Three-Point Difference Schemes of High Order of Accuracy for Solving the Sturm-Liouville Problem

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Preface

This thesis is based on the papers [46, 47, 45, 39, 40]. Here, Chapter 2 mainly contains the theory and results of the works [46, 47, 45]. The content of [39, 40] can be found in Chapter 3.

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Abstract

The dissertation is devoted to the construction and justification of three-point difference schemes of high order of accuracy for solving the Sturm-Liouville problem. A new algorithmic realization of the exact three-point difference scheme on a non-uniform grid has been developed. We show that to compute the coefficients of the exact scheme in an arbitrary grid node x_j, it is necessary to solve two auxiliary Cauchy problems for the system of three linear ordinary differential equations of the first order: one problem on the interval $[x_{j-1}, x_j]$ (forward) and one problem on the interval $[x_j, x_{j+1}]$ (backward). The coefficient stability of the exact three-point difference scheme is proved. If the Cauchy problems are solved numerically using any one-step method, we obtain the truncated three-point difference scheme of rank $\bar{m} = 2[(m+1)/2]$ (m is a given natural number, $[\cdot]$ is the integer part of the argument in brackets). An estimate of the accuracy of three-point difference schemes was obtained and an algorithm for finding their solution was developed.

We also developed a new algorithmic realization of the exact three-point difference scheme for the Sturm-Liouville problem with singularities at the ends of the interval. As in the case of the classical Sturm-Liouville problem, to find the coefficients of the exact three-point difference scheme, it is necessary to solve two auxiliary Cauchy problems for each grid node. The coefficient stability of the exact three-point difference scheme is proved. Since the Cauchy problems for the first and last grid nodes are singular, the Taylor series method has been developed to solve them. An accuracy estimate of truncated three-point difference schemes was obtained. To solve the difference scheme, Newton's iterative method is used.

Numerical experiments are presented which confirm the efficiency of the proposed approach.

Contents

Glossary

In this work, we use notations introduced by A. A. Samarskii in [73]. Here, we list the most important symbols of the thesis.

Regular grid:

$\omega_h := \{x_j = jh,\ j = 1, 2, \dots N-1,\ h = l/N\}$ – a regular (uniform) grid on the interval $(0, l)$;

$\bar{\omega}_h := \{x_j = jh,\ j = 0, 1, \dots N,\ h = l/N\}$ – a regular (uniform) grid on the segment $[0, l]$;

$\omega_h^+ := \omega_h \cup x_N$;

h – step of the grid ω_h;

x_j – a node of the grid ω_h;

$y = y_j = y(x_j)$ – a function defined on the grid ω_h;

$y_{\bar{x}} = y_{\bar{x},j} := (y_j - y_{j-1})/h$ – the left difference derivative at a point x_j;

$y_x = y_{x,j} := (y_{j+1} - y_j)/h$ – the right difference derivative at a point x_j;

$y_{\bar{x}x} = y_{\bar{x}x,j} := (y_{j+1} - 2y_j + y_{j-1})/h^2$ – the second difference derivative at a point x_j.

Irregular grid:

$\hat{\omega}_h := \{x_j \in (0, l),\ j = 1, 2, ..., N-1\}$ – an irregular (non-uniform) grid on the interval $(0, l)$;

$\hat{\bar{\omega}}_h := \{x_j \in [0, l],\ j = 0, 1, 2, ..., N,\ x_0 = 0,\ x_N = l\}$ – an irregular (non-uniform) grid on the segment $[0, l]$;

$\hat{\omega}_h^+ := \hat{\omega}_h \cup x_N$;

$h_j := x_j - x_{j-1}$ – step of the grid $\hat{\omega}_h$;

$\hbar_j := 0.5(h_j + h_{j+1})$;

$h := \max\limits_{1 \le j \le N} h_j$;

$y_{\bar{x},j} := (y_j - y_{j-1})/h_j$ – the left difference derivative at a point x_j on irregular grid $\hat{\omega}_h$;

$y_{x,j} := (y_{j+1} - y_j)/h_{j+1}$, $y_{\hat{x},j} := (y_{j+1} - y_j)/\hbar_j$ – the right difference derivatives at a point x_j on irregular grid $\hat{\omega}_h$;

$y_{\bar{x}\hat{x},j} := \dfrac{1}{\hbar_j}\left(\dfrac{y_{j+1} - y_j}{h_{j+1}} - \dfrac{y_j - y_{j-1}}{h_j}\right)$ – the second difference derivative at a point x_j on irregular grid $\hat{\omega}_h$.

H_h – space of the grid function $y_j, j = 1, 2, ..., N-1$ given on the grid $\hat{\omega}_h$.

List of inner products and associated norms on the grid $\hat{\omega}_h$:

$$(y, v) := \sum_{j=1}^{N-1} y_j v_j \hbar_j, \quad \|y\| := \sqrt{(y, y)}, \quad (y, v] := \sum_{j=1}^{N} y_j v_j h_j,$$
$$\|y]| := \sqrt{(y, y]}, \quad \|y\|_{C(\hat{\omega}_h)} := \max_{x \in \hat{\omega}_h} |y(x)| = \max_{1 \le j \le N-1} |y_j| .$$

$Q^{(m)}[a, b]$ – class of functions having m piecewise continuous derivatives and finite number of discontinuity points of first kind.

Structure of the Thesis

This thesis consists of three chapters, conclusions and the list of references.

In **Chapter 1**, we justify the relevance of the topic and give an overview of the related literature. We give an overview of different approaches to the numerical solution of Sturm-Liouville problems as well as describe and analyze the approach of constructing exact and truncated three-point difference schemes for solving boundary value problems for ordinary differential equations and Sturm-Liouville problems.

In **Chapter 2**, we develop high-order three-point difference schemes on an arbitrary non-uniform grid for the classical Sturm-Liouville problem. In Section 2.1, we present the formulation of the problem and its main properties. In Section 2.2, an exact three-point difference scheme is constructed. In Section 2.3, a theorem on the coefficient stability of the exact three-point difference scheme is presented. As opposed to the approach in Section 1.2.2, a new algorithmic realization of the exact three-point difference scheme is proposed in Section 2.4. Section 2.5 is devoted to the construction of high-order truncated three-point difference schemes, including the development of a Newton's iterative method for finding the solution of these schemes. Section 2.6 presents the results of numerical experiments.

Chapter 3 is devoted to the development of high-order difference schemes for solving the Sturm-Liouville problem with singularities at the endpoints of the interval. In Section 3.1, the problem formulation and the construction of an exact three-point difference scheme are discussed. In Section 3.2, a theorem on the coefficient stability of the exact three-point difference scheme is proven. In contrast to the approach in Section 1.2.3, a new algorithmic realization of the exact three-point difference scheme is introduced in Section 3.3. In Section 3.4, a high-order three-point difference scheme is constructed and justified, and Newton's iterative method for finding its solution is developed. Section 3.5 presents numerical examples.

Chapter 1

Introduction

1.1 Motivation and Aims of the Thesis

Eigenvalue problems for ordinary differential equations arise in the study of problems of quantum mechanics and astrophysics, chemistry and mechanics, etc. (see [19]). For example, the determination of the eigenvibrations of a string leads to the eigenvalue problem for a second-order linear differential equation (Sturm-Liouville problem). The quantum mechanical problem of energy levels of particles moving in a given one-dimensional field and the problem of finding the spectrum of a hydrogen atom are reduced to this problem. In addition, such problems arise when applying the method of separation of variables to solve boundary value problems for partial differential equations. It is mostly impossible to find analytical solutions to eigenvalue problems, and therefore numerical methods, in particular, the finite difference method, are used to solve them.

Among the difference schemes for ordinary differential equations, the compact difference schemes play an important role. A difference scheme for ordinary differential equations of the order k is called compact if it uses only $k+1$ values of the grid function. It is known (see [38]) that such schemes are stable. The implementation of compact schemes requires a small number of arithmetic operations. In addition, in the case of schemes of high order of accuracy, these schemes can be used to find a solution to the original problem on grids with large steps [28].

In the works of A. N. Tikhonov and A. A. Samarskii [82, 83], exact three-point (compact) difference schemes and three-point difference schemes of arbitrary (predefined) order of accuracy were constructed for numerical solution of boundary value problems for linear ordinary differential equations of the second order with piecewise smooth coefficients and general two-point boundary conditions. For the Sturm-Liouville problem and the eigenvalue problem with a singular differential operator, exact three-point difference schemes and three-point difference schemes of arbitrary order of accuracy were proposed in [68, 60]. However, the algorithms of implementation of truncated three-point difference schemes of high order of accuracy for eigenvalue problems proposed in these articles contain an unconstructive procedure for calculating multiple integrals of the coefficients of a differential equation. In the works of A. A. Samarskii and V. L. Makarov [76, 75], it was shown that for linear inhomogeneous ordinary differential equations of the second order, the coefficients of the exact three-point difference scheme and the right-hand side at an arbitrary grid node can be expressed through the solutions of four auxiliary Cauchy problems, each of which is approximately solved in one step by a one-step numerical method (Taylor series expansion or Runge-Kutta method). This approach was later extended to the case of nonlinear boundary value problems in the works of A. A. Samarskii, V. L. Makarov, M. V. Kutniv, I. P. Gavrilyuk, M. Hermann, L. B. Hnativ, O. I. Pazdriy, A. V. Kunynets [63, 54, 48, 50, 51, 49, 28, 53, 43, 44, 52], etc. A logical extension is to apply these ideas to the Sturm-Liouville problem.

Thus, one of the current challenges in computational mathematics is the construction and justification of a new algorithmic realization of an exact three-point difference scheme using truncated high-order three-point difference schemes. This approach would enable the development of an efficient numerical algorithm for solving eigenvalue problems for second-order ordinary differential equations.

The aim of this work is to construct and justify three-point difference schemes of arbitrary accuracy order for solving eigenvalue problems for linear second-order ordinary differential equations.

To achieve the goal of the dissertation, the following tasks need to be accomplished:

- Develop a new algorithmic realization of exact three-point difference schemes using truncated three-point difference schemes of arbitrary accuracy order for solving the Sturm-Liouville problem.
- Develop a new algorithmic realization of exact three-point difference schemes through truncated three-point difference schemes of arbitrary accuracy order for solving the eigenvalue problem for a singular differential operator.
- Construct and justify three-point difference schemes of high accuracy order for solving eigenvalue problems for linear second-order ordinary differential equations.

- Prove the convergence and obtain an error estimate for truncated three-point difference schemes.
- Develop a Newton's iterative method for solving of difference schemes.
- Validate the difference schemes on test examples and confirm theoretical conclusions through numerical experiments.

1.2 Overview of the Literature

Numerical methods for solving the Sturm-Liouville problems have been the subject of a large number of works: monographs and textbooks, scientific articles, conference papers, and online resources. The review presented in this chapter includes only information directly related to the topic of the dissertation and does not claim to be complete. It discusses some of the most commonly used numerical methods, as well as exact and truncated three-point difference schemes for solving the Sturm-Liouville problems.

1.2.1 Numerical Methods for Solving the Sturm-Liouville Problem

Numerical methods for solving the Sturm-Liouville problem were created by Babuska I. [12], Gould S. H. [32], Osborn J. E. [66], Pryce J. D. [70], Zettl A. [87], Andrew A. L. [4, 5, 6, 8, 67], Gould S. H. [32], Babenko K. I. [11], Prikazchikov V. G. [69, 68], Makarov V. L. [58, 59, 64, 29], Quarteroni A. [71], Weinstein A. [86], Algazin S. D. [2, 3], Kreiss H.-O. [42], Strang G. [80], Fix G. I. [26], Marchuk G. I., Shaidurov V. V. [65], Collatz L. [19], Samarskii A. A. [72, 84, 85] and others.

The most commonly used numerical methods for solving the Sturm-Liouville problem are the shooting method, the finite difference method, variational methods (Ritz, Galerkin), and spectral and pseudospectral methods.

Consider the classic Sturm-Liouville problem

$$\frac{d}{dx}\left[k(x)\frac{du}{dx}\right] - q(x)u(x) = -\lambda r(x)u(x),\ x \in (0,1),\ u(0) = u(1) = 0. \quad (1.1)$$

For the problem (1.1), the shooting method is reduced to solving a sequence of Cauchy problems for this differential equation with the initial conditions $u(0) = 0, u'(0) = 1$. By solving the Cauchy problem numerically, we obtain the solution $u(x; \lambda)$, which satisfies left boundary condition and depends on the parameter λ. Generally speaking, $u(1;\lambda) \neq 0$, that means, this solution does not satisfy the right boundary condition. Then we will change the parameter λ until we get $u(1;\lambda) = 0$ with the desired accuracy. To do this, we use the usual methods of finding the roots of an algebraic equation. The shooting method is difficult to apply if the corresponding Cauchy problem is not stable. Then a small change in λ can significantly change the solution $u(x)$. In this case, it is impossible to organize the process of solving the algebraic equation $u(1;\lambda) = 0$.

In an attempt to overcome the difficulties encountered in the shooting method, the Prüfer transformation and the Riccati equation were used in the article [30]. Let us introduce the phase function $\varphi(x)$, which satisfies the equations

$$u(x) = \rho(x)\sin\varphi(x), \quad k(x)u'(x) = \rho(x)\cos\varphi(x).$$

Then we can show that $\varphi(x)$ is the solution of the differential equation

$$\varphi'(x) = \frac{1}{k(x)}\cos^2\varphi(x) + (\lambda r(x) - q(x))\sin^2\varphi(x) \tag{1.2}$$

with boundary conditions

$$\varphi(0) = 0, \quad \varphi(1) = \pi n.$$

If we choose $n = 1, 2, ...$ and integrate the equation (1.2) with the initial condition $\varphi(0) = 0$, we will get the eigenvalues $\lambda_1, \lambda_2, ...$ in turn as a result of solving the nonlinear equation $\varphi(1) = \pi n$ with respect to λ. In particular, the well-known SLEIG code [13] and the program [15] from the NAG library are based on the Prüfer transformation.

The Prüfer transformation was also used to solve the singular Sturm-Liouville problems [14, 16].

Later, this approach was generalized to the case of regular self-adjoint Sturm-Liouville problems with matrix coefficients [9], and the algorithm was further developed in the works [23, 24].

The finite difference method is usually used when the Cauchy problem in the shooting method is not stable. Using the simplest linear finite difference approximations of the derivatives for the problem (1.1) on a uniform grid $\bar{\omega}_h = \{x_j = jh, j = 0, 1, ..., N, hN = 1\}$, a difference scheme of the second order of accuracy can be constructed (see, for example, [72])

$$(ay_{\bar{x}})_{x,j} - d_j y_j + \lambda \rho_j y_j = 0, \quad j = 1, 2, ..., N-1, \quad y_0 = y_N = 0, \tag{1.3}$$

where

$$y_j \approx u(x_j), \quad y_{\bar{x},j} := \frac{y_j - y_{j-1}}{h}, \quad y_{x,j} := \frac{y_{j+1} - y_j}{h},$$

$$a_j = k\left(x_j - h/2\right), \quad d_j = q(x_j), \quad \rho_j = r(x_j).$$

Then we get an algebraic problem on eigenvalues and eigenvectors. In general, the matrix of the algebraic eigenvalue problem, although tridiagonal, is too large to calculate all its eigenvalues. In addition, the eigenvalues of a large number of the matrix do not approximate well the corresponding eigenvalues of the differential operator (see, e.g., [7, pp. 61 – 63]). Therefore, a partial eigenvalue problem is usually solved, i.e., usually the first (smallest) eigenvalues are calculated. In particular, to solve the system (1.3), which is a system of nonlinear equations due to the term $\lambda\rho_j y_j$, one can apply Newton's iterative method, which in the case of (1.1) coincides with the Derwidue method [22].

To solve the Sturm-Liouville problem

$$\frac{d^2u}{dx^2} - q(x)u = -\lambda u(x), \quad x \in (0,\pi), \quad u(0) = u(\pi) = 0$$

one can also use the Numerov's scheme [21], which has the 4th order of accuracy, as well as difference schemes of higher order of accuracy, which are a multi-step generalization of the Numerov's scheme [1]. However, these schemes cannot be applied to the more general problem (1.1) and they have the appropriate order of accuracy only on a uniform grid.

If we use variational methods [32, 80] (e.g., the Galerkin method), the original problem also reduces to an algebraic eigenvalue problem for the unknown coefficients of the expansion of the approximate solution by some system of basis functions. In the Galerkin method, the approximate solution of (1.1) is sought as a linear combination of the complete system of functions $\varphi_j(x), j = 1, 2, ..., n,$

$$y_n(x) = \sum_{j=1}^{n} c_j\varphi_j(x), \quad 0 \le x \le 1,$$

which are chosen so that the boundary conditions are satisfied. The coefficients $c_j, j = 1, 2, ..., n$ are determined from the orthogonality conditions of the residual function for $y_n(x)$ to the basis functions $\varphi_j(x), j = 1, 2, ..., n$

$$\int_0^1 \left[\frac{d}{dx}\left(k(x)\frac{dy_n(x)}{dx}\right) - (q(x) - \lambda r(x))y_n(x)\right]\varphi_j(x)dx = 0,$$
$$j = 1, 2, ..., n.$$

These conditions form a system of n linear algebraic equations with $n+1$ unknowns $c_1, c_2, ..., c_n, \lambda$. Another (additional) equation can be obtained from one of the boundary conditions.

Let us now consider an eigenvalue problem for a singular system of first-order ordinary differential equations

$$u'(x) - \frac{M(x)}{x^{\alpha}} u(x) = \lambda u(x), \quad x \in (0,1), \tag{1.4}$$

$$B_0 u(0) + B_1 u(1) = 0, \tag{1.5}$$

where $u = (u_1, u_2, \dots, u_N)^T$, $M(x)$ is a sufficiently smooth matrix, and $\alpha \geq 1$. The normalization condition

$$v(x) = \int_0^x \sum_{k=1}^{N} |u_k(\xi)|^2 \, d\xi, \quad v(1) = 1,$$

ensures the uniqueness of the eigenfunction.

In [10], the following approach was proposed to solve this problem. The problem (1.4), (1.5) is supplemented with an ordinary differential equation

$$\lambda'(x) = 0,$$

with an additional equation derived from the normalization condition

$$v'(x) = \sum_{k=1}^{N} |u_k(x)|^2,$$

as well as with two new boundary conditions

$$v(0) = 0, \quad v(1) = 1.$$

To solve the resulting boundary value problem with the unknowns (λ, $u(x)$, $v(x)$), the polynomial collocation method is applied (see [41]).

However, all of these methods have a significant drawback: their convergence rate depends on the serial number of the corresponding eigenvalue, and the higher the number, the worse the accuracy.

To overcome these disadvantages, asymptotic correction of the calculated eigenvalues is used both by the finite difference method and variational methods, namely the finite element method (see, e.g., [4]). However, this combined approach is inefficient for obtaining high accuracy except for very small or very large eigenvalues. In addition, it does not allow refinement of eigenfunctions. It is also possible to combine asymptotic correction with Richardson extrapolation, but this has the same disadvantages.

A method with an exponential convergence rate based on the approximation of the eigenfunction by an interpolation polynomial with Chebyshev nodes was proposed in [17, 2, 11]. However, this method is also effective only for obtaining very small or very large eigenvalues.

Another approach is to approximate the coefficients of the differential equation. The coefficients are replaced by piecewise constant functions because in this case an exact general solution of the equation can be obtained [4, 25, 67, 70]. This method is numerically analytical and is often used to obtain two-sided estimates for the eigenvalues. In [31], it is proposed to use piecewise linear and piecewise quadratic approximation. The disadvantage of this approach is that in order to achieve high accuracy, it is necessary to choose a fine division of the integration interval into subintervals.

The FD-method proposed in [55], which is recurrent, is free from many of the above disadvantages. At the zero iteration, this method coincides with the method of approximation of a differential equation, and at each next iteration, a boundary value problem is solved for a second-order differential equation with piecewise constant coefficients and a variable right-hand side, which is built through previous iterative solutions. The FD-method has also been successfully applied to singular Sturm-Liouville problems [57, 56]. However, this method is numerically analytical and involves the use of computer algebra systems.

Since the topic of this dissertation is directly related to exact and truncated three-point difference schemes for solving boundary value problems for second-order ODEs, we will further elaborate on the construction and study of three-point difference schemes of high order of accuracy for solving the Sturm-Liouville problems.

Definition 1.1. *A difference scheme is called exact if its solution* $\{y_j\}_{j=0}^N$ *coincides with the grid function of the exact solution* $u(x)$ *of the given BVP, i.e.,* $y_j = u_j = u(x_j)$.

1.2.2 Exact and Truncated Three-Point Difference Schemes for the Sturm-Liouville Problem

In [82, 83], for linear ordinary differential equations of the second order the exact three-point difference scheme (ETDS) is constructed, and the algorithmic realization of the exact scheme by truncated three-point difference schemes (TDS) of any order of accuracy is developed and substantiated. In [74], these results are extended to the case when the coefficients of the differential equation are generalized functions.

For the Sturm-Liouville problem

$$\frac{d}{dx}\left[k(x)\frac{du}{dx}\right] - q(x)u(x) = -\lambda r(x)u(x), \quad x \in (0,1), \tag{1.6}$$

$$u(0) = 0, \quad u(1) = 0, \tag{1.7}$$

where $k(x), q(x), r(x) \in Q[0,1]$ are piecewise continuous functions satisfying the conditions $0 < C_1 \leq k(x) \leq C_2,\ 0 \leq q(x) \leq C_3,\ 0 < C_4 \leq r(x) \leq C_5$, exact three-point difference scheme and three-point difference schemes of arbitrary order of accuracy have been proposed in [68]. Let us dwell on these results in more detail.

Let us consider the irregular grid

$$\hat{\omega}_h := \left\{ x_j \in (0,1), j = 1, 2, ..., N-1, x_j - x_{j-1} =: h_j > 0, \sum_{j=1}^{N} h_j = 1 \right\},$$
$$h := \max_{1 \leq j \leq N} h_j.$$

By analogy with [82, 83, 68], we will construct the exact scheme for the problem (1.6), (1.7). To construct the exact scheme, it is sufficient to obtain a relation which relates the values of eigenfunction at the three points x_{j-1}, x_j, x_{j+1}. Let us express the solution of equation (1.6) at any internal point of the interval (x_{j-1}, x_{j+1}) in terms of the values u_{j-1}, u_{j+1}. The general solution of the differential equation (1.6) can be written as

$$u(x) = A_j v_1^j(x) + B_j v_2^j(x), \quad x_{j-1} \leq x \leq x_{j+1}, \tag{1.8}$$

where $A_j, B_j, j = 1, 2, ..., N-1$ are constants, $v_\alpha^j(x, \lambda), \alpha = 1, 2$ are linearly independent solutions of Cauchy problems (pattern functions)

$$\frac{d}{dx}\left[k(x)\frac{dv_\alpha^j}{dx}\right] - q(x)v_\alpha^j(x,\lambda) + \lambda r(x) v_\alpha^j(x,\lambda) = 0, \quad x \in (x_{j-1}, x_{j+1}), \tag{1.9}$$

$$v_\alpha^j(x_{j+(-1)^\alpha}, \lambda) = 0, \quad k(x)\frac{dv_\alpha^j(x,\lambda)}{dx}\bigg|_{x = x_{j+(-1)^\alpha}} = (-1)^{\alpha+1}, \quad \alpha = 1, 2, \tag{1.10}$$

$$j = 1, 2, ..., N-1.$$

Pattern functions have the following properties.

1. The ratios are true:

$$\begin{aligned} v_1^j(x_{j+1},\lambda) =& v_2^j(x_{j-1},\lambda), \quad v_2^j(x_j,\lambda) = v_1^{j+1}(x_{j+1},\lambda), \\ v_1^j(x_{j+1},\lambda) =& v_1^j(x_j,\lambda) + v_2^j(x_j,\lambda) \\ & + v_2^j(x_j,\lambda)\int_{x_{j-1}}^{x_j} v_1^j(\xi,\lambda)\left[q(\xi) - \lambda r(\xi)\right] d\xi \\ & + v_1^j(x_j,\lambda)\int_{x_j}^{x_{j+1}} v_2^j(\xi,\lambda)\left[q(\xi) - \lambda r(\xi)\right] d\xi. \end{aligned}$$

2. $v^j_\alpha(x,\lambda) > 0, \alpha = 1,2.$

Let us put in equation (1.8) $x = x_{j-1}$ and find the constant

$$B_j = \frac{u_{j-1}}{v^j_2(x_{j-1},\lambda)}.$$

If $x = x_{j+1}$, then from this equation we obtain

$$A_j = \frac{u_{j+1}}{v^j_1(x_{j+1},\lambda)}.$$

So,

$$u(x) = \frac{v^j_1(x,\lambda)}{v^j_1(x_{j+1},\lambda)} u_{j+1} + \frac{v^j_2(x,\lambda)}{v^j_2(x_{j-1},\lambda)} u_{j-1}, \quad x_{j-1} \leq x \leq x_j.$$

Then at $x = x_j$ we have

$$u(x_j) = \frac{v^j_1(x_j,\lambda)}{v^j_1(x_{j+1},\lambda)} u_{j+1} + \frac{v^j_2(x_j,\lambda)}{v^j_2(x_{j-1},\lambda)} u_{j-1}. \tag{1.11}$$

Using further the properties of pattern functions and replacing u_j with y_j, the last equality can be written as exact three-point difference scheme

$$(ay_{\bar{x}})_{\hat{x},j} - d_j y_j + \lambda \rho_j y_j = 0, \quad j = 1,2,...,N-1, \quad y_0 = y_N = 0, \tag{1.12}$$

where

$$y_{\bar{x},j} := \frac{y_j - y_{j-1}}{h_j}, \quad y_{\hat{x},j} := \frac{y_{j+1} - y_j}{\hbar_j}, \quad \hbar_j := \frac{h_j + h_{j+1}}{2},$$

$$\begin{aligned} a_j &= \left[\frac{1}{h_j} v^j_1(x_j,\lambda)\right]^{-1}, \\ d_j &= \frac{1}{\hbar_i v^j_1(x_j,\lambda)} \int_{x_{j-1}}^{x_j} v^j_1(\xi,\lambda) q(\xi) d\xi \\ &\qquad + \frac{1}{\hbar_j v^j_2(x_j,\lambda)} \int_{x_j}^{x_{j+1}} v^j_2(\xi,\lambda) q(\xi) d\xi, \\ \rho_j &= \frac{1}{\hbar_i v^j_1(x_j,\lambda)} \int_{x_{j-1}}^{x_j} v^j_1(\xi,\lambda) r(\xi) d\xi \\ &\qquad + \frac{1}{\hbar_j v^j_2(x_j,\lambda)} \int_{x_j}^{x_{j+1}} v^j_2(\xi,\lambda) r(\xi) d\xi. \end{aligned} \tag{1.13}$$

The solution y_j of the difference problem (1.12) coincides with the solution of the original problem u_j at the grid nodes up to a multiplicative constant.

At the point $x = x_j$, let us switch to the local coordinate system

$$\begin{aligned} &x = x_j + (s - \Delta_j)\hbar_j = \bar{x}_j + s\hbar_j, \quad -1 \le s \le 1, \\ &\Delta_j = \frac{h_{j+1} - h_j}{2\hbar_j}, \quad \bar{x}_j = x_j + \Delta_j \hbar_j. \end{aligned}$$

Then the segment $[x_{j-1}, x_{j+1}]$ maps to the segment $-1 \le s \le 1$, and the point $x = x_j$ corresponds to $s = \Delta_j$. Let us assume

$$v_1^j(x, \lambda) = v_1^j(x_j + (s - \Delta_j)\hbar_j, \lambda) = \hbar_j \alpha^j(s, \lambda),$$

$$v_2^j(x, \lambda) = v_2^j(x_j + (s - \Delta_j)\hbar_j, \lambda) = \hbar_j \beta^j(s, \lambda), \quad -1 \le s \le 1.$$

The pattern functions $\alpha^j(s, \lambda), \beta^j(s, \lambda)$ satisfy the conditions

$$\begin{aligned} &\frac{d}{ds}\left[\bar{k}(s)\frac{d\alpha^j(s,\lambda)}{ds}\right] - \hbar_j^2\left(\bar{q}(s) - \lambda\bar{r}(s)\right)\alpha^j(s,\lambda) = 0, \quad -1 < s < 1, \\ &\alpha^j(-1, \lambda) = 0, \quad \bar{k}(x)\frac{d\alpha^j}{ds}\bigg|_{s=-1} = 1, \\ &\frac{d}{ds}\left[\bar{k}(s)\frac{d\beta^j(s,\lambda)}{ds}\right] - \hbar_j^2\left(\bar{q}(s) - \lambda\bar{r}(s)\right)\beta^j(s,\lambda) = 0, \quad -1 < s < 1, \\ &\beta^j(1, \lambda) = 0, \quad \bar{k}(x)\frac{d\beta^j}{ds}\bigg|_{s=1} = -1, \end{aligned}$$

where

$$\begin{aligned} &\bar{k}(s) = k(x_j + \hbar_j(s - \Delta_j)), \quad \bar{q}(s) = q(x_j + \hbar_j(s - \Delta_j)), \\ &\bar{r}(s) = r(x_j + \hbar_j(s - \Delta_j)). \end{aligned}$$

Then the coefficients a, d, ρ are calculated by the formulas

$$a(x_j) = \left[\frac{\hbar_j}{h_j}\alpha^j(0, \lambda)\right]^{-1}, \tag{1.14}$$

$$d(x_j) = \frac{1}{\alpha^j(0,\lambda)}\int_{-1}^{\Delta_j} \alpha^j(s,\lambda)\bar{q}(s)ds + \frac{1}{\beta^j(0,\lambda)}\int_{\Delta_j}^{1} \beta^j(s,\lambda)\bar{q}(s)ds, \tag{1.15}$$

$$\rho(x_j) = \frac{1}{\alpha^j(0,\lambda)}\int_{-1}^{\Delta_j} \alpha^j(s,\lambda)\bar{r}(s)ds + \frac{1}{\beta^j(0,\lambda)}\int_{\Delta_j}^{1} \beta^j(s,\lambda)\bar{r}(s)ds. \tag{1.16}$$

In general, the coefficients of an exact scheme cannot be directly expressed using quadratures, so we will use truncated schemes of high order of accuracy. These schemes are derived from the exact one, and the coefficients are represented as multiple integrals of $k(x), q(x), r(x)$.

It can be shown that $\alpha^j(s,\lambda)$ and $\beta^j(s,\lambda)$ are analytic functions of the arguments h_j^2 and h_{j+1}^2, i.e. they can be represented as convergent series

$$\alpha^j(s,\lambda)=\alpha_0+\sum_{k=1}^{\infty}\alpha_k^j(s,\lambda)h_j^{2k},\quad \beta^j(s,\lambda)=\beta_0+\sum_{k=1}^{\infty}\beta_k^j(s,\lambda)h_{j+1}^{2k},\quad (1.17)$$

and the coefficients of the decompositions are calculated by the recurrence formulas

$$\alpha_k^j(s,\lambda)=\int_{-1}^{s}\frac{1}{\bar{k}(t)}\left[\int_{-1}^{t}(\bar{q}(\mu)-\lambda\bar{r}(\mu))\alpha_{k-1}^j(\mu,\lambda)d\mu\right]dt,\quad k=1,2,...,$$

$$\beta_k^j(s,\lambda)=\int_{s}^{1}\frac{1}{\bar{k}(t)}\left[\int_{t}^{1}(\bar{q}(\mu)-\lambda\bar{r}(\mu))\beta_{k-1}^j(\mu,\lambda)d\mu\right]dt,\quad k=1,2,...,$$

$$\alpha_0^j(s,\lambda)=\int_{-1}^{s}\frac{1}{\bar{k}(t)}dt,\quad \beta_0^j(s,\lambda)=\int_{s}^{1}\frac{1}{\bar{k}(t)}dt.$$

If we take a finite number of terms in (1.17)

$$\alpha^{(m)j}(s,\lambda)=\alpha_0+\sum_{k=1}^{m}\alpha_k^j(s,\lambda)h_j^{2k},\quad \beta^{(m)j}(s,\lambda)=\beta_0+\sum_{k=1}^{m}\beta_k^j(s,\lambda)h_{j+1}^{2k}$$

and calculate by formulas (1.14) — (1.16) the coefficients $a^{(m)}, d^{(m)}, \rho^{(m)}$ replacing in these formulas $\alpha^j(s,\lambda)$, $\beta^j(s,\lambda)$ by polynomials $\alpha^{(m)j}(s,\lambda)$, $\beta^{(m)j}(s,\lambda)$, then we obtain a truncated three-point difference scheme of rank m

$$\begin{aligned}&\left(a^{(m)}y_{\bar{x}}^{(m)}\right)_{\hat{x},j}-d_j^{(m)}y_j^{(m)}+\lambda^{(m)}\rho_j^{(m)}y_j^{(m)}=0,\quad j=1,2,...,N-1,\\&y_0^{(m)}=y_N^{(m)}=0.\end{aligned}\qquad(1.18)$$

Theorem 1.1. *The truncated difference scheme (1.18) of rank m has the $2m+2$th order of accuracy at $h\to 0$ in the class $Q[0,1]$ of piecewise continuous functions $k(x), q(x), r(x)$ on an arbitrary irregular grid, i.e.*

$$\left|\lambda_n-\lambda_n^{(m)}\right|\le M_1h^{2m+2},$$

$$\left\|y_n-y_n^{(m)}\right\|_{C(\hat{\omega}_h)}\le M_2h^{2m+2},$$

where M_1, M_2 are constants independent of h, $\|y(x)\|_{C(\hat{\omega}_h)}=\max\limits_{1\le j\le N-1}|y_j|$.

1.2.3 Exact and Truncated Three-Point Difference Schemes for the Eigenvalue Problem with Some Singular Differential Operator

In the articles [58, 59] are considered difference schemes for a certain class of singular differential equations of the form

$$\frac{d}{dx}\left[k(x)\frac{du}{dx}\right] - q(x)u(x) = -\lambda r(x)u(x), \quad x \in (-1,1), \tag{1.19}$$

$$u(-1) \neq \infty, \quad u(1) \neq \infty, \tag{1.20}$$

where

$$\begin{aligned} &k(x) = (1-x^2)k_1(x), \quad 0 < C_1 \leq k_1(x) \leq C_2, \\ &0 < C_3 \leq q(x) \leq C_4, \quad 0 < C_5 \leq r(x) \leq C_6, \end{aligned} \tag{1.21}$$

$C_i, i = 1, 2, ..., 6$ are constants. Exact and truncated three-point difference schemes for this problem are developed in [60]. Let us dwell on these results in more detail.

Let us introduce the uniform grid

$$\omega_h := \{x_j = -1 + (j-0.5)h,\ h = 2/N,\ j = 1, 2, \ldots, N\}$$

$$x_0 = -1,\ x_{N+1} = 1,$$

and select the pattern functions $v_\alpha^j(x,\lambda)$, $\alpha = 1, 2$, $j = 1, 2, ..., N$ as solutions to the Cauchy problems

$$\begin{aligned} &\frac{d}{dx}\left[k(x)\frac{dv_1^1}{dx}\right] - q(x)v_1^1(x,\lambda) + \lambda r(x)v_1^1(x,\lambda) = 0, \quad x \in (x_0, x_2), \\ &v_1^1(x_0,\lambda) = 1, \quad k(x)\frac{dv_1^1(x,\lambda)}{dx}\bigg|_{x=x_0} = 0, \end{aligned} \tag{1.22}$$

$$\begin{aligned} &\frac{d}{dx}\left[k(x)\frac{dv_\alpha^j}{dx}\right] - q(x)v_\alpha^j(x,\lambda) + \lambda r(x)v_\alpha^j(x,\lambda) = 0, \quad x \in (x_{j-1}, x_{j+1}), \\ &v_\alpha^j(x_{j+(-1)^\alpha},\lambda) = 0, \quad k(x)\frac{dv_\alpha^j(x,\lambda)}{dx}\bigg|_{x=x_{j+(-1)^\alpha}} = (-1)^{\alpha+1}, \\ &\alpha = 1, 2, \quad j = 3-\alpha, 4-\alpha, \ldots, N+1-\alpha, \end{aligned} \tag{1.23}$$

$$\begin{aligned} &\frac{d}{dx}\left[k(x)\frac{dv_2^N}{dx}\right] - q(x)v_2^N(x,\lambda) + \lambda r(x)v_2^N(x,\lambda) = 0, \quad x \in (x_{N-1}, x_{N+1}), \\ &v_2^N(x_{N+1},\lambda) = 1, \quad k(x)\frac{dv_2^N(x,\lambda)}{dx}\bigg|_{x=x_{N+1}} = 0. \end{aligned} \tag{1.24}$$

These functions have properties:

1. $v_\alpha^j(x,\lambda) > 0,\ \alpha = 1,2$ for all $x \in (x_{j-1}, x_{j+1})$, $j = 1,2,...,N$;
2. These functions satisfy the relations

$$\begin{aligned} v_1^j(x_{j+1},\lambda) &= v_2^j(x_{j-1},\lambda), \quad j = 2,3,...,N-1, \\ v_2^j(x_j,\lambda) &= v_1^{j+1}(x_{j+1},\lambda), \quad j = 1,2,...,N-1, \end{aligned}$$

$$\begin{aligned} v_1^1(x_2,\lambda) = v_1^1(x_1,\lambda) &+ v_2^1(x_1,\lambda)\int_{x_0}^{x_1} v_1^1(\xi,\lambda)[q(\xi)-\lambda r(\xi)]\,d\xi \\ &+ v_1^1(x_1,\lambda)\int_{x_1}^{x_2} v_2^1(\xi,\lambda)[q(\xi)-\lambda r(\xi)]\,d\xi, \end{aligned}$$

$$\begin{aligned} v_1^j(x_{j+1},\lambda) &= v_1^j(x_j,\lambda) + v_2^j(x_j,\lambda) \\ &+ v_2^j(x_j,\lambda)\int_{x_{j-1}}^{x_j} v_1^j(\xi,\lambda)[q(\xi)-\lambda r(\xi)]\,d\xi \\ &+ v_1^j(x_j,\lambda)\int_{x_j}^{x_{j+1}} v_2^j(\xi,\lambda)[q(\xi)-\lambda r(\xi)]\,d\xi, \quad j = 2,...,N-1, \end{aligned}$$

$$\begin{aligned} v_2^N(x_{N-1},\lambda) = v_2^N(x_N,\lambda) &+ v_2^N(x_{N-1},\lambda)\int_{x_{N-1}}^{x_N} v_1^N(\xi,\lambda)[q(\xi)-\lambda r(\xi)]\,d\xi \\ &+ v_1^N(x_N,\lambda)\int_{x_N}^{x_{N+1}} v_2^N(\xi,\lambda)[q(\xi)-\lambda r(\xi)]\,d\xi. \end{aligned}$$

Let us move on to the construction of the exact three-point difference scheme. At the nodes $x_j, j = 2,3,...,N-1$, this is done similarly to the case of the regular Sturm-Liouville problem, i.e., the relation that connects the values of eigenfunction at three points x_{j-1}, x_j, x_{j+1} is given by the formula (1.11). Since $v_2^1(x)$ is unbounded at $x \to -1$, choosing A_2, on the interval $[-1, x_2]$ the solution to the original problem is represented as

$$u(x) = A_1 v_1^1(x).$$

Let us put $x = x_2$ in this equation and find $A_1 = \frac{u(x_2)}{v_1^1(x_2)}$. Then at $x = x_1$ we have

$$u(x_1) = \frac{v_1^1(x_1)}{v_1^1(x_2)} u(x_2).$$

A similar relation can be obtained at the point x_N, namely

$$u(x_N) = \frac{v_2^N(x_N)}{v_2^N(x_{N-1})} u(x_{N-1}).$$

Using the properties of pattern functions, we get the exact three-point difference scheme

$$\begin{aligned}&(ay_{\bar{x}})_{x,j} - d_j y_j + \lambda \rho_j y_j = 0, \quad j = 1, 2, ..., N,\\ &y_0 \neq \infty, \quad y_{N+1} \neq \infty,\end{aligned} \tag{1.25}$$

where

$$\begin{aligned}
&y_{\bar{x},j} := \frac{y_j - y_{j-1}}{h}, \quad y_{x,j} := \frac{y_{j+1} - y_j}{h},\\
&a_1 = a_{N+1} = 0, \quad a_j = \left[\frac{1}{h} v_1^j(x_j, \lambda)\right]^{-1}, \quad j = 2, 3, ..., N,\\
&d_j = \frac{1}{h v_1^j(x_j, \lambda)} \int_{x_{j-1}}^{x_j} v_1^j(\xi, \lambda) q(\xi)\, d\xi + \frac{1}{h v_2^j(x_j, \lambda)} \int_{x_j}^{x_{j+1}} v_2^j(\xi, \lambda) q(\xi) d\xi,\\
&\quad j = 1, 2, ..., N,\\
&\rho_j = \frac{1}{h v_1^j(x_j, \lambda)} \int_{x_{j-1}}^{x_j} v_1^j(\xi, \lambda) r(\xi)\, d\xi + \frac{1}{h v_2^j(x_j, \lambda)} \int_{x_j}^{x_{j+1}} v_2^j(\xi, \lambda) r(\xi)\, d\xi,\\
&\quad j = 1, 2, ..., N.
\end{aligned} \tag{1.26}$$

At the point $x = x_j$, let us move to the local coordinate system using the formulas $x = x_j + sh, s = \frac{x - x_j}{h}$. Then the segment $[x_{j-1}, x_{j+1}], j = 2, 3, ..., N-1$ maps to the segment $-1 \le s \le 1$, the segment $[-1, x_2]$ maps to the segment $-\frac{1}{2} \le s \le 1$, and the segment $[x_{N-1}, 1]$ maps to the segment $-1 \le s \le \frac{1}{2}$. Let us put

$$\begin{aligned}
&v_1^1(x, \lambda) = v_1^1(x_1 + sh, \lambda) = \alpha^1(s, \lambda),\\
&v_1^j(x, \lambda) = v_1^j(x_j + sh, \lambda) = h\alpha^j(s, \lambda), \quad j = 2, 3, ..., N,
\end{aligned}$$

$$\begin{aligned}
&v_2^j(x, \lambda) = v_2^j(x_j + sh, \lambda) = h\beta^j(s, \lambda), \quad j = 1, 2, ..., N-1,\\
&v_2^N(x, \lambda) = v_2^N(x_N + sh, \lambda) = \beta^N(s, \lambda).
\end{aligned}$$

The pattern functions $\alpha^j(s, \lambda), \beta^j(s, \lambda)$ satisfy the conditions

$$\begin{aligned}
&\frac{d}{ds}\left[\bar{k}(s) \frac{d\alpha^1(s, \lambda)}{ds}\right] - h^2 \left(\bar{q}(s) - \lambda \bar{r}(s)\right) \alpha^1(s, \lambda) = 0, \quad -\frac{1}{2} < s < 1,\\
&\alpha^1(-1, \lambda) = 1, \quad \bar{k}(s) \frac{d\alpha^1}{ds}\bigg|_{s=-1} = 0,
\end{aligned}$$

$$\begin{aligned}
&\frac{d}{ds}\left[\bar{k}(s) \frac{d\alpha^j(s, \lambda)}{ds}\right] - h^2 \left(\bar{q}(s) - \lambda \bar{r}(s)\right) \alpha^j(s, \lambda) = 0, \quad -1 < s < 1,\\
&\alpha^j(-1, \lambda) = 0, \quad \bar{k}(s) \frac{d\alpha^j}{ds}\bigg|_{s=-1} = 1, \quad j = 2, 3, ..., N,
\end{aligned}$$

$$\frac{d}{ds}\left[\bar{k}(s)\frac{d\beta^j(s,\lambda)}{ds}\right] - h^2\left(\bar{q}(s) - \lambda\bar{r}(s)\right)\beta^j(s,\lambda) = 0, \quad -1 < s < 1,$$

$$\beta^j(1,\lambda) = 0, \quad \bar{k}(s)\frac{d\beta^j}{ds}\bigg|_{s=1} = -1, \quad j = 1, 2, ..., N-1,$$

$$\frac{d}{ds}\left[\bar{k}(s)\frac{d\beta^N(s,\lambda)}{ds}\right] - h^2\left(\bar{q}(s) - \lambda\bar{r}(s)\right)\beta^N(s,\lambda) = 0, \quad -1 < s < \frac{1}{2},$$

$$\beta^N(1,\lambda) = 1, \quad \bar{k}(s)\frac{d\beta^N}{ds}\bigg|_{s=1} = 0,$$

where

$$\bar{k}(s) = k(x_j + sh), \quad \bar{q}(s) = q(x_j + sh) \quad \bar{r}(s) = r(x_j + sh).$$

Then the coefficients a, d, ρ are calculated by the formulas

$$a(x_j) = \left[\alpha^j(0,\lambda)\right]^{-1}, \quad j = 2, 3, ..., N,$$

$$d(x_1) = \frac{1}{\alpha^1(0,\lambda)}\int_{-\frac{1}{2}}^{0}\alpha^1(s,\lambda)\bar{q}(s)ds + \frac{1}{\beta^1(0,\lambda)}\int_0^1\beta^1(s,\lambda)\bar{q}(s)ds,$$

$$d(x_j) = \frac{1}{\alpha^j(0,\lambda)}\int_{-1}^{0}\alpha^j(s,\lambda)\bar{q}(s)ds + \frac{1}{\beta^j(0,\lambda)}\int_0^1\beta^j(s,\lambda)\bar{q}(s)ds,$$
$$j = 2, 3, ..., N-1,$$

$$d(x_N) = \frac{1}{\alpha^N(0,\lambda)}\int_{-1}^{0}\alpha^N(s,\lambda)\bar{q}(s)ds + \frac{1}{\beta^N(0,\lambda)}\int_0^{\frac{1}{2}}\beta^N(s,\lambda)\bar{q}(s)ds,$$

$$\rho(x_1) = \frac{1}{\alpha^1(0,\lambda)}\int_{-\frac{1}{2}}^{0}\alpha^1(s,\lambda)\bar{r}(s)ds + \frac{1}{\beta^1(0,\lambda)}\int_0^1\beta^1(s,\lambda)\bar{r}(s)ds,$$

$$\rho(x_j) = \frac{1}{\alpha^j(0,\lambda)}\int_{-1}^{0}\alpha^j(s,\lambda)\bar{r}(s)ds + \frac{1}{\beta^j(0,\lambda)}\int_0^1\beta^j(s,\lambda)\bar{r}(s)ds,$$
$$j = 2, 3, ..., N-1,$$

$$\rho(x_N) = \frac{1}{\alpha^N(0,\lambda)}\int_{-1}^{0}\alpha^N(s,\lambda)\bar{r}(s)ds + \frac{1}{\beta^N(0,\lambda)}\int_0^{\frac{1}{2}}\beta^N(s,\lambda)\bar{r}(s)ds.$$

In general, the coefficients of this exact scheme cannot be directly expressed using quadratures, so we will use truncated schemes of high order of accuracy. These schemes are derived from the exact one, and the coefficients are represented as multiple integrals of $k(x), q(x), r(x)$.

The approximate values of the functions $\alpha^j(s,\lambda)$ and $\beta^j(s,\lambda)$ will be sought in the form

$$\alpha^{(m)j}(s,\lambda) = \alpha_0^j + \sum_{k=1}^{m} \alpha_k^j(s,\lambda) h_j^{2k}, \quad \beta^{(m)j}(s,\lambda) = \beta_0^j + \sum_{k=1}^{m} \beta_k^j(s,\lambda) h_{j+1}^{2k},$$

the coefficients $\alpha_k^j(s,\lambda), \beta_k^j(s,\lambda)$ are calculated by recurrence formulas similar to those in the regular case and require the calculation of multiple integrals, for example, by the Monte-Carlo method. If we calculate the coefficients a, d, ρ by replacing $\alpha^j(s,\lambda), \beta^j(s,\lambda)$ with polynomials $\alpha^{(m)j}(s,\lambda)$, $\beta^{(m)j}(s,\lambda)$ in these formulas, we obtain a truncated three-point difference scheme of rank m

$$\begin{aligned} &\left(a^{(m)} y_{\bar{x}}^{(m)}\right)_{x,j} - d_j^{(m)} y_j^{(m)} + \lambda^{(m)} \rho_j^{(m)} y_j^{(m)} = 0, \quad j = 1, 2, ..., N, \\ &y_0 \neq \infty, \quad y_{N+1} \neq \infty. \end{aligned} \tag{1.27}$$

Theorem 1.2. *(see [60]) The truncated difference scheme (1.27) of rank m has $m+1$ th order of accuracy at $h \to 0$ in the class $Q[-1,1]$ of piecewise continuous functions $k(x), q(x), r(x)$, i.e.*

$$\left|\lambda_n - \lambda_n^{(m)}\right| \leq M_1 h^{m+1},$$

$$\left\|y_n - y_n^{(m)}\right\|_{C(\omega_h)} \leq M_2 h^{m+1},$$

where M_1, M_2 are constants independent of h, $\|y(x)\|_{C(\omega_h)} = \max\limits_{1 \leq j \leq N} |y_j|$.

Thus, the implementation algorithms of truncated three-point difference schemes of high order of accuracy for eigenvalue problems proposed in [68, 60] contain an unconstructive procedure for calculating multiple integrals of the coefficients of differential equation, albeit over a small integration domain.

The topic of this dissertation is related to the ideas proposed in [76, 75], where it was first shown that for linear inhomogeneous ordinary differential equations of the second order, the coefficients of the exact three-point difference scheme and the right-hand side at any grid node can be expressed through solutions of four auxiliary Cauchy problems, each of which can be solved approximately in one step by a one-step numerical method (Taylor series expansion or Runge-Kutta method). This approach has been further extended to the case of nonlinear boundary value problems (see, e.g., [63, 54, 48, 50, 51, 49, 28, 53, 43, 44, 52]) and has found wide application in practical calculations. A logical extension is to apply these ideas to the Sturm-Liouville problem.

Chapter 2

High-Order Difference Schemes for the Sturm-Liouville Problem

In this chapter, we develop and justify a new algorithmic realization of an exact three-point difference scheme on a non-uniform grid for the Sturm-Liouville problem. We demonstrate that the coefficients of this scheme at any grid node can be expressed in terms of the solutions of two auxiliary Cauchy problems for the system of three linear ordinary differential equations of the first order linear ordinary differential equations, each of which can be numerically solved in a single step using any one-step method. Furthermore, we construct and substantiate a three-point difference scheme of arbitrary accuracy order and provide an accuracy estimate for this scheme. To compute the solution of this difference scheme with accuracy order $\bar{m} = 2[(m+1)/2]$ (m is a given natural number, $[\cdot]$ is the integer part of the argument in brackets), we develop a Newton's iterative method. At each iteration, this method requires solving a system of linear algebraic equations with a matrix that is nearly tridiagonal. Moreover, to achieve the same accuracy, the matrix of the system of linear algebraic equations will be smaller than in the case of difference schemes of lower order of accuracy. Additionally, the proposed difference scheme is constructed on an arbitrary non-uniform grid and enables more precise computation of eigenvalues and eigenfunctions, including those of higher indices. Numerical examples confirm these conclusions and the effectiveness of our approach.

2.1 The Sturm-Liouville Differential Problem and Its Properties

The Sturm-Liouville problem lies in finding the values of parameter λ (eigenvalues), for which the boundary value problem

$$\begin{aligned} &\frac{d}{dx}\left[k(x)\frac{du}{dx}\right] - q(x)u(x) = -\lambda r(x)u(x), \quad x \in (0,1), \\ &u(0) = u(1) = 0 \end{aligned} \tag{2.1}$$

has nontrivial solutions $u(x)$ (eigenfunctions). Here $k(x), q(x), r(x) \in Q[0,1]$ are piecewise continuous functions that satisfy the conditions

$$0 < C_1 \le k(x) \le C_2, \quad 0 \le q(x) \le C_3, \quad 0 < C_4 \le r(x) \le C_5, \tag{2.2}$$

where C_1, C_2, C_3, C_4, C_5 are constants.

If $k\,(x)$ has a discontinuity of the first kind at the point $x = \xi$ $(0 < \xi < 1)$, then the continuity conditions (continuity of $u\,(x)$ and $k\,(x)\,\frac{du}{dx}$) must be satisfied at this point:

$$u(\xi - 0) = u(\xi + 0), \quad k\,(x)\,\frac{du}{dx}\bigg|_{x=\xi-0} = k\,(x)\,\frac{du}{dx}\bigg|_{x=\xi+0}. \tag{2.3}$$

Since the equation (2.1) defines the function only up to an arbitrary constant multiplier, we will assume that the eigenfunction satisfies the normalization condition

$$\int_0^1 r(x)u^2(x)dx = 1$$

and

$$\frac{du(0)}{dx} > 0.$$

We multiply the differential equation (2.1) by $u(x)$ and integrate from 0 to 1. Then, taking into account the boundary conditions and Green's formula, we obtain

$$\int_0^1 k(x)\left(\frac{du}{dx}\right)^2 dx + \int_0^1 q(x)u^2(x)dx = \lambda \int_0^1 r(x)u^2(x)dx.$$

From here we find

$$\lambda = R(u) = \frac{\displaystyle\int_0^1 k(x)\left(\frac{du}{dx}\right)^2 dx + \int_0^1 q(x)u^2(x)dx}{\displaystyle\int_0^1 r(x)u^2(x)dx}.$$

The problem (2.1) as known (see [20, 18]) is equivalent to the variational problem:
1) In the class of piecewise-smooth functions $u(x)$ that satisfy the conditions

$$\int_0^1 r\left(x\right) u^2\left(x\right) dx = 1, \quad u(0) = u(1) = 0 \tag{2.4}$$

is the minimum of the functional $R(u)$ to find. This minimum defines the first eigenvalue

$$\lambda_1 = \min_u R(u) = R(u_1).$$

2) Other eigenvalues $\lambda_n, n > 1$ are found as the minimum of functional $R(u)$ in class of piecewise-smooth functions $u(x)$, that satisfy additional conditions

$$\int_0^1 r\left(x\right) u^2\left(x\right) dx = 1, \quad \int_0^1 r(x)u(x)u_m(x)dx = 0 \quad \text{for} \quad m < n,$$
$$u(0) = u(1) = 0,$$

where $u_m\left(x\right)$ is the eigenfunction of number m. This minimum defines the nth eigenvalue

$$\lambda_n = \min_u R(u) = R(u_n)$$

and is achieved at the nth eigenfunction $u_n\left(x\right)$.

The Sturm-Liouville problem (2.1) for piecewise continuous coefficients $k, q, r \in Q^{(0)}[0, 1]$ has a countable set of eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \leq ...$, which correspond to eigenfunctions $u_1\left(x\right), u_2\left(x\right), ..., u_n\left(x\right), ...$.

Let us specify some known [20] properties of the eigenfunctions and eigenvalues.

1. Each eigenvalue corresponds to only one eigenfunction.
2. The eigenfunctions $\{u_n\left(x\right)\}$ form an orthogonal and with respect to the weight function $r(x)$ normalized system:

$$\int_0^1 r(x)u_n(x)u_m(x)dx = 0 \quad \text{for} \quad m \neq n, \qquad \int_0^1 r\left(x\right) u_n^2\left(x\right) dx = 1.$$

3. All eigenvalues are positive: $\lambda_n > 0, n = 1, 2, ...$
4. For the eigenvalues holds $\lambda_n \to \infty$ as $n \to \infty$, or more precisely

$$C_6 n^2 \leq \lambda_n \leq C_7 n^2, \tag{2.5}$$

where C_6 and C_7 are positive constants, independent of number n and dependent only on $C_j, j = 1, ..., 5$.

5. Eigenfunctions and their first derivatives are bounded, more precisely,
$$|u_n(x)| < C_8, \quad |u_n'(x)| \le C_9'\sqrt{\lambda_n} \le C_9 n, \tag{2.6}$$
where C_8 and C_9 are positive constants dependent only on $C_j, j = 1, ..., 5$.

For the case $k, q, r \in C^{(2)}[0,1]$ the proof of (2.6) is provided in [20]. This proof, with some modifications, can be applied to the case of $k, r \in Q^{(1)}[0,1], q \in Q[0,1]$ (see [72, 85]).

To prove these estimates, we introduce a new variable
$$t = \int_0^x r(x)\,dx.$$
Then the equation (2.1) will take the form
$$\frac{d}{dt}\left[\bar{k}(t)\frac{d\bar{u}}{dt}\right] - \bar{q}(t)\,\bar{u} + \lambda\bar{u} = 0, \quad 0 < t < l, \tag{2.7}$$
where
$$\bar{k}(t) = k(x)\,r(x)\,,\ \bar{q}(t) = q(x)/r(x),\ \bar{u}(t) = u(x)\,,\ l = \int_0^1 r(x)\,dx.$$

We multiply equation (2.7) by $\bar{u}'(t)$ and integrate from 0 to t. Taking into account that
$$(\bar{k}\bar{u}')'\bar{u}' = \frac{1}{2\bar{k}}\left[(\bar{k}\bar{u}')^2\right]', \quad \bar{u}\bar{u}' = 0,5\left(\bar{u}^2\right)',$$
we get
$$\int_0^t \frac{1}{2\bar{k}(\tau)}\left[(\bar{k}(\tau)\bar{u}'(\tau))^2\right]' d\tau - \int_0^t \bar{q}(\tau)\bar{u}(\tau)\bar{u}'(\tau)d\tau + \frac{\lambda}{2}\int_0^t \left(\bar{u}^2(\tau)\right)' d\tau = 0.$$
After integration by parts we have
$$\begin{aligned}\bar{k}(0)\left[\bar{u}'(0)\right]^2 + \lambda\bar{u}^2(0) &= \bar{k}(t)\left[\bar{u}'(t)\right]^2 + \lambda\bar{u}^2(t)\\ &- 2\int_0^t \bar{q}\bar{u}(\tau)\bar{u}'(\tau)d\tau - \int_0^t \bar{k}(\tau)(\bar{u}'(\tau))^2\frac{\bar{k}'(\tau)}{\bar{k}(\tau)}d\tau.\end{aligned} \tag{2.8}$$
Integrate again with respect to t from 0 to l, then we have
$$\begin{aligned}l\bar{k}(0)\left[\bar{u}'(0)\right]^2 + l\lambda\bar{u}^2(0) &= \int_0^l \bar{k}(t)\left[\bar{u}'(t)\right]^2 dt + \lambda\int_0^l \bar{u}^2(t)\,dt\\ &- 2\int_0^l dt\int_0^t \bar{q}\bar{u}(\tau)\bar{u}'(\tau)d\tau - \int_0^l dt\int_0^t \bar{k}(\tau)(\bar{u}'(\tau))^2\frac{\bar{k}'(\tau)}{\bar{k}(\tau)}d\tau.\end{aligned}$$

Taking into account that

$$
\begin{gathered}
\int_0^l \bar{u}^2(t)dt = 1, \qquad \int_0^l \bar{k}(t)\left[\bar{u}'(t)\right]^2 dt \le \lambda, \\
2\left|\int_0^l dt \int_0^t \bar{q}\bar{u}\bar{u}' d\tau\right| \le 2l\frac{C_3}{C_4}\left(\int_0^l \bar{u}^2 dt \int_0^l \left[\bar{u}'(t)\right]^2 dt\right)^{\frac{1}{2}} \\
\le 2l\frac{C_3}{C_4}\left(\int_0^l \frac{\bar{k}(t)}{C_1 C_4}\left[\bar{u}'(t)\right]^2 dt\right)^{\frac{1}{2}} \le \frac{2lC_3}{C_1^{1/2}C_4^{3/2}}\sqrt{\lambda} \le C_{10}\sqrt{\lambda}, \\
\left|\int_0^l dt \int_0^t \bar{k}(\tau)(\bar{u}'(\tau))^2 \frac{\bar{k}'(\tau)}{\bar{k}(\tau)} d\tau\right| \le l\left\|\frac{\bar{k}'}{\bar{k}}\right\|_{C[0,l]}\lambda \le C_{11}\lambda,
\end{gathered}
\tag{2.9}
$$

we obtain

$$
\bar{k}(0)\left[\bar{u}'(0)\right]^2 + \lambda\bar{u}^2(0) \le (2 + C_{11})\lambda + C_{10}\sqrt{\lambda}. \tag{2.10}
$$

Now going back to (2.8) and taking into account inequalities (2.9) and (2.10), we get

$$
\bar{k}(t)\left[\bar{u}'(t)\right]^2 + \lambda\bar{u}^2(t) \le \tilde{C}_{11}\lambda + \tilde{C}_{10}\sqrt{\lambda}.
$$

Therefore,

$$
\bar{u}^2(t) \le \tilde{C}_{11} + \frac{\tilde{C}_{10}}{\sqrt{\lambda}} = C_{12}^2, \quad |\bar{u}(t)| = |u(x)| \le C_{12},
$$

$$
|\bar{u}'(t)| = \frac{1}{r(x)}|u'(x)| \le \left(\frac{\tilde{C}_{11}\lambda + \tilde{C}_{10}\sqrt{\lambda}}{\bar{k}(t)}\right)^{1/2} \le C_{13}\sqrt{\lambda}, \quad |u'(x)| \le C_{14}\sqrt{\lambda}.
$$

From here and from inequality (2.5) the estimates (2.6) follow.

2.2 Exact Three-Point Difference Scheme

We introduce the non-uniform grid

$$
\hat{\omega}_h := \left\{x_j \in (0,1), j = 1, 2, ..., N-1, x_j - x_{j-1} =: h_j > 0, \sum_{j=1}^{N} h_j = 1\right\},
$$
$$
h := \max_{1 \le j \le N} h_j
$$

in such a way that the discontinuity points of the functions $k(x), q(x), r(x)$ coincide with the nodes of the grid $\hat{\omega}_h$. We denote the set of all discontinuity points by σ and assume that N is such that $\sigma \subset \hat{\omega}_h$. We will assume that at the points of discontinuity the solution of problem (2.1) satisfies the continuity conditions

$$u(x_j - 0) = u(x_j + 0), \quad k(x)\frac{du}{dx}\bigg|_{x=x_j-0} = k(x)\frac{du}{dx}\bigg|_{x=x_j+0}. \tag{2.11}$$

We define the pattern functions $v_\alpha^j(x,\lambda), \alpha = 1,2$ as solutions to the Cauchy problems

$$\frac{d}{dx}\left[k(x)\frac{dv_\alpha^j}{dx}\right] - q(x)v_\alpha^j(x,\lambda) + \lambda r(x)v_\alpha^j(x,\lambda) = 0, \quad x \in (x_{j-1}, x_{j+1}), \tag{2.12}$$

$$v_\alpha^j(x_{j+(-1)^\alpha}, \lambda) = 0, \quad k(x)\frac{dv_\alpha^j(x,\lambda)}{dx}\bigg|_{x=x_{j+(-1)^\alpha}} = (-1)^{\alpha+1}, \tag{2.13}$$

$$\alpha = 1,2, \quad j = 1,2,...,N-1,$$

for which we also require the fulfillment of conditions (2.11).

Lemma 2.1. *The functions* $v_\alpha^j(x,\lambda)$ *have the following properties:*

$$v_1^j(x_{j+1},\lambda) = v_2^j(x_{j-1},\lambda), \tag{2.14}$$

$$v_2^j(x_j,\lambda) = v_1^{j+1}(x_{j+1},\lambda), \tag{2.15}$$

$$\begin{aligned} v_1^j(x_{j+1},\lambda) &= v_1^j(x_j,\lambda) + v_2^j(x_j,\lambda) \\ &+ v_2^j(x_j,\lambda)\int_{x_{j-1}}^{x_j} v_1^j(\xi,\lambda)\left[q(\xi) - \lambda r(\xi)\right]d\xi \\ &+ v_1^j(x_j,\lambda)\int_{x_j}^{x_{j+1}} v_2^j(\xi,\lambda)\left[q(\xi) - \lambda r(\xi)\right]d\xi, \end{aligned} \tag{2.16}$$

$$\begin{aligned} &v_\alpha^j(x,\lambda) > 0, \ \alpha = 1,2 \textit{ for all } x \in (x_{j-1},x_{j+1}), \ j = 1,2,...,N, \\ &v_\alpha^j(x,\lambda), \ \alpha = 1,2 \textit{ are linearly independent.} \end{aligned} \tag{2.17}$$

Proof. The proof is analogous to the proof of corresponding properties from [73, p. 208]. We first prove properties (2.14) — (2.16).

Let us introduce the notation

$$Lv_\alpha^j(x,\lambda) = \frac{d}{dx}\left[k(x)\frac{dv_\alpha^j}{dx}\right] - \left(q(x) - \lambda r(x)\right)v_\alpha^j(x,\lambda).$$

Then taking into account (2.12), (2.13), we get

$$\begin{aligned} 0 &= \int_{x_{j-1}}^{x_{j+1}} \left(v_1^j L v_2^j - v_2^j L v_1^j\right)dx = \left(v_1^j k(x)\frac{dv_2^j}{dx} - v_2^j k(x)\frac{dv_1^j}{dx}\right)\Bigg|_{x_{j-1}}^{x_{j+1}} \\ &= -v_1^j(x_{j+1},\lambda) + v_2^j(x_{j-1},\lambda). \end{aligned}$$

Since $v_1^{j+1}(x,\lambda)$ satisfies the conditions

$$Lv_1^{j+1}(x,\lambda)=0,\quad x_j<x<x_{j+2},$$
$$v_1^{j+1}(x_j,\lambda)=0,\quad k(x)\frac{dv_1^{j+1}}{dx}\bigg|_{x=x_j}=1,$$

we obtain

$$0=\int_{x_j}^{x_{j+1}}\left(v_1^{j+1}Lv_2^j-v_2^jLv_1^{j+1}\right)dx$$
$$=\left(v_1^{j+1}k(x)\frac{dv_2^j}{dx}-v_2^jk(x)\frac{dv_1^{j+1}}{dx}\right)\bigg|_{x_j}^{x_{j+1}}=-v_1^{j+1}(x_{j+1},\lambda)+v_2^j(x_j,\lambda).$$

We now write the Green's formula on the interval $[x_{j-1},x_j]$

$$0=\int_{x_{j-1}}^{x_j}\left(v_1^jLv_2^j-v_2^jLv_1^j\right)dx=\left(v_1^jk(x)\frac{dv_2^j}{dx}-v_2^jk(x)\frac{dv_1^j}{dx}\right)\bigg|_{x_{j-1}}^{x_j}$$
$$=k(x_j)\frac{dv_2^j(x_j,\lambda)}{dx}v_1^j(x_j,\lambda)-k(x_j)\frac{dv_1^j(x_j,\lambda)}{dx}v_2^j(x_j,\lambda)+v_2^j(x_{j-1},\lambda)$$

and substitute

$$k(x_j)\frac{dv_1^j(x_j,\lambda)}{dx}=1+\int_{x_{j-1}}^{x_j}\left(q(\xi)-\lambda r(\xi)\right)v_1^j(\xi,\lambda)d\xi,$$
$$k(x_j)\frac{dv_2^j(x_j,\lambda)}{dx}=-1-\int_{x_j}^{x_{j+1}}\left(q(\xi)-\lambda r(\xi)\right)v_2^j(\xi,\lambda)d\xi.$$

Then we get the equality (2.16).

We prove that the functions $v_\alpha^j(x,\lambda),\alpha=1,2$ are linearly independent. Suppose the opposite, then the Wronskian $W\left[v_1^j(x,\lambda),v_2^j(x,\lambda)\right]$ is identically equal to zero on the segment $[x_{j-1},x_{j+1}]$. Calculating Wronskian at the points $x_{j+(-1)^\alpha},\alpha=1,2$ and taking into account that $v_2^j(x_{j-1},\lambda)=v_1^j(x_{j+1},\lambda)$, we get

$$W\left[v_1^j(x,\lambda),v_2^j(x,\lambda)\right]_{x=x_{j+(-1)^\alpha}}=-\frac{v_1^j(x_{j+1},\lambda)}{k(x_{j+(-1)^\alpha})}.$$

Hence follows that $v_1^j(x_{j-1},\lambda)=v_1^j(x_{j+1},\lambda)=0$, i.e., $v_1^j(x,\lambda)$ is the solution of the boundary value problem

$$\frac{d}{dx}\left[k(x)\frac{dv_1^j}{dx}\right]-q(x)v_1^j(x,\lambda)+\lambda r(x)v_1^j(x,\lambda)=0,\quad x\in(x_{j-1},x_{j+1}),$$
$$v_1^j(x_{j-1},\lambda)=v_1^j(x_{j+1},\lambda)=0. \tag{2.18}$$

We now show that at sufficiently small $h < h_0$ and at $\lambda = \lambda_n, 1 \leq n \leq k, k << N$, the problem (2.18) has only a trivial solution. To do this, it is enough to show that at $h < h_0$, the inequality

$$-[q(x) - \lambda r(x)] \leq \lambda r(x) < \underline{\mu}_1, \quad \forall x \in [x_{j-1}, x_{j+1}], \tag{2.19}$$

is fulfilled, where $\underline{\mu}_1$ is the lower bound estimate of the smallest eigenvalue of the problem

$$\frac{d}{dx}\left[k(x)\frac{dv}{dx}\right] + \mu v(x) = 0, \quad x \in (x_{j-1}, x_{j+1}), \quad v(x_{j-1}) = v(x_{j+1}) = 0.$$

From an equivalent variational problem (see [20]) under the condition

$$\|v\|^2 = \int_{x_{j-1}}^{x_{j+1}} v^2(x)dx = 1$$

follows

$$\begin{aligned}\mu_1 &= \min \int_{x_{j-1}}^{x_{j+1}} k(\xi)\left[v'(\xi)\right]^2 d\xi > C_1 \min \int_{x_{j-1}}^{x_{j+1}} \left[v'(\xi)\right]^2 d\xi \\ &= \frac{C_1\pi^2}{4\hbar_j^2} \geq \frac{C_1\pi^2}{4h^2} = \underline{\mu}_1.\end{aligned}$$

Thus, there exists h_0 such that for all $h < h_0$ the inequality

$$h^2\lambda r(x) < \frac{C_1\pi^2}{4} \quad \forall x \in (x_{j-1}, x_{j+1}), \quad j = 1, 2, ..., N-1,$$

holds. It follows that at $h < h_0 = \frac{\pi}{2}\sqrt{\frac{C_1}{\lambda C_5}}$, $v_1^j(x,\lambda) \equiv 0, x \in [x_{j-1}, x_{j+1}]$, and this contradicts the condition $k(x)\frac{dv_1^j(x,\lambda)}{dx}\Big|_{x=x_{j-1}} = 1$. That is, $v_1^j(x_{j+1},\lambda) \neq 0$. Thus, $v_1^j(x,\lambda)$ and $v_2^j(x,\lambda)$ are linearly independent functions.

By reasoning analogously, it is easy to show that $v_1^j(x,\lambda) \neq 0$ at any point of the interval $(x_{j-1}, x_{j+1}]$, i.e., the function is of constant sign on this interval. Moreover, we show that $v_1^j(x,\lambda) > 0,\ x \in (x_{j-1}, x_{j+1}]$. To do this, it is sufficient to prove that this function is positive on the interval $(x_{j-1}, x_{j-1} + \delta)$ for an arbitrarily small $\delta > 0$. From (2.12), it follows that

$$k(x)\frac{dv_1^j(x,\lambda)}{dx} = 1 + \int_{x_{j-1}}^{x} \left[q(\xi) - \lambda r(\xi)\right] v_1^j(\xi,\lambda)d\xi, \tag{2.20}$$

$$\begin{aligned} v_1^j(x,\lambda) = &\int_{x_{j-1}}^{x} \frac{dt}{k(t)} \\ &+ \int_{x_{j-1}}^{x} \frac{1}{k(t)} \int_{x_{j-1}}^{t} \left[q(\xi) - \lambda r(\xi)\right] v_1^j(\xi,\lambda) d\xi dt. \end{aligned} \tag{2.21}$$

The relation (2.21) under the assumptions (2.2) leads to inequality

$$|v_1^j(x,\lambda)| \le (x - x_{j-1}) \left[\frac{1}{C_1} + \frac{C_3 + \lambda C_5}{C_1} \int_{x_{j-1}}^{x} |v_1^j(t,\lambda)| dt\right]$$

at any bounded λ. Let us make a substitution

$$\bar{v}_1^j(x,\lambda) = \frac{|v_1^j(x,\lambda)|}{x - x_{j-1}},$$

then

$$\bar{v}_1^j(x,\lambda) \le \frac{1}{C_1} + \frac{C_3 + \lambda C_5}{C_1} \int_{x_{j-1}}^{x} (t - x_{j-1}) \bar{v}_1^j(t,\lambda) dt.$$

Applying Gronwall's inequality (see, e.g., [34, p. 37]), we get

$$\bar{v}_1^j(x,\lambda) \le \frac{1}{C_1} e^{\frac{(C_3+\lambda C_5)(x-x_{j-1})^2}{2C_1}}$$

or

$$|v_1^j(x,\lambda)| \le \frac{x - x_{j-1}}{C_1} e^{\frac{(C_3+\lambda C_5)(x-x_{j-1})^2}{2C_1}} \quad \forall x \in [x_{j-1}, x_{j+1}]. \tag{2.22}$$

From the equation (2.21) and the inequality (2.22), an estimate follows

$$\begin{aligned} v_1^j(x,\lambda) &\ge \int_{x_{j-1}}^{x} \frac{dt}{k(t)} - \int_{x_{j-1}}^{x} \frac{1}{k(t)} \int_{x_{j-1}}^{t} |\lambda r(\xi) - q(\xi)| |v_1^j(\xi,\lambda)| d\xi dt \\ &\ge \int_{x_{j-1}}^{x} \frac{dt}{k(t)} \left[1 - (C_3 + \lambda C_5) \int_{x_{j-1}}^{x} |v_1^j(\xi,\lambda)| d\xi\right] \\ &\ge \int_{x_{j-1}}^{x} \frac{dt}{k(t)} \left[1 - \frac{C_3 + \lambda C_5}{C_1} \int_{x_{j-1}}^{x} (\xi - x_{j-1}) e^{\frac{(C_3+\lambda C_5)(\xi-x_{j-1})^2}{2C_1}} d\xi\right] \\ &\ge \frac{1}{C_2}\left[2 - e^{\frac{(C_3+\lambda C_5)(x-x_{j-1})^2}{2C_1}}\right]. \end{aligned}$$

So, $v_1^j(x,\lambda) > 0$ for x sufficiently close to x_{j-1}, and since $v_1^j(x,\lambda)$ is a sign-constant function, it is positive over the entire interval (x_{j-1}, x_{j+1}). The proof for $v_2^j(x,\lambda) > 0$ is analogous. □

Lemma 2.2. *Let $k(x) \in Q^1[0,1], q(x), r(x) \in Q[0,1]$, then at*

$$h = \max_{1 \le j \le N} h_j \le h_0 = \frac{1}{2}\sqrt{\frac{C_1}{C_3 + \lambda C_5}} \tag{2.23}$$

the following statements are true:

1. *For all $j = 1, 2, ..., N-1$ the pattern functions $v_\alpha^j(x,\lambda), \alpha = 1,2$ have the properties: $v_1^j(x,\lambda)$ increases monotonically by $(x_{j-1}, x_{j+1}]$, and $v_2^j(x,\lambda)$ decreases monotonically by $[x_{j-1}, x_{j+1})$;*
2. *The estimates hold:*

$$\frac{2}{3C_2} \le \frac{v_\alpha^j(x,\lambda)}{|x - x_{j+(-1)^\alpha}|} \le \frac{2}{C_1}, \quad \alpha = 1,2, \quad j = 1,2,...,N-1. \tag{2.24}$$

Proof. We carry out the proof only for the first pattern function $v_1^j(x,\lambda)$ (for $v_2^j(x,\lambda)$ it is similar). From the estimate (2.22) and equality (2.20), inequality follows

$$\begin{aligned} k(x)\frac{dv_1^j(x,\lambda)}{dx} &\ge 1 - \int_{x_{j-1}}^{x} |\lambda r(\xi) - q(\xi)| v_1^j(\xi,\lambda) d\xi \ge \\ &\ge 1 - \frac{C_3 + \lambda C_5}{C_1}\int_{x_{j-1}}^{x} (\xi - x_{j-1}) e^{\frac{(C_3+\lambda C_5)(\xi - x_{j-1})^2}{2C_1}} d\xi \\ &\ge 2 - e^{\frac{(C_3+\lambda C_5)(x - x_{j-1})^2}{2C_1}} \ge 2 - e^{\frac{2(C_3+\lambda C_5)h^2}{C_1}}, \end{aligned}$$

which, taking into account the fact that the function $g(t) = 2 - e^t$ monotonically decreases and $g(1/2) > 0$, proves under condition (2.23) the statement 1.

Going back to (2.21), with the help of the proven statement 1. we get

$$\begin{aligned} v_1^j(x,\lambda) &\le \frac{x - x_{j-1}}{C_1} + \frac{C_3 + \lambda C_5}{C_1} v_1^j(x,\lambda)\frac{(x - x_{j-1})^2}{2}, \\ v_1^j(x,\lambda) &\ge \frac{x - x_{j-1}}{C_2} - \frac{C_3 + \lambda C_5}{C_1} v_1^j(x,\lambda)\frac{(x - x_{j-1})^2}{2}. \end{aligned}$$

Hence, it follows

$$\begin{aligned} \frac{v_1^j(x,\lambda)}{x - x_{j-1}}\left(1 - \frac{2(C_3 + \lambda C_5)h^2}{C_1}\right) &\le \frac{1}{C_1}, \\ \frac{v_1^j(x,\lambda)}{x - x_{j-1}}\left(1 + \frac{2(C_3 + \lambda C_5)h^2}{C_1}\right) &\ge \frac{1}{C_2}, \end{aligned}$$

which, together with (2.23) proves (2.24). □

Lemma 2.3. *Let the conditions of lemma 2.2 be fulfilled. Then, for the problem (2.1) there exists the ETDS of the form*

$$\Lambda y + \lambda \rho y \triangleq (a y_{\bar{x}})_{\hat{x}} - dy + \lambda \rho y = 0, \quad x \in \hat{\omega}_h, \quad y_0 = y_N = 0, \tag{2.25}$$

the solution of which $y(x)$ coincides with the solution of the original problem $u(x)$ in the nodes of the grid $\hat{\omega}_h$ up to a constant multiplier, where

$$y_{\bar{x},j} := \frac{y_j - y_{j-1}}{h_j}, \quad y_{\hat{x},j} := \frac{y_{j+1} - y_j}{\hbar_j}, \quad \hbar_j := \frac{h_j + h_{j+1}}{2},$$

$$\begin{aligned} a_j &= a(x_j,\lambda) = \left[\frac{1}{h_j} v_1^j(x_j,\lambda)\right]^{-1}, \\ d_j &= d(x_j,\lambda) = \hat{T}^{x_j}(q,\lambda), \quad \rho_j = \rho(x_j,\lambda) = \hat{T}^{x_j}(r,\lambda), \end{aligned} \tag{2.26}$$

$$\begin{aligned} \hat{T}^{x_j}(w(\xi),\lambda) = &\frac{1}{\hbar_j v_1^j(x_j,\lambda)} \int_{x_{j-1}}^{x_j} v_1^j(\xi,\lambda) w(\xi) d\xi \\ &+ \frac{1}{\hbar_j v_2^j(x_j,\lambda)} \int_{x_j}^{x_{j+1}} v_2^j(\xi,\lambda) w(\xi) d\xi, \end{aligned}$$

$$0 < C_1' \le a_j \le C_2', \quad C_1' = \frac{C_1}{2}, \quad C_2' = \frac{3C_2}{2}, \tag{2.27}$$

$$0 \le d_j \le C_3', \quad C_3' = 2C_3, \tag{2.28}$$

$$0 < C_4' \le \rho_j \le C_5', \quad C_4' = \frac{C_1 C_4}{3C_2} \quad C_5' = 2C_5. \tag{2.29}$$

Proof. Let's consider the problem

$$\begin{aligned} &\frac{d}{dx}\left[k(x)\frac{du}{dx}\right] - q(x)u(x) = -\lambda r(x) u(x), \quad x \in (x_{j-1}, x_{j+1}), \\ &u(x_{j-1}) = u_{j-1}, \quad u(x_{j+1}) = u_{j+1}, \quad j = 1, 2, ..., N-1, \end{aligned} \tag{2.30}$$

for which the Green's function has the form

$$G^j(x,\xi) = \frac{1}{v_1^j(x_{j+1},\lambda)} \begin{cases} v_1^j(x,\lambda) v_2^j(\xi,\lambda), & x_{j-1} \le x \le \xi, \\ v_1^j(\xi,\lambda) v_2^j(x,\lambda), & \xi \le x \le x_{j+1}. \end{cases}$$

We construct the exact three-point difference scheme. To do this, let's write the obvious corollary of (2.30). Then we obtain

$$\int\limits_{x_{j-1}}^{x_{j+1}} G^j(x,\xi) \frac{d}{d\xi}\left[k(\xi)\frac{du}{d\xi}\right] d\xi - \int\limits_{x_{j-1}}^{x_{j+1}} G^j(x,\xi) \left[q(\xi) - \lambda r(\xi)\right] u(\xi) d\xi = 0. \tag{2.31}$$

If we integrate by parts the integral on the left side of (2.31), then taking into account (2.12) we get

$$\begin{aligned}&\frac{v_1^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\left[u_{j+1}+k(x)\frac{dv_2^j(x,\lambda)}{dx}u(x)\right]\\&\quad+\frac{v_2^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\left[u_{j-1}-k(x)\frac{dv_1^j(x,\lambda)}{dx}u(x)\right]=0.\end{aligned}$$

Since from (2.12) follows

$$\begin{aligned}k(x)\frac{dv_\alpha^j(x,\lambda)}{dx}&=(-1)^{\alpha+1}\\&\quad+\int_{x_{j+(-1)^\alpha}}^{x}\left[q(\xi)-\lambda r(\xi)\right]v_\alpha^j(\xi,\lambda)d\xi,\quad \alpha=1,2,\end{aligned}\tag{2.32}$$

then

$$\begin{aligned}&\frac{v_1^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\left[u_{j+1}-\left(1+\int\limits_{x}^{x_{j+1}}\left(q(\xi)-\lambda r(\xi)\right)v_2^j(\xi,\lambda)d\xi\right)u(x)\right]+\frac{v_2^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\\&\quad\times\left[u_{j-1}-\left(1+\int\limits_{x_{j-1}}^{x}\left(q(\xi)-\lambda r(\xi)\right)v_1^j(\xi,\lambda)d\xi\right)u(x)\right]=0.\end{aligned}\tag{2.33}$$

Let us put in (2.33) $x=x_j$ and multiply the resulting equality by

$$\frac{v_1^j(x_{j+1},\lambda)}{\hbar_j v_1^j(x_j,\lambda)v_2^j(x_j,\lambda)}.$$

Using the properties of pattern functions $v_1^j(x_{j+1},\lambda)=v_2^j(x_{j-1},\lambda)$, $v_2^j(x_j,\lambda)=v_1^{j+1}(x_{j+1},\lambda)$, we get the exact three-point difference scheme (2.25).

The inequality (2.27) follows from (2.24). Let's prove the estimate (2.28). Since

$$d_j=\frac{1}{\hbar_j v_1^j(x_j,\lambda)}\int_{x_{j-1}}^{x_j}v_1^j(\xi,\lambda)q(\xi)d\xi+\frac{1}{\hbar_j v_2^j(x_j,\lambda)}\int_{x_j}^{x_{j+1}}v_2^j(\xi,\lambda)q(\xi)d\xi,$$

then, taking into account the positivity and monotonicity of the functions $v_1^j(x,\lambda)$ and $v_2^j(x,\lambda)$, we obtain

$$0\le d_j\le\frac{2C_3}{h_j+h_{j+1}}\left[\int_{x_{j-1}}^{x_j}\frac{v_1^j(\xi,\lambda)}{v_1^j(x_j,\lambda)}d\xi+\int_{x_j}^{x_{j+1}}\frac{v_2^j(\xi,\lambda)}{v_2^j(x_j,\lambda)}d\xi\right]\le 2C_3.$$

Similarly, the inequality $\rho_j \leq 2C_5$ can be proven. In addition, using estimates (2.24), we get

$$\begin{aligned}
\rho_j &= \frac{1}{\hbar_j v_1^j(x_j,\lambda)} \int_{x_{j-1}}^{x_j} v_1^j(\xi,\lambda) r(\xi) d\xi + \frac{1}{\hbar_j v_2^j(x_j,\lambda)} \int_{x_j}^{x_{j+1}} v_2^j(\xi,\lambda) r(\xi) d\xi \\
&\geq \frac{2C_4}{h_j + h_{j+1}} \left[\frac{1}{v_1^j(x_j,\lambda)} \int_{x_{j-1}}^{x_j} v_1^j(\xi,\lambda) d\xi + \frac{1}{v_2^j(x_j,\lambda)} \int_{x_j}^{x_{j+1}} v_2^j(\xi,\lambda) d\xi \right] \\
&\geq \frac{4C_4}{3C_2(h_j + h_{j+1})} \left[\frac{1}{v_1^j(x_j,\lambda)} \int_{x_{j-1}}^{x_j} (\xi - x_{j-1}) d\xi \right. \\
&\qquad \left. + \frac{1}{v_2^j(x_j,\lambda)} \int_{x_j}^{x_{j+1}} (x_{j+1} - \xi) d\xi \right] \\
&\geq \frac{2C_4}{3C_2(h_j + h_{j+1})} \left[\frac{h_j^2}{v_1^j(x_j,\lambda)} + \frac{h_{j+1}^2}{v_2^j(x_j,\lambda)} \right] \geq \frac{C_1 C_4}{3C_2} > 0.
\end{aligned}$$

□

Note that if the solution of the problem (2.1) is normalized by the condition

$$\int_0^1 r(x) u^2(x) dx = 1,$$

then, for the exact normalization on the grid we have

$$\sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} r(x) \left[\frac{v_1^j(x,\lambda)}{v_1^j(x_j,\lambda)} y_j + \frac{v_2^{j-1}(x,\lambda)}{v_2^{j-1}(x_{j-1},\lambda)} y_{j-1} \right]^2 dx = 1.$$

2.3 Coefficient Stability of the Exact Three-Point Difference Scheme

When calculating the coefficients of difference schemes, an error will be admitted. Therefore, it is natural to require the coefficient stability of difference schemes, i.e., the stability to perturbations of the coefficients. We say that a scheme is stable with respect to coefficients if a solution of the boundary value problem has slight variations under small perturbations of the scheme coefficients (see [82]). In the following, we prove the coefficient stability of the constructed difference schemes. The property of coefficient stability of a difference scheme allows us to prove the convergence of truncated three-point difference schemes.

We consider the difference problem (2.25) in the space H_h of the grid functions y such that $y(0) = y(1) = 0$, with scalar products and norms

$$(y, v) := \sum_{\xi \in \hat{\omega}_h} \hbar(\xi) y(\xi) v(\xi), \quad (y, v] := \sum_{\xi \in \hat{\omega}_h^+} h(\xi) y(\xi) v(\xi), \quad \hat{\omega}_h^+ := \hat{\omega}_h \cup x_N,$$

$$\|y\| := (y, y)^{1/2}, \quad \|y]| := (y, y]^{1/2}, \quad \|y\|_C := \max_{\xi \in \hat{\omega}_h} |y(\xi)|.$$

Let $\lambda^h = \lambda_n^h$ be the eigenvalue of this problem of the number n, and $y = y_n$ be the normalized eigenfunction. There are $N - 1$ real eigenvalues $0 < \lambda_1^h < \lambda_2^h < ...\lambda_{N-1}^h$, which correspond to eigenfunctions $y_1, y_2, ..., y_{N-1}$, orthonormalized with weight ρ, i.e. $(\rho y_n, y_m) = 0$ at $n \neq m$ and $(\rho y_n, y_n) = 1$.

Multiplying (2.25) scalarwise by y and taking into account the difference Green's formula (see [78, p. 26]), we find

$$\lambda^h = R_N(y) = \frac{(a, y_{\bar{x}}^2] + (d, y^2)}{(\rho, y^2)}.$$

It is easy to see that the difference problem (2.25) is equivalent to the following variational problem: in the class of grid functions H_h, which satisfy the conditions

$$(\rho, y^2) = 1, \quad y_0 = y_N = 0$$

is minimum of the functional $R_N(y)$ to find. Then, the value

$$\lambda_1^h = \min_y R_N(y) = R_N(y_1)$$

is the smallest eigenvalue, and $y_1(x)$ is the corresponding eigenfunction of problem (2.25) according to the maximum principle. The eigenvalues λ_n^h for $n > 1$ are found as the minimum of the functional $R_N(y)$, that is,

$$\lambda_n^h = \min_y R_N(y) = R_N(y_n)$$

in class of functions y, which satisfy the conditions

$$(\rho, y^2) = 1, \quad (\rho y, y_m) = 0, \quad m = 1, 2, ..., n - 1, \quad y_0 = y_N = 0.$$

Lemma 2.4. *For the eigenfunctions of the problem (2.25) – (2.29), the following estimates are fulfilled:*

$$\|y\|_C \leq M_1 \left(\lambda^h\right)^{1/4}, \quad ||y_{\bar{x}}]|_C \leq M_2 \left(\lambda^h\right)^{3/4}, \tag{2.34}$$

where the constants M_1, M_2 depend on constants $C_i', i = 1, 2, ..., 5$, which are included in (2.27) – (2.29).

Proof. Let x, x' be arbitrary grid nodes of $\hat{\omega}_h$. Consider identities

$$y^2(x) = \sum_{\xi=x_1}^{x} h(\xi)[y^2(\xi)]_{\bar{\xi}} = \sum_{\xi=x_1}^{x} h(\xi)[y(\xi) + y(\xi - h(\xi))]y_{\bar{\xi}}(\xi), \tag{2.35}$$

$$\begin{aligned}
(a(x)y_{\bar{x}}(x))^2 - (a(x')y_{\bar{x}}(x'))^2 &= \sum_{\xi=x'}^{x-h(x)} \hbar(\xi) \left[\left(a(\xi)y_{\bar{\xi}}(\xi)\right)^2\right]_{\hat{\xi}} \\
= \sum_{\xi=x'}^{x-h(x)} \hbar(\xi) \left[a(\xi)y_{\bar{\xi}}(\xi) + a(\xi + h(\xi))y_{\xi}(\xi)\right] \left(a(\xi)y_{\bar{\xi}}(\xi)\right)_{\hat{\xi}} \qquad (2.36) \\
= \sum_{\xi=x'}^{x-h(x)} \hbar(\xi) \left[a(\xi)y_{\bar{\xi}}(\xi) + a(\xi + h(\xi))y_{\xi}(\xi)\right] \left(d(\xi) - \lambda^h \rho(\xi)\right) y(\xi).
\end{aligned}$$

Applying to the right-hand side of (2.35) the Cauchy-Bunyakovsky-Schwarz inequality and considering that

$$(\rho, y^2) = 1, \quad (a, y_{\bar{x}}^2] \leq \lambda^h, \tag{2.37}$$

we get

$$\begin{aligned}
y^2(x) &\leq \frac{2}{\sqrt{C_1' C_4'}} \left(\sum_{\xi=x_1}^{x} h(\xi)a(\xi)y_{\bar{\xi}}^2\right)^{1/2} \left(\sum_{\xi=x_1}^{x} \hbar(\xi)\rho(\xi)y^2(\xi)\right)^{1/2} \\
&\leq \frac{2}{\sqrt{C_1' C_4'}} (a, y_{\bar{x}}^2]^{1/2} (\rho, y^2)^{1/2} \leq \frac{2}{\sqrt{C_1' C_4'}} (\lambda^h)^{1/2}.
\end{aligned}$$

This leads to the first estimate (2.34).

It follows from (2.37) that there exists a node $x' \in \hat{\omega}_h$ at which $a(x')y_{\bar{x}}^2(x') \leq \lambda^h$ holds and, therefore, $\left(a(x')y_{\bar{x}}(x')\right)^2 \leq C_2' \lambda^h$ follows. Using the Cauchy-Bunyakovsky-Schwarz inequality to transform the right part of the identity (2.36) and taking into account (2.34) and the inequality $\hbar_j + \hbar_{j-1} \leq C h_j$, we obtain

$$\begin{aligned}
y_{\bar x}^2(x) &\le \frac{C_2'}{(C_1')^2}\lambda^h + \frac{1}{(C_1')^2}\left(\sum_{\xi=x'}^{x}(\hbar(\xi)+\hbar(\xi-h(\xi)))\left(a(\xi)y_{\bar\xi}(\xi)\right)^2\right)^{1/2} \\
&\times\left[\left(\sum_{\xi=x'}^{x-h(x)}\hbar(\xi)(d(\xi)y(\xi))^2\right)^{1/2} + \lambda^h\left(\sum_{\xi=x'}^{x-h(x)}\hbar(\xi)(\rho(\xi)y(\xi))^2\right)^{1/2}\right] \\
&\le \frac{C_2'}{(C_1')^2}\lambda^h + \frac{C\sqrt{C_2'}}{(C_1')^2}(a, y_{\bar x}^2(x)]^{1/2}\left(\frac{C_3'}{\sqrt{C_4'}}+\lambda^h\sqrt{C_5'}\right)(\rho, y^2)^{1/2} \\
&\le \frac{C_2'}{(C_1')^2}\lambda^h + \frac{C\sqrt{C_2'}}{(C_1')^2}(\lambda^h)^{1/2}\left(\frac{C_3'}{\sqrt{C_4'}}+\lambda^h\sqrt{C_5'}\right) \\
&\le (\lambda^h)^{3/2}\left[\frac{C_2'}{(C_1')^2(\lambda^h)^{1/2}} + \frac{CC_3'\sqrt{C_2'}}{(C_1')^2\sqrt{C_4'}\lambda^h} + \frac{C\sqrt{C_2'C_5'}}{(C_1')^2}\right]. \qquad (2.38)
\end{aligned}$$

Since $\|y_{\bar x}]|^2 \ge 4\|y\|^2$ (see, e.g., [73, p. 111]), then for λ^h we get an estimate

$$\lambda^h \ge (a, y_{\bar x}^2] \ge C_1'(1, y_{\bar x}^2] \ge 4C_1'(1, y^2) \ge \frac{4C_1'(\rho, y^2)}{C_5'} = \frac{4C_1'}{C_5'}.$$

Then from (2.38) follows the second estimate (2.34). ☐

Note that for the case of a uniform grid and piecewise continuous coefficients the estimates (2.34) were proved for the first time in [72, p. 170].

We now consider the perturbed problem

$$\tilde\Lambda\tilde y + \tilde\lambda^h\tilde\rho\tilde y = 0, \quad x\in\hat\omega_h, \quad \tilde y(0) = \tilde y(1) = 0, \qquad (2.39)$$

where

$$\tilde\Lambda\tilde y = (\tilde a\tilde y_{\bar x})_{\hat x} - \tilde d\tilde y.$$

We introduce the function $z = y - \tilde y$. Then for z we have the boundary value problem

$$\Lambda z + \lambda^h\rho z = -\Psi(x), \quad x\in\hat\omega_h, \quad z(0) = z(1) = 0, \qquad (2.40)$$

where

$$\Psi(x) = \Lambda\tilde y + \lambda^h\rho\tilde y.$$

Using the equation (2.39), the function $\Psi(x)$ can be transformed to the form

$$\Psi(x) = \psi(x) + \left(\lambda^h - \tilde\lambda^h\right)\rho\tilde y,$$

where

$$\psi(x) = \eta_{\hat{x}} + \psi^*(x),$$
$$\eta = (a - \tilde{a})\, \tilde{y}_{\bar{x}}, \quad \psi^*(x) = -\left(d - \tilde{d}\right) \tilde{y} + \tilde{\lambda}^h (\rho - \tilde{\rho})\, \tilde{y}. \tag{2.41}$$

Since the parameter λ^h is eigenvalue for the difference operator of the problem (2.40), then the inhomogeneous equation (2.40) is solvable only if the eigenfunction $y(x)$ is orthogonal to the right-hand side of this equation, or, more precisely, the equality must hold:

$$(\Psi, y) = (\psi, y) + \left(\lambda^h - \tilde{\lambda}^h\right) (\rho\tilde{y}, y) = 0. \tag{2.42}$$

The eigenvalue λ^h corresponds to only one eigenfunction which is determined to an arbitrary multiplier C_0. Let us choose this multiplier such that the function $\bar{y} = C_0 y$ is orthogonal to the difference $\bar{z} = \bar{y} - \tilde{y}$:

$$(\rho\bar{y}, \bar{z}) = 0. \tag{2.43}$$

Hence, taking into account the normalization condition $(\rho y, y) = 1$ we get

$$(\rho\tilde{y}, y) = (\rho y, \bar{y} - \bar{z}) = (\rho y, \bar{y}) - (\rho y, \bar{z}) = C_0\, (\rho y, y) = C_0.$$

If $\tilde{y} \to y$ at $h \to 0$, then we can assume that $C_0 > 0$.

Further,

$$\begin{aligned}\left(\rho, \tilde{y}^2\right) &= \left(\rho, (\bar{y} - \bar{z})^2\right) = \left(\rho, \bar{y}^2\right) - 2\,(\rho, \bar{z}\bar{y}) + \left(\rho, \bar{z}^2\right) = \\ &= C_0^2\, (\rho y, y) + \left(\rho, \bar{z}^2\right) = C_0^2 - (\rho, \bar{z}\tilde{y})\,,\end{aligned}$$

so that

$$1 - C_0^2 = -\,(\rho, \bar{z}\tilde{y}) - \left[\left(\rho, \tilde{y}^2\right) - \left(\tilde{\rho}, \tilde{y}^2\right)\right]. \tag{2.44}$$

We use the equality (2.42) to find $\lambda^h - \tilde{\lambda}^h$:

$$\lambda^h - \tilde{\lambda}^h = -\frac{(\psi, y)}{(\rho\tilde{y}, y)} = -\frac{(\psi, \bar{y})}{C_0^2}. \tag{2.45}$$

Let us convert the right-hand side of the formula (2.45), taking into account (2.41) and the summation by parts formula (see, e.g., [78, p. 25]):

$$(\psi, \bar{y}) = -\,(\eta, \bar{y}_{\bar{x}}] + (\psi^*, \bar{y})\,.$$

From here, and from the estimates (2.34) for $\bar{y}$ and $\bar{y}_{\bar{x}}$, it follows that

$$\begin{aligned}|\lambda^h - \tilde{\lambda}^h| &\le \frac{|\,(\psi, \bar{y})\,|}{C_0^2} \le \frac{|\,(\eta, \bar{y}_{\bar{x}}]\,| + |\,(\psi^*, \bar{y})\,|}{C_0^2} \\ &\le \frac{\|\bar{y}_{\bar{x}}]|_C(1, |\eta|) + \|\bar{y}\|_C(1, |\psi^*|)}{C_0^2} \le M(\lambda^h)^{3/4}\left[(1, |\eta|] + (1, |\psi^*|)\right].\end{aligned}$$

Lemma 2.5. *Let the conditions (2.27) – (2.29) be satisfied for the difference Sturm-Liouville problem (2.25), then the following estimate will hold*

$$\left|\lambda_n^h - \tilde{\lambda}_n^h\right| \leq M(\lambda^h)^{3/4}\left[(1,|\eta|] + (1,|\psi^*|)\right], \tag{2.46}$$

where $M = const > 0$ depends on the constants $C_i', i = 1, 2, ..., 5$ and C_0.

We move on to the estimate for $\bar{z}$. Since $\bar{y} = C_0 y$, then $\bar{y}$ satisfies the equation (2.25) and $(\rho, \bar{y}^2) = C_0^2$, and for $\bar{z} = \bar{y} - \tilde{y}$ we get the problem (2.40).

Let us reduce this problem to a discrete analogue of the integral equation

$$\bar{z}(x) = \lambda^h\left(G(x,\xi), \rho(\xi)\bar{z}(\xi)\right) + \left(G(x,\xi), \Psi(\xi)\right), \tag{2.47}$$

where $G(x,\xi) = G^h(x,\xi)$ is the difference Green's function of the operator $\Lambda y = (ay_{\bar{x}})_{\hat{x}} - dy$ with boundary conditions $y(0) = y(1) = 0$.

The eigenfunction $\bar{y}$ of the problem (2.25) satisfies the equation

$$\bar{y}(x) = \lambda^h\left(G(x,\xi), \rho(\xi)\bar{y}(\xi)\right). \tag{2.48}$$

Let us transform equations (2.47) and (2.48) into a form where the kernels become symmetric. To achieve this, we make the following substitution

$$\begin{aligned}&v(x) = \sqrt{\rho(x)}\bar{z}(x), \quad \varphi(x) = \sqrt{\rho(x)}\bar{y}(x),\\ &K(x,\xi) = \sqrt{\rho(x)\rho(\xi)}G(x,\xi).\end{aligned}$$

Then the equations (2.47) and (2.48) will take the form

$$v_n(x) = \lambda_n^h\left(K(x,\xi), v_n(\xi)\right) + f(x), \tag{2.49}$$

$$f(x) = \left(K(x,\xi), \bar{\Psi}(\xi)\right), \quad \bar{\Psi}(\xi) = \Psi(\xi)/\sqrt{\rho(\xi)},$$

$$\varphi_n(x) = \lambda_n^h\left(K(x,\xi), \varphi_n(\xi)\right). \tag{2.50}$$

The orthogonality condition of $f(x)$ to the functions $\varphi_n(x)$ is satisfied in view of the condition (2.42):

$$\begin{aligned}(\varphi_n(x), f(x)) &= \left(\varphi_n(x), \left(K(x,\xi), \bar{\Psi}(\xi)\right)\right) = \left(\bar{\Psi}(\xi), \left(K(x,\xi), \varphi_n(x)\right)\right)\\ &= \frac{1}{\lambda_n^h}\left(\bar{\Psi}(\xi), \varphi_n(\xi)\right) = \frac{1}{\lambda_n^h}\left(\frac{\Psi}{\sqrt{\rho}}, \sqrt{\rho}\bar{y}\right) = \frac{1}{\lambda_n^h}\left(\Psi, \bar{y}\right) = 0.\end{aligned}$$

The condition (2.43) we rewrite as follows:

$$(\varphi_n, v_n) = 0. \tag{2.51}$$

We will seek the solution $v(x) = v_n(x)$ of equation (2.49) in the form

$$v_n(x) = f(x) + \sum_{\substack{k=1\\k\neq n}}^{N-1} c_k \varphi_k(x) \tag{2.52}$$

with an additional condition (2.51).

We substitute this expression into the right-hand side of the equation (2.49):

$$v_n(x) = f(x) + \lambda_n^h \sum_{\substack{k=1\\k\neq n}}^{N-1} c_k \left(K(x,\xi), \varphi_k(\xi)\right) + \lambda_n^h \left(K(x,\xi), f(\xi)\right). \tag{2.53}$$

Expanding $f(x)$ in terms of the eigenfunctions $\{\varphi_k(x)\}$

$$f(x) = \sum_{\substack{k=1\\k\neq n}}^{N-1} f_k \varphi_k(x), \quad f_k = \left(f, \varphi_k\right),$$

we obtain

$$\left(K(x,\xi), f(\xi)\right) = \sum_{\substack{k=1\\k\neq n}}^{N-1} \frac{f_k}{\lambda_k^h} \varphi_k(x).$$

Then, taking into account (2.50), we can express the equality (2.53) in the form of

$$v_n(x) = f(x) + \lambda_n^h \sum_{\substack{k=1\\k\neq n}}^{N-1} \left[\frac{c_k}{\lambda_k^h} + \frac{f_k}{\lambda_k^h}\right] \varphi_k(x). \tag{2.54}$$

From the equality (2.54) it follows that

$$c_k = \left(v_n - f, \varphi_k\right) = \frac{\lambda_n^h}{\lambda_k^h} c_k + \frac{\lambda_n^h}{\lambda_k^h} \left(f, \varphi_k\right).$$

Substituting $c_k = \dfrac{\lambda_n^h \left(f, \varphi_k\right)}{\lambda_k^h - \lambda_n^h}$ in (2.52) we get

$$v_n(x) = f(x) + \sum_{\substack{k=1\\k\neq n}}^{N-1} \frac{\lambda_n^h \left(f, \varphi_k\right)}{\lambda_k^h - \lambda_n^h} \varphi_k(x). \tag{2.55}$$

We now estimate the second term in the right-hand side of (2.55). We obtain

$$\left|\sum_{\substack{k=1\\k\neq n}}^{N-1} \frac{\lambda_n^h\,(f,\varphi_k)}{\lambda_k^h-\lambda_n^h}\varphi_k(x)\right| \le \|f\|\|\varphi_k\|\lambda_n^h \sum_{\substack{k=1\\k\neq n}}^{N-1} \frac{|\varphi_k|}{\left|\lambda_k^h-\lambda_n^h\right|}$$
$$\le M\|f\|\lambda_n^h \sum_{\substack{k=1\\k\neq n}}^{N-1} \frac{(\lambda_k^h)^{1/4}}{\left|\lambda_k^h-\lambda_n^h\right|}.$$

Let $\varepsilon > 0$ be any number independent of h. We choose the number n_0 such that $\lambda_{n_0}^h \ge (1+\varepsilon)\lambda_n^h$. Then

$$\sum_{k=n_0}^{N-1} \frac{(\lambda_k^h)^{1/4}}{\left|\lambda_k^h-\lambda_{n_0}^h\right|} \le \frac{1+\varepsilon}{\varepsilon}\sum_{k=n_0}^{N-1} \frac{(\lambda_k^h)^{1/4}}{\lambda_k^h} \le \frac{M'}{\varepsilon}\sum_{k=n_0}^{N-1} \frac{1}{(\lambda_k^h)^{3/4}} \le M,$$

where $M = const > 0$ does not depend on h.

Since $\lambda_k^h \to \lambda_k$ for $k \le n_0$ at $h \to 0$, then at a sufficiently small $h \le h_0$ holds

$$\sum_{k=1}^{n_0-1} \frac{(\lambda_k^h)^{1/4}}{\left|\lambda_k^h-\lambda_n^h\right|} \le M,$$

where M does not depend on h.

This proves the estimate

$$\|v_n\|_C \le M(n)\|f\|_C. \tag{2.56}$$

Let us transform the expression for $f(x)$:

$$\begin{aligned} f(x) &= \big(K(x,\xi),\bar\Psi(\xi)\big) = \left(\sqrt{\rho(x)}\sqrt{\rho(\xi)}G(x,\xi),\frac{\Psi(\xi)}{\sqrt{\rho(\xi)}}\right)\\ &= \sqrt{\rho(x)}\,(G(x,\xi),\Psi(\xi)) = \left(\lambda^h-\tilde\lambda^h\right)\sqrt{\rho(x)}\,(G(x,\xi),\rho(\xi)\tilde y(\xi))\\ &+ \sqrt{\rho(x)}\left(G(x,\xi),\eta_{\hat\xi}(\xi)+\psi^*(\xi)\right)\\ &= \left(\lambda^h-\tilde\lambda^h\right)\sqrt{\rho(x)}\,(G(x,\xi),\rho(\xi)\tilde y(\xi))\\ &+ \sqrt{\rho(x)}\left\{-\left(G_{\bar\xi}(x,\xi),\eta(\xi)\right] + (G(x,\xi),\psi^*(\xi))\right\}. \end{aligned}$$

Hence, considering the boundedness of $G(x,\xi)$ and $G_{\bar\xi}(x,\xi)$ (see [73, p. 204]) such that

$$|G(x,\xi)| \le \frac{1}{C_1}, \quad \left|G_{\bar\xi}(x,\xi)\right| \le \frac{2}{C_1},$$

we get the estimate

$$\|f\|_C \leq M_1 \left\{(1, |\eta|] + (1, |\psi^*|)\right\} + M_2 \left|\lambda^h - \tilde{\lambda}^h\right|.$$

Let us substitute this estimate in (2.56), return to the function $\bar{z}(x) = \dfrac{v(x)}{\sqrt{\rho(x)}}$ and consider the inequality (2.34) and the Lemma 2.5. Then we obtain

$$\|z\|_C \leq M(n) \left\{(1, |\eta|] + (1, |\psi^*|)\right\}.$$

We are interested in the difference $z = y - \tilde{y}$, which is expressed as follows:

$$z = \frac{\bar{z}}{C_0} + \frac{1 - C_0}{C_0}\tilde{y} = \frac{\bar{z}}{C_0} + \frac{1 - C_0^2}{C_0(1 + C_0)}\tilde{y}.$$

Since $\|\tilde{y}\|_C$ is bounded, it follows that at a sufficiently small h holds

$$\|z\|_C \leq \frac{\|\bar{z}\|_C}{C_0} + \left|\frac{1 - C_0^2}{C_0(1 + C_0)}\right| \|y\|_C \leq M(C_0) \left(\|\bar{z}\|_C + \left|1 - C_0^2\right|\right).$$

From formula (2.44), we see that

$$\left|1 - C_0^2\right| \leq \left(\rho, \bar{z}^2\right)^{1/2} \left(\rho, \tilde{y}^2\right)^{1/2} + \left|\left(\rho, \tilde{y}^2\right) - \left(\tilde{\rho}, \tilde{y}^2\right)\right| \leq M_1 \|\bar{z}\| + \left(\tilde{\rho}, \tilde{y}^2\right).$$

So,

$$\|z\|_C \leq M \left(\|\bar{z}\|_C + \left|\left(\rho, \tilde{y}^2\right) - \left(\tilde{\rho}, \tilde{y}^2\right)\right|\right).$$

By substituting the estimate for $\|\bar{z}\|_C$, we confirm that the following Theorem holds.

Theorem 2.1. *Let the conditions of the Lemma 2.5 be fulfilled, then at a sufficiently small $h \leq h_0$, the estimates will hold*

$$\|y_n - \tilde{y}_n\|_C \leq M_1 \left\{(1, |\eta|] + (1, |\psi^*|)\right\} + M_2 \left|\left(\rho, \tilde{y}^2\right) - \left(\tilde{\rho}, \tilde{y}^2\right)\right|, \qquad (2.57)$$

$$\left|\lambda_n^h - \tilde{\lambda}_n^h\right| \leq M_3 \left\{(1, |\eta|] + (1, |\psi^*|)\right\}, \qquad (2.58)$$

where the constants M_i, $i = 1, 2, 3$ depend only on C_i', $i = 1, 2, ..., 5$ and C_0.

This Theorem proves the continuous dependence of the solution of the problem (2.25) on the coefficients, i.e. the coefficient stability (see [82]).

2.4 Algorithmic Realization of the Exact Three-Point Difference Scheme

We move on to the algorithmic realization of the ETDS (2.25). According to the main idea, it is necessary to express the coefficients a_j, d_j, ρ_j of the ETDS in terms of solutions of the Cauchy problems (2.12), (2.13). First of all, we note that the problem (2.12), (2.13) is equivalent to the Cauchy problem for the system of ODE

$$\begin{aligned}
&\frac{dv_\alpha^j(x,\lambda)}{dx} = \frac{w_\alpha^j(x,\lambda) - \lambda z_\alpha^j(x,\lambda)}{k(x)},\\
&\frac{dw_\alpha^j(x,\lambda)}{dx} = q(x)v_\alpha^j(x,\lambda),\\
&\frac{dz_\alpha^j(x,\lambda)}{dx} = r(x)v_\alpha^j(x,\lambda), \quad x \in (x_{j-2+\alpha}, x_{j-1+\alpha}),
\end{aligned} \tag{2.59}$$

$$\begin{aligned}
&v_\alpha^j(x_{j+(-1)^\alpha},\lambda) = 0, \quad w_\alpha^j(x_{j+(-1)^\alpha},\lambda) = (-1)^{\alpha+1},\\
&z_\alpha^j(x_{j+(-1)^\alpha},\lambda) = 0, \quad j = 1,2,...,N+1-\alpha, \quad \alpha = 1,2.
\end{aligned} \tag{2.60}$$

Indeed, if we multiply the first equation of the system (2.59) by $k(x)$ and differentiate the both parts of the resulting equality, then taking into account the second and third equations of the system, we get the equation (2.12). From the first equation and the initial conditions (2.60) the second initial condition (2.13) follows.

For calculation of the coefficients a_j of the ETDS (2.25), we already have the necessary representation (see (2.26)). Note that taking into account (2.59) and (2.60), we have

$$\begin{aligned}
(-1)^{\alpha+1}\int_{x_{j+(-1)^\alpha}}^{x_j} v_\alpha^j(\xi,\lambda)q(\xi)d\xi &= (-1)^{\alpha+1}\int_{x_{j+(-1)^\alpha}}^{x_j} \frac{dw_\alpha^j(\xi,\lambda)}{d\xi}d\xi\\
= (-1)^{\alpha+1}\left[w_\alpha^j(x_j,\lambda) - w_\alpha^j(x_{j+(-1)^\alpha},\lambda)\right] &= (-1)^{\alpha+1}\left[w_\alpha^j(x_j,\lambda) + (-1)^\alpha\right],
\end{aligned}$$

$$\begin{aligned}
(-1)^{\alpha+1}\int_{x_{j+(-1)^\alpha}}^{x_j} v_\alpha^j(\xi,\lambda)r(\xi)d\xi &= (-1)^{\alpha+1}\int_{x_{j+(-1)^\alpha}}^{x_j} \frac{dz_\alpha^j(\xi,\lambda)}{d\xi}d\xi\\
= (-1)^{\alpha+1}\left[z_\alpha^j(x_j,\lambda) - z_\alpha^j(x_{j+(-1)^\alpha},\lambda)\right] &= (-1)^{\alpha+1}z_\alpha^j(x_j,\lambda).
\end{aligned}$$

Hence, the coefficients d_j and ρ_j of the difference scheme (2.25) can be represented as

$$\begin{aligned}
d_j &= \frac{1}{\hbar_j v_1^j(x_j,\lambda)}\int_{x_{j-1}}^{x_j} v_1^j(\xi,\lambda)q(\xi)d\xi + \frac{1}{\hbar_j v_2^j(x_j,\lambda)}\int_{x_j}^{x_{j+1}} v_2^j(\xi,\lambda)q(\xi)d\xi\\
&= \hbar_j^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[v_\alpha^j(x_j,\lambda)\right]^{-1}\left[w_\alpha^j(x_j,\lambda) + (-1)^\alpha\right],
\end{aligned}$$

$$\begin{aligned}\rho_j &= \frac{1}{\hbar_j v_1^j(x_j,\lambda)}\int_{x_{j-1}}^{x_j} v_1^j(\xi,\lambda)r(\xi)d\xi + \frac{1}{\hbar_j v_2^j(x_j,\lambda)}\int_{x_j}^{x_{j+1}} v_2^j(\xi,\lambda)r(\xi)d\xi \\ &= \hbar_j^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[v_\alpha^j(x_j,\lambda)\right]^{-1} z_\alpha^j(x_j,\lambda).\end{aligned}$$

Then we rewrite the ETDS as

$$(ay_{\bar{x}})_{\hat{x}} - (d-\lambda\rho)y = 0, \quad x\in\hat{\omega}_h, \quad y_0 = y_N = 0, \tag{2.61}$$

where

$$a_j = \left[\frac{1}{h_j}v_1^j(x_j,\lambda)\right]^{-1}, \quad j = 1,2,...,N, \tag{2.62}$$

$$d_j = \hbar_j^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[v_\alpha^j(x_j,\lambda)\right]^{-1}\left[w_\alpha^j(x_j,\lambda)+(-1)^\alpha\right], \tag{2.63}$$

$$\rho_j = \hbar_j^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[v_\alpha^j(x_j,\lambda)\right]^{-1} z_\alpha^j(x_j,\lambda). \tag{2.64}$$

Therefore, to calculate the coefficients a_j, d_j, ρ_j of the ETDS at any node x_j of the grid $\hat{\omega}_h$ we need to solve two Cauchy problems (2.59), (2.60) with smooth coefficients: at $\alpha = 1$ on the interval $[x_{j-1}, x_j]$ (forward) and at $\alpha = 2$ on the interval $[x_j, x_{j+1}]$ (backward).

2.5 Three-Point Difference Schemes of High Order of Accuracy

2.5.1 Difference Schemes of Rank $\bar{m}$

Each of the Cauchy problems (2.59), (2.60) we will solve numerically in one step by any one-step method (Taylor series expansion or the Runge-Kutta method) of the order of accuracy $\bar{m} = 2[(m+1)/2]$ (m is a given natural number, $[\cdot]$ is the integer part of the argument in brackets). Then

$$\begin{aligned}&v_\alpha^{(\bar{m})j}(x_j,\lambda) \\ &\quad = (-1)^{\alpha+1}h_{j-1+\alpha}\Phi_1\left(x_{j+(-1)^\alpha}, 0, (-1)^{\alpha+1}, 0, (-1)^{\alpha+1}h_{j-1+\alpha}\right),\end{aligned} \tag{2.65}$$

$$\begin{aligned}&w_\alpha^{(\bar{m})j}(x_j,\lambda) = (-1)^{\alpha+1} \\ &\quad + (-1)^{\alpha+1}h_{j-1+\alpha}\Phi_2\left(x_{j+(-1)^\alpha}, 0, (-1)^{\alpha+1}, 0, (-1)^{\alpha+1}h_{j-1+\alpha}\right),\end{aligned} \tag{2.66}$$

$$\begin{aligned}&z_\alpha^{(\bar{m})j}(x_j,\lambda) \\ &\quad = (-1)^{\alpha+1}h_{j-1+\alpha}\Phi_3\left(x_{j+(-1)^\alpha}, 0, (-1)^{\alpha+1}, 0, (-1)^{\alpha+1}h_{j-1+\alpha}\right),\end{aligned} \tag{2.67}$$

where $\Phi_1(x,u,y,z,h)$, $\Phi_2(x,u,y,z,h)$, $\Phi_3(x,u,y,z,h)$ are the increment functions of the one-step method, and $v_\alpha^{(\bar m)j}(x_j,\lambda), w_\alpha^{(\bar m)j}(x_j,\lambda), z_\alpha^{(\bar m)j}(x_j,\lambda)$ approximate the corresponding values $v_\alpha^j(x_j,\lambda)$, $w_\alpha^j(x_j,\lambda)$, $z_\alpha^j(x_j,\lambda)$ with order of accuracy $\bar m$.

In the case of the Taylor series method we get

$$\Phi_1\left(x_{j+(-1)^\alpha},0,(-1)^{\alpha+1},0,(-1)^{\alpha+1}h_{j-1+\alpha}\right)=\frac{(-1)^{\alpha+1}}{k_{j+(-1)^\alpha}}$$

$$+\frac{h_{j-1+\alpha}}{2}\frac{d}{dx}\left(\frac{1}{k(x)}\right)\bigg|_{x=x_{j+(-1)^\alpha}}+\sum_{p=3}^{\bar m}\frac{\left[(-1)^{\alpha+1}h_{j-1+\alpha}\right]^{p-1}}{p!}\frac{d^p v_\alpha^j(x_{j+(-1)^\alpha},\lambda)}{dx^p},$$

$$\Phi_2\left(x_{j+(-1)^\alpha},0,(-1)^{\alpha+1},0,(-1)^{\alpha+1}h_{j-1+\alpha}\right)=\frac{h_{j-1+\alpha}}{2}\left.\frac{q(x)}{k(x)}\right|_{x=x_{j+(-1)^\alpha}}$$

$$+\frac{h_{j-1+\alpha}^2}{6}(-1)^{\alpha+1}\left(\frac{2}{k(x)}\frac{dq(x)}{dx}+\frac{d}{dx}\left(\frac{1}{k(x)}\right)q(x)\right)\bigg|_{x=x_{j+(-1)^\alpha}}$$

$$+\sum_{p=4}^{\bar m}\frac{\left[(-1)^{\alpha+1}h_{j-1+\alpha}\right]^{p-1}}{p!}\frac{d^p w_\alpha^j(x_{j+(-1)^\alpha},\lambda)}{dx^p},$$

$$\Phi_3\left(x_{j+(-1)^\alpha},0,(-1)^{\alpha+1},0,(-1)^{\alpha+1}h_{j-1+\alpha}\right)=\frac{h_{j-1+\alpha}}{2}\left.\frac{r(x)}{k(x)}\right|_{x=x_{j+(-1)^\alpha}}$$

$$+\frac{h_{j-1+\alpha}^2}{6}(-1)^{\alpha+1}\left(\frac{2}{k(x)}\frac{dr(x)}{dx}+\frac{d}{dx}\left(\frac{1}{k(x)}\right)r(x)\right)\bigg|_{x=x_{j+(-1)^\alpha}}$$

$$+\sum_{p=4}^{\bar m}\frac{\left[(-1)^{\alpha+1}h_{j-1+\alpha}\right]^{p-1}}{p!}\frac{d^p z_\alpha^j(x_{j+(-1)^\alpha},\lambda)}{dx^p},$$

and in the case of the Runge-Kutta methods

$$\Phi_1\left(x_{j+(-1)^\alpha},0,(-1)^{\alpha+1},0,(-1)^{\alpha+1}h_{j-1+\alpha}\right)=b_1g_1+b_2g_2+...+b_sg_s,$$

$$\Phi_2\left(x_{j+(-1)^\alpha},0,(-1)^{\alpha+1},0,(-1)^{\alpha+1}h_{j-1+\alpha}\right)=b_1\bar g_1+b_2\bar g_2+...+b_s\bar g_s,$$

$$\Phi_3\left(x_{j+(-1)^\alpha},0,(-1)^{\alpha+1},0,(-1)^{\alpha+1}h_{j-1+\alpha}\right)=b_1\tilde g_1+b_2\tilde g_2+...+b_s\tilde g_s,$$

$$g_i=\frac{(-1)^{\alpha+1}\left(1+h_{j-1+\alpha}\sum_{p=1}^{i-1}a_{ip}(\bar g_p-\tilde g_p)\right)}{k\left(x_{j+(-1)^\alpha}+c_i(-1)^{\alpha+1}h_{j-1+\alpha}\right)},$$

$$\bar g_i=(-1)^{\alpha+1}h_{j-1+\alpha}\sum_{p=1}^{i-1}a_{ip}g_pq\left(x_{j+(-1)^\alpha}+c_i(-1)^{\alpha+1}h_{j-1+\alpha}\right),$$

$$\tilde{g}_i = (-1)^{\alpha+1} h_{j-1+\alpha} \sum_{p=1}^{i-1} a_{ip} g_p r\left(x_{j+(-1)^\alpha} + c_i (-1)^{\alpha+1} h_{j-1+\alpha}\right),$$
$$i = 1, 2, ..., s.$$

If $k(x), q(x), r(x)$ are sufficiently smooth functions and methods (2.65) – (2.67) have the order of accuracy $\bar{m}$, then the equalities hold (see, e.g., [33, p. 168])

$$\begin{aligned} v_\alpha^j(x_j, \lambda) &= v_\alpha^{(\bar{m})j}(x_j, \lambda) \\ &\quad + \left[(-1)^{\alpha+1} h_{j-1+\alpha}\right]^{\bar{m}+1} \psi_\alpha^j(x_{j+(-1)^\alpha}) + O\left(h_{j-1+\alpha}^{\bar{m}+2}\right), \end{aligned} \tag{2.68}$$

$$\begin{aligned} w_\alpha^j(x_j, \lambda) &= w_\alpha^{(\bar{m})j}(x_j, \lambda) \\ &\quad + \left[(-1)^{\alpha+1} h_{j-1+\alpha}\right]^{\bar{m}+1} \bar{\psi}_\alpha^j(x_{j+(-1)^\alpha}) + O\left(h_{j-1+\alpha}^{\bar{m}+2}\right), \end{aligned} \tag{2.69}$$

$$\begin{aligned} z_\alpha^j(x_j, \lambda) &= z_\alpha^{(\bar{m})j}(x_j, \lambda) \\ &\quad + \left[(-1)^{\alpha+1} h_{j-1+\alpha}\right]^{\bar{m}+1} \tilde{\psi}_\alpha^j(x_{j+(-1)^\alpha}) + O\left(h_{j-1+\alpha}^{\bar{m}+2}\right). \end{aligned} \tag{2.70}$$

Lemma 2.6. *Let the conditions (2.2), $k(x) \in Q^{(m+1)}[0,1]$, $q(x), r(x) \in Q^{(m)}[0,1]$ be fulfilled and for the numerical method (2.65) – (2.67) the equalities (2.68) – (2.70) are satisfied. Then the following relations hold*

$$v_\alpha^j(x_j, \lambda) = v_\alpha^{(\bar{m})j}(x_j, \lambda) + h_{j-1+\alpha}^{\bar{m}+1} \psi_1^{j-1+\alpha}(x_{j+(-1)^\alpha}) + O\left(h_{j-1+\alpha}^{\bar{m}+2}\right), \tag{2.71}$$

$$w_\alpha^j(x_j, \lambda) = w_\alpha^{(\bar{m})j}(x_j, \lambda) + h_{j-1+\alpha}^{\bar{m}+1} \bar{\psi}_1^{j-1+\alpha}(x_{j+(-1)^\alpha}) + O\left(h_{j-1+\alpha}^{\bar{m}+2}\right), \tag{2.72}$$

$$z_\alpha^j(x_j, \lambda) = z_\alpha^{(\bar{m})j}(x_j, \lambda) + h_{j-1+\alpha}^{\bar{m}+1} \tilde{\psi}_1^{j-1+\alpha}(x_{j+(-1)^\alpha}) + O\left(h_{j-1+\alpha}^{\bar{m}+2}\right), \tag{2.73}$$
$$\alpha = 1, 2, \quad j = 1, 2, ..., N-1.$$

Proof. We note that the functions $n_2^j(x_j, \lambda) = -v_2^j(x_j, \lambda)$, $l_2^j(x_j, \lambda) = -w_2^j(x_j, \lambda)$, $p_2^j(x_j, \lambda) = -z_2^j(x_j, \lambda)$ are the solution of the Cauchy problem

$$\begin{aligned} \frac{dn_2^j(x, \lambda)}{dx} &= \frac{l_2^j(x, \lambda) - \lambda p_2^j(x, \lambda)}{k(x)}, \\ \frac{dl_2^j(x, \lambda)}{dx} &= q(x) n_2^j(x, \lambda), \\ \frac{dp_2^j(x, \lambda)}{dx} &= r(x) n_2^j(x, \lambda), \quad x \in (x_j, x_{j+1}), \end{aligned}$$

$$n_2^j(x_{j+1}, \lambda) = 0, \quad l_2^j(x_{j+1}, \lambda) = 1, \quad p_2^j(x_{j+1}, \lambda) = 0,$$
$$j = 1, 2, ..., N-1.$$

For the numerical solving of this problem we apply the one-step method

$$n_2^{(\bar m)j}(x_j,\lambda) = -h_{j+1}\Phi_1(x_{j+1},0,1,0,-h_{j+1}), \tag{2.74}$$

$$l_2^{(\bar m)j}(x_j,\lambda) = 1 - h_{j+1}\Phi_2(x_{j+1},0,1,0,-h_{j+1}), \tag{2.75}$$

$$p_2^{(\bar m)j}(x_j,\lambda) = -h_{j+1}\Phi_3(x_{j+1},0,1,0,-h_{j+1}), \tag{2.76}$$

According to (2.68) – (2.70) and the fact that $\bar m$ is an even number, for $n_2^j(x_j,\lambda), l_2^j(x_j,\lambda), p_2^j(x_j,\lambda)$ hold the equalities

$$n_2^j(x_j,\lambda) = -v_2^{(\bar m)j}(x_j,\lambda) + h_{j+1}^{\bar m+1}\psi_2^j(x_{j+1}) + O\left(h_{j+1}^{\bar m+2}\right), \tag{2.77}$$

$$l_2^j(x_j,\lambda) = -w_2^{(\bar m)j}(x_j,\lambda) + h_{j+1}^{\bar m+1}\bar\psi_2^j(x_{j+1}) + O\left(h_{j+1}^{\bar m+2}\right), \tag{2.78}$$

$$p_2^j(x_j,\lambda) = -z_2^{(\bar m)j}(x_j,\lambda) + h_{j+1}^{\bar m+1}\tilde\psi_2^j(x_{j+1}) + O\left(h_{j+1}^{\bar m+2}\right). \tag{2.79}$$

If in formulas (2.65) – (2.67) for $\alpha = 1$ we replace the index j by $j+1$, then we get the adjoint method (see [33, p. 220]) to the method (2.74) – (2.76). Applying the Theorem 8.5 [33, p. 220] to these methods and considering that the principal error terms for $n_2^j(x_j,\lambda), l_2^j(x_j,\lambda), p_2^j(x_j,\lambda)$ and for $v_2^j(x_j,\lambda), w_2^j(x_j,\lambda), z_2^j(x_j,\lambda)$ differ only in signs, we obtain

$$\begin{aligned}\psi_2^j(x_{j+1}) &= -(-1)^{\bar m}\psi_1^{j+1}(x_j) = -(-1)^{\bar m}\psi_1^{j+1}(x_{j+1}) + O(h_{j+1})\\ &= -\psi_1^{j+1}(x_{j+1}) + O(h_{j+1}),\end{aligned}$$

$$\begin{aligned}\bar\psi_2^j(x_{j+1}) &= -(-1)^{\bar m}\bar\psi_1^{j+1}(x_{j+1}) + O(h_{j+1}) = -\bar\psi_1^{j+1}(x_{j+1}) + O(h_{j+1}),\\ \tilde\psi_2^j(x_{j+1}) &= -(-1)^{\bar m}\tilde\psi_1^{j+1}(x_{j+1}) + O(h_{j+1}) = -\tilde\psi_1^{j+1}(x_{j+1}) + O(h_{j+1}).\end{aligned}$$

From this and the equalities (2.68) – (2.70), the statement of the Lemma follows. □

Instead of the ETDS (2.61) – (2.64), we can now use the three-point difference scheme of rank $\bar m$ of the form

$$\left(a^{(\bar m)}y_{\bar x}^{(\bar m)}\right)_{\hat x} - (d^{(\bar m)} - \lambda^{(\bar m)}\rho^{(\bar m)})y^{(\bar m)} = 0, \quad x \in \hat\omega_h, \quad y_0^{(\bar m)} = y_N^{(\bar m)} = 0, \tag{2.80}$$

where

$$a^{(\bar m)}(x_j,\lambda^{(\bar m)}) = \left[\frac{1}{h_j}v_1^{(\bar m)j}(x_j,\lambda^{(\bar m)})\right]^{-1}, \quad j = 1,2,...,N, \tag{2.81}$$

$$\begin{aligned}&d^{(\bar m)}(x_j,\lambda^{(\bar m)})\\ &\quad = \hbar_j^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[v_\alpha^{(\bar m)j}(x_j,\lambda^{(\bar m)})\right]^{-1}\left[w_\alpha^{(\bar m)j}(x_j,\lambda^{(\bar m)}) + (-1)^\alpha\right],\end{aligned} \tag{2.82}$$

$$\rho^{(\bar{m})}(x_j, \lambda^{(\bar{m})}) = \hbar_j^{-1} \sum_{\alpha=1}^{2} (-1)^{\alpha+1} \left[v_\alpha^{(\bar{m})j}(x_j, \lambda^{(\bar{m})}) \right]^{-1} z_\alpha^{(\bar{m})j}(x_j, \lambda^{(\bar{m})}). \tag{2.83}$$

Hereafter, we will denote by M a general constant that does not depend on h.

Note that the coefficients $a^{(\bar{m})}(x_j, \lambda), d^{(\bar{m})}(x_j, \lambda), \rho^{(\bar{m})}(x_j, \lambda)$ weakly depend (with the second order of smallness relative to h) on the eigenvalue λ.

Lemma 2.7. *Let the conditions of Lemma 2.6 be satisfied. Then*

$$\left| a^{(\bar{m})}(x_j, \lambda) - a(x_j, \lambda) \right| \le M h^{\bar{m}}, \tag{2.84}$$

$$\begin{aligned} d(x_j, \lambda) - d^{(\bar{m})}(x_j, \lambda) = &- \left\{ h_j^{\bar{m}} \; k(x) \bar{\psi}_1^j(x) \big|_{x=x_j-0} \right\}_{\hat{x}} \\ &+ O\left(\frac{h_{j+1}^{\bar{m}+1} + h_j^{\bar{m}+1}}{\hbar_j} \right), \end{aligned} \tag{2.85}$$

$$\begin{aligned} \rho(x_j, \lambda) - \rho^{(\bar{m})}(x_j, \lambda) = &- \left\{ h_j^{\bar{m}} \; k(x) \tilde{\psi}_1^j(x) \Big|_{x=x_j-0} \right\}_{\hat{x}} \\ &+ O\left(\frac{h_{j+1}^{\bar{m}+1} + h_j^{\bar{m}+1}}{\hbar_j} \right), \end{aligned} \tag{2.86}$$

$$\left| a^{(\bar{m})}(x_j, \lambda) - a^{(\bar{m})}(x_j, \tilde{\lambda}) \right| \le M h^2 \cdot |\lambda - \tilde{\lambda}|, \tag{2.87}$$

$$\left| d^{(\bar{m})}(x_j, \lambda) - d^{(\bar{m})}(x_j, \tilde{\lambda}) \right| \le M h^2 \cdot |\lambda - \tilde{\lambda}|, \tag{2.88}$$

$$\left| \rho^{(\bar{m})}(x_j, \lambda) - \rho^{(\bar{m})}(x_j, \tilde{\lambda}) \right| \le M h^2 \cdot |\lambda - \tilde{\lambda}|. \tag{2.89}$$

Proof. The inequality (2.84) follows from (2.71):

$$a^{(\bar{m})}(x_j, \lambda) - a(x_j, \lambda) = \frac{h_j \left[v_1^j(x_j, \lambda) - v_1^{(\bar{m})j}(x_j, \lambda) \right]}{v_1^j(x_j, \lambda)\, v_1^{(\bar{m})j}(x_j, \lambda)} = O\left(h_j^{\bar{m}} \right).$$

Let us prove now (2.85) and (2.86). First of all, we note that

$$\begin{aligned} &d(x_j, \lambda) - d^{(\bar{m})}(x_j, \lambda) \\ &= \frac{1}{\hbar_j} \sum_{\alpha=1}^{2} (-1)^{\alpha+1} \left[\frac{w_\alpha^j(x_j, \lambda) + (-1)^\alpha}{v_\alpha^j(x_j, \lambda)} - \frac{w_\alpha^{(\bar{m})j}(x_j, \lambda) + (-1)^\alpha}{v_\alpha^{(\bar{m})j}(x_j, \lambda)} \right], \end{aligned} \tag{2.90}$$

$$\rho(x_j,\lambda) - \rho^{(\bar{m})}(x_j,\lambda) = \frac{1}{\hbar_j}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[\frac{z_\alpha^j\left(x_j,\lambda\right)}{v_\alpha^j\left(x_j,\lambda\right)} - \frac{z_\alpha^{(\bar{m})j}\left(x_j,\lambda\right)}{v_\alpha^{(\bar{m})j}\left(x_j,\lambda\right)}\right]. \quad (2.91)$$

Using (2.71), (2.72) and the relation

$$v_\alpha^j\left(x_j,\lambda\right) = v_\alpha^{(\bar{m})j}\left(x_j,\lambda\right) + O\left(h_{j-1+\alpha}^{\bar{m}+1}\right) = \frac{h_{j-1+\alpha}}{k\left(x_{j+(-1)^\alpha}\right)} + O\left(h_{j-1+\alpha}^2\right),$$

we get

$$\begin{aligned}
&\frac{w_\alpha^j\left(x_j,\lambda\right)+(-1)^\alpha}{v_\alpha^j\left(x_j,\lambda\right)} - \frac{w_\alpha^{(\bar{m})j}\left(x_j,\lambda\right)+(-1)^\alpha}{v_\alpha^{(\bar{m})j}\left(x_j,\lambda\right)}\\
&= \frac{v_\alpha^{(\bar{m})j}\left(x_j,\lambda\right)\left[w_\alpha^j\left(x_j,\lambda\right) - w_\alpha^{(\bar{m})j}\left(x_j,\lambda\right)\right]}{v_\alpha^{(\bar{m})j}\left(x_j,\lambda\right)v_\alpha^j\left(x_j,\lambda\right)}\\
&+\frac{\left[v_\alpha^{(\bar{m})j}\left(x_j,\lambda\right) - v_\alpha^j\left(x_j,\lambda\right)\right]\left[w_\alpha^{(\bar{m})j}\left(x_j,\lambda\right)+(-1)^\alpha\right]}{v_\alpha^{(\bar{m})j}\left(x_j,\lambda\right)v_\alpha^j\left(x_j,\lambda\right)}\\
&= \frac{h_{j-1+\alpha}^{\bar{m}+1}\bar{\psi}_1^{j-1+\alpha}\left(x_{j+(-1)^\alpha}\right) + O\left(h_{j-1+\alpha}^{\bar{m}+2}\right)}{\dfrac{h_{j-1+\alpha}}{k\left(x_{j+(-1)^\alpha}\right)} + O\left(h_{j-1+\alpha}^2\right)}+\\
&+\left[-(-1)^{\alpha+1}h_{j-1+\alpha}^{\bar{m}+1}\psi_1^{j-1+\alpha}\left(x_{j+(-1)^\alpha}\right) + O\left(h_{j-1+\alpha}^{\bar{m}+2}\right)\right]\times\\
&\times\frac{\dfrac{(-1)^{\alpha+1}h_{j-1+\alpha}^2}{2}\left.\dfrac{q(x)}{k(x)}\right|_{x=x_{j+(-1)^\alpha}} + O\left(h_{j-1+\alpha}^3\right)}{\left[\dfrac{h_{j-1+\alpha}}{k\left(x_{j+(-1)^\alpha}\right)} + O\left(h_{j-1+\alpha}^2\right)\right]^2} =\\
&= h_{j-1+\alpha}^{\bar{m}}k\left(x_{j+(-1)^\alpha}\right)\bar{\psi}_1^{j-1+\alpha}\left(x_{j+(-1)^\alpha}\right) + O\left(h_{j-1+\alpha}^{\bar{m}+1}\right).
\end{aligned}$$

Similarly, taking into account (2.71) and (2.73), we have

$$\begin{aligned}
\frac{z_\alpha^j\left(x_j,\lambda\right)}{v_\alpha^j\left(x_j,\lambda\right)} - \frac{z_\alpha^{(\bar{m})j}\left(x_j,\lambda\right)}{v_\alpha^{(\bar{m})j}\left(x_j,\lambda\right)} &= h_{j-1+\alpha}^{\bar{m}}k\left(x_{j+(-1)^\alpha}\right)\tilde{\psi}_1^{j-1+\alpha}\left(x_{j+(-1)^\alpha}\right)\\
&+ O\left(h_{j-1+\alpha}^{\bar{m}+1}\right).
\end{aligned}$$

Substituting the last two equalities in (2.90) and (2.91), we obtain the relation

$$\begin{aligned}
& d(x_j,\lambda) - d^{(\bar m)}(x_j,\lambda) \\
&\quad = \hbar_j^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}h_{j-1+\alpha}^{\bar m}k\left(x_{j+(-1)^\alpha}\right)\bar\psi_1^{j-1+\alpha}\left(x_{j+(-1)^\alpha}\right) \\
&\quad + O\left(\frac{h_{j+1}^{\bar m+1}+h_j^{\bar m+1}}{\hbar_j}\right) = \frac{1}{\hbar_j}\left[h_j^{\bar m}k(x_{j-1}+0)\bar\psi_1^j(x_{j-1}+0)-\right. \\
&\quad \left.-h_{j+1}^{\bar m}k(x_{j+1}-0)\bar\psi_1^{j+1}(x_{j+1}-0)\right] + O\left(\frac{h_{j+1}^{\bar m+1}+h_j^{\bar m+1}}{\hbar_j}\right),
\end{aligned} \tag{2.92}$$

$$\begin{aligned}
& \rho(x_j,\lambda) - \rho^{(\bar m)}(x_j,\lambda) \\
&\quad = \hbar_j^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}h_{j-1+\alpha}^{\bar m}k\left(x_{j+(-1)^\alpha}\right)\tilde\psi_1^{j-1+\alpha}\left(x_{j+(-1)^\alpha}\right) \\
&\quad + O\left(\frac{h_{j+1}^{\bar m+1}+h_j^{\bar m+1}}{\hbar_j}\right) = \frac{1}{\hbar_j}\left[h_j^{\bar m}k(x_{j-1}+0)\tilde\psi_1^j(x_{j-1}+0)\right. \\
&\quad \left.-h_{j+1}^{\bar m}k(x_{j+1}-0)\tilde\psi_1^{j+1}(x_{j+1}-0)\right] + O\left(\frac{h_{j+1}^{\bar m+1}+h_j^{\bar m+1}}{\hbar_j}\right).
\end{aligned} \tag{2.93}$$

Therefore, since

$$k(x_{j-1}+0)\bar\psi_1^j(x_{j-1}+0) = k(x_j-0)\bar\psi_1^j(x_j-0) + O(h_j),$$
$$k(x_{j-1}+0)\tilde\psi_1^j(x_{j-1}+0) = k(x_j-0)\tilde\psi_1^j(x_j-0) + O(h_j),$$

then the relations (2.85), (2.86) follow from (2.92), (2.93).

Let us prove the inequalities (2.87) – (2.89). Considering the equalities (2.65) – (2.67) and the theorem on finite increments, we get

$$\begin{aligned}
\left|a^{(\bar m)}(x_j,\lambda) - a^{(\bar m)}(x_j,\tilde\lambda)\right| &\le \left|\frac{h_j\left[v_1^{(\bar m)j}(x_j,\tilde\lambda) - v_1^{(\bar m)j}(x_j,\lambda)\right]}{v_1^{(\bar m)j}(x_j,\lambda)v_1^{(\bar m)j}(x_{j-1},\tilde\lambda)}\right| \\
&= \left|\frac{h_j\sum_{p=3}^{\bar m}\frac{h_j^p}{p!}\left[\frac{d^pv_\alpha^j(x_{j-1},\tilde\lambda)}{dx^p} - \frac{d^pv_\alpha^j(x_{j-1},\lambda)}{dx^p}\right]}{\left[h_j/k_{j-1}+O(h_j^2)\right]^2}\right| \\
&\le \frac{h_j\sum_{p=3}^{\bar m}\frac{h_j^p}{p!}\left|\frac{\partial}{\partial\lambda}\left(\frac{d^pv_\alpha^j(x_{j+(-1)^\alpha},\lambda)}{dx^p}\right)\Big|_{\lambda=\bar\lambda}\right||\lambda-\tilde\lambda|}{h_j^2/k_{j-1}^2+O(h_j^3)} \\
&\le Mh^2|\lambda-\tilde\lambda|, \quad \bar\lambda = \lambda+\theta(\lambda-\tilde\lambda), \quad 0<\theta<1.
\end{aligned}$$

$$\begin{aligned}
&\left|d^{(\bar m)}(x_j,\lambda)-d^{(\bar m)}(x_j,\tilde\lambda)\right| \\
&\quad\le \hbar_j^{-1}\sum_{\alpha=1}^{2}\left|\frac{w_\alpha^{(\bar m)j}(x_j,\lambda)+(-1)^\alpha}{v_\alpha^{(\bar m)j}(x_j,\lambda)}-\frac{w_\alpha^{(\bar m)j}(x_j,\tilde\lambda)+(-1)^\alpha}{v_\alpha^{(\bar m)j}(x_j,\tilde\lambda)}\right| \\
&\quad= \hbar_j^{-1}\sum_{\alpha=1}^{2}\left|\frac{\sum_{p=4}^{\bar m}\frac{\left[(-1)^{\alpha+1}h_{j-1+\alpha}\right]^p}{p!}\left[\frac{d^p w_\alpha^j(x_{j+(-1)^\alpha},\lambda)}{dx^p}-\frac{d^p w_\alpha^j(x_{j+(-1)^\alpha},\tilde\lambda)}{dx^p}\right]}{h_{j-1+\alpha}/k_{j+(-1)^\alpha}+O(h_{j-1+\alpha}^2)}\right| \\
&\quad\le \hbar_j^{-1}\sum_{\alpha=1}^{2}\frac{\sum_{p=4}^{\bar m}\frac{h_{j-1+\alpha}^p}{p!}\left|\frac{\partial}{\partial\lambda}\left(\frac{d^p w_\alpha^j(x_{j+(-1)^\alpha},\lambda)}{dx^p}\right)\Big|_{\lambda=\bar\lambda}\right||\lambda-\tilde\lambda|}{h_{j-1+\alpha}/k_{j+(-1)^\alpha}+O(h_{j-1+\alpha}^2)} \\
&\quad\le Mh^2|\lambda-\tilde\lambda|,\quad \bar\lambda=\lambda+\theta(\lambda-\tilde\lambda),\quad 0<\theta<1.
\end{aligned}$$

$$\begin{aligned}
&\left|\rho^{(\bar m)}(x_j,\lambda)-\rho^{(\bar m)}(x_j,\tilde\lambda)\right| \le \hbar_j^{-1}\sum_{\alpha=1}^{2}\left|\frac{z_\alpha^{(\bar m)j}(x_j,\lambda)}{v_\alpha^{(\bar m)j}(x_j,\lambda)}-\frac{z_\alpha^{(\bar m)j}(x_j,\tilde\lambda)}{v_\alpha^{(\bar m)j}(x_j,\tilde\lambda)}\right| \\
&\quad= \hbar_j^{-1}\sum_{\alpha=1}^{2}\left|\frac{\sum_{p=4}^{\bar m}\frac{\left[(-1)^{\alpha+1}h_{j-1+\alpha}\right]^p}{p!}\left[\frac{d^p z_\alpha^j(x_{j+(-1)^\alpha},\lambda)}{dx^p}-\frac{d^p z_\alpha^j(x_{j+(-1)^\alpha},\tilde\lambda)}{dx^p}\right]}{h_{j-1+\alpha}/k_{j+(-1)^\alpha}+O(h_{j-1+\alpha}^2)}\right| \\
&\quad\le \hbar_j^{-1}\sum_{\alpha=1}^{2}\frac{\sum_{p=4}^{\bar m}\frac{h_{j-1+\alpha}^p}{p!}\left|\frac{\partial}{\partial\lambda}\left(\frac{d^p z_\alpha^j(x_{j+(-1)^\alpha},\lambda)}{dx^p}\right)\Big|_{\lambda=\bar\lambda}\right||\lambda-\tilde\lambda|}{h_{j-1+\alpha}/k_{j+(-1)^\alpha}+O(h_{j-1+\alpha}^2)} \\
&\quad\le Mh^2|\lambda-\tilde\lambda|,\quad \bar\lambda=\lambda+\theta(\lambda-\tilde\lambda),\quad 0<\theta<1.
\end{aligned}$$

□

Theorem 2.2. *Let the conditions of the Lemma 2.6 be fulfilled. Then* $\exists h_0>0$ *such that at* $h\le h_0$ *for the error of the difference scheme* (2.80) – (2.83), *the inequalities will hold*

$$\left\|y_n-y_n^{(\bar m)}\right\|_C\le M_1h^{\bar m}, \tag{2.94}$$

$$\left|\lambda_n-\lambda_n^{(\bar m)}\right|\le M_2h^{\bar m}, \tag{2.95}$$

where M_1, M_2 *are constants independent of* h.

Proof. In view of the conditions of the Theorem, we apply to the ETDS (2.61) – (2.64) and to the difference scheme (2.80) – (2.83) the Theorem 2.1 on coefficient stability where we put $\tilde y(x)=y^{(\bar m)}(x)$, $\tilde a(x,\tilde\lambda)=a^{(\bar m)}(x,\lambda^{(\bar m)})$, $\tilde d(x,\tilde\lambda)=d^{(\bar m)}(x,\lambda^{(\bar m)})$, $\tilde\rho(x,\tilde\lambda)=\rho^{(\bar m)}(x,\lambda^{(\bar m)})$. Let us write in these terms the values η, ψ^* (see (2.41))

$$\eta=\left(a(x,\lambda)-a^{(\bar m)}(x,\lambda^{(\bar m)})\right)y_{\bar x}^{(\bar m)},$$

$$\psi^* = -\left(d(x,\lambda) - d^{(\bar m)}(x,\lambda^{(\bar m)})\right) y^{(\bar m)} + \lambda^{(\bar m)}\left(\rho(x,\lambda) - \rho^{(\bar m)}(x,\lambda^{(\bar m)})\right) y^{(\bar m)}.$$

Then we get an estimate

$$\begin{aligned}(1,|\eta|] = \sum_{\xi\in\hat\omega_h^+} & h(\xi)\left|\left(a(\xi,\lambda) - a^{(\bar m)}(\xi,\lambda)\right) + \left(a^{(\bar m)}(\xi,\lambda) - a^{(\bar m)}(\xi,\lambda^{(\bar m)})\right)\right| \\ & \times \left|y_{\bar\xi}^{(\bar m)}(\xi)\right| \le M_3 h^{\bar m} + Mh^2|\lambda - \lambda^{(\bar m)}|, \end{aligned} \tag{2.96}$$

as well as in view of the summation by parts formula we get the estimate

$$\begin{aligned}(1,|\psi^*|) = & \left|\sum_{\xi\in\hat\omega_h} \hbar(\xi)\left[d^{(\bar m)}(\xi,\lambda) - d(\xi,\lambda) - \lambda^{(\bar m)}\left(\rho^{(\bar m)}(\xi,\lambda) - \rho(\xi,\lambda)\right)\right] y^{(\bar m)}(\xi)\right. \\ & \left. + \sum_{\xi\in\hat\omega_h} \hbar(\xi)\left[d^{(\bar m)}(\xi,\lambda^{(\bar m)}) - d^{(\bar m)}(\xi,\lambda) - \lambda^{(\bar m)}\left(\rho^{(\bar m)}(\xi,\lambda^{(\bar m)}) - \rho^{(\bar m)}(\xi,\lambda)\right)\right] y^{(\bar m)}(\xi)\right| \\ & \le \sum_{\xi\in\hat\omega_h^+} h^{\bar m+1}(\xi)k(\xi)\left[\left|\bar\psi_1^j(\xi)\right| + \left|\lambda^{(\bar m)}\right|\left|\tilde\psi_1^j(\xi)\right|\right]\left|y_{\bar\xi}^{(\bar m)}(\xi)\right| + Mh^2|\lambda - \lambda^{(\bar m)}| \\ & + O\left(h^{\bar m}\right) \le M_4 h^{\bar m} + Mh^2|\lambda - \lambda^{(\bar m)}|. \end{aligned} \tag{2.97}$$

From this and from the Theorem 2.1 follows the inequality

$$\left(1 - Mh^2\right)\left|\lambda - \lambda^{(\bar m)}\right| \le M_2 h^{\bar m},$$

from which, for sufficiently small h, the estimate (2.95) is derived.

To prove the estimate (2.94), let us initially note that according to the summation by parts formula and to the relations (2.86), (2.95) we obtain

$$\begin{aligned} & \left|\left(\rho(\xi,\lambda), \left(y^{(\bar m)}\right)^2\right) - \left(\rho^{(\bar m)}(\xi,\lambda^{(\bar m)}), \left(y^{(\bar m)}\right)^2\right)\right| \\ & \quad \le \left|\left(\rho(\xi,\lambda) - \rho^{(\bar m)}(\xi,\lambda), \left(y^{(\bar m)}\right)^2\right)\right| \\ & \quad + \left|\left(\rho^{(\bar m)}(\xi,\lambda) - \rho^{(\bar m)}(\xi,\lambda^{(\bar m)}), \left(y^{(\bar m)}\right)^2\right)\right| \\ & \quad \le \sum_{\xi\in\hat\omega_h^+} h^{\bar m+1}(\xi)k(\xi)\left|\tilde\psi_1^j(\xi)\right|\left(y^{(\bar m)}(\xi)\right)^2_{\bar\xi} + Mh^2|\lambda - \lambda^{(\bar m)}| \\ & \quad + O\left(h^{\bar m}\right) \le M_5 h^{\bar m}. \end{aligned} \tag{2.98}$$

Thus, based on the inequalities (2.96) – (2.98) and the Theorem 2.1, we get the estimate (2.94). □

2.5.2 Newton's Iterative Method for Finding the Eigenvalues and Eigenvectors

The problem (2.80) – (2.83) can be considered as a system of equations with N unknowns $y_j^{(\bar{m})}$, $j = 1, 2, ..., N-1$, $\lambda^{(\bar{m})}$. This system is non-linear due to the terms $\lambda^{(\bar{m})} y_j^{(\bar{m})} \rho_j^{(\bar{m})}$. To find the solution of difference scheme (2.80) – (2.83), we apply the Newton's iterative method. Linearizing (2.80), we will write the Newton's iterative method as

$$\begin{aligned}&\left(a^{(\bar{m})} \nabla y_{\bar{x}}^{(\bar{m},s)}\right)_{\hat{x},j} - \left(d_j^{(\bar{m})} - \lambda^{(\bar{m},s)} \rho_j^{(\bar{m})}\right) \nabla y_j^{(\bar{m},s)} + \nabla\lambda^{(\bar{m},s)} \rho_j^{(\bar{m})} y_j^{(\bar{m},s-1)} \\ &= -\left(a^{(\bar{m})} y_{\bar{x}}^{(\bar{m},s-1)}\right)_{\hat{x},j} + \left(d_j^{(\bar{m})} - \lambda^{(\bar{m},s-1)} \rho_j^{(\bar{m})}\right) y_j^{(\bar{m},s-1)}, \end{aligned} \tag{2.99}$$

$$\begin{aligned}&\nabla y_0^{(\bar{m},s)} = \nabla y_N^{(\bar{m},s)} = 0, \quad \lambda^{(\bar{m},s)} = \lambda^{(\bar{m},s-1)} + \nabla\lambda^{(\bar{m},s)}, \\ &y_j^{(\bar{m},s)} = y_j^{(\bar{m},s-1)} + \nabla y_j^{(\bar{m},s)}, \quad j = 1, 2, ..., N-1, \quad s = 1, 2, ...,\end{aligned} \tag{2.100}$$

where

$$\begin{aligned}&a_j^{(\bar{m})} = a^{(\bar{m})}(x_j, \lambda^{(\bar{m},s-1)}), \\ &d_j^{(\bar{m})} = d^{(\bar{m})}(x_j, \lambda^{(\bar{m},s-1)}), \quad \rho_j^{(\bar{m})} = \rho^{(\bar{m})}(x_j, \lambda^{(\bar{m},s-1)}).\end{aligned}$$

The convergence of Newton's iterative method (2.99) – (2.100) is proved in [37, 36].

The initial approximation $\lambda^{(\bar{m},0)}, y_j^{(\bar{m},0)}, j = 1, 2, ..., N-1$ can be found using approximate methods, for example, the Galerkin method, or in applied problems, it can often be obtained from physical considerations.

The system (2.99) – (2.100) contains $N-1$ equations, linear with respect to N unknown $\nabla y_j^{(\bar{m},s)}$, $j = 1, 2, ..., N-1$, $\nabla\lambda^{(\bar{m},s)}$. Since the eigenvector is defined up to a constant multiplier, then in order for the system (2.99) – (2.100) to have a unique solution, one can assume, for example, $\nabla y_1^{(\bar{m},s)} = 0$ or $\nabla y_{N-1}^{(\bar{m},s)} = 0$.

If we divide each component of the found solution $y_j^{(\bar{m})}$, $j = 1, 2, ..., N-1$ by the value

$$\pm\left(\rho^{(\bar{m})}, (y^{(\bar{m})})^2\right)^{1/2} = \pm\left[\sum_{j=1}^{N-1} \hbar_j \rho_j^{(\bar{m})} \left(y_j^{(\bar{m})}\right)^2\right]^{1/2},$$

then we get the normalized eigenfunctions of the problem (2.80) – (2.83). The normalization condition $\left(\rho^{(\bar{m})}, (y^{(\bar{m})})^2\right) = 1$ defines the eigenfunction up to the sign. To uniquely determine the eigenfunctions, we introduce an additional condition for selecting the sign, for example, $y_{x,0}^{(\bar{m})} > 0$.

The proposed iterative process (2.99) – (2.100) is especially advantageous for the case of three-point difference schemes, since the matrix of the system (2.99) – (2.100) in structure differs from a tridiagonal matrix by adding only one non-zero column. The elimination method for complicated systems (see [77, p. 81]) can be used to solve the system of linear equations (2.99) – (2.100). Solving this system of linear equations by the elimination method requires $15N-19$ arithmetic operations. The iterative process (2.99) – (2.100) is useful for differential equations, for which the Cauchy problems (in shooting method) are unstable, as it corrects this instability.

2.6 Numerical Examples

Example 2.1. *Let us solve the Sturm-Liouville problem*

$$u'' + \lambda x u = 0, \quad u(0) = u(1) = 0. \tag{2.101}$$

Note that for this problem $k(x) = 1$, $q(x) = 0$, $r(x) = x$. *The exact solution of this problem (see [61]) are the eigenvalues*

$$\lambda_n = \left(\frac{3}{2} j_{1/3,n}\right)^2, \quad n = 1, 2, \ldots$$

and the corresponding eigenfunctions

$$u_n(x) = \sqrt{x} I_{1/3}\left(\frac{2}{3}\sqrt{\lambda_n} x^{3/2}\right), \quad n = 1, 2, \ldots,$$

where

$$I_v(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{v+2k}}{k!\Gamma(v+k+1)}$$

are Bessel functions of the first kind, $j_{v,n}$ *are zeros of the Bessel function* $I_v(x)$ *(see, for example, [35]).*

For numerical solving of the problem (2.101)*, we use the three-point difference scheme of the 6th order of accuracy (*$m = \bar{m} = 6$*) on a uniform grid* $\omega_h = \{x_j = jh,\ j = 1, 2, \ldots, N-1,\ h = 1/N\}$. *Auxiliary Cauchy problems* (2.59), (2.60) *will be solved by the Runge-Kutta method of the 6th order of accuracy (see, for example, [33, p. 202]). The results of solving the Sturm-Liouville problem* (2.101) *are given in Table 2.1. For practical estimation of the convergence rate, the following values are used*

$$err_h = \max\left\{ \left\|y^{(\bar{m})} - u\right\|_{C(\omega_h)}, \left|\lambda^{(\bar{m})h} - \lambda\right| \right\}, \quad p = \log_2 \frac{err_h}{err_{h/2}}.$$

Table 2.1: The results of solving the Sturm-Liouville problem (2.101) by three-point difference scheme of the 6th order of accuracy

n	N	err_h	p	n	N	err_h	p
1	8	$1.9474 \cdot 10^{-5}$		4	8	$8.5503 \cdot 10^{-1}$	
	16	$2.4132 \cdot 10^{-7}$	6.3		16	$1.0831 \cdot 10^{-2}$	6.3
	32	$3.5412 \cdot 10^{-9}$	6.1		32	$1.7645 \cdot 10^{-4}$	5.9
	64	$5.4396 \cdot 10^{-11}$	6.0		64	$2.7577 \cdot 10^{-6}$	6.0
	128	$6.9988 \cdot 10^{-13}$	6.3		128	$4.3071 \cdot 10^{-8}$	6.0
2	8	$9.7884 \cdot 10^{-4}$		5	8	$1.3747 \cdot 10^{1}$	
	16	$1.0359 \cdot 10^{-5}$	6.6		16	$7.7833 \cdot 10^{-2}$	7.5
	32	$2.3159 \cdot 10^{-7}$	5.5		32	$1.1891 \cdot 10^{-3}$	6.0
	64	$3.8675 \cdot 10^{-9}$	5.9		64	$1.8220 \cdot 10^{-5}$	6.0
	128	$6.1036 \cdot 10^{-11}$	6.0		128	$2.8315 \cdot 10^{-7}$	6.0
3	8	$1.9499 \cdot 10^{-2}$					
	16	$7.9954 \cdot 10^{-4}$	4.6				
	32	$1.3962 \cdot 10^{-5}$	5.8				
	64	$2.2239 \cdot 10^{-7}$	6.0				
	128	$3.4913 \cdot 10^{-9}$	6.0				

For comparison, see Table 2.2, which shows the results of solving the problem (2.101) *using a difference scheme* (1.3) *of second order of accuracy (see, e.g., [72]). Note that for* $n = 4, 5$ *this scheme failed to solve the problem.*

Example 2.2. *Consider the Sturm-Liouville problem*

$$u'' - e^{x} \cdot u + \lambda u = 0, \quad u(0) = u(\pi) = 0 \tag{2.102}$$

with its known eigenvalues $\lambda_1 = 4.896669380$, $\lambda_2 = 10.045189893$, $\lambda_3 = 16.019267250$, $\lambda_4 = 23.266270940$, $\lambda_5 = 32.263707046$, $\lambda_6 = 43.220019641, \ldots$ *(see [70, p. 278]). For numerical solving of the problem we use the three-point difference scheme of the 6th order of accuracy (*$\bar{m} = 6$*) on a uniform grid* $\bar{\omega}_h$*. The results of solving the problem* (2.102) *are given in Table 2.3, where* $p = \log_2 \frac{|\lambda^{(\bar{m})h} - \lambda|}{|\lambda^{(\bar{m})h/2} - \lambda|}$*. Table 2.4 shows the results of solving the problem* (2.102) *using a difference scheme of the second order of accuracy.*

Table 2.2: The results of solving the Sturm-Liouville problem (2.101) by a scheme of the second order of accuracy

n	N	err_h	p
1	8	$2.7885 \cdot 10^{-1}$	
	16	$7.0072 \cdot 10^{-2}$	2.0
	32	$1.7539 \cdot 10^{-2}$	2.0
	64	$4.3860 \cdot 10^{-3}$	2.0
	128	$1.0966 \cdot 10^{-3}$	2.0
2	8	5.1583	
	16	1.3050	2.0
	32	$3.2713 \cdot 10^{-1}$	2.0
	64	$8.1835 \cdot 10^{-2}$	2.0
	128	$2.0462 \cdot 10^{-2}$	2.0
3	8	$2.7165 \cdot 10^{1}$	
	16	6.9476	2.0
	32	1.7454	2.0
	64	$4.3689 \cdot 10^{-1}$	2.0
	128	$1.0926 \cdot 10^{-1}$	2.0

Example 2.3. *Consider the problem*

$$\begin{aligned} &\frac{d}{dx}\left[(2\alpha x+1)^2\frac{du}{dx}\right] = -\lambda u, \quad x \in (0,1), \\ &u(0) = u(1) = 0, \quad \alpha = (e^2-1)/2, \end{aligned} \tag{2.103}$$

with its exact solution

$$\lambda_n = (1+\pi^2 n^2)\alpha^2, \quad u_n(x) = \sqrt{\frac{2\alpha}{2\alpha x+1}}\sin\left(\frac{n\pi}{2}\log(2\alpha x+1)\right)$$

as described in [65, p. 131]. For the numerical solution of the problem with a given tolerance ε, we use a three-point difference scheme of the 6th order of accuracy on a uniform grid $\omega_h = \{x_j = jh, j = 1,2,...,N-1, h = 1/N\}$. For the practical assessment of accuracy, the Richardson extrapolation is used.

Table 2.3: The results of solving the Sturm-Liouville problem (2.102) by the three-point difference scheme of the 6th order of accuracy

n	N	$\left\|\lambda^{(6)h}-\lambda\right\|$	p	n	N	$\left\|\lambda^{(6)h}-\lambda\right\|$	p
1	8	$4.7367\cdot10^{-5}$		4	8	$3.0186\cdot10^{-2}$	
	16	$1.0382\cdot10^{-6}$	5.5		16	$3.3307\cdot10^{-4}$	6.5
	32	$1.7357\cdot10^{-8}$	5.9		32	$4.8131\cdot10^{-6}$	6.1
	64	$2.7563\cdot10^{-10}$	6.0		64	$7.3784\cdot10^{-8}$	6.0
	128	$4.3583\cdot10^{-12}$	6.0		128	$1.1474\cdot10^{-9}$	6.0
2	8	$4.1773\cdot10^{-4}$		5	8	$1.9775\cdot10^{-1}$	
	16	$7.4713\cdot10^{-6}$	5.8		16	$1.6911\cdot10^{-3}$	6.9
	32	$1.1915\cdot10^{-7}$	6.0		32	$2.3241\cdot10^{-5}$	6.2
	64	$1.8695\cdot10^{-9}$	6.0		64	$3.5205\cdot10^{-7}$	6.0
	128	$2.9191\cdot10^{-11}$	6.0		128	$5.4586\cdot10^{-9}$	6.0
3	8	$3.9796\cdot10^{-3}$		6	8	1.1452	
	16	$5.3783\cdot10^{-5}$	6.2		16	$7.0488\cdot10^{-3}$	7.3
	32	$8.0971\cdot10^{-7}$	6.0		32	$9.1218\cdot10^{-5}$	6.3
	64	$1.2533\cdot10^{-8}$	6.0		64	$1.3628\cdot10^{-6}$	6.1
	128	$1.9539\cdot10^{-10}$	6.0		128	$2.1058\cdot10^{-8}$	6.0

Specifically, if the condition

$$\max\left\{\left\|y_N^{(6)}-y_{2N}^{(6)}\right\|_{C(\omega_h)},\ \frac{\left|\lambda_N^{(6)}-\lambda_{2N}^{(6)}\right|}{\lambda_{2N}^{(6)}}\right\}\leq 63\varepsilon$$

is satisfied, then the required tolerance ε *is considered as achieved. Otherwise, the number of grid points* N *is doubled. Here,* $y_N^{(6)}, \lambda_N^{(6)}$ *denote the solution of the 6th-order difference scheme on the grid* $\{x_1, x_2, \ldots, x_N\}$*, while* $y_{2N}^{(6)}, \lambda_{2N}^{(6)}$ *denote the solution of the 6th-order difference scheme on the grid* $\{x_1, x_2, \ldots, x_{2N}\}$*. If the accuracy is achieved, the solution can be further*

Table 2.4: The results of solving the Sturm-Liouville problem (2.102) by a scheme of the second order of accuracy

n	N	$\left\|\lambda^{(2)h} - \lambda\right\|$	p	n	N	$\left\|\lambda^{(2)h} - \lambda\right\|$	p
1	8	$7.2665 \cdot 10^{-2}$		3	8	1.4123	
	16	$1.8033 \cdot 10^{-2}$	2.0		16	$3.4350 \cdot 10^{-1}$	2.0
	32	$4.4997 \cdot 10^{-3}$	2.0		32	$8.5409 \cdot 10^{-2}$	2.0
	64	$1.1244 \cdot 10^{-3}$	2.0		64	$2.1325 \cdot 10^{-2}$	2.0
	128	$2.8106 \cdot 10^{-4}$	2.0		128	$5.3294 \cdot 10^{-3}$	2.0
2	8	4.4530		4	8	3.5956	
	16	$1.0857 \cdot 10^{-1}$	2.0		16	$8.9482 \cdot 10^{-1}$	2.0
	32	$2.6981 \cdot 10^{-2}$	2.0		32	$2.2431 \cdot 10^{-1}$	2.0
	64	$6.7354 \cdot 10^{-3}$	2.0		64	$5.6125 \cdot 10^{-2}$	2.0
	128	$1.6832 \cdot 10^{-3}$	2.0		128	$1.4034 \cdot 10^{-2}$	2.0

refined using the Richardson extrapolation

$$\hat{y}_N(x_j) = y^{(6)}_{2N}(x_{2j}) + \frac{y^{(6)}_{2N}(x_{2j}) - y^{(6)}_N(x_j)}{63}, \quad j = 1, 2, ..., N,$$

$$\hat{\lambda}_N = \lambda^{(6)}_{2N} + \frac{\lambda^{(6)}_{2N} - \lambda^{(6)}_N}{63}.$$

The iterations in Newton's method stopped if

$$\max\left\{ \left\| y_N^{(6,s)} - y_N^{(6,s-1)} \right\|_{C(\omega_h)}, \frac{\left| \lambda_N^{(6,s)} - \lambda_N^{(6,s-1)} \right|}{\lambda_N^{(6,s)}} \right\} \leq 0.5\varepsilon$$

where $s = 1, 2, ..., 7$ *is the iteration number. In addition, at each iteration, it is checked whether the Euclidean norm of the residual decreases. The results of solving the problem* (2.103) *with a given tolerance* ε *using difference scheme of the 6th order of accuracy for the fourth eigenvalue and the corresponding eigenfunction are presented in Table 2.5, where*

$$err_1 = \frac{\left|\lambda^{(6)h} - \lambda\right|}{\lambda}, \qquad err_2 = \left\| y^{(6)} - u \right\|_{C(\omega_h)}.$$

Table 2.5: The results of solving the problem (2.103) using the difference scheme of the 6th order of accuracy with a given tolerance ε

n	ε	N	err_1	err_2	$time(sec)$
4	10^{-4}	32	$0.321 \cdot 10^{-7}$	$0.242 \cdot 10^{-5}$	0.4
	10^{-6}	64	$0.665 \cdot 10^{-9}$	$0.100 \cdot 10^{-7}$	0.8
	10^{-8}	256	$0.561 \cdot 10^{-13}$	$0.623 \cdot 10^{-12}$	2.7

Example 2.4. *We consider the Sturm-Liouville problem*

$$u'' - \frac{1}{(x+0.1)^2} \cdot u = -\lambda u, \quad x \in (0,\pi), \quad u(0) = u(\pi) = 0, \tag{2.104}$$

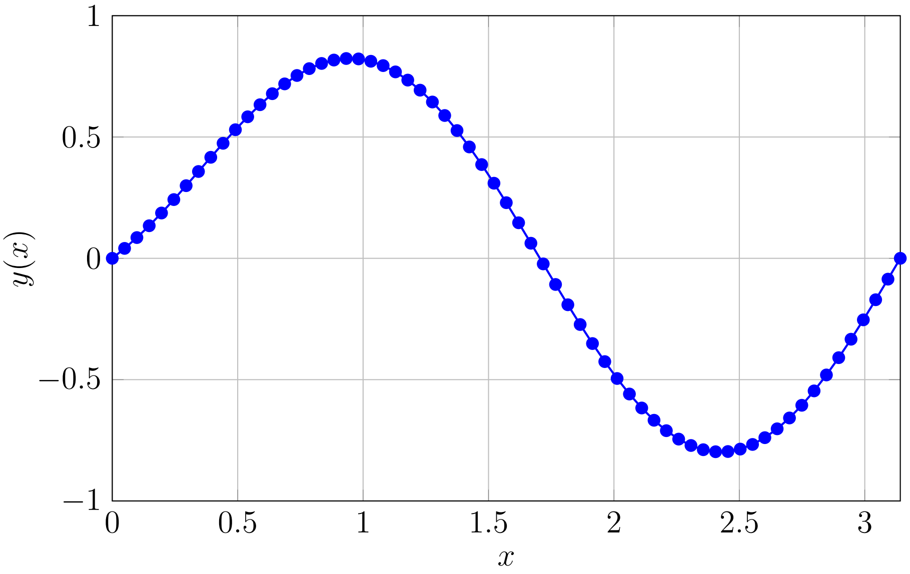
with its known eigenvalues: $\lambda_1 = 1.5198658211$, $\lambda_2 = 4.9433098221$, $\lambda_3 = 10.284662645$, $\lambda_4 = 17.559957746$, $\lambda_5 = 26.782863158$, $\lambda_6 = 37.964425862$, $\ldots$ *(see [70, p. 278]). The results of solving the problem* (2.104) *with a given tolerance* ε *using the difference scheme of the 6th order of accuracy are presented in Table 2.6.*

Table 2.6: The results of solving the problem (2.104) using the difference scheme of the 6th order of accuracy with a given tolerance ε

n	ε	N	err_1	$time(sec)$
2	10^{-4}	16	$0.190 \cdot 10^{-5}$	0.15
	10^{-6}	32	$0.154 \cdot 10^{-7}$	0.3
	10^{-8}	64	$0.112 \cdot 10^{-10}$	0.67

Thus, the numerical results confirm the theoretical conclusions about the 6th order of accuracy of the difference scheme and also confirm the effectiveness of the proposed approach.

Figure 2.1: The second eigenfunction of the problem (2.104) for $N = 64$



Chapter 3

High-Order Difference Schemes for Singular Sturm-Liouville Problem

In this chapter, we develop a new algorithmic realization of the exact three-point difference schemes for the Sturm-Liouville problem with the singularity at the ends of the segment $[-1, 1]$. We show that to compute the coefficients of the exact difference scheme in an arbitrary grid node x_j, it is necessary to solve two auxiliary Cauchy problems for the system of three linear ordinary differential equations of the first order: one problem on the interval $[x_{j-1}, x_j]$ (forward) and one problem on the interval $[x_j, x_{j+1}]$ (backward). The coefficient stability of the exact three-point difference scheme is proved. Three-point difference schemes of high-order of accuracy have been constructed and justified. Numerical experiments were conducted that confirm the theoretical conclusions.

3.1 Exact Three-Point Difference Scheme

We consider the following singular Sturm-Liouville problem

$$\frac{d}{dx}\left[k(x)\frac{du}{dx}\right] - q(x)u(x) = -\lambda r(x)u(x), \quad x \in (-1, 1), \tag{3.1}$$

$$u(-1) \neq \infty, \quad u(1) \neq \infty, \tag{3.2}$$

where

$$\begin{aligned} &k(x) = (1 - x^2)k_1(x), \quad 0 < C_1 \leq k_1(x) \leq C_2, \\ &0 < C_3 \leq q(x) \leq C_4, \quad 0 < C_5 \leq r(x) \leq C_6, \end{aligned} \tag{3.3}$$

C_i, $i = 1, 2, ..., 6$ are constants, $k(x), q(x), r(x)$ are piecewise continuous functions with a finite number of discontinuity points of the first kind.

Problems of this kind arise in applications when partial differential equations in spherical coordinates are solved using separation of variables (see, e.g., [79, pp. 167 – 170]).

Remark 3.1. *Boundary conditions* (3.2) *can be replaced (see [81, pp. 628 – 631]) with conditions*

$$\lim_{x\to -1} k(x)\frac{du}{dx} = 0, \quad \lim_{x\to 1} k(x)\frac{du}{dx} = 0.$$

Indeed, the general solution of the equation (3.1) *has the following form*

$$u(x) = Au_1(x) + Bu_2(x),$$

where u_1, u_2 *are any linearly independent solutions of the equation* (3.1)*,* A, B *are arbitrary constants. If the conditions* (3.2) *are satisfied, then* $u_2(x)$ *for* $x \to \pm 1$ *becomes infinite (see Lemma 1, [81, p. 628]). Therefore, it follows that* $B = 0$*. Integrating equation* (3.1) *from some value* x_1 $(x < x_1)$*, we obtain*

$$k(x)\frac{du_1}{dx} = k(x)\frac{du_1}{dx}\bigg|_{x=x_1} - \int_x^{x_1} [q(t) - \lambda r(t)]u_1(t)dt = Q(x).$$

Hence it follows that $Q(x)$ *is continuous function on* $[-1, x_1]$*. Passing to a limit as* $x \to -1$*, we see that there exists the limit*

$$\lim_{x\to -1} k(x)\frac{du_1}{dx} = C = Q(-1).$$

Let us prove that $C = Q(-1) = 0$*. Expressing function* $u_1(x)$ *by* $Q(x)$*, we find*

$$\begin{aligned} u_1(x) &= u_1(x_2) - \int_x^{x_2} \frac{Q(t)}{k(t)}dt \\ &= u_1(x_2) - \int_x^{x_2} \frac{Q(t)}{(1-t^2)k_1(t)}dt, \quad -1 \le x \le x_2. \end{aligned}$$

From this formula we see immediately that if $Q(-1) \neq 0$*, then as* $x \to -1, u_1(x) \to \infty$*, which contradicts the boundary condition of* $u_1(x)$ *at* $x = -1$*. Thus,* $C = Q(-1) = 0$ *and*

$$\lim_{x\to -1} k(x)\frac{du_1}{dx} = 0.$$

Similarly, it can be proved that

$$\lim_{x\to 1} k(x)\frac{du_1}{dx} = 0.$$

If function $k(x)$ has a discontinuity of the first kind at the point $x = \xi, -1 < \xi < 1$, then the following continuity conditions are satisfied at this point:

$$u(\xi+0) - u(\xi-0) = 0, \quad k(x)\frac{du}{dx}\bigg|_{x=\xi-0} = k(x)\frac{du}{dx}\bigg|_{x=\xi+0}.$$

For simplicity, we restrict ourselves to the uniform grid

$$\omega_h := \left\{x_j = -1 + (j-0.5)h,\ h = 2/N,\ j = 1,2,\ldots,N\right\},$$

$$x_0 = -1,\ x_{N+1} = 1,$$

and one discontinuity point ξ at the jth node x_j of the grid.

Let us define the pattern functions $v_\alpha^j(x,\lambda)$, $\alpha = 1,2$, $j = 1,2,...,N$ as the solutions of the following Cauchy problems

$$\begin{aligned}
&\frac{d}{dx}\left[k(x)\frac{dv_1^1}{dx}\right] - q(x)v_1^1(x,\lambda) + \lambda r(x)v_1^1(x,\lambda) = 0, \quad x \in (x_0, x_2),\\
&v_1^1(x_0,\lambda) = 1, \quad k(x)\frac{dv_1^1(x,\lambda)}{dx}\bigg|_{x=x_0} = 0,
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&\frac{d}{dx}\left[k(x)\frac{dv_\alpha^j}{dx}\right] - q(x)v_\alpha^j(x,\lambda) + \lambda r(x)v_\alpha^j(x,\lambda) = 0, \quad x \in (x_{j-1}, x_{j+1}),\\
&v_\alpha^j(x_{j+(-1)^\alpha},\lambda) = 0, \quad k(x)\frac{dv_\alpha^j(x,\lambda)}{dx}\bigg|_{x=x_{j+(-1)^\alpha}} = (-1)^{\alpha+1},\\
&\alpha = 1,2, \quad j = 3-\alpha, 4-\alpha, \ldots, N+1-\alpha,
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\frac{d}{dx}\left[k(x)\frac{dv_2^N}{dx}\right] - q(x)v_2^N(x,\lambda) + \lambda r(x)v_2^N(x,\lambda) = 0,\ x \in (x_{N-1}, x_{N+1}),\\
&v_2^N(x_{N+1},\lambda) = 1, \quad k(x)\frac{dv_2^N(x,\lambda)}{dx}\bigg|_{x=x_{N+1}} = 0.
\end{aligned} \tag{3.6}$$

Similarly to [60], we also define the properties of the pattern functions.

Lemma 3.1. *The functions $v_\alpha^j(x,\lambda)$, $\alpha = 1,2$ have the following properties:*

$$v_1^j(x_{j+1},\lambda) = v_2^j(x_{j-1},\lambda), \quad j = 2,3,...,N-1, \tag{3.7}$$

$$v_2^j(x_j,\lambda) = v_1^{j+1}(x_{j+1},\lambda), \quad j = 1,2,...,N-1, \tag{3.8}$$

$$\begin{aligned}
v_1^1(x_2,\lambda) = v_1^1(x_1,\lambda) &+ v_2^1(x_1,\lambda)\int_{x_0}^{x_1} v_1^1(\xi,\lambda)[q(\xi) - \lambda r(\xi)]\,d\xi\\
&+ v_1^1(x_1,\lambda)\int_{x_1}^{x_2} v_2^1(\xi,\lambda)[q(\xi) - \lambda r(\xi)]\,d\xi,
\end{aligned} \tag{3.9}$$

$$\begin{aligned} v_1^j(x_{j+1},\lambda) &= v_1^j(x_j,\lambda) + v_2^j(x_j,\lambda) \\ &\quad + v_2^j(x_j,\lambda)\int_{x_{j-1}}^{x_j} v_1^j(\xi,\lambda)[q(\xi)-\lambda r(\xi)]\,d\xi \\ &\quad + v_1^j(x_j,\lambda)\int_{x_j}^{x_{j+1}} v_2^j(\xi,\lambda)[q(\xi)-\lambda r(\xi)]\,d\xi, \\ &\quad j = 2,...,N-1, \end{aligned} \tag{3.10}$$

$$\begin{aligned} v_2^N(x_{N-1},\lambda) &= v_2^N(x_N,\lambda) + v_2^N(x_N,\lambda)\int_{x_{N-1}}^{x_N} v_1^N(\xi,\lambda)[q(\xi)-\lambda r(\xi)]\,d\xi \\ &\quad + v_1^N(x_N,\lambda)\int_{x_N}^{x_{N+1}} v_2^N(\xi,\lambda)[q(\xi)-\lambda r(\xi)]\,d\xi. \end{aligned} \tag{3.11}$$

$$\begin{aligned} &v_\alpha^j(x,\lambda) > 0,\ \alpha = 1,2\ \textit{for all}\ x \in (x_{j-1},x_{j+1}),\ j = 1,2,...,N, \\ &v_\alpha^j(x,\lambda),\ \alpha = 1,2\ \textit{are linearly independent.} \end{aligned} \tag{3.12}$$

Proof. The proof of the equalities (3.7) – (3.11) is similar to the regular case (see Lemma 2.1).

We now prove that the functions $v_\alpha^j(x,\lambda), \alpha = 1,2$ for $j = 2,3,...,N-1$ are linearly independent. As it is known, for the linear independence of the solutions of the problem (3.5) it is necessary and sufficient, that the Wronskian should be different from zero at least in one point of the interval $[x_{j-1},x_{j+1}]$. Let us assume the contrary for $j = 2,3,...,N-1$. Then the Wronskian $W[v_1^j(x,\lambda), v_2^j(x,\lambda)]$ is identically equal to zero on the interval $[x_{j-1},x_{j+1}]$. Calculating the Wronskian at the points $x_{j+(-1)^\alpha}$, $\alpha = 1,2$ and taking into account the fact that $v_2^j(x_{j-1},\lambda) = v_1^j(x_{j+1},\lambda)$, we obtain

$$W[v_1^j(x,\lambda), v_2^j(x,\lambda)]_{x=x_{j+(-1)^\alpha}} = -\frac{v_1^j(x_{j+1},\lambda)}{k(x_{j+(-1)^\alpha})}.$$

It follows that $v_1^j(x_{j+1},\lambda) = 0$, i.e., $v_1^j(x,\lambda)$ is the solution of boundary value problem

$$\begin{aligned} &\frac{d}{dx}\left[k(x)\frac{dv_1^j}{dx}\right] - q(x)v_1^j(x,\lambda) + \lambda r(x)v_1^j(x,\lambda) = 0, \quad x \in (x_{j-1},x_{j+1}), \\ &v_1^j(x_{j-1},\lambda) = v_1^j(x_{j+1},\lambda) = 0, \quad j = 2,3,\dots,N-1. \end{aligned} \tag{3.13}$$

We now show that for sufficiently small $h < h_0$ and for $\lambda = \lambda_m$, $1 \le m \le k$, $k \ll N$, problem (3.13) has only the trivial solution. For this purpose, it is sufficient to show that for $h < h_0$ the following inequality is satisfied:

$$-[q(x) - \lambda r(x)] \le \lambda r(x) < \underline{\mu}_1 \quad \forall x \in [x_{j-1},x_{j+1}],$$

where $\underline{\mu}_1$ is the lower estimate of the smallest eigenvalue of the problem

$$\begin{aligned}&\frac{d}{dx}\left[k(x)\frac{dv}{dx}\right]+\mu v(x)=0, \quad x\in(x_{j-1},x_{j+1}),\\ &v(x_{j-1})=v(x_{j+1})=0.\end{aligned}$$

This problem is known (see [20, 18]) to be equivalent to the variational problem of finding the minimum of the functional

$$\min\int_{x_{j-1}}^{x_{j+1}} k(\xi)[v'(\xi)]^2\,d\xi$$

under condition

$$\|v\|^2=\int_{x_{j-1}}^{x_{j+1}} v^2(x)\,dx=1.$$

Considering that $h<1$, $x_{j-1}\le\xi\le x_{j+1}$ for $j=2,3,...,N-1$,

$$k(\xi)\ge C_1(1-\xi^2)\ge C_1[1-(-1+0,5h)^2]=C_1\left[h-\frac{h^2}{4}\right]>\frac{3}{4}hC_1$$

and $\min\int_{x_{j-1}}^{x_{j+1}}[v'(\xi)]^2\,d\xi=\frac{\pi^2}{4h^2}$, we obtain $\mu_1>\frac{3C_1\pi^2}{16h}=\underline{\mu}_1$. Hence, there exists h_0 such that for all $h<h_0$ the inequality

$$h\lambda r(x)<\frac{3C_1\pi^2}{16}\quad\forall x\in[x_{j-1},x_{j+1}],\quad j=2,3,\ldots,N-1$$

is satisfied. Consequently, it follows that $v_1^j(x,\lambda)\equiv 0$, $x\in[x_{j-1},x_{j+1}]$ for $h<h_0=\frac{3C_1\pi^2}{16\lambda C_6}$, which contradicts the condition $k(x)\frac{dv_1^j(x,\lambda)}{dx}\Big|_{x=x_{j-1}}=1$.

Similar considerations show that $v_1^j(x,\lambda)\ne 0$ holds in any point of the interval $(x_{j-1},x_{j+1}]$, i.e., our function is of constant-sign on this interval. Thus, $v_1^j(x,\lambda)$, $v_2^j(x,\lambda)$, $j=2,3,...,N-1$ are linearly independent on the interval $[x_{j-1},x_{j+1}]$.

For $j=1$ at $\lambda=\lambda_m$, $m=1,2,...,k$, $k<<N$, we have $v_1^1(x,\lambda)=c_m u_m(x)$ where $u_m(x)$ is the eigenfunction of the problem (3.1), (3.2) which corresponds to the eigenvalue λ_m. We denote by $x_{\min}^m$ the minimum, and by $x_{\max}^m$ the maximum zero of the function $u_m(x)$ on the interval $(-1,1)$. If we choose $h<h_1=\frac{2}{3}(1+x_{\min}^m)$, then $x_2=-1+1.5h<-1+(1+x_{\min}^m)=x_{\min}^m$ is obtained. Hence, $v_1^1(x,\lambda)\ne 0$, $x\in[-1,x_2]$. From the fact that $v_1^1(x_2,\lambda)\ne 0$, follows the linear independence of the functions $v_1^1(x,\lambda)$, $v_2^1(x,\lambda)$. Similarly, for $j=N$ we get that for $h<h_2=\frac{2}{3}(1-x_{\max}^m)$ the inequality $v_2^N(x,\lambda)\ne 0$, $x\in[x_{N-1},1]$ is satisfied (then $x_{N-1}=-1+(N-1.5)h=1-1.5h>$

$1-(1-x_{\max}^m) = x_{\max}^m$) and therefore $v_1^N(x,\lambda)$ and $v_2^N(x,\lambda)$ are linearly independent.

Since from (3.4), (3.5)

$$v_1^1(x,\lambda) = 1 + \int_{x_0}^{x} \frac{1}{k(t)} \int_{x_0}^{t} (q(\xi) - \lambda r(\xi)) v_1^1(\xi,\lambda)\, d\xi\, dt,$$

$$\begin{aligned} v_1^j(x,\lambda) &= \int_{x_{j-1}}^{x} \frac{1}{k(t)} \left(1 + \int_{x_{j-1}}^{t} (q(\xi) - \lambda r(\xi)) v_1^j(\xi,\lambda)\, d\xi\right) dt, \\ j &= 2, ..., N, \end{aligned} \tag{3.14}$$

then according to (3.2) and to the mean value theorem, there exists a point $\bar{x} \in (x_{j-1}, x)$ such that

$$\begin{aligned} v_1^1(x,\lambda) &\geq 1 - \int_{x_0}^{x} \frac{1}{k(t)} \int_{x_0}^{t} |q(\xi) - \lambda r(\xi)| \left|v_1^1(\xi,\lambda)\right| d\xi\, dt \\ &\geq 1 - \frac{C_4 + \lambda C_6}{C_1(1-\bar{x}^2)}(x - x_0) \int_{x_0}^{\bar{x}} \left|v_1^1(\xi,\lambda)\right| d\xi, \end{aligned}$$

$$\begin{aligned} v_1^j(x,\lambda) &\geq \int_{x_{j-1}}^{x} \frac{1}{k(t)} \left(1 - \int_{x_{j-1}}^{t} |q(\xi) - \lambda r(\xi)| \left|v_1^j(\xi,\lambda)\right| d\xi\right) dt \\ &\geq \int_{x_{j-1}}^{x} \frac{dt}{k(t)} \left(1 - (C_4 + \lambda C_6) \int_{x_{j-1}}^{x} \left|v_1^j(\xi,\lambda)\right| d\xi\right) \\ &= \int_{x_{j-1}}^{x} \frac{dt}{k(t)} \left(1 - (C_4 + \lambda C_6)(x - x_{j-1}) \left|v_1^j(\bar{x},\lambda)\right|\right). \end{aligned}$$

From these inequalities, it follows that $v_1^j(x,\lambda) > 0$, $j = 1, 2, ..., N$ on the interval $(x_{j-1}, x_{j-1} + \delta)$ for any small $\delta > 0$. Since the functions $v_1^j(x,\lambda)$, $j = 1, 2, ..., N$ are of constant-sign, they are positive on the entire interval (x_{j-1}, x_{j+1}). □

The following statement is true.

Lemma 3.2. *Suppose that the assumptions* (3.3) *are satisfied. Then, at*

$$h \leq h_0 = \frac{1}{2}\sqrt{\frac{C_1(1 - x_{j+1}^2)}{C_4 + \lambda C_6}} \tag{3.15}$$

the following assertions hold:

1. *The pattern functions have the properties: the functions* $v_1^j(x,\lambda)$, $j=2,3,...,N$ *increase monotonically on* $(x_{j-1},x_{j+1}]$, *and the functions* $v_2^j(x,\lambda)$, $j=1,2,...,N-1$ *decrease monotonically on* $[x_{j-1},x_{j+1})$;
2. *For all* $j=3-\alpha,4-\alpha,\dots,N+1-\alpha,\ \alpha=1,2,$ *it holds that*

$$\begin{aligned}\frac{2}{3C_2(1+x)(1-x_{j+(-1)^\alpha})} &\le \frac{v_\alpha^j(x,\lambda)}{\left|x-x_{j+(-1)^\alpha}\right|}\\ &\le \frac{2}{C_1(1-x)(1+x_{j+(-1)^\alpha})}.\end{aligned} \tag{3.16}$$

Proof. We only prove the assertions for the pattern function $v_1^j(x,\lambda)$ since those for the $v_2^j(x,\lambda)$ follow analogously.

Note that equation (3.14) together with assumptions (3.3) lead, for any bounded λ, to the inequality

$$\begin{aligned}v_1^j(x,\lambda) &\le \int_{x_{j-1}}^x \frac{dt}{1-t^2}\left[\frac{1}{C_1}+\frac{C_4+\lambda C_6}{C_1}\int_{x_{j-1}}^x v_1^j(t,\lambda)dt\right]\\ &= \frac{1}{2}\ln\left(1+\frac{2(x-x_{j-1})}{(1-x)(1+x_{j-1})}\right)\left[\frac{1}{C_1}+\frac{C_4+\lambda C_6}{C_1}\int_{x_{j-1}}^x v_1^j(t,\lambda)dt\right].\end{aligned}$$

Using the well-known inequality

$$\frac{r}{r+1}\le \ln(1+r)\le r, \tag{3.17}$$

which is true for $r\ge 0$, we thus obtain

$$v_1^j(x,\lambda)\le \frac{x-x_{j-1}}{(1-x)(1+x_{j-1})}\left[\frac{1}{C_1}+\frac{C_4+\lambda C_6}{C_1}\int_{x_{j-1}}^x v_1^j(t,\lambda)dt\right].$$

We now make the substitution

$$\bar{v}_1^j(x,\lambda)=\frac{v_1^j(x,\lambda)(1-x)(1+x_{j-1})}{x-x_{j-1}}$$

in order to obtain

$$\bar{v}_1^j(x,\lambda)\le \frac{1}{C_1}+\frac{C_4+\lambda C_6}{C_1(1+x_{j-1})}\int_{x_{j-1}}^x \frac{t-x_{j-1}}{1-t}\bar{v}_1^j(t,\lambda)dt.$$

Applying the Gronwall inequality (see, e.g., [34, p. 37]), we obtain

$$\bar{v}_1^j(x,\lambda)\le \frac{1}{C_1}e^{\frac{C_4+\lambda C_6}{C_1(1+x_{j-1})}\int_{x_{j-1}}^x \frac{t-x_{j-1}}{1-t}dt}$$

or, equivalently,

$$v_1^j(x,\lambda) \le \frac{1}{C_1}\frac{x-x_{j-1}}{(1-x)(1+x_{j-1})}e^{\frac{C_4+\lambda C_6}{C_1(1+x_{j-1})}\int_{x_{j-1}}^{x}\frac{t-x_{j-1}}{1-t}dt} \qquad \forall x \in [x_{j-1},x_{j+1}].$$

From the last estimate and the equality

$$k(x)\frac{dv_1^j(x,\lambda)}{dx} = 1 + \int_{x_{j-1}}^{x}(q(\xi)-\lambda r(\xi))v_1^j(\xi,\lambda)\,d\xi,$$

follows the inequality

$$\begin{aligned} k(x)\frac{dv_1^j(x,\lambda)}{dx} &\ge 1 - \int_{x_{j-1}}^{x}|q(\xi)-\lambda r(\xi)|\,v_1^j(\xi,\lambda)\,d\xi \\ &\ge 1 - \frac{C_4+\lambda C_6}{C_1(1+x_{j-1})}\int_{x_{j-1}}^{x}\frac{t-x_{j-1}}{1-t}e^{\frac{C_4+\lambda C_6}{C_1(1+x_{j-1})}\int_{x_{j-1}}^{t}\frac{\xi-x_{j-1}}{1-\xi}d\xi}dt \\ &\ge 2 - e^{\frac{(C_4+\lambda C_6)}{C_1(1+x_{j-1})}\int_{x_{j-1}}^{x}\frac{t-x_{j-1}}{1-t}dt}. \end{aligned}$$

Since

$$\int_{x_{j-1}}^{x}\frac{t-x_{j-1}}{1-t}dt = -(x-x_{j-1}) + (1-x_{j-1})\ln\left(1+\frac{x-x_{j-1}}{1-x}\right),$$

and due to inequality (3.17), we get

$$\int_{x_{j-1}}^{x}\frac{t-x_{j-1}}{1-t}dt \le -(x-x_{j-1}) + \frac{(x-x_{j-1})(1-x_{j-1})}{1-x} = \frac{(x-x_{j-1})^2}{1-x}.$$

Hence,

$$k(x)\frac{dv_1^j(x,\lambda)}{dx} \ge 2 - e^{\frac{(C_4+\lambda C_6)(x-x_{j-1})^2}{C_1(1+x_{j-1})(1-x)}} \ge 2 - e^{\frac{4(C_4+\lambda C_6)h^2}{C_1(1-x_{j+1}^2)}}$$

holds, which proves that under condition (3.15) the function $v_1^j(x,\lambda)$ increases monotonically on $(x_{j-1},x_{j+1}]$ given that the function $g(t)=2-e^t$ decreases monotonically and $g(1/2)>0$.

Returning to the equality (3.14), we obtain with the help of the proved assertion 1 that

$$\begin{aligned} v_1^j(x,\lambda) &\le \frac{1}{C_1}\int_{x_{j-1}}^{x}\frac{dt}{1-t^2} + \frac{C_4+\lambda C_6}{2C_1}v_1^j(x,\lambda)\int_{x_{j-1}}^{x}\frac{t-x_{j-1}}{1-t^2}dt \\ &= \frac{1}{2C_1}\ln\left(1+\frac{2(x-x_{j-1})}{(1-x)(1+x_{j-1})}\right) + \frac{C_4+\lambda C_6}{4C_1}v_1^j(x,\lambda) \\ &\times (x-x_{j-1})\left[\ln\left(1+\frac{x-x_{j-1}}{1-x}\right) - \ln\left(1+\frac{x-x_{j-1}}{1-x_{j-1}}\right)\right]. \end{aligned}$$

Using the inequality (3.17), we get

$$v_1^j(x,\lambda) \leq \frac{x-x_{j-1}}{C_1(1-x)(1+x_{j-1})} + \frac{C_4+\lambda C_6}{2C_1} v_1^j(x,\lambda)\frac{(x-x_{j-1})^2}{1-x^2},$$

and

$$\begin{aligned} v_1^j(x,\lambda) &\geq \frac{1}{2C_2}\ln\left(1+\frac{2(x-x_{j-1})}{(1-x)(1+x_{j-1})}\right) - \frac{C_4+\lambda C_6}{2C_1} v_1^j(x,\lambda)\frac{(x-x_{j-1})^2}{1-x^2} \\ &\geq \frac{x-x_{j-1}}{C_2(1+x)(1-x_{j-1})} - \frac{C_4+\lambda C_6}{2C_1} v_1^j(x,\lambda)\frac{(x-x_{j-1})^2}{1-x^2}. \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{v_1^j(x,\lambda)}{x-x_{j-1}}\left(1-\frac{2(C_4+\lambda C_6)h^2}{C_1(1-x_{j+1}^2)}\right) \leq \frac{1}{C_1(1-x)(1+x_{j-1})}, \\ &\frac{v_1^j(x,\lambda)}{x-x_{j-1}}\left(1+\frac{2(C_4+\lambda C_6)h^2}{C_1(1-x_{j+1}^2)}\right) \geq \frac{1}{C_2(1+x)(1-x_{j-1})}, \end{aligned}$$

which, in view of the condition (3.15), proves the estimate (3.16). □

Lemma 3.3. *Suppose that the assumptions of Lemma 3.2 are satisfied. Then, for the problem* (3.1) – (3.3) *there exists the ETDS of the form*

$$\begin{aligned} &\Lambda y_j + \lambda\rho_j y_j \triangleq (a y_{\bar{x}})_{x,j} - d_j y_j + \lambda\rho_j y_j = 0, \quad j=1,2,...,N, \\ &y_0 \neq \infty, \quad y_{N+1} \neq \infty, \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} &y_{\bar{x},j} := \frac{y_j - y_{j-1}}{h}, \quad y_{x,j} := \frac{y_{j+1}-y_j}{h}, \\ &a_j = \left[\frac{1}{h}v_1^j(x_j,\lambda)\right]^{-1}, \quad j=2,3,...,N, \quad a_1 = a_{N+1} = 0, \\ &d_j = T^{x_j}(q,\lambda), \quad \rho_j = T^{x_j}(r,\lambda), \quad j=1,2,...,N, \\ &T^{x_j}(w(\xi),\lambda) = \frac{1}{h v_1^j(x_j,\lambda)}\int_{x_{j-1}}^{x_j} v_1^j(\xi,\lambda)w(\xi)\,d\xi \\ &\qquad + \frac{1}{h v_2^j(x_j,\lambda)}\int_{x_j}^{x_{j+1}} v_2^j(\xi,\lambda)w(\xi)\,d\xi, \end{aligned} \tag{3.19}$$

and for sufficiently small h

$$\begin{aligned} &0 < \left(1-x_{j-1/2}^2\right)C_1' \leq a_j \leq \left(1-x_{j-1/2}^2\right)C_2', \\ &C_1' = \frac{C_1}{2}, \quad C_2' = \frac{3C_2}{2}, \quad x_{j-1/2} = x_j - \frac{h}{2}, \end{aligned} \tag{3.20}$$

$$0 < C_3' \le d_j \le C_4', \quad C_4' = 2C_4, \quad 0 < C_5' \le \rho_j \le C_6', \quad C_6' = 2C_6. \tag{3.21}$$

The solution $y(x)$ *of problem* (3.18) *coincides with the solution* $u(x)$ *of the original problem* (3.1), (3.2) *at nodes of the grid* ω_h *up to a constant multiplier.*

Proof. First of all, we note that the problem (3.1), (3.2) is equivalent to the sequence of problems

$$\begin{aligned} &\frac{d}{dx}\left[k(x)\frac{du}{dx}\right] - q(x)u(x) = -\lambda r(x)u(x), \quad x \in (x_0, x_2), \\ &k(x)\frac{du}{dx}\bigg|_{x=x_0} = 0, \quad u(x_2) = u_2, \end{aligned} \tag{3.22}$$

$$\begin{aligned} &\frac{d}{dx}\left[k(x)\frac{du}{dx}\right] - q(x)u(x) = -\lambda r(x)u(x), \quad x \in (x_{j-1}, x_{j+1}), \\ &u(x_{j-1}) = u_{j-1}, \quad u(x_{j+1}) = u_{j+1}, \quad j = 2, 3, ..., N-1, \end{aligned} \tag{3.23}$$

$$\begin{aligned} &\frac{d}{dx}\left[k(x)\frac{du}{dx}\right] - q(x)u(x) = -\lambda r(x)u(x), \quad x \in (x_{N-1}, x_{N+1}), \\ &u(x_{N-1}) = u_{N-1}, \quad k(x)\frac{du}{dx}\bigg|_{x=x_{N+1}} = 0, \end{aligned} \tag{3.24}$$

whose Green's functions have the form

$$G^j(x,\xi) = \frac{1}{v_1^j(x_{j+1},\lambda)} \begin{cases} v_1^j(x,\lambda)v_2^j(\xi,\lambda), & x_{j-1} \le x \le \xi, \\ v_1^j(\xi,\lambda)v_2^j(x,\lambda), & \xi \le x \le x_{j+1}, \end{cases} \quad j = 1, 2, ..., N.$$

We construct the exact three-point difference scheme. For this purpose, we write the obvious integral representation of (3.22) – (3.24). Then, we have

$$\begin{aligned} &\int_{x_{j-1}}^{x_{j+1}} G^j(x,\xi)\frac{d}{d\xi}\left[k(\xi)\frac{du}{d\xi}\right] d\xi \\ &- \int_{x_{j-1}}^{x_{j+1}} G^j(x,\xi)[q(\xi) - \lambda r(\xi)]u(\xi)\, d\xi = 0, \quad j = 1, 2, ..., N. \end{aligned} \tag{3.25}$$

Calculating the integral in the left-hand side of (3.25) by integration by parts and using (3.4) – (3.6), we get

$$\int_{x_{j-1}}^{x_{j+1}} G^j(x,\xi)\frac{d}{d\xi}\left[k(\xi)\frac{du}{d\xi}\right] d\xi = \frac{v_1^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)} \int_x^{x_{j+1}} v_2^j(\xi,\lambda)\frac{d}{d\xi}\left[k(\xi)\frac{du}{d\xi}\right] d\xi$$

$$
\begin{aligned}
&+\frac{v_2^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)} \int_{x_{j-1}}^{x} v_1^j(\xi,\lambda)\frac{d}{d\xi}\left[k(\xi)\frac{du}{d\xi}\right] d\xi \\
&= \frac{v_1^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\left[v_2^j(\xi,\lambda)k(\xi)\frac{du}{d\xi}\bigg|_{x}^{x_{j+1}} - \int_{x}^{x_{j+1}} k(\xi)\frac{dv_2^j(\xi,\lambda)}{d\xi}\frac{du}{d\xi}d\xi\right] \\
&+\frac{v_2^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\left[v_1^j(\xi,\lambda)k(\xi)\frac{du}{d\xi}\bigg|_{x_{j-1}}^{x} - \int_{x_{j-1}}^{x} k(\xi)\frac{dv_1^j(\xi,\lambda)}{d\xi}\frac{du}{d\xi}d\xi\right] \\
&= \frac{v_1^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\left[-v_2^j(x,\lambda)k(x)\frac{du}{dx} - \int_{x}^{x_{j+1}} k(\xi)\frac{dv_2^j(\xi,\lambda)}{d\xi}\frac{du}{d\xi}d\xi\right] \\
&+\frac{v_2^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\left[v_1^j(x,\lambda)k(x)\frac{du}{dx} - \int_{x_{j-1}}^{x} k(\xi)\frac{dv_1^j(\xi,\lambda)}{d\xi}\frac{du}{d\xi}d\xi\right] \\
&= \frac{v_1^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\left[-k(\xi)\frac{dv_2^j(\xi,\lambda)}{d\xi}u(\xi)\bigg|_{x}^{x_{j+1}} + \int_{x}^{x_{j+1}} \frac{d}{d\xi}\left[k(\xi)\frac{dv_2^j(\xi,\lambda)}{d\xi}\right]u(\xi)d\xi\right] \\
&+\frac{v_2^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\left[-k(\xi)\frac{dv_1^j(\xi,\lambda)}{d\xi}u(\xi)\bigg|_{x_{j-1}}^{x} + \int_{x_{j-1}}^{x} \frac{d}{d\xi}\left[k(\xi)\frac{dv_1^j(\xi,\lambda)}{d\xi}\right]u(\xi)d\xi\right] \\
&= -\frac{v_1^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\, k(\xi)\frac{dv_2^j(\xi,\lambda)}{d\xi}u(\xi)\bigg|_{x}^{x_{j+1}} - \frac{v_2^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\, k(\xi)\frac{dv_1^j(\xi,\lambda)}{d\xi}u(\xi)\bigg|_{x_{j-1}}^{x} \\
&+\frac{v_1^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)} \int_{x}^{x_{j+1}} \left[q(\xi)-\lambda r(\xi)\right] v_2^j(\xi,\lambda)u(\xi)d\xi \\
&+\frac{v_2^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)} \int_{x_{j-1}}^{x} \left[q(\xi)-\lambda r(\xi)\right] v_1^j(\xi,\lambda)u(\xi)d\xi \\
&= -\frac{v_1^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\, k(\xi)\frac{dv_2^j(\xi,\lambda)}{d\xi}u(\xi)\bigg|_{x}^{x_{j+1}} - \frac{v_2^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\, k(\xi)\frac{dv_1^j(\xi,\lambda)}{d\xi}u(\xi)\bigg|_{x_{j-1}}^{x} \\
&+\int_{x_{j-1}}^{x_{j+1}} G^j(x,\xi)\left[q(\xi)-\lambda r(\xi)\right]u(\xi)d\xi.
\end{aligned}
$$

Then the equality (3.25) will have the following form

$$
\begin{aligned}
&-\frac{v_1^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\, k(\xi)\frac{dv_2^j(\xi,\lambda)}{d\xi}u(\xi)\bigg|_{x}^{x_{j+1}} \\
&-\frac{v_2^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\, k(\xi)\frac{dv_1^j(\xi,\lambda)}{d\xi}u(\xi)\bigg|_{x_{j-1}}^{x} = 0.
\end{aligned}
\tag{3.26}
$$

For $j = 1$, we have

$$\frac{v_1^1(x,\lambda)}{v_1^1(x_2,\lambda)}\left[u_2 + k(x)\frac{dv_2^1(x,\lambda)}{dx}u(x)\right] - \frac{v_2^1(x,\lambda)}{v_1^1(x_2,\lambda)}k(x)\frac{dv_1^1(x,\lambda)}{dx}u(x) = 0.$$

From (3.4) and (3.5), it follows that

$$\begin{aligned} k(x)\frac{dv_1^1(x,\lambda)}{dx} &= \int_{x_0}^{x}[q(\xi)-\lambda r(\xi)]v_1^1(\xi,\lambda)d\xi, \\ k(x)\frac{dv_2^1(x,\lambda)}{dx} &= -1 - \int_{x}^{x_2}[q(\xi)-\lambda r(\xi)]v_2^1(\xi,\lambda)d\xi. \end{aligned} \tag{3.27}$$

Thus,

$$\begin{aligned} &\frac{v_1^1(x,\lambda)}{v_1^1(x_2,\lambda)}\left[u_2 - u(x)\left(1 + \int_{x}^{x_2}[q(\xi)-\lambda r(\xi)]v_2^1(\xi,\lambda)d\xi\right)\right] \\ &- \frac{v_2^1(x,\lambda)}{v_1^1(x_2,\lambda)}u(x)\int_{x_0}^{x}[q(\xi)-\lambda r(\xi)]v_1^1(\xi,\lambda)d\xi = 0. \end{aligned} \tag{3.28}$$

For $x = x_1$, let us multiply equality (3.28) by $\frac{v_1^1(x_2,\lambda)}{hv_1^1(x_1,\lambda)v_2^1(x_1,\lambda)}$. Note that due to $v_2^1(x_1,\lambda) = v_1^2(x_2,\lambda)$, we have

$$\begin{aligned} &\frac{u_2-u_1}{hv_1^2(x_2,\lambda)} - u_1\left[\frac{1}{hv_2^1(x_1,\lambda)}\int_{x_1}^{x_2}[q(\xi)-\lambda r(\xi)]v_2^1(\xi,\lambda)d\xi \right. \\ &\left. + \frac{1}{hv_1^1(x_1,\lambda)}\int_{x_0}^{x_1}[q(\xi)-\lambda r(\xi)]v_1^1(\xi,\lambda)d\xi\right] = 0, \end{aligned}$$

or, equivalently, in view of $a_1 = 0$,

$$\frac{1}{h}\left(a_2u_{x,1} - a_1u_{\bar{x},1}\right) - d_1u_1 + \lambda\rho_1u_1 = 0.$$

For $j = 2, 3, ..., N-1$, equality (3.26) has the form

$$\begin{aligned} &\frac{v_1^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\left[u_{j+1} + k(x)\frac{dv_2^j(x,\lambda)}{dx}u(x)\right] \\ &+ \frac{v_2^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\left[u_{j-1} - k(x)\frac{dv_1^j(x,\lambda)}{dx}u(x)\right] = 0. \end{aligned}$$

Since it follows from (3.5) that

$$\begin{aligned} k(x)\frac{dv_\alpha^j(x,\lambda)}{dx} &= (-1)^{\alpha+1} \\ &+ \int_{x_{j+(-1)^\alpha}}^{x}[q(\xi)-\lambda r(\xi)]v_\alpha^j(\xi,\lambda)\,d\xi, \quad \alpha = 1, 2, \end{aligned} \tag{3.29}$$

we have

$$\frac{v_1^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}\left[u_{j+1}-\left(1+\int_x^{x_{j+1}}[q(\xi)-\lambda r(\xi)]v_2^j(\xi,\lambda)d\xi\right)u(x)\right]+\frac{v_2^j(x,\lambda)}{v_1^j(x_{j+1},\lambda)}$$
$$\times\left[u_{j-1}-\left(1+\int_{x_{j-1}}^{x}[q(\xi)-\lambda r(\xi)]v_1^j(\xi,\lambda)d\xi\right)u(x)\right]=0. \tag{3.30}$$

Taking in (3.30) $x = x_j$, multiplying the obtained equality by $\frac{v_1^j(x_{j+1},\lambda)}{hv_1^j(x_j,\lambda)v_2^j(x_j,\lambda)}$, and using the properties of the pattern functions $v_1^j(x_{j+1},\lambda) = v_2^j(x_{j-1},\lambda)$, $v_2^j(x_j,\lambda) = v_1^{j+1}(x_{j+1},\lambda)$, we arrive at the exact three-point difference scheme (3.18) for $j = 2, 3, ..., N-1$.

Let us rewrite (3.26) for $j = N$ by

$$\frac{v_1^N(x,\lambda)}{v_1^N(x_{N+1},\lambda)}k(x)\frac{dv_2^N(x,\lambda)}{dx}u(x)$$
$$+\frac{v_2^N(x,\lambda)}{v_1^N(x_{N+1},\lambda)}\left[u_{N-1}-k(x)\frac{dv_1^N(x,\lambda)}{dx}u(x)\right]=0.$$

Then, considering the equalities

$$\begin{aligned}k(x)\frac{dv_1^N(x,\lambda)}{dx}&=1+\int_{x_{N-1}}^{x}[q(\xi)-\lambda r(\xi)]v_1^N(\xi,\lambda)d\xi,\\ k(x)\frac{dv_2^N(x,\lambda)}{dx}&=-\int_x^{x_{N+1}}[q(\xi)-\lambda r(\xi)]v_2^N(\xi,\lambda)d\xi,\end{aligned} \tag{3.31}$$

which follow from (3.5) and (3.6), we obtain

$$\begin{aligned}&-\frac{v_1^N(x,\lambda)}{v_1^N(x_{N+1},\lambda)}\int_x^{x_{N+1}}[q(\xi)-\lambda r(\xi)]v_2^N(\xi,\lambda)d\xi\cdot u(x)+\frac{v_2^N(x,\lambda)}{v_1^N(x_{N+1},\lambda)}\\ &\times\left[u_{N-1}-\left(1+\int_{x_{N-1}}^{x}[q(\xi)-\lambda r(\xi)]v_1^N(\xi,\lambda)d\xi\right)u(x)\right]=0.\end{aligned} \tag{3.32}$$

Taking $x = x_N$ and multiplying the obtained equality by $\frac{v_1^N(x_{N+1},\lambda)}{hv_1^N(x_N,\lambda)v_2^j(x_N,\lambda)}$, we obtain

$$-\frac{u_N-u_{N-1}}{hv_1^N(x_N,\lambda)}-u_N\left[\frac{1}{hv_2^N(x_N,\lambda)}\int_{x_N}^{x_{N+1}}[q(\xi)-\lambda r(\xi)]v_2^N(\xi,\lambda)d\xi\right.$$
$$\left.+\frac{1}{hv_1^N(x_N,\lambda)}\int_{x_{N-1}}^{x_N}[q(\xi)-\lambda r(\xi)]v_1^N(\xi,\lambda)d\xi\right]=0,$$

which due to $a_{N+1} = 0$, can be written as

$$\frac{1}{h}\left(a_{N+1}u_{x,N} - a_N u_{\bar{x},N}\right) - d_N u_N + \lambda \rho_N u_N = 0.$$

Inequality (3.20) follows from (3.16). Indeed,

$$\begin{aligned} a_j &= \frac{h}{v_1^j(x_j,\lambda)} \le \frac{3}{2}C_2(1+x_j)(1-x_{j-1}) \\ &= \frac{3}{2}C_2\left(1+x_{j-1/2}+\frac{h}{2}\right)\left(1-x_{j-1/2}+\frac{h}{2}\right) \\ &= \frac{3}{2}C_2\left[1-x_{j-1/2}^2+h+\frac{h^2}{4}\right], \end{aligned}$$

$$\begin{aligned} a_j &\ge \frac{1}{2}C_1(1-x_j)(1+x_{j-1}) = \frac{1}{2}C_1\left(1-x_{j-1/2}-\frac{h}{2}\right)\left(1+x_{j-1/2}-\frac{h}{2}\right) \\ &= \frac{1}{2}C_1\left[1-x_{j-1/2}^2-h+\frac{h^2}{4}\right]. \end{aligned}$$

It follows from this that for sufficiently small h the inequalities (3.20) are valid.

We now prove the estimate (3.21). Since

$$d_j = \frac{1}{hv_1^j(x_j,\lambda)}\int_{x_{j-1}}^{x_j} v_1^j(\xi,\lambda)q(\xi)\,d\xi + \frac{1}{hv_2^j(x_j,\lambda)}\int_{x_j}^{x_{j+1}} v_2^j(\xi,\lambda)q(\xi)\,d\xi,$$

in view of the positivity and monotonicity of the functions $v_1^j(x,\lambda)$, $v_2^j(x,\lambda)$ we have

$$d_j \le \frac{C_4}{h}\left[\int_{x_{j-1}}^{x_j}\frac{v_1^j(\xi,\lambda)}{v_1^j(x_j,\lambda)}\,d\xi + \int_{x_j}^{x_{j+1}}\frac{v_2^j(\xi,\lambda)}{v_2^j(x_j,\lambda)}\,d\xi\right] \le 2C_4.$$

In addition, using the estimates (3.16), we obtain

$$d_j \ge \frac{C_3}{h}\left[\frac{1}{v_1^j(x_j,\lambda)}\int_{x_{j-1}}^{x_j} v_1^j(\xi,\lambda)\,d\xi + \frac{1}{v_2^j(x_j,\lambda)}\int_{x_j}^{x_{j+1}} v_2^j(\xi,\lambda)\,d\xi\right] > C_3'.$$

Analogously, the inequality $0 < C_5' \le \rho_j \le 2C_6$ can be proven. □

Note that if the solution of the problem (3.1) is normalized by the condition

$$\int_{-1}^{1} r(x)u^2(x)\,dx = 1,$$

then, for the exact normalization on the grid, we have

$$\sum_{j=2}^{N}\int_{x_{j-1}}^{x_j} r(x)\left[\frac{v_1^j(x,\lambda)}{v_1^j(x_j,\lambda)}y_j + \frac{v_2^{j-1}(x,\lambda)}{v_2^{j-1}(x_{j-1},\lambda)}y_{j-1}\right]^2 dx$$
$$+\int_{-1}^{x_1} r(x)\left[\frac{v_1^1(x,\lambda)}{v_1^1(x_j,\lambda)}y_1\right]^2 dx + \int_{x_N}^{1} r(x)\left[\frac{v_2^N(x,\lambda)}{v_2^N(x_N,\lambda)}y_N\right]^2 dx = 1.$$

3.2 Coefficient Stability of the Exact Three-Point Difference Scheme

We consider the difference problem (3.18) in the space H_h of the grid functions y with the following scalar product and norms:

$$(y,v) := \sum_{\xi\in\omega_h} hy(\xi)v(\xi), \quad \|y\| := (y,y)^{1/2}, \quad \|y\|_C := \max_{\xi\in\omega_h}|y(\xi)|\,.$$

Suppose that $\lambda^h = \lambda^h_n$ is the nth eigenvalue of this problem, and that $y = y_n$ is the normalized eigenfunction. There exist N real eigenvalues $\lambda_1^h, \lambda_2^h, \ldots, \lambda_N^h$, to which the appropriate eigenfunctions $y_1, y_2, \ldots, y_N$ correspond. The eigenfunctions are orthonormalized with weight ρ, such that $(\rho y_n, y_m) = 0$ holds for $n \neq m$ and $(\rho y_n, y_n) = 1$.

Multiplying (3.18) scalarwise by y and taking the difference Green's formula (see [78, p. 26]) and the equalities $a_1 = a_{N+1} = 0$ into account, we find

$$\lambda^h = R_N(y) = \frac{(a, y_{\bar{x}}^2) + (d, y^2)}{(\rho, y^2)}.$$

It is easy to see that the difference problem (3.18) is equivalent to the variational problem

$$\begin{aligned}
&\lambda_1^h = \min_y R_N(y), \quad \text{provided that} \quad (\rho, y^2) = 1,\\
&\lambda_n^h = \min_y R_N(y), \quad \text{provided that} \quad (\rho, y^2) = 1, \quad (\rho y, y_m) = 0,\\
&m = 1, 2, \ldots, n-1, \quad n = 2, 3, \ldots, N.
\end{aligned}$$

The following assertion holds:

Lemma 3.4. *For the eigenvalues and the eigenfunctions of problem* (3.18) – (3.21) *the following estimates are satisfied:*

$$M_1 n^2 \leq \lambda_n^h \leq M_2 n^2, \tag{3.33}$$

$$\left\|\sqrt{a}y_n\right\|_C \leq M_3\sqrt{n}, \quad \|a(y_n)_{\bar{x}}\|_C \leq M_4 n^{3/2}, \tag{3.34}$$

where the constants M_1, M_2, M_3, M_4 do not depend on h and n, $n = 1, 2, ..., N$.

Proof. First, let us note that for the difference scheme (3.18) – (3.21) with coefficients $a_j = 1 - x_{j-1/2}^2$, $d_j \equiv 0$, $\rho_j \equiv 1$, the eigenvalues are $\overset{\circ}{\lambda}{}_n^h = n(n+1)$. Therefore, the following estimate holds

$$n^2 \leq \overset{\circ}{\lambda}{}_n^h \leq 2n^2. \tag{3.35}$$

Next, taking into account that $C_1'(1 - x_{j-1/2}^2) \leq a_j \leq C_2'(1 - x_{j-1/2}^2)$ (see (3.20)), the inequality

$$\frac{C_1'(1-(x-h/2)^2, y_{\bar{x}}^2)}{C_6'(1, y^2)} \leq \frac{(a, y_{\bar{x}}^2) + (d, y^2)}{(\rho, y^2)} \leq \frac{C_2'(1-(x-h/2)^2, y_{\bar{x}}^2)}{C_5'(1, y^2)} + \frac{C_4'}{C_5'}$$

is satisfied, from which it follows that

$$\frac{C_1'}{C_6'}\overset{\circ}{\lambda}{}_n^h \leq \lambda_n^h \leq \frac{C_2'}{C_5'}\overset{\circ}{\lambda}{}_n^h + \frac{C_4'}{C_5'}.$$

Substituting the estimate (3.35), we obtain the inequality (3.33).

Let us move on to proving the inequality (3.34). Consider the identities

$$y_i^2 - y_1^2 = \sum_{j=2}^{i} h\left(y^2\right)_{\bar{x},j} = \sum_{j=2}^{i} h y_{\bar{x},j}\left(y_{j-1} + y_j\right), \tag{3.36}$$

$$\begin{aligned}(a_i y_{\bar{x},i})^2 &= \sum_{j=1}^{i-1} h\left[(a y_{\bar{x}})^2\right]_{x,j} = \sum_{j=1}^{i-1} h\left(a y_{\bar{x}}\right)_{x,j}\left(a_j y_{\bar{x},j} + a_{j+1} y_{x,j}\right)\\ &= \sum_{j=1}^{i-1} h\left(d_j - \lambda^h \rho_j\right) y_j\left(a_j y_{\bar{x},j} + a_{j+1} y_{x,j}\right).\end{aligned} \tag{3.37}$$

From the normalization condition $(\rho, y^2) = 1$, it follows that $\rho_1 y_1^2 \leq 1$, and thus $y_1^2 \leq 1/C_5'$. Applying to the right-hand side of the identity

$$a_i y_i^2 - a_i y_1^2 = a_i \sum_{j=2}^{i} h y_{\bar{x},j}\left(y_{j-1} + y_j\right)$$

the Cauchy-Bunyakovsky-Schwarz inequality and taking into account that

$$(\rho, y^2) = 1, \quad (a, y_{\bar{x}}^2) \leq \lambda^h,$$
$$\tilde{C}_1 = C_1'(1 - x_{3/2}^2) \leq C_1'(1 - x_{i-1/2}^2) \leq a_i \leq C_2'(1 - x_{i-1/2}^2) \leq C_2',$$

as well as the estimate (3.33), we obtain

$$
\begin{aligned}
a_i y_i^2 &\le \frac{C_2'}{C_4'} + \frac{2C_2'}{\sqrt{\tilde{C}_1 C_5'}} \left(\sum_{j=2}^{i} h a_j y_{\bar{x},j}^2 \right)^{1/2} \left(\sum_{j=2}^{i} h \rho_j y_j^2 \right)^{1/2} \\
&\le \frac{C_2'}{C_4'} + \frac{2C_2'}{\sqrt{\tilde{C}_1 C_5'}} \left(a, y_{\bar{x}}^2\right)^{1/2} \left(\rho, y^2\right)^{1/2} \le \frac{C_2'}{C_4'} + \frac{2C_2'}{\sqrt{\tilde{C}_1 C_5'}} (\lambda^h)^{1/2} \le M_3^2 n.
\end{aligned}
$$

Using the Cauchy-Bunyakovsky-Schwarz inequality to transform the right-hand side of the identity (3.37), we obtain

$$
\begin{aligned}
(a_i y_{\bar{x},i})^2 &\le 2 \left(\sum_{j=1}^{i} h \left(a y_{\bar{x},j}\right)^2 \right)^{1/2} \left[\left(\sum_{j=1}^{i-1} h (d_j y_j)^2 \right)^{1/2} + \lambda^h \left(\sum_{j=1}^{i-1} h (\rho_j y_j)^2 \right)^{1/2} \right] \\
&\le 2\sqrt{C_2'} \left(a, y_{\bar{x}}^2\right)^{1/2} \left(\frac{C_4'}{\sqrt{C_5'}} + \lambda^h \sqrt{C_6'} \right) \left(\rho, y^2\right)^{1/2} \\
&\le 2\sqrt{C_2'} (\lambda^h)^{1/2} \left(\frac{C_4'}{\sqrt{C_5'}} + \lambda^h \sqrt{C_6'} \right) \le M_4^2 n^3.
\end{aligned}
$$

□

Together with the ETDS (3.18) – (3.21), we consider the perturbed three-point difference scheme

$$
\tilde{\Lambda} \tilde{y} + \tilde{\lambda}^h \tilde{\rho} \tilde{y} = 0, \quad x \in \omega_h, \quad \tilde{y}_0 \ne \infty, \quad \tilde{y}_{N+1} \ne \infty, \tag{3.38}
$$

where

$$
\tilde{\Lambda} \tilde{y} = (\tilde{a} \tilde{y}_{\bar{x}})_x - \tilde{d} \tilde{y}, \quad x \in \omega_h, \quad \tilde{a}_1 = \tilde{a}_{N+1} = 0.
$$

Introducing a function $z = y - \tilde{y}$, we obtain the boundary value problem

$$
\Lambda z + \lambda^h \rho z = -\Psi(x), \quad x \in \omega_h, \quad z_0 \ne \infty, \quad z_{N+1} \ne \infty, \tag{3.39}
$$

where

$$
\Psi(x) = \Lambda \tilde{y} + \lambda^h \rho \tilde{y}.
$$

Using the equation (3.38), we can rewrite the function $\Psi(x)$ into

$$
\begin{aligned}
\Psi(x) = \Lambda \tilde{y} + \lambda^h \rho \tilde{y} - \tilde{\Lambda} \tilde{y} - \tilde{\lambda}^h \tilde{\rho} \tilde{y} &= ((a - \tilde{a}) \tilde{y}_{\bar{x}})_x - (d - \tilde{d}) \tilde{y} \\
&+ \tilde{\lambda}^h (\rho - \tilde{\rho}) \tilde{y} + (\lambda^h - \tilde{\lambda}^h) \rho \tilde{y} = \psi(x) + (\lambda^h - \tilde{\lambda}^h) \rho \tilde{y},
\end{aligned}
$$

where

$$\begin{aligned}&\psi(x) = \eta_x + \psi^*(x), \quad \eta = (a - \tilde{a})\tilde{y}_{\bar{x}},\\ &\psi^* = -(d - \tilde{d})\tilde{y} + \tilde{\lambda}^h(\rho - \tilde{\rho})\tilde{y}.\end{aligned} \tag{3.40}$$

The parameter λ^h is an eigenvalue for the difference operator of problem (3.39). Thus, the inhomogeneous equation (3.39) is solvable only if the eigenfunction $y(x)$ is orthogonal to the right-hand side of equation (3.39), or, more precisely, if the equality

$$(\Psi, y) = (\psi, y) + (\lambda^h - \tilde{\lambda}^h)(\rho\tilde{y}, y) = 0 \tag{3.41}$$

is satisfied.

Only a single eigenfunction, determined accurately up to an arbitrary multiplier C_0, corresponds to the eigenvalue λ^h. We choose this multiplier in a way such that the function $\bar{y} = C_0 y$ is orthogonal to the difference $\bar{z} = \bar{y} - \tilde{y}$:

$$(\rho\bar{y}, \bar{z}) = 0. \tag{3.42}$$

Due to the normalization condition $(\rho y, y) = 1$, we thus obtain

$$(\rho\tilde{y}, y) = (\rho y, \bar{y} - \bar{z}) = (\rho y, \bar{y}) - (\rho y, \bar{z}) = C_0(\rho y, y) = C_0.$$

If $\tilde{y} \to y$ as $h \to 0$, we can assume that $C_0 > 0$.

Further,

$$\begin{aligned}(\rho, \tilde{y}^2) &= (\rho, (\bar{y} - \bar{z})^2) = (\rho, \bar{y}^2) - 2(\rho, \bar{z}\bar{y}) + (\rho, \bar{z}^2)\\ &= C_0^2(\rho y, y) + (\rho, \bar{z}^2) = C_0^2 - (\rho, \bar{z}\tilde{y}),\end{aligned}$$

is valid and, hence,

$$1 - C_0^2 = -(\rho, \bar{z}\tilde{y}) - [(\rho, \tilde{y}^2) - (\tilde{\rho}, \tilde{y}^2)]. \tag{3.43}$$

We use equality (3.41) for determining $\lambda^h - \tilde{\lambda}^h$:

$$\lambda^h - \tilde{\lambda}^h = -\frac{(\psi, y)}{(\rho\tilde{y}, y)} = -\frac{(\psi, \bar{y})}{C_0^2}. \tag{3.44}$$

We transform the right-hand side of the equation (3.44) by taking (3.40), the summation by parts formula (see, e.g., [73, p. 99]), and the equalities $a_1 = a_{N+1} = 0$ into account

$$(\psi, \bar{y}) = -(\eta, \bar{y}_{\bar{x}}) + (\psi^*, \bar{y}).$$

From this and the estimates (3.34) for $\bar{y}$, $\bar{y}_{\bar{x}}$, we find

$$\begin{aligned}\left|\lambda^h-\tilde{\lambda}^h\right| &\leq \frac{|(\psi,\bar{y})|}{C_0^2} \leq \frac{|(\eta,\bar{y}_{\bar{x}})|+|(\psi^*,\bar{y})|}{C_0^2}\\ &\leq \frac{\|\bar{y}_{\bar{x}}\|_C(1,|\eta|)+\|\bar{y}\|_C(1,|\psi^*|)}{C_0^2} \leq Mn^{3/2}[(1,|\eta|)+(1,|\psi^*|)].\end{aligned}$$

We arrive at the following assertion.

Lemma 3.5. *Suppose that the conditions* (3.20), (3.21) *for the difference Sturm-Liouville problem* (3.18) *are satisfied. Then, the estimate*

$$\left|\lambda_n^h-\tilde{\lambda}_n^h\right| \leq Mn^{3/2}[(1,|\eta|)+(1,|\psi^*|)] \tag{3.45}$$

holds, where the constant $M>0$ *depends on* C_i', $i=1,2,\ldots,6$, *and* C_0.

We now find an estimate for $\bar{z}$. Since $\bar{y}=C_0y$, we see that $\bar{y}$ satisfies the equation (3.18) and $(\rho,\bar{y}^2)=C_0^2$, and for $\bar{z}=\bar{y}-\tilde{y}$ we arrive at the problem (3.39).

This problem is reduced to a discrete analogue of the integral equation

$$\bar{z}(x)=\lambda^h(G(x,\xi),\rho(\xi)\bar{z}(\xi))+(G(x,\xi),\Psi(\xi)), \tag{3.46}$$

where $G(x,\xi)=G^h(x,\xi)$ is the difference Green function of the operator $\Lambda y=(ay_{\bar{x}})_x-dy$ with boundary conditions $y_0\neq\infty$, $y_{N+1}\neq\infty$ (see [60]).

The eigenfunction $\bar{y}$ of the problem (3.18) satisfies the equation

$$\bar{y}(x)=\lambda^h(G(x,\xi),\rho(\xi)\bar{y}(\xi)). \tag{3.47}$$

We transform the equations (3.46) and (3.47) into such a form such that the corresponding kernels become symmetric. For this purpose, we use the substitutions

$$v(x)=\sqrt{\rho(x)}\bar{z}(x), \quad \varphi(x)=\sqrt{\rho(x)}\bar{y}(x), \quad K(x,\xi)=\sqrt{\rho(x)\rho(\xi)}G(x,\xi).$$

Then the equations (3.46) and (3.47) take the form

$$v_n(x)=\lambda_n^h(K(x,\xi),v_n(\xi))+f(x), \quad f(x)=(K(x,\xi),\bar{\Psi}(\xi)), \tag{3.48}$$

$$\bar{\Psi}(\xi)=\frac{\Psi(\xi)}{\sqrt{\rho(\xi)}}, \quad \varphi_n(x)=\lambda_n^h(K(x,\xi),\varphi_n(\xi)). \tag{3.49}$$

The condition of orthogonality of the function $f(x)$ to the functions $\varphi_n(x)$ is satisfied in view of the condition (3.41):

$$\begin{aligned}(\varphi_n(x), f(x)) &= (\varphi_n(x), (K(x,\xi), \bar{\Psi}(\xi))) = (\bar{\Psi}(\xi), (K(x,\xi), \varphi_n(x))) \\ &= \frac{1}{\lambda_n^h}(\bar{\Psi}(\xi), \varphi_n(\xi)) = \frac{1}{\lambda_n^h}\left(\frac{\Psi}{\sqrt{\rho}}, \sqrt{\rho}\bar{y}\right) = \frac{1}{\lambda_n^h}(\Psi, \bar{y}) = 0.\end{aligned}$$

We rewrite the condition (3.42) as

$$(\varphi_n, v_n) = 0. \tag{3.50}$$

Searching for the solution $v(x) = v_n(x)$ of the equation (3.48) of the form

$$v_n(x) = f(x) + \sum_{\substack{k=1,\\ k\neq n}}^{N-1} c_k\varphi_k(x) \tag{3.51}$$

under the additional condition (3.50), we substitute this expression to the right-hand side of the equation (3.48) to obtain

$$v_n(x) = f(x) + \lambda_n^h \sum_{\substack{k=1,\\ k\neq n}}^{N-1} c_k(K(x,\xi), \varphi_k(\xi)) + \lambda_n^h(K(x,\xi), f(\xi)). \tag{3.52}$$

Expanding $f(x)$ by the eigenfunctions $\{\varphi_k(x)\}$

$$f(x) = \sum_{\substack{k=1,\\ k\neq n}}^{N-1} f_k\varphi_k(x), \quad f_k = (f, \varphi_k),$$

it follows that

$$(K(x,\xi), f(\xi)) = \sum_{\substack{k=1,\\ k\neq n}}^{N-1} \frac{f_k}{\lambda_k^h}\varphi_k(x).$$

Thus, in view of (3.49), we can rewrite the equality (3.52) into

$$v_n(x) = f(x) + \lambda_n^h \sum_{\substack{k=1,\\ k\neq n}}^{N-1} \left[\frac{c_k}{\lambda_k^h} + \frac{f_k}{\lambda_k^h}\right]\varphi_k(x). \tag{3.53}$$

Due to the equality (3.53), we have

$$c_k = (v_n - f, \varphi_k) = \frac{\lambda_n^h}{\lambda_k^h}c_k + \frac{\lambda_n^h}{\lambda_k^h}(f, \varphi_k),$$

and substituting $c_k = \lambda_n^h(f,\varphi_k)/(\lambda_k^h - \lambda_n^h)$ into (3.51), we obtain

$$v_n(x) = f(x) + \sum_{\substack{k=1,\\ k\neq n}}^{N-1} \frac{\lambda_n^h(f,\varphi_k)}{\lambda_k^h - \lambda_n^h}\varphi_k(x). \tag{3.54}$$

Multiplying the equation (3.54) by $a^\mu(x)$, $0 < \mu \leq 1$, we can estimate the second term on the right-hand side of this equation by

$$\begin{aligned}\left|\sum_{\substack{k=1,\\ k\neq n}}^{N-1} \frac{\lambda_n^h(f,\varphi_k)}{\lambda_k^h - \lambda_n^h} a^\mu(x)\varphi_k(x)\right| &\leq \|f\|\,\|a^\mu\varphi_k\|\,\lambda_n^h \sum_{\substack{k=1,\\ k\neq n}}^{N-1} \frac{|\varphi_k|}{|\lambda_k^h - \lambda_n^h|}\\ &\leq M\|f\|\lambda_n^h \sum_{\substack{k=1,\\ k\neq n}}^{N-1} \frac{(\lambda_k^h)^{1/4}}{|\lambda_k^h - \lambda_n^h|}.\end{aligned}$$

Let $\varepsilon > 0$ be an arbitrary number independent of h. We choose the number n_0 in a way such that $\lambda_{n_0}^h \geq (1+\varepsilon)\lambda_n^h$. Then,

$$\sum_{k=n_0}^{N-1} \frac{(\lambda_k^h)^{1/4}}{|\lambda_k^h - \lambda_n^h|} \leq \frac{1+\varepsilon}{\varepsilon} \sum_{k=n_0}^{N-1} \frac{(\lambda_k^h)^{1/4}}{\lambda_k^h} \leq \frac{M'}{\varepsilon} \sum_{k=n_0}^{N-1} \frac{1}{(\lambda_k^h)^{3/4}} \leq M,$$

where the constant $M > 0$ is independent of h.

Since $\lambda_k^h \to \lambda_k$ for $k \leq n_0$ as $h \to 0$, we obtain for a sufficiently small $h \leq h_0$ the estimate

$$\sum_{k=1}^{n_0-1} \frac{(\lambda_k^h)^{1/4}}{|\lambda_k^h - \lambda_n^h|} \leq M,$$

where M does not depend on h.

Hence, the estimate

$$\|a^\mu v_n\|_C \leq M\,\|a^\mu f\|_C \tag{3.55}$$

is satisfied.

We transform the expression for $f(x)$ into

$$\begin{aligned} f(x) = (K(x,\xi), \bar{\Psi}(\xi)) &= \left(\sqrt{\rho(x)}\sqrt{\rho(\xi)}G(x,\xi), \frac{\Psi(\xi)}{\sqrt{\rho(\xi)}}\right)\\ &= \sqrt{\rho(x)}(G(x,\xi),\Psi(\xi)) = (\lambda^h - \tilde{\lambda}^h)\sqrt{\rho(x)}(G(x,\xi),\rho(\xi)\tilde{y}(\xi))\\ &+ \sqrt{\rho(x)}(G(x,\xi),\eta_\xi(\xi) + \psi^*(\xi)) = (\lambda^h - \tilde{\lambda}^h)\sqrt{\rho(x)}(G(x,\xi),\rho(\xi)\tilde{y}(\xi))\\ &+ \sqrt{\rho(x)}\{-(G_{\bar{\xi}}(x,\xi),\eta(\xi)) + (G(x,\xi),\psi^*(\xi))\}.\end{aligned}$$

Hence, taking the estimates (see [60, 27])

$$\|a^{\mu}(\xi)G(x,\xi)\|_C \le C_7, \quad \left\|\tilde{a}(\xi)G_{\bar{\xi}}(x,\xi)\right\|_C \le C_8$$

into account, where the constants C_7, C_8 do not depend on h and n, we obtain

$$\begin{aligned}\|a^{\mu}f\|_C \le & \left\|\left(a^{\mu}(x)\tilde{a}(\xi)G_{\bar{\xi}}(x,\xi), \frac{\eta(\xi)}{\tilde{a}(\xi)}\right)\right\|_C \\ & + \left\|\sqrt{\rho(x)}\left(a^{\mu}(x)G(x,\xi), \psi^{*}(\xi)\right)\right\|_C \\ & + \left\|\sqrt{\rho(x)}\left(a^{\mu}G(x,\xi), \rho(\xi)\tilde{y}(\xi)\right)\right\|_C |\lambda^h - \tilde{\lambda}^h| \\ \le & M_1\left\{\left(1, \left|\frac{\eta}{\tilde{a}}\right|\right) + (1, |\psi^{*}|)\right\} + M_2|\lambda^h - \tilde{\lambda}^h|.\end{aligned}$$

Substituting this estimate into (3.55), returning back to the function $\bar{z}(x) = v(x)/\sqrt{\rho(x)}$ and taking the inequality (3.34) as well as Lemma 3.5 into account, we get

$$\|a^{\mu}\bar{z}\|_C \le M\left\{\left(1, \left|\frac{\eta}{\tilde{a}}\right|\right) + (1, |\psi^{*}|)\right\}.$$

We are interested in the difference $z = y - \tilde{y}$ which is expressed by

$$z = \frac{\bar{z}}{C_0} + \frac{1 - C_0}{C_0}\tilde{y} = \frac{\bar{z}}{C_0} + \frac{1 - C_0^2}{C_0(1 + C_0)}\tilde{y}.$$

Since $\|a^{\mu}\tilde{y}\|_C$ is bounded, it follows that for a sufficiently small h, we have

$$\|a^{\mu}z\|_C \le \frac{\|a^{\mu}\bar{z}\|_C}{C_0} + \left|\frac{1 - C_0^2}{C_0(1 + C_0)}\right| \|a^{\mu}\tilde{y}\|_C \le M(C_0)\left(\|a^{\mu}\bar{z}\|_C + \left|1 - C_0^2\right|\right).$$

It is obvious from formula (3.43), that

$$\begin{aligned}\left|1 - C_0^2\right| & \le (\rho, \bar{z}^2)^{1/2}(\rho, \tilde{y}^2)^{1/2} + \left|(\rho, \tilde{y}^2) - (\tilde{\rho}, \tilde{y}^2)\right| \\ & \le M_1 \|a^{\mu}\bar{z}\|_C + \left|(\rho, \tilde{y}^2) - (\tilde{\rho}, \tilde{y}^2)\right|.\end{aligned}$$

If μ is chosen as $\mu = 0.5 + \varepsilon$, where $0 < \varepsilon \le 0.5$, we get

$$\left\|a^{0.5+\varepsilon}z\right\|_C \le M\left(\left\|a^{0.5+\varepsilon}\bar{z}\right\|_C + \left|(\rho, \tilde{y}^2) - (\tilde{\rho}, \tilde{y}^2)\right|\right).$$

Inserting the estimate for $\|a^{0.5+\varepsilon}\bar{z}\|_C$, we make sure that by $\varepsilon \to 0$ the following statement is true:

Theorem 3.1. *Suppose that the assumptions of Lemma 3.5 are satisfied. Then, for sufficiently small $h \leq h_0$, we have the following estimates:*

$$\left\|\sqrt{a}(y_n - \tilde{y}_n)\right\|_C \leq M_1 \left\{\left(1, \left|\frac{\eta}{\tilde{a}}\right|\right) + (1, |\psi^*|)\right\} + M_2 \left|(\rho, \tilde{y}^2) - (\tilde{\rho}, \tilde{y}^2)\right|,$$

$$\left|\lambda_n^h - \tilde{\lambda}_n^h\right| \leq M_3 \left\{(1, |\eta|) + (1, |\psi^*|)\right\},$$

where the constants M_i, $i = 1, 2, 3$ depend only on C_i', $i = 1, 2, \ldots, 6$, and C_0.

This Theorem proves the continuous dependence of the solution of the problem (3.18) on the coefficients, that is, the coefficient stability.

3.3 Algorithmic Realization of the Exact Three-Point Difference Scheme

We pass to the algorithmic realization of the ETDS (3.18). According to the basic idea, it is necessary to express the coefficients a_j, d_j, ρ_j of the ETDS in terms of solutions of the Cauchy problems (3.4) – (3.6). First of all, note that the problem (3.5) is equivalent to the Cauchy problem for the system of ODE

$$\begin{aligned} \frac{dv_\alpha^j(x,\lambda)}{dx} &= \frac{w_\alpha^j(x,\lambda) - \lambda z_\alpha^j(x,\lambda)}{k(x)}, \\ \frac{dw_\alpha^j(x,\lambda)}{dx} &= q(x) v_\alpha^j(x,\lambda), \\ \frac{dz_\alpha^j(x,\lambda)}{dx} &= r(x) v_\alpha^j(x,\lambda), \quad x \in (x_{j-2+\alpha}, x_{j-1+\alpha}), \end{aligned} \tag{3.56}$$

$$\begin{aligned} &v_\alpha^j(x_{j+(-1)^\alpha}, \lambda) = 0, \quad w_\alpha^j(x_{j+(-1)^\alpha}, \lambda) = (-1)^{\alpha+1}, \\ &z_\alpha^j(x_{j+(-1)^\alpha}, \lambda) = 0, \quad j = 3-\alpha, 4-\alpha, \ldots, N+1-\alpha, \quad \alpha = 1, 2. \end{aligned} \tag{3.57}$$

Indeed, if the first equation of the system (3.56) is multiplied by $k(x)$ and the left and right parts of the resulting equality are differentiated, then taking into account the second and third equations of the system, we get the equation (3.5). From the first equation and the initial conditions (3.57) the second initial condition (3.5) follows.

To calculate the coefficient a_j of the ETDS (3.18), we already have the necessary representation (see (3.19)). Note that taking into account (3.56) and (3.57), we have

$$\begin{aligned} &(-1)^{\alpha+1} \int_{x_{j+(-1)^\alpha}}^{x_j} v_\alpha^j(\xi,\lambda) q(\xi) d\xi = (-1)^{\alpha+1} \int_{x_{j+(-1)^\alpha}}^{x_j} \frac{dw_\alpha^j(\xi,\lambda)}{d\xi} d\xi \\ &= (-1)^{\alpha+1} \left[w_\alpha^j(x_j,\lambda) - w_\alpha^j(x_{j+(-1)^\alpha},\lambda)\right] = (-1)^{\alpha+1} \left[w_\alpha^j(x_j,\lambda) + (-1)^\alpha\right], \end{aligned}$$

$$(-1)^{\alpha+1}\int_{x_{j+(-1)^{\alpha}}}^{x_j} v_{\alpha}^{j}(\xi,\lambda)r(\xi)d\xi = (-1)^{\alpha+1}\int_{x_{j+(-1)^{\alpha}}}^{x_j} \frac{dz_{\alpha}^{j}(\xi,\lambda)}{d\xi}d\xi$$
$$= (-1)^{\alpha+1}\left[z_{\alpha}^{j}(x_j,\lambda) - z_{\alpha}^{j}(x_{j+(-1)^{\alpha}},\lambda)\right] = (-1)^{\alpha+1}z_{\alpha}^{j}(x_j,\lambda).$$

Hence, the coefficients d_j and ρ_j, $j = 2, 3, ..., N-1$, of the difference scheme (3.18), (3.19) can be represented as

$$d_j = \frac{1}{hv_1^j(x_j,\lambda)}\int_{x_{j-1}}^{x_j} v_1^j(\xi,\lambda)q(\xi)d\xi + \frac{1}{hv_2^j(x_j,\lambda)}\int_{x_j}^{x_{j+1}} v_2^j(\xi,\lambda)q(\xi)d\xi$$
$$= h^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[v_{\alpha}^{j}(x_j,\lambda)\right]^{-1}\left[w_{\alpha}^{j}(x_j,\lambda) + (-1)^{\alpha}\right],$$

$$\rho_j = \frac{1}{hv_1^j(x_j,\lambda)}\int_{x_{j-1}}^{x_j} v_1^j(\xi,\lambda)r(\xi)d\xi + \frac{1}{hv_2^j(x_j,\lambda)}\int_{x_j}^{x_{j+1}} v_2^j(\xi,\lambda)r(\xi)d\xi$$
$$= h^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[v_{\alpha}^{j}(x_j,\lambda)\right]^{-1}z_{\alpha}^{j}(x_j,\lambda).$$

The first singular Cauchy problem (3.4) is equivalent to the system

$$\begin{aligned}\frac{dv_1^1(x,\lambda)}{dx} &= \frac{w_1^1(x,\lambda) - \lambda z_1^1(x,\lambda)}{k(x)},\\ \frac{dw_1^1(x,\lambda)}{dx} &= q(x)v_1^1(x,\lambda),\\ \frac{dz_1^1(x,\lambda)}{dx} &= r(x)v_1^1(x,\lambda), \quad x \in (x_0, x_1),\end{aligned} \tag{3.58}$$

$$v_1^1(x_0,\lambda) = 1, \quad w_1^1(x_0,\lambda) = 0, \quad z_1^1(x_0,\lambda) = 0. \tag{3.59}$$

Let us calculate the coefficients

$$d_1 = \frac{1}{hv_1^1(x_1,\lambda)}\int_{x_0}^{x_1} v_1^1(\xi,\lambda)q(\xi)d\xi + \frac{1}{hv_2^1(x_1,\lambda)}\int_{x_1}^{x_2} v_2^1(\xi,\lambda)q(\xi)d\xi,$$

$$\rho_1 = \frac{1}{hv_1^1(x_1,\lambda)}\int_{x_0}^{x_1} v_1^1(\xi,\lambda)r(\xi)d\xi + \frac{1}{hv_2^1(x_1,\lambda)}\int_{x_1}^{x_2} v_2^1(\xi,\lambda)r(\xi)d\xi.$$

For this we find

$$\int_{x_1}^{x_2} v_2^1(\xi,\lambda)q(\xi)d\xi = \int_{x_1}^{x_2}\frac{dw_2^1(\xi,\lambda)}{d\xi}d\xi$$
$$= w_2^1(x_2,\lambda) - w_2^1(x_1,\lambda) = -1 - w_2^1(x_1,\lambda),$$

$$\int_{x_0}^{x_1} v_1^1(\xi,\lambda)q(\xi)d\xi = \int_{x_0}^{x_1}\frac{dw_1^1(\xi,\lambda)}{d\xi}d\xi = w_1^1(x_1,\lambda) - w_1^1(x_0,\lambda) = w_1^1(x_1,\lambda),$$

$$\int_{x_1}^{x_2} v_2^1(\xi,\lambda)r(\xi)d\xi = \int_{x_1}^{x_2} \frac{dz_2^1(\xi,\lambda)}{d\xi}d\xi = z_2^1(x_2,\lambda) - z_2^1(x_1,\lambda) = -z_2^1(x_1,\lambda),$$

$$\int_{x_0}^{x_1} v_1^1(\xi,\lambda)r(\xi)d\xi = \int_{x_0}^{x_1} \frac{dz_1^1(\xi,\lambda)}{d\xi}d\xi = z_1^1(x_1,\lambda) - z_1^1(x_0,\lambda) = z_1^1(x_1,\lambda).$$

Then

$$\begin{aligned} d_1 &= h^{-1}\left[v_1^1(x_1,\lambda)\right]^{-1} w_1^1(x_1,\lambda) - h^{-1}\left[v_2^1(x_1,\lambda)\right]^{-1}\left[w_2^1(x_1,\lambda)+1\right] \\ &= h^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[v_\alpha^1(x_1,\lambda)\right]^{-1}\left[w_\alpha^1(x_1,\lambda)+\alpha-1\right], \end{aligned}$$

$$\begin{aligned} \rho_1 &= h^{-1}\left[v_1^1(x_1,\lambda)\right]^{-1} z_1^1(x_1,\lambda) - h^{-1}\left[v_2^1(x_1,\lambda)\right]^{-1} z_2^1(x_1,\lambda) \\ &= h^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[v_\alpha^1(x_1,\lambda)\right]^{-1} z_\alpha^1(x_1,\lambda). \end{aligned}$$

The last singular Cauchy problem (3.6) is equivalent to the system

$$\begin{aligned} \frac{dv_2^N(x,\lambda)}{dx} &= \frac{w_2^N(x,\lambda)-\lambda z_2^N(x,\lambda)}{k(x)}, \\ \frac{dw_2^N(x,\lambda)}{dx} &= q(x)v_2^N(x,\lambda), \\ \frac{dz_2^N(x,\lambda)}{dx} &= r(x)v_2^N(x,\lambda), \quad x\in(x_N,x_{N+1}), \end{aligned} \tag{3.60}$$

$$v_2^N(x_{N+1},\lambda)=1, \quad w_2^N(x_{N+1},\lambda)=0, \quad z_2^N(x_{N+1},\lambda)=0. \tag{3.61}$$

The coefficients d_N, ρ_N can be written as

$$\begin{aligned} d_N &= \frac{1}{hv_1^N(x_N,\lambda)}\int_{x_{N-1}}^{x_N} v_1^N(\xi,\lambda)q(\xi)d\xi \\ &+ \frac{1}{hv_2^N(x_N,\lambda)}\int_{x_N}^{x_{N+1}} v_2^N(\xi,\lambda)q(\xi)d\xi, \end{aligned}$$

$$\begin{aligned} \rho_N &= \frac{1}{hv_1^N(x_N,\lambda)}\int_{x_{N-1}}^{x_N} v_1^N(\xi,\lambda)r(\xi)d\xi \\ &+ \frac{1}{hv_2^N(x_N,\lambda)}\int_{x_N}^{x_{N+1}} v_2^N(\xi,\lambda)r(\xi)d\xi. \end{aligned}$$

Because

$$\begin{aligned} \int_{x_{N-1}}^{x_N} v_1^N(\xi,\lambda)q(\xi)d\xi &= \int_{x_{N-1}}^{x_N} \frac{dw_1^N(\xi,\lambda)}{d\xi}d\xi \\ &= w_1^N(x_N,\lambda) - w_1^N(x_{N-1},\lambda) = w_1^N(x_N,\lambda) - 1, \end{aligned}$$

$$\begin{aligned}\int_{x_N}^{x_{N+1}} v_2^N(\xi,\lambda)q(\xi)d\xi &= \int_{x_N}^{x_{N+1}} \frac{dw_2^N(\xi,\lambda)}{d\xi}d\xi \\ &= w_2^N(x_{N+1},\lambda) - w_2^N(x_N,\lambda) = -w_2^N(x_N,\lambda),\\ \int_{x_{N-1}}^{x_N} v_1^N(\xi,\lambda)r(\xi)d\xi &= \int_{x_{N-1}}^{x_N} \frac{dz_1^N(\xi,\lambda)}{d\xi}d\xi \\ &= z_1^N(x_N,\lambda) - z_1^N(x_{N-1},\lambda) = z_1^N(x_N,\lambda),\\ \int_{x_N}^{x_{N+1}} v_2^N(\xi,\lambda)r(\xi)d\xi &= \int_{x_N}^{x_{N+1}} \frac{dz_2^N(\xi,\lambda)}{d\xi}d\xi \\ &= z_2^N(x_{N+1},\lambda) - z_2^N(x_N,\lambda) = -z_2^N(x_N,\lambda),\end{aligned}$$

then

$$\begin{aligned}d_N &= h^{-1}\left[v_1^N(x_N,\lambda)\right]^{-1}\left[w_1^N(x_N,\lambda)-1\right] - h^{-1}\left[v_2^N(x_N,\lambda)\right]^{-1}w_2^N(x_N,\lambda)\\ &= h^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[v_\alpha^N(x_N,\lambda)\right]^{-1}\left[w_\alpha^N(x_N,\lambda)+\alpha-2\right],\end{aligned}$$

$$\begin{aligned}\rho_N &= h^{-1}\left[v_1^N(x_N,\lambda)\right]^{-1}z_1^N(x_N,\lambda) - h^{-1}\left[v_2^N(x_N,\lambda)\right]^{-1}z_2^N(x_N,\lambda)\\ &= h^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[v_\alpha^N(x_N,\lambda)\right]^{-1}z_\alpha^N(x_N,\lambda).\end{aligned}$$

Thus, let us rewrite the ETDS as

$$\begin{aligned}&(ay_{\bar{x}})_{x,j} - (d_j-\lambda\rho_j)y_j = 0, \quad j=2,3,...,N-1,\\ &\frac{1}{h}a_2y_{x,1} - (d_1-\lambda\rho_1)y_1 = 0, \quad -\frac{1}{h}a_Ny_{\bar{x},N} - (d_N-\lambda\rho_N)y_N = 0,\end{aligned} \tag{3.62}$$

where

$$y_{\bar{x},j} := \frac{y_j-y_{j-1}}{h}, \quad y_{x,j} := \frac{y_{j+1}-y_j}{h}, \tag{3.63}$$

$$a_j = a(x_j,\lambda) = \left[\frac{1}{h}v_1^j(x_j,\lambda)\right]^{-1}, \quad j=2,3,...,N, \tag{3.64}$$

$$\begin{aligned}&d_j = d(x_j,\lambda) = h^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[v_\alpha^j(x_j,\lambda)\right]^{-1}\left[w_\alpha^j(x_j,\lambda)+(-1)^\alpha\right],\\ &\quad j=2,3,...,N-1,\end{aligned} \tag{3.65}$$

$$d_1 = d(x_1,\lambda) = h^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[v_\alpha^1(x_1,\lambda)\right]^{-1}\left[w_\alpha^1(x_1,\lambda)+\alpha-1\right], \tag{3.66}$$

$$\begin{aligned} d_N &= d(x_N, \lambda) \\ &= h^{-1} \sum_{\alpha=1}^{2} (-1)^{\alpha+1} \left[v_\alpha^N(x_N, \lambda) \right]^{-1} \left[w_\alpha^N(x_N, \lambda) + \alpha - 2 \right], \end{aligned} \tag{3.67}$$

$$\begin{aligned} \rho_j &= \rho(x_j, \lambda) \\ &= h^{-1} \sum_{\alpha=1}^{2} (-1)^{\alpha+1} \left[v_\alpha^j(x_j, \lambda) \right]^{-1} z_\alpha^j(x_j, \lambda), \quad j = 1, 2, ..., N. \end{aligned} \tag{3.68}$$

Therefore, to calculate the coefficients a_j, d_j, ρ_j of the ETDS for any node x_j of the grid ω_h one needs to solve two Cauchy problems (3.58), (3.59), (3.56), (3.57), (3.60), (3.61) with smooth coefficients: at $\alpha = 1$ on the interval $[x_{j-1}, x_j]$ (forward) and at $\alpha = 2$ on the interval $[x_j, x_{j+1}]$ (backward).

3.4 Three-Point Difference Schemes of High Order of Accuracy

3.4.1 Difference Schemes of Rank $\bar{m}$

Each of the Cauchy problems (3.56), (3.57) will be solved numerically in one step by any one-step method (Taylor series expansion or the Runge-Kutta method) of the order of accuracy $\bar{m} = 2[(m+1)/2]$ (m is a given natural number, $[\cdot]$ is the integer part of the argument in brackets). Then

$$v_\alpha^{(\bar{m})j}(x_j, \lambda) = (-1)^{\alpha+1} h \; \Phi_1 \left(x_{j+(-1)^\alpha}, 0, (-1)^{\alpha+1}, 0, (-1)^{\alpha+1} h \right), \tag{3.69}$$

$$\begin{aligned} w_\alpha^{(\bar{m})j}(x_j, \lambda) &= (-1)^{\alpha+1} \\ &\quad + (-1)^{\alpha+1} h \; \Phi_2 \left(x_{j+(-1)^\alpha}, 0, (-1)^{\alpha+1}, 0, (-1)^{\alpha+1} h \right), \end{aligned} \tag{3.70}$$

$$z_\alpha^{(\bar{m})j}(x_j, \lambda) = (-1)^{\alpha+1} h \; \Phi_3 \left(x_{j+(-1)^\alpha}, 0, (-1)^{\alpha+1}, 0, (-1)^{\alpha+1} h \right), \tag{3.71}$$

where $\Phi_1(x, u, y, z, h)$, $\Phi_2(x, u, y, z, h)$, $\Phi_3(x, u, y, z, h)$ are the increment functions of the one-step method, and $v_\alpha^{(\bar{m})j}(x_j, \lambda), w_\alpha^{(\bar{m})j}(x_j, \lambda), z_\alpha^{(\bar{m})j}(x_j, \lambda)$ approximate the corresponding values $v_\alpha^j(x_j, \lambda)$, $w_\alpha^j(x_j, \lambda), z_\alpha^j(x_j, \lambda)$ with the order of accuracy $\bar{m}$.

In the case of the Taylor series method

$$\begin{aligned} \Phi_1 \left(x_{j+(-1)^\alpha}, 0, (-1)^{\alpha+1}, 0, (-1)^{\alpha+1} h \right) &= \frac{(-1)^{\alpha+1}}{k_{j+(-1)^\alpha}} \\ + \frac{h}{2} \frac{d}{dx} \left(\frac{1}{k(x)} \right) \Bigg|_{x = x_{j+(-1)^\alpha}} &+ \sum_{p=3}^{\bar{m}} \frac{\left[(-1)^{\alpha+1} h \right]^{p-1}}{p!} \left. \frac{d^p v_\alpha^j(x, \lambda)}{dx^p} \right|_{x = x_{j+(-1)^\alpha}}, \end{aligned}$$

$$\Phi_2\left(x_{j+(-1)^\alpha},0,(-1)^{\alpha+1},0,(-1)^{\alpha+1}h\right)=\frac{h}{2}\left.\frac{q(x)}{k(x)}\right|_{x=x_{j+(-1)^\alpha}}$$
$$+\frac{h^2}{6}(-1)^{\alpha+1}\left.\left(\frac{2}{k(x)}\frac{dq(x)}{dx}+\frac{d}{dx}\left(\frac{1}{k(x)}\right)q(x)\right)\right|_{x=x_{j+(-1)^\alpha}}$$
$$+\sum_{p=4}^{\bar{m}}\frac{\left[(-1)^{\alpha+1}h\right]^{p-1}}{p!}\left.\frac{d^p w_\alpha^j(x,\lambda)}{dx^p}\right|_{x=x_{j+(-1)^\alpha}},$$
$$\Phi_3\left(x_{j+(-1)^\alpha},0,(-1)^{\alpha+1},0,(-1)^{\alpha+1}h\right)=\frac{h}{2}\left.\frac{r(x)}{k(x)}\right|_{x=x_{j+(-1)^\alpha}}$$
$$+\frac{h^2}{6}(-1)^{\alpha+1}\left.\left(\frac{2}{k(x)}\frac{dr(x)}{dx}+\frac{d}{dx}\left(\frac{1}{k(x)}\right)r(x)\right)\right|_{x=x_{j+(-1)^\alpha}}$$
$$+\sum_{p=4}^{\bar{m}}\frac{\left[(-1)^{\alpha+1}h\right]^{p-1}}{p!}\left.\frac{d^p z_\alpha^j(x,\lambda)}{dx^p}\right|_{x=x_{j+(-1)^\alpha}},$$

and in the case of the Runge-Kutta methods

$$\Phi_1\left(x_{j+(-1)^\alpha},0,(-1)^{\alpha+1},0,(-1)^{\alpha+1}h\right)=b_1g_1+b_2g_2+...+b_sg_s,$$
$$\Phi_2\left(x_{j+(-1)^\alpha},0,(-1)^{\alpha+1},0,(-1)^{\alpha+1}h\right)=b_1\bar{g}_1+b_2\bar{g}_2+...+b_s\bar{g}_s,$$
$$\Phi_3\left(x_{j+(-1)^\alpha},0,(-1)^{\alpha+1},0,(-1)^{\alpha+1}h\right)=b_1\tilde{g}_1+b_2\tilde{g}_2+...+b_s\tilde{g}_s,$$
$$g_i=\frac{(-1)^{\alpha+1}\left(1+h\sum_{p=1}^{i-1}a_{ip}(\bar{g}_p-\tilde{g}_p)\right)}{k\left(x_{j+(-1)^\alpha}+c_i(-1)^{\alpha+1}h\right)},$$
$$\bar{g}_i=(-1)^{\alpha+1}h\sum_{p=1}^{i-1}a_{ip}g_pq\left(x_{j+(-1)^\alpha}+c_i(-1)^{\alpha+1}h\right),$$
$$\tilde{g}_i=(-1)^{\alpha+1}h\sum_{p=1}^{i-1}a_{ip}g_pr\left(x_{j+(-1)^\alpha}+c_i(-1)^{\alpha+1}h\right),\quad i=1,2,...,s.$$

We will solve the singular Cauchy problem (3.58), (3.59) by the Taylor series method. To do this, at first we use the substitutions $w_1^1(x,\lambda)=(1+x)\tilde{w}_1^1(x,\lambda)$, $z_1^1(x,\lambda)=(1+x)\tilde{z}_1^1(x,\lambda)$. Taking into account

$$\tilde{w}_1^1(x_0,\lambda)=\lim_{x\to x_0}\frac{w_1^1(x,\lambda)}{1+x}=\lim_{x\to x_0}\frac{dw_1^1(x,\lambda)}{dx}=q_0$$

and

$$\tilde{z}_1^1(x_0,\lambda)=\lim_{x\to x_0}\frac{z_1^1(x,\lambda)}{1+x}=\lim_{x\to x_0}\frac{dz_1^1(x,\lambda)}{dx}=r_0,$$

we reduce this problem to the form

$$\begin{aligned}
\frac{dv_1^1(x,\lambda)}{dx} &= \frac{\tilde{w}_1^1(x,\lambda)-\lambda\tilde{z}_1^1(x,\lambda)}{(1-x)k_1(x)},\\
\frac{d\tilde{w}_1^1(x,\lambda)}{dx} &= \frac{-\tilde{w}_1^1(x,\lambda)+q(x)v_1^1(x,\lambda)}{1+x},\\
\frac{d\tilde{z}_1^1(x,\lambda)}{dx} &= \frac{-\tilde{z}_1^1(x,\lambda)+r(x)v_1^1(x,\lambda)}{1+x},\\
v_1^1(x_0,\lambda) &= 1, \quad \tilde{w}_1^1(x_0,\lambda)=q_0, \quad \tilde{z}_1^1(x_0,\lambda)=r_0.
\end{aligned} \tag{3.72}$$

Applying the L'hopital's rule, we pass to the limit as $x \to x_0$ in the second equation of the system (3.72), then we get

$$\begin{aligned}
\frac{d\tilde{w}_1^1(x_0,\lambda)}{dx} &= \lim_{x\to x_0}\frac{-\tilde{w}_1^1+q(x)v_1^1(x,\lambda)}{1+x}\\
&= -\frac{d\tilde{w}_1^1(x_0,\lambda)}{dx}+\frac{d}{dx}\left(q(x)v_1^1(x,\lambda)\right)\Big|_{x=x_0}.
\end{aligned}$$

It follows

$$\frac{d\tilde{w}_1^1(x_0,\lambda)}{dx} = \frac{1}{2}\frac{d}{dx}\left(q(x)v_1^1(x,\lambda)\right)\Big|_{x=x_0}.$$

Differentiating the second equation of the system (3.72) and taking into account the equality

$$-\tilde{w}_1^1+q(x)v_1^1(x,\lambda) = (1+x)\frac{d\tilde{w}_1^1(x,\lambda)}{dx},$$

we obtain

$$\begin{aligned}
\frac{d^2\tilde{w}_1^1(x,\lambda)}{dx^2} &= \frac{\left(-\frac{d\tilde{w}_1^1(x,\lambda)}{dx}+\frac{d}{dx}\left(q(x)v_1^1(x,\lambda)\right)\right)(1+x)-\left(-\tilde{w}_1^1+q(x)v_1^1(x,\lambda)\right)}{(1+x)^2}\\
&= \frac{1}{1+x}\left(-2\frac{d\tilde{w}_1^1(x,\lambda)}{dx}+\frac{d}{dx}\left(q(x)v_1^1(x,\lambda)\right)\right).
\end{aligned} \tag{3.73}$$

By passing to the limit as $x \to x_0$, we get

$$\begin{aligned}
\frac{d^2\tilde{w}_1^1(x_0,\lambda)}{dx^2} &= \lim_{x\to x_0}\frac{-2\frac{d\tilde{w}_1^1(x,\lambda)}{dx}+\frac{d}{dx}\left(q(x)v_1^1(x,\lambda)\right)}{1+x}\\
&= -2\frac{d^2\tilde{w}_1^1(x_0,\lambda)}{dx^2}+\frac{d^2}{dx^2}\left(q(x)v_1^1(x,\lambda)\right)\Big|_{x=x_0},
\end{aligned}$$

from where follows

$$\frac{d^2\tilde{w}_1^1(x_0,\lambda)}{dx^2} = \frac{1}{3}\frac{d^2}{dx^2}\left(q(x)v_1^1(x,\lambda)\right)\Big|_{x=x_0}.$$

Using the method of mathematical induction, we show that the derivatives of the function $\tilde{w}_1^1(x,\lambda)$ of order i have the form

$$\frac{d^i\tilde{w}_1^1(x,\lambda)}{dx^i} = \frac{1}{1+x}\left(-i\frac{d^{i-1}\tilde{w}_1^1(x,\lambda)}{dx^{i-1}} + \frac{d^{i-1}}{dx^{i-1}}\left(q(x)v_1^1(x,\lambda)\right)\right), \quad i = 1,2,... \tag{3.74}$$

and

$$\frac{d^i\tilde{w}_1^1(x_0,\lambda)}{dx^i} = \frac{1}{i+1}\frac{d^i}{dx^i}\left(q(x)v_1^1(x,\lambda)\right)\Big|_{x=x_0}, \quad i = 1,2,... \tag{3.75}$$

Let the equalities (3.74), (3.75) hold for $i = j$, i.e.

$$\frac{d^j\tilde{w}_1^1(x,\lambda)}{dx^j} = \frac{1}{1+x}\left(-j\frac{d^{j-1}\tilde{w}_1^1}{dx^{j-1}} + \frac{d^{j-1}}{dx^{j-1}}\left(q(x)v_1^1(x,\lambda)\right)\right) \tag{3.76}$$

and

$$\frac{d^j\tilde{w}_1^1(x_0,\lambda)}{dx^j} = \frac{1}{j+1}\frac{d^j}{dx^j}\left(q(x)v_1^1(x,\lambda)\right)\Big|_{x=x_0}. \tag{3.77}$$

Let us prove the formulas (3.74), (3.75) for $i = j+1$.

We differentiate (3.76), then we get

$$\frac{d^{j+1}\tilde{w}_1^1(x,\lambda)}{dx^{j+1}} =$$

$$= \frac{\left[-j\frac{d^j\tilde{w}_1^1}{dx^j} + \frac{d^j}{dx^j}\left(q(x)v_1^1(x,\lambda)\right)\right](1+x) - \left(-j\frac{d^{j-1}\tilde{w}_1^1}{dx^{j-1}} + \frac{d^{j-1}}{dx^{j-1}}\left(q(x)v_1^1(x,\lambda)\right)\right)}{(1+x)^2}.$$

From the equality (3.76) we find

$$-j\frac{d^{j-1}\tilde{w}_1^1}{dx^{j-1}} + \frac{d^{j-1}}{dx^{j-1}}\left(q(x)v_1^1(x,\lambda)\right) = (1+x)\frac{d^j\tilde{w}_1^1(x,\lambda)}{dx^j}.$$

So,

$$\frac{d^{j+1}\tilde{w}_1^1(x,\lambda)}{dx^{j+1}} = \frac{1}{1+x}\left(-(j+1)\frac{d^j\tilde{w}_1^1(x,\lambda)}{dx^j} + \frac{d^j}{dx^j}\left(q(x)v_1^1(x,\lambda)\right)\right).$$

In the last equation, we find the limit as $x \to x_0$. Taking into account (3.77), for calculating the limit we apply the L'hopital's rule

$$\begin{aligned}\frac{d^{j+1}\tilde{w}_1^1(x_0,\lambda)}{dx^{j+1}} &= \lim_{x\to x_0} \frac{-(j+1)\frac{d^j\tilde{w}_1^1(x,\lambda)}{dx^j} + \frac{d^j}{dx^j}\left(q(x)v_1^1(x,\lambda)\right)}{1+x} = \\ &= -(j+1)\frac{d^{j+1}\tilde{w}_1^1(x_0,\lambda)}{dx^{j+1}} + \frac{d^{j+1}}{dx^{j+1}}\left(q(x)v_1^1(x,\lambda)\right)\Big|_{x=x_0}.\end{aligned}$$

Then

$$\frac{d^{j+1}\tilde{w}_1^1(x_0,\lambda)}{dx^{j+1}} = \frac{1}{j+2}\frac{d^{j+1}}{dx^{j+1}}\left(q(x)v_1^1(x,\lambda)\right)\Big|_{x=x_0}.$$

Therefore, the validity of the equalities (3.74), (3.75) has been proved.

Taking into account that the third equation of the system (3.72) has the same form as the second, the analogous equalities hold for the function $\tilde{z}_1^1(x,\lambda)$

$$\begin{aligned}\frac{d^i\tilde{z}_1^1(x,\lambda)}{dx^i} &= \frac{1}{1+x}\left(-i\frac{d^{i-1}\tilde{z}_1^1(x,\lambda)}{dx^{i-1}} + \frac{d^{i-1}}{dx^{i-1}}\left(r(x)v_1^1(x,\lambda)\right)\right), \\ &\quad i = 1, 2, ...\end{aligned} \tag{3.78}$$

and

$$\frac{d^i\tilde{z}_1^1(x_0,\lambda)}{dx^i} = \frac{1}{i+1}\frac{d^i}{dx^i}\left(r(x)v_1^1(x,\lambda)\right)\Big|_{x=x_0}, \quad i = 1, 2, ... \tag{3.79}$$

The method of Taylor series for the problem (3.72) in the vicinity of the singularity point $x = x_0$ has the form

$$\begin{aligned}\tilde{w}_1^{(\bar{m})1}(x_1,\lambda) &= \tilde{w}_1^1(x_0,\lambda) + \sum_{i=1}^{\bar{m}-1}\frac{h^i}{2^i i!}\frac{d^i\tilde{w}_1^1(x,\lambda))}{dx^i}\Big|_{x=x_0} \\ &= q_0 + \sum_{i=1}^{\bar{m}-1}\frac{h^i}{2^i(i+1)!}\frac{d^i(q(x)v_1^1(x,\lambda))}{dx^i}\Big|_{x=x_0}, \\ \tilde{z}_1^{(\bar{m})1}(x_1,\lambda) &= \tilde{z}_1^1(x_0,\lambda) + \sum_{i=1}^{\bar{m}-1}\frac{h^i}{2^i i!}\frac{d^i\tilde{z}_1^1(x,\lambda))}{dx^i}\Big|_{x=x_0} \\ &= r_0 + \sum_{i=1}^{\bar{m}-1}\frac{h^i}{2^i(i+1)!}\frac{d^i(r(x)v_1^1(x,\lambda))}{dx^i}\Big|_{x=x_0}.\end{aligned}$$

Returning to the functions $w_1^1(x,\lambda) = (1+x)\tilde{w}_1^1(x,\lambda)$, $z_1^1(x,\lambda) = (1+x)\tilde{z}_1^1(x,\lambda)$, we get

$$w_1^{(\bar{m})1}(x_1,\lambda) = \frac{h}{2}q_0 + \sum_{i=2}^{\bar{m}}\frac{h^i}{2^i i!}\frac{d^{i-1}(q(x)v_1^1(x,\lambda))}{dx^{i-1}}\Big|_{x=x_0}. \tag{3.80}$$

$$z_1^{(\bar{m})1}(x_1,\lambda)=\frac{h}{2}r_0+\sum_{i=2}^{\bar{m}}\frac{h^i}{2^i i!}\left.\frac{d^{i-1}(r(x)v_1^1(x,\lambda))}{dx^{i-1}}\right|_{x=x_0}. \tag{3.81}$$

The series expansion of the function $v_1^1(x,\lambda)$ will be of the form

$$v_1^{(\bar{m})1}(x_1,\lambda)=v_1^1(x_0,\lambda)+\frac{h}{2}\left.\frac{dv_1^1(x,\lambda)}{dx}\right|_{x=x_0}+\sum_{i=2}^{\bar{m}}\frac{h^i}{2^i i!}\left.\frac{d^i v_1^1(x,\lambda)}{dx^i}\right|_{x=x_0}. \tag{3.82}$$

It follows

$$\Phi_1(x_0,1,0,0,h)=\frac{q(x_0)-\lambda r(x_0)}{4k_1(x_0)}+\sum_{i=2}^{\bar{m}}\frac{h^{i-1}}{2^i i!}\left.\frac{d^i v_1^1(x,\lambda)}{dx^i}\right|_{x=x_0},$$

$$\Phi_2(x_0,1,0,0,h)=\frac{1}{2}q(x_0)+\sum_{i=2}^{\bar{m}}\frac{h^{i-1}}{2^i i!}\left.\frac{d^{i-1}(q(x)v_1^1(x,\lambda))}{dx^{i-1}}\right|_{x=x_0},$$

$$\Phi_3(x_0,1,0,0,h)=\frac{1}{2}r(x_0)+\sum_{i=2}^{\bar{m}}\frac{h^{i-1}}{2^i i!}\left.\frac{d^{i-1}(r(x)v_1^1(x,\lambda))}{dx^{i-1}}\right|_{x=x_0}.$$

We now consider the Cauchy problem (3.60), (3.61). Using the substitution $w_2^N(x,\lambda)=(1-x)\tilde{w}_2^N(x,\lambda)$ and $z_2^N(x,\lambda)=(1-x)\tilde{z}_2^N(x,\lambda)$, we reduce this problem to the system

$$\begin{aligned}
&\frac{dv_2^N(x,\lambda)}{dx}=\frac{\tilde{w}_2^N(x,\lambda)-\lambda\tilde{z}_2^N(x,\lambda)}{(1+x)k_1(x)},\\
&\frac{d\tilde{w}_2^N(x,\lambda)}{dx}=\frac{\tilde{w}_2^N(x,\lambda)+q(x)v_2^N(x,\lambda)}{1-x},\\
&\frac{d\tilde{z}_2^N(x,\lambda)}{dx}=\frac{\tilde{z}_2^N(x,\lambda)+r(x)v_2^N(x,\lambda)}{1-x},\\
&v_2^N(x_{N+1},\lambda)=1,\quad \tilde{w}_2^N(x_{N+1},\lambda)=-q_{N+1},\quad \tilde{z}_2^N(x_{N+1},\lambda)=-r_{N+1}.
\end{aligned} \tag{3.83}$$

We pass to the limit in the second equation of the system as $x\to x_{N+1}$

$$\begin{aligned}
\frac{d\tilde{w}_2^N(x_{N+1},\lambda)}{dx}&=\lim_{x\to x_{N+1}}\frac{\tilde{w}_2^N(x,\lambda)+q(x)v_2^N(x,\lambda)}{1-x}\\
&=-\frac{d\tilde{w}_2^N(x_{N+1},\lambda)}{dx}-\frac{d}{dx}\left.\left(q(x)v_2^N(x,\lambda)\right)\right|_{x=x_{N+1}}.
\end{aligned}$$

Then

$$\frac{d\tilde{w}_2^N(x_{N+1},\lambda)}{dx}=-\frac{1}{2}\frac{d}{dx}\left.\left(q(x)v_2^N(x,\lambda)\right)\right|_{x=x_{N+1}}.$$

Differentiating the second equation of the system (3.83) and taking into account the equality

$$\tilde{w}_2^N(x,\lambda) + q(x)v_2^N(x,\lambda) = (1-x)\frac{d\tilde{w}_2^N(x,\lambda)}{dx},$$

we get

$$\begin{aligned}\frac{d^2\tilde{w}_2^N(x,\lambda)}{dx^2} &= \frac{\left(\frac{d\tilde{w}_2^N(x,\lambda)}{dx} + \frac{d}{dx}\left(q(x)v_2^N(x,\lambda)\right)\right)(1-x) + \left(\tilde{w}_2^N(x,\lambda) + q(x)v_2^N(x,\lambda)\right)}{(1-x)^2} \\ &= \frac{1}{1-x}\left(2\frac{d\tilde{w}_2^N(x,\lambda)}{dx} + \frac{d}{dx}\left(q(x)v_2^N(x,\lambda)\right)\right).\end{aligned}$$

It follows

$$\frac{d^2\tilde{w}_2^N(x_{N+1},\lambda)}{dx^2} = -\frac{1}{3}\frac{d^2}{dx^2}\left(q(x)v_2^N(x,\lambda)\right)\Big|_{x=x_{N+1}}.$$

Using the method of mathematical induction, we prove that the derivatives of the function $\tilde{w}_2^N(x_{N+1})$ of order i have the form

$$\begin{aligned}\frac{d^i\tilde{w}_2^N(x,\lambda)}{dx^i} &= \frac{1}{1-x}\left(i\frac{d^{i-1}\tilde{w}_2^N(x,\lambda)}{dx^{i-1}} + \frac{d^{i-1}}{dx^{i-1}}\left(q(x)v_2^N(x,\lambda)\right)\right), \\ & i = 1,2,...\end{aligned} \tag{3.84}$$

and

$$\frac{d^i\tilde{w}_2^N(x_{N+1},\lambda)}{dx^i} = -\frac{1}{i+1}\frac{d^i}{dx^i}\left(q(x)v_2^N(x,\lambda)\right)\Big|_{x=x_{N+1}}, \quad i = 1,2,... \tag{3.85}$$

Let (3.84) and (3.85) hold for $i = j$, then

$$\frac{d^j\tilde{w}_2^N(x,\lambda)}{dx^j} = \frac{1}{1-x}\left(j\frac{d^{j-1}\tilde{w}_2^N(x,\lambda)}{dx^{j-1}} + \frac{d^{j-1}}{dx^{j-1}}\left(q(x)v_2^N(x,\lambda)\right)\right), \tag{3.86}$$

$$\frac{d^j\tilde{w}_2^N(x_{N+1},\lambda)}{dx^j} = -\frac{1}{j+1}\frac{d^j}{dx^j}\left(q(x)v_2^N(x,\lambda)\right)\Big|_{x=x_{N+1}}. \tag{3.87}$$

Let us prove the formulas (3.84), (3.85) for $i = j+1$.

Differentiate (3.86), then

$$\begin{aligned}&\frac{d^{j+1}\tilde{w}_2^N(x,\lambda)}{dx^{j+1}} = \\ &= \frac{\left[j\frac{d^j\tilde{w}_2^N(x,\lambda)}{dx^j} + \frac{d^j}{dx^j}\left(q(x)v_2^N(x,\lambda)\right)\right](1-x) + \left(j\frac{d^{j-1}\tilde{w}_2^N(x,\lambda)}{dx^{j-1}} + \frac{d^{j-1}}{dx^{j-1}}\left(q(x)v_2^N(x,\lambda)\right)\right)}{(1-x)^2}.\end{aligned}$$

Taking into account

$$j\frac{d^{j-1}\tilde{w}_2^N(x,\lambda)}{dx^{j-1}}+\frac{d^{j-1}}{dx^{j-1}}\left(q(x)v_2^N(x,\lambda)\right)=(1-x)\frac{d^j\tilde{w}_2^N}{dx^j},$$

we get

$$\frac{d^{j+1}\tilde{w}_2^N(x,\lambda)}{dx^{j+1}}=\frac{1}{1-x}\left((j+1)\frac{d^j\tilde{w}_2^N(x,\lambda)}{dx^j}+\frac{d^j}{dx^j}\left(q(x)v_2^N(x,\lambda)\right)\right).$$

So,

$$\frac{d^{j+1}\tilde{w}_2^N(x_{N+1},\lambda)}{dx^{j+1}}=-\frac{1}{j+2}\frac{d^{j+1}}{dx^{j+1}}\left(q(x)v_2^N(x,\lambda)\right)\Big|_{x=x_{N+1}}.$$

Thus, the equalities (3.84), (3.85) are proved.

Since the third equation has a similar form to the second, similar equalities hold for the function $\tilde{z}_2^N(x_{N+1},\lambda)$, i.e.

$$\frac{d^i\tilde{z}_2^N(x,\lambda)}{dx^i}=\frac{1}{1-x}\left(i\frac{d^{i-1}\tilde{z}_2^N(x,\lambda)}{dx^{i-1}}+\frac{d^{i-1}}{dx^{i-1}}\left(r(x)v_2^N(x,\lambda)\right)\right),\quad i=1,2,... \tag{3.88}$$

and

$$\frac{d^i\tilde{z}_2^N(x_{N+1},\lambda)}{dx^i}=-\frac{1}{i+1}\frac{d^i}{dx^i}\left(r(x)v_2^N(x,\lambda)\right)\Big|_{x=x_{N+1}},\quad i=1,2,... \tag{3.89}$$

The method of Taylor series for the problem (3.83) in the vicinity of the point $x=x_{N+1}$ will have the form

$$\tilde{w}_2^{(\bar{m})N}(x_N,\lambda)=-q_{N+1}-\sum_{i=1}^{\bar{m}-1}\frac{(-h)^i}{2^i(i+1)!}\left.\frac{d^i(q(x)v_2^N(x,\lambda))}{dx^i}\right|_{x=x_{N+1}},$$

$$\tilde{z}_2^{(\bar{m})N}(x_N,\lambda)=-r_{N+1}-\sum_{i=1}^{\bar{m}-1}\frac{(-h)^i}{2^i(i+1)!}\left.\frac{d^i(r(x)v_2^N(x,\lambda))}{dx^i}\right|_{x=x_{N+1}}.$$

Let us now write the Taylor series method for the functions $w_2^N(x,\lambda)=\tilde{w}_2^N(x,\lambda)(1-x)$ and $z_2^N(x,\lambda)=\tilde{z}_2^N(x,\lambda)(1-x)$ as

$$w_2^{(\bar{m})N}(x_N,\lambda)=-\frac{h}{2}q_{N+1}+\sum_{i=2}^{\bar{m}}\frac{(-h)^i}{2^i i!}\left.\frac{d^{i-1}(q(x)v_2^N(x,\lambda))}{dx^{i-1}}\right|_{x=x_{N+1}}, \tag{3.90}$$

$$z_2^{(\bar{m})N}(x_N,\lambda) = -\frac{h}{2}r_{N+1} + \sum_{i=2}^{\bar{m}} \frac{(-h)^i}{2^i i!}\left.\frac{d^{i-1}(r(x)v_2^N(x,\lambda))}{dx^{i-1}}\right|_{x=x_{N+1}}. \qquad (3.91)$$

The Taylor series method for the function $v_2^N(x,\lambda)$ will be

$$v_2^{(\bar{m})N}(x_N,\lambda) = 1 - \frac{h}{2}\left.\frac{dv_2^N(x,\lambda)}{dx}\right|_{x=x_{N+1}} + \sum_{i=2}^{\bar{m}} \frac{(-h)^i}{2^i i!}\left.\frac{d^i v_2^N(x,\lambda)}{dx^i}\right|_{x=x_{N+1}}. \qquad (3.92)$$

It follows

$$\Phi_1(x_{N+1},1,0,0,-h) = -\frac{q(x_{N+1}) - \lambda r(x_{N+1})}{4k_1(x_{N+1})} + \sum_{i=2}^{\bar{m}} \frac{(-h)^{i-1}}{2^i i!}\left.\frac{d^i v_2^N(x,\lambda)}{dx^i}\right|_{x=x_{N+1}},$$

$$\Phi_2(x_{N+1},1,0,0,-h) = \frac{1}{2}q(x_{N+1}) + \sum_{i=2}^{\bar{m}} \frac{(-h)^{i-1}}{2^i i!}\left.\frac{d^{i-1}(q(x)v_2^N(x,\lambda))}{dx^{i-1}}\right|_{x=x_{N+1}},$$

$$\Phi_3(x_{N+1},1,0,0,-h) = \frac{1}{2}r(x_{N+1}) + \sum_{i=2}^{\bar{m}} \frac{(-h)^{i-1}}{2^i i!}\left.\frac{d^{i-1}(r(x)v_2^N(x,\lambda))}{dx^{i-1}}\right|_{x=x_{N+1}}.$$

If $k(x), q(x), r(x)$ are sufficiently smooth functions and the methods (3.69) – (3.71) have the order of accuracy $\bar{m}$, then for $\alpha = 1,2,\ j = 1,2,...,N$ the equalities hold (see, e.g., [33, p. 168])

$$v_\alpha^j(x_j,\lambda) = v_\alpha^{(\bar{m})j}(x_j,\lambda) + \left[(-1)^{\alpha+1}h\right]^{\bar{m}+1}\psi_\alpha^j(x_{j+(-1)^\alpha}) + O\left(h^{\bar{m}+2}\right), \qquad (3.93)$$

$$w_\alpha^j(x_j,\lambda) = w_\alpha^{(\bar{m})j}(x_j,\lambda) + \left[(-1)^{\alpha+1}h\right]^{\bar{m}+1}\bar{\psi}_\alpha^j(x_{j+(-1)^\alpha}) + O\left(h^{\bar{m}+2}\right), \qquad (3.94)$$

$$z_\alpha^j(x_j,\lambda) = z_\alpha^{(\bar{m})j}(x_j,\lambda) + \left[(-1)^{\alpha+1}h\right]^{\bar{m}+1}\tilde{\psi}_\alpha^j(x_{j+(-1)^\alpha}) + O\left(h^{\bar{m}+2}\right). \qquad (3.95)$$

Lemma 3.6. *Let the conditions (3.3), $k(x) \in Q^{(m+1)}[-1,1]$, $q(x), r(x) \in Q^{(m)}[-1,1]$ hold and for the numerical method (3.69) – (3.71) the equalities (3.93) – (3.95) are satisfied. Then the relations will hold*

$$v_\alpha^j(x_j,\lambda) = v_\alpha^{(\bar{m})j}(x_j,\lambda) + h^{\bar{m}+1}\psi_1^{j-1+\alpha}(x_{j+(-1)^\alpha}) + O\left(h^{\bar{m}+2}\right), \qquad (3.96)$$

$$w_\alpha^j(x_j,\lambda) = w_\alpha^{(\bar{m})j}(x_j,\lambda) + h^{\bar{m}+1}\bar{\psi}_1^{j-1+\alpha}(x_{j+(-1)^\alpha}) + O\left(h^{\bar{m}+2}\right), \qquad (3.97)$$

$$z_\alpha^j(x_j,\lambda) = z_\alpha^{(\bar{m})j}(x_j,\lambda) + h^{\bar{m}+1}\tilde{\psi}_1^{j-1+\alpha}(x_{j+(-1)^\alpha}) + O\left(h^{\bar{m}+2}\right), \qquad (3.98)$$

$$j = 3-\alpha, 4-\alpha, \ldots, N+1-\alpha, \quad \alpha = 1,2.$$

Proof. Note that the functions $n_2^j(x_j,\lambda) = -v_2^j(x_j,\lambda)$, $l_2^j(x_j,\lambda) = -w_2^j(x_j,\lambda)$, $p_2^j(x_j,\lambda) = -z_2^j(x_j,\lambda)$ are the solution of the Cauchy problem

$$\begin{aligned} \frac{dn_2^j(x,\lambda)}{dx} &= \frac{l_2^j(x,\lambda) - \lambda p_2^j(x,\lambda)}{k(x)}, \\ \frac{dl_2^j(x,\lambda)}{dx} &= q(x)n_2^j(x,\lambda), \\ \frac{dp_2^j(x,\lambda)}{dx} &= r(x)n_2^j(x,\lambda), \quad x \in (x_j, x_{j+1}), \end{aligned}$$

$$n_2^j(x_{j+1},\lambda) = 0, \quad l_2^j(x_{j+1},\lambda) = 1, \quad p_2^j(x_{j+1},\lambda) = 0,$$
$$j = 1, 2, ..., N-1,$$

for the numerical solution of which we apply the one-step method

$$n_2^{(\bar{m})j}(x_j,\lambda) = -h\Phi_1(x_{j+1},0,1,0,-h), \tag{3.99}$$

$$l_2^{(\bar{m})j}(x_j,\lambda) = 1 - h\Phi_2(x_{j+1},0,1,0,-h), \tag{3.100}$$

$$p_2^{(\bar{m})j}(x_j,\lambda) = -h\Phi_3(x_{j+1},0,1,0,-h). \tag{3.101}$$

Considering (3.93) – (3.95) and the fact that $\bar{m}$ is an even number, for $n_2^j(x_j,\lambda), l_2^j(x_j,\lambda), p_2^j(x_j,\lambda)$ the equalities hold

$$n_2^j(x_j,\lambda) = -v_2^{(\bar{m})j}(x_j,\lambda) + h^{\bar{m}+1}\psi_2^j(x_{j+1}) + O\left(h^{\bar{m}+2}\right), \tag{3.102}$$

$$l_2^j(x_j,\lambda) = -w_2^{(\bar{m})j}(x_j,\lambda) + h^{\bar{m}+1}\bar{\psi}_2^j(x_{j+1}) + O\left(h^{\bar{m}+2}\right), \tag{3.103}$$

$$p_2^j(x_j,\lambda) = -z_2^{(\bar{m})j}(x_j,\lambda) + h^{\bar{m}+1}\tilde{\psi}_2^j(x_{j+1}) + O\left(h^{\bar{m}+2}\right). \tag{3.104}$$

If in formulas (3.69) – (3.71) for $\alpha = 1$ we replace the index j by $j+1$, then we get the adjoint method to the method (3.99) – (3.101). Applying the Theorem 8.5 [33, p. 220] to these methods and considering that the principal error terms for $n_2^j(x_j,\lambda), l_2^j(x_j,\lambda), p_2^j(x_j,\lambda)$ and for $v_2^j(x_j,\lambda), w_2^j(x_j,\lambda), z_2^j(x_j,\lambda)$ differ only in signs, we obtain

$$\begin{aligned} \psi_2^j(x_{j+1}) &= -(-1)^{\bar{m}}\psi_1^{j+1}(x_j) = -(-1)^{\bar{m}}\psi_1^{j+1}(x_{j+1}) + O(h) \\ &= -\psi_1^{j+1}(x_{j+1}) + O(h), \end{aligned}$$

$$\bar{\psi}_2^j(x_{j+1}) = -(-1)^{\bar{m}}\bar{\psi}_1^{j+1}(x_{j+1}) + O(h) = -\bar{\psi}_1^{j+1}(x_{j+1}) + O(h),$$
$$\tilde{\psi}_2^j(x_{j+1}) = -(-1)^{\bar{m}}\tilde{\psi}_1^{j+1}(x_{j+1}) + O(h) = -\tilde{\psi}_1^{j+1}(x_{j+1}) + O(h).$$

From this and the equalities (3.93) – (3.95), the statement of the Lemma follows. □

Instead of the ETDS (3.62) – (3.68), we can now use the three-point difference scheme of rank $\bar{m}$ of the form

$$\begin{aligned} &\left(a^{(\bar{m})} y_{\bar{x}}^{(\bar{m})}\right)_{x,j} - (d_j^{(\bar{m})} - \lambda^{(\bar{m})} \rho_j^{(\bar{m})}) y_j^{(\bar{m})} = 0, \quad j = 2, 3, ..., N-1, \\ &\frac{1}{h} a_2^{(\bar{m})} y_{x,1}^{(\bar{m})} - (d_1^{(\bar{m})} - \lambda^{(\bar{m})} \rho_1^{(\bar{m})}) y_1^{(\bar{m})} = 0, \\ &-\frac{1}{h} a_N^{(\bar{m})} y_{\bar{x},N}^{(\bar{m})} - (d_N^{(\bar{m})} - \lambda^{(\bar{m})} \rho_N^{(\bar{m})}) y_N^{(\bar{m})} = 0, \end{aligned} \tag{3.105}$$

where

$$a^{(\bar{m})}(x_j, \lambda^{(\bar{m})}) = \left[\frac{1}{h} v_1^{(\bar{m})j}(x_j, \lambda^{(\bar{m})})\right]^{-1}, \quad j = 2, 3, ..., N, \tag{3.106}$$

$$\begin{aligned} d^{(\bar{m})}(x_j, \lambda^{(\bar{m})}) = h^{-1} \sum_{\alpha=1}^{2} (-1)^{\alpha+1} \left[v_\alpha^{(\bar{m})j}(x_j, \lambda^{(\bar{m})})\right]^{-1} \left[w_\alpha^{(\bar{m})j}(x_j, \lambda^{(\bar{m})}) + (-1)^\alpha\right], \\ j = 2, 3, ..., N-1, \end{aligned} \tag{3.107}$$

$$d^{(\bar{m})}(x_1, \lambda^{(\bar{m})}) = h^{-1} \sum_{\alpha=1}^{2} (-1)^{\alpha+1} \left[v_\alpha^{(\bar{m})1}(x_1, \lambda^{(\bar{m})})\right]^{-1} \left[w_\alpha^{(\bar{m})1}(x_1, \lambda^{(\bar{m})}) + \alpha - 1\right], \tag{3.108}$$

$$d^{(\bar{m})}(x_N, \lambda^{(\bar{m})}) = h^{-1} \sum_{\alpha=1}^{2} (-1)^{\alpha+1} \left[v_\alpha^{(\bar{m})N}(x_N, \lambda^{(\bar{m})})\right]^{-1} \left[w_\alpha^{(\bar{m})N}(x_N, \lambda^{(\bar{m})}) + \alpha - 2\right], \tag{3.109}$$

$$\begin{aligned} \rho^{(\bar{m})}(x_j, \lambda^{(\bar{m})}) = h^{-1} \sum_{\alpha=1}^{2} (-1)^{\alpha+1} \left[v_\alpha^{(\bar{m})j}(x_j, \lambda^{(\bar{m})})\right]^{-1} z_\alpha^{(\bar{m})j}(x_j, \lambda^{(\bar{m})}), \\ j = 1, 2, ..., N. \end{aligned} \tag{3.110}$$

In the following, we will denote by M a general constant, which does not depend on h.

Lemma 3.7. *Let the conditions of Lemma 3.6 be satisfied. Then*

$$\left|a^{(\bar{m})}(x_j, \lambda) - a(x_j, \lambda)\right| \le M h^{\bar{m}}, \tag{3.111}$$

$$d(x_j, \lambda) - d^{(\bar{m})}(x_j, \lambda) = -\left\{h^{\bar{m}} \, k(x) \bar{\psi}_1^j(x)\Big|_{x=x_j-0}\right\}_x + O\left(h^{\bar{m}}\right), \tag{3.112}$$

$$\rho(x_j, \lambda) - \rho^{(\bar{m})}(x_j, \lambda) = -\left\{h^{\bar{m}} \, k(x) \tilde{\psi}_1^j(x)\Big|_{x=x_j-0}\right\}_x + O\left(h^{\bar{m}}\right), \tag{3.113}$$

$$\left|a^{(\bar{m})}(x_j, \lambda) - a^{(\bar{m})}(x_j, \tilde{\lambda})\right| \le M h^2 |\lambda - \tilde{\lambda}|, \tag{3.114}$$

$$\left|d^{(\bar{m})}(x_j,\lambda)-d^{(\bar{m})}(x_j,\tilde{\lambda})\right|\leq Mh|\lambda-\tilde{\lambda}|, \tag{3.115}$$

$$\left|\rho^{(\bar{m})}(x_j,\lambda)-\rho^{(\bar{m})}(x_j,\tilde{\lambda})\right|\leq Mh|\lambda-\tilde{\lambda}|. \tag{3.116}$$

Proof. From (3.61) and (3.69), we get

$$a^{(\bar{m})}(x_j,\lambda)-a(x_j,\lambda)=\frac{h\left[v_1^j\left(x_j,\lambda\right)-v_1^{(\bar{m})j}\left(x_j,\lambda\right)\right]}{v_1^j\left(x_j,\lambda\right)v_1^{(\bar{m})j}\left(x_j,\lambda\right)}=O\left(h^{\bar{m}}\right).$$

Hence, the inequality (3.111) follows.

Now, we prove (3.112), (3.113). First of all, let us note that

$$\begin{aligned}&d(x_j,\lambda)-d^{(\bar{m})}(x_j,\lambda)\\&=\frac{1}{h}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[\frac{w_\alpha^j\left(x_j,\lambda\right)+(-1)^\alpha}{v_\alpha^j\left(x_j,\lambda\right)}-\frac{w_\alpha^{(\bar{m})j}\left(x_j,\lambda\right)+(-1)^\alpha}{v_\alpha^{(\bar{m})j}\left(x_j,\lambda\right)}\right],\\&j=2,3,...,N-1,\end{aligned} \tag{3.117}$$

$$\begin{aligned}&d(x_1,\lambda)-d^{(\bar{m})}(x_1,\lambda)\\&=\frac{1}{h}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[\frac{w_\alpha^1\left(x_1,\lambda\right)+\alpha-1}{v_\alpha^1\left(x_1,\lambda\right)}-\frac{w_\alpha^{(\bar{m})1}\left(x_1,\lambda\right)+\alpha-1}{v_\alpha^{(\bar{m})1}\left(x_1,\lambda\right)}\right],\end{aligned} \tag{3.118}$$

$$\begin{aligned}&d(x_N,\lambda)-d^{(\bar{m})}(x_N,\lambda)\\&=\frac{1}{h}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[\frac{w_\alpha^N\left(x_N,\lambda\right)+\alpha-2}{v_\alpha^N\left(x_N,\lambda\right)}-\frac{w_\alpha^{(\bar{m})N}\left(x_N,\lambda\right)+\alpha-2}{v_\alpha^{(\bar{m})N}\left(x_N,\lambda\right)}\right],\end{aligned} \tag{3.119}$$

$$\begin{aligned}&\rho(x_j,\lambda)-\rho^{(\bar{m})}(x_j,\lambda)=\frac{1}{h}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}\left[\frac{z_\alpha^j\left(x_j,\lambda\right)}{v_\alpha^j\left(x_j,\lambda\right)}-\frac{z_\alpha^{(\bar{m})j}\left(x_j,\lambda\right)}{v_\alpha^{(\bar{m})j}\left(x_j,\lambda\right)}\right],\\&j=1,2,...,N.\end{aligned} \tag{3.120}$$

Using (3.96), (3.97) and the relations

$$\begin{aligned}v_1^1\left(x_1,\lambda\right)&=v_1^{(\bar{m})1}\left(x_1,\lambda\right)+O\left(h^{\bar{m}+1}\right)=1+O(h),\\v_\alpha^j\left(x_j,\lambda\right)&=v_\alpha^{(\bar{m})j}\left(x_j,\lambda\right)+O\left(h^{\bar{m}+1}\right)=\frac{h}{k\left(x_{j+(-1)^\alpha}\right)}+O\left(h^2\right),\\&j=3-\alpha,4-\alpha,...,N+1-\alpha,\quad\alpha=1,2,\\v_2^N\left(x_N,\lambda\right)&=v_2^{(\bar{m})N}\left(x_N,\lambda\right)+O\left(h^{\bar{m}+1}\right)=1+O(h),\end{aligned}$$

we get

$$\frac{w_1^1(x_1,\lambda)}{v_1^1(x_1,\lambda)} - \frac{w_1^{(\bar{m})1}(x_1,\lambda)}{v_1^{(\bar{m})1}(x_1,\lambda)} = \frac{v_1^{(\bar{m})1}(x_1,\lambda)\left[w_1^1(x_1,\lambda) - w_1^{(\bar{m})1}(x_1,\lambda)\right]}{v_1^{(\bar{m})1}(x_1,\lambda)\, v_1^1(x_1,\lambda)}$$

$$+\frac{\left[v_1^{(\bar{m})1}(x_1,\lambda) - v_1^1(x_1,\lambda)\right] w_1^{(\bar{m})1}(x_1,\lambda)}{v_1^{(\bar{m})1}(x_1,\lambda)\, v_1^1(x_1,\lambda)} = \frac{h^{\bar{m}+1}\bar{\psi}_1^1(x_0) + O(h^{\bar{m}+2})}{1+O(h)}$$

$$+\frac{\left[-h^{\bar{m}+1}\psi_1^1(x_0) + O(h^{\bar{m}+2})\right]\left[hq_0/2 + O(h^2)\right]}{\left[1+O(h)\right]^2}$$

$$= h^{\bar{m}} k(x_0)\, \bar{\psi}_1^1(x_0) + O\left(h^{\bar{m}+1}\right),$$

$$\frac{w_\alpha^j(x_j,\lambda) + (-1)^\alpha}{v_\alpha^j(x_j,\lambda)} - \frac{w_\alpha^{(\bar{m})j}(x_j,\lambda) + (-1)^\alpha}{v_\alpha^{(\bar{m})j}(x_j,\lambda)}$$

$$= \frac{v_\alpha^{(\bar{m})j}(x_j,\lambda)\left[w_\alpha^j(x_j,\lambda) - w_\alpha^{(\bar{m})j}(x_j,\lambda)\right]}{v_\alpha^{(\bar{m})j}(x_j,\lambda)\, v_\alpha^j(x_j,\lambda)}$$

$$+\frac{\left[v_\alpha^{(\bar{m})j}(x_j,\lambda) - v_\alpha^j(x_j,\lambda)\right]\left[w_\alpha^{(\bar{m})j}(x_j,\lambda) + (-1)^\alpha\right]}{v_\alpha^{(\bar{m})j}(x_j,\lambda)\, v_\alpha^j(x_j,\lambda)}$$

$$= \frac{h^{\bar{m}+1}\bar{\psi}_1^{j-1+\alpha}\left(x_{j+(-1)^\alpha}\right) + O(h^{\bar{m}+2})}{\dfrac{h}{k\left(x_{j+(-1)^\alpha}\right)} + O\left(h^2\right)}$$

$$+\left[(-1)^\alpha h^{\bar{m}+1}\psi_1^{j-1+\alpha}\left(x_{j+(-1)^\alpha}\right) + O\left(h^{\bar{m}+2}\right)\right]$$

$$\times\frac{\dfrac{(-1)^{\alpha+1}h^2}{2}\left.\dfrac{q(x)}{k(x)}\right|_{x=x_{j+(-1)^\alpha}} + O\left(h^3\right)}{\left[\dfrac{h}{k\left(x_{j+(-1)^\alpha}\right)} + O\left(h^2\right)\right]^2}$$

$$= h^{\bar{m}} k\left(x_{j+(-1)^\alpha}\right)\bar{\psi}_1^{j-1+\alpha}\left(x_{j+(-1)^\alpha}\right) + O\left(h^{\bar{m}+1}\right),$$

$$j = 3-\alpha, 4-\alpha, \ldots, N+1-\alpha, \quad \alpha = 1, 2,$$

$$\frac{w_2^N(x_N,\lambda)}{v_2^N(x_N,\lambda)} - \frac{w_2^{(\bar{m})N}(x_N,\lambda)}{v_2^{(\bar{m})N}(x_N,\lambda)} = \frac{v_2^{(\bar{m})N}(x_N,\lambda)\left[w_2^N(x_N,\lambda) - w_2^{(\bar{m})N}(x_N,\lambda)\right]}{v_2^{(\bar{m})N}(x_N,\lambda)\, v_2^N(x_N,\lambda)}$$

$$+\frac{\left[v_2^{(\bar{m})N}(x_N,\lambda) - v_2^N(x_N,\lambda)\right] w_2^{(\bar{m})N}(x_N,\lambda)}{v_2^{(\bar{m})N}(x_N,\lambda)\, v_2^N(x_N,\lambda)} = \frac{-h^{\bar{m}+1}\bar{\psi}_2^N(x_{N+1}) + O(h^{\bar{m}+2})}{1+O(h)}$$

$$
\begin{aligned}
&+\frac{\left[h^{\bar m+1}\psi_2^N\left(x_{N+1}\right)+O\left(h^{\bar m+2}\right)\right]\left[-hq_{N+1}/2+O\left(h^2\right)\right]}{\left[1+O(h)\right]^2}\\
&=h^{\bar m}k\left(x_{N+1}\right)\bar\psi_1^{N+1}\left(x_{N+1}\right)+O\left(h^{\bar m+1}\right).
\end{aligned}
$$

Similarly, taking into account (3.96) and (3.98), we obtain

$$
\frac{z_1^1\left(x_1,\lambda\right)}{v_1^1\left(x_1,\lambda\right)}-\frac{z_1^{(\bar m)1}\left(x_1,\lambda\right)}{v_1^{(\bar m)1}\left(x_1,\lambda\right)}=h^{\bar m}k\left(x_0\right)\tilde\psi_1^1\left(x_0\right)+O\left(h^{\bar m+1}\right),
$$

$$
\frac{z_\alpha^j\left(x_j,\lambda\right)}{v_\alpha^j\left(x_j,\lambda\right)}-\frac{z_\alpha^{(\bar m)j}\left(x_j,\lambda\right)}{v_\alpha^{(\bar m)j}\left(x_j,\lambda\right)}=h^{\bar m}k\left(x_{j+(-1)^\alpha}\right)\tilde\psi_1^{j-1+\alpha}\left(x_{j+(-1)^\alpha}\right)+O\left(h^{\bar m+1}\right),
$$

$$
\frac{z_2^N\left(x_N,\lambda\right)}{v_2^N\left(x_N,\lambda\right)}-\frac{z_2^{(\bar m)N}\left(x_N,\lambda\right)}{v_2^{(\bar m)N}\left(x_N,\lambda\right)}=h^{\bar m}k\left(x_{N+1}\right)\tilde\psi_1^{N+1}\left(x_{N+1}\right)+O\left(h^{\bar m+1}\right).
$$

Substituting the last equations in (3.117) – (3.120), for $j=1,2,...,N$ we get

$$
\begin{aligned}
&d(x_j,\lambda)-d^{(\bar m)}(x_j,\lambda)=h^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}h^{\bar m}k\left(x_{j+(-1)^\alpha}\right)\bar\psi_1^{j-1+\alpha}\left(x_{j+(-1)^\alpha}\right)\\
&+O\left(h^{\bar m}\right)=\frac{1}{h}\left[h^{\bar m}k(x_{j-1}+0)\bar\psi_1^j(x_{j-1}+0)-h^{\bar m}k(x_{j+1}-0)\bar\psi_1^{j+1}(x_{j+1}-0)\right]\\
&+O\left(h^{\bar m}\right),
\end{aligned}
\tag{3.121}
$$

$$
\begin{aligned}
&\rho(x_j,\lambda)-\rho^{(\bar m)}(x_j,\lambda)=h^{-1}\sum_{\alpha=1}^{2}(-1)^{\alpha+1}h^{\bar m}k\left(x_{j+(-1)^\alpha}\right)\tilde\psi_1^{j-1+\alpha}\left(x_{j+(-1)^\alpha}\right)\\
&+O\left(h^{\bar m}\right)=\frac{1}{h}\left[h^{\bar m}k(x_{j-1}+0)\tilde\psi_1^j(x_{j-1}+0)-h^{\bar m}k(x_{j+1}-0)\tilde\psi_1^{j+1}(x_{j+1}-0)\right]\\
&+O\left(h^{\bar m}\right).
\end{aligned}
\tag{3.122}
$$

So, since

$$
k(x_{j-1}+0)\bar\psi_1^j(x_{j-1}+0)=k(x_j-0)\bar\psi_1^j(x_j-0)+O(h),
$$

$$
k(x_{j-1}+0)\tilde\psi_1^j(x_{j-1}+0)=k(x_j-0)\tilde\psi_1^j(x_j-0)+O(h),
$$

then the relations (3.112), (3.113) follow from (3.121), (3.122).

Let us prove the inequality (3.114). Taking into account the equalities (3.69) – (3.71) and the finite increment theorem, we get

$$\begin{aligned}
\left|a^{(\bar m)}(x_j,\lambda)-a^{(\bar m)}(x_j,\tilde\lambda)\right| &\le \left|\frac{h\left[v_1^{(\bar m)j}(x_j,\tilde\lambda)-v_1^{(\bar m)j}(x_j,\lambda)\right]}{v_1^{(\bar m)j}(x_j,\lambda)v_1^{(\bar m)j}(x_j,\tilde\lambda)}\right| \\
&= \left|\frac{h\sum_{p=3}^{\bar m}\frac{h^p}{p!}\left[\left.\frac{d^p v_1^j(x,\tilde\lambda)}{dx^p}\right|_{x=x_{j-1}}-\left.\frac{d^p v_1^j(x,\lambda)}{dx^p}\right|_{x=x_{j-1}}\right]}{\left[h/k_{j-1}+O(h^2)\right]^2}\right| \\
&\le \frac{h\sum_{p=3}^{\bar m}\frac{h^p}{p!}\left|\frac{\partial}{\partial\lambda}\left[\left.\frac{d^p v_1^j(x,\lambda)}{dx^p}\right|_{x=x_{j-1}}\right]\right|_{\lambda=\bar\lambda}\left|\lambda-\tilde\lambda\right|}{h^2/k_{j-1}^2+O(h^3)} \\
&\le Mh^2|\lambda-\tilde\lambda|,\quad \bar\lambda=\lambda+\theta(\lambda-\tilde\lambda),\quad 0<\theta<1.
\end{aligned}$$

The inequality (3.115) follows from the estimates

$$\begin{aligned}
&\left|d^{(\bar m)}(x_j,\lambda)-d^{(\bar m)}(x_j,\tilde\lambda)\right| \\
&\quad\le \frac{1}{h}\sum_{\alpha=1}^{2}\left|\frac{w_\alpha^{(\bar m)j}(x_j,\lambda)+(-1)^\alpha}{v_\alpha^{(\bar m)j}(x_j,\lambda)}-\frac{w_\alpha^{(\bar m)j}(x_j,\tilde\lambda)+(-1)^\alpha}{v_\alpha^{(\bar m)j}(x_j,\tilde\lambda)}\right| \\
&\quad= \frac{1}{h}\sum_{\alpha=1}^{2}\left|\frac{\sum_{p=4}^{\bar m}\frac{\left[(-1)^{\alpha+1}h\right]^p}{p!}\left[\left.\frac{d^p w_\alpha^j(x,\lambda)}{dx^p}\right|_{x=x_{j+(-1)^\alpha}}-\left.\frac{d^p w_\alpha^j(x,\tilde\lambda)}{dx^p}\right|_{x=x_{j+(-1)^\alpha}}\right]}{h/k_{j+(-1)^\alpha}+O(h^2)}\right| \\
&\quad\le \frac{1}{h}\sum_{\alpha=1}^{2}\sum_{p=4}^{\bar m}\frac{h^{p-1}}{p!}\left|\frac{\partial}{\partial\lambda}\left[\left.\frac{d^p w_\alpha^j(x,\lambda)}{dx^p}\right|_{x=x_{j+(-1)^\alpha}}\right]\right|_{\lambda=\bar\lambda}\left|\lambda-\tilde\lambda\right| \\
&\quad\le Mh^2|\lambda-\tilde\lambda|,\quad \bar\lambda=\lambda+\theta(\lambda-\tilde\lambda),\quad 0<\theta<1,
\end{aligned}$$

$$
\begin{aligned}
\left|d^{(\bar{m})}(x_1,\lambda)-d^{(\bar{m})}(x_1,\tilde{\lambda})\right| &\le \frac{1}{h}\left\{\left|\frac{w_1^{(\bar{m})1}(x_1,\lambda)}{v_1^{(\bar{m})1}(x_1,\lambda)}-\frac{w_1^{(\bar{m})1}(x_1,\tilde{\lambda})}{v_1^{(\bar{m})1}(x_1,\tilde{\lambda})}\right|\right.\\
&\quad\left.+\left|\frac{w_2^{(\bar{m})1}(x_1,\lambda)+1}{v_2^{(\bar{m})1}(x_1,\lambda)}-\frac{w_2^{(\bar{m})1}(x_1,\tilde{\lambda})+1}{v_2^{(\bar{m})1}(x_1,\tilde{\lambda})}\right|\right\}\\
&= \frac{1}{h}\left\{\left|\frac{\sum_{p=2}^{\bar{m}}\frac{h^p}{2^p p!}\left[\left.\frac{d^{p-1}\big(q(x)v_1^1(x,\lambda)\big)}{dx^{p-1}}\right|_{x=x_0}-\left.\frac{d^{p-1}\big(q(x)v_1^1(x,\tilde{\lambda})\big)}{dx^{p-1}}\right|_{x=x_0}\right]}{1+O(h)}\right|\right.\\
&\quad\left.+\left|\frac{\sum_{p=4}^{\bar{m}}\frac{h^p}{p!}\left[\left.\frac{d^p w_2^1(x,\lambda)}{dx^p}\right|_{x=x_2}-\left.\frac{d^p w_2^j(x,\tilde{\lambda})}{dx^p}\right|_{x=x_2}\right]}{h/k_2+O(h^2)}\right|\right\}\\
&\le \frac{1}{h}\left\{\sum_{p=2}^{\bar{m}}\frac{h^p}{2^p p!}\left|\frac{\partial}{\partial\lambda}\left[\left.\frac{d^{p-1}\left(q(x)v_1^1(x,\lambda)\right)}{dx^{p-1}}\right|_{x=x_0}\right]\right|_{\lambda=\bar{\lambda}}|\lambda-\tilde{\lambda}|\right.\\
&\quad\left.+\sum_{p=4}^{\bar{m}}\frac{h^{p-1}}{p!}\left|\frac{\partial}{\partial\lambda}\left[\left.\frac{d^p w_2^1(x,\lambda)}{dx^p}\right|_{x=x_2}\right]\right|_{\lambda=\bar{\lambda}}\Bigg|\,|\lambda-\tilde{\lambda}|\right\}\\
&\le Mh|\lambda-\tilde{\lambda}|, \quad \bar{\lambda}=\lambda+\theta(\lambda-\tilde{\lambda}), \quad 0<\theta<1,
\end{aligned}
$$

$$
\begin{aligned}
\left|d^{(\bar{m})}(x_N,\lambda)-d^{(\bar{m})}(x_N,\tilde{\lambda})\right| &\le \frac{1}{h}\left\{\left|\frac{w_1^{(\bar{m})N}(x_N,\lambda)-1}{v_1^{(\bar{m})N}(x_N,\lambda)}-\frac{w_1^{(\bar{m})N}(x_N,\tilde{\lambda})-1}{v_1^{(\bar{m})N}(x_N,\tilde{\lambda})}\right|\right.\\
&\quad\left.+\left|\frac{w_2^{(\bar{m})N}(x_N,\lambda)}{v_2^{(\bar{m})N}(x_N,\lambda)}-\frac{w_2^{(\bar{m})N}(x_N,\tilde{\lambda})}{v_2^{(\bar{m})N}(x_N,\tilde{\lambda})}\right|\right\}\\
&= \frac{1}{h}\left\{\left|\frac{\sum_{p=4}^{\bar{m}}\frac{h^p}{p!}\left[\left.\frac{d^p w_1^N(x,\lambda)}{dx^p}\right|_{x=x_{N-1}}-\left.\frac{d^p w_1^N(x,\tilde{\lambda})}{dx^p}\right|_{x=x_{N-1}}\right]}{h/k_{N-1}+O(h^2)}\right|\right.
\end{aligned}
$$

$$+\left|\frac{\sum_{p=2}^{\bar m}\frac{(-h)^p}{2^p p!}\left[\left.\frac{d^{p-1}\big(q(x)v_2^N(x,\lambda)\big)}{dx^{p-1}}\right|_{x=x_{N+1}}-\left.\frac{d^{p-1}\big(q(x)v_2^N(x,\tilde\lambda)\big)}{dx^{p-1}}\right|_{x=x_{N+1}}\right]}{1+O(h)}\right|\Bigg\}$$

$$\le\frac{1}{h}\left\{\sum_{p=4}^{\bar m}\frac{h^{p-1}}{p!}\left|\frac{\partial}{\partial\lambda}\left[\left.\frac{d^p w_1^N(x,\lambda)}{dx^p}\right|_{x=x_{N-1}}\right]\right|_{\lambda=\bar\lambda}\right||\lambda-\tilde\lambda|$$

$$+\sum_{p=2}^{\bar m}\frac{h^p}{p!}\left|\frac{\partial}{\partial\lambda}\left[\left.\frac{d^{p-1}\left(q(x)v_2^N(x,\lambda)\right)}{dx^{p-1}}\right|_{x=x_{N+1}}\right]\right|_{\lambda=\bar\lambda}\left||\lambda-\tilde\lambda|\right\}$$

$$\le Mh|\lambda-\tilde\lambda|,\quad \bar\lambda=\lambda+\theta(\lambda-\tilde\lambda),\quad 0<\theta<1.$$

Similarly, the inequality (3.116) is proved.

□

Theorem 3.2. *Let the conditions of the Lemma 3.7 be satisfied. Then* $\exists h_0>0$ *such that at* $h\le h_0$ *for the error of the difference scheme* (3.105) – (3.110), *the inequalities*

$$\left\|\sqrt{a}\left(y_n-y_n^{(\bar m)}\right)\right\|_C\le M_1h^{\bar m-1}, \tag{3.123}$$

$$\left|\lambda_n-\lambda_n^{(\bar m)}\right|\le M_2h^{\bar m} \tag{3.124}$$

will be fulfilled, where M_1, M_2 *are constants independent of* h.

Proof. Taking into account the conditions of the Theorem, we apply to the ETDS (3.62) – (3.68) and to the difference scheme (3.105) – (3.110) the coefficient stability Theorem 3.1 by putting $\tilde y(x)=y^{(\bar m)}(x)$, $\tilde a(x,\tilde\lambda)=a^{(\bar m)}(x,\lambda^{(\bar m)})$, $\tilde d(x,\tilde\lambda)=d^{(\bar m)}(x,\lambda^{(\bar m)})$, $\tilde\rho(x,\tilde\lambda)=\rho^{(\bar m)}(x,\lambda^{(\bar m)})$. Let us write in these terms the values η, ψ^* (see (3.40))

$$\eta=\left(a(x,\lambda)-a^{(\bar m)}(x,\lambda^{(\bar m)})\right)y_{\bar x}^{(\bar m)},$$

$$\psi^*=-\left(d(x,\lambda)-d^{(\bar m)}(x,\lambda^{(\bar m)})\right)y^{(\bar m)}+\lambda^{(\bar m)}\left(\rho(x,\lambda)-\rho^{(\bar m)}(x,\lambda^{(\bar m)})\right)y^{(\bar m)}.$$

Then, in view of the estimates (3.111), (3.114) we get

$$\begin{aligned}(1,|\eta|)&=\sum_{\xi\in\omega_h}h\left|a(\xi,\lambda)-a^{(\bar m)}(\xi,\lambda)+a^{(\bar m)}(\xi,\lambda)-a^{(\bar m)}(\xi,\lambda^{(\bar m)})\right|\left|y_{\bar\xi}^{(\bar m)}(\xi)\right|\\&\le\sum_{\xi\in\omega_h}h\left[\left|a(\xi,\lambda)-a^{(\bar m)}(\xi,\lambda)\right|+\left|a^{(\bar m)}(\xi,\lambda)-a^{(\bar m)}(\xi,\lambda^{(\bar m)})\right|\right]\left|y_{\bar\xi}^{(\bar m)}(\xi)\right|\\&\le M_3h^{\bar m}+Mh^2|\lambda-\lambda^{(\bar m)}|,\end{aligned} \tag{3.125}$$

as well as in view of the summation by parts formula and relations (3.112), (3.113), (3.115), (3.116) we obtain

$$\begin{aligned}
(1, |\psi^*|) &= \Bigg| \sum_{\xi \in \omega_h} h \left[d^{(\bar{m})}(\xi, \lambda) - d(\xi, \lambda) - \lambda^{(\bar{m})} \left(\rho^{(\bar{m})}(\xi, \lambda) - \rho(\xi, \lambda) \right) \right] y^{(\bar{m})}(\xi) \\
&+ \sum_{\xi \in \omega_h} h \left[d^{(\bar{m})}(\xi, \lambda^{(\bar{m})}) - d^{(\bar{m})}(\xi, \lambda) - \lambda^{(\bar{m})} \left(\rho^{(\bar{m})}(\xi, \lambda^{(\bar{m})}) - \rho^{(\bar{m})}(\xi, \lambda) \right) \right] y^{(\bar{m})}(\xi) \Bigg| \\
&\le \sum_{\xi \in \omega_h^+} h^{\bar{m}+1} k(\xi) \left[\left| \bar{\psi}_1^j(\xi) \right| + \left| \lambda^{(\bar{m})} \right| \left| \tilde{\psi}_1^j(\xi) \right| \right] \left| y_{\bar{\xi}}^{(\bar{m})}(\xi) \right| + Mh |\lambda - \lambda^{(\bar{m})}| + O(h^{\bar{m}}) \\
&\le M_4 h^{\bar{m}} + Mh |\lambda - \lambda^{(\bar{m})}|. \qquad (3.126)
\end{aligned}$$

Hence and from the Theorem 3.1 follows the inequality

$$(1 - Mh) \left| \lambda - \lambda^{(\bar{m})} \right| \le M_2 h^{\bar{m}},$$

from which, for sufficiently small h, the estimate (3.124) follows.

To prove the estimate (3.123), let us at first note that since

$$\begin{aligned}
a^{(\bar{m})}(x_j, \lambda) &= \frac{h}{v_1^{(\bar{m})j}(x_j, \lambda)} = \frac{1}{\frac{1}{k_{j-1}} + O(h)} = \frac{1}{\frac{1}{(1 - x_{j-1}^2) k_1(x_{j-1})} + O(h)} \\
&= \frac{(j - 1.5)h \left(2 - (j - 1.5)h\right) k_1(x_{j-1})}{1 + (j - 1.5)h \left(2 - (j - 1.5)h\right) \cdot O(h)} = O(h),
\end{aligned}$$

then

$$\left| \frac{a^{(\bar{m})}(x_j, \lambda) - a(x_j, \lambda)}{a^{(\bar{m})}(x_j, \lambda)} \right| \le Mh^{\bar{m}-1},$$

$$\left| \frac{a^{(\bar{m})}(x_j, \lambda) - a^{(\bar{m})}(x_j, \tilde{\lambda})}{a^{(\bar{m})}(x_j, \lambda)} \right| \le Mh |\lambda - \tilde{\lambda}|.$$

It follows

$$\begin{aligned}
\left(1, \left| \frac{\eta}{a^{(\bar{m})}} \right| \right) &= \sum_{\xi \in \omega_h} h \left| \frac{a(\xi, \lambda) - a^{(\bar{m})}(\xi, \lambda)}{a^{(\bar{m})}(\xi, \lambda)} + \frac{a^{(\bar{m})}(\xi, \lambda) - a^{(\bar{m})}(\xi, \lambda^{(\bar{m})})}{a^{(\bar{m})}(\xi, \lambda)} \right| \\
&\times \left| y_{\bar{\xi}}^{(\bar{m})}(\xi) \right| \le M_6 h^{\bar{m}-1} + Mh |\lambda - \lambda^{(\bar{m})}| \le M_7 h^{\bar{m}-1}. \qquad (3.127)
\end{aligned}$$

Also, according to the summation by parts formula and relations (3.113), (3.116) follows

$$\begin{aligned}
&\left|\left(\rho(\xi,\lambda),\left(y^{(\bar m)}\right)^2\right)-\left(\rho^{(\bar m)}(\xi,\lambda^{(\bar m)}),\left(y^{(\bar m)}\right)^2\right)\right|\\
&\quad\le\left|\left(\rho(\xi,\lambda)-\rho^{(\bar m)}(\xi,\lambda),\left(y^{(\bar m)}\right)^2\right)\right|+\left|\left(\rho^{(\bar m)}(\xi,\lambda)-\rho^{(\bar m)}(\xi,\lambda^{(\bar m)}),\left(y^{(\bar m)}\right)^2\right)\right|\\
&\quad\le\sum_{\xi\in\omega_h^+}h^{\bar m+1}k(\xi)\left|\tilde\psi_1^j(\xi)\right|\left(y^{(\bar m)}(\xi)\right)^2_{\bar\xi}+Mh|\lambda-\lambda^{(\bar m)}|+O(h^{\bar m})\le M_8h^{\bar m}.
\end{aligned}\tag{3.128}$$

So, based on the inequalities (3.126) – (3.128) and the Theorem 3.1, we get the estimate (3.123). □

3.4.2 Newton's Iterative Method for Finding the Eigenvalues and Eigenvectors

The problem (3.105) – (3.110) can be considered as a system of equations with $N+1$ unknowns $y_j^{(\bar m)}$, $j=1,2,...,N$, $\lambda^{(\bar m)}$. This system is nonlinear due to the terms $\lambda^{(\bar m)}y_j^{(\bar m)}\rho_j^{(\bar m)}$. To find the solution of difference scheme (3.105) – (3.110), we apply the Newton's iterative method. Linearized (3.105), the Newton's iterative method is written as

$$\begin{aligned}
&\frac{1}{h}a_2^{(\bar m)}\nabla y_{x,1}^{(\bar m,s)}-(d_1^{(\bar m)}-\lambda^{(\bar m,s-1)}\rho_1^{(\bar m)})\nabla y_1^{(\bar m,s)}+\nabla\lambda^{(\bar m,s)}\rho_1^{(\bar m)}y_1^{(\bar m,s-1)}\\
&\qquad=-\frac{1}{h}a_2^{(\bar m)}y_{x,1}^{(\bar m,s-1)}+(d_1^{(\bar m)}-\lambda^{(\bar m,s-1)}\rho_1^{(\bar m)})y_1^{(\bar m,s-1)},
\end{aligned}\tag{3.129}$$

$$\begin{aligned}
&\left(a^{(\bar m)}\nabla y_{\bar x}^{(\bar m,s)}\right)_{x,j}-\left(d_j^{(\bar m)}-\lambda^{(\bar m,s-1)}\rho_j^{(\bar m)}\right)\nabla y_j^{(\bar m,s)}+\nabla\lambda^{(\bar m,s)}\rho_j^{(\bar m)}y_j^{(\bar m,s-1)}\\
&\qquad=-\left(a^{(\bar m)}y_{\bar x}^{(\bar m,s-1)}\right)_{x,j}+\left(d_j^{(\bar m)}-\lambda^{(\bar m,s-1)}\rho_j^{(\bar m)}\right)y_j^{(\bar m,s-1)},\\
&\qquad j=2,3,...,N-1,
\end{aligned}\tag{3.130}$$

$$\begin{aligned}
&-\frac{1}{h}a_N^{(\bar m)}\nabla y_{\bar x,N}^{(\bar m,s)}-(d_N^{(\bar m)}-\lambda^{(\bar m,s-1)}\rho_N^{(\bar m)})\nabla y_N^{(\bar m,s)}+\nabla\lambda^{(\bar m,s)}\rho_N^{(\bar m)}y_N^{(\bar m,s-1)}\\
&\qquad=\frac{1}{h}a_N^{(\bar m)}y_{\bar x,N}^{(\bar m,s-1)}+(d_N^{(\bar m)}-\lambda^{(\bar m,s-1)}\rho_N^{(\bar m)})y_N^{(\bar m,s-1)},
\end{aligned}\tag{3.131}$$

$$\lambda^{(\bar m,s)}=\lambda^{(\bar m,s-1)}+\nabla\lambda^{(\bar m,s)},\quad y_j^{(\bar m,s)}=y_j^{(\bar m,s-1)}+\nabla y_j^{(\bar m,s)},\tag{3.132}$$

$$j=1,2,...,N,\quad s=1,2,...,$$

where $\lambda^{(\bar m,0)},y_j^{(\bar m,0)},j=1,2,...,N$ is the initial approximation,

$$\begin{aligned}
&a_j^{(\bar m)}=a^{(\bar m)}(x_j,\lambda^{(\bar m,s-1)}),\\
&d_j^{(\bar m)}=d^{(\bar m)}(x_j,\lambda^{(\bar m,s-1)}),\quad\rho_j^{(\bar m)}=\rho^{(\bar m)}(x_j,\lambda^{(\bar m,s-1)}).
\end{aligned}$$

The system (3.129) – (3.131) contains N equations linear with respect to $N+1$ of unknown $\nabla y_j^{(\bar{m},s)}$, $j=1,2,...,N$, $\nabla\lambda^{(\bar{m},s)}$. Since the eigenvector is defined up to a constant multiplier, in order to find a unique solution of the system (3.129) – (3.131), one can put, for example, $\nabla y_1^{(\bar{m},s)}=0$ or $\nabla y_N^{(\bar{m},s)}=0$.

If each component of the found solution $y_j^{(\bar{m})}$, $j=1,2,...,N$ is divided by the value

$$(\rho y^{(\bar{m})}, y^{(\bar{m})})^{1/2}=\left[\sum_{j=1}^{N} h\rho_j^{(\bar{m})}\left(y_j^{(\bar{m})}\right)^2\right]^{1/2},$$

then we get the normalized eigenfunctions.

3.5 Numerical Examples

Example 3.1. *Solve the Sturm-Liouville problem*

$$\begin{aligned}&\frac{d}{dx}\left[(1-x^2)\frac{du}{dx}\right]-\frac{1}{4}u(x)=-\lambda u(x), \quad x\in(-1,1),\\ &u(-1)\neq\infty, \quad u(1)\neq\infty.\end{aligned} \tag{3.133}$$

Note that for this problem $k(x)=1-x^2$, $q(x)=\frac{1}{4}$, $r(x)=1$. *The exact solution of the problem (see [20]) are the eigenvalues*

$$\lambda_n=(n+1/2)^2, \quad n=0,1,2,\dots$$

and the corresponding eigenfunctions, which are the Legendre polynomials

$$u_0(x)=P_0(x)=1, \quad u_n(x)=P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n, \quad n=1,2,...$$

For the numerical solving of the problem (3.133) *we use the three-point difference scheme of the 3rd order of accuracy on a uniform grid* $\omega_h=\{x_j=-1+(j-0.5)h,\ h=2/N,\ j=1,2,\dots,N\}$. *The auxiliary Cauchy problems* (3.56), (3.57) *are solved by the Runge-Kutta method of the 4th order of accuracy (see, e.g., [33, p. 202]), and the auxiliary singular Cauchy problems* (3.58), (3.59) *and* (3.60), (3.61) *– by the Taylor series method. In the case of the problem* (3.133), *the Taylor series method* (3.80) *–* (3.82) *will take the form*

$$v_1^{(4)1}(x_1,\lambda)=1+\sum_{i=1}^{4}\frac{h^i}{2^i i!}\left.\frac{d^i v_1^1(x,\lambda)}{dx^i}\right|_{x=x_0},$$

$$w_1^{(4)1}(x_1,\lambda)=\frac{h}{8}+\sum_{i=2}^{4}\frac{h^i}{2^{i+2}i!}\left.\frac{d^{i-1}v_1^1(x,\lambda)}{dx^{i-1}}\right|_{x=x_0},$$

$$z_1^{(4)1}(x_1,\lambda)=\frac{h}{2}+\sum_{i=2}^{4}\frac{h^i}{2^{i}i!}\left.\frac{d^{i-1}v_1^1(x,\lambda)}{dx^{i-1}}\right|_{x=x_0},$$

where

$$\left.\frac{d^iv_1^1(x,\lambda)}{dx^i}\right|_{x=x_0}=\frac{\left((i-\frac{1}{2})^2-\lambda\right)}{2i}\left.\frac{d^{i-1}v_1^1(x,\lambda)}{dx^{i-1}}\right|_{x=x_0},\quad i=1,2,3,4,$$
$$v_1^1(x_0,\lambda)=1,$$

and the Taylor series method (3.90) – (3.92) *will take the form*

$$v_2^{(4)N}(x_N,\lambda)=1+\sum_{i=1}^{4}\frac{(-h)^i}{2^{i}i!}\left.\frac{d^{i}v_2^N(x,\lambda)}{dx^{i}}\right|_{x=x_{N+1}},$$

$$w_2^{(4)N}(x_N,\lambda)=-\frac{h}{8}+\sum_{i=2}^{4}\frac{(-h)^i}{2^{i+2}i!}\left.\frac{d^{i-1}v_2^N(x,\lambda)}{dx^{i-1}}\right|_{x=x_{N+1}},$$

$$z_2^{(4)N}(x_N,\lambda)=-\frac{h}{2}+\sum_{i=2}^{4}\frac{(-h)^i}{2^{i}i!}\left.\frac{d^{i-1}v_2^N(x,\lambda)}{dx^{i-1}}\right|_{x=x_{N+1}},$$

where

$$\left.\frac{d^iv_2^N(x,\lambda)}{dx^i}\right|_{x=x_{N+1}}=(-1)^i\frac{\left((i-\frac{1}{2})^2-\lambda\right)}{2i}\left.\frac{d^{i-1}v_2^N(x,\lambda)}{dx^{i-1}}\right|_{x=x_{N+1}},\quad i=1,2,3,4,$$
$$v_2^N(x_{N+1},\lambda)=1.$$

The results of solving the Sturm-Liouville problem (3.133) *are shown in Table 3.1. For practical estimate of the convergence rate, the following values are used*

$$err_h=\max\left\{\left\|\sqrt{a^{(4)}}\left(y^{(4)}-u\right)\right\|_{C(\omega_h)},|\lambda^{(4)h}-\lambda|\right\},\quad p=\log_3\frac{err_h}{err_{h/3}}.$$

Thus, the numerical results confirm the theoretical conclusions about the 3rd order of accuracy of the difference scheme.

Table 3.1: The results of solving the Sturm-Liouville problem (3.133) by the three-point difference scheme of the 3rd order of accuracy

n	N	err_h	p	n	N	err_h	p
1	8	$3.4461 \cdot 10^{-4}$		3	8	$3.2988 \cdot 10^{-1}$	
	24	$4.2657 \cdot 10^{-5}$	1.9		24	$2.6843 \cdot 10^{-2}$	2.3
	72	$2.1058 \cdot 10^{-6}$	2.7		72	$1.7749 \cdot 10^{-3}$	2.5
	216	$8.6483 \cdot 10^{-8}$	2.9		216	$8.4362 \cdot 10^{-5}$	2.8
	648	$3.3332 \cdot 10^{-9}$	3.0		648	$3.4750 \cdot 10^{-6}$	2.9
					1944	$1.3457 \cdot 10^{-7}$	3.0
2	8	$2.8502 \cdot 10^{-2}$		4	16	$3.1398 \cdot 10^{-1}$	
	24	$3.0581 \cdot 10^{-3}$	2.0		48	$2.4707 \cdot 10^{-2}$	2.3
	72	$1.6666 \cdot 10^{-4}$	2.7		144	$1.4722 \cdot 10^{-3}$	2.6
	216	$7.2205 \cdot 10^{-6}$	2.9		216	$2.1751 \cdot 10^{-4}$	2.8
	648	$2.8537 \cdot 10^{-7}$	2.9		648	$1.0932 \cdot 10^{-5}$	2.9
	1944	$1.0852 \cdot 10^{-8}$	3.0		1944	$4.5599 \cdot 10^{-7}$	3.0

Example 3.2. *We consider the Sturm-Liouville problem*

$$\begin{aligned} &\frac{d}{dx}\left[(1-x^2)\frac{du}{dx}\right] - x^2 u(x) = -\lambda u(x), \quad x \in (-1,1), \\ &u(-1) \neq \infty, \quad u(1) \neq \infty \end{aligned} \tag{3.134}$$

with its known eigenvalues $\lambda_2 = 6.533471867$, $\lambda_3 = 12.51446215$, $\lambda_4 = 20.50827436$, $\lambda_5 = 30.50540463, \ldots$ *(see [62]). For the numerical solution of the problem with a given tolerance* ε*, we use the three-point difference scheme of third order of accuracy, that is, the scheme of rank* $\bar{m} = m = 4$ *on the uniform grid* $\omega_h = \{x_j = -1 + (j-0.5)h,\ h = 2/N,\ j = 1, 2, \ldots, N\}$*. The auxiliary Cauchy problems* (3.56), (3.57) *are solved by the Runge-Kutta method of the 4th order of accuracy, and the auxiliary singular Cauchy problems* (3.58), (3.59) *and* (3.60), (3.61) *– by the 4th-order Taylor series methods* (3.80) *–* (3.82) *and* (3.90) *–* (3.92)*.*

For the practical estimate of the accuracy of the difference scheme, the

Richardson extrapolation is used. If the condition

$$\max\left\{\left\|\sqrt{a^{(4)}}\left(y_N^{(4)}-y_{3N}^{(4)}\right)\right\|_{C(\omega_h)},\ \frac{\left|\lambda_N^{(4)}-\lambda_{3N}^{(4)}\right|}{\lambda_{3N}^{(4)}}\right\}\le 26\varepsilon$$

is satisfied, then the required tolerance ε *is considered as achieved. Otherwise, the number of grid points* N *is increased threefold. Here,* $y_N^{(4)}, \lambda_N^{(4)}$ *denote the solution of the difference scheme on the grid* $\{x_1, x_2, ..., x_N\}$*, while* $y_{3N}^{(4)}, \lambda_{3N}^{(4)}$ *denote the solution of the difference scheme on the grid* $\{x_1, x_2, ..., x_{3N}\}$*. If the accuracy is achieved, then the solution can be refined using the formulas*

$$\hat{y}_N(x_j) = y_{3N}^{(4)}(x_{3j}) + \frac{y_{3N}^{(4)}(x_{3j}) - y_N^{(4)}(x_j)}{26}, \quad j = 1, 2, ..., N,$$
$$\hat{\lambda}_N = \lambda_{3N}^{(4)} + \frac{\lambda_{3N}^{(4)} - \lambda_N^{(4)}}{26}.$$

The iterations in Newton's method stop if

$$\max\left\{\left\|y_N^{(4,s)}-y_N^{(4,s-1)}\right\|_{C(\omega_h)},\ \frac{\left|\lambda_N^{(4,s)}-\lambda_N^{(4,s-1)}\right|}{\lambda_N^{(4,s)}}\right\}\le 0.5\varepsilon$$

where $s = 1, 2, ..., 7$ *is the iteration number. The results of solving the problem* (3.134) *with a given tolerance* ε *using the third-order difference scheme for the third and fourth eigenvalues are presented in the Table 3.2, where*

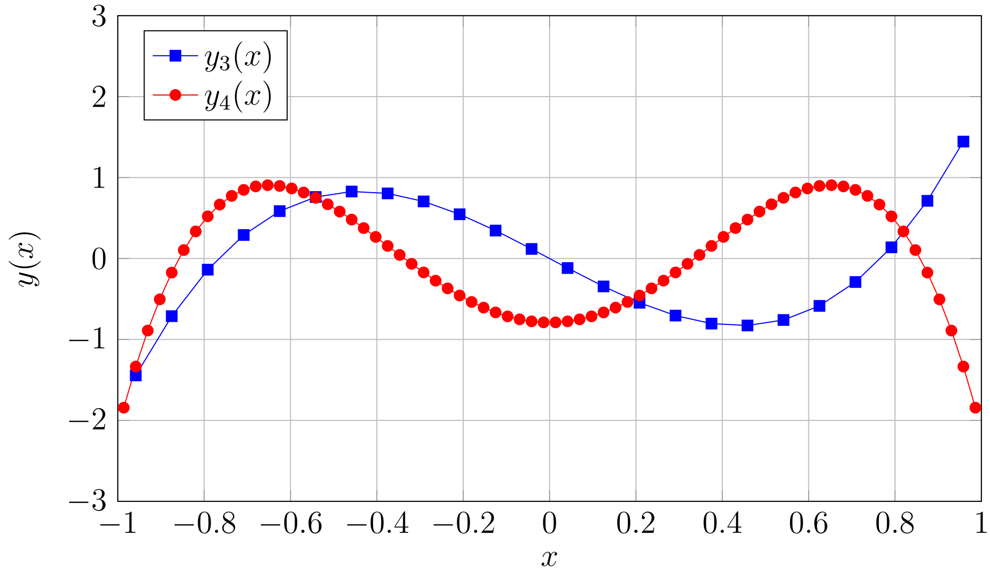
$$err = \frac{\left|\lambda^{(4)h} - \lambda\right|}{\lambda}.$$

In conclusion, we note that all the results presented in this section can be easily transferred to the case of non-uniform grids.

Table 3.2: The results of solving the problem (3.134) using the difference scheme of 3rd order of accuracy with a given tolerance ε

n	ε	N	err	$time(sec)$
3	10^{-4}	24	$0.533 \cdot 10^{-4}$	0.07
	10^{-6}	216	$0.238 \cdot 10^{-7}$	0.7
	10^{-8}	648	$0.781 \cdot 10^{-9}$	2.1
4	10^{-4}	72	$0.628 \cdot 10^{-5}$	0.2
	10^{-6}	216	$0.131 \cdot 10^{-6}$	0.7
	10^{-8}	1944	$0.878 \cdot 10^{-10}$	5.6

Figure 3.1: The 3rd and the 4th eigenfunctions of the problem (3.134) for $\varepsilon = 10^{-4}$



Conclusions

In this thesis, we developed three-point difference schemes of high order of accuracy for the numerical solution of Sturm-Liouville problems.

Main contributions:

1. A new algorithmic realization of the exact three-point difference scheme on a non-uniform grid is developed and justified for the Sturm-Liouville problem. It is shown that the coefficients of the exact three-point difference scheme at any grid node can be expressed in terms of the solutions of two auxiliary Cauchy problems for the system of three linear ordinary differential equations, each of which can be numerically solved in a single step using any one-step method. A theorem on the coefficient stability of the exact three-point difference scheme is proved. Three-point difference schemes of arbitrary accuracy order are constructed, and an accuracy estimate for these schemes is provided. The Newton's iterative method is developed for solving these schemes with accuracy order $\bar{m} = 2[(m+1)/2]$ (m is a given natural number, $[\cdot]$ is the integer part of the argument in brackets). Each iteration of this method requires solving a system of linear algebraic equations, whose matrix differs from a tridiagonal form by only one additional nonzero column.

2. An algorithmic realization of the exact three-point difference scheme is developed for solving the Sturm-Liouville problem with singularities at the endpoints of the interval. It is shown that computing the coefficients of the exact scheme at any grid node x_j requires solving two auxiliary Cauchy problems for a system of three linear ordinary differential equations of the first order (including singular cases): one on the interval $[x_{j-1}, x_j]$ (forward) and one on $[x_j, x_{j+1}]$ (backward). A theorem on the coefficient stability of the exact scheme is proved. High-order three-point difference schemes are developed, and their accuracy order is established. The Newton's iterative method is proposed for solving these schemes efficiently.

3. Extensive numerical experiments confirm the theoretical accuracy es-

timates of the presented difference schemes. The results of solving Sturm-Liouville problems with a specified tolerance ε demonstrate the effectiveness of the proposed approach. It is shown that the proposed sixth-order difference scheme provides significantly higher accuracy in computing eigenfunctions and eigenvalues compared to the classical second-order finite difference scheme. Additionally, it enables the computation of higher-index eigenvalues with greater precision.

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