cdot R_\alpha(\sigma(1,u)) - (L_\alpha - \zeta(\alpha)) \leq K_\alpha(\sigma(1,u)) \leq \gamma_\alpha \cdot R_\alpha(\sigma(1,u)) - \gamma_\alpha \pi^{\alpha-1} \quad (u \in \mathbb{S}).$$

Thus, by the substitution $u = w\overline{z}$, for $z, w \in \mathbb{S}$ and the rotation invariance of σ, we obtain

$$\gamma_\alpha \cdot R_\alpha(\sigma(z,w)) - (L_\alpha - \zeta(\alpha)) \leq K_\alpha(\sigma(z,w)) \leq \gamma_\alpha \cdot R_\alpha(\sigma(z,w)) - \gamma_\alpha \pi^{\alpha-1} \ (z, w \in \mathbb{S}). \tag{3.22}$$

In particular, the kernels K_α and (R_α, σ) are equivalent on $\mathbb{S}$.

For $\alpha \in (0, 1]$, we define the capacity

$$\dot{C}_\alpha := c_{(R_\alpha, \sigma)}.$$

We call $\dot{C}_\alpha$ **spherical Riesz α-capacity** or spherical α-capacity.

Theorem 3.2.8 *Let $\alpha \in (0, 1)$. Then, for any finite $\mu \in \boldsymbol{\mathcal{M}}(\mathbb{S})$*

$$\gamma_\alpha I_{(R_\alpha,\sigma)}(\mu) - (L_\alpha - \zeta(\alpha))\mu(\mathbb{S})^2 \leq I_{K_\alpha}(\mu) \leq \gamma_\alpha I_{(R_\alpha,\sigma)}(\mu) - \gamma_\alpha \pi^{\alpha-1}\mu(\mathbb{S})^2. \tag{3.23}$$

In particular, the Li_α-capacity and the spherical Riesz α-capacity are equivalent on $\mathbb{S}$.

Proof: The estimate in (3.23) is an immediate consequence of (3.22) and the definition of the energy of a measure.

The equivalence of the Li_α-capacity and the spherical α-capacity follows by the equivalence of their respective kernels on $\mathbb{S}$ (see Remark 3.2.7) and Proposition 3.1.11. □

Remark 3.2.9 Let $\alpha \in (0, 1)$. We observe

$$|1 - e^{it}| = |2\sin(t/2)| \quad (t \in [-\pi, \pi])$$

to obtain that

$$(0, \pi] \ni t \mapsto R_\alpha(|1 - e^{it}|) - R_\alpha(\sigma(1, e^{it})) = R_\alpha\Big(2\sin(t/2)\Big) - R_\alpha(t)$$

is a continuous and increasing function which vanishes as $t \to 0$. Thus, the function $R_\alpha \circ d_{|\cdot|}(1, \cdot) - R_\alpha \circ \sigma(1, \cdot)$ is continuous on $\mathbb{S}$ with minimum 0 and maximum $R_\alpha(2) - R_\alpha(\pi)$. By the substitution $u = w\overline{z}$ for $z, w \in \mathbb{S}$ and observing that $|1 - w\overline{z}| = |z - w|$, we obtain

$$R_\alpha(\sigma(z,w)) \leq R_\alpha(|z-w|) \leq R_\alpha(\sigma(z,w)) + R_\alpha(2) - R_\alpha(\pi) \quad (z, w \in \mathbb{S}). \tag{3.24}$$

For $\alpha = 1$ one can show that

$$R_1(\sigma(z,w)) \leq R_1(|z-w|) \leq R_1(\sigma(z,w)) + \ln(\pi/2) \quad (z,w \in \mathbb{S}).$$

In particular, for any $\alpha \in (0,1]$, the kernels (R_α, σ) and $(R_\alpha, d_{|\cdot|})$ are equivalent. By the same arguments as in Theorem 3.2.8 one obtains for any finite $\mu \in \boldsymbol{\mathcal{M}}(\mathbb{S})$

$$I_{(R_\alpha,\sigma)}(\mu) \leq I_{(R_\alpha,d_{|\cdot|})}(\mu) \leq I_{(R_\alpha,\sigma)}(\mu) + (R_\alpha(2) - R_\alpha(\pi))\mu(\mathbb{S})^2$$

if $\alpha \in (0,1)$ and

$$I_{(R_1,\sigma)}(\mu) \leq I_{(R_1,d_{|\cdot|})}(\mu) \leq I_{(R_1,\sigma)}(\mu) + \ln(\pi/2)\mu(\mathbb{S})^2.$$

In particular, for any $\alpha \in (0,1]$, the spherical α-capacity and the α-capacity are equivalent.

For any $\alpha \in (0,1]$, Theorem 3.2.8, Remark 3.2.9 and (3.18) imply that the Li_α-capacity c_α, the α-capacity C_α, and the spherical α-capacity $\dot{C}_\alpha$ are equivalent on $\mathbb{S}$. This motivates the following definition.

Definition 3.2.10 We call a set **α-polar** if it is c_α-polar. Moreover, we say that a property holds **α-quasieverywhere** if it holds c_α-quasieverywhere.

To finish this section, we return to Frostman's theorem, which provides essential estimates related to the potential function of an equilibrium measure for a compact set E and the energy of E. However, Frostman's theorem only gives an upper bound for the potential function on the support of the equilibrium measure. In the following definition, we recall a concept that addresses the global boundedness of potential functions.

Definition 3.2.11 Let (X,d) be a compact metric space. A kernel K is said to satisfy **Frostman's maximum principle** (on X) if for any finite $\mu \in \boldsymbol{\mathcal{M}}(X)$

$$K\mu \leq M \quad \text{on } \operatorname{supp}(\mu)$$

already implies that

$$K\mu \leq M \quad \text{on } X.$$

Theorem 3.2.12 (Frostman's maximum principle)
If K is a convex kernel function, then the kernel (K,σ) satisfies Frostman's maximum principle. In particular, for any $\alpha \in (0,1]$, the kernels (K_α,σ) and (R_α,σ) satisfy Frostman's maximum principle.

Proof: Let $\mu \in \boldsymbol{\mathcal{M}}(\mathbb{S})$ be finite and suppose that $(K,\sigma)\mu \le M$. We may assume without loss of generality that $M < \infty$ and that $\operatorname{supp}(\mu) \neq \mathbb{S}$.

We have to show that the potential function $(K,\sigma)\mu$ is bounded by M on $\mathbb{S} \setminus \operatorname{supp}(\mu)$. Since $\operatorname{supp}(\mu)$ is closed by definition, it is enough to show that $(K,\sigma)\mu$ is bounded by M on each arc in $\mathbb{S} \setminus \operatorname{supp}(\mu)$ with boundary points in $\operatorname{supp}(\mu)$, that means for any $u, v \in \partial A$ with $s := \arg(u) \le \arg(v) =: t$ and $A_u^v \cap \operatorname{supp}(\mu) = \{u, v\}$, we have to verify

$$(K,\sigma)\mu \le M \quad \text{on } A_u^v.$$

Let us assume that $z \in A_u^v$. With the transformation

$$T : [t, t+2\pi) \to \mathbb{S}, \quad T(\omega) := e^{i\omega}$$

we derive by observing $\operatorname{supp}(\mu^{T^{-1}}) \subset [t, s+2\pi]$ the subsequent identity

$$(K,\sigma)\mu(z) = \int_{\mathbb{S}} K(\sigma(z,w))d\mu(w) = \int_{[t,s+2\pi]} K(\sigma(z,e^{i\omega}))d\mu^{T^{-1}}(\omega) \quad (z \in A_u^v).$$

The function

$$[s,t] \ni \theta \mapsto \sigma(e^{i\theta}, w) \in [0,\pi] \tag{3.25}$$

is piecewise linear and *concave* for all $w \in \mathbb{S} \setminus A_u^v$. As K is convex and nonnegative, we infer that

$$[s,t] \ni \theta \mapsto K(\sigma(e^{i\theta}, w)) \in [0,\infty]$$

defines a convex function for all $w \in \mathbb{S} \setminus A_u^v$. As convexity is preserved under integration (see for example [8, p. 79]), we deduce that

$$[s,t] \ni \theta \mapsto (K,\sigma)\mu(e^{i\theta}) \quad (z \in A_u^v)$$

is convex. A well-known result of convex optimization states that convex functions on compact intervals attain their maximum at the boundary. In our situation, this means

$$(K,\sigma)\mu(z) \le \max\left\{(K,\sigma)\mu(u),\ (K,\sigma)\mu(v)\right\} \le M,$$

where the last inequality is due to $u, v \in \operatorname{supp}(\mu)$.

Since the kernel functions R_α and K_α are convex for any $\alpha \in (0,1]$, we obtain the assertion. □

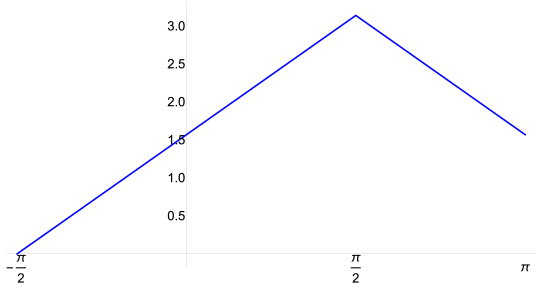

Figure 3.2: Plot of (3.25) in case that $w = -i$, $s = -\pi/2$ and $t = \pi$.

3.3 An energy formula for the Li_α-kernel

In this section, we aim to establish an energy formula in terms of the Fourier coefficients of a measure. It is inspired by the result of Hare and Roginskaya [29, Theorem 2.2] who were able to show a multidimensional energy estimate for finite, positive Borel measures μ on $\mathbb{S}^n$ relating the Riesz energy of μ to the Fourier coefficients of μ. However, we are able to prove an energy formula involving the Li_α-kernel which is equal to the logarithmic kernel with the Euclidean metric in the case where $\alpha = 1$ as we discovered in (3.18). Furthermore, we will show that the formula holds not only for positive Borel measures but also for complex measures. It will turn out that the energy formula is a natural extension of the energy formula for the logarithmic kernel (see e.g. [16, Theorem 2.4.4]).

In case (X, Σ) is a measurable space, $\boldsymbol{\mathcal{M}}_{\mathbb{C}}(X, \Sigma)$ denotes the set of complex measures on (X, Σ) and if X is a topological space, we write $\boldsymbol{\mathcal{M}}_{\mathbb{C}}(X)$ for the set of all complex Borel measures on X. For $\mu \in \boldsymbol{\mathcal{M}}_{\mathbb{C}}(X, \Sigma)$, $|\mu|$ denotes the corresponding total variation measure. Note that we have gathered some results on complex measures in Appendix A. We will refer to them in the following course of this section if the need arises. For $\mu \in \boldsymbol{\mathcal{M}}_{\mathbb{C}}(X, \Sigma)$, we write $\overline{\mu}$ for the **complex conjugated measure** of μ which is defined by

$$\overline{\mu}(A) := \overline{\mu(A)} \quad (A \in \Sigma).$$

This prepares us to extend Definition 3.1.1 to the mutual energy of two complex measures as follows.

Definition 3.3.1 Let (X, d) be a separable metric space and let $K = (K, d)$ be a kernel. Suppose that μ_1 and μ_2 in $\boldsymbol{\mathcal{M}}_{\mathbb{C}}(X)$ such that

$$\int\int K(d(x, y))d|\mu_1|(y)d|\mu_2|(x) < \infty$$

or that μ_1 and μ_2 in $\boldsymbol{\mathcal{M}}(X)$. Then the **mutual energy** of μ_1 and μ_2 is defined by

$$I_K(\mu_1, \mu_2) := I_{(K,d)}(\mu_1, \mu_2) := \int\int K(d(x, y))d\mu_1(y)d\overline{\mu_2}(x) = \int K(d(x, y))d(\mu_1 \otimes \overline{\mu_2})(y, x).$$

In the special case that $\mu_1 = \mu_2 =: \mu$, we define in accordance with Definition 3.1.1

$$I_K(\mu) := I_K(\mu, \mu) \tag{3.26}$$

which is called the energy of the measure μ.

We note that

$$I_K(\mu_1, \mu_2) = \overline{I_K(\mu_2, \mu_1)}$$

which is an immediate consequence of Fubini's theorem for complex measures (see Remark A.6). Moreover, (3.26) is an extension of Definition 3.1.1 since positive measures satisfy the condition $\mu = \overline{\mu}$.

In the sequel, we shift our attention to the case $(X, d) = (\mathbb{S}, \sigma)$. Note that Definition 2.1.7 already encompasses the definition of the k-th Fourier coefficient of a measure $\mu \in \boldsymbol{\mathcal{M}}_{\mathbb{C}}(\mathbb{S})$ which is due to the Riesz representation theorem for complex measures (see [65, Theorem 6.19]).

We observe that any $\mu \in \boldsymbol{\mathcal{M}}_{\mathbb{C}}(\mathbb{S})$ fulfills the following identity

$$\widehat{\overline{\mu}}(k) = \overline{\widehat{\mu}(-k)} \quad (k \in \mathbb{Z}) \tag{3.27}$$

which is a consequence of the polar decomposition $\mu = hd|\mu|$ (see also (A.2)) and

$$\overline{\mu} = \overline{hd|\mu|} = \overline{h}\, d|\mu|.$$

Due to (3.27), we have in particular

$$\widehat{\mu}(k)\widehat{\overline{\mu}}(-k) = |\widehat{\mu}(k)|^2 \quad (k \in \mathbb{Z}). \tag{3.28}$$

To establish the energy formula, we require some estimates.

Remark 3.3.2 Let $\alpha \in (0, 1]$. By the integral representation in (3.11) we know that Li_α is a holomorphic function on $\mathbb{C} \setminus [1, \infty)$. Moreover, $z \mapsto (-\log(z))^{\alpha-1}$ is a holomorphic function on $\mathbb{C}_- \setminus [1, \infty)$ whose real part can be continuously extended to $\mathbb{C} \setminus [1, \infty)$ by (C.10). Thus, by (C.5), we have that $f_\alpha : \mathrm{cl}(\mathbb{D}) \to \mathbb{R}$

$$f_\alpha(z) := -\zeta(\alpha) + \begin{cases} \mathrm{Re}\Big(\mathrm{Li}_\alpha(z) - \Gamma(1-\alpha)\big(-\log(z)\big)^{\alpha-1}\Big), & z \in \mathrm{cl}(\mathbb{D}) \setminus \{0\} \\ 0, & z = 0 \end{cases} \tag{3.29}$$

is a continuous function which is harmonic on $\mathbb{D} \setminus (-1, 0]$. By the maximum principle for harmonic functions (see e.g. [61, Theorem 1.1.8]), we obtain that f_α takes its maximum and minimum on $\mathbb{S} \cup [-1, 0]$. By (C.9) and (C.5), we get that $f_\alpha|_{[-1,\,0]}$ is increasing. Consequently, $f_\alpha|_{[-1,\,0]}$ takes its maximum at 0 with $f_\alpha(0) = -\zeta(\alpha)$ and its minimum at -1 with $f_\alpha(-1) = L_\alpha - \gamma_\alpha \pi^{\alpha-1} - \zeta(\alpha)$ where γ_α was defined in (3.20). Since $f_\alpha(e^{it}) = h_\alpha(t)$ for $t \in [-\pi, \pi]$ (see Remark 3.2.7 for the definition of h_α), we infer that $f_\alpha|_{\mathbb{S}}$ attains its minimum at 1 and consequently

$$0 \leq f_\alpha(z) \leq -\zeta(\alpha) \quad (z \in \mathrm{cl}(\mathbb{D})).$$

By the definition of the Riesz kernel function R_α and the latter estimate, we obtain by invoking (C.9) for any $r \in [0, 1]$ and $u \in \mathbb{S}$

$$L_\alpha \leq \mathrm{Re}\Big(\mathrm{Li}_\alpha(ru)\Big) \leq \Gamma(1-\alpha)\Big|\ln(1/r) + i\arg(u)\Big|^{\alpha-1} \leq \Gamma(1-\alpha) \cdot R_\alpha(\sigma(1, u)). \tag{3.30}$$

We are now prepared to present a formula that relates the mutual energy concerning the Li_α-kernel of two complex measures to their respective Fourier coefficients.

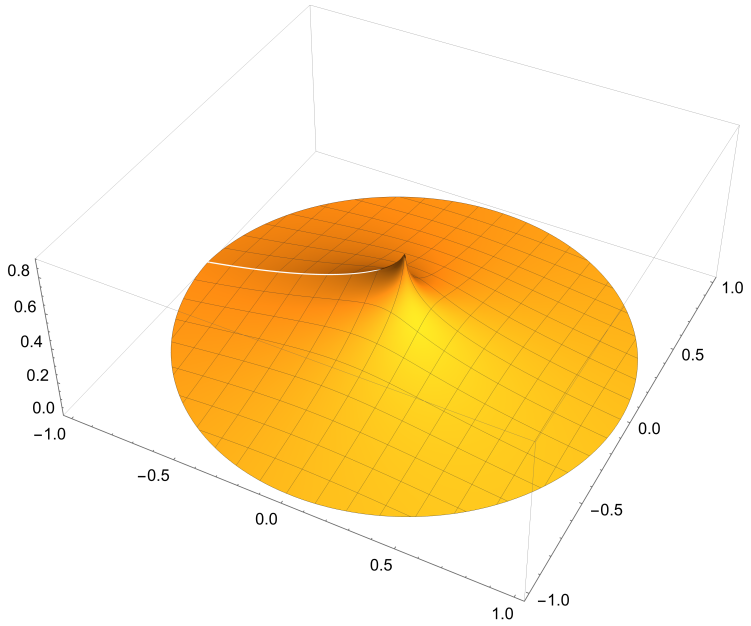

Figure 3.3: Plot of f_α (see (3.29)) for $\alpha = 0.3$.

Theorem 3.3.3 (The energy formula for the Li_α-kernel)
Let $\alpha \in (0,1]$ and suppose that $\mu_1, \mu_2 \in \boldsymbol{\mathcal{M}}_{\mathbb{C}}(\mathbb{S})$ such that $I_{K_\alpha}(|\mu_1|, |\mu_2|) < \infty$. Then

$$I_{K_\alpha}(\mu_1, \mu_2) = \lim_{r \to 1^-} \frac{1}{2} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{r^{|k|}}{|k|^\alpha} \widehat{\mu_1}(-k) \widehat{\overline{\mu_2}}(k) - L_\alpha \mu_1(\mathbb{S}) \overline{\mu_2}(\mathbb{S}). \tag{3.31}$$

If $\mu_1 = \mu_2 =: \mu$, then

$$I_{K_\alpha}(\mu) = \frac{1}{2} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{1}{|k|^\alpha} |\widehat{\mu}(k)|^2 - L_\alpha |\mu(\mathbb{S})|^2. \tag{3.32}$$

Moreover, if $\mu_1, \mu_2 \in \boldsymbol{\mathcal{M}}(\mathbb{S})$ are finite and if $I_{K_\alpha}(\mu_1, \mu_2) = \infty$, then

$$\sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{1}{|k|^\alpha} \widehat{\mu_1}(-k) \widehat{\mu_2}(k) = \infty. \tag{3.33}$$

Proof: We divide the proof into two parts. We first consider the case where the mutual energy $I_{K_\alpha}(|\mu_1|, |\mu_2|)$ is finite. Hereafter, we discuss the case $I_{K_\alpha}(\mu_1, \mu_2) = \infty$ for positive measures μ_1 and μ_2.

Let us assume that $I_{K_\alpha}(|\mu_1|, |\mu_2|) < \infty$. First, by the definition of the Li_α-kernel function in combination with (3.9), we have for all $u \in \mathbb{S}$ the identity

$$K_\alpha(\sigma(1,u)) = \mathrm{Re}(\mathrm{Li}_\alpha(u)) - L_\alpha.$$

We define

$$R(r,u) := 2\mathrm{Re}\Big(\mathrm{Li}_\alpha(ru)\Big) = \mathrm{Li}_\alpha(ru) + \overline{\mathrm{Li}_\alpha(ru)} = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{r^{|k|}}{|k|^\alpha}\, u^k \qquad (r\in[0,1], u\in\mathbb{S}),$$

where we used the power series of the polylogarithm on $\mathbb{D}$ (see (C.1)). With Fubini's theorem and the dominated convergence theorem, we obtain via the substitution $u := w\overline{z}$, where $z, w \in \mathbb{S}$, for fixed $r \in [0,1)$ the following identity

$$\begin{aligned}\int R(r, w\overline{z}) d(\mu_1 \otimes \overline{\mu_2})(w,z) &= \iint R(r,w\overline{z}) d\mu_1(w) d\overline{\mu_2}(z) \\ &= \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{r^{|k|}}{|k|^\alpha}\bigg(\iint (w\overline{z})^k d\mu_1(w) d\overline{\mu_2}(z)\bigg).\end{aligned} \tag{3.34}$$

Now, we employ the orthogonality relation in (2.6) which yields for any $k \in \mathbb{Z}\setminus\{0\}$

$$\iint (w\overline{z})^k \, d\mu_1(w) d\overline{\mu_2}(z) = \int (\overline{w})^{-k}\, d\mu_1(w) \int \overline{z}^k \, d\overline{\mu_2}(z) = \widehat{\mu_1}(-k)\widehat{\overline{\mu_2}}(k). \tag{3.35}$$

We go ahead by substituting (3.35) into (3.34) and obtain

$$\int R(r, w\overline{z})\, d(\mu_1 \otimes \overline{\mu_2})(w,z) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{r^{|k|}}{|k|^\alpha} \widehat{\mu_1}(-k)\widehat{\overline{\mu_2}}(k). \tag{3.36}$$

It remains to show that the limit and the integral can be interchanged as r tends to 1. To address this, we invoke (3.22) which implies that $I_{K_\alpha}(|\mu_1|,|\mu_2|)$ is finite if and only if the Riesz energy with the spherical metric $I_{(R_\alpha,\sigma)}(|\mu_1|,|\mu_2|)$ is finite. Additionally, we can utilize (3.30) which yields for all $r\in[0,1)$ and $z,w\in\mathbb{S}$

$$L_\alpha \leq \mathrm{Re}\Big(\mathrm{Li}_\alpha(rw\overline{z})\Big) \leq \Gamma(1-\alpha)\cdot R_\alpha(\sigma(z,w)).$$

We have identified an upper bound for the function R which remains independent of r. Moreover, this majorant is integrable with respect to the product measure $|\mu_1|\otimes|\mu_2|$. By applying the dominated convergence theorem, we can interchange the order of limit and integral. Furthermore, since the function $R(\cdot,u)$ is continuous for all $u\in\mathbb{S}\setminus\{1\}$, it follows that $R(1,u) = \lim_{r\to 1} R(r,u)$. We obtain

$$\begin{aligned}\int R(1,w\overline{z})\; d(\mu_1\otimes\overline{\mu_2})(w,z) &= \iint \lim_{r\to 1} R(r,w\overline{z})\; d(\mu_1\otimes\overline{\mu_2})(w,z) \\ &= \lim_{r\to 1} \iint R(r,w\overline{z})\; d(\mu_1\otimes\overline{\mu_2})(w,z) \\ &= \lim_{r\to 1} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{r^{|k|}}{|k|^\alpha}\widehat{\mu_1}(-k)\widehat{\overline{\mu_2}}(k).\end{aligned}$$

Finally, by the definition of K_α, it only remains to divide the latter equality by 2 and add the constant

$$\iint -L_\alpha d\mu_1(w)d\overline{\mu_2}(z) = -L_\alpha \mu_1(\mathbb{S})\overline{\mu_2}(\mathbb{S})$$

which yields

$$I_{K_\alpha}(\mu_1,\mu_2) = \int K_\alpha(\sigma(1,w\overline{z}))d(\mu_1\otimes\overline{\mu_2})(w,z) = \lim_{r\to 1}\frac{1}{2}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{r^{|k|}}{|k|^\alpha}\widehat{\mu_1}(-k)\widehat{\overline{\mu_2}}(k) - L_\alpha\mu_1(\mathbb{S})\overline{\mu_2}(\mathbb{S}).$$

Now, let us consider the energy of the measure μ. Due to (3.26), we have the identity $I_{K_\alpha}(\mu) = I_{K_\alpha}(\mu,\mu)$. Thus, we can apply (3.31). By (3.28) and the monotone convergence theorem, we obtain

$$I_{K_\alpha}(\mu) = \lim_{r\to 1}\frac{1}{2}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{r^{|k|}}{|k|^\alpha}|\widehat{\mu}(k)|^2 - L_\alpha|\mu(\mathbb{S})|^2 = \frac{1}{2}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{1}{|k|^\alpha}|\widehat{\mu}(k)|^2 - L_\alpha|\mu(\mathbb{S})|^2.$$

It remains to discuss the case $I_{K_\alpha}(\mu_1,\mu_2) = \infty$ where $\mu_1,\mu_2 \in \boldsymbol{\mathcal{M}}(\mathbb{S})$ are finite. As a consequence of (3.30), the family

$$\Big\{R(r,\cdot) : r\in(0,1)\Big\} \tag{3.37}$$

is uniformly bounded below by L_α. An application of Fatou's lemma yields

$$\begin{aligned}
2I_{K_\alpha}(\mu_1,\mu_2) + 2L_\alpha\mu_1(\mathbb{S})\mu_2(\mathbb{S}) &= \int \liminf_{r\to 1} R(r,w\overline{z})\; d(\mu_1\otimes\mu_2)(w,z)\\
&\le \liminf_{r\to 1}\int R(r,w\overline{z})\; d(\mu_1\otimes\mu_2)(w,z)\\
&= \liminf_{r\to 1}\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{r^{|k|}}{|k|^\alpha}\widehat{\mu_1}(-k)\widehat{\mu_2}(k).
\end{aligned}$$

Since the left-hand side of the above estimate is infinite, we get that the right-hand side is infinite as well. In particular, we deduce

$$\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{1}{|k|^\alpha}\widehat{\mu_1}(-k)\widehat{\mu_2}(k) = \infty$$

which yields the desired result. □

Remark 3.3.4 In case $\mu\in\boldsymbol{\mathcal{M}}(\mathbb{S})$ is a finite measure, we have

$$\widehat{\mu}(k) = \overline{\widehat{\mu}(-k)} \quad (k\in\mathbb{Z}).$$

In particular, by (3.32) and (3.33), we obtain

$$I_{K_\alpha}(\mu) = \sum_{k=1}^{\infty} \frac{1}{k^\alpha} |\widehat{\mu}(k)|^2 - L_\alpha \mu(\mathbb{S})^2 \tag{3.38}$$

which is an extension of the energy formula for the logarithmic kernel on $\mathbb{S}$ (see for instance [16, Theorem 2.4.4]).

We now present an application of the energy formula. Typically, finding an equilibrium measure of a compact set is not straightforward. It becomes even more difficult to deduce uniqueness. We address this issue for the Li_α-kernel and the set $\mathbb{S}$ by leveraging our previous result.

Example 3.3.5 Let $\alpha \in (0, 1]$. The energy formula allows us to determine the K_α-equilibrium measure of the set $\mathbb{S}$. Due to (2.6) the arc measure m satisfies the identity

$$\widehat{m}(k) = \int_{\mathbb{S}} p_{-k} dm = \delta_{0,k} \quad (k \in \mathbb{Z}).$$

The energy formula yields

$$I_{K_\alpha}(m) = \sum_{k=1}^{\infty} \frac{1}{k^\alpha} |\widehat{m}(k)|^2 - L_\alpha m(\mathbb{S})^2 = -L_\alpha$$

for all $\alpha \in (0, 1]$. Moreover, due to the injectivity of the Fourier transform, the arc measure m is the unique minimizer of (3.38) which results in the conclusion that m must be the unique K_α-equilibrium measure of the set $\mathbb{S}$. In particular, we have

$$c_\alpha(\mathbb{S}) = -\frac{1}{L_\alpha}.$$

3.4 Complement-equivalence of capacities

In the first section of this chapter, we dealt in particular with sets of vanishing (outer) capacity. Now, we consider compact sets $E \subset X$ such that $X \setminus E$ has the same (inner) capacity as X. We start with the following definition.

Definition 3.4.1 Let d, d' be two metrics on X such that (X, d) and (X, d') are two compact metric spaces and suppose that $K = (K, d)$ and $L = (L, d')$ are two kernels. We say that the corresponding capacities c_K and c_L are **complement-equivalent** if for all $E \subset X$ compact

$$c_K(X \setminus E) = c_K(X) \quad \text{if and only if} \quad c_L(X \setminus E) = c_L(X).$$

Next, we want to give a sufficient condition for the complement-equivalence of two capacities.

Proposition 3.4.2 *Let d, d' be two metrics on X such that (X, d) and (X, d') are two compact metric spaces and suppose that $K = (K, d)$ and $L = (L, d')$ are two kernels. Then the corresponding capacities c_K and c_L are complement-equivalent if the following two conditions hold.*

i) There exists a measure $\nu \in \mathrm{Prob}(X)$ which is the unique K-equilibrium measure and the unique L-equilibrium measure for X.

ii) There exist a constant $A \in (0, \infty)$ and a function $R : X \times X \to \mathbb{R}$ which is continuous with respect to the metrics d and d' such that

$$A \cdot K(d(x, y)) + R(x, y) = L(d'(x, y)) \quad (x, y) \in X \times X.$$

Proof: Let A and R be the required constant and function, respectively, as specified above and assume that $E \subset X$ is a compact set such that

$$c_K(X \setminus E) = c_K(X)$$

holds. As E is compact and thus $X \setminus E$ is open, Definition 3.1.6 gives us a sequence of compact sets $F_n \nearrow X \setminus E,\ n \in \mathbb{N}$ such that

$$c_K(F_n) \to c_K(X \setminus E) = c_K(X) \quad (n \to \infty). \tag{3.39}$$

Let us pick K-equilibrium measures, denoted by ν_n, for the sets F_n which exist due to Remark 3.1.5. Since $\{\nu_n : n \in \mathbb{N}\}$ is a tight family (this is obvious since X is compact), we conclude by Prokhorov's theorem (see Theorem A.8) that there exists a set $J \subset \mathbb{N}$ such that

$$\nu_n \to \nu \text{ weak}^* \quad (J \ni n \to \infty) \tag{3.40}$$

for a probability measure ν on X. Due to (3.39), ν is a K-equilibrium measure for the set X, and since ν is unique, it is also the L-equilibrium measure for the set X. Without loss of generality, we assume $J = \mathbb{N}$ to simplify the notation. The monotonicity of the energy and assumption ii) yields

$$I_L(X \setminus E) \leq I_L(F_n) \leq \int\int A \cdot K(d(x, y)) + R(x, y)\, d\nu_n(y) d\nu_n(x) \quad (n \in \mathbb{N})$$

and as R is continuous we go ahead by using the weak* convergence of the sequence (ν_n) (observe that $\nu_n \otimes \nu_n \to \nu \otimes \nu$ weak*) and establish

$$\begin{aligned} I_L(X \setminus E) &\leq \lim_{n\to\infty} \int\int A \cdot K(d(x, y))\, d\nu_n(y) d\nu_n(x) + \lim_{n\to\infty} \int\int R(x, y)\, d(\nu_n \otimes \nu_n)(y, x) \\ &= A \cdot I_K(\nu) + \int\int R(x, y)\, d(\nu \otimes \nu)(y, x) \\ &= \int\int A \cdot K(d(x, y)) + R(x, y)\, d\nu(y) d\nu(x) \\ &= I_L(\nu). \end{aligned}$$

Now, we utilize that ν is the unique L-equilibrium measure of X and that $X \setminus E$ is open, hence capacitable. Thus,

$$c_L(X \setminus E) = c_L^*(X \setminus E) = \frac{1}{I_L(X \setminus E)} \geq \frac{1}{I_L(\nu)} = c_L(X).$$

The converse inequality follows by the monotonicity of outer capacities (see Remark 3.1.7) and therefore

$$c_L(X \setminus E) = c_L(X).$$

The reverse direction is analogous to the proof of the "if" part by considering the function

$$K(d(x,y)) = A^{-1}\Big(L(d'(x,y)) - R(x,y)\Big).$$

□

We show that the Li_α-capacity is complement-equivalent to the Riesz α-capacity and to the spherical Riesz α-capacity. Again, there is no need to consider the case $\alpha = 1$ due to (3.18). Thus, let us suppose $\alpha \in (0,1)$ throughout the following remark where we show that the assumptions of Proposition 3.4.2 are satisfied for the kernels (R_α, σ) and $(R_\alpha, d_{|\cdot|})$ as well as K_α and (R_α, σ).

Remark 3.4.3 Let $\alpha \in (0,1)$. First, we observe that

$$\mathbb{S} \times \mathbb{S} \ni (z,w) \mapsto \begin{cases} R_\alpha(|z-w|) - R_\alpha(\sigma(z,w)) & z \neq w \\ 0 & z = w \end{cases} \tag{3.41}$$

is continuous due to Remark 3.2.9. By recalling the definition of the function h_α and the constant γ_α in Remark 3.2.7, we obtain by the rotation invariance of σ that

$$h_\alpha(\sigma(z,w)) - L_\alpha + \zeta(\alpha) = K_\alpha(\sigma(z,w)) - \gamma_\alpha \cdot R_\alpha(\sigma(z,w))$$

defines a continuous function on $\mathbb{S} \times \mathbb{S}$. Consequently, the kernels (R_α, σ) and $(R_\alpha, d_{|\cdot|})$ as well as K_α and (R_α, σ) fulfill the second condition of Proposition 3.4.2.

It remains to verify that the kernels satisfy the first condition of Proposition 3.4.2. Thanks to Example 3.3.5, we know that m is the unique K_α-equilibrium measure of $\mathbb{S}$. The result in [24, Corollary 6] indicates that m is also the unique $(R_\alpha, d_{|\cdot|})$-equilibrium measure on $\mathbb{S}$. Moreover, since (R_α, σ) fulfills Frostman's maximum principle (see Theorem 3.2.12) and since $(R_\alpha, \sigma)m$ is constant on $\mathbb{S}$, Frostman's theorem yields that m is the unique (R_α, σ)-equilibrium measure of $\mathbb{S}$.

Applying Proposition 3.4.2 yields the following result.

Proposition 3.4.4 *For all $\alpha \in (0,1]$, the capacities c_α, C_α, and $\dot{C}_\alpha$ are complement-equivalent on $\mathbb{S}$.*

We introduce a terminology that is naturally related to the concept of polar sets (see Definition 3.1.8)

Definition 3.4.5 Suppose that (X, d) is a compact metric space and that K is a kernel. We call a compact set $E \subset X$ c_K**-complement-polar** if

$$c_K(X \setminus E) = c_K(X).$$

Given that $X = \mathbb{S}$, $\alpha \in (0, 1]$ and $K = K_\alpha$, then we refer to a set E that fulfills the aforementioned condition as α**-complement-polar**.

Proposition 3.4.4 yields that a set is α-complement-polar if and only if it is C_α- or, equivalently, $\dot{C}_\alpha$-complement-polar.

The property that a compact set $E \subset X$ is c_K-complement-polar proves to be quite abstract. It is challenging to provide a more geometric interpretation of this condition. To conclude this section, let us take a closer look at the properties of compact sets that are complement-polar.

Remark 3.4.6 Assume that (X, d) is a compact metric space and that K is a kernel. Moreover, let ν be a K-equilibrium measure of X and suppose that $E \subset X$ is a compact set such that $\nu(E) = 0$. Since ν is a *finite* Borel measure and since (X, d) is compact, ν is a Radon measure (see e.g. [7, Theorem 7.1.7]). Consequently, we are able to find an increasing sequence of compact sets $B_n \subset X \setminus E$ with the property

$$\nu(B_n) > 1 - \frac{1}{n}.$$

Define

$$\mu_n := \frac{1}{\nu(B_n)} \cdot \nu|_{B_n}.$$

Clearly, $\operatorname{supp}(\mu_n) \subset B_n$ and the monotonicity of the energy leads to

$$\begin{aligned} I_K(X) \leq I_K(X \setminus E) \leq I_K(B_n) &\leq \int_{B_n} \int_{B_n} K(d(x, y)) d\mu_n(y) d\mu_n(x) \\ &= \frac{1}{\nu(B_n)^2} \int_{B_n} \int_{B_n} K(d(x, y)) d\nu(y) d\nu(x). \end{aligned}$$

The latter integral can be estimated from above by replacing the underlying set B_n with X. This, combined with the previous estimate leads to

$$I_K(X) \leq I_K(X \setminus E) \leq \frac{1}{\nu(B_n)^2} I_K(X) \quad (n \in \mathbb{N}).$$

Since $\nu(B_n) \to 1$ as $n \to \infty$, we obtain

$$I_K(X) = I_K(X \setminus E).$$

In particular, if $X = \mathbb{S}$, then m is the K_α-equilibrium measure of $\mathbb{S}$ and we deduce that for all $\alpha \in (0, 1]$, any compact m-null set E is α-complement-polar!

However, this condition is not necessary for the α-complement-polarity of a set E as the following example shows.

Example 3.4.7 Let $\alpha \in (0,1)$. Throughout this example, we consider the spherical Riesz α-capacity $\dot{C}_\alpha$ which is complement-equivalent to the Li_α-capacity c_α due to Proposition 3.4.4. We aim to construct a sequence of compact sets (E_j) in $\mathbb{S}$ independent of α which meets the two subsequent properties

i) For all $j \in \mathbb{N}$ and $\alpha \in (0,1)$, E_j is α-complement-polar,

ii) $m(E_j) > 1 - 2^{-j}$ $(j \in \mathbb{N})$.

For this purpose let

$$B_{n,\varepsilon} := \left\{ z \in \mathbb{S} : \left| \arg(z^n) \right| < \frac{\pi}{2} \cdot \varepsilon \right\} \qquad \Big(n \in 2\mathbb{N},\ \varepsilon \in (0,1] \Big),$$

and let $\mu_{n,\varepsilon}$ be the uniform distribution on $B_{n,\varepsilon}$. One can show that the set $B_{n,\varepsilon}$ has n connected components which we denote by $B_{n,\varepsilon}^{(k)}$ counterclockwise for $k \in \{0,\dots,n-1\}$ where we begin the numeration with the arc that contains the element 1. Throughout this example let $\zeta_t := e^{2\pi i/t}$ for $t \in [1,\infty)$. This enables us to express the n connected components of $B_{n,\varepsilon}$ as follows

$$B_{n,\varepsilon}^{(k)} = \zeta_n^k \cdot B_{n,\varepsilon}^{(0)}.$$

Given that the corresponding arcs of the set $B_{n,\varepsilon}$ are pairwise disjoint, we infer

$$\int_{B_{n,\varepsilon}} \int_{B_{n,\varepsilon}} \sigma(z,w)^{\alpha-1} d\mu_{n,\varepsilon}(z) d\mu_{n,\varepsilon}(w) = \sum_{l=0}^{n-1} \sum_{k=0}^{n-1} \int_{B_{n,\varepsilon}^{(l)}} \int_{B_{n,\varepsilon}^{(k)}} \sigma(z,w)^{\alpha-1} d\mu_{n,\varepsilon}(z) d\mu_{n,\varepsilon}(w),$$

and applying the rotation invariance of the metric σ yields

$$\int_{B_{n,\varepsilon}} \int_{B_{n,\varepsilon}} \sigma(z,w)^{\alpha-1} d\mu_{n,\varepsilon}(z) d\mu_{n,\varepsilon}(w) = n \sum_{k=0}^{n-1} \int_{B_{n,\varepsilon}^{(0)}} \int_{B_{n,\varepsilon}^{(k)}} \sigma(z,w)^{\alpha-1} d\mu_{n,\varepsilon}(z) d\mu_{n,\varepsilon}(w). \quad (3.42)$$

We will consider the first summand on the right-hand side in (3.42). This requires some supplementary computations. Consider the arc $A_t := A_{\zeta_{-t}}^{\zeta_t}$ for $2 < t < \infty$ (recall (3.7)). By the definition of σ and the arc measure, we obtain

$$\begin{aligned} \int_{A_t} \int_{A_t} \sigma(z,w)^{\alpha-1} dm(z) dm(w) &= \int_{A_t} \int_{A_t} |\arg(\overline{z}w)|^{\alpha-1} dm(z) dm(w) \\ &= \frac{1}{(2\pi)^2} \int_{-2\pi/t}^{2\pi/t} \int_{-2\pi/t}^{2\pi/t} |x-y|^{\alpha-1}\, dy\, dx \qquad (3.43) \\ &= \frac{2^{2\alpha+1}}{\pi^{1-\alpha}\, \alpha(\alpha+1)} t^{-(1+\alpha)}. \end{aligned}$$

Furthermore, observe that $m(B_{n,\varepsilon}) = \varepsilon/2$ which yields

$$m|_{B_{n,\varepsilon}} = \frac{\varepsilon}{2}\mu_{n,\varepsilon}.$$

Thus, we get the following identity by applying (3.43) with $t = n/\varepsilon$ to the first summand of (3.42)

$$n\int_{B_{n,\varepsilon}^{(0)}}\int_{B_{n,\varepsilon}^{(0)}}\sigma(z,w)^{\alpha-1}d\mu_{n,\varepsilon}(z)d\mu_{n,\varepsilon}(w) = n\cdot(2/\varepsilon)^2\cdot\frac{2^{2\alpha+1}}{\pi^{1-\alpha}\,\alpha(\alpha+1)}\cdot(\varepsilon/n)^{1+\alpha} = M_\alpha\cdot n^{-\alpha}\varepsilon^{\alpha-1}, \tag{3.44}$$

where $M_\alpha := 2^{2\alpha+3}/(\pi^4\alpha(\alpha+1))$ is a constant only depending on α.

Let us now consider the remainder sum in (3.42). First, for all $k \in \{1,\dots,n-1\}$, we have

$$\mu_{n,\varepsilon}(B_{n,\varepsilon}^{(k)}) = 1/n,$$

and therefore we estimate

$$\begin{aligned}\int_{B_{n,\varepsilon}^{(0)}}\int_{B_{n,\varepsilon}^{(k)}}\sigma(z,w)^{\alpha-1}d\mu_{n,\varepsilon}(z)d\mu_{n,\varepsilon}(w) &\le 1/n^2\cdot\sup\left\{\sigma(z,w)^{\alpha-1}:\ z\in B_{n,\varepsilon}^{(0)},\ w\in\zeta_n^k\cdot B_{n,\varepsilon}^{(0)}\right\}\\ &= 1/n^2\cdot\sup\left\{\sigma(z,\zeta_n^k\cdot w)^{\alpha-1}:\ z,w\in B_{n,\varepsilon}^{(0)},\right\}\\ &= 1/n^2\cdot\begin{cases}\sigma\left(\zeta_n^{\varepsilon/2},\zeta_n^{k-\varepsilon/2}\right)^{\alpha-1}, & k\le\frac{n}{2}\\ \sigma\left(\zeta_n^{-\varepsilon/2},\zeta_n^{k+\varepsilon/2}\right)^{\alpha-1}, & k>\frac{n}{2}.\end{cases}\end{aligned}$$

Recall that $(R_\alpha,\sigma)m$ is the potential function of (R_α,σ) with respect to the arc measure m. By the rotation invariance of m and σ, we deduce that $(R_\alpha,\sigma)m$ is constant on $\mathbb{S}$. Since m is the (R_α,σ)-equilibrium measure for the set $\mathbb{S}$, we obtain

$$(R_\alpha,\sigma)m(1) = I_{(R_\alpha,\sigma)}(m) = \frac{1}{\dot{C}_\alpha(\mathbb{S})}.$$

Thus, for any $\varepsilon\in(0,1)$

$$n\sum_{k=1}^{n-1}\int_{B_{n,\varepsilon}^{(0)}}\int_{B_{n,\varepsilon}^{(k)}}\sigma(z,w)^{\alpha-1}d\mu_{n,\varepsilon}(z)d\mu_{n,\varepsilon}(w) \le \sum_{k=1}^{n-1}\frac{1}{n}\sigma\left(1,\zeta_n^{k-\varepsilon}\right)^{\alpha-1}\to(R_\alpha,\sigma)m(1) = \frac{1}{\dot{C}_\alpha(\mathbb{S})} \tag{3.45}$$

as n tends to infinity.

Let us now consider the sequence

$$A_j := B_{2^{(j^2)},2^{-j}}\quad(j\in\mathbb{N}).$$

Define $\mu_j := \mu_{2^{(j^2)},2^{-j}}$. First, by observing (3.44), we have for all $\alpha\in(0,1)$

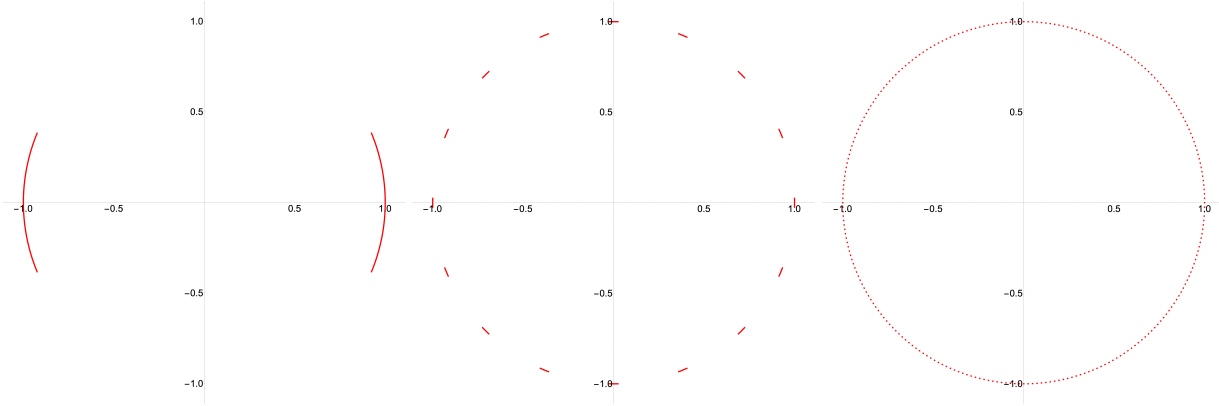

Figure 3.4: Plot of the sets A_1, A_2 and A_3 from left to right.

$$2^{j^2}\int_{A_j^{(0)}}\int_{A_j^{(0)}}\sigma(z,w)^{\alpha-1}d\mu_j(z)d\mu_j(w) = M_\alpha(2^{j^2})^{-\alpha}(2^{-j})^{\alpha-1} \to 0 \quad (j\to\infty).$$

Moreover, A_j is open in $\mathbb{S}$ for all $j\in\mathbb{N}$ and therefore, any union of elements of the family $\{A_j : j\in\mathbb{N}\}$ is open, hence capacitable. Combined with the monotonicity of outer capacities and (3.45), we have for all $j\in\mathbb{N}$ and $k\geq j$

$$\dot{C}_\alpha\left(\bigcup_{l\geq j}A_l\right) = \dot{C}^*_\alpha\left(\bigcup_{l\geq j}A_l\right) \geq \dot{C}^*_\alpha(A_k) = \frac{1}{I_{(R_\alpha,\sigma)}(A_k)} \geq \frac{1}{I_{(R_\alpha,\sigma)}(\mu_k)} \to \dot{C}_\alpha(\mathbb{S}) \quad (k\to\infty),$$

and consequently

$$\dot{C}_\alpha\left(\bigcup_{k\geq j}A_k\right) = \dot{C}_\alpha(\mathbb{S}) \quad (j\in\mathbb{N}).$$

Now, let

$$E_j := \mathbb{S}\setminus\bigcup_{k\geq j}A_k = \bigcap_{k\geq j}(\mathbb{S}\setminus A_k) \quad (j\in\mathbb{N}).$$

Observe that E_j is compact as the intersection of compact sets. The arc measure of E_j can be estimated by using σ-subadditivity and the geometric series as follows

$$m(E_j) = 1 - m\left(\bigcup_{k\geq j}A_k\right) \geq 1 - \sum_{k=j}^{\infty}m(A_k) = 1-\sum_{k=j}^{\infty}2^{-k-1} = 1-2^{-j}.$$

We have finished the construction since the sequence (E_j) fulfills i) and ii).

3.5 The Bessel capacity

In this section, our objective is to recall a more general concept of capacities that encompasses the specific capacities discussed in the first section of this chapter in case that

$(X,d) = (\mathbb{R}^N, d_{|\cdot|}^{(N)})$ where $d_{|\cdot|}^{(N)} = d_{|\cdot|}$ is the Euclidean distance on $\mathbb{R}^N$. Among other things, we will see that these capacities satisfy all the properties mentioned in Remark 3.1.7. Our approach primarily follows the work of Adams and Hedberg [1].

In what follows, let $\psi : \mathbb{R}^N \times \mathbb{R}^N \to [0,\infty)$ be a function such that $x \mapsto \psi(x,y)$ is lower semicontinuous for all $y \in \mathbb{R}^N$, and $y \mapsto \psi(x,y)$ is Borel measurable for all $x \in \mathbb{R}^N$. For a finite Borel measure $\mu \in \boldsymbol{\mathcal{M}}(\mathbb{R}^N)$, we define

$$(\Psi\mu)(x) := \int \psi(x,y)\, d\mu(y) \qquad (x \in \mathbb{R}^N). \tag{3.46}$$

In case μ has a density function f with respect to the N-dimensional Lebesgue measure λ_N, we occasionally write Ψf.

Moreover, we use the notation $L_+^p(\mathbb{R}^N)$ for the set of nonnegative functions in $L^p(\mathbb{R}^N)$. The aforementioned conventions and notation lay the groundwork for the following notion of capacities.

Definition 3.5.1 Let $E \subset \mathbb{R}^N$, $p \in (1,\infty)$, and

$$\Omega_E := \Big\{ f \in L_+^p(\mathbb{R}^N) \,:\, \Psi f \geq 1 \text{ on } E \Big\}.$$

The corresponding (ψ,p)-capacity is defined by

$$C_{\psi,p}(E) := \inf \left\{ \int f^p \, d\lambda_N \,:\, f \in \Omega_E \right\}, \tag{3.47}$$

where $\inf \emptyset = \infty$.

A well-known result of (ψ,p)-capacities is that they fulfill the properties in Remark 3.1.7 which is also known as capacitability theorem (see e.g. [1, Theorem 2.3.11]). In the sequel, we will examine the relationship between (ψ,p)-capacities and α-capacities. More precisely, we will see that (ψ,p)-capacities correspond to α-capacities for suitable ψ and p. The idea of choosing the appropriate ψ leads to the following definition.

Definition 3.5.2 Let $\alpha > 0$. The function $G_\alpha^{(N)} : (0,\infty) \to [0,\infty)$,

$$G_\alpha^{(N)}(t) := M_\alpha \int_0^\infty s^{\frac{\alpha-N}{2}} e^{-\frac{\pi t^2}{s}} \cdot e^{-\frac{s}{4\pi}} \frac{ds}{s} \qquad (t \in (0,\infty))$$

and $G_\alpha^{(N)}(0) := \lim_{t\to 0} G_\alpha^{(N)}(t) = +\infty$ is called **Bessel kernel function** of order α, where $M_\alpha^{-1} := (4\pi)^{\alpha/2}\Gamma(\alpha/2)$. Moreover, we define

$$G_\alpha^{(N)} := (G_\alpha^{(N)}, d_{|\cdot|})$$

which is called **Bessel kernel** of degree α. In cases where the dimension is implicitly understood from the context, we write G_α instead of $G_\alpha^{(N)}$.

We also extend the definition of the Riesz kernel function by $R_\alpha^{(N)} : (0,\infty) \to [0,\infty)$,

$$R_\alpha^{(N)}(t) := \begin{cases} 1/t^{N-\alpha}, & \alpha \in (0,N) \\ \max\{\ln(2/t), 0\}, & \alpha = N \end{cases} \qquad (t \in (0,\infty))$$

and $R_\alpha^{(N)}(0) := \lim_{t\to 0} R_\alpha^{(N)}(t) = +\infty$. Keep in mind that we have $R_{\alpha-1} = R_\alpha^{(2)}$ according to Definition 3.2.1.

By [1, (1.2.6)], [1, (1.2.14)] and [1, (1.2.23)], for any $0 < \alpha \le N$ and $r \in (0,\infty)$, there are constants $A_{\alpha,r} \in [1,\infty)$ and $M_{\alpha,r} \in [0,\infty)$ such that

$$\frac{1}{A_\alpha} \cdot R_\alpha^{(N)}(t) - M_{\alpha,r} \le G_\alpha^{(N)}(t) \le A_\alpha \cdot R_\alpha^{(N)}(t) + M_{\alpha,r} \quad (t \in [0,r]). \tag{3.48}$$

Thus, according to Definition 3.1.10, the kernels $(G_\alpha^{(N)}, d_{|\cdot|})$ and $(R_\alpha^{(N)}, d_{|\cdot|})$ are equivalent on any compact subset of $\mathbb{R}^N$.

In the sequel, we recall the definition of Riesz and Bessel (α,p)-capacities motivated by [1, p. 19].

Definition 3.5.3 Let $\alpha \in (0,\infty)$ and $p \in (1,\infty)$ such that $\alpha p \le N$. If

$$\psi(x,y) = (G_\alpha^{(N)} \circ d_{|\cdot|})(x,y) \quad (x,y \in \mathbb{R}^N), \tag{3.49}$$

then the corresponding (ψ,p)-capacity (see (3.47)) is denoted as **Bessel** (α,p)**-capacity**. For this capacity, we use the symbol $\mathrm{Cap}_{\alpha,p}^{(N)}$ or $\mathrm{Cap}_{\alpha,p}$ if the dimension is clear from the context.

In case

$$\psi(x,y) = (R_\alpha^{(N)} \circ d_{|\cdot|})(x,y) \quad (x,y \in \mathbb{R}^N), \tag{3.50}$$

the corresponding (ψ,p)-capacity is denoted as **Riesz** (α,p)**-capacity**. For this capacity, we use the symbol $C_{\alpha,p}^{(N)}$ or $C_{\alpha,p}$ if the dimension is clear from the context.

Observe that it is feasible to consider Bessel and Riesz capacities for $\alpha p > N$. However, in this case, every singleton possesses positive capacity, leading to the conclusion that only the empty set has vanishing capacity (see e.g. [1, p. 38]). Bessel capacities play an important role in determining the size of a set. Among its various uses, we will see that Bessel capacities allow a more refined categorization of the size of a set compared to what can be achieved with α-capacities.

A significant finding related to Bessel and Riesz capacities is that for $0 < \alpha p < N$ and any $r \in (0,\infty)$ there is a constant $A_r \in [1,\infty)$ such that

$$\mathrm{Cap}_{\alpha,p}(E) \le C_{\alpha,p}(E) \le A_r \cdot \mathrm{Cap}_{\alpha,p}(E) \quad \Big(E \subset \mathbb{R}^N,\ \mathrm{diam}(E) \le r\Big). \tag{3.51}$$

This result can be found in [1, Proposition 5.1.4]. Owing to (3.51), we infer that the bounded sets of vanishing Bessel capacity and the bounded sets of vanishing Riesz capacity correspond.

In alignement with [1, p. 50], we refer to sets of vanishing Bessel (α, p)-capacity as (α, p)-**polar**. Accordingly, we say that a property holds (α, p)-**quasieverywhere** if it holds up to a set of vanishing Bessel (α, p)-capacity.

Note that the Bessel kernel and its corresponding capacity are better suited to the Euclidean space $\mathbb{R}^N$, primarily due to its rapid decay that prevents potential integrability issues at infinity.

Similarly to the second section of this chapter, our primary attention is focused on compact sets $E \subset \mathbb{S}$. We are especially interested in sets of vanishing Bessel and Riesz capacity.

In the sequel, we will link Bessel capacities and α-capacities. For this purpose, we quote the following result which is one of the central theorems in the theory of (ψ, p)-capacities. A proof can be found in [1, Theorem 2.5.1]. Be advised that q plays the role of the conjugate exponent.

Theorem 3.5.4 *Let $p \in (1, \infty)$ and let $E \subset \mathbb{R}^N$ be a compact set. Then we have*

$$C_{\psi,p}(E)^{1/p} = \sup\left\{\mu(E) : \mu \in \boldsymbol{\mathcal{M}}(\mathbb{R}^N) \text{ finite, } \operatorname{supp}(\mu) \subset E, \ \|\Psi\mu\|_{L^q(\mathbb{R}^N)} \leq 1\right\} \quad (3.52)$$

The theorem is essential to establish a connection between α-capacities and Bessel capacities. However, first, we recall that the convolution of a Borel measurable function $F : \mathbb{R}^N \to [0, \infty)$ and $\mu \in \boldsymbol{\mathcal{M}}(\mathbb{R}^N)$ is given by

$$(F * \mu)(x) := \int F(x - y)\, d\mu(y) \quad (x \in \mathbb{R}^N).$$

In the case $h \in \mathcal{M}(\mathbb{R}^N)$ such that $\mu = h d\lambda_N$, we also write $F * h$ instead of $F * \mu$. Finally, if $K = (K, d_{|\cdot|})$ is a kernel, we write $K * \mu$ instead of $K \circ d_{|\cdot|}(0, \cdot) * \mu$ and $K * h$ instead of $K \circ d_{|\cdot|}(0, \cdot) * h$.

Remark 3.5.5 Assume that $p = q = 2$, $0 < \alpha \leq N/2$, ψ as specified in (3.49) and that $\mu \in \boldsymbol{\mathcal{M}}(\mathbb{R}^N)$ finite with $\operatorname{supp}(\mu) \subset E$ where $E \subset \mathbb{R}^N$ is a compact set. By (3.46) and the definition of the convolution for kernels, we have

$$(\Psi\mu)(x) = \int G_\alpha(|x - y|) d\mu(y) = \Big(G_\alpha * \mu\Big)(x) \quad (x \in \mathbb{R}^N).$$

Consequently, by utilizing Fubini's theorem and observing $\operatorname{supp}(\mu) \subset E$, we derive

$$\|\Psi\mu\|^2_{L^2(\mathbb{R}^N)} = \int_E \int_E \left(\int G_\alpha(|x - y|) G_\alpha(|z - y|) d\lambda_N(y) \right) d\mu(x) d\mu(z).$$

By the linear transformation $\mathbb{R}^N \ni y \mapsto y + x \in \mathbb{R}^N$, we can represent the braced term on the right-hand side as well as

$$\int G_\alpha(|y|) G_\alpha\Big(\Big|(z - x) - y\Big|\Big) d\lambda_N(y) = (G_\alpha * G_\alpha)\Big(|z - x|\Big).$$

Furthermore, by recalling the definition of the energy of a measure (see Definition 3.1.1), and due to the identity $(G_\alpha * G_\alpha)(|x-z|) = G_{2\alpha}(|x-z|)$ (compare [1, p. 21]), as a consequence of the group property of Bessel kernels (see [1, p. 10, (1.2.9)]), we arrive at the following identity

$$\|\Psi\mu\|^2_{L^2(\mathbb{R}^N)} = \int_E \int_E \Big(G_{2\alpha} \circ d_{|\cdot|}\Big)(z,x)\, d\mu(x) d\mu(z) = I_{G_{2\alpha}}(\mu). \tag{3.53}$$

As $\mu \mapsto \|\Psi\mu\|_{L^2(\mathbb{R}^N)}$ is homogeneous of degree 1 and since $p = 2$, we can rewrite (3.52) as follows

$$\mathrm{Cap}_{\alpha,2}(E)^{1/2} = \sup\left\{ \frac{\mu(E)}{\|\Psi\mu\|_{L^2(\mathbb{R}^N)}} : \mu \in \boldsymbol{\mathcal{M}}(\mathbb{R}^N) \setminus \{0\} \text{ finite, } \mathrm{supp}(\mu) \subset E \right\}. \tag{3.54}$$

Again, by the homogeneity of $\mu \mapsto \|\Psi\mu\|_{L^2(\mathbb{R}^N)}$, we can replace the condition $\mu \in \boldsymbol{\mathcal{M}}(\mathbb{R}^N) \setminus \{0\}$ finite by requiring $\mu \in \mathrm{Prob}(\mathbb{R}^N)$ in (3.54). Taking the square on both sides and observing (3.53) finally yields

$$\mathrm{Cap}_{\alpha,2}(E) = \sup\left\{ \frac{1}{I_{G_{2\alpha}}(\mu)} : \ \mu \in \mathrm{Prob}(\mathbb{R}^N), \ \mathrm{supp}(\mu) \subset E \right\}.$$

Hence, we deduce that

$$\mathrm{Cap}_{\alpha,2}(E) = c_{G_{2\alpha}}(E), \tag{3.55}$$

where the expression on the right-hand side is interpreted as in Definition 3.1.2.

In the sequel, we use the notation

$$B(a,r) := B^{(N)}(a,r) := \Big\{ x \in \mathbb{R}^N : |a-x| < r \Big\}$$

for the (open) ball with center $a \in \mathbb{R}^N$ and radius $r \in (0,\infty)$ in the N-dimensional Euclidean space.

Proposition 3.5.6 *Let $X = (\mathrm{cl}(B(a,r)), d_{|\cdot|})$ for some $a \in \mathbb{R}^N$ and $r \in (0,\infty)$. Then for all $0 < \alpha \le N/2$*

$$\mathrm{Cap}_{\alpha,2}(E) = c^*_{G_{2\alpha}}(E) \quad (E \subset X) \tag{3.56}$$

Proof: Due to Remark 3.5.5, we know that (3.56) is valid for any compact $E \subset X$. We show that the above identity transfers to arbitrary $E \subset X$. For any open set $U \subset X$ there exists a sequence of compact sets F_k such that $F_k \nearrow U$. By [1, Proposition 2.3.12] and (3.55) as well as the definition of inner capacities (see Definition 3.1.6), we obtain

$$\mathrm{Cap}_{\alpha,2}(U) = \lim_{k\to\infty} \mathrm{Cap}_{\alpha,2}(F_k) = \lim_{k\to\infty} c_{G_{2\alpha}}(F_k) = c_{G_{2\alpha}}(U).$$

Since $\mathrm{Cap}_{\alpha,2}$ is outer due to [1, Proposition 2.3.5], we get

$$\mathrm{Cap}_{\alpha,2}(E) = c^*_{G_{2\alpha}}(E) \quad (E \subset X).$$

□

If $N = 2$, then we obtain as an immediate consequence of Proposition 3.5.6

$$\mathrm{Cap}^{(2)}_{\alpha,2}(E) = c^*_{G^{(2)}_{2\alpha}}(E) \quad (E \subset \mathbb{S}). \tag{3.57}$$

Before proving the equivalence of $(\alpha, 2)$-capacities and α-capacities on $\mathbb{S} \subset \mathbb{R}^2$, our objective is to contrast Bessel capacities across different dimensions, specifically when $p = 2$. At this point, the reader is advised that we will extend this result in the ensuing section which will incorporate the theory of Wolff potentials.

Proposition 3.5.7 *Let $M, N \in \mathbb{N}$ with $M \leq N$ and let $0 < \alpha \leq M/2$. Moreover, let $\beta := \alpha - (M - N)/2$. Define $\iota : \mathbb{R}^M \to \mathbb{R}^N$ as the canonical embedding of $\mathbb{R}^M$ to $\mathbb{R}^N$. Then*

$$\mathrm{Cap}^{(M)}_{\alpha,2}(E) = \mathrm{Cap}^{(N)}_{\beta,2}(\iota(E)) \quad (E \subset \mathbb{R}^M). \tag{3.58}$$

Proof: We first assume that $E \subset \mathbb{R}^M$ is a compact set. Due to Remark 3.5.5, we have the identities

$$\mathrm{Cap}^{(M)}_{\alpha,2}(E) = c_{G^{(M)}_{2\alpha}}(E) \qquad \text{and} \qquad \mathrm{Cap}^{(N)}_{\beta,2}(\iota(E)) = c_{G^{(N)}_{2\beta}}(\iota(E)). \tag{3.59}$$

Beyond this, any $\mu' \in \mathrm{Prob}(\mathbb{R}^N)$ with $\mathrm{supp}(\mu') \subset \iota(E)$ can be uniquely identified with a measure $\mu \in \mathrm{Prob}(\mathbb{R}^M)$ supported on E with

$$\mu'|_{\mathbb{R}^M} = \mu. \tag{3.60}$$

Since $\beta = \alpha - (M - N)/2$, we have $G^{(M)}_{2\alpha} = G^{(N)}_{2\beta}$ and thus in particular

$$G^{(M)}_{2\alpha}(|x|) = G^{(N)}_{2\beta}(|\iota(x)|) \quad (x \in \mathbb{R}^M).$$

As $E \subset \mathbb{R}^M$ is a compact set, we have for the corresponding energies

$$I_{G^{(M)}_{2\alpha}}(\mu) = I_{G^{(N)}_{2\beta}}(\mu').$$

This, together with (3.59) and (3.60) yields the assertion for $E \subset \mathbb{R}^M$ compact.

Let us extend (3.58) to $U \subset \mathbb{R}^M$ open. There is an increasing sequence of compact sets $F_k \nearrow U$ and by [1, Proposition 2.3.12], we obtain

$$\mathrm{Cap}^{(M)}_{\alpha,2}(U) = \lim_{k\to\infty} \mathrm{Cap}^{(M)}_{\alpha,2}(F_k) = \lim_{k\to\infty} \mathrm{Cap}^{(N)}_{\beta,2}\Big(\iota(F_k)\Big) = \mathrm{Cap}^{(N)}_{\beta,2}\Big(\iota(U)\Big).$$

Finally, (3.58) holds for arbitrary $E \subset \mathbb{R}^M$ since Bessel capacities are outer. □

In the setting of the foregoing proposition, we find in particular that $\mathrm{Cap}^{(M)}_{\alpha,2}$ and $\mathrm{Cap}^{(N)}_{\beta,2} \circ \iota$ are equivalent on $\mathbb{R}^M$. As announced, we will later obtain the equivalence for $p \in (1, \infty)$ with the theory of Wolff potentials.

Proposition 3.5.8 (Equivalence of Li_α- and $(\alpha,2)$-capacities on $\mathbb{S}$)
Let $N \geq 2$. Define $\iota : \mathbb{C} \to \mathbb{R}^N$ as the canonical embedding of $\mathbb{C}$ to $\mathbb{R}^N$. Then for all $(N-1)/2 < \alpha \leq N/2$ there is a constant $A \in [1,\infty)$ such that

$$\frac{1}{A} \cdot \mathrm{Cap}^{(N)}_{\alpha,2}(\iota(E)) \leq c^*_{2\alpha+1-N}(E) \leq A \cdot \mathrm{Cap}^{(N)}_{\alpha,2}(\iota(E)) \quad (E \subset \mathbb{S}).$$

Proof: By Proposition 3.5.7 and Equation (3.57), we have for $\beta := \alpha - (N-2)/2$

$$\mathrm{Cap}^{(N)}_{\alpha,2}(\iota(E)) = \mathrm{Cap}^{(2)}_{\beta,2}(E) = c^*_{G^{(2)}_{2\beta}}(E) \quad (E \subset \mathbb{S}).$$

Since the kernels $(G^{(2)}_{2\beta}, d_{|\cdot|})$ and $(R^{(2)}_{2\beta}, d_{|\cdot|})$ are equivalent on $\mathbb{S}$ (see (3.48)) and since $R^{(2)}_{2\beta} = R_{2\beta-1}$, we infer the existence of a constant $A_1 \in [1,\infty)$ such that

$$c^*_{G^{(2)}_{2\beta}}(E) \leq A_1 c^*_{(R^{(2)}_{2\beta}, d_{|\cdot|})}(E) = A_1 C^*_{2\beta-1}(E) \quad (E \subset \mathbb{S}).$$

Due to Theorem 3.2.8 and Remark 3.2.9, the capacities $C_{2\beta-1}$ and $c_{2\beta-1}$ are equivalent on $\mathbb{S}$ and consequently, there is a constant $A_2 \in [1,\infty)$ such that (observe $2\beta-1 = 2\alpha+1-N$)

$$C^*_{2\beta-1}(E) \leq A_2 c^*_{2\alpha+1-N}(E) \quad (E \subset \mathbb{S}).$$

Thus, by $A := A_1 A_2$, we obtain

$$\frac{1}{A} \cdot \mathrm{Cap}^{(N)}_{\alpha,2}(\iota(E)) \leq c^*_{2\alpha+1-N}(E) \quad (E \subset \mathbb{S}).$$

In complete analogy, one can show the second inequality. □

3.6 Relations between capacities and metric quantities

Numerous queries can be effectively addressed when viewed through the lens of capacities. However, the geometrical meaning of capacities is not transparent. As a result, we devote a significant part of this section to contrasting (α,p)-capacities with metric outer measures. We will subsequently observe that (α,p)-capacities exhibit an inherent order. Finally, we explore the connection of (α,p)-capacities across various dimensions.

For $\alpha \in [0,N]$ we recall that Λ_α denotes the α-dimensional outer Hausdorff measure defined on the power set of $\mathbb{R}^N$. If Λ_α is restricted to the σ-algebra of the Λ_α-measurable sets, then Λ_α is called the α-dimensional Hausdorff measure. More details and information are provided in Appendix B.

The first part of the ensuing theorem was given in [1, Theorem 5.1.9]. The second part can be found in [1, Theorem 5.1.13]. We prefer to state the theorem by making an assertion concerning the α-dimensional outer Hausdorff measure. It should be noted that the assertions of the previously mentioned theorems are somewhat stronger.

Theorem 3.6.1 *Let $N \in \mathbb{N}$, $p \in (1,\infty)$ and $0 < \alpha p \leq N$. Moreover, let $E \subset \mathbb{R}^N$. Then the following statements hold*

1)

$$\Lambda_{N-\alpha p}(E) < \infty \quad \textit{implies} \quad \mathrm{Cap}_{\alpha,p}(E) = 0. \tag{3.61}$$

2) For all $\varepsilon > 0$

$$\mathrm{Cap}_{\alpha,p}(E) = 0 \quad \textit{implies} \quad \Lambda_{N-\alpha p+\varepsilon}(E) = 0. \tag{3.62}$$

For both of the aforementioned statements, it is unfeasible to derive the converse direction which, in the first case, is inferred by considering the function

$$h_q(t) := t^{N-\alpha p}(-\ln(t))^{-1/(q-1)}$$

and applying [1, Theorem 5.4.2]. For the second part we apply [1, Theorem 5.4.1] to the function

$$h_\varepsilon(t) := \begin{cases} t^{N-\alpha p+\varepsilon}, & \varepsilon > 0 \\ -t^{N-\alpha p}\ln(t)^{-1}, & \varepsilon = 0. \end{cases}$$

The above theorem and Proposition 3.5.8 enable us to establish relations between the α-capacity and the Hausdorff dimension (see Definition B.2) of sets $E \subset \mathbb{S}$. We recall that $\dim(E)$ denotes the Hausdorff dimension of the set E.

In the setting $p = 2$ and $(N-1)/2 < \alpha \leq N/2$, we obtain for $E \subset \mathbb{S}$ the following two implications:

i)

$$\Lambda_{N-2\alpha}(E) = 0 \quad \text{implies} \quad 0 = \mathrm{Cap}^{(N)}_{\alpha,2}(E) = c^*_{2\alpha+1-N}(E),$$

in particular, for $\beta \in (0,1]$

$$\Lambda_{1-\beta}(E) = 0 \quad \text{implies} \quad c^*_\beta(E) = 0.$$

ii)

$$0 = c^*_{2\alpha+1-N}(E) = \mathrm{Cap}^{(N)}_{\alpha,2}(E) \quad \text{implies} \quad \Lambda_{N-2\alpha+\varepsilon}(E) = 0 \qquad (\varepsilon > 0),$$

in particular, for $\beta \in (0,1]$

$$c^*_\beta(E) = 0 \quad \text{implies} \quad \Lambda_{1-\beta+\varepsilon}(E) = 0 \qquad (\varepsilon > 0).$$

As seen above, it is known that the reverse implications do not apply to the statements i) and ii). However, by combining i) and ii) and recalling $\sup\emptyset = 0$, we find for $E \subset \mathbb{S}$

$$\dim(E) = \sup\Big\{\beta \in [0,1) : c^*_{1-\beta}(E) > 0\Big\}. \tag{3.63}$$

The following theorem illustrates that (α,p)-capacities fulfill a natural ordering. Moreover, we will see that capacities with different parameters are not equivalent. A proof of this result can be found in [1, Theorem 5.5.1].

Theorem 3.6.2 *Let $E \subset \mathbb{R}^N$ be a bounded set. Then we get the following order of Bessel capacities:*

1)
$$\mathrm{Cap}_{\alpha,p}(E) = 0 \quad \textit{implies} \quad \mathrm{Cap}_{\beta,q}(E) = 0 \qquad 0 < \beta q < \alpha p < N.$$

2)
$$\mathrm{Cap}_{\alpha,p}(E) = 0 \quad \textit{implies} \quad \mathrm{Cap}_{\beta,q}(E) = 0 \qquad 0 < \beta q = \alpha p < N,\ p < q.$$

3)
$$\mathrm{Cap}_{\alpha,p}(E) = 0 \quad \textit{implies} \quad \mathrm{Cap}_{\beta,q}(E) = 0 \qquad 0 < \beta q < \alpha p = N.$$

4)
$$\mathrm{Cap}_{\alpha,p}(E) = 0 \quad \textit{implies} \quad \mathrm{Cap}_{\beta,q}(E) = 0 \qquad \beta q = \alpha p = N,\ p < q.$$

In all four cases above the contrary direction is generally false, i.e. there are bounded sets $E \subset \mathbb{R}^N$ such that $\mathrm{Cap}_{\beta,q}(E) = 0$ but $\mathrm{Cap}_{\alpha,p}(E) > 0$.

Within the context of Proposition 3.5.8, we have delineated an equivalence between Bessel (α, p)-capacities and α-capacities, in case $p = 2$. Moreover, Theorem 3.6.2 says that for all $\alpha \in (1/2, 1)$, there lacks a corresponding pair (β, q) where $\beta > 0$ and $q \in (1, \infty)\setminus\{2\}$ such that the two capacities $\mathrm{Cap}^{(2)}_{\beta,q}$ and $c_{2\alpha-1}$ are equivalent. This observation underscores that Bessel capacities are an extension of α-capacities.

To complete this section, we establish some relations between (α, p)-capacities defined on spaces $\mathbb{R}^M$ and $\mathbb{R}^N$. In this context, we will compare the capacity of sets E that are contained in the lower dimensional space $\mathbb{R}^M$ and its embedding $\iota(E)$ in $\mathbb{R}^N$. The result is an extension of Proposition 3.5.7 for arbitrary $p \in (1, \infty)$.

In the following discussion, it becomes essential to apply Wolff's inequality, originally stated in [32]. In simple terms, Wolff's inequality gives an estimate concerning the function $\Psi\mu$, with ψ serving as $G_\alpha \circ d_{|\cdot|}$ (see (3.46)), and the Wolff potential (cf. [1, Definition 4.5.1]) which is defined by $W^{(N)}_{\alpha,p}\mu : \mathbb{R}^N \to [0, \infty]$,

$$W^{(N)}_{\alpha,p}\mu(x) := \sum_{k=0}^{\infty} \Big(2^{k(N-\alpha p)}\mu\Big(B^{(N)}(x, 2^{-k})\Big)\Big)^{q-1}$$

where μ is a finite measure in $\boldsymbol{\mathcal{M}}(\mathbb{R}^N)$, $p \in (1, \infty)$, $0 < \alpha p \leq N$ and q denotes the conjugate exponent of p. It proves advantageous to employ Wolff's inequality in the form presented in [1, Theorem 4.5.2].

Besides results on Wolff potentials, we also need another finding on Bessel capacities which can be seen as a generalization of the concept of equilibrium measures. For $E \subset \mathbb{R}^N$, there exists a finite measure $\mu^E \in \boldsymbol{\mathcal{M}}(\mathbb{R}^N)$ supported on $\mathrm{cl}(E)$ such that

$$\mu^E(\mathrm{cl}(E)) = \int \Big(G_\alpha * \mu^E\Big)^q d\lambda_N = \mathrm{Cap}_{\alpha,p}(E). \tag{3.64}$$

We will denote the measure μ^E as (G_α, p)-equilibrium measure of E. The foregoing assertion is a consequence of [1, Theorem 2.5.6].

Proposition 3.6.3 *Let $M, N \in \mathbb{N}$ with $M \leq N$. Furthermore, assume $\alpha, \beta > 0$ and $p \in (1, \infty)$ such that $M - \alpha p = N - \beta p$ holds. Then there exists a constant $A \in [1, \infty)$ which only depends on α, p, M and N such that for all $E \subset \mathbb{R}^M$ the following estimate holds*

$$\frac{1}{A} \cdot \mathrm{Cap}^{(M)}_{\alpha,p}(E) \leq \mathrm{Cap}^{(N)}_{\beta,p}(\iota(E)) \leq A \cdot \mathrm{Cap}^{(M)}_{\alpha,p}(E), \tag{3.65}$$

where $\iota : \mathbb{R}^M \to \mathbb{R}^N$ is the canonical embedding of $\mathbb{R}^M$ to $\mathbb{R}^N$. In particular, we have

$$\mathrm{Cap}^{(M)}_{\alpha,p}(E) = 0 \qquad \text{if and only if} \qquad \mathrm{Cap}^{(N)}_{\beta,p}(\iota(E)) = 0. \tag{3.66}$$

Proof: First of all, thanks to Wolff's inequality (see [1, Theorem 4.5.2]), there exists a constant $A \in [1, \infty)$ only depending on α, p and M such that

$$\frac{1}{A} \int \Big(G^{(M)}_\alpha * \mu\Big)^q d\lambda_M \leq \int W^{(M)}_{\alpha,p} \mu \, d\mu \leq A \int \Big(G^{(M)}_\alpha * \mu\Big)^q d\lambda_M \quad (\mu \in \boldsymbol{\mathcal{M}}(\mathbb{R}^M)).$$

Let $E \subset \mathbb{R}^M$. Due to (3.64), we can infer the existence of a $(G^{(M)}_\alpha, p)$-equilibrium measure μ^E for E supported on $\mathrm{cl}(E) \subset \mathbb{R}^M$ such that

$$\int \Big(G^{(M)}_\alpha * \mu^E\Big)^q d\lambda_M = \mathrm{Cap}^{(M)}_{\alpha,p}(E).$$

Together with the above estimate, we obtain by μ^E in the role of μ

$$\frac{1}{A} \cdot \mathrm{Cap}^{(M)}_{\alpha,p}(E) \leq \int W^{(M)}_{\alpha,p} \mu^E \, d\mu^E \leq A \cdot \mathrm{Cap}^{(M)}_{\alpha,p}(E). \tag{3.67}$$

The same techniques can be applied to get (3.67) with M replaced by N and E replaced by $\iota(E)$. Here, the constant will depend on α, p and N. We go ahead by considering the maximum of these two constants.

Our approach is to work out a connection between Wolff's potential in different dimensions by the following argumentation. As μ^E is an equilibrium measure of E, we find that the support of μ^E is contained in $\mathrm{cl}(E)$. Since $\mathrm{cl}(\iota(E)) = \iota(\mathrm{cl}(E))$, an equilibrium measure for $\iota(E)$ is given by the trivial continuation of μ^E in $\mathbb{R}^N$ which we subsequently denote as μ'^E. Consequently, for any $x \in E$

$$\mu^E\Big(B^{(M)}(x,r)\Big) = \mu'^E\Big(\iota(B^{(M)}(x,r))\Big) = \mu'^E\Big(B^{(N)}(x,r)\Big) \qquad (r > 0),$$

where we note that $B^{(M)}(x,r) = \iota^{-1}(B^{(N)}(x,r))$ and $E \subset \mathbb{R}^M$.

Now, we use the assumption $M - \alpha p = N - \beta p$ and the above identity to conclude that for any $x \in E$ the following equations hold

$$\begin{aligned} W^{(M)}_{\alpha,p}\mu^E(x) &= \sum_{k=0}^{\infty} \Big(2^{k(M-\alpha p)} \mu^E\Big(B^{(M)}(x, 2^{-k})\Big)\Big)^{q-1} = \sum_{k=0}^{\infty} \Big(2^{k(N-\beta p)} \mu'^E\Big(B^{(N)}(x, 2^{-k})\Big)\Big)^{q-1} \\ &= W^{(N)}_{\beta,p}\mu'^E(x). \end{aligned} \tag{3.68}$$

Combining (3.67) with (3.68) finishes the proof. □

Chapter 4

Uniqueness Results

4.1 Motivation and known results

This section aims to provide a concise introduction and summary of findings in the theory of sets of uniqueness. It is important to note that our interpretation of sets of uniqueness differs from the version utilized e.g. by Menshov (see [52]). As we will discover later, this modified version is linked to the issue of simultaneous approximation which will be of significant importance in the subsequent discussion.

Definition 4.1.1 Let $E \subset \mathbb{C}$ be a closed set and let $\mathcal{C}$ be a class of functions defined on an open set $U \subset \mathbb{C} \setminus E$. In this situation, E is called a **set of uniqueness** for the class $\mathcal{C}$ if

$$f \in H(\mathbb{C}_\infty \setminus E) \qquad \text{and} \qquad f_{|_U} \in \mathcal{C}$$

already implies that $f = 0$.

In the following discussion, we revisit several established theorems fundamental to this theory. We begin with a famous result which traces back to Havin.

Theorem 4.1.2 (Havin, [31])
Let $p \in [1, \infty]$ and let $E \subset \mathbb{S}$ be a closed set. Then E is an m-null set if and only if E is a set of uniqueness for the class $\mathcal{H}^p$.

The proof of this theorem can be found in the work of Khrushchev [35, Theorem A]. The "only if" part of the theorem is derived primarily from Fatou's theorem together with an application of the F. and M. Riesz theorem.

The "if" part can be shown by contradiction. The central idea is to deduce the existence of a nontrivial and bounded analytic function on $\mathbb{C}_\infty \setminus E$ by observing that the analytic capacity is positive for all compact sets $E \subset \mathbb{S}$ with positive arc measure (compare [22, p. 31, Theorem 7.1]).

Interestingly, the sets of uniqueness do not depend on the parameter p and the theorem holds true even if $p = \infty$.

Next, we like to consider the class $\mathcal{A}^2$, the space of holomorphic functions on the unit disk that are square integrable. Our focus is on the sets of uniqueness for this class. Fortunately, Khrushchev and Peller provided an answer that enables the characterization of these sets through the logarithmic capacity. The proof involves the use of intricate and far-reaching results.

Theorem 4.1.3 (Khrushchev and Peller, [36, Theorem 3.17])
Let $E \subset \mathbb{S}$ be a closed set. Then E is polar with respect to the logarithmic capacity if and only if E is a set of uniqueness for the class of all Cauchy transforms of a complex Borel measure μ supported on E which satisfy

$$C\mu_{|_{\mathbb{D}}} \in \mathcal{A}^2.$$

This theorem is quite remarkable since no restriction is imposed on the behavior of the Cauchy transforms on the outer unit disk. It is noteworthy that Koosis managed to present a significantly shorter proof of Theorem 4.1.3. His proof relies mainly on the equivalence

$$\sum_{k=1}^{\infty} \frac{|\widehat{\mu}(k)|^2}{k} < \infty \qquad \text{if and only if} \qquad \sum_{k=1}^{\infty} \frac{|\widehat{\mu}(-k)|^2}{k} < \infty$$

for all $\mu \in \boldsymbol{\mathcal{M}}_{\mathbb{C}}(\mathbb{S})$ which is a consequence of the fact that the function

$$z \mapsto \operatorname{Im}(\operatorname{Li}_1(z)) = \sum_{k=1}^{\infty} \frac{\operatorname{Im}(z^k)}{k}$$

is bounded on $\mathbb{S}$.

Unfortunately, the proof of Koosis cannot be generalized to the class $(D_{-\alpha})_+$ for $\alpha \in (0,1)$ by applying the same techniques (see as well [42, 3. A concluding Remark]) since for $\alpha \in (0,1)$ the function

$$z \mapsto \operatorname{Im}(\operatorname{Li}_\alpha(z)) = \sum_{k=1}^{\infty} \frac{\operatorname{Im}(z^k)}{k^\alpha}$$

is no longer bounded on $\mathbb{S}$.

The previously revisited theorems share a common feature: They only enforce a growth condition within the unit disk. Since the findings of Khrushchev and Peller cannot be extended to $\alpha \in (0,1)$ using the same methods as Koosis, in the subsequent section, it is our declared aim to concentrate on classes where a growth condition is applied both inside and outside the unit circle. This will lead to uniqueness results for the class $\mathcal{D}_\alpha$.

However, first, we want to recall a finding attributed to Adams and Hedberg. This can be viewed as a result related to sets of uniqueness with growth conditions enforced on both sides, provided that E is a closed subset of $\mathbb{S}$. As this result will be pivotal in the upcoming chapter, we wish to present it below. Bear in mind that q denotes the conjugate exponent of p.

Theorem 4.1.4 (Adams and Hedberg, [1, Proposition 11.1.1])
Let $p \in [2, \infty)$ and suppose that $E \subset \mathbb{C}$ is a closed set. If E is $\mathrm{Cap}^{(2)}_{1,q}$-polar, then E is a set of uniqueness for the class $L^p(\mathbb{C} \setminus E)$. In particular, any closed and $\mathrm{Cap}^{(2)}_{1,q}$-polar set $E \subset \mathbb{S}$ is a set of uniqueness for $\mathcal{A}^p \oplus \mathcal{A}^p_{\mathrm{ext}}$.

4.2 Uniqueness results for the class $\mathcal{D}_\alpha$

As previously delineated, we aim to achieve uniqueness results for the class $(D_\alpha)^* = \mathcal{D}_{-\alpha}$. Our proof's framework is inspired by [16, p. 178] where an outline for the case $\alpha = 1$ is presented. Building upon this foundation, we will prove an extension of this result for $\alpha \in (0, 1]$. The energy formula for the Li_α-kernel proves to be a crucial tool in the corresponding proof.

Theorem 4.2.1 *Let $0 < \alpha \leq 1$ and suppose that $E \subset \mathbb{S}$ is a closed set. If E is α-polar, then E is a set of uniqueness for the class $\mathcal{D}_{-\alpha}$.*

Proof: First, for $n \in \mathbb{N}$, we define

$$E_n := \Big\{\zeta \in \mathbb{S} : \mathrm{dist}(\zeta, E) \leq 1/n\Big\}.$$

For any $n \in \mathbb{N}$, the set E_n encompasses an open subarc of $\mathbb{S}$. Specifically, the Hausdorff dimension of E_n is 1. Consequently, by leveraging (3.63), we deduce that E_n does not form an α-polar set. Since E_n is closed and thus a compact set, we can invoke Remark 3.1.5 to infer the existence of a K_α-equilibrium measure for each set E_n which will be denoted by ν_n in the sequel of this proof.

Let us assume that $f \in H(\mathbb{C}_\infty \setminus E)$ and $f|_{\mathbb{C}_\infty \setminus \mathbb{S}} \in \mathcal{D}_{-\alpha}$. In particular, there exists a sequence $(a_k)_{k\in\mathbb{Z}} \in D_{-\alpha}$ such that

$$f|_{\mathbb{D}} = \sum_{k=0}^{\infty} a_k\, p_k|_{\mathbb{D}} \qquad \text{and} \qquad f|_{\mathbb{D}_{\mathrm{ext}}} = \sum_{k=-\infty}^{-1} (-a_k)\, p_k|_{\mathbb{D}_{\mathrm{ext}}}\,.$$

For $r \in (0, 1)$ let us consider the function $h_r : \mathbb{S} \to \mathbb{C}$ defined by

$$h_r(\zeta) := f(r\zeta) - f(\zeta/r) \quad (\zeta \in \mathbb{S}).$$

Given that the function f is continuous on $\mathbb{S} \setminus E_n$, we can draw upon the definition of h_r to infer that $h_r \to 0$ pointwise on $\mathbb{S} \setminus E_n$ for any $n \in \mathbb{N}$ as r tends to 1. Moreover, by Frostman's maximum principle (see Theorem 3.2.12), we know that

$$K_\alpha \nu_n \leq I_{K_\alpha}(\nu_n) \quad \text{on } \mathbb{S}.$$

As a result, we obtain for any $n \in \mathbb{N}$

$$\left|\int_{\mathbb{S}\setminus E_n} \Big(I_{K_\alpha}(\nu_n) - K_\alpha \nu_n(\zeta)\Big) h_r(\zeta) dm(\zeta)\right| \leq I_{K_\alpha}(\nu_n) \cdot \left\|h_r \cdot \mathbb{1}_{\mathbb{S}\setminus E_n}\right\|_{L^1(\mathbb{S})} \to 0 \tag{4.1}$$

as r tends to 1. Furthermore, since ν_n is a K_α-equilibrium measure of E_n, we deduce by Frostman's maximum principle combined with Frostman's theorem (cf. Theorem 3.1.9) that

$$K_\alpha \nu_n(\zeta) - c_\alpha(E_n)^{-1} = K_\alpha \nu_n(\zeta) - I_{K_\alpha}(\nu_n) = 0 \tag{4.2}$$

holds α-quasieverywhere on E_n. Specifically, by invoking (3.63), we deduce that the set where (4.2) is not satisfied has Hausdorff dimension less than 1, and thus, it is an m-null set. We obtain

$$\left| \int_{E_n} \Big(c_\alpha(E_n)^{-1} - K_\alpha \nu_n(\zeta) \Big) h_r(\zeta) dm(\zeta) \right| = 0 \quad (r \in (0,1)). \tag{4.3}$$

Putting (4.1) and (4.3) together yields

$$\left| \int_{\mathbb{S}} \Big(c_\alpha(E_n)^{-1} - K_\alpha \nu_n(\zeta) \Big) h_r(\zeta) dm(\zeta) \right| \to 0 \quad (r \to 1). \tag{4.4}$$

Since h_r is continuous for all $r \in (0,1)$, we are justified in applying Fubini's theorem in conjunction with the orthogonality relation for the unit circle (see (2.6)) which results in

$$\int h_r \, dm = \sum_{k=0}^{\infty} a_k r^k \int_{\mathbb{S}} \zeta^k dm(\zeta) + \sum_{k=1}^{\infty} a_{-k} r^k \int_{\mathbb{S}} \overline{\zeta}^k dm(\zeta) = a_0.$$

For this reason, we can infer from (4.4) together with the latter equation that

$$\int_{\mathbb{S}} K_\alpha \nu_n(\zeta) h_r(\zeta) dm(\zeta) \to \frac{f(0)}{c_\alpha(E_n)} \quad (r \to 1), \tag{4.5}$$

where we take into account that $f(0) = a_0$.

By definition of the Li_α-kernel function and due to the invariance of the spherical metric under rotation, we have

$$K_\alpha(\sigma(\zeta, w)) = \mathrm{Re}(\mathrm{Li}_\alpha(\zeta \overline{w})) - L_\alpha \quad (\zeta, w \in \mathbb{S}).$$

Moreover, since $K_\alpha \nu_n$ is bounded by $I_{K_\alpha}(\nu_n)$ on $\mathbb{S}$, we are allowed to apply Fubini's theorem on the left-hand side in (4.5) which combined with the latter identity results in

$$\int_{\mathbb{S}} K_\alpha \nu_n(\zeta) h_r(\zeta) dm(\zeta) = \int_{\mathbb{S}} \int_{\mathbb{S}} \mathrm{Re}\Big(\mathrm{Li}_\alpha(\zeta \overline{w}) - L_\alpha \Big) h_r(\zeta) dm(\zeta) d\nu_n(w).$$

We shall now consider the integrand of the right-hand side. We rewrite the integrand by employing the power series of the polylogarithm. This gives us for all $w, \zeta \in \mathbb{S},\ w \neq \zeta$

$$\begin{aligned}
2 \cdot \mathrm{Re}(\mathrm{Li}_\alpha(\zeta \overline{w})) h_r(\zeta) &= \left(\sum_{j=1}^{\infty} \frac{(\overline{\zeta} w)^j + (\zeta \overline{w})^j}{j^\alpha} \right) \left(\left(\sum_{k=0}^{\infty} a_k r^k \zeta^k \right) + \left(\sum_{k=1}^{\infty} a_{-k} r^k \overline{\zeta}^k \right) \right) \\
&= \left(\sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} \frac{(\overline{\zeta} w)^j}{|j|^\alpha} \right) \left(\sum_{k=-\infty}^{\infty} a_k r^{|k|} \zeta^k \right) \\
&= \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} \sum_{k=-\infty}^{\infty} \frac{a_k r^{|k|}}{|j|^\alpha} \cdot \zeta^{k-j} w^j.
\end{aligned}$$

For all $j \in \mathbb{Z} \setminus \{0\}$ and $k \in \mathbb{Z}$, the orthogonality relation (see (2.6)) yields

$$\int_{\mathbb{S}} \zeta^{k-j} dm(\zeta) = \delta_{k,j}.$$

Moreover, due to Example 3.3.5, we get

$$\int_{\mathbb{S}} \int_{\mathbb{S}} -L_\alpha dm \, d\nu_n = c_\alpha(\mathbb{S})^{-1}.$$

By interchanging the order of the sum and the integral, we can rewrite the left-hand side of (4.5) in the following way

$$\int_{\mathbb{S}} K_\alpha \nu_n(\zeta) h_r(\zeta) dm(\zeta) = \frac{1}{2} \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{a_j r^{|j|}}{|j|^\alpha} \int_{\mathbb{S}} w^j d\nu_n(w) + \frac{f(0)}{c_\alpha(\mathbb{S})} = \frac{1}{2} \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{a_j r^{|j|}}{|j|^{\alpha/2}} \cdot \frac{\overline{\widehat{\nu_n}(j)}}{|j|^{\alpha/2}} + \frac{f(0)}{c_\alpha(\mathbb{S})}.$$

We go ahead by an application of the Cauchy-Schwarz inequality followed by employing the energy formula (cf. Theorem 3.3.3) and observing the identity $I_{K_\alpha}(\nu_n) = c_\alpha(E_n)^{-1}$. Thus,

$$\begin{aligned} \left| \int_{\mathbb{S}} K_\alpha \nu_n(\zeta) h_r(\zeta) dm(\zeta) \right| &\leq \left(\frac{1}{2} \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{|a_j|^2 r^{2|j|}}{|j|^\alpha} \right)^{1/2} \cdot \left(\frac{1}{2} \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{1}{|j|^\alpha} \cdot |\widehat{\nu_n}(j)|^2 \right)^{1/2} + \frac{|f(0)|}{c_\alpha(\mathbb{S})} \\ &\leq \frac{\|f\|_{\mathcal{D}_{-\alpha}}}{c_\alpha(E_n)^{1/2}} + \frac{|f(0)|}{c_\alpha(\mathbb{S})}. \end{aligned}$$

Note that the right-hand side of the preceding inequality does not depend on r. By referring back to (4.5), we can conclude for all $n \in \mathbb{N}$

$$\frac{|f(0)|}{c_\alpha(E_n)} = \left| \lim_{r\to 1} \int_{\mathbb{S}} K_\alpha \nu_n(\zeta) h_r(\zeta) dm(\zeta) \right| \leq \frac{\|f\|_{\mathcal{D}_{-\alpha}}}{c_\alpha(E_n)^{1/2}} + \frac{|f(0)|}{c_\alpha(\mathbb{S})}. \tag{4.6}$$

Now, we utilize the fact that E_n is a decreasing sequence of compact sets and that

$$E = \bigcap_{n=1}^{\infty} E_n$$

to infer with (3.3) that

$$\lim_{n\to\infty} c_\alpha(E_n) = c_\alpha\left(\bigcap_{n=1}^{\infty} E_n \right) = c_\alpha(E) = 0.$$

Hence, $c_\alpha(E_n)$ vanishes as n tends to ∞. This immediately implies that (4.6) can only be valid for any $n \in \mathbb{N}$ if

$$f(0) = 0.$$

In particular, $p_{-1} \cdot f \in H(\mathbb{C}_\infty \setminus E)$. By definition of $\mathcal{D}_{-\alpha}$, it follows that $p_{-1} \cdot f \in \mathcal{D}_{-\alpha}$. Consequently, we can reapply the above procedure to deduce that

$$(p_{-1} \cdot f)(0) = 0,$$

in particular, $a_1 = 0$.

Let us continue this procedure inductively. It follows

$$a_k = 0 \quad (k \in \mathbb{N}_0).$$

Consequently, we obtain that $f_{|\mathbb{D}} = 0$. We observe that Example 3.3.5 implies in particular $E \neq \mathbb{S}$. Thus, $\mathbb{C}_\infty \setminus E$ is connected and by the identity theorem, we obtain $f = 0$. $\square$

Later, we will see that closed sets $E \subset \mathbb{S}$ which are sets of uniqueness for the class $\mathcal{D}_{-\alpha}$ have necessarily to be α-polar.

For the sake of completeness, we discuss the case $\alpha = 0$. Since $\mathcal{D}_0 = (D_0)_+ \oplus (D_0)_-$ and since $(D_0)_+$ corresponds to the Hardy space $\mathcal{H}^2$, we can draw upon Theorem 4.1.2 which implies in particular that any closed set $E \subset \mathbb{S}$ of vanishing arc measure is a set of uniqueness for the class $\mathcal{D}_0$.

In the subsequent step, our aim is to derive a uniqueness result for the class $(D_{-\alpha})^* = \mathcal{D}_\alpha$ for $\alpha \in (0, 1]$. The case $\alpha = 1$ and $E \subset \mathbb{S}$ closed was previously addressed by Ahlfors and Beurling in [3, Theorem 14]. By recalling Definition 3.4.5, the theorem of Ahlfors and Beurling can be stated as follows.

Theorem 4.2.2 (Ahlfors and Beurling, [3, Theorem 14])
Suppose that $E \subset \mathbb{S}$ is a closed set. Then E is a 1-complement-polar set if and only if E is a set of uniqueness for the class $\mathcal{D}_1$.

We now extend the uniqueness result of Ahlfors and Beurling for $\alpha \in (0, 1]$. We will be guided by the proof in [9, p. 82, Theorem 2].

Theorem 4.2.3 *Let $0 < \alpha \leq 1$ and suppose that $E \subset \mathbb{S}$ is a closed set. Then E is an α-complement-polar set if and only if E is a set of uniqueness for the class $\mathcal{D}_\alpha$.*

Proof: Throughout this proof, we shall always assume that $E \subset \mathbb{S}$ is a closed set and that $0 < \alpha \leq 1$.

First, let us suppose that E is an α-complement-polar set which means by Definition 3.4.5 that

$$c_\alpha(\mathbb{S} \setminus E) = c_\alpha(\mathbb{S}) \tag{4.7}$$

holds. For $n \in \mathbb{N}$, we define

$$B_n := \Big\{ \zeta \in \mathbb{S} : \operatorname{dist}(\zeta, E) \geq 1/n \Big\} \subset \mathbb{S} \setminus E.$$

Since E is compact and $E \neq \mathbb{S}$, we find that B_n is a nonempty compact set which contains an open arc if n is large enough. For those B_n we can find corresponding K_α-equilibrium measures (cf. Remark 3.1.5) which we denote by ν_n. Moreover, (B_n) is increasing with

$$B_n \to \mathbb{S} \setminus E \quad (n \to \infty).$$

Since (4.7) holds, we obtain by recalling the definition of inner capacities (see Definition 3.1.6) that

$$c_\alpha(B_n) \to c_\alpha(\mathbb{S}) \quad (n \to \infty),$$

or equivalently

$$I_{K_\alpha}(\nu_n) \to I_{K_\alpha}(m) \quad (n \to \infty). \tag{4.8}$$

We use Prokhorov's theorem (note that the family $\{\nu_n : n \in \mathbb{N}\}$ is tight since $(\mathbb{S}, \sigma)$ is a compact metric space) which allows us to infer the existence of a measure $\nu \in \mathrm{Prob}(\mathbb{S})$ and a $J \subset \mathbb{N}$ such that

$$\nu_n \to \nu \quad \text{weak}^* \qquad (J \ni n \to \infty).$$

Throughout this proof, we assume that $J = \mathbb{N}$ is valid. Due to Example 3.3.5, we know that m is the *unique* K_α-equilibrium measure for the unit circle and by (4.8), we deduce that

$$\nu = m.$$

Since $I_{K_\alpha}(m) = -L_\alpha$ due to Example 3.3.5, the energy formula (cf. Theorem 3.3.3) together with (4.8) implies that the following limit holds

$$\frac{1}{2} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{1}{|k|^\alpha} |\widehat{\nu_n}(k)|^2 = I_{K_\alpha}(\nu_n) - I_{K_\alpha}(m) \to 0 \quad (n \to \infty). \tag{4.9}$$

Let us assume that $f \in H(\mathbb{C}_\infty \setminus E)$ and $f|_{\mathbb{C}_\infty \setminus \mathbb{S}} \in \mathcal{D}_\alpha$. In particular, there exists a sequence $(a_k)_{k \in \mathbb{Z}} \in D_\alpha$ such that

$$f|_{\mathbb{D}} = \sum_{k=0}^{\infty} a_k \, p_k|_{\mathbb{D}} \qquad \text{and} \qquad f|_{\mathbb{D}_{\mathrm{ext}}} = \sum_{k=-\infty}^{-1} (-a_k) \, p_k|_{\mathbb{D}_{\mathrm{ext}}} \, .$$

For $r \in (0,1)$, let us define the function $h_r : \mathbb{S} \to \mathbb{C}$

$$h_r(\zeta) := f(r\zeta) - f(\zeta/r) = \sum_{k=-\infty}^{\infty} a_k r^{|k|} \zeta^k \quad (\zeta \in \mathbb{S}).$$

Since f is in particular continuous on $\mathbb{S} \setminus E$, we obtain for all $\zeta \in B_n \subset \mathbb{S} \setminus E$ that the subsequent limit holds

$$h_r(\zeta) \to 0 \quad (r \to 1). \tag{4.10}$$

The family $\Big\{ h_r|_{B_n} : r \in (0,1)\Big\}$ is uniformly bounded as a consequence of the latter limit and the fact that B_n is compact. Hence, the dominated convergence theorem and (4.10) yield

$$\lim_{r\to 1}\int_{B_n} h_r(\zeta)\,d\nu_n(\zeta) = \int_{B_n}\lim_{r\to 1} h_r(\zeta)\,d\nu_n(\zeta) = 0 \quad (n\in\mathbb{N}). \tag{4.11}$$

Moreover, by the Cauchy-Schwarz inequality and the energy formula, we estimate

$$\sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty}\Big|a_k\,\overline{\widehat{\nu_n}(k)}\Big| \le \left(\sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty}|k|^{\alpha}|a_k|^2\right)^{1/2}\cdot\left(\sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty}\frac{1}{|k|^{\alpha}}\,|\widehat{\nu_n}(k)|^2\right)^{1/2} \le \|f\|_{\mathcal{D}_\alpha}\cdot\sqrt{2I_{K_\alpha}(\nu_n)},$$

and therefore, by the dominated convergence theorem,

$$\lim_{r\to 1}\sum_{k=-\infty}^{\infty} a_k r^{|k|}\overline{\widehat{\nu_n}(k)} = \sum_{k=-\infty}^{\infty} a_k\overline{\widehat{\nu_n}(k)}. \tag{4.12}$$

By recalling the series representation of h_r, combining (4.11) and (4.12), and using the orthogonality relation established in (2.6) we find

$$\sum_{k=-\infty}^{\infty} a_k\,\overline{\widehat{\nu_n}(k)} = 0.$$

We take $a_0\widehat{\nu_n}(0)$ to the right-hand side and observe that $\widehat{\nu_n}(0) = 1$. This permits us to establish for any $n\in\mathbb{N}$ the following estimate

$$\begin{aligned}|a_0|^2 &= \left|\sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} a_k\,\overline{\widehat{\nu_n}(k)}\right|^2 \le \left(\sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty}|k|^{\alpha}|a_k|^2\right)\cdot\left(\sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty}\frac{1}{|k|^{\alpha}}|\widehat{\nu_n}(k)|^2\right)\\ &\le \|f\|_{\mathcal{D}_\alpha}^2\cdot 2\Big(I_{K_\alpha}(\nu_n) - I_{K_\alpha}(m)\Big).\end{aligned}$$

Since (4.9) holds, we infer that the right-hand side vanishes as $n\to\infty$. We obtain

$$f(0) = a_0 = 0.$$

By the same arguments as in Theorem 4.2.1, we conclude that $f = 0$.

Let us turn to the "only if" part which we want to prove by contradiction. So, let us assume that (4.7) does not hold. In particular,

$$I_{K_\alpha}(\mathbb{S}) < I_{K_\alpha}(\mathbb{S}\setminus E). \tag{4.13}$$

For $n\in\mathbb{N}$, let B_n and ν_n be defined as above. We apply Frostman's theorem and Frostman's maximum principle which allow us for n large enough, say $n\ge N\in\mathbb{N}$, to find numbers $\gamma_n\in(0,\infty)$ such that

$$K_\alpha\nu_n = I_{K_\alpha}(B_n) =: \gamma_n \tag{4.14}$$

α-quasieverywhere on B_n. For $n \in \mathbb{N}$, we define the functions

$$f_n(z) := \int_{\mathbb{S}} \frac{1}{1 - z\overline{\zeta}} \Big(K_\alpha \nu_n(\zeta) - \gamma_n \Big) dm(\zeta)$$

which will play a key role in constructing a nontrivial function $f \in H(\mathbb{C}_\infty \setminus E)$ with $f_{|_{\mathbb{C}_\infty \setminus \mathbb{S}}} \in \mathcal{D}_\alpha$. To derive the contradiction, the following four statements shall be outlined:

1) $f_n \in H(\mathbb{C}_\infty \setminus (\mathbb{S} \setminus B_n))$ for any $n \in \mathbb{N}$,

2) $(f_n(0))_n$ is increasing and bounded from above by $I_{K_\alpha}(\mathbb{S}) - I_{K_\alpha}(\mathbb{S} \setminus E) < 0$,

3) $(f_n)_{n \in \mathbb{N}}$ is a bounded sequence in $\mathcal{D}_\alpha$,

4) there exists a subsequence f_{n_j} which converges locally uniformly to a nontrivial $f \in H(\mathbb{C}_\infty \setminus E)$ and $f_{|_{\mathbb{C}_\infty \setminus \mathbb{S}}} \in \mathcal{D}_\alpha$.

Let us turn to the first statement. For $n \in \mathbb{N}$, we define $h_n : \mathbb{S} \to \mathbb{R}$

$$h_n(\zeta) := K_\alpha \nu_n(\zeta) - \gamma_n \quad \text{and} \quad \mu_n := h_n dm.$$

For any $n \in \mathbb{N}$, the function f_n is the Cauchy transform of the measure μ_n, that means

$$f_n = C\mu_n. \tag{4.15}$$

The support of μ_n is a subset of $\mathbb{S} \setminus B_n$ which follows immediately from (4.14), and the fact that an α-polar Borel set is in particular an m-null set. This gives us

$$f_n \in H(\mathbb{C}_\infty \setminus (\mathbb{S} \setminus B_n)) \quad (n \in \mathbb{N}).$$

Let us proceed with the second statement. Since $K_\alpha m$ is a constant function on the unit circle which is a direct consequence of the rotation invariance of the spherical metric σ and the arc measure m it follows by Fubini's theorem that

$$\int_{\mathbb{S}} K_\alpha \nu_n \, dm = \int_{\mathbb{S}} K_\alpha m \, d\nu_n = I_{K_\alpha}(\mathbb{S}).$$

By definition of the functions f_n, the constants γ_n, and the functions h_n, we obtain

$$f_n(0) = \int_{\mathbb{S}} h_n \, dm = \int_{\mathbb{S}} K_\alpha \nu_n(\zeta) - \gamma_n \, dm(\zeta) = I_{K_\alpha}(\mathbb{S}) - I_{K_\alpha}(B_n). \tag{4.16}$$

Now, since $B_n \subset \mathbb{S} \setminus E$ for all $n \in \mathbb{N}$, the monotonicity of energies yields $I_{K_\alpha}(\mathbb{S} \setminus E) \leq I_{K_\alpha}(B_n)$ and consequently by (4.13)

$$f_n(0) \leq I_{K_\alpha}(\mathbb{S}) - I_{K_\alpha}(\mathbb{S} \setminus E) < 0 \quad (n \in \mathbb{N}).$$

As a consequence of (4.16), the sequence $(f_n(0))$ is increasing since the sequence (B_n) is increasing.

Let us prove the third statement. Due to the representation in (4.15), we obtain by applying Remark 2.3.2 ii) that f_n can also be represented by

$$f_n(z) = \begin{cases} \sum_{j=0}^{\infty} \widehat{h_n}(j) z^j & z \in \mathbb{D} \\ -\sum_{j=1}^{\infty} \widehat{h_n}(-j) z^{-j} & z \in \mathbb{D}_{\text{ext}} \end{cases}.$$

Now, by the definition of $\mathcal{D}_\alpha$, we only need to show that $(\widehat{h_n})_{n\in\mathbb{N}}$ is a bounded sequence in D_α. Due to the orthogonality relation (see (2.6)), we can infer that the subsequent identity holds true for all $k \in \mathbb{N}$ and $j \in \mathbb{Z} \setminus \{0\}$,

$$\int_{\mathbb{S}} \int_{\mathbb{S}} w^k \zeta^{-(k+j)} + w^{-k} \zeta^{k-j} dm(\zeta) d\nu_n(w) = \int_{\mathbb{S}} w^k \delta_{k,-j} + w^{-k} \delta_{k,j} d\nu_n(w) = (\delta_{k,-j} + \delta_{k,j}) \cdot \widehat{\nu_n}(j).$$

By the definition of h_n, the power series representation of the Li_α-kernel function, Fubini's theorem, and the latter calculation, we obtain for $n \in \mathbb{N}$ and $j \in \mathbb{Z} \setminus \{0\}$, the following identity

$$\widehat{h_n}(j) = \int_{\mathbb{S}} \int_{\mathbb{S}} \frac{1}{2} \sum_{k=1}^{\infty} \frac{w^k \zeta^{-(k+j)} + w^{-k} \zeta^{k-j}}{k^\alpha} dm(\zeta) d\nu_n(w) = \frac{\widehat{\nu_n}(j)}{2|j|^\alpha}. \tag{4.17}$$

Now, we can deploy the energy formula for the Li_α-kernel together with the definition of the $\|\cdot\|_{D_\alpha}$ norm and the observation that $f_n(0) = \widehat{h_n}(0)$ to deduce the inequality

$$\begin{aligned} \|\widehat{h_n}\|_{D_\alpha}^2 - |f_n(0)|^2 &= \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} (|j|+1)^\alpha |\widehat{h_n}(j)|^2 = \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{(|j|+1)^\alpha}{2|j|^\alpha} \cdot \frac{|\widehat{\nu_n}(j)|^2}{2|j|^\alpha} \leq \frac{1}{2} \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty} \frac{|\widehat{\nu_n}(j)|^2}{|j|^\alpha} \\ &= I_{K_\alpha}(\nu_n) + L_\alpha. \end{aligned}$$

Since the sequence $(f_n(0))_n$ is increasing and bounded above by zero because of 2) and since $I_{K_\alpha}(\nu_n)$ is decreasing, we conclude that

$$\|\widehat{h_n}\|_{D_\alpha}^2 \leq I_{K_\alpha}(\nu_N) + L_\alpha + |f_N(0)|^2 < \infty \quad (n \geq N).$$

Therefore, the third statement is fulfilled.

It only remains to show the last statement. As a consequence of Banach Alaoglu's theorem, we obtain that there exists a subsequence $(\widehat{h_{n_j}})_{j\in\mathbb{N}}$ which converges weakly to an $a \in D_\alpha$ and by definition

$$f_{n_j} \to f \quad \text{weak in } \mathcal{D}_\alpha \quad (j \to \infty) \tag{4.18}$$

for an element $f \in \mathcal{D}_\alpha$. Without loss of generality, let us assume that the initial sequence $(\widehat{h_n})_{n\in\mathbb{N}}$, and therefore f_n, converges weakly to a and f, respectively.

We show that f is the Cauchy transform of a complex measure μ with support on E. This means in particular $f \in H(\mathbb{C}_\infty \setminus E)$. By Frostman's maximum principle, the family $\{\mu_n : n \geq N\}$ is bounded with respect to the total variation norm $\|\cdot\|_{\mathrm{TV}}$ since

$$\|\mu_n\|_{\mathrm{TV}} = |\mu_n|(\mathbb{S}) \leq \int_{\mathbb{S}} |h_n|\, dm \leq \sup_{\zeta \in \mathbb{S}} |K_\alpha \nu_n(\zeta) - \gamma_n| \cdot m(\mathbb{S}) \leq I_{K_\alpha}(\nu_N) < \infty \quad (n \geq N).$$

Thus, by Prokhorov's theorem, we deduce the existence of a set $J \subset \mathbb{N}$ and a complex measure $\mu \in \boldsymbol{\mathcal{M}}_{\mathbb{C}}(\mathbb{S})$ such that

$$\mu_n \to \mu \quad \text{weak}^* \quad (J \ni n \to \infty). \tag{4.19}$$

Let us consider the support of μ. Due to the Portmanteau theorem and by the definition of the increasing sequence $(B_n)_{n\in\mathbb{N}}$, we infer

$$\operatorname{supp}(\mu) \subset \liminf_{n\to\infty} \operatorname{supp}(\mu_n) \subset \liminf_{n\to\infty} \mathbb{S} \setminus B_n = \bigcap_{n=1}^{\infty} \operatorname{cl}(\mathbb{S} \setminus B_n) = E,$$

where lim inf denotes the Kuratowski limit inferior. In particular, by recalling (4.18), we obtain that

$$f_n \to f \quad \text{weak in } \mathcal{D}_\alpha \quad (J \ni n \to \infty)$$

and (4.19) yields that f has the representation

$$f(z) = \int_E \frac{d\mu(\zeta)}{1 - z\overline{\zeta}} \quad (z \in \mathbb{C} \setminus E).$$

This contradicts the assumption that E is a set of uniqueness for the class $\mathcal{D}_\alpha$. □

Remark 4.2.4 Let $0 < \alpha \leq 1$ and suppose that $E \subset \mathbb{S}$ is a closed set. The proof of the forgoing theorem reveals that the following three statements are equivalent.

i) There exists a complex measure $\mu \neq 0$ with support in E such that $C\mu|_{\mathbb{C}_\infty \setminus \mathbb{S}} \in \mathcal{D}_\alpha$.

ii) There is a function $f \in H(\mathbb{C}_\infty \setminus \mathbb{S}) \setminus \{0\}$ such that $f_{|\mathbb{C}_\infty \setminus \mathbb{S}} \in \mathcal{D}_\alpha$.

iii) The set E is *not* α-complement-polar.

Chapter 5

Universal Fourier series

In the initial chapter, we recalled the notion of universality, the current chapter is dedicated to presenting universality results for two-sided sequences. To do this, we will primarily rely on the uniqueness results that we summarized in the foregoing chapter and the theory of capacities. To this aim, we shall first recall some particular tools needed; more precisely, we will address the universality criterion and the technique of simultaneous approximation which are important in this regard. After this, we move on to investigate the spaces D_α and D^p_{harm} for universality with respect to uniform convergence. We will also obtain sharpness results where we mainly draw on results by Carleson [9] and Twomey [68].

5.1 Some tools for universality

In the beginning, let us revisit an instrumental tool that will be utilized in the proof of the existence of universal elements and which allows us even to ensure the existence of a large number of universal elements in the sense of Baire categories. The theorem traces back to Grosse-Erdmann who formulated it in 1987 [25, Satz 1.2.2]. We will present the theorem in a somewhat restricted form as it is exclusively applied to spaces where the topology is induced by a norm.

Theorem 5.1.1 (The Universality Criterion, (cf. [26, Theorem 1]))
Let $(X, \|\cdot\|_X)$ be a Banach space and $(Y, \|\cdot\|_Y)$ be a separable normed vector space. Moreover, let $T_n : X \to Y$, $(n \in \mathbb{N})$ be a sequence of bounded operators. Then the following four statements are equivalent:

1) *There exists a residual set of points $x \in X$ such that x is $(Y, \|\cdot\|_Y)$-universal.*

2) *There exists a dense G_δ-set of points $x \in X$ such that x is $(Y, \|\cdot\|_Y)$-universal.*

3) *There exists a dense set of points $x \in X$ such that x is $(Y, \|\cdot\|_Y)$-universal.*

4) *For all nonempty open sets $U \subset X$ and $V \subset Y$ there exists an $N \in \mathbb{N}$ such that*

$$T_N(U) \cap V \neq \emptyset.$$

In alignment with [27, Definition 1.56], a family of operators that satisfies the fourth assertion in the latter theorem is referred to as **topologically transitive**.

Within our framework, the sequence of operators T_n will take the role of S_n where we recall that $S_n : \mathbb{C}^{\mathbb{Z}} \to \mathcal{P}$ is given by

$$S_n(a) = \sum_{k=-n}^{n} a_k p_k \qquad (a \in \mathbb{C}^{\mathbb{Z}},\ n \in \mathbb{N})$$

according to Definition 1.2.12. While it is difficult to find universal elements, it is all the more remarkable that the aforementioned theorem provides us with a criterion that allows us to infer the existence of a residual set of universal elements.

Having a look at the structure of the universality criterion, our strategy is to validate that the family $\{S_n : n \in \mathbb{N}\}$ is topologically transitive in case $X = D_\alpha$ or $X = \mathcal{D}^p_{\mathrm{harm}}$ and $Y = C(E)$ for an appropriate set $E \subset \mathbb{S}$ in order to deduce the existence of a residual set of universal elements. In the ensuing proposition, we will highlight a connection between the intersection of the Cauchy dual of two normed vector spaces and the topological transitivity of a family of operators. The cornerstone for substantiating this statement will be the universality criterion and the technique of simultaneous approximation.

Proposition 5.1.2 *Let $(X, \|\cdot\|_X)$ be a Banach space, $(Y, \|\cdot\|_Y)$ be a normed vector space and $T_n : X \to Y$, $n \in \mathbb{N}$ be a sequence of bounded operators. Suppose that there is a sequence $(x_k)_{k\in\mathbb{Z}} \in X^{\mathbb{Z}}$ such that $M_X := \operatorname{span}\{x_k : k \in \mathbb{Z}\}$ is dense in X. Moreover, assume that there is a set $J \subset \mathbb{N}$ such that*

$$y_k := \lim_{J \ni n \to \infty} T_n(x_k) \tag{5.1}$$

exists for any $k \in \mathbb{Z}$ and that $M_Y := \operatorname{span}\{y_k : k \in \mathbb{Z}\}$ is dense in Y. If

$$X^{*,M_X} \cap Y^{*,M_Y} = \{0\},$$

then there exists a residual set of elements in X which are $(Y, \|\cdot\|_Y)$-universal for the family $\{T_n : n \in \mathbb{N}\}$.

Proof: Since $X^{*,M_X} \cap Y^{*,M_Y} = \{0\}$, Proposition 2.3.6 yields that

$$M := \operatorname{span}\Big\{(x_k, y_k) : k \in \mathbb{Z}\Big\} \tag{5.2}$$

forms a dense set in $X \oplus Y$. In particular, for any $x \in X$, any $y \in Y$, and any $\varepsilon > 0$, there exists a pair $(u, v) \in M$ such that

$$\|x - u\|_X < \varepsilon \quad \text{and} \quad \|y - v\|_Y < \varepsilon/2.$$

Moreover, by (5.1), we obtain the existence of an $N \in J$ such that $\|T_N(u) - v\|_Y < \varepsilon/2$. Hence, for any nonempty open $U \subset X$ and $V \subset Y$, the existence of a $u \in M_X$ and an $N \in J$ with the property

$$u \in U \quad \text{and} \quad T_N(u) \in V$$

can be inferred and consequently,

$$T_N(U) \cap V \neq \emptyset.$$

Thus, the family $\{T_n : n \in \mathbb{N}\}$ is topologically transitive and, by the universality criterion, we obtain the desired result. □

Definition 5.1.3 Let $a := (a_k)_{k\in\mathbb{Z}}$ be a two-sided sequence of complex numbers.

1) We say that $E \subset \mathbb{S}$ is a **set of divergence** for a if the trigonometric series

$$\Big((S_n a)(z)\Big)_{n\in\mathbb{N}} \tag{5.3}$$

is divergent for all $z \in E$. Consequently, if (5.3) is convergent for all $z \in E$, we call E a **set of convergence** for a.

Moreover, if X is a collection of two-sided complex sequences, we call E a set of divergence for X if there exists an $a \in X$ such that E is a set of divergence for the sequence a.

2) If $E \subset \mathbb{S}$ and if $Y \subset \mathbb{C}^E$ such that $(Y, \mathcal{T})$ is a topological space, then we say that E is a **set of universality** for the sequence a with respect to $(Y, \mathcal{T})$ in case that the sequence a is $(Y, \mathcal{T})$-universal for the family $\{S_n : n \in \mathbb{N}\}$.

Moreover, if X is a collection of two-sided complex sequences, we call E a set of universality for X with respect to $(Y, \mathcal{T})$ if there exists an $a \in X$ such that E is a set of universality for the sequence a with respect to $(Y, \mathcal{T})$.

In case that the topology $\mathcal{T}$ on Y is induced by a metric d or a norm $\|\cdot\|$, we replace $\mathcal{T}$ by d or $\|\cdot\|$, respectively.

Note that a set of universality is always a set of divergence which is readily apparent upon recalling the definition of universality (see Definition 1.2.1). However, the converse statement is *not* true as we already figured out in Remark 1.2.8.

The following proposition highlights a connection between sets of uniqueness (see Definition 4.1.1) and sets of universality. Suppose that $X \subset \mathbb{C}^{\mathbb{Z}}$ such that $(X, \|\cdot\|)$ is a normed space. If $c_{00}(\mathbb{Z})$ is dense in X, then we write X^* for the Cauchy dual of X with respect to $c_{00}(\mathbb{Z})$ in the sequel.

Proposition 5.1.4 *Let $X \subset \mathbb{C}^{\mathbb{Z}}$ such that $(X, \|\cdot\|)$ is a Banach space and suppose that $c_{00}(\mathbb{Z})$ is dense in X. If $E \subset \mathbb{S}$ closed is a set of uniqueness for X^*, then E is a set of universality for X with respect to $(C(E), \|\cdot\|_\infty)$. Moreover, in this case, the set of universal elements is residual in X.*

Proof: We show that the assumptions of Proposition 5.1.2 are satisfied. Firstly, for all $k \in \mathbb{Z}$, we have

$$S_n e_k = p_k \quad (n \geq k).$$

Moreover, since the space of trigonometric polynomials $\mathcal{P} = \operatorname{span}\{p_k : k \in \mathbb{Z}\}$ is dense in $C(\mathbb{S})$ due to Fejér's theorem, this also holds true for $C(E)$ by Tietze's extension theorem (see e.g. [64, 20.4 Theorem]).

As seen in Remark 2.3.2, the Riesz representation theorem yields that every element in $C(E)^{*,\mathcal{P}}$ corresponds to the Cauchy transform of a complex measure supported on E. In particular

$$C(E)^{*,\mathcal{P}} \subset H(\mathbb{C}_\infty \setminus E).$$

Since E is a set of uniqueness for X^* by assumption, it follows that

$$X^* \cap C(E)^{*,\mathcal{P}} = \{0\}.$$

Thus, the assumptions of Proposition 5.1.2 are satisfied. We deduce that E is a set of universality for X with respect to $(C(E), \|\cdot\|_\infty)$ and that the set of universal elements is residual in X. □

Remark 5.1.5 We summarize that every closed set of uniqueness for X^* is a set of universality for X with respect to $(C(E), \|\cdot\|_\infty)$ and, by definition, every set of universality is a set of divergence.

5.2 Sets of universality for the classes D_α

In this section, we discuss universality results for the class D_α. Let us briefly consider the case $\alpha > 1$. In this case, we obtain by an application of Hölder's inequality that D_α is contained in the **Wiener algebra** which is defined as the space of all sequences $a \in \mathbb{C}^{\mathbb{Z}}$ satisfying

$$\sum_{k\in\mathbb{Z}} |a_k| < \infty.$$

In particular, $S_n a$ is uniformly convergent on $\mathbb{S}$ for any $a \in D_\alpha$.

Remark 5.2.1 Let $\lambda := (\lambda_n)_{n\in\mathbb{N}_0}$ be a sequence of positive real numbers and let us denote D_λ as the set of all two-sided sequences (a_k) of complex numbers which satisfy

$$\sum_{k=-\infty}^{\infty} \lambda_{|k|} |a_k|^2 < \infty.$$

We consider the following question: What is the size of the sets of convergence for $a \in D_\lambda$? If $\lambda_n = 1$ for $n \in \mathbb{N}_0$, then $D_\lambda = D_0$ and Carleson's theorem for $L^2(\mathbb{S})$ yields that $(S_n a)_{n\in\mathbb{N}}$ converges m-a.e. on $\mathbb{S}$.

Now, we consider the case where the sequence λ is given by

$$\lambda_n^{-1} := \frac{2}{\pi}\int_0^\pi K(t)\cos(nt)dt \quad (n \in \mathbb{N}_0) \tag{5.4}$$

where K is a kernel function as specified in Definition 3.1.1. In this case, Carleson [9, p. 50, Theorem 2] has shown that $(S_n a)$ converges $c_{(\dot{K},\sigma)}$-quasieverywhere for any $a \in D_\lambda$ where $\dot{K}$ is given by

$$\dot{K}(t) := \frac{2}{\pi t}\int_0^t K(s)ds. \tag{5.5}$$

Furthermore, he established that if E is both closed and $c_{(K,\sigma)}$-polar and if $K \in O(K(2\cdot))$, then there exists a sequence $a \in D_\lambda$ for which the corresponding trigonometric series $(S_n a)_{n\in\mathbb{N}}$ diverges on E. In other words: Every closed and $c_{(K,\sigma)}$-polar set is a set of divergence for the class D_λ in the case that K fulfills the aforementioned conditions.

By the definition of the Riesz kernel function and the fundamental theorem of calculus we have $\dot{R}_\alpha \sim R_\alpha$, that means $R_\alpha \in O(\dot{R}_\alpha)$ and $\dot{R}_\alpha \in O(R_\alpha)$. Since $K_\alpha \sim R_\alpha$ as a consequence of Remark 3.2.7, we obtain $\dot{K}_\alpha \sim K_\alpha$. In particular, the corresponding capacities are equivalent on $\mathbb{S}$. If λ is defined via K_α (cf. (5.4)), then Remark 5.2.1 implies that $(S_n a)_{n\in\mathbb{N}}$ converges α-quasieverywhere on $\mathbb{S}$ for any $a \in D_\lambda$.

We will determine the sequence λ in the case λ is defined by K_α.

Proposition 5.2.2 *Suppose that $\alpha \in (0,1]$ and let $(\lambda_n)_{n\in\mathbb{N}_0}$ be defined as in (5.4) where $K = K_\alpha$. Then*

$$\lambda_n = n^\alpha \quad (n \in \mathbb{N}) \qquad \text{and} \qquad \lambda_0 = -(2L_\alpha)^{-1}.$$

In particular, we have

$$D_\lambda = D_\alpha.$$

Proof: First, we observe that

$$(-\pi,\pi)\setminus\{0\} \ni t \mapsto K_\alpha(|t|) = \mathrm{Re}\Big(\mathrm{Li}_\alpha(e^{it})\Big) \in [0,\infty) \tag{5.6}$$

is an even function. Thus, we obtain

$$\frac{2}{\pi}\int_0^\pi K_\alpha(t)\cos(nt)dt = \frac{1}{\pi}\int_{-\pi}^\pi \mathrm{Re}\Big(\mathrm{Li}_\alpha(e^{it})\Big)\mathrm{Re}(e^{int})dt - \frac{L_\alpha}{\pi}\int_{-\pi}^\pi \cos(nt)dt \quad (n\in\mathbb{N}). \tag{5.7}$$

Observe that the second integral on the right-hand side drops out. Together with the definition of the arc measure m and the fact that $t \mapsto K_\alpha(t)$ is an integrable majorant for the integrand on the left-hand side in (5.7), the dominated convergence theorem yields for $n \in \mathbb{N}$ and $\alpha \in (0,1]$

$$\begin{aligned}\frac{2}{\pi}\int_0^\pi K_\alpha(t)\cos(nt)dt &= 2\int_{\mathbb{S}}\left(\frac{1}{2}\sum_{k=1}^\infty \frac{1}{k^\alpha}(z^k + z^{-k})\right)\left(\frac{1}{2}\Big(z^n + z^{-n}\Big)\right)dm(z)\\ &= \frac{1}{2}\sum_{k=1}^\infty \frac{1}{k^\alpha}\left(\int_{\mathbb{S}} z^{k+n} + z^{-k-n}dm(z) + \int_{\mathbb{S}} z^{n-k} + z^{k-n}dm(z)\right).\end{aligned} \tag{5.8}$$

We now employ the orthogonality relation (2.6) to deduce that the first integral on the right-hand side vanishes for any $n \in \mathbb{N}$, and the second one reduces to $2\delta_{k,n}$. This finally shrinks the term on the right-hand side in (5.8) to $n^{-\alpha}$. Thus,

$$\lambda_n = n^{\alpha} \quad (n \in \mathbb{N}).$$

By recalling (5.7), we deduce that $\lambda_0 = -(2L_\alpha)^{-1}$. □

We will now present a characterization of the sets of universality for the class D_α Specifically, the uniqueness result for the class $\mathcal{D}_\alpha$ will play a pivotal role in the proof.

Theorem 5.2.3 *Let $\alpha \in (0, 1]$ and suppose that $E \subset \mathbb{S}$ is a closed set. Then the following three statements are equivalent*

1) E is α-polar,

2) E is a set of universality with respect to $(C(E), \|\cdot\|_\infty)$ for the class D_α,

3) E is a set of divergence for the class D_α.

If one of these conditions holds, then the set of universal elements is residual in D_α.

Proof: We show that 1) implies 2). By Proposition 2.3.4, we have that the Cauchy dual of D_α is given by $\mathcal{D}_{-\alpha}$. Since E is closed and α-polar Theorem 4.2.1 implies that E is a set of uniqueness for $\mathcal{D}_{-\alpha} = (D_\alpha)^*$ and, finally, Proposition 5.1.4 yields 2).

The implication 2) to 3) is an immediate consequence of Definition 5.1.3.

For the implication 3) to 1) we observe that $(S_n a)_{n\in\mathbb{N}}$ converges α-quasieverywhere on $\mathbb{S}$ for any $a \in D_\alpha$ due to Remark 5.2.1 and Proposition 5.2.2. Thus, any set of divergence is α-polar. □

By Theorem 5.2.3 and Remark 5.1.5, we obtain that the closed sets of uniqueness for the class $\mathcal{D}_{-\alpha}$ are necessarily α-polar. In particular, Theorem 4.2.1 is sharp.

For the sake of completeness, we shift our attention to the case $\alpha = 0$ which corresponds to the space $L^2(\mathbb{S})$. Here, our argument is based upon the use of Theorem 1.2.7 which implies in particular that E is a set of universality for the class D_0 provided E is an m-null set. The reverse implication is derived from Carleson's theorem for $L^2(\mathbb{S})$ (cf. Remark 5.2.1).

We investigate sets of universality for the class $D_{-\alpha}$ for $\alpha \in (0, 1]$.

Theorem 5.2.4 *Let $\alpha \in (0, 1]$ and suppose that $E \subset \mathbb{S}$ is a closed set. If E is α-complement-polar, then E is a set of universality with respect to $(C(E), \|\cdot\|_\infty)$ for the class $D_{-\alpha}$. Moreover, in this case, the set of universal elements is residual in $D_{-\alpha}$.*

Proof: By Proposition 2.3.4, we find that the Cauchy dual of $D_{-\alpha}$ is given by $\mathcal{D}_\alpha$. Since E is a closed and α-complement-polar set, Theorem 4.2.3 implies that E is a set of uniqueness for $\mathcal{D}_\alpha = (D_{-\alpha})^*$ and, finally, Proposition 5.1.4 yields the assertion. □

Remark 5.2.5 In Example 3.4.7, we have seen that for any $\alpha \in (0,1)$ and for any $\delta > 0$ there is a closed and α-complement-polar set $E \subset \mathbb{S}$ such that

$$m(E) > 1 - \delta.$$

However, due to Shkarin's theorem (see Theorem 1.2.14), any set $E \subset \mathbb{S}$ that contains a nontrivial arc cannot be a set of universality with respect to $(C(E), \mathcal{T}_{\mathrm{pw}})$ for the class D_α for any $\alpha \in \mathbb{R}$.

5.3 Sets of universality for the classes $D^p_{\mathbf{harm}}$

In the present section, we characterize the closed sets of universality for the class $\mathcal{D}^p_{\mathrm{harm}}$. The proof will rely on the representation of the Cauchy dual of D^p_{harm} as well as a statement about the sets of uniqueness for the class $\mathcal{A}^q \oplus \mathcal{A}^q_{\mathrm{ext}}$ where $q \in [2, \infty)$.

However, first, we want to revisit a result which traces back to Twomey [68]. For $p \in (1, \infty)$, we recall that the Hardy space $\mathcal{H}^p$ can be interpreted as the space of all functions $f \in H(\mathbb{D})$ for which there exists an $h \in L^p(\mathbb{S})$ satisfying the property

$$f(z) = \int_{\mathbb{S}} \frac{h(\zeta)}{1 - z\overline{\zeta}} dm(\zeta).$$

In alignment with this observation, $\mathcal{H}^p_\alpha$ is defined as the space of all functions $f \in H(\mathbb{D})$ which can be represented via a function $h \in L^p(\mathbb{S})$ and the fractional Cauchy kernel by

$$f(z) = \int_{\mathbb{S}} \frac{h(\zeta)}{(1 - z\overline{\zeta})^{1-\alpha}} dm(\zeta),$$

where $\alpha \in [0, 1)$ (see also [68, Equation (1.2)]). As previously mentioned, $\mathcal{H}^p_0 = \mathcal{H}^p$ is the classical Hardy space. Moreover, Girela and Pelaez (cf. [23]) showed that for any $1 < p \leq 2$

$$\mathcal{D}^p \subset \mathcal{H}^p_{1-1/p}. \tag{5.9}$$

The functions of the class $\mathcal{H}^p_\alpha$ have been the subject of extensive research regarding boundary behavior (cf. [55]). A significant outcome of this research is that the nontangential limit of functions in the class $\mathcal{H}^p_\alpha$ exists $\mathrm{Cap}^{(1)}_{\alpha,p}$-quasieverywhere (compare [55, Theorem 3.13 (i)]). Within this framework, the result by Twomey provides a strengthening for [55, Theorem 3.13 (i)]. We recall that $T_n f$ is the n-th partial Taylor sum of a holomorphic function f on $\mathbb{D}$.

Theorem 5.3.1 (Twomey, [68, Theorem 1])
Let $p \in (1,\infty)$ and $\alpha \in (0,1)$. For all $f \in \mathcal{H}^p_\alpha$ the series $(T_n f)_{n\in\mathbb{N}}$ converges $\mathrm{Cap}^{(1)}_{\alpha,p}$-quasieverywhere on $\mathbb{S}$.

We will investigate the sets of convergence for elements $a \in D^p_{\mathrm{harm}}$. For this purpose, we observe that Proposition 3.6.3 with parameters $M = 1$, $\alpha = 1 - 1/p$, $N = 2$ and $\beta = 1$ yields

$$\mathrm{Cap}^{(1)}_{1-1/p,\,p}(E) = 0 \qquad \text{if and only if} \qquad \mathrm{Cap}^{(2)}_{1,p}(\iota(E)) = 0. \tag{5.10}$$

Here, ι is the canonical embedding of $\mathbb{R}$ in $\mathbb{R}^2$.

As a consequence of Theorem 5.3.1 as well as (5.9) and (5.10), we obtain the following result.

Remark 5.3.2 Let $p \in (1,2]$. For all $f \in \mathcal{D}^p$ the series $(T_n f)_{n\in\mathbb{N}}$ converges $\mathrm{Cap}^{(2)}_{1,p}$-quasieverywhere on $\mathbb{S}$, or equivalently, $(S_n^+ a)_{n\in\mathbb{N}}$ converges $\mathrm{Cap}^{(2)}_{1,p}$-quasieverywhere on $\mathbb{S}$ for any $a \in D^p$.

In the sequel, we always write $\mathrm{Cap}_{1,p}$ instead of $\mathrm{Cap}^{(2)}_{1,p}$. Recall that we denote sets of vanishing $\mathrm{Cap}_{1,p}$-capacity as $(1,p)$-polar and that a property is said to hold $(1,p)$-quasieverywhere if the property holds $\mathrm{Cap}_{1,p}$-quasieverywhere.

Proposition 5.3.3 *Let $p \in (1,2]$. For all $a \in D^p_{\mathrm{harm}}$, the trigonometric series $(S_n a)_{n\in\mathbb{N}}$ converges $(1,p)$-quasieverywhere on $\mathbb{S}$. In particular, the sets of divergence for the class D^p_{harm} are $(1,p)$-polar.*

Proof: Consider a function $f \in \mathcal{D}^p_{\mathrm{harm}}$ and let a be the corresponding sequence in D^p_{harm} (recall that $\mathcal{D}^p_{\mathrm{harm}}$ and D^p_{harm} are isometrically isomorphic). Owing to Remark 2.2.6, the Bergman projection can be used to decompose the function f in the sense of (2.4) into its holomorphic and antiholomorphic part $f_{\mathrm{H}} \in \mathcal{D}^p_{\mathrm{harm}}$ and $f_{\mathrm{AH}} \in \mathcal{D}^p_{\mathrm{harm}}$. Let a_{H} and a_{AH} be the corresponding sequences of f_{H} and f_{AH}. By the mapping $f \mapsto \overline{f}$, Remark 5.3.2 can be extended to a_{AH} if S_n^+ is substituted by S_n^-. Therefore, we can conclude that $(S_n^+ a_{\mathrm{H}})_{n\in\mathbb{N}}$ and $(S_n^- a_{\mathrm{AH}})_{n\in\mathbb{N}}$ converge on $\mathbb{S}$ up to two sets E_+ and E_-, respectively, which are $(1,p)$-polar. In particular, for any $z \in \mathbb{S} \setminus (E_+ \cup E_-)$

$$\lim_{n\to\infty}\Big(S_n^+ a_{\mathrm{H}}\Big)(z) + \lim_{n\to\infty}\Big(S_n^- a_{\mathrm{AH}}\Big)(z) = \lim_{n\to\infty}\Big(S_n a_{\mathrm{H}}\Big)(z) + \lim_{n\to\infty}\Big(S_n a_{\mathrm{AH}}\Big)(z) = \lim_{n\to\infty}\Big(S_n a\Big)(z).$$

Since the union of two polar sets is polar, the assertion follows. □

We will now present a characterization of the sets of universality for the class D^p_{harm}.

Theorem 5.3.4 *Let $p \in (1,2]$ and suppose that $E \subset \mathbb{S}$ is a closed set. Then the following three statements are equivalent*

1) E is $(1,p)$-polar,

2) E is a set of universality with respect to $(C(E), \|\cdot\|_\infty)$ for the class D^p_{harm},

3) E is a set of divergence for the class D^p_{harm}.

If one of these conditions holds, then the set of universal elements is residual in D^p_{harm}.

Remark: In case $p = 2$, the class D^p_{harm} corresponds to the class D_1 which we treated in Theorem 5.2.3. In this case, a set is $(1,2)$-polar if and only if the set is of vanishing logarithmic capacity (see Proposition 3.5.8).

Proof: We show that 1) implies 2). By Proposition 2.3.5, the Cauchy dual of D^p_{harm} is given by $\mathcal{A}^q \oplus \mathcal{A}^q_{\mathrm{ext}}$. Since $E \subset \mathbb{S}$ is closed and $(1,p)$-polar, we obtain by Theorem 4.1.4 that E is a set of uniqueness for $\mathcal{A}^q \oplus \mathcal{A}^q_{\mathrm{ext}} = (D^p_{\mathrm{harm}})^*$ and Proposition 5.1.4 yields 2).

The implication 2) to 3) is an immediate consequence of Definition 5.1.3.

For the implication 3) to 1) we invoke Proposition 5.3.3 which says that $(S_n a)_{n\in\mathbb{N}}$ converges $(1,p)$-quasieverywhere on $\mathbb{S}$ for any $a \in D^p_{\mathrm{harm}}$. Thus, any set of divergence for the class D^p_{harm} is $(1,p)$-polar. □

In the following remark, we consider the class D^p_{harm} for $p > 2$.

Remark 5.3.5 If $p > 2$, it is apparent from a result by Beatrous and Burbea [4, Corollary 5.5] and the decomposition of $\mathcal{D}^p_{\mathrm{harm}}$ (see Remark 2.2.6) that any radial limit function of $f \in \mathcal{D}^p_{\mathrm{harm}}$ is contained in $\mathrm{Lip}_{1-1/p}(\mathbb{S})$ where $\mathrm{Lip}_{1-1/p}(\mathbb{S})$ is the space of all $h \in C(\mathbb{S})$ with

$$|h(z) - h(w)| \in O\Big(|z-w|^{1-1/p}\Big) \quad \Big(z, w \in \mathbb{S}\Big).$$

Since $1 - 1/p > 1/2$, Bernstein's theorem (see e.g. [34, 6.3 Theorem]) yields that the class $\mathrm{Lip}_{1-1/p}(\mathbb{S})$ is contained in the Wiener algebra. Consequently, $(S_n a)$ is uniformly convergent on $\mathbb{S}$ for any $a \in D^p_{\mathrm{harm}}$.

In the final part of this chapter, we investigate the closed sets of divergence for the class D^p. Recall that Remark 5.3.2 says in particular that $(S^+_n a)_{n\in\mathbb{N}}$ converges $(1,p)$-quasieverywhere on $\mathbb{S}$ for any $a \in D^p$. The possibility of obtaining larger sets of convergence for $a \in D^p$ is not immediately evident. We will address this query in the following proposition.

Proposition 5.3.6 *Let $p \in (1,2]$ and suppose that $E \subset \mathbb{S}$ is a closed set. If E is $(1,p)$-polar, then E is a set of divergence for the class D^p.*

Proof: Since E is closed and $(1,p)$-polar, Theorem 5.3.4 yields the existence of an element $a \in D^p_{\mathrm{harm}}$ such that for any real-valued function $h \in C(E)$, there exists a subset $J \subset \mathbb{N}$ such that

$$S_n a \to h \quad (J \ni n \to \infty). \tag{5.11}$$

Given that h only takes values in $\mathbb{R}$, we can substitute $S_n a$ by its real part $\mathrm{Re}(S_n a)$ and the limit in (5.11) remains valid. Furthermore, Remark 2.2.6 permits us to decompose the sequence a continuously in D^p_{harm}, that means

$$a^+(k) := a(k) \cdot \mathbb{1}_{\mathbb{N}_0}(k) \qquad \text{and} \qquad a^-(k) := a(k) \cdot \mathbb{1}_{-\mathbb{N}}(k) \quad (k \in \mathbb{Z})$$

with $a^+, a^- \in D^p_{\mathrm{harm}}$. Hence, the sequence

$$b_k := a^+(k) + \overline{a^-(-k)} \quad (k \in \mathbb{N}_0)$$

is contained in D^p. Finally, by establishing

$$\mathrm{Re}\Big(S_n a\Big) = \mathrm{Re}\Big(S_n\Big(a^+ + a^-\Big)\Big) = \mathrm{Re}\Big(S_n a^+\Big) + \mathrm{Re}\Big(\overline{S_n a^-}\Big) = \mathrm{Re}\Big(S_n^+ b\Big) \quad (n \in \mathbb{N})$$

and utilizing (5.11), we conclude that

$$\Big((S_n^+ b)(z)\Big)_{n\in\mathbb{N}}$$

diverges for any $z \in E$. □

Let us consider the class D^2. According to [54, Theorem 4], each closed and $(1,2)$-polar set $E \subset \mathbb{S}$ (that means polar with respect to the logarithmic capacity) is a set of universality for the class D^2. Note that [54, Theorem 4] mainly relies on a uniqueness result by Khrushchev and Peller for the class $\mathcal{A}^2$ (see Theorem 4.1.3). Unfortunately, if $p \in (1,2)$ and if $E \subset \mathbb{S}$ is a closed and $(1,p)$-polar set, the possibility of extending Theorem 4.1.3 to the class of all Cauchy transforms of a complex measure $\mu \in \boldsymbol{\mathcal{M}}_{\mathbb{C}}(E)$ with

$$C\mu_{|_{\mathbb{D}}} \in \mathcal{A}^q$$

remains uncertain. Therefore, to the best of our knowledge, it remains an open question whether the closed sets of universality for the class D^p are characterized by $\mathrm{Cap}_{1,p}$-polarity.

Appendix A

Measure and Integration Theory

This part of the appendix pursues the goal of recalling fundamental results of measure and integration theory that play a crucial role in this thesis. The reader is referred to [7] for an introduction to measure and integration theory, as well as to [65, Chapter 6] for insights into the theory of complex measures.

Let X be a set and let Σ be a σ-algebra on X. The pair (X, Σ) is a measurable space. A set function $\mu : \Sigma \to [0, \infty]$ is a **positive measure** (on (X, Σ)) if $\mu(\emptyset) = 0$ and if μ is countable additive. The triple (X, Σ, μ) is a measure space.

Definition A.1 Let (X, Σ) be a measurable space. In the sequel, the notation $\boldsymbol{\mathcal{M}}(X, \Sigma)$ is used for the set of all σ-finite positive measures μ on the measurable space (X, Σ). In case X is equipped with a topology and if Σ is the Borel σ-algebra $\mathcal{B}(X)$, we will occasionally write $\boldsymbol{\mathcal{M}}(X)$ instead of $\boldsymbol{\mathcal{M}}(X, \mathcal{B}(X))$. In this case the elements of $\boldsymbol{\mathcal{M}}(X)$ are called σ-finite Borel measures.

Moreover, we write $\mathrm{Prob}(X, \Sigma)$ for the set of all probability measures contained in $\boldsymbol{\mathcal{M}}(X, \Sigma)$. Again, if X is a topological space we use the shorter notation $\mathrm{Prob}(X)$ in the event of $\Sigma = \mathcal{B}(X)$.

Definition A.2 Let X be a separable metric space and $\mu \in \boldsymbol{\mathcal{M}}(X)$. The **support** of μ is the intersection of all closed sets $E \subset X$ such that the complement is a μ-null set, that means

$$\mathrm{supp}(\mu) = \bigcap_{\substack{E \subset X \text{ closed} \\ \mu(E^c)=0}} E.$$

In particular, $\mathrm{supp}(\mu)$ is always a closed set.

In the sequel, our main focus will be on complex measures on measurable spaces (X, Σ). Their definition will be given below following [65, Chapter 6]. Before, note that a countable family of sets $\{E_n : n \in \mathbb{N}\} \subset \Sigma$ is a partition of the set $E \in \Sigma$ if the following conditions hold

$$\bigcup_{n \in \mathbb{N}} E_n = E \qquad \text{and} \qquad E_n \cap E_k = \emptyset \quad (n, k \in \mathbb{N},\ n \neq k).$$

Definition A.3 Let (X, Σ) be a measurable space. A set function $\mu : \Sigma \to \mathbb{C}$ is called a **complex measure** if

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E_n) \quad (E \in \Sigma)$$

for every partition $\{E_n : n \in \mathbb{N}\}$ of E.

We write $\boldsymbol{\mathcal{M}}_{\mathbb{C}}(X, \Sigma)$ for the space of complex measures on (X, Σ) and, if X is a topological space and $\Sigma = \mathcal{B}(X)$, then we occasionally write $\boldsymbol{\mathcal{M}}_{\mathbb{C}}(X)$ instead of $\boldsymbol{\mathcal{M}}_{\mathbb{C}}(X, \mathcal{B}(X))$. The elements of $\boldsymbol{\mathcal{M}}_{\mathbb{C}}(X)$ are denoted as complex Borel measures.

Definition A.4 Let (X, Σ) be a measurable space and let μ be a complex measure on (X, Σ). The set function $|\mu| : \Sigma \to [0, \infty)$ is defined by

$$|\mu|(E) := \sup \left\{ \sum_{n \in \mathbb{N}} |\mu(E_n)| : \{E_n : n \in \mathbb{N}\} \text{ is a partition of } E \right\} \quad (E \in \Sigma).$$

Remark A.5 In the setting of the latter definition, the function $|\mu|$ defines a positive measure on (X, Σ), the so-called **total variation measure** of the complex measure μ. Moreover, $|\mu|$ is of bounded variation, that means

$$\|\mu\|_{\mathrm{TV}} := |\mu|(X) < \infty \tag{A.1}$$

which is a result of [65, Theorem 6.4]. One can show that the **total variation norm** $\|\cdot\|_{\mathrm{TV}}$ defines a complete norm on $\boldsymbol{\mathcal{M}}_{\mathbb{C}}(X, \Sigma)$.

A further result of complex measures is the existence of a $|\mu|$-almost unique function $h : X \to \mathbb{S}$ such that

$$\mu(E) = \int_E h \, d|\mu| \quad (E \in \Sigma). \tag{A.2}$$

In this case, we also write $\mu = h d|\mu|$ which is sometimes referred to as the polar decomposition of the complex measure μ (see e.g. [65, Theorem 6.12]).

Finally, if X is a separable metric space, then the support of a complex measure μ is given by the support of its corresponding total variation measure.

Due to the polar decomposition, it is possible to obtain a generalization of Fubini's theorem for complex measures. If $\mu_1 \in \boldsymbol{\mathcal{M}}(X_1, \Sigma_1)$ and $\mu_2 \in \boldsymbol{\mathcal{M}}(X_2, \Sigma_2)$, then $\mu_1 \otimes \mu_2$ is the product measure on the product σ-algebra $\Sigma_1 \otimes \Sigma_2$. If $\mu_1 = h_1 d|\mu_1|$ and $\mu_2 = h_2 d|\mu_2|$ are two complex measures on (X_1, Σ_1) and (X_2, Σ_2), respectively, then the product measure $\mu_1 \otimes \mu_2 : \Sigma_1 \otimes \Sigma_2 \to \mathbb{C}$ is defined by

$$(\mu_1 \otimes \mu_2)(E) := \int_E h_1(x) h_2(y) \, d(|\mu_1| \otimes |\mu_2|)(x, y) \quad (E \in \Sigma_1 \otimes \Sigma_2).$$

Remark A.6 Suppose that $\mu_1 \in \boldsymbol{\mathcal{M}}_{\mathbb{C}}(X_1, \Sigma_1)$ and $\mu_2 \in \boldsymbol{\mathcal{M}}_{\mathbb{C}}(X_2, \Sigma_2)$. If $f : X_1 \times X_2 \to \mathbb{C}$ is an integrable function with respect to $|\mu_1| \otimes |\mu_2|$, i.e.

$$\int_{X_1 \times X_2} |f| \, d(|\mu_1| \otimes |\mu_2|) < \infty,$$

then the following equations hold

$$\int_{X_1\times X_2} f\, d(\mu_1\otimes\mu_2) = \int_{X_2}\int_{X_1} f(x,y)\, d\mu_1(x)d\mu_2(y) = \int_{X_1}\int_{X_2} f(x,y)\, d\mu_2(y)d\mu_1(x).$$

Finally, we recall the concept of tightness for a family of complex Borel measures.

Definition A.7 Let X be a separable metric space and $\mathcal{F}\subset \boldsymbol{\mathcal{M}}_{\mathbb{C}}(X)$ be a family of complex Borel measures. The family $\mathcal{F}$ is called **tight** (or sometimes uniformly tight) if for every $\varepsilon>0$ there exists a compact set $E\subset X$ such that

$$\sup_{\mu\in\mathcal{F}} |\mu|(X\setminus E)\leq\varepsilon.$$

In particular, if X is a compact metric space, the aforementioned definition implies that *every* family of complex Borel measures $\mathcal{F}\subset\boldsymbol{\mathcal{M}}_{\mathbb{C}}(X)$ is tight.

In the sequel, we suppose that (X,d) is a complete and separable metric space. The subsequent theorem, originally proved by Prokhorov in 1953 for probability measures, establishes a link between the tightness of a family of complex Borel measures $\mathcal{F}\subset\boldsymbol{\mathcal{M}}_{\mathbb{C}}(X)$, and the relative compactness of $\mathcal{F}$ with respect to the weak* topology on $\boldsymbol{\mathcal{M}}_{\mathbb{C}}(X)$. A proof for families of finite signed measures, which can be generalized to families of complex measures, can for example be found in [7, 8.6.2 Theorem].

Theorem A.8 Let (X,d) be a complete and separable metric space and suppose that $\mathcal{F}\subset\boldsymbol{\mathcal{M}}_{\mathbb{C}}(X)$ is a family of complex Borel measures. Then the following two statements are equivalent:

1) Every sequence $(\mu_n)_{n\in\mathbb{N}}\in\mathcal{F}^{\mathbb{N}}$ contains a weak* convergent subsequence.

2) The family $\mathcal{F}$ is tight and uniformly bounded in variation.

We observe that the condition "uniformly bounded in variation" drops out in the second part of the foregoing theorem in case that $\mathcal{F}\subset\mathrm{Prob}(X)$.

Appendix B

Hausdorff Measures

We recall the notion of Hausdorff measures on $\mathbb{R}^N$ where we follow [48, Chapter 4] and [60, Section 10.2]. In the sequel, $\mathcal{P}(\mathbb{R}^N)$ is the power set of $\mathbb{R}^N$ and the diameter of $A \subset \mathbb{R}^N$ is given by

$$\operatorname{diam}(A) = \sup\{|x - y| : x, y \in A\}.$$

Let $\delta > 0$. Then $\mathcal{E} \subset \mathcal{P}(\mathbb{R}^N)$ is called a δ-cover of the set $A \subset \mathbb{R}^N$, if $\mathcal{E}$ contains countable many elements whose diameter is bounded by δ and if

$$A \subset \bigcup_{E \in \mathcal{E}} E.$$

Let $h : (0, \infty) \to (0, \infty)$ be an increasing function and $\delta \in (0, \infty]$. Define the function $\Lambda_h^{(\delta)} : \mathcal{P}(\mathbb{R}^N) \to [0, \infty]$,

$$\Lambda_h^{(\delta)}(A) := \inf \left\{ \sum_{E \in \mathcal{E}} h(\operatorname{diam}(E)) : \mathcal{E} \text{ is a } \delta\text{-cover of the set } A \right\}.$$

For fixed $\delta > 0$ the set function $\Lambda_h^{(\delta)}$ is monotone and subadditive. Moreover, in case $0 < \delta_1 \leq \delta_2 \leq \infty$, we have

$$\Lambda_h^{(\delta_2)}(A) \leq \Lambda_h^{(\delta_1)}(A) \quad (A \subset \mathbb{R}^N).$$

Thus, the function $\Lambda_h^{(\delta)}$ possesses a limit function if δ tends to 0. We shall write

$$\Lambda_h(A) := \lim_{\delta \to 0} \Lambda_h^{(\delta)}(A) = \sup_{\delta > 0} \Lambda_h^{(\delta)}(A) \quad (A \subset \mathbb{R}^N).$$

Let us gather some results on the function Λ_h. As a consequence of the translation invariance of the Euclidean norm, we also get the translation invariance of Λ_h, that means

$$\Lambda_h(\cdot + x) = \Lambda_h \quad (x \in \mathbb{R}^N).$$

Moreover, it is known that Λ_h defines a metric outer measure on the power set of $\mathbb{R}^N$. Consequently, by the extension theorem of Carathéodory, the set function Λ_h restricted to the σ-algebra of all Λ_h-measurable sets, denoted by Σ_{Λ_h} and defined by the collection of sets $A \subset \mathbb{R}^N$ which fulfill

$$\Lambda_h(T) = \Lambda_h(T \cap A) + \Lambda_h(T \setminus A) \quad (T \subset \mathbb{R}^N),$$

defines a measure on Σ_{Λ_h}. Since Λ_h is a metric outer measure one can show that the Borel sets are contained in Σ_{Λ_h} (see e.g. [14, Theorem 5.4.2]). In case Λ_h is restricted to Σ_{Λ_h} the set function Λ_h is called h-**Hausdorff measure**.

It is obvious by definition that h-Hausdorff measures have a natural order. More precisely, we have

$$\Lambda_{h_1}(A) \leq \Lambda_{h_2}(A) \quad (A \subset \mathbb{R}^N)$$

if $h_1 \leq h_2$ in a neighborhood of 0. The following statement can be seen as an intensification of this observation. A proof can for example be found in [1, Proposition 5.1.6].

Proposition B.1 Let $h_1, h_2 : (0, \infty) \to (0, \infty)$ be two increasing functions such that $h_1(t) \in o(h_2(t))$ as $t \to 0$. Then

$$\Lambda_{h_1}(A) > 0 \quad \text{implies} \quad \Lambda_{h_2}(A) = \infty \qquad (A \subset \mathbb{R}^N).$$

In the remainder of this appendix, we shall focus on a special class of Hausdorff measures.

Definition B.2

1) Let $\alpha \in [0, \infty)$. If $h : (0, \infty) \to (0, \infty)$, $h(t) = t^\alpha$, then we write Λ_α instead of Λ_h. Moreover, in accordance with [48, p. 55], Λ_α restricted to Σ_{Λ_α} is called α-**dimensional Hausdorff measure**.

2) The **Hausdorff dimension** of a set $A \subset \mathbb{R}^N$ is defined by

$$\dim(A) := \sup\{\alpha : \Lambda_\alpha(A) > 0\}.$$

Due to Proposition B.1, we also have

$$\dim(A) = \inf\{\alpha : \Lambda_\alpha(A) = 0\}.$$

Furthermore, the homogeneity of the function $t \mapsto t^\alpha$ implies the following formula for dilations (see e.g. [48, p. 57])

$$\Lambda_\alpha(rA) = r^\alpha \Lambda_\alpha(A) \quad (A \subset \mathbb{R}^N),$$

where $r \in (0, \infty)$ and $\alpha \in [0, \infty)$.

We compile the key findings on the Hausdorff dimension and the α-dimensional Hausdorff measure.

Proposition B.3 Let $A \subset \mathbb{R}^N$.

i) $\dim(A) \leq N$.

ii) If $\operatorname{int}(A) \neq \emptyset$, then $\dim(A) = N$.

iii) If A is a countable set and if $h(t) \in o(1)$ as $t \to 0$, then $\Lambda_h(A) = 0$. In this case, we have in particular $\dim(A) = 0$.

Additionally, we find that the unit ball has positive and finite N-dimensional Hausdorff measure, and therefore there exists a constant $M \in (0, \infty)$ such that

$$\Lambda_N(B) = M \cdot \lambda_N(B) \quad (B \in \mathcal{B}(\mathbb{R}^N)).$$

Appendix C

The Polylogarithm

In the sequel, we concentrate on certain special holomorphic functions and their characteristics. The polylogarithm, also known as the Jonquière function, is our primary focus. We begin with its definition.

Definition C.1 Let $\alpha \in \mathbb{R}$. The **polylogarithm** $\mathrm{Li}_\alpha : \mathbb{D} \to \mathbb{C}$ of order α is defined by

$$\mathrm{Li}_\alpha(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^\alpha} \qquad (z \in \mathbb{D}). \tag{C.1}$$

In the subsequent remark, we will explore the principal properties of the polylogarithm.

Remark C.2 Let $\alpha \in \mathbb{R}$. The polylogarithm defines a holomorphic function on the unit disk. With the help of contour integrals, it is possible to obtain an analytic continuation on $\mathbb{C} \setminus [1, \infty)$. In case $\alpha > 0$, we have the integral representation (see [57, 25.12.11])

$$\mathrm{Li}_\alpha(z) = \frac{z}{\Gamma(\alpha)} \int_0^\infty \frac{s^{\alpha-1}}{e^s - z} ds \quad \Big(z \in \mathbb{C} \setminus [0, \infty)\Big), \tag{C.2}$$

and if $\alpha < 0$, we obtain by relations between the polylogarithm with the Hurwitz zeta function and an application of the Abel–Plana formula that the integral representation

$$\mathrm{Li}_\alpha(z) = \int_0^\infty \frac{s^{-\alpha} \sin(\alpha\pi/2 - s\log(-z))}{\sinh(\pi s)} ds \quad \Big(z \in \mathbb{C} \setminus [0, \infty)\Big) \tag{C.3}$$

holds.

Our main objective is to analyze how the polylogarithm of order $\alpha \in (0, 1)$ behaves near the point 1. We examine the limit cases $\alpha = 0$ and $\alpha = 1$. The first case corresponds to the geometric series which implies

$$\mathrm{Li}_0(z) = \frac{z}{1 - z} \quad (z \in \mathbb{D}).$$

The second case leads to

$$\mathrm{Li}_1(z) = \log\left(\frac{1}{1-z}\right) \quad (z \in \mathbb{D}), \tag{C.4}$$

where log is the principle branch of the logarithm.

The main reason why we look at the polylogarithm is the peculiarity of its pole or branch point at 1. For $\alpha = 0$ we derive that Li_0 has a pole of order 1 and if $\alpha = 1$, Li_1 has a logarithmic branch point. But what happens in the case $\alpha \in (0,1)$? The following theorem specifies the behavior of the polylogarithm at 1 in case that α is not an integer. In order to formulate the theorem, we recall that Γ denotes the gamma function defined by

$$\Gamma(z) := \int_0^\infty s^{z-1} e^{-s} ds \quad (\mathrm{Re}(z) > 0),$$

as well as the definition of the Riemann zeta function ζ given by

$$\zeta(z) := \sum_{k=1}^{\infty} 1/k^z \quad (\mathrm{Re}(z) > 1),$$

which can be extended analytically to $\mathbb{C} \setminus \{1\}$.

Theorem C.3 Let $\alpha \in \mathbb{R} \setminus \mathbb{N}$. According to Paulsen [59, Proposition 3], there exists a sequence of complex numbers $(a_k)_{k\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ such that

$$\mathrm{Li}_\alpha(z) - \Gamma(1-\alpha)(-\log(z))^{\alpha-1} - \zeta(\alpha) = \sum_{k=1}^{\infty} a_k (z-1)^k \quad (|z-1| < 1). \tag{C.5}$$

We note that the sequence (a_k) is more precisely specified in [59].

Alternatively, according to Wood (compare [69, (9.3)]), we have

$$\mathrm{Li}_\alpha(z) - \Gamma(1-\alpha)(-\log(z))^{\alpha-1} - \zeta(\alpha) = \sum_{k=1}^{\infty} \frac{\zeta(\alpha-k)}{k!} \log^k(z) \quad \Big(z \in \mathbb{C}_-,\ |\log(z)| < 2\pi\Big). \tag{C.6}$$

In the course of this thesis, we consider the derivative of the polylogarithm. By the power series representation of the polylogarithm on the unit disk and the identity theorem, we find that the polylogarithm meets the following differential equation

$$z \cdot \mathrm{Li}'_\alpha(z) = \mathrm{Li}_{\alpha-1}(z) \qquad \Big(z \in \mathbb{C} \setminus [1,\infty)\Big).$$

Inductively, we have for any $w \in \mathbb{C}$ with $e^w \notin [1,\infty)$ and any $k \in \mathbb{N}_0$

$$(\mathrm{Li}_\alpha \circ \exp)^{(k)}(w) = (\mathrm{Li}_{\alpha-k} \circ \exp)(w) \tag{C.7}$$

Remark C.4 Suppose that $\alpha \in (0,1)$. Recall that arg is the argument of a complex number. For all $z = r \cdot u \in \mathbb{C}_- \setminus [1,\infty)$ with $r \in (0,\infty)$ and $u \in \mathbb{S} \setminus \{-1\}$ we have

$$\begin{aligned}(-\log(ru))^{\alpha-1} &= \exp\Big((\alpha-1)\log\big(\ln(1/r) - i\arg(u)\big)\Big) \qquad \text{(C.8)}\\ &= \exp\bigg((\alpha-1)\Big(\ln\Big|\ln(1/r) - i\arg(u)\Big| + i\arg\Big(\big(\ln(1/r) - i\arg(u)\big)\Big)\bigg).\end{aligned}$$

Since

$$\operatorname{Re}(\exp(x+iy)) = \exp(x)\cos(y) \quad (x,y \in \mathbb{R}),$$

we infer by applying the real part in (C.8)

$$\operatorname{Re}\Big((-\log(ru))^{\alpha-1}\Big) = \Big|\ln(1/r) - i\arg(u)\Big|^{\alpha-1}\cos\Big((\alpha-1)\arg\big(\ln(1/r) - i\arg(u)\big)\Big). \quad \text{(C.9)}$$

Since $\arg(u) = -\arg(\overline{u})$, $\ln(1/r) \to \infty$ as $r \to 0$ and cosine is an even function, we obtain that the right-hand side of (C.9) can be continuously extended to $\mathbb{C} \setminus [1,\infty)$. We deduce that

$$\mathbb{C} \setminus [1,\infty) \ni z \mapsto \begin{cases} \operatorname{Re}\Big(\big(-\log(z)\big)^{\alpha-1}\Big), & z \in \mathbb{C}^* \setminus [1,\infty) \\ 0, & z = 0 \end{cases} \qquad \text{(C.10)}$$

is a continuous function on $\mathbb{C} \setminus [1,\infty)$.

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List of Symbols

Capacities and potentials

(R_α, d)	Riesz kernel of degree α with respect to the metric d,	43
(R_1, d)	logarithmic kernel with respect to the metric d,	43
$\mathrm{Cap}_{\alpha,p}^{(N)}$, $\mathrm{Cap}_{\alpha,p}$	(N-dimensional) Bessel (α, p)-capacity,	65
$\dot{C}_\alpha$	spherical Riesz α-capacity with parameter $\alpha \in (0, 1]$,	49
C_1	logarithmic capacity,	43
C_α	Riesz α-capacity with parameter $\alpha \in (0, 1)$,	43
c_α	Li_α-capacity with parameter $\alpha \in (0, 1]$ and kernel (K_α, σ),	46
c_K, $c_{(K,d)}$	capacity with respect to the kernel $K = (K, d)$,	39
c_K^*, $c_{(K,d)}^*$	outer capacity with respect to the kernel $K = (K, d)$,	40
$C_{\alpha,p}^{(N)}$, $C_{\alpha,p}$	(N-dimensional) Riesz (α, p)-capacity,	65
$C_{\psi,p}$	(ψ, p)-capacity,	64
$G_\alpha^{(N)} = (G_\alpha^{(N)}, d_{\lvert\cdot\rvert})$	N-dimensional Bessel kernel of order $\alpha > 0$,	64
$G_\alpha = (G_\alpha, d_{\lvert\cdot\rvert})$	Bessel kernel of order $\alpha > 0$ if the dimension is clear from the context,	64
$G_\alpha^{(N)}$, G_α	(N-dimensional) Bessel kernel function of order $\alpha > 0$,	64
$I_K(\mu)$, $I_{(K,d)}(\mu)$	energy of μ with respect to the kernel $K = (K, d)$,	38
$I_K(\mu_1, \mu_2)$	mutual energy of μ_1 and μ_2 with respect to the kernel K,	52
$K\mu$, $(K, d)\mu$	potential function of the measure μ with respect to the kernel $K = (K, d)$,	38
K_α	Li_α-kernel function,	46
$K_\alpha = (K_\alpha, \sigma)$	Li_α-kernel,	46
R_1	logarithmic kernel function,	43
R_α	Riesz kernel function of degree α for $\alpha \in (0, 1)$,	43
$R_\alpha^{(N)}$	N-dimensional Riesz kernel function with parameter $\alpha > 0$,	65

Functions

arg	argument function,	44
B	Bergman kernel,	27
P	Bergman projection,	27
P_-	$P_- = I - P$,	30
Γ	gamma function,	101
Li_α	polylogarithm of order α,	100
log	principal branch of the complex logarithm,	44
$\mathcal{F}$	Fourier transform,	18
$\mathbb{1}_A$	indicator function to the set A,	18
$\widehat{\varphi}$	Fourier transform of the distribution φ,	20
$\widehat{h}$	Fourier transform of the function h,	18
ζ	Riemann zeta function,	101
C	Cauchy transform,	22
$F * \mu$	convolution of a nonnegative function F and a measure μ,	66
p_k	k-th monomial,	9

Measures

m_2	normalized area measure on the unit disk $\mathbb{D}$,	10
λ_N	N-dimensional Lebesgue measure on $\mathbb{R}^N$,	18
$\mu_1 \otimes \mu_2$	product of two σ-finite or two complex measures μ_1 and μ_2,	95
$\overline{\mu}$	complex conjugated measure of μ,	52
$\mu\vert_A$	μ restricted to the trace σ-algebra Σ_A,	13
$\vert\mu\vert$	total variation measure of a complex measure μ,	95
$hd\mu$	measure with μ-density h,	33
m	arc measure,	10

Norms and metrics

$d_{\vert\cdot\vert}$	Euclidean distance,	41
σ	spherical metric on $\mathbb{S}$,	44
$\vert\cdot\vert$	Euclidean norm,	8
$\Vert\cdot\Vert_{\mathrm{TV}}$	total variation norm,	95
$\Vert\cdot\Vert_\infty$	maximum norm,	9

$\|\cdot\|_{\mathrm{op}}$	operator norm,	19

Other symbols

$\operatorname{diam}(A)$	diameter of A,	38
$\dim(A)$	Hausdorff dimension of A,	98
γ_α	$\gamma_\alpha = \Gamma(1-\alpha)\cos(\pi(\alpha-1)/2)$ for $\alpha \in (0,1)$,	48
L_α	$L_\alpha = \mathrm{Li}_\alpha(-1)$,	46
$\mathcal{T}_{\mathrm{pw}}$	topology of pointwise convergence,	14
∇f	(weak) gradient of a function f,	24
$\operatorname{supp}(\mu)$	support of a measure μ,	94
$\widehat{\varphi}(k)$	k-th Fourier coefficient of the distribution φ,	20
$\widehat{h}(k)$	k-th Fourier coefficient of the function h,	18
e_k	k-th unit vector in $\mathbb{C}^{\mathbb{Z}}$,	34

Spaces

$W^{1,p}(\Omega)$	Sobolev space,	24
A^p_{ext}	set of all sequences $(a_k)_{k\in\mathbb{N}_0}$ in $\mathbb{C}^{-\mathbb{N}}$ such that $(a_{-k})_{k\in-\mathbb{N}} \in A^p/\mathbb{C}$,	31
$\mathcal{A}^p_{\mathrm{ext}}$	outer Bergman space with parameter p,	31
$\boldsymbol{\mathcal{M}}_{\mathbb{C}}(X,\Sigma)$	space of complex measures on (X,Σ),	95
$\boldsymbol{\mathcal{M}}_{\mathbb{C}}(X)$	space of complex Borel measures on $(X,\mathcal{B}(X))$,	95
$\boldsymbol{\mathcal{M}}(X)$	space of σ-finite Borel measures on $(X,\mathcal{B}(X))$,	94
$\boldsymbol{\mathcal{M}}(X,\Sigma)$	space of σ-finite positive measures on (X,Σ),	94
$\mathcal{B}^p_{1-1/p}(\mathbb{S})$	Sobolev-Slobodeckij space on $\mathbb{S}$ with parameter p,	25
$\mathcal{D}^p_{\mathrm{harm}}$	harmonic Dirichlet space with parameter p,	25
D^p_{harm}	$\left\{a \in \mathbb{C}^{\mathbb{Z}} : \sum_{k=0}^{\infty} a_k\, p_k\big\vert_{\mathbb{D}} + \sum_{k=1}^{\infty} a_{-k}\, \overline{p_{-k}}\big\vert_{\mathbb{D}} \in \mathcal{D}^p_{\mathrm{harm}}\right\}$,	26
$\mathrm{Lip}_\alpha(\mathbb{S})$	space of α-Lipschitz continuous functions on $\mathbb{S}$,	92
$\mathcal{A}^p$	Bergman space on $\mathbb{D}$ with parameter p,	12
$\mathcal{A}^p_\beta$	Bergman space on $\mathbb{D}$ with parameter p and weight $z \mapsto (1-\lvert z\rvert^2)^\beta$,	23
$\mathcal{D}^p$	L^p-type Dirichlet space on $\mathbb{D}$ with parameter p,	24
$\mathcal{H}^\infty$	space of all bounded holomorphic functions in $\mathbb{D}$,	11
$\mathcal{H}^p$	Hardy space on $\mathbb{D}$ with parameter p,	10

$\mathcal{H}^p_\alpha$	fractional Hardy space with parameters $p \in (1,\infty)$ and $\alpha \in (0,1]$,	90
$\mathcal{M}(\Omega)$	$\mathcal{M}(\Omega, \mathcal{B}(\Omega))$,	8
$\mathcal{M}(\Omega,\Sigma)$	space of all measurable functions $f : (\Omega,\Sigma) \to (\mathbb{C}, \mathcal{B}(\mathbb{C}))$,	8
$\mathcal{P}$	space of trigonometric polynomials,	13
D_α	$\left\{a \in \mathbb{C}^{\mathbb{Z}} : \sum\limits_{k=-\infty}^{\infty} (\lvert k\rvert + 1)^\alpha \lvert a_k\rvert^2 < \infty\right\}$,	17
$\mathcal{D}_\alpha$	$(D_\alpha)_\pm$,	17
$A(E)$	space of the continuous functions on the compact set E holomorphic in $\mathrm{int}(E)$,	9
A^p	$\left\{a \in \mathbb{C}^{\mathbb{N}_0} : \sum\limits_{k=0}^{\infty} a_k\, p_k\vert_{\mathbb{D}} \in \mathcal{A}^p\right\}$,	15
$C(E)$	space of the continuous functions on the compact set E,	9
$C^\infty(\mathbb{S})$	Schwartz space on $\mathbb{S}$,	19
$C^\infty(\mathbb{S})'$	space of tempered distributions on $\mathbb{S}$,	20
$c_0(\mathbb{Z})$	space of all two-sided zero tending sequences,	14
$c_{00}(\mathbb{Z})$	space of all two-sided sequences with finite support,	21
D^p	$\left\{a \in \mathbb{C}^{\mathbb{N}_0} : \sum\limits_{k=0}^{\infty} a_k\, p_k\vert_{\mathbb{D}} \in \mathcal{D}^p\right\}$,	25
$H(\Omega)$	space of all functions holomorphic on Ω for $\Omega \subset \mathbb{C}_\infty$ open,	8
$L^0(\mu)$	space of measurable functions equipped with the topology of convergence in measure with respect to μ,	12
$L^p(\mu)$	space of p-integrable functions with respect to the measure μ,	18
$s(\mathbb{Z})$	space of sequences of rapid decay,	19
$U \oplus V$	direct sum of U and V,	16
U/W	quotient of the vector space V by a subspace W,	16
X'	dual space of X,	19
$X^{*,M}$	Cauchy dual of X relative to M,	32
X_+	$\left\{a_+ \in H(\mathbb{D}) : a_+ = \sum\limits_{k=0}^{\infty} a_k\, p_k\vert_{\mathbb{D}},\ a \in X\right\}$ for $X \subset \mathbb{C}^{\mathbb{N}_0}$,	16
X_-	$\left\{a_- \in H(\mathbb{D}_{\mathrm{ext}}) : a_- = \sum\limits_{k=1}^{\infty} a_{-k}\, p_{-k}\vert_{\mathbb{D}_{\mathrm{ext}}},\ a \in X\right\}$ for $X \subset \mathbb{C}^{-\mathbb{N}}$,	17
$X_\pm$	$X_+ \oplus X_-$,	17

Sets

$\mathbb{C}$	set of complex numbers,	8
$\mathbb{C}^*$	$\mathbb{C} \setminus \{0\}$,	8
$\mathbb{C}_-$	$\mathbb{C} \setminus (-\infty, 0]$,	44
$\mathbb{C}_\infty$	extended plane of complex numbers,	8
$\mathbb{D}$	(open) unit disk, $\{z \in \mathbb{C} : \lvert z\rvert < 1\}$,	8
$\mathbb{D}_{\text{ext}}$	$\{z \in \mathbb{C}_\infty : \lvert z\rvert > 1\}$,	8
$\mathcal{B}(\Omega)$	Borel σ-algebra of the topological space Ω,	8
$\mathbb{N}$	set of positive integers,	8
$\mathbb{N}_0$	set of nonnegative integers,	8
$\text{Prob}(X)$	set of probability measures on $(X, \mathcal{B}(X))$,	94
$\text{Prob}(X, \Sigma)$	set of probability measures on (X, Σ),	94
$\mathbb{Q}$	set of rational numbers,	8
$\mathbb{R}$	set of real numbers,	8
$\mathbb{S}$	unit circle, $\{z \in \mathbb{C} : \lvert z\rvert = 1\}$,	8
$\Sigma_1 \otimes \Sigma_2$	product σ-algebra of Σ_1 and Σ_2,	95
$\widecheck{X}$	$\widecheck{X} := \{\varphi \in C^\infty(\mathbb{S})' : \widehat{\varphi} \in X\}$,	21
$\widehat{D}$	range of $D \subset C^\infty(\mathbb{S})'$ under the Fourier transform,	21
$\mathbb{Z}$	set of integers,	8
A_z^w	$\big\{\zeta \in \mathbb{S} : \arg(\zeta) \in [\arg(z), \arg(w)]\big\}$,	44
$B^{(N)}(a, r),\ B(a, r)$	(open) ball with center a and radius r in $\mathbb{R}^N$,	67

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