De Rham and Čech-de Rham Cohomologies of Smooth Foliated Manifolds

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0 Introduction

In differential geometry, smooth manifolds can be classified according to their de Rham cohomology classes, which also provide topological information of the manifold. The aim of this dissertation is to study the de Rham cohomology for foliated differential forms of a smooth foliated manifold and to develop tools for the computation of the so called **foliated de Rham cohomology**. Among these results, there is a Künneth formula for foliated de Rham cohomology ([Ber11]), which states a connection of the foliated de Rham cohomology of the product foliation on the product manifold of foliated manifolds with the cohomology of its factors. Trying to relax the requirements of this theorem leads us to the SK-Theory, developed by D. Vogt to investigate the splitting of exact sequences in the category of Fréchet spaces. We show that the Cartan-differential is an SK-homomorphism by using strictness of projective spectra. Unfortunately, this proof does not transfer to the foliated Cartan-differential in general, which would be required for a generalization of the Künneth formula. At least, there is a sufficient condition to guarantee that the foliated Cartan-differential is an SK-homomorphism.
To achieve these results, an interaction of different mathematical areas is needed. The reader should have a basic understanding of category theory and functors. Apart from that, knowledge about topology, differential geometry, homological algebra and functional analysis is recommended. Anyhow, we are making an effort to explain everything for a reader, who has no extensive background.

Let us begin with a motivation and an overview of each chapter. Some definitions are mentioned, without being written out in detail, to give the reader a first impression of the presented theories. For details we always refer to the considered chapter.
We start with the definition of smooth manifolds and smooth maps between smooth manifolds in the preliminaries. Without further saying, all manifolds are assumed to be connected. After that, we introduce smooth vector bundles and smooth sections in chapter 2, which are frequently used in differential geometry. The most elementary smooth vector bundle is the tangent bundle of a smooth manifold. Smooth sections of the tangent bundle are precisely vector fields. Further important examples of vector bundles are constructed out of the tangent bundle, for example the cotangent bundle or the alternating k-tensor bundle of the tangent bundle. The sections of this bundle are differential k-forms, which are probably the most interesting objects in differential geometry. One can imagine differential k-forms at each point of a smooth manifold as a signed (sub-)volume function acting on k many tangent vectors of the tangent space at the considered point. There is also the Cartan differential d, which is a linear map that sends a differential k-form to some $k+1$-form. Since the composition $d \circ d$ is always zero, the Cartan differential is a coboundary operator of the cochain complex given by differential k-forms. It therefore allows the definition of the de Rham cohomology.
Thus, concerned by the wideness of applications of smooth vector bundles and sections, we study them mostly abstractly in chapter 2. A smooth $\mathbb{R}$-vector bundle of rank N consists of a smooth surjective map $\pi : E \to M$ between smooth manifolds such that all

preimages $E_x := \pi^{-1}(\{x\})$ are N-dimensional $\mathbb{R}$-vector spaces and all points $x \in M$ have a smooth local trivialization, which is a certain diffeomorphism, defined on some open neighbourhood U of x onto $U \times \mathbb{R}^N$. In 2.1, we define morphisms between smooth vector bundles and form the category of smooth vector bundles. We deduce a local representation of vector bundle morphisms and show that a transition function between smooth local trivializations is also smooth.
Next, we introduce smooth sections and frames of smooth vector bundles in 2.2. A smooth local section of a vector bundle $\pi : E \to M$ is a smooth right inverse $\sigma : U \to E$ of the projection map π, defined on some open $U \subseteq M$, i.e. $\pi \circ \sigma = \mathrm{id}_U$. An ordered tuple $(\sigma_1, ..., \sigma_N)$ of smooth local sections $\sigma_j : U \to E$ is called a frame, if $\sigma_1(x), ..., \sigma_N(x)$ form a basis of E_x for all $x \in U$. An equivalent condition for smooth vector bundles on frames instead of smooth local trivializations follows. A frame allows locally a unique representation of a local section and similar to a change of basis in linear algebra, there is change of frames between frames over the same open set. Locally, smooth sections are isomorphic as vector spaces to smooth functions. Later, we will use this isomorphism to define a topology on the local (and also global) sections, which will make the space of sections into a Fréchet space.
Chapter 2.3 is dedicated to the constructions of smooth vector bundles. The main tool for the verification of a smooth vector bundle is the Vector Bundle Construction Lemma. We introduce the subbundle of a smooth vector bundle and the quotient bundle of a smooth vector bundle by a subbundle. Whereas the definition of a subbundle easily implies that it is a smooth vector bundle, the verification of the quotient bundle as a smooth vector bundle is more subtle and involves the Vector Bundle Construction Lemma. To construct more vector bundle from old ones, we adopt an abstract approach from [Eur] and [Tan14], where the vector bundles are induced by so called **manifold-enriched functors**. As an application, we obtain that the dual bundle and the (alternating) k-tensor bundle of a smooth vector bundle is also a smooth vector bundle.
We consider the category $\mathbf{VB}_M$ of smooth vector bundles over a fixed base space M in 2.4 and remark that it is not an abelian category. After classifying the homomorphisms in $\mathbf{VB}_M$, we define a short exact sequence of smooth vector bundle over M. If one of the equivalent assertions of the Splitting Lemma is satisfied, we say that the exact sequence splits. By the previous work, we can show that every short exact sequences of vector bundles over the same base space M splits.
In 2.5 we want to build a functor from $\mathbf{VB}_M$ to **Fréchet**, the category of Fréchet spaces with continuous and linear maps by assigning the space of smooth sections to a given smooth vector bundle over M. So first of all, we equip the space of smooth sections $\Gamma(M, E)$ with a directed system of seminorms such that it becomes a Fréchet space. This structure is induced locally by the Fréchet space of smooth functions in Euclidean space. We also provide certain continuity criteria to obtain the continuity of the pushforward $f_* : \Gamma(M, E) \to \Gamma(M, F)$, $\sigma \mapsto f \circ \sigma$ of a smooth vector bundle M-morphism $f : E \to F$. Therefore, the section functor is indeed a covariant functor from $\mathbf{VB}_M$ to **Fréchet**. It is also an exact functor, i.e. a short exact sequence of vector bundles is send to a short exact sequence of Fréchet spaces, which also splits by functoriality since the short exact

sequences of vector bundles over M splits. We end up this chapter by an isomorphism between the section of a quotient bundle and the quotient space of the spaces of sections, which will be used later to show that two alternative definitions of foliated differential forms are isomorphic.

We proceed with the main chapter 3 about foliated differential forms and foliated cohomologies. Foliated manifolds generalize the concept of manifolds in the sense that each manifold M can be made into a foliated manifold by its one leaf foliation $\mathcal{F} = \{M\}$. In general, a **smooth** (p, q)**-foliated manifold** is a smooth manifold M together with a foliation $\mathcal{F} = \{\mathcal{L}_\alpha\}_{\alpha \in \mathscr{A}}$ on M, which is a decomposition of M into connected and disjoint p-dimensional submanifolds $\mathcal{L}_\alpha$, called **leaves**. The foliation is locally modelled on the decomposition of an open $V \times W \subseteq \mathbb{R}^p \times \mathbb{R}^q$ into its cosets $V \times \{c\}$ where c runs through W. Besides the trivial one leaf foliation, there is also always the discrete foliation $\mathcal{F} = \{\{x\} : x \in M\}$ by points of M.
Foliations are closely related to the solvability of systems of first-order differential equations. For instance, the *integral manifolds* of a non-vanishing vector field X, which are the images of the *integral curves* of X, are forming a $(1, n-1)$-foliation on an n-dimensional manifold M. Here, an *integral curve* of a vector field X on M (passing through $x \in M$) is a smooth map $u : I \to M$ satisfying $u'(t) = X_{u(t)}$ on some open interval I (and $u(t_0) = x$ for some $t_0 \in I$). Hence, integral curves are the solutions of a system of first-order ordinary differential equations and their existence is guaranteed by the *Picard-Lindelöf Theorem*. A non-vanishing vector field X on M is a global frame of an one-dimensional subbundle of the tangent bundle of M, such that this subbundle is equal to the so called **foliated tangential subbundle** of the foliation induced by X, given by the disjoint union of the tangent spaces of the leaves of the foliation. More general, consider pointwise linearly independent vector fields $X^1, ..., X^p$ of M, where $p \leq n$. These vector fields span a p-dimensional subbundle of TM, denoted by $\text{span}\{X^1, ..., X^p\}$. The *Frobenius Theorem* states that there is an underlying smooth $(p, n-p)$-foliation $\mathcal{F}$ of M such that its foliated tangential subbundle is equal to $\text{span}\{X^1, ..., X^p\}$ if and only if this subbundle is *involutive*. That means, for all smooth sections X and Y in the subbundle $E = \text{span}\{X^1, ..., X^p\}$ of TM, their *Lie-bracket* $[X, Y] : M \to TM$, which is pointwise defined to be the *derivation*[1] $[X, Y]_x(f) = X_x(Y_x(f)) - Y_x(X_x(f))$ for $x \in M$ and $f \in C^\infty(M)$, is also a section in the subbundle E of TM or equivalently $[X^i, X^j]$ can be written as a linear combination of $X^1, ..., X^p$ for all $i \neq j$. The involutivity condition arises as a generalization of the commutativity of partial derivatives. In that case, the underlying foliation $\mathcal{F}$ consists of the *integral manifolds of* $X^1, ..., X^p$, which are the images of smooth maps $u : W \subseteq \mathbb{R}^p \to M$ on open domains $W \subseteq \mathbb{R}^p$ solving the system of first-order partial differential equations $\frac{\partial u}{\partial x^i}(x) = X^i_{u(x)}$ $(i = 1, ..., p)$. These connections are illustrated in Examples 3.2.2.
We start in 3.1 with the definition of foliations and continue with transverse maps to a foliation, which allows to pullback a foliation of the target manifold to produce more

[1]See 2.1.4 (3) for a definition of the tangent space as derivations.

examples of foliations by old ones. In particular, a smooth submersion is transverse to any foliation on the target manifold and the pullback foliation of the discrete foliation yields a foliation of the domain manifold, given by the connected components of the level sets of the submersion. Locally, every foliation is of that type. For the existence of the pullback foliation, we use the *Local Submersion Theorem*, which is based on the *Inverse Function Theorem*. We list some examples of foliations.

Next, we want to define the foliated differential forms of a foliated manifold in 3.2. Recall, the disjoint union of the tangent spaces of all leaves gives rise to a smooth subbundle $T\mathcal{F}$ of the tangent bundle of M, which is called the **foliated tangential subbundle**. Using the constructions of chapter 2, we can build further vector bundles as the normal bundle, which is the quotient bundle of the tangent bundle by the foliated tangential subbundle. Moreover, we can define the dual bundle and more general the alternating k-tensor bundle of the foliated tangential subbundle, which we call the **foliated alternating k-tensor bundle**. As we clarify, it can be seen as a quotient bundle of the alternating k-tensor bundle of the tangent bundle by a subbundle, called the k-annihilator of $T\mathcal{F}$. This observation gives rise to two alternative definitions of foliated differential forms, namely the smooth sections of the foliated alternating k-tensor bundle and the quotient space of differential forms by the subspace of sections of the annihilator of $T\mathcal{F}$, respectively. We show that both constructions end up with isomorphic Fréchet spaces.

As differential forms can be better understood by the Cartan-differential, which allows a local representation, we are also interested to obtain a similar exterior derivative for foliated differential forms in 3.3. Aside from that, the Cartan-differential is used to build the de Rham cohomology of manifolds, which we want to generalize for foliated manifolds. The definition of the so called **foliated Cartan-differential** $d_{\mathcal{F}}$ is based on the differential of an $\mathbb{R}$-valued smooth function defined on a smooth manifold and the inclusion morphism $i_{T\mathcal{F}} : T\mathcal{F} \to TM$ of smooth vector bundles over M. The existence and uniqueness of the foliated Cartan-differential is shown including the most important properties of $d_{\mathcal{F}}$ such as the commutativity with pullbacks of foliated maps, a local representation and the coboundary property $d_{\mathcal{F}} \circ d_{\mathcal{F}} = 0$.

This allows the definition of the so called **foliated de Rham cohomology** in 3.4 similar to the definition of the (classical) de Rham cohomology. By the coboundary operator property, $\mathrm{Im}(d_{\mathcal{F}}^{k-1}) \subseteq \Omega^k(M, \mathcal{F})$ is a subspace of the vector space $\mathrm{Ker}(d_{\mathcal{F}}^{k}) \subseteq \Omega^k(M, \mathcal{F})$, such that we can define the **kth foliated de Rham cohomology group** of a foliated manifold as the quotient space $H^k(M, \mathcal{F}) = {}^{\mathrm{Ker}(d_{\mathcal{F}}^{k})}/_{\mathrm{Im}(d_{\mathcal{F}}^{k-1})}$. The term *group* is common since in homological algebra (co)homologies classes are principally just groups, but we are actually dealing with vector spaces. Because of dimensional reasons, we can conclude that $H^k(M, \mathcal{F}) = 0$ if $k < 0$ or $k > p$ for a (p, q)-foliated manifold. Further, $H^0(M, \mathcal{F})$ is the space of leafwise constant functions, i.e. the space of smooth $\mathbb{R}$-valued functions on M such that the restriction to any leaf is constant. For the product manifold of connected smooth manifolds $F \times T$ foliated by $\mathcal{F}_T(F) = \{F \times \{t\} : t \in T\}$, we obtain $H^0(F \times T, \mathcal{F}_T(F)) \cong C^\infty(T)$, which is of course not a finite dimensional vector space. The leafwise constant functions can vary on the *transversal manifold*. We proceed with induced cohomology maps of smooth foliated maps, defined as the pullback of smooth foliated maps on the equivalence classes. Together with the assignment $(M, \mathcal{F}) \to H^k(M, \mathcal{F})$, we obtain a functor from the category **FolMfld** of smooth foliated manifolds with smooth foliated maps to the category $\mathbf{Vect}_{\mathbb{R}}$ of $\mathbb{R}$-vector spaces with linear maps, the so called **foliated de Rham functor**. This yields the foliated diffeomorphism invariance of the foliated de Rham cohomology, i.e. if $f : (M, \mathcal{F}) \to (N, \mathcal{G})$ is a smooth foliated diffeomorphism between smooth manifolds, then $H^k(M, \mathcal{F}) \cong H^k(N, \mathcal{G})$ for all $k \in \mathbb{N}_0$. The classical de Rham cohomology is even a homotopy invariance. This is not true for the foliated de Rham cohomology as we see in an example, even if the homotopy map is foliated. Nevertheless, there is a stronger type of homotopy, called **integrable homotopy**, which yields an invariance of the foliated de Rham cohomology. Two smooth foliated maps $f, g : (M, \mathcal{F}) \to (N, \mathcal{G})$ are integrable homotopic, if there is a smooth foliated map $H : (M \times \mathbb{R}, \pi^*(\mathcal{F})) \to (N, \mathcal{G})$ such that $H(x, t) = f(x)$ if $t \leq 0$ and $H(x, t) = g(x)$ if $t \geq 1$ for all $x \in M$. The important point is to consider the pullback foliation $\pi^*(\mathcal{F})$ on $M \times \mathbb{R}$, where $\pi : M \times \mathbb{R} \to M$ is the projection map, such that the leaves are given by $\{\mathcal{L} \times \mathbb{R}\}$ for $\mathcal{L} \in \mathcal{F}$. As we have seen in the example before, the foliation $\{\mathcal{L} \times \{t\} : \mathcal{L} \in \mathcal{F}, t \in \mathbb{R}\}$ on $M \times \mathbb{R}$ does not yield a homotopy invariance. The proof of the integrable homotopy invariance is then quite similar to the homotopy invariance of the de Rham cohomology. At first, we show the existence of a *homotopy operator* $h : \Omega^k(M \times \mathbb{R}, \pi^*(\mathcal{F})) \to \Omega^{k-1}(M, \mathcal{F})$. Then, we deduce that two smooth foliated maps which are integrable homotopic induce the same cohomology maps between foliated de Rham cohomologies. Together with the foliated de Rham functor, this gives the integrable homotopy invariance. We can use this fact by proving a Poincaré Lemma for a star-shaped foliation by points. To be more precise, let M be a star-shaped open subset of $\mathbb{R}^p$ with center $c \in M$ and N a smooth manifold. Consider the M-foliation $\mathcal{F}_N(M) = \{M \times \{y\} : y \in N\}$ by points of N on $M \times N$, then $\{c\} \times N$ is an *integrable deformation retract* of $(M \times N, \mathcal{F}_N(M))$ and the only non-zero foliated cohomology group is $H^0(M \times N, \mathcal{F}_N(M)) \cong C^\infty(N)$.

A very powerful tool to compute de Rham cohomologies is the *Mayer-Vietoris Theorem*, which states a connection between the de Rham cohomology of a union $U \cup V$ and the cohomologies of submanifolds U, V of some smooth manifold M as well as their intersection $U \cap V$ ordered in a long exact sequence. We have to assume some transversality conditions on the inclusion maps such that they become foliated maps and can conclude a similar result for the foliated de Rham cohomology. However, open subsets satisfy these assumptions. The long exact sequence is built from short exact sequences for each $k \in \mathbb{N}$ together with connecting maps obtained by the *ZigZag Lemma* (also known as *Snake Lemma*). Once again, we consider the M-foliation $\mathcal{F}_N(M)$ by points of N on the product manifold $M \times N$, with the only assumption for M that it has a *finite* **good cover**. A good cover consists of open sets such that all open sets and all finite non-empty intersections of the cover are diffeomorphic to $\mathbb{R}^m$, if M is a manifold of dimension m. Every smooth manifold has a good cover. If the manifold is compact, the good cover may be chosen to be finite. The de Rham cohomology of a manifold, which has a finite good cover, is finite dimensional. As we have seen, this is not the case for foliated de Rham cohomology. Returning to the foliation $\mathcal{F}_N(M)$ on $M \times N$, we can show with an application of the Mayer-Vietoris Theorem that $H^k(M \times N, \mathcal{F}_N(M)) \cong H^k(M) \otimes C^\infty(N)$

for each $k \in \mathbb{N}$, which generalizes the Poincaré Lemma for a star-shaped foliation by points, since $H^0(M) \cong \mathbb{R}$ and otherwise $H^k(M) = 0$ for a star-shaped M by the (classical) Poincaré Lemma. Since $H^k(S^1) \cong \mathbb{R}$ for $k = 0, 1$, we can easily compute the foliated the Rahm cohomology of the torus $\mathbb{T} = S^1 \times S^1$, foliated by points of a circle, namely $H^k(\mathbb{T}^2, \mathcal{F}_{S^1}(S^1)) = C^\infty(S^1)$ for $k = 0, 1$.
To achieve a generalization of the Mayer-Vietoris sequence that deals with more than two open sets, we introduce the **Čech complex**, which is a cochain complex with a so called **cover operator** δ as coboundary operator, depending on an open cover of (countably many) sets and a presheaf. In short, a ($\mathbf{Vect}_{\mathbb{R}}$-valued) presheaf $\mathscr{G}$ on a topological space X is contravariant functor from the category $\mathbf{Open}(X)$ of open sets in X with inclusions as morphisms to the category $\mathbf{Vect}_{\mathbb{R}}$ of $\mathbb{R}$-vector spaces with linear maps. For instance, on a smooth (foliated) manifold, there is the presheaf of (foliated) differential k-forms and the presheaf of the kth (foliated) de Rham cohomology. Here, we need for the foliated versions that the inclusion $i_U : U \to M$ of an open subset $U \subseteq M$ is transversal to any foliation $\mathcal{F}$ on M, such that it becomes a foliated map $i_U : (U, \mathcal{F}|_U) \to (M, \mathcal{F})$ with respect to the pullback foliation $\mathcal{F}|_U = i_U^*(\mathcal{F})$. The Čech complex of an open cover $\mathcal{U} = (U_\alpha)_{\alpha \in J}$ with values in the presheaf $\mathscr{G}$ is defined by

$$\check{C}^\ell(\mathcal{U}, \mathscr{G}) = \prod_{\alpha_0 < \cdots < \alpha_\ell} \mathscr{G}\left(\bigcap_{j=0}^{\ell} U_{\alpha_j}\right) \text{ for each } \ell \in \mathbb{N}_0.$$

Similar to the classical case of just smooth manifolds, we can show that the augmented sequence of the Čech complex with values in the presheaf $\Omega^k_{\mathcal{F}}$ of foliated differential k-forms is exact, such that the Čech cohomology vanishes identically for this presheaf. The two cochain complexes, the foliated de Rham complex and the Čech complex with values in the presheaf $\Omega^k_{\mathcal{F}}$, give rise to a double cochain complex, called the **foliated Čech-de Rham complex**. As specified, one can build also a single cochain complex by forming the discrete sums $A^j = \bigoplus_{k+\ell=j} \check{C}^\ell(\mathcal{U}, \Omega^k_{\mathcal{F}})$ over the anti-diagonal lines of the double complex and setting $D\omega = \delta^\ell \omega + (-1)^k d^k_{\mathcal{F}} \omega$ for $\omega \in \check{C}^\ell(\mathcal{U}, \Omega^k_{\mathcal{F}})$. We proceed with the generalized Mayer-Vietoris principle, which tells us that the cohomology of the foliated Čech-de Rham complex is isomorphic to the foliated de Rham cohomology. Analysing the proof and augmenting the double complex by an initial row, we get by a consequence of the Poincaré Lemma for a star-shaped foliation by points a further result for an underlying good cover: The Čech cohomology with values in the presheaf of leafwise constant functions $\mathscr{G}^{\mathrm{lc}}_{\mathcal{F}}$ of a good cover is isomorphic to the Čech-de Rham cohomology and therefore also isomorphic to the foliated de Rham cohomology. These isomorphisms does not seem to be explicitly discussed in the literature. If one defines the Čech cohomology $\check{H}^*(M, \mathscr{G})$ of the whole manifold M as the direct limit of the Čech cohomology of all covers, preordered by refinements, then we can conclude by the previous result that $\check{H}^*(M, \mathscr{G}^{\mathrm{lc}}_{\mathcal{F}})$ is isomorphic to the foliated de Rham cohomology since good covers are cofinal in the set of all covers of a manifold. Considering the one leaf foliation of a smooth manifold, we also obtain these results for the (classical) de Rham and Čech cohomologies.

In 3.6 we discuss the prospects of a Künneth formula for the foliated de Rham cohomology. Formulas of the (co)homology of the product of two objects related to the (co)homologies of the single objects are called a **Künneth formula**. The formula for the foliated the Rham cohomology of $(M \times N, \mathcal{F}_N(M))$ suggests for foliated manifolds $(M, \mathcal{F})$ and $(N, \mathscr{G})$:

$$H^k(M \times N, \mathcal{F} \times \mathcal{G}) \cong \bigoplus_{i+j=k} H^i(M, \mathcal{F}) \otimes H^j(N, \mathcal{G}) \text{ for } k \in \mathbb{N}_0.$$

But as we remark, if the foliated cohomologies of both factors are not finite dimensional, we have to substitute the algebraic tensor product $\otimes$ by a topological tensor product $\hat{\otimes}$ in order to expect a valid formula. Recall that the foliated de Rham cohomology was defined as a quotient space of the kernel by the image according to the foliated Cartan-differential. Only if $d_{\mathcal{F}}$ has a closed image, the cohomology groups are Hausdorff and in particular a Fréchet space, such that a topological tensor product can be defined. Fortunately, we are actually dealing with *nuclear* Fréchet spaces in that case, such that the (in general different) injective and projective topologies coincide on the algebraic tensor product. Nevertheless, the image of $d_{\mathcal{F}}$ is not closed in general as we see for the example of the *Kronecker foliation* $\mathcal{F}_\alpha$ *of the torus* $\mathbb{T}^2$ for a slope α which is a *Liouville number*. We compute the foliated de Rham cohomologies in the cases of an irrational slope, which is a Liouville number or not. We obtain $H^1(\mathbb{T}^2, \mathcal{F}_\alpha) \cong \mathbb{R}$ if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is not a Liouville number, but if α is a Liouville number, $H^1(\mathbb{T}^2, \mathcal{F}_\alpha)$ is infinite-dimensional and non-Hausdorff, such that the image of $d_{\mathcal{F}_\alpha}$ is not closed, or equivalently $d_{\mathcal{F}_\alpha}$ is no homomorphism. We present the Künneth formula in two situations, which are shown in [Ber11]. The second version requires finite dimensionality of the foliated de Rham cohomology of one factor and that the manifold of that factor is compact. The proof uses a continuous right inverse of $d_{\mathcal{F}}$, which is guaranteed by a Splitting Theorem in that case of an underlying compact manifold. $d_{\mathcal{F}}$ would also have a continuous right inverse, if one can show, that it is a so called **SK-homomorphism**. In that case, the compactness of the underlying manifold is not required for the Künneth formula. We end this chapter by presenting the Splitting theorems and the involved invariants (DN), (Ω), $(\mathrm{DN}_{\mathrm{loc}})$ and (Ω_{loc}).
As motivated in the third chapter, we are interested in the property of $d_{\mathcal{F}}$ being an SK-homomorphism in Chapter 4. In 4.1 we will introduce the SK-theory, developed by D. Vogt, to investigate the splitting of exact sequences in the category **Fréchet**. In particular, we are interested in the existence of a continuous linear right inverse of some continuous and linear map between locally convex spaces. If that is the case, we observe a certain condition in which the *seminorm kernels* play a central role. Hence, one is interested the induced topology of the system of seminorm kernels, called the **SK-topology**. The observation in the case of the existence of a continuous linear right inverse is then just the condition to be a homomorphism with respect to the SK-topologies of the locally convex spaces, which we will call an SK-homomorphism. We proceed with some compatibility properties of the SK-topology and see in an example that the SK-topology of the quotient topology can be strictly coarser than the quotient topology of the SK-topology. If the topologies coincide on the quotient space formed by a closed subspace, this subspace is called an SK-subspace and we give a characterization of that property. We prove that the k-annihilator

of a foliated manifold is an SK-subspace of the space of differential k-forms, such that the corresponding quotient map is an SK-homomorphism. Using the construction of the foliated differential forms as quotient spaces, gives rise to an alternative foliated Cartan-differential. Due to the fact that the composition $g \circ f$ of SK-homomorphisms requires $\operatorname{Ker}(g) \subseteq \operatorname{Im}(f)$ to be also an SK-homomorphism, we can conclude, that the foliated Cartan-differential $d_{\mathcal{F}} : \Omega^k(M, \mathcal{F}) \to \Omega^{k+1}(M, \mathcal{F})$ is an SK-homomorphism if the $(k+1)$-annihilator $\Omega^{k+1}_{\perp\!\!\perp}(M, \mathcal{F})$ is a subset of $\operatorname{Im}(d) \subseteq \Omega^{k+1}(M)$, the image of the (classical) Cartan-differential and if d is an SK-homomorphism. Then we characterize d being an SK-homomorphism and obtain a so called *strictness condition* on the kernel spectrum of d, which we will verify later.

In 4.2 we introduce strictness of a general projective spectrum and also their projective limits. A **projective spectrum** $\mathscr{X} = (X_n, \varrho^n_m)$ is a sequence $(X_n)_{n\in\mathbb{N}}$ of linear spaces together with linear maps $\varrho^n_m : X_m \to X_n$, the **linking maps**, satisfying $\varrho^n_n = \mathrm{id}_{X_n}$ and $\varrho^k_n \circ \varrho^n_m = \varrho^k_m$ for $k \le n \le m$. $\mathscr{X}$ is called **strict**, if

$$\forall\, n \in N\; \exists\, m \ge n\; \forall\, \ell \ge m : \varrho^n_m(X_m) \subseteq \varrho^n_\ell(X_\ell).$$

Note that in that case, the subsets are actually equal by the linking map property. Hence, one can think of the strictness condition to be a kind of *eventual surjectivity* of the linking maps. The **projective limit** of a projective spectrum $\mathscr{X}$ is defined by

$$\operatorname{Proj}(\mathscr{X}) = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n : \varrho^n_m(x_m) = x_n \text{ for all } m \ge n \right\}$$

and we can characterize the strictness of $\mathscr{X}$ to the condition

$$\forall\, n \in \mathbb{N}\; \exists\, m \ge n : \varrho^n_m(X_m) \subseteq \varrho^n(\operatorname{Proj}(\mathscr{X})),$$

where $\varrho^n : \operatorname{Proj}(\mathscr{X}) \to X_n$ denotes the projection onto the nth-component. Next we investigate the effects of strict projective spectra in an exact sequence of projective spectra for the remaining spectra. The image spectrum of a strict spectrum is also strict and if the left and right projective spectrum is strict, then also the spectrum in the middle is strict. To finally conclude, that the kernel spectrum of d is strict, it suffices to show the strictness of the image spectrum of d and the strictness of the projective spectrum of de Rham cohomology. The last strictness follows by a discrete Mittag-Leffler condition based on the fact that for an open and relatively compact subset N of a smooth manifold M, the restriction map $\varrho : H^k(M) \to H^k(N)$ has finite dimensional image for each $k \in \mathbb{N}_0$. Here we use a good cover of M, such that the compact closure $\overline{N}$ has a finite good cover and the restriction ϱ factorizes through a finite dimensional de Rham cohomology. It should be remarked that we can not transfer this result to foliated de Rham cohomology. Putting everything together, we finally obtain that the kernel spectrum of d, the projective spectrum of closed forms, is strict, such that d is indeed an SK-homomorphism. As explained before, we can state the corollary, that $d_{\mathcal{F}} : \Omega^k(M, \mathcal{F}) \to \Omega^{k+1}(M, \mathcal{F})$ is an SK-homomorphism if $\Omega^{k+1}_{\perp\!\!\perp}(M, \mathcal{F}) \subseteq d(\Omega^k(M))$ holds. We finish with a remark about the Čech cohomology of leafwise constant functions, where we reject a nearby replacement for the finite dimensional image of the restriction map ϱ between foliated de Rham cohomologies.

1 Preliminaries

Let X be a topological space. If X has a countable basis of topology, it is called *second countable*. X is said to be *locally Euclidean of dimension* $n \in \mathbb{N}$ if for every point $x \in X$ there is an open set $U \subseteq X$ containing x, together with a homeomorphism $\varphi : U \to V$ onto an open subset V of $\mathbb{R}^n$. In that case, (U, φ) is called a *chart*. A second countable, topological space with the Hausdorff property (different points can be separated by disjunct neighbourhoods) that is locally Euclidean of dimension n is called a *topological manifold of dimension* n. If (U_1, φ_1) and (U_2, φ_2) are charts with $U_1 \cap U_2 \neq \emptyset$, the so called *transition functions*, $\varphi_1^2 := \varphi_2 \circ \varphi_1^{-1}|_{\varphi_1(U_1 \cap U_2)} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ and φ_2^1, defined analogously, are homeomorphisms between open subsets of $\mathbb{R}^n$. We say two charts $(U_1, \varphi_1), (U_2, \varphi_2)$ are C^∞-*compatible* if their transition functions φ_1^2 and φ_2^1 are of class C^∞. A C^∞-*atlas* on a topological manifold M is a collection $\{(U_\alpha, \varphi_\alpha) : \alpha \in I\}$ of pairwise C^∞-compatible charts such that $M = \bigcup_{\alpha \in I} U_\alpha$. If the union of two C^∞-atlases is a C^∞-atlas we say the atlases are C^∞-*compatible*. This defines an equivalence relation on the C^∞-atlases and an equivalence class of a C^∞-atlas is called a *smooth differential structure* on the manifold. A smooth manifold M of dimension n is an n-dimensional topological manifold with a smooth differentiable structure and the charts of a C^∞-atlas which represents the differentiable structure, are called *smooth charts* of M. We could also require only C^k-compatibility ($k \in \mathbb{N}$) of the charts such that we get a C^k-atlas. But a famous Theorem of Whitney tells us, that for each $k \in \mathbb{N}$ every C^k-atlas contains a C^∞-subatlas.[2] Therefore it is convenient to consider always smooth manifolds. In general, a manifold needs not to be connected. But every manifold is a disjoint union of its connected components, which are connected. If not other mentioned, we assume a manifold to be connected.

A map $f : M \to \tilde{M}$ between smooth manifolds is called *smooth*, if for every $x \in M$ there exist smooth charts (U, φ) of M and $(\tilde{U}, \tilde{\varphi})$ of $\tilde{M}$ with $x \in U$ and $f(U) \subseteq \tilde{U}$ such that the *coordinate representation* $\tilde{\varphi} \circ f \circ \varphi^{-1} : \varphi(U) \to \tilde{\varphi}(\tilde{U})$ between open sets of Euclidean spaces is smooth. The space of smooth maps $f : M \to \tilde{M}$ will be denoted by $C^\infty(M, \tilde{M})$ and we simply write $C^\infty(M)$ for $C^\infty(M, \mathbb{R})$.

[2]See [Whi36, Theorem 1].

2 Smooth Vector Bundles and Sections

Our aim in this chapter is to introduce smooth vector bundles and smooth sections, which generalize smooth functions and provide therefore more applications in differential geometry. First of all, we need to define a smooth vector bundle over a smooth manifold. The reader should have in mind the example of the tangent bundle or the vector bundle of alternating k-tensors on the tangent spaces of a manifold, which is also called vector bundle of exterior k-forms. The sections of these bundles are precisely vector fields and differential k-forms, respectively. After defining the space of smooth sections, we equip it with a system of seminorms such that it becomes a Fréchet space.

2.1 Introduction of Smooth Vector Bundles

2.1.1 Definition (Smooth Vector Bundle)

Let M be an n-dimensional smooth manifold. A **smooth $\mathbb{R}$-vector bundle ξ of rank N over M** is a triple (E, π, M), where E is a smooth manifold and $\pi : E \to M$ is a surjective smooth map (called **projection**) such that for every $x \in M$:

(VB1) The set $E_x := \pi^{-1}(\{x\}) \subseteq E$ (called **fibre of E over x**) has the structure of an N-dimensional $\mathbb{R}$-vector space and

(VB2) there exists a **smooth local trivialization** (U, Φ) for x, i.e. U is an open neighbourhood of x and $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^N$ is a diffeomorphism such that $\mathrm{pr}_I \circ \Phi = \pi|_{\pi^{-1}(U)}$ holds for the projection $\mathrm{pr}_I : U \times \mathbb{R}^N \to U$ and further for every $y \in U$, the restriction $\Phi|_{E_y}$ is an isomorphism of E_y onto $\{y\} \times \mathbb{R}^N \cong \mathbb{R}^N$.

In that case, E is called the **total space** and M the **base space** of the bundle.
If there is a smooth local trivialization (M, Φ), this Φ is a **smooth global trivialization** and we call ξ a **trivial vector bundle**.

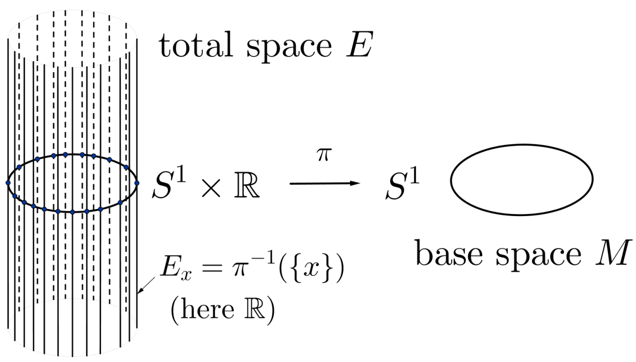


Figure 1: Trivial $\mathbb{R}$-vector bundle of rank 1 over $M = S^1$.

2.1.2 Remark (Smooth Vector Bundles)

(1) $\mathbb{C}$-vector bundles are defined similarly by replacing $\mathbb{R}$ with $\mathbb{C}$ everywhere in the definition, but we will treat only real vector bundles and henceforth we omit the field in the notation of a vector bundle.

(2) If (U, Φ) is a smooth local trivialization and $V \neq \emptyset$ is an open subset of U, then $(V, \Phi|_{\pi^{-1}(V)})$ is also a smooth local trivialization, which we will denote, by abuse of terminology, simply as (V, Φ). If $\{U_\alpha\}_{\alpha \in \mathscr{A}}$ is an open cover of M, we just write E_α for $\pi^{-1}(U_\alpha)$, such that $\{E_\alpha\}_{\alpha \in \mathscr{A}}$ is an open cover of E.

(3) The dimension of the manifold E has to be $n + N$ since the open set $\pi^{-1}(U)$ in E is diffeomorphic to an open subset of $\mathbb{R}^n \times \mathbb{R}^N$ by the composition of smooth maps

$$\pi^{-1}(U) \xrightarrow{\Phi} U \times \mathbb{R}^N \xrightarrow{\varphi \times \mathrm{id}_{\mathbb{R}^N}} \varphi(U) \times \mathbb{R}^N,$$

where (U, φ) is a smooth chart of M and (U, Φ) a smooth trivialization.

(4) The projection $\pi : E \to M$ of a smooth vector bundle is necessarily a submersion by (VB2), i.e. its differential[3] is surjective at each point. (For each $x \in M$, there is a smooth local trivialization (U, Φ) such that for every $\nu_x \in E_x$, we obtain $d\pi_{\nu_x} = d\left(\pi|_{\pi^{-1}(U)}\right)_{\nu_x} = d(\mathrm{pr}_I \circ \Phi)_{\nu_x}$, which is a surjective map since $\mathrm{pr}_I \circ \Phi$ is a submersion.)

(5) One may ask why (VB1) is required since (VB2) allows a way to equip the fibres with a vector space structure induced by a trivialization Φ. Concretely, define on a fibre E_x the operations $u + v = \Phi^{-1}(\Phi(u) + \Phi(v))$ and $r \cdot v = \Phi^{-1}(r\Phi(v))$, where $u, v \in E_x$, $r \in \mathbb{R}$. This makes E_x into a vector space but this vector space structure will be different to the structure induced by another trivialization Ψ since Φ and Ψ are not linear as diffeomorphisms in general. However, Lemma 2.1.7 shows, that two smooth trivializations with overlapping domains transform in terms of so called **transition functions**.

(6) In many examples, the total space is defined as a disjoint union of vector spaces over all points of the base space M. The disjoint union of a collection $\{V_x\}_{x \in M}$ of sets is defined to be the set

$$\bigsqcup_{x \in M} V_x = \bigcup_{x \in M} \{x\} \times V_x. \tag{2.1}$$

Thus, for every $v \in \bigsqcup_{x \in M} V_x$ there is a unique $v_x \in V_x$ corresponding to a unique $x \in M$ with $v = (x, v_x)$, such that we can identify v with v_x. The map $\pi : \bigsqcup_{x \in M} V_x \to M$, defined

[3] See Example 2.1.4, (3) for the definition.

by $v_x \mapsto x$, is clearly surjective and is the canonical candidate for the projection map of the bundle. Note that, even if all V_x are vector spaces, the total space has not yet a structure of a smooth manifold and talking about smoothness or the differential of π is meaningless. It is part of the *Vector Bundle Construction Lemma* (2.3.2) to specify these things.

Before giving examples, we introduce morphisms between smooth vector bundles.

2.1.3 Definition (Smooth Vector Bundle Morphism)

Let $\xi_E = (E, \pi_E, M)$ and $\xi_F = (F, \pi_F, M')$ be two smooth vector bundles. A pair (f, g) of smooth maps $f : E \to F$ and $g : M \to M'$ is called a **smooth vector bundle morphism from ξ_E to ξ_F** if $f_x = f|_{E_x} : E_x \to F_{g(x)}$ is a linear map between the fibres for each $x \in M$ and $g \circ \pi_E = \pi_F \circ f$ holds, i.e. the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ M & \xrightarrow[g]{} & M' \end{array}$$

The composition $(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1)$ of smooth vector bundle morphisms (f_1, g_1) from ξ_E to ξ_F and (f_2, g_2) from ξ_F to ξ_G defines a smooth vector bundle morphism from ξ_E to ξ_G. Together with this composition of smooth vector bundle morphisms, the class of smooth vector bundles is a category which will be denoted by **VB**. So we have the notion of isomorphisms and we use a characterization for the definition: A smooth vector bundle morphism (f, g) from ξ_E to ξ_F is a **smooth vector bundle isomorphism** if f, g are diffeomorphisms and f_x is an isomorphism of vector spaces for each $x \in M$. (ξ_E and ξ_F are isomorphic smooth vector bundles if and only if the manifolds E and F, M and M', respectively, are isomorphic as smooth manifolds and the corresponding fibres are isomorphic as vector spaces. In that case, the rank of ξ_E and ξ_F has to be the same.)

If (f, id_M) is a smooth vector bundle morphism between smooth vector bundles (E, π_E, M) and (F, π_F, M) over the same base space M, we call f a **smooth vector bundle M-morphism**. The category of smooth vector bundles over a fixed base space M and smooth vector bundle M-morphisms will be denoted by $\mathbf{VB}_M$.
The **rank** of a smooth vector bundle M-morphism f is a function $\mathrm{rank}(f) : M \to \mathbb{N}_0$, which assigns to each $x \in M$ the dimension of the linear space $f_x(E_x)$.
A smooth vector bundle M-morphism f (of necessarily constant rank) is called **injective** (**surjective**, **bijective**, respectively) if f_x is injective (surjective, bijective, respectively) for all $x \in M$.

2.1.4 Examples (Smooth Vector Bundles)

In the following, let M be a smooth n-dimensional manifold.

(1) **Trivial Vector Bundle:**
The projection $\mathrm{pr}_I : M \times \mathbb{R}^N \to M$ is smooth and $(M \times \mathbb{R}^N, \mathrm{pr}_I, M)$ is a trivial vector bundle of rank N since the fibres are $\mathrm{pr}_I^{-1}(\{x\}) = \{x\} \times \mathbb{R}^N \cong \mathbb{R}^N$ for each $x \in M$ and the identity on $M \times \mathbb{R}^N$ yields a smooth global trivialization. Moreover, a smooth vector bundle $\xi = (E, \pi_E, M)$ of rank N is trivial if and only if it is isomorphic to $(M \times \mathbb{R}^N, \mathrm{pr}_I, M)$. (For a smooth global trivialization $\Phi : E \to M \times \mathbb{R}^N$ the pair (Φ, id_M) defines a smooth vector bundle isomorphism and vice versa, if (f, g) is a smooth vector bundle isomorphism from ξ to $(M \times \mathbb{R}^N, \mathrm{pr}_I, M)$, a smooth global trivialization of E is given by f.)
Let $A : M \to \mathbb{R}^{L \times N}$ be a smooth matrix-valued map, i.e. all entries are smooth functions on M. Then, $f_A : M \times \mathbb{R}^N \to M \times \mathbb{R}^L$, $(x, \lambda) \mapsto (x, A(x)\lambda)$ defines a smooth vector bundle M-morphism between the trivial vector bundles.

(2) **Restricted Vector Bundle:**
Let $\xi = (E, \pi_E, M)$ be a smooth vector bundle of rank N and $\emptyset \neq U \subseteq M$ be open. Define $E_U = \pi_E^{-1}(U)$ and $\pi_{E_U} = \pi|_{E_U} : E_U \to U$, then we obtain a smooth vector bundle $\xi|_U = (E_U, \pi_{E_U}, U)$ of rank N, called the **restriction of ξ to U**. The pair of inclusions $i_{E_U} : E_U \to E$ and $i_U : U \to M$ defines a smooth vector bundle morphism from $\xi|_U$ to ξ since the following diagram commutes:

$$\begin{array}{ccc} E_U & \xrightarrow{i_{E_U}} & E \\ {\scriptstyle \pi_{E_U}}\downarrow & & \downarrow{\scriptstyle \pi_E} \\ U & \xrightarrow[i_U]{} & M \end{array}$$

For another non-empty open subset $V \subseteq U$ of M, we obtain $(\xi|_U)|_V = \xi|_V$ by $\pi_E^{-1}(V) \subseteq \pi_E^{-1}(U)$. Note that for a smooth local trivialization (U, Φ) of E, the restricted vector bundle on U is a trivial vector bundle. This relation enables an auspicious strategy for working with vector bundles. More precisely, given an open cover $(U_\alpha)_{\alpha \in \mathscr{A}}$ of M such that there are smooth local trivializations (U_α, Φ_α) of E, we can work on the restricted trivial vector bundles, where things are typically easier to handle. Hopefully, we can use a partition of unity subordinate to the open cover to get global results. But, in order to obtain well defined objects for the original vector bundle, we have to verify that the constructions are independent of the choice of trivializations. For that purpose, the *transition functions between local trivializations* (Lemma 2.1.7) are significant.
Given a smooth vector bundle M-morphism between vector bundles (E, π_E, M) and (F, π_F, M), we simply write f_U for the restriction $f|_{E_U} : E_U \to F_U$, which is a smooth vector bundle U-morphism between the restricted vector bundles. Further, if $\{U_\alpha\}_{\alpha \in \mathscr{A}}$ is an open cover of M, we write f_α for the restriction of f to $E_\alpha = \pi^{-1}(U_\alpha)$.

(3) **Tangent Bundle:**
The tangent bundle of a smooth manifold is a prime example of a smooth vector bundle in differential geometry. Other important vector bundles are constructed out of it, e.g. (4) and (5). For a detailed introduction including alternative definitions, we refer to [Lee13, Chapter 3, p. 50ff., p. 65ff., p. 71f. and Proposition 10.4, p. 252].
A linear function $\nu_x : C^\infty(M) \to \mathbb{R}$ is called a **derivation at** $x \in M$ if it satisfies for all $f, g \in C^\infty(M)$

$$\nu_x(fg) = f(x)\nu_x(g) + g(x)\nu_x(f). \tag{2.2}$$

The set of all derivations at $x \in M$ is defined to be the tangent space T_xM of x. It has the structure of an n-dimensional $\mathbb{R}$-vector space. Define

$$TM = \bigsqcup_{x \in M} T_xM \quad \text{and} \quad \tau_M : TM \to M, (x, \nu_x) \mapsto x.$$

To see that TM is a smooth manifold, note that once we have chosen a smooth chart $(U, \varphi = (x^1, ..., x^n))$ of M, we get canonically a basis of the tangent space T_xM for each $x \in U$ by the partial derivatives $\frac{\partial}{\partial x^i}\big|_p$ $(i = 1, ..., n)$. The ith partial derivative is defined for $f \in C^\infty(M)$ by $\left.\dfrac{\partial}{\partial x^i}\right|_x (f) = \left.\dfrac{\partial(f \circ \varphi^{-1})}{\partial x^i}\right|_{\varphi(x)}$. Define for $\pi^{-1}(U) = \bigsqcup_{x \in U} T_xM$:

$$\tilde{\varphi} : \pi^{-1}(U) \to \varphi(U) \times \mathbb{R}^n, \quad \left(x, \sum_{i=1}^n v_i(x) \left.\frac{\partial}{\partial x^i}\right|_x\right) \mapsto (\varphi(x), v_1(x), ..., v_n(x)), \tag{2.3}$$

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^n, \quad \left(x, \sum_{i=1}^n v_i(x) \left.\frac{\partial}{\partial x^i}\right|_x\right) \mapsto (x, v_1(x), ..., v_n(x)). \tag{2.4}$$

By running through a smooth atlas $(U_\alpha, \varphi_\alpha)_{\alpha \in \mathscr{A}}$ of M, we receive a smooth atlas $(\pi^{-1}(U_\alpha), \tilde{\varphi_\alpha})$ of TM and a family of smooth local trivializations $(U_\alpha, \Phi_\alpha)_{\alpha \in \mathscr{A}}$. The coordinate representations of τ_M corresponding to these charts are obviously smooth. Hence, (TM, τ_M, M) is a smooth $\mathbb{R}$-vector bundle of rank n, called the **tangent bundle** of M.
For a smooth map $F : M \to N$ between smooth manifolds and $x \in M$ define

$$dF_x : T_xM \to T_{F(x)}N, \text{ by } dF_x(\nu_x)(f) = \nu_x(f \circ F), \tag{2.5}$$

where $\nu_x \in T_xM$ and $f \in C^\infty(N)$. This even defines a smooth map $dF : TM \to TN$, called the **differential, derivative** or **pushforward** of F and is sometimes denoted by F_*. Moreover, the diagram

$$\begin{array}{ccc} TM & \xrightarrow{dF} & TN \\ {\scriptstyle \tau_M}\downarrow & & \downarrow{\scriptstyle \tau_N} \\ M & \xrightarrow[F]{} & N \end{array}$$

commutes, such that (dF, F) is a smooth vector bundle morphism from (TM, τ_M, M) to (TN, τ_N, N). The smooth map F is called an **immersion** (a **submersion**, resp.), if dF_x is injective (surjective, resp.) for every $x \in M$.

(4) **Cotangent Bundle**
The cotangent bundle is the dual bundle[4] of the tangent bundle. For each $x \in M$, the cotangent space T_x^*M is defined as $(T_xM)^*$, the (algebraic) dual space of T_xM, namely the set of linear functions from T_xM to $\mathbb{R}$.[5] Let T^*M be the disjoint union of T_x^*M over $x \in M$ and $\pi : T^*M \to M$ the canonical projection. Then (T^*M, π, M) is a smooth vector bundle of rank n, called the **cotangent bundle** of M. For a detailed introduction, we refer to [Lee13, Chapter 11].

(5) **Vector Bundle of Exterior k-Forms /Alternating k-Tensor Bundle:**
Let $k \in \mathbb{N}$. For $x \in M$ denote by $\Lambda^k(T_x^*M)$ the space of multilinear alternating[6] functions $\omega_x : (T_xM)^k \to \mathbb{R}$ and by $\Lambda^k(T^*M)$ the disjoint union of all $\Lambda^k(T_x^*M)$. Together with the canonical projection $\pi : \Lambda^k(T^*M) \to M$, we will see in Corollary 2.3.13 (c) that this defines a smooth vector bundle of rank $\binom{n}{k}$, called **vector bundle of exterior k-forms** or **alternating k-tensor bundle**. We list some special cases:

$$\Lambda^0(T^*M) = M \times \mathbb{R}, \qquad \text{since } \Lambda^0(T_x^*M) = \mathbb{R} \text{ for all } x \in M; \tag{2.6}$$

$$\Lambda^1(T^*M) = T^*M, \qquad \text{since alternation in one argument is no constraint;} \tag{2.7}$$

$$\Lambda^k(T^*M) = M \times \{0\}, \qquad \text{if } k > n, \text{ since then} \Lambda^k(T_x^*M) = \{0\} \text{ for all } x \in M. \tag{2.8}$$

As we have seen in example 2.1.4 (1), a smooth matrix-valued map induces a smooth vector bundle M-morphism between trivial vector bundles. Vice versa, we will show in Lemma 2.1.6, every smooth vector bundle M-morphism is locally induced by a smooth matrix-valued map. The following Lemma allows one to assume an open cover consisting of domains of charts and also local trivializations.

2.1.5 Lemma (Domains of Local Trivializations and Charts)

Let (E, π, M) be a smooth vector bundle of rank N. Then there is a countable locally finite open covering $(U_i)_{i\in I}$ of M such that:

(DTC) For each $i \in I$ there are a smooth local trivialization $\Phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^N$ of E over U_i and a smooth chart $\varphi_i : U_i \to W_i \subseteq \mathbb{R}^n$ of M.

Proof. The (VB2)-property yields an open covering $(V_x)_{x\in M}$ of M together with local trivializations Φ_x of E over V_x. Since M is a separable locally compact metrizable space as a manifold, there exists a countable locally finite open covering $(U_i)_{i\in I}$ of M, which is

[4] See Corollary 2.3.13 (a).

[5] The notation T_x^*M for $(T_xM)^*$ seems unnecessary but turns out to be useful to keep further expressions (as in the alternating k-tensor bundle) simpler without the brackets.

[6] The value of the function changes its sign whenever two arguments are permuted, see Remark 2.3.8.

finer than $(V_x)_{x\in M}$, i.e. for every $i \in I$ there is some V_x with $U_i \subseteq V_x$.[7] Further, we can assume that each V_i is the domain of a chart of M. (For each open covering of a differential manifold there exists a countable locally finite finer open covering consisting of domains of charts.)[8] So, for each $i \in I$, there is a chart $\varphi_i : U_i \to W_i \subseteq \mathbb{R}^n$. Because $(U_i)_{i\in I}$ is finer than $(V_x)_{x\in M}$, there is for each $i \in I$ a V_x containing U_i. Hence $\pi^{-1}(U_i) \subseteq \pi^{-1}(V_x)$ and $\Phi_i = \Phi_x|_{\pi^{-1}(U_i)} : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^N$ defines a smooth local trivialization of E over U_i. □

2.1.6 Lemma (Local Representations of Vector Bundle M-Morphisms)

Let (E, π_E, M) and (F, π_F, M) be smooth vector bundles of rank N and L, respectively. Moreover, let $U \subseteq M$ be open such that there are

(1) a chart (U, φ) of M,

(2) a smooth local trivialization (U, Φ_E) of E over U and

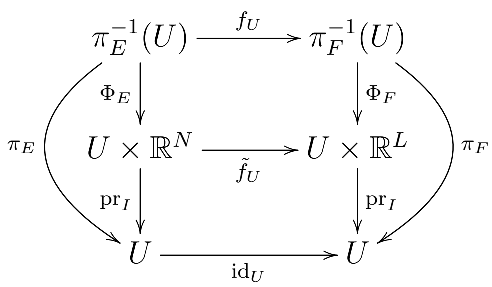
(3) a smooth local trivialization (U, Φ_F) of F over U.

For a smooth vector bundle M-morphism $f : E \to F$, there is a smooth matrix-valued map $A_f : U \to \mathbb{R}^{L\times N}$ satisfying

$$\Phi_F \circ f \circ \Phi_E^{-1}(x, \lambda) = (x, A_f(x)\lambda) \text{ on } U \times \mathbb{R}^N, \tag{2.9}$$

where each $A_f(x)$ is unique according to the standard basis of $\mathbb{R}^N$ and $\mathbb{R}^L$, respectively.

Proof.
We have the following commutative diagram for $f_U = f|_{\pi_E^{-1}(U)}$ and $\tilde{f}_U = \Phi_F \circ f_U \circ \Phi_E^{-1}$:



In particular, $\mathrm{pr}_I \circ \tilde{f}_U = \mathrm{pr}_I$ holds, such that there is a smooth function $\gamma : U \times \mathbb{R}^N \to \mathbb{R}^L$ satisfying $\tilde{f}_U(x, \lambda) = (x, \gamma(x, \lambda))$. For every fixed $x \in U$, the map $\lambda \mapsto \gamma(x, \lambda)$ from $\mathbb{R}^N$ to $\mathbb{R}^L$ is linear since $\tilde{f}_U$ is a composition of fibrewise linear maps. Hence, according to the standard basis of $\mathbb{R}^N$ and $\mathbb{R}^L$, respectively, there is a unique matrix $A_f(x) \in \mathbb{R}^{L\times N}$ representing $\lambda \mapsto \gamma(x, \lambda)$, i.e. $\tilde{f}_U(x, \lambda) = (x, A_f(x)\lambda)$. The entries $(A_f(x))_{i,j} = \pi_i(\gamma(x, e_j))$ are smooth maps from U to $\mathbb{R}$, since each projection $\pi_i : \mathbb{R}^L \to \mathbb{R}$ onto the i-th coordinate is smooth, such that $A_f : U \to \mathbb{R}^{L\times N}$ is indeed a smooth map. □

[7] This general topological result can be found in [Die76, 12.6.1].
[8] See [Die72, 16.1.4].

2.1.7 Lemma (Existence of Transition Functions between Localizations)

Let (E, π_E, M) be a smooth vector bundle of rank N. For any two smooth trivializations (U, Φ) and (V, Ψ) with $U \cap V \neq \emptyset$ there exists a unique smooth map $\tau : U \cap V \to \mathrm{GL}(\mathbb{R}, N)$ such that the composition $\Phi \circ \Psi^{-1} : (U \cap V) \times \mathbb{R}^N \to (U \cap V) \times \mathbb{R}^N$ is given by

$$\Phi \circ \Psi^{-1}(x, \lambda) = (x, \tau(x)\lambda), \tag{2.10}$$

In that case, we call τ the **transition function between Φ and Ψ**.

Proof. Since $U \cap V \neq \emptyset$ is open and the restrictions of Φ and Ψ to $U \cap V$ are smooth local trivializations of E, we can assume $U = V = U \cap V$ without loss of generality. The identity $\mathrm{id}_E : E \to E$ is clearly a vector bundle M-morphism. Hence, by Lemma 2.1.6, there is a unique smooth map $\tau : U \to \mathbb{R}^{N \times N}$ satisfying:

$$\Phi \circ \Psi^{-1}(x, \lambda) = \Phi \circ \mathrm{id}_E \circ \Psi^{-1}(x, \lambda) = (x, \tau(x)\lambda)$$

Smooth local trivializations are fibrewise linear isomorphisms, such that the representing matrix $\tau(x)$ is invertible for each $x \in U$. □

2.1.8 Remark (Transition Function)

(1) Note that the transition function between Ψ and Φ is given by the pointwise inverse matrix $\tau(x)^{-1}$, if τ is the transition function between Φ and Ψ.

(2) For any finite dimensional $\mathbb{R}$-vector space V of dimension N, the space $\mathrm{End}(V)$ of endomorphisms on V is a finite dimensional $\mathbb{R}$-vector space of dimension N^2 and hence a smooth manifold.[9] The set $\mathrm{Aut}(V)$ of automorphisms, the isomorphisms from V into V, is an open subset of $\mathrm{End}(V)$ and therefore also a smooth manifold. A choice of basis of V induces a diffeomorphism between $\mathrm{Aut}(V)$ and $\mathrm{GL}(\mathbb{R}, N)$ by building the representation matrix w.r.t. the chosen basis. Hence, we can replace the target space of a transition function τ by $\mathrm{Aut}(V)$ and write $\tau(x)(\lambda)$ instead of the matrix vector multiplication in (2.10).

2.2 Smooth Sections and Frames of Vector Bundles

2.2.1 Definition (Smooth Sections and Frames of Vector Bundles)

Let $\xi = (E, \pi, M)$ be a smooth vector bundle of rank N over a smooth n-dimensional manifold M and let $U \subseteq M$ be an open subset.

(a) A **smooth local section** $\sigma : U \to E$ **of** ξ is a smooth right inverse of the projection π, i.e. $\pi \circ \sigma = \mathrm{id}_U$. Addition and scalar multiplication, defined pointwise in the corresponding fibre, make the set of smooth local sections of E over U into an $\mathbb{R}$-vector

[9] See Example 2.3.10 (1) for more details.

space, which will be denoted by $\Gamma(U,E)$. Further, the pointwise multiplication in the corresponding fibre with a smooth function $\varphi : U \to \mathbb{R}$ is again a smooth local section, such that $\Gamma(U,E)$ can also be seen as a module over the ring of smooth functions $C^\infty(U)$. Elements of $\Gamma(M,E)$ are called **smooth global sections of** ξ.

(b) A **frame of** E **over** U is an ordered tuple $(\sigma_j)_{j=1}^N = (\sigma_1, ..., \sigma_N)$ of smooth local sections $\sigma_j \in \Gamma(U,E)$, such that $\sigma_1(x), ..., \sigma_N(x)$ form a basis of E_x for each $x \in U$. In that case, we will write $(U, (\sigma_j)_{j=1}^N)$ for a frame of E.

(c) Let $(U, (\sigma_j)_{j=1}^N)$ and $(V, (\varrho_j)_{j=1}^N)$ be frames of E with non-empty intersection $U \cap V$. If the map $\tau : U \cap V \to \mathrm{GL}(\mathbb{R}, N)$, given pointwise by the transition matrix from the basis $(\sigma_j(x))_{j=1}^N$ to $(\varrho_j(x))_{j=1}^N$ for $x \in U \cap V$, is a smooth map, then we call τ **a change of frames from** $(\sigma_j)_{j=1}^N$ **to** $(\varrho_j)_{j=1}^N$. For every $x \in U \cap V$, the transition matrix yields for each vector $v = \sum_{j=1}^N \lambda^j \sigma_j(x) \in E_x$ a representation

$$v = \sum_{k=1}^N \mu^k \varrho_k(x), \text{ where } \begin{bmatrix} \mu^1 \\ \vdots \\ \mu^N \end{bmatrix} = \tau(x) \begin{bmatrix} \lambda^1 \\ \vdots \\ \lambda^N \end{bmatrix}. \tag{2.11}$$

Local trivializations are naturally connected with local frames such that we can express conditions on trivializations equivalently in terms of frames:

2.2.2 Lemma (Equivalent Conditions on Frames Instead of Trivializations)

Let (E, π, M) be a smooth vector bundle of rank N.

(a) (1) If (U, Φ) is a smooth local trivialization of E, then $\sigma_j = \Phi^{-1}(\cdot, e_j)$ defines a frame of E over U, the so called **associated frame of** Φ, where e_j denotes the jth unitvector in $\mathbb{R}^N$.

(2) If $(U, (\sigma_j)_{j=1}^N)$ is a frame of E, then

$$\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^N, \ \sum_{j=1}^N \lambda^j \sigma_j(x) \mapsto (x, \lambda^1, ..., \lambda^N) \tag{2.12}$$

defines a smooth local trivialization over U, which we call the **associated trivialization of** $(\sigma_j)_{j=1}^N$.

In particular, the condition (VB2) of 2.1.1 is equivalent to

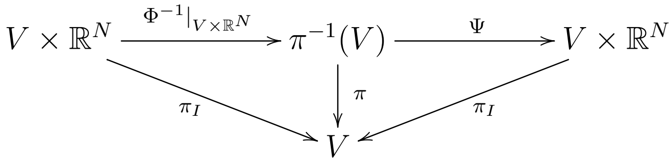
(VB2') For every $x \in M$ there is a frame $(\sigma_j)_{j=1}^N$ of E over some open neighbourhood $U \subseteq M$ of x.

Moreover, this shows that a vector bundle (E, π, M) is trivial if and only if there is a global frame of E over M.

(b) Let (U, Φ) and (V, Ψ) be smooth local trivializations of E with $U \cap V \neq \emptyset$ and denote their associated frames by $(V, (\sigma_j = \Psi^{-1}(\cdot, e_j))_{j=1}^N)$ and $(U, (\varrho_j = \Phi^{-1}(\cdot, e_j))_{j=1}^N)$. Then $\tau : U \cap V \to \mathrm{GL}(\mathbb{R}, N)$ is the transition function between Φ and Ψ if and only if τ is the change of their associated frames from $(\sigma_j)_{j=1}^N$ to $(\varrho_j)_{j=1}^N$. In particular, a change of frames with overlapping domains always exists and is unique.

Proof. (a) (1) The smoothness of $\sigma_j : U \to E$, $\sigma_j(x) = \Phi^{-1}(x, e_j)$ is inherited from Φ^{-1} and since $\Phi^{-1}(x, e_j) \in E_x$ for each $x \in U$, it follows that $\pi \circ \sigma_j = \mathrm{id}_U$. The restriction $\Phi|_{E_x}$ is a linear isomorphism for fixed $x \in U$, such that $(\sigma_j(x))_{j=1}^N$ is a basis of E_x if and only if $\Phi|_{E_x}$ maps it to a basis of $\{x\} \times \mathbb{R}^N \cong \mathbb{R}^N$, which is true since $(\Phi|_{E_x}(\sigma_j(x)) = (x, e_j))_{j=1}^N$ is a basis of $\{x\} \times \mathbb{R}^N$.

(2) $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^N$, $\sum\limits_{j=1}^N \lambda^j \sigma_j(x) \mapsto (x, \lambda^1, ..., \lambda^N)$ is bijective because $\sigma_1(x), ..., \sigma_N(x)$ is a basis of E_x for each $x \in U$. In addition, $\Phi|_{E_y}$ is a linear isomorphism of E_y onto $\{y\} \times \mathbb{R}^N \cong \mathbb{R}^N$ for each $y \in U$ and $\pi_I \circ \Phi = \pi$ holds for the projection $\pi_I : U \times \mathbb{R}^N \to U$. So if we can show, that Φ is a diffeomorphism, it is already a local trivialization of E. Due to bijectivity, it suffices to show that Φ is locally a diffeomorphism. Thus, we choose for $x \in U$ a local trivialization (V, Ψ) of E such that $x \in V \subseteq U$ and we will verify that $\Psi \circ \Phi^{-1}|_{V \times \mathbb{R}^N}$ is a diffeomorphism from $V \times \mathbb{R}^N$ to itself. Since each composition $\Psi \circ \sigma_j : V \to V \times \mathbb{R}^N$ is smooth, there are smooth functions $\sigma_j^1, ..., \sigma_j^N : V \to \mathbb{R}$ such that $\Psi \circ \sigma_j(y) = (y, \sigma_j^1(y), ..., \sigma_j^N(y))$. The map $S : V \to \mathrm{GL}(\mathbb{R}, N)$, defined by $(S(y))_{j,\ell} = \sigma_j^\ell(y)$ is well defined because $\Psi \circ \sigma_1(y), ..., \Psi \circ \sigma_N(y)$ is a basis of $\{y\} \times \mathbb{R}^N \cong \mathbb{R}^N$ and S is also smooth because all entries are smooth. Since $\sigma_j(y) = \Phi^{-1}(x, e_j)$, the commutative diagram



shows, that on $V \times \mathbb{R}^N$,

$$\Psi \circ \Phi^{-1}(y, \lambda^1, ..., \lambda^N) = (y, (S(y)\lambda)^T), \tag{2.13}$$

where $\lambda = [\lambda^1, ..., \lambda^N]^T$ and $(\cdot)^T$ denotes the (smooth) matrix transposition, such that the smoothness follows.
Matrix inversion is also smooth (Cramer´s rule), thus we obtain on $V \times \mathbb{R}^N$ smoothness of the inverse

$$(\Psi \circ \Phi^{-1})^{-1}(y, \mu^1, ..., \mu^N) = (y, \left(S^{-1}(y)\mu\right)^T),$$

where $\mu = [\mu^1, ..., \mu^N]^T$.
(b) Let $\tau : U \cap V \to \mathrm{GL}(\mathbb{R}, N)$ be the transition function between Φ and Ψ, hence $\Phi \circ \Psi^{-1}(x, \lambda) = (x, \tau(x)\lambda)$ on $(U \cap V) \times \mathbb{R}^N$. Fix $x \in U \cap V$ and $v = \sum\limits_{j=1}^N \lambda^j \sigma_j(x)$. Since

$\Psi|_{E_x}^{-1}$ is linear and $\sigma_j(x) = \Psi^{-1}(x, e_j)$ by definition, we obtain

$$v = \Psi|_{E_x}^{-1}\left(x, \sum_{j=1}^{N} \lambda^j e_j\right) = \Psi^{-1}(x, \lambda). \tag{2.14}$$

For $\mu^k = (\tau(x)\lambda)^k$, the kth-coordinate of $\tau(x)\lambda$, we get

$$v = \Phi^{-1} \circ \left(\Phi \circ \Psi^{-1}\right)(x, \lambda) = \Phi^{-1}(x, \tau(x)\lambda) = \sum_{k=1}^{N} (\tau(x)\lambda)^k \varrho_k(x) = \sum_{k=1}^{N} \mu^k \varrho_k(x). \tag{2.15}$$

Otherwise, let τ be a change of frames from $(\sigma_j)_{j=1}^N$ to $(\varrho_j)_{j=1}^N$. Then, for $x \in U \cap V$, we have $\Psi^{-1}(x, e_j) = \sigma_j(x) = \sum_{k=1}^{N} (\tau(x)e_j)^k \varrho_k(x)$. By linearity of $\Phi|_{E_x}$, this gives

$$\Phi \circ \Psi^{-1}(x, e_j) = \left(x, \sum_{k=1}^{N} (\tau(x)e_j)^k \Phi|_{E_x}(\varrho_k(x))\right) = \left(x, \sum_{k=1}^{N} (\tau(x)e_j)^k e_k\right). \tag{2.16}$$

It already implies $\Phi \circ \Psi^{-1}(x, \lambda) = (x, \tau(x)\lambda)$ on $(U \cap V) \times \mathbb{R}^N$, since $\Phi_{E_x} \circ \Psi|_{E_x}^{-1}$ is linear from $\{x\} \times \mathbb{R}^N$ to itself. In particular, existence and uniqueness of change of frames with overlapping domains follows by Lemma 2.1.7. □

2.2.3 Lemma (Representation of Smooth Local Sections)

Let (E, π, M) be a smooth vector bundle of rank N.

(a) A frame $(U, (\sigma_j)_{j=1}^N)$ of E is a basis of the $C^\infty(U)$-module $\Gamma(U, E)$, i.e. every $\gamma \in \Gamma(U, E)$ has a unique representation

$$\gamma = \sum_{j=1}^{N} f_j \sigma_j \text{ with smooth functions } f_j : U \to \mathbb{R}. \tag{2.17}$$

(b) Let $\tau : U \cap V \to \mathrm{GL}(\mathbb{R}, N)$ be a change of frames from $((\sigma_j)_{j=1}^N, U)$ to $((\varrho_j)_{j=1}^N, V)$ with $U \cap V \neq \emptyset$. Every $\gamma = \sum_{j=1}^{N} f_j \sigma_j \in \Gamma(U \cap V, E)$ can be written as $\gamma = \sum_{j=1}^{N} g_j \varrho_j$, where $\begin{bmatrix} g_1 \\ \vdots \\ g_N \end{bmatrix} = \tau \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}$ is defined pointwise as matrix-vector-multiplication and consists of smooth functions $g_j : U \cap V \to \mathbb{R}$.

Proof. (a) Let (U, Φ) be the associated trivialization of $(U, (\sigma_j)_{j=1}^N)$ and $\mathrm{pr}_j : U \times \mathbb{R}^N \to \mathbb{R}$, $(x, \lambda^1, \cdots \lambda^N) \to \lambda^j$ be the projection onto the jth-coordinate. For $\gamma \in \Gamma(U, E)$ define $f_j = \mathrm{pr}_j \circ \Phi \circ \gamma$, such that f_j is smooth as composition of smooth functions. Furthermore,

for each $x \in U$ there are unique $\lambda^1, ..., \lambda^N \in \mathbb{R}$ with $\gamma(x) = \sum_{j=1}^{N} \lambda^j \sigma_j(x)$. By linearity of $\Phi|_{E_x}$ it follows, that $\Phi \circ \gamma(x) = \sum \lambda^j \Phi|_{E_x}(\sigma_j(x)) = (x, \lambda^1, ..., \lambda^N)$. Hence, we obtain $f_j(x) = \mathrm{pr}_j(x, \lambda^1, ..., \lambda^N) = \lambda^j$ as required.
(b) This follows immediately by definition of a change of frames and part (a). □

2.2.4 Lemma (Smooth Sections are Locally Isomorphic to Smooth Functions)

Let (E, π, M) be a smooth vector bundle of rank N and let $U \subseteq M$ be open such that there are

(1) a chart (φ, U) of M and

(2) a smooth local trivialization (Φ_E, U) of E over U.

Then $\Gamma(U, E)$ and $C^\infty(\varphi(U), \mathbb{R}^N)$ are isomorphic $\mathbb{R}$-vector spaces by the linear maps

$$T_E : \Gamma(U, E) \to C^\infty(\varphi(U), \mathbb{R}^N), \qquad \sigma \mapsto \mathrm{pr}_{II} \circ \Phi_E \circ \sigma \circ \varphi^{-1} \quad \text{and its inverse} \tag{2.18}$$

$$T_E^{-1} : C^\infty(\varphi(U), \mathbb{R}^N) \to \Gamma(U, E), \qquad g \mapsto \Phi_E^{-1} \circ (\mathrm{id}_U \times (g \circ \varphi)), \tag{2.19}$$

where pr_{II} denotes the projection of $U \times \mathbb{R}^N$ onto $\mathbb{R}^N$.

Proof. Both functions are well defined, since all involved functions are smooth and

$$\pi_E \circ T_E^{-1}(g) = \mathrm{pr}_I \circ (\mathrm{id}_U \times (g \circ \varphi)) = \mathrm{id}_U \tag{2.20}$$

holds. The vector space structure on $\Gamma(U, E)$ is defined pointwise in each fibre such that the linearity of T_E and T_E^{-1} already follows by the fibrewise linearity of Φ_E. So it remains to show the inversion identities:

$$T_E \circ T_E^{-1}(g) = \mathrm{pr}_{II} \circ \Phi_E \circ (\Phi_E^{-1} \circ (\mathrm{id}_U \times (g \circ \varphi))) \circ \varphi^{-1} = \mathrm{pr}_{II} \circ (\varphi^{-1}, g) = g; \tag{2.21}$$

$$T_E^{-1} \circ T_E(\sigma) = \Phi_E^{-1} \circ (\mathrm{id}_U \times (\mathrm{pr}_{II} \circ \Phi_E \circ \sigma \circ \varphi^{-1} \circ \varphi) \tag{2.22}$$

$$= \Phi_E^{-1} \circ ((\pi_E \circ \sigma), \mathrm{pr}_{II} \circ \Phi_E \circ \sigma) \overset{\pi_E = \mathrm{pr}_I \circ \Phi_E}{=} \Phi_E^{-1} \circ \Phi_E \circ \sigma = \sigma. \tag{2.23}$$

□

2.3 Constructions of Smooth Vector Bundles

In this subsection we present some useful tools to construct smooth vector bundles.

The following lemma can be found in [Lee13, Lemma 1.35, p.21] and will be used in the proof of the *Vector Bundle Construction Lemma* to equip the total space with the structure of a smooth manifold. We will skip the proof.

2.3.1 Smooth Manifold Chart Lemma

Let M be a set and $\{U_\alpha\}_{\alpha\in\mathscr{A}}$ a collection of subsets of M together with injective maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ satisfying:

(1) $\varphi_\alpha(U_\alpha)$ and $\varphi_\alpha(U_\alpha \cap U_\beta)$ are open in $\mathbb{R}^n$ for each $\alpha, \beta \in \mathscr{A}$.

(2) $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is a diffeomorphism whenever $U_\alpha \cap U_\beta \neq \emptyset$.

(3) M is covered by countably many of the sets U_α.

(4) For all distinct $x, y \in M$, either there exists some U_α containing x and y or there exist disjoint sets U_α, U_β with $x \in U_\alpha$ and $y \in U_\beta$.

Then M has a unique smooth manifold structure such that $\mathcal{A} = \{(U_\alpha, \varphi_\alpha) : \alpha \in \mathscr{A}\}$ is a C^∞-atlas.

The next lemma can be found in [Lee13, Lemma 10.6, p.253] and is a very important tool to construct smooth vector bundles.

2.3.2 Vector Bundle Construction Lemma

Let M be a smooth manifold and $E = \bigsqcup_{x\in M} E_x$, where E_x is a p-dimensional $\mathbb{R}$-vector space for each $x \in M$. Define $\pi : E \to M$ by $\pi(v_x) = x$ if $v_x \in E_x$. Further, suppose there are

(1) an open cover $\{U_\alpha\}_{\alpha\in\mathscr{A}}$ of M;

(2) a bijective map $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^p$ for each $\alpha \in \mathscr{A}$ such that the restriction to each E_x ($x \in U_\alpha$) is a linear isomorphism from E_x to $\{x\} \times \mathbb{R}^p \cong \mathbb{R}^p$;

(3) for each $\alpha, \beta \in \mathscr{A}$ with $U_{\alpha,\beta} = U_\alpha \cap U_\beta \neq \emptyset$, a smooth map $\tau : U_{\alpha,\beta} \to \mathrm{GL}(\mathbb{R}, p)$ such that the composite map $\Phi_\alpha \circ \Phi_\beta^{-1}$ from $(U_{\alpha,\beta}) \times \mathbb{R}^p$ to itself has the form

$$\Phi_\alpha \circ \Phi_\beta^{-1}(x, \lambda) = (x, \tau(x)\lambda). \tag{2.24}$$

Then E has a unique smooth manifold structure such that (E, π, M) becomes a smooth vector bundle of rank p with smooth local trivializations given by Φ_α ($\alpha \in \mathscr{A}$).

Proof. For each $x \in M$, choose some U_α containing x and choose a smooth chart (V_x, φ_x) of M such that $x \in V_x \subseteq U_\alpha$. Set $\tilde{V}_x = \varphi_x(V_x) \subseteq \mathbb{R}^n$, where n is the dimension of M and define $\tilde{\varphi}_x : \pi^{-1}(V_x) \to \tilde{V}_x \times \mathbb{R}^p$ by $\tilde{\varphi}_x = (\varphi_x \times \mathrm{id}_{\mathbb{R}^p}) \circ \Phi_\alpha$, i.e.

$$\pi^{-1}(V_x) \xrightarrow{\Phi_\alpha} V_x \times \mathbb{R}^p \xrightarrow{\varphi_x \times \mathrm{id}_{\mathbb{R}^p}} \tilde{V}_x \times \mathbb{R}^p.$$

We will verify the conditions of the Smooth Manifold Chart Lemma 2.3.1 to show, that the collection $\{(\pi^{-1}(V_x), \tilde{\varphi}_x) : x \in M\}$ makes E into a smooth manifold.
$\tilde{\varphi}_x$ is bijective as a composition of bijective maps. The images $\tilde{\varphi}_x(\pi^{-1}(V_x) = \varphi_x(V_x) \times \mathbb{R}^p$ and $\tilde{\varphi}_x(\pi^{-1}(V_x) \cap \pi^{-1}(V_y)) = \varphi_x(V_x \cap V_y) \times \mathbb{R}^p$ are open for all $x, y \in M$. Further, if $\pi^{-1}(V_x) \cap \pi^{-1}(V_y) = \pi^{-1}(V_x \cap V_y) \neq \emptyset$, we obtain for $\beta \in \mathscr{A}$ with $V_y \subseteq U_\beta$:

$$\tilde{\varphi}_x \circ \tilde{\varphi}_y^{-1} = (\varphi_x \times \mathrm{id}_{\mathbb{R}^p}) \circ (\Phi_\alpha \circ \Phi_\beta^{-1}) \circ (\varphi_x \times \mathrm{id}_{\mathbb{R}^p})^{-1}. \tag{2.25}$$

Hence, $\tilde{\varphi}_x \circ \tilde{\varphi}_y^{-1}$ is smooth as a composition of smooth maps. Since $\{V_x : x \in M\}$ has a countable subcover $\{V_x : x \in I\}$, $\{\pi^{-1}(V_x) : x \in I\}$ is a countable subcover of E. If $u, v \in E$ lie in the same fibre E_x, then it follows that already $u, v \in \pi^{-1}(V_x)$. Otherwise $u \in E_x$ and $v \in E_y$ for distinct $x, y \in M$. Then, by the Hausdorff property of M, we can choose disjoint open neighbourhoods V_x of x and V_y of y, respectively, such that $\pi^{-1}(V_x)$ and $\pi^{-1}(V_y)$ are disjoint neighbourhoods of u and v, respectively. Thus E has the structure of a smooth manifold induced by the C^∞-atlas $\{(\pi^{-1}(V_x), \tilde{\varphi}_x) : x \in M\}$. Note that the coordinate representation of Φ_α with respect to the charts $(\pi^{-1}(V_x), \tilde{\varphi}_x)$ of E and $(V_x \times \mathbb{R}^p, \varphi_p \times \mathrm{id}_{\mathbb{R}^p})$ of $V_x \times \mathbb{R}^p$, is the identity map, such that Φ_α is smooth. π is also smooth since the coordinate representation $\varphi_x \circ \pi \circ \tilde{\varphi}_x^{-1}$ maps $(\lambda, \mu) \in \varphi_x(V_x) \times \mathbb{R}^p$ to $\lambda \in \varphi_x(V_x)$, which is smooth. Moreover, Φ_α maps E_x to $\{x\} \times \mathbb{R}^p$, such that $\pi_I \circ \Phi_\alpha = \pi$ follows and Φ_α is by hypothesis linear on the fibres. Hence Φ_α is indeed a smooth local trivialization of E. □

For a given smooth vector bundle (E, π, M), we have seen in Example 2.1.4 (2) how we can get by restriction a bundle over a smaller base space. The definition of a subbundle tells us when we can get a bundle over a smaller total space. It is indeed a smooth vectorbundle as clarified in Lemma 2.3.5.

2.3.3 Definition (Subbundle)

Let $\xi = (E, \pi, M)$ be a smooth vector bundle of rank N and $G \subseteq E$. The triple $(G, \pi|_G, M)$ is called a (smooth) p-**subbundle** of ξ if:

(SB1) For each $x \in M$, the set $G_x = E_x \cap G$ is a p-dimensional $\mathbb{R}$-subspace of E_x.

(SB2) Each $x \in M$ has a smooth local trivialization $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^N$ of E over an open neighbourhood U of x such that $\Phi(\pi^{-1}(U) \cap G) = U \times \mathbb{R}^p \times \{0 \in \mathbb{R}^{N-p}\}$.

Note that by Lemma 2.2.2 (a), (SB2) is equivalent to the following condition in terms of local frames:

(SB2') For each $x \in M$, there is frame $\{\sigma_j\}_{j=1}^N$ of E over some open neighbourhood $U \subseteq M$ of x such that $\{\sigma_j\}_{j=1}^p$ is a frame of G over U.

In order to give a few examples, we introduce the kernel and image vector bundle of a smooth vector bundle M-morphism, which is only well-defined if the rank of the morphism, defined in Definition 2.1.3, is constant:

2.3.4 Definition (Kernel and Image Vector Bundle)

Consider a smooth vector bundle M-morphism f from an N-ranked vector bundle ξ_E to ξ_F. Then, $\mathrm{Ker}(f)$ defined as the disjoint union of all $\mathrm{Ker}(f_x)$ with the canonical projection defines an $(N-r)$-subbundle of ξ_E if and only if $\mathrm{rank}(f)$ is constant of value r. Similarly, $\mathrm{Im}(f)$ defined as the disjoint union of all $\mathrm{Im}(f_x)$ together with the canonical projection is an r-subbundle of ξ_F if and only if $\mathrm{rank}(f)$ is constant of value r.

2.3.5 Lemma (Properties of a Subbundle)

Let $(G, \pi|_G, M)$ be a p-subbundle of a smooth vector bundle (E, π, M) of rank N over the smooth n-dimensional manifold M. Then

(a) G is a smooth $(n+p)$-dimensional embedded submanifold of the smooth $(n+N)$-dimensional manifold E, i.e.:

  (SM) Each $g \in G$ has a chart (V, ψ) in E with $g \in V$ such that $\psi : V \to \mathbb{R}^{n+N}$ satisfies $\psi(V \cap G) = \psi(V) \cap (\mathbb{R}^{n+p} \times \{0 \in \mathbb{R}^{N-p}\})$

(b) $(G, \pi|_G, M)$ is a smooth vector bundle of rank p.

(c) The inclusion $i : G \to E$, defined fibrewise by $i_x : G_x \to E_x$, $y \to y$, is an injective smooth vector bundle M-morphism and has a left inverse smooth vector bundle M-morphism $\ell : E \to G$, i.e. $\ell \circ i = \mathrm{id}_G$.

(d) The transition function $\tau : U \cap V \to \mathrm{GL}(\mathbb{R}, N)$ between local trivializations Φ and Ψ according to (SB2) has the form

$$\tau(x) = \begin{bmatrix} A(x) & C(x) \\ 0 & B(x) \end{bmatrix}, \tag{2.26}$$

where $A : U \cap V \to \mathrm{GL}(\mathbb{R}, p)$, $B : U \cap V \to \mathrm{GL}(\mathbb{R}, N-p)$ and $C : U \cap V \to \mathbb{R}^{p \times (N-p)}$ are smooth maps.

Proof. (a) As mentioned in Remark 2.1.2 (1), E has necessarily dimension $n+N$. Imitating the proof of 2.1.5 using (SB2) instead of (VB2), one gets a countable open covering $(U_\alpha)_{\alpha \in \mathscr{A}}$ of M such that for each $\alpha \in \mathscr{A}$ there are a smooth chart $\varphi_\alpha : U_\alpha \to W_\alpha \subseteq \mathbb{R}^n$ of M and a diffeomorphism $\Phi_\alpha : E_\alpha \to U_\alpha \times \mathbb{R}^N$ with $\Phi_\alpha(G_\alpha) = U_\alpha \times \mathbb{R}^p \times \{0 \in \mathbb{R}^{N-p}\}$, where $E_\alpha = \pi^{-1}(U_\alpha)$ and $G_\alpha = \pi|_G^{-1}(U_\alpha)$. Since U_α is diffeomorphic by the chart φ_α to some open set $W_\alpha \subseteq \mathbb{R}^n$, we obtain a diffeomorphism $\Psi_\alpha = (\varphi_\alpha \times \mathrm{id}_{\mathbb{R}^N}) \circ \Phi_\alpha : E_\alpha \to W_\alpha \times \mathbb{R}^N \subseteq \mathbb{R}^{n+N}$, such that $\Psi_\alpha(G_\alpha) = \Psi_\alpha(E_\alpha) \cap (\mathbb{R}^{n+p} \times \{0 \in \mathbb{R}^{N-p}\})$. $\{E_\alpha\}_{\alpha \in \mathscr{A}}$ is an open covering of E since π is continuous. In particular, each $g \in G \subseteq E$ is contained in some E_α and the corresponding diffeomorphism Ψ_α is a chart with the required property.
(b) (SB1) gives (VB1) with fibre dimension p and (VB2') follows immediately by (SB2').
(c) The inclusion $i : G \to E$ is clearly continuous because of $\pi \circ i = \pi|_G$, such that

$i^{-1}(\pi^{-1}(U)) = \pi^{-1}|_G(U)$ for every open $U \subseteq M$. Moreover, i is smooth since the coordinate representations are inclusions between open subsets of $\mathbb{R}^{n+p}$ and $\mathbb{R}^{n+N}$, which are smooth. Further, i is injective because the restrictions $i_x = i|_{G_x} : G_x \to E_x$ are injective and linear as well for all $x \in M$, such that i is indeed an injective smooth vector bundle M-morphism. For the construction of a left inverse, consider an open cover $\{U_\alpha\}_{\alpha\in\mathscr{A}}$ of M, such that $\{G_\alpha = \pi|_G^{-1}(U_\alpha)\}$ is an open cover of G. Define $i_I : \mathbb{R}^p \to \mathbb{R}^p \times \{0 \in \mathbb{R}^{N-p}\}$, $\lambda \mapsto (\lambda, 0)$ and $\mathrm{pr}_I : \mathbb{R}^N \to \mathbb{R}^p$, $(\mu_1, ..., \mu_N) \mapsto (\mu_1, ..., \mu_p)$ such that

$$i_I \circ \mathrm{pr}_I : \mathbb{R}^N \to \mathbb{R}^p \times \{0 \in \mathbb{R}^{N-p}\} \text{ and } i_I \circ \mathrm{pr}_I |_{\mathbb{R}^p\times\{0\}} = \mathrm{id}_{\mathbb{R}^p\times\{0\}} \,. \tag{2.27}$$

For $\alpha \in \mathscr{A}$, there is a smooth local trivialization $\Phi_\alpha : E_\alpha \to U_\alpha \times \mathbb{R}^N$ according to $(SB2)$, hence $\Phi_\alpha(G_\alpha) = U_\alpha \times \mathbb{R}^p \times \{0\}$. Define $\ell_\alpha = \Phi_\alpha^{-1} \circ (\mathrm{id}_{U_\alpha} \times (i_I \circ \mathrm{pr}_I)) \circ \Phi_\alpha)$. Then, by (2.27), ℓ_α is a map from E_α to G_α satisfying $\ell_\alpha \circ i_\alpha = \ell_\alpha|_{G_\alpha} = \mathrm{id}_{G_\alpha}$. As a composition of continuous and smooth maps, ℓ_α is also continuous and smooth. Moreover, ℓ_α is linear on each fibre E_x for $x \in U_\alpha$ and $\pi|_G \circ \ell_\alpha = \pi$ holds. For a partition of unity $\{\chi_\alpha\}_{\alpha\in\mathscr{A}}$ subordinate to the open cover $\{U_\alpha\}_{\alpha\in\mathscr{A}}$, define $\ell = \sum\limits_{\alpha\in\mathscr{A}} \chi_\alpha \ell_\alpha$. It is easy to check by the properties of each ℓ_α, that ℓ is a smooth vector bundle M-morphism and $\ell \circ i = \sum\limits_{\alpha\in\mathscr{A}} \chi_\alpha \,\mathrm{id}_{G_\alpha} = \mathrm{id}_G$.

(d) For $x \in U \cap V$, consider $\tau(x)$ as a block matrix

$$\tau(x) = \begin{bmatrix} A(x) & C(x) \\ D(x) & B(x) \end{bmatrix}, \tag{2.28}$$

where $A(x) \in \mathbb{R}^{p\times p}$, $B(x) \in \mathbb{R}^{(N-p)\times(N-p)}$, $C(x) \in \mathbb{R}^{p\times(N-p)}$ and $D(x) \in \mathbb{R}^{(N-p)\times p}$. All block matrices are smooth functions on $U\cap V$ because all entries are smooth and if $D(x) = 0$ for all $x \in U \cap V$, it follows that $A(x)$ and $B(x)$ are regular on $U \cap V$ since $\tau(x)$ is regular. Note that for local trivializations Φ and Ψ according to (SB2), the restriction of $\Phi \circ \Psi^{-1}$ to $(U\cap V) \times \mathbb{R}^p \times \{0\}$ maps to $(U\cap V) \times \mathbb{R}^p \times \{0\}$. Hence, because of $\Phi\circ\Psi^{-1}(x, \lambda) = (x, \tau(x)\lambda)$, this already implies $D(x) = 0$ on $U \cap V$. □

Given a subbundle of a smooth vector bundle, we can form a quotient bundle, defined fibrewise by the quotient vector spaces. It is indeed a smooth vector bundle as we see in Lemma 2.3.7.

2.3.6 Definition (Quotient Bundle)

Let $(G, \pi|_G, M)$ be a p-subbundle of a smooth vector bundle (E, π, M) of rank N.
Define $E/G = \bigsqcup\limits_{x\in M} E_x/G_x$ and $\pi_{E/G} : E/G \to M$, $v_x \mapsto x$ if $v_x \in E_x/G_x$. Moreover, define $q : E \to E/G$, where $q|_{E_x}$ is the quotient map of E_x onto E_x/G_x.
$(E/G, \pi_{E/G}, M)$ is called the quotient bundle of (E, π, M) by (the subbundle) $(G, \pi|_G, M)$ and q is called the quotient bundle map. This is justified by the following result.

2.3.7 Lemma (Properties of a Quotient Bundle)

Let $(G, \pi|_G, B)$ be a smooth p-subbundle of a smooth N-ranked vector bundle (E, π, B). Then:

(a) The quotient bundle $(E/G, \pi_{E/G}, M)$ is a smooth $\mathbb{R}$-vector bundle of rank $N - p$.

(b) The quotient bundle map q is a surjective smooth vector bundle M-morphism and it has a smooth vector bundle M-morphism right inverse $r : E/G \to E$, i.e. $q \circ r = \mathrm{id}_{E/G}$.

Proof.
(a) We will verify the conditions of the Vector Bundle Construction Lemma. We choose an open cover $\{U_\alpha\}_{\alpha \in \mathscr{A}}$ of M together with smooth local trivializations $\Phi_\alpha : E_\alpha \to U_\alpha \times \mathbb{R}^N$ according to (SB2), i.e.

$$\Phi_\alpha(G_\alpha) = U_\alpha \times \mathbb{R}^p \times \{0 \in \mathbb{R}^{N-p}\}. \tag{2.29}$$

Define $\mathrm{pr}_{II} : \mathbb{R}^N \to \mathbb{R}^{N-p}$, $(\lambda_1, ..., \lambda_N) \mapsto (\lambda_{p+1}, ..., \lambda_N)$ and $i_{II} : \mathbb{R}^{N-p} \to \mathbb{R}^N$, $(\mu_1, ..., \mu_{N-p}) \mapsto (0, ..., 0, \mu_1, ..., \mu_{N-p})$, such that

$$\mathrm{pr}_{II} \circ i_{II} = \mathrm{id}_{\mathbb{R}^{N-p}} \qquad \text{and} \tag{2.30}$$

$$i_{II} \circ \mathrm{pr}_{II} : \mathbb{R}^N \to \{0 \in \mathbb{R}^p\} \times \mathbb{R}^{N-p}, (\lambda_1, ..., \lambda_N) \mapsto (0, ..., 0, \lambda_{p+1}, ..., \lambda_N). \tag{2.31}$$

For fixed $\alpha \in \mathscr{A}$, define the map $f_\alpha = (\mathrm{id}_{U_\alpha} \times \mathrm{pr}_{II}) \circ \Phi_\alpha : E_\alpha \to U_\alpha \times \mathbb{R}^{N-p}$. By (2.29), the restriction $f_x = f_\alpha|_{E_x}$ is constant on equivalence classes of E_x/G_x for each $x \in U_\alpha$, such that there is a linear map $\Psi_{\alpha,x} : E_x/G_x \to \{x\} \times \mathbb{R}^{N-p}$ satisfying $\Psi_{\alpha,x} \circ q|_{E_x} = f_\alpha|_{E_x}$ for each $x \in U_\alpha$. This defines a map $(\Psi_\alpha : E/G)_\alpha \to U_\alpha \times \mathbb{R}^{N-p}$ by $\Psi_\alpha|_{E_x/G_x} = \Psi_{\alpha,x}$ such that

$$\Psi_\alpha \circ q_\alpha = f_\alpha = (\mathrm{id}_{U_\alpha} \times \mathrm{pr}_{II}) \circ \Phi_\alpha \text{ on } E_\alpha. \tag{2.32}$$

Moreover, Ψ_α is bijective, where its inverse is given by $\Psi_\alpha^{-1} = q_\alpha \circ \Phi_\alpha^{-1} \circ (\mathrm{id}_{U_\alpha} \times i_{II})$. Indeed,

$$\begin{aligned} \Psi_\alpha \circ \Psi_\alpha^{-1} &= (\Psi_\alpha \circ q) \circ \Phi_\alpha^{-1} \circ (\mathrm{id}_{U_\alpha} \times i_{II}) \\ &\overset{(2.32)}{=} (\mathrm{id}_{U_\alpha} \times \mathrm{pr}_{II}) \circ (\Phi_\alpha \circ \Phi_\alpha^{-1}) \circ (\mathrm{id}_{U_\alpha} \times i_{II}) \\ &= \mathrm{id}_{U_\alpha} \times (\mathrm{pr}_{II} \circ i_{II}) \overset{(2.30)}{=} \mathrm{id}_{U_\alpha \times \mathbb{R}^{N-p}}. \end{aligned}$$

For the other identiy, it sufficies to show $\Psi_\alpha^{-1} \circ \Psi_\alpha \circ q_\alpha = \mathrm{id}_{(E/G)_\alpha} \circ q_\alpha$ on E_α since q_x is surjective for all $x \in U_\alpha$. Therefore, define $g_\alpha = \Phi_\alpha^{-1} \circ (\mathrm{id}_{U_\alpha} \times (i_{II} \circ \mathrm{pr}_{II})) \circ \Phi_\alpha$, such that $\Psi_\alpha^{-1} \circ \Psi_\alpha \circ q_\alpha = q_\alpha \circ g_\alpha$ on E_α. For $x \in U_\alpha$ and $v_x \in E_x$, we obtain by fibrewise linearity,

$$g_\alpha(v_x) - v_x = \Phi_\alpha^{-1}((\mathrm{id}_{U_\alpha} \times (i_{II} \circ \mathrm{pr}_{II})) \circ \Phi_\alpha(v_x) - \Phi_\alpha(v_x)) \tag{2.33}$$

$$= \Phi_\alpha^{-1}(\mathrm{id}_{U_\alpha} \times (i_{II} \circ \mathrm{pr}_{II} - \mathrm{id}_{\mathbb{R}^N}))(\Phi_\alpha(v_x)) \in G_x, \tag{2.34}$$

because $(i_{II} \circ \mathrm{pr}_{II} - \mathrm{id}_{\mathbb{R}^N})$ maps to $\mathbb{R}^p \times \{0\}$ by (2.31) and $\Phi_\alpha^{-1}(U_\alpha \times \mathbb{R}^p \times \{0\}) = G_\alpha$. Hence, we obtain $\Psi_\alpha^{-1} \circ \Psi_\alpha \circ q_\alpha = q_\alpha \circ g_\alpha = \mathrm{id}_{(E/G)_\alpha} \circ q_\alpha$ on E_α. In addition, $g_x = g_\alpha|_{E_x}$

is constant on equivalence classes of ${}^{E_x}\!/\!{}_{G_x}$ for each $x \in U_\alpha$ by (2.29) such that there is a fibrewise linear map $r_\alpha : ({}^{E}\!/\!{}_{G})_\alpha \to E_\alpha$ satisfying $r_\alpha \circ q_\alpha = g_\alpha$ on E_α, which implies $q_\alpha \circ r_\alpha = \mathrm{id}_{({}^{E}\!/\!{}_{G})_\alpha}$.[10] To sum up, for each $\alpha \in \mathscr{A}$, we have a bijective map Ψ_α such that for all $x \in U_\alpha$, the restriction to ${}^{E_x}\!/\!{}_{G_x}$ is a linear isomorphism onto $\{x\} \times \mathbb{R}^{N-p} \cong \mathbb{R}^{N-p}$. Now, for $\alpha, \beta \in \mathscr{A}$ with $U_{\alpha,\beta} = U_\alpha \cap U_\beta \neq \emptyset$, the transition function τ between Φ_α and Φ_β according to (SB2) is of the form $\tau(x) = \begin{bmatrix} A(x) & C(x) \\ 0 & B(x) \end{bmatrix}$ by Lemma 2.3.5 (d). Hence, we obtain for $(x, \lambda) \in U_{\alpha,\beta} \times \mathbb{R}^{N-p}$,

$$\Psi_\alpha \circ \Psi_\beta^{-1}(x, \lambda) = (\Psi_\alpha \circ q) \circ \Phi_\beta^{-1}((\mathrm{id}_{U_{\alpha,\beta}} \times i_{II})(x, \lambda) \tag{2.35}$$

$$= (\mathrm{id}_{U_{\alpha,\beta}} \times \mathrm{pr}_{II}) \circ \Phi_\alpha \circ \Phi_\beta^{-1}(x, (0, \lambda)) \tag{2.36}$$

$$= (\mathrm{id}_{U_{\alpha,\beta}} \times \mathrm{pr}_{II}) \left(x, \tau(x) \begin{bmatrix} 0 \\ \lambda \end{bmatrix} \right) = (x, B(x)\lambda), \tag{2.37}$$

where $B : U_{\alpha,\beta} \to \mathrm{GL}(\mathbb{R}, N-p)$ is a smooth map by Lemma 2.3.5 (d) such that all conditions of the Vector Bundle Construction Lemma are satisfied and therefore $({}^{E}\!/\!{}_{G}, \pi_{{}^{E}\!/\!{}_{Q}}, M)$ is a smooth vector bundle of rank $N-p$ with smooth local trivializations $\{(U_\alpha, \Psi_\alpha)\}_{\alpha \in \mathscr{A}}$.
(b) The quotient bundle map $q : E \to {}^{E}\!/\!{}_{G}$ is surjective since it is surjective on each fibre. Consider an open cover $\{U_\alpha\}_{\alpha \in \mathscr{A}}$ of M and for each $\alpha \in \mathscr{A}$ the map r_α as in part (a), then we have the following commutative diagrams for each $\alpha \in \mathscr{A}$:

$$\begin{array}{ccc} E & \xrightarrow{q} & {}^{E}\!/\!{}_{G} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_{E/G}} \\ M & \xrightarrow[\mathrm{id}_M]{} & M \end{array} \qquad \qquad \begin{array}{ccc} ({}^{E}\!/\!{}_{G})_\alpha & \xrightarrow{r_\alpha} & E_\alpha \\ {\scriptstyle (\pi_{E/G})_\alpha}\downarrow & & \downarrow{\scriptstyle \pi} \\ U_\alpha & \xrightarrow[\mathrm{id}_{U_\alpha}]{} & U_\alpha \end{array}$$

Hence, q is continuous because of $q^{-1}(\pi_{E/G}^{-1}(U)) = (\pi_{E/G} \circ q)^{-1}(U) = \pi^{-1}(U)$ for any open $U \subseteq M$. Analogously, r_α is also continuous. To check smoothness, we recall that for a C^∞-atlas of M, consisting of charts $\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{R}^n$, a C^∞-atlas of E is given by charts $\tilde{\varphi}_\alpha = (\varphi_\alpha \times \mathrm{id}_{\mathbb{R}^N}) \circ \Phi_\alpha$. As we can see in the proof of the Vector Bundle Construction Lemma, $\tilde{\psi}_\alpha = (\varphi_\alpha \times \mathrm{id}_{\mathbb{R}^{N-p}}) \circ \Psi_\alpha$ defines a chart of an C^∞-atlas of ${}^{E}\!/\!{}_{G}$, where Φ_α and Ψ_α are smooth local trivializations of the underlying bundles as in part (a). The corresponding coordinate representation of q becomes

$$\tilde{\psi}_\alpha \circ q_\alpha \circ \tilde{\varphi}_\alpha^{-1} = (\varphi_\alpha \times \mathrm{id}_{\mathbb{R}^{N-p}}) \circ (\Psi_\alpha \circ q_\alpha) \circ ((\varphi_\alpha \times \mathrm{id}_{\mathbb{R}^N}) \circ \Phi_\alpha)^{-1} \tag{2.38}$$

$$\overset{(2.32)}{=} (\varphi_\alpha \times \mathrm{id}_{\mathbb{R}^{N-p}}) \circ (\mathrm{id}_{U_\alpha} \times \mathrm{pr}_{II}) \circ \Phi_\alpha \circ \Phi_\alpha^{-1} \circ (\varphi_\alpha^{-1} \times \mathrm{id}_{\mathbb{R}^N}) \tag{2.39}$$

$$= (\varphi_\alpha \circ \varphi_\alpha^{-1}) \times \mathrm{pr}_{II} = \mathrm{id}_{V_\alpha} \times \mathrm{pr}_{II}, \tag{2.40}$$

which is truly a smooth map from $V_\alpha \times \mathbb{R}^N$ to $V_\alpha \times \mathbb{R}^{N-p}$.

[10] We will construct a right inverse r of q in (ii) using r_α.

Since $r_\alpha \circ q_\alpha = g_\alpha = \Phi_\alpha^{-1} \circ (\mathrm{id}_{U_\alpha} \times (i_{II} \circ \mathrm{pr}_{II})) \circ \Phi_\alpha$, we obtain,

$$\begin{aligned}
\tilde{\varphi}_\alpha \circ r_\alpha \circ \tilde{\psi}_\alpha^{-1} &= \tilde{\varphi}_\alpha \circ r_\alpha \circ \Psi_\alpha^{-1} \circ (\varphi_\alpha^{-1} \times \mathrm{id}_{\mathbb{R}^{N-p}}) && (2.41)\\
&= \tilde{\varphi}_\alpha \circ r_\alpha \circ \Psi_\alpha^{-1} \circ (\varphi_\alpha^{-1} \times \mathrm{id}_{\mathbb{R}^{N-p}}) && (2.42)\\
&= \tilde{\varphi}_\alpha \circ (r_\alpha \circ q_\alpha) \circ \Phi_\alpha^{-1} \circ (\mathrm{id}_{U_\alpha} \times i_{II}) \circ (\varphi_\alpha^{-1} \times \mathrm{id}_{\mathbb{R}^{N-p}}) && (2.43)\\
&= \tilde{\varphi}_\alpha \circ \Phi_\alpha^{-1} \circ (\varphi_\alpha^{-1} \times (i_{II} \circ (\mathrm{pr}_{II} \circ i_{II}))) && (2.44)\\
&= (\varphi_\alpha \times \mathrm{id}_{\mathbb{R}^N}) \circ \Phi_\alpha \circ \Phi_\alpha^{-1} \circ (\varphi_\alpha^{-1} \times i_{II}) = \mathrm{id}_{V_\alpha} \times i_{II}, && (2.45)
\end{aligned}$$

which is a smooth map from $V_\alpha \times \mathbb{R}^{N-p}$ to $V_\alpha \times \mathbb{R}^N$. Let $(\chi_\alpha)_{\alpha \in \mathscr{A}}$ be a smooth partition of unity subordinate to $\{U_\alpha\}_{\alpha \in \mathscr{A}}$, then $r = \sum_{\alpha \in A} \chi_\alpha r_\alpha$ defines a smooth map $r : E/G \to E$ such that $q \circ r = \mathrm{id}_{E/G}$ Indeed, since the partition of unity is locally finite, the identity follows pointwise by linearity of q on each fibre and $q_\alpha \circ r_\alpha = \mathrm{id}_{(E/G)_\alpha}$ for each $\alpha \in \mathscr{A}$. □

In order to verify that the dual bundle or (alternating) k-tensor bundle is indeed a vector bundle, we want to use an abstract approach to construct vector bundles from old ones, induced by so called *manifold-enriched functors*. This procedure is adopted from [Eur], which is based on lecture notes by Hiro Lee Tanaka [Tan14]. We start with a remark about the vector space of alternating tensors, including definitions and an induced basis.

2.3.8 Remark (Alternating Tensors)

For $k \in \mathbb{N}_0$ and an $\mathbb{R}$-vector space V of dimension N, the vector space of **alternating k-tensors** $\Lambda^k(V^*)$ is the set of alternating multilinear functions $\nu : V^k \to \mathbb{R}$. Alternating means that the function changes its sign whenever two arguments are interchanged. The $\mathbb{R}$-vector space structure is defined by pointwise addition and pointwise scalar multiplication. Note that $\Lambda^0(V^*) = \mathbb{R}$ and $\Lambda^k(V^*) = \{0\}$ for $k > N$. Further, $\Lambda^1(V^*) = V^*$ is just the (algebraic) dual space of V. We refer to [Lee13, p. 350 ff.] for details and summarize a few facts. Let $I^k_{\mathrm{inc}}(N)$ be the set of increasing multi-indices $I = (i_1, ..., i_k)$ of length $k \leq N$, i.e. $1 \leq i_1 < ... < i_k \leq N$. If $\{b_1, ..., b_N\}$ is a basis of V and $\{b_1^*, .., b_N^*\}$ is its dual basis, we define for $I = (i_1, ..., i_k) \in I^k_{\mathrm{inc}}(N)$ a so called **elementary alternating tensor** $\beta^I \in \Lambda^k(V^*)$ by

$$\beta^I(v_1, ..., v_k) = \det \begin{bmatrix} b_{i_1}^*(v_1) & \cdots & b_{i_1}^*(v_k) \\ \vdots & \ddots & \vdots \\ b_{i_k}^*(v_1) & \cdots & b_{i_k}^*(v_k) \end{bmatrix}. \tag{2.46}$$

The family $\{\beta^I\}_{I \in I^k_{\mathrm{inc}}(N)}$ constitutes a basis of $\Lambda^k(V^*)$, hence $\Lambda^k(V^*)$ is of dimension $\binom{N}{k}$.

2.3.9 Definition (Manifold-Enriched Category and Functor)

(a) A **category C** is called **manifold-enriched** if for any two objects $A, B \in \mathrm{Obj}(\mathbf{C})$, the set of morphisms $\mathrm{Mor}(A, B)$ has a structure of a smooth manifold.

(b) A **functor** $\mathscr{F}$ between two manifold-enriched categories $\mathbf{C}$ and $\mathbf{C}'$ is called **manifold-enriched** if for any two objects $A, B \in \mathrm{Obj}(\mathbf{C})$, the mapping $f \mapsto \mathscr{F}(f)$

$$\mathscr{F}_{A,B} : \mathrm{Mor}(A,B) \to \mathrm{Mor}(\mathscr{F}(A), \mathscr{F}(B)) \text{ if } \mathscr{F} \text{ is covariant,} \tag{2.47}$$

$$\mathscr{F}_{A,B} : \mathrm{Mor}(A,B) \to \mathrm{Mor}(\mathscr{F}(B), \mathscr{F}(A)) \text{ if } \mathscr{F} \text{ is contravariant, respectively,} \tag{2.48}$$

is smooth.

2.3.10 Example (Manifold-Enriched Categories and Functors)

(1) For our purpose, the category of interest is $\mathbf{Vect}_{\mathbb{R}}^{\mathbf{f.d.}}$ consisting of finite dimensional $\mathbb{R}$-vector spaces as objects and linear maps as morphisms with the usual composition. Note that $\mathrm{Mor}(V,W) = \{f : V \to W \text{ linear }\}$ is again a finite dimensional vector space and therefore it has the structure of a smooth manifold. To be more precise, let $(a_1, ..., a_n)$ be a basis of V and $(b_1, ..., b_m)$ a basis of W. Then, the linear functions $\gamma_{i,j} : V \to W$, $\gamma_{i,j}(v) = a_j^*(v) b_i$ for $i = 1, ..., n$ and $j = 1, ..., m$ define a basis of $\mathrm{Mor}(V,W)$, where a_j^* denotes the jth basis vector of the dual basis of $(a_1, ..., a_n)$. Hence, $\dim(\mathrm{Mor}(V,W)) = \dim(V) \cdot \dim(W)$. Further, $f \in \mathrm{Mor}(V,W)$ has the unique representation $f = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{i,j} \gamma_{i,j}$ with $\gamma_{i,j}^*(f) = \lambda_{i,j} \in \mathbb{R}$. The map

$$\varphi : \mathrm{Mor}(V,W) \to \mathbb{R}^{n \cdot m},\ \varphi(f) = (\gamma_{1,1}^*(f), ..., \gamma_{n,1}^*(f), ..., \gamma_{1,m}^*(f), ..., \gamma_{n,m}^*(f)) \tag{2.49}$$

defines a linear bijection. Starting with different bases of V and W, one obtains a C^∞-compatible chart for φ since the transition functions are linear, hence continuous and smooth. Thus, $\mathrm{Mor}(V,W)$ is indeed a smooth manifold of dimension $n \cdot m$.

(2) Let V, W be finite dimensional $\mathbb{R}$-vector spaces and $f : V \to W$ be a linear map. The following (contravariant) functors from $\mathbf{Vect}_{\mathbb{R}}^{\mathbf{f.d.}}$ to itself are manifold-enriched:

(i) **Dual Functor** $\mathscr{F}^*$
$\mathscr{F}^*(V) = V^* = \{\varphi : V \to \mathbb{R} \text{ linear}\}$ (dual vector space), $\dim(V^*) = \dim(V)$;
$\mathscr{F}^*(f) = f^* : W^* \to V^*$, $f^*(\psi) = \psi \circ f$ (transposed, pullback).

(ii) **k-Tensor Functor** $\mathscr{F}_{\otimes}^k$
$\mathscr{F}_{\otimes}^k(V) = \bigotimes^k(V^*) = \{\mu : V^k \to \mathbb{R} \text{ multilinear}\}$, $\dim(\bigotimes^k(V^*)) = \dim(V)^k$;
$\mathscr{F}_{\otimes}^k(f) = f^* : \bigotimes^k(W^*) \to \bigotimes^k(V^*)$, $f^*(\omega) : V^k \to \mathbb{R}$,
defined by $f^*(\omega)(v_1, ..., v_k) = \omega(f(v_1), ..., f(v_k))$ (pullback).

(iii) **Alternating k-Tensor Functor**$\mathscr{F}_{\wedge}^k$
$\mathscr{F}_{\wedge}^k(V) = \Lambda^k(V^*) = \{\mu : V^k \to \mathbb{R} \text{ alternating multilinear}\}$, $\dim(\Lambda^k(V^*)) = \binom{\dim(V)}{k}$;
$\mathscr{F}_{\wedge}^k(f) = f^* : \Lambda^k(W^*) \to \Lambda^k(V^*)$, $f^*(\omega) : V^k \to \mathbb{R}$,
defined by $f^*(\omega)(v_1, ..., v_k) = \omega(f(v_1), ..., f(v_k))$ as in (ii) (pullback).

The pullbacks are linear and the (global) representative of the mapping $f \mapsto \mathscr{F}(f)$ from $\mathrm{Mor}(V,W)$ to $\mathrm{Mor}(\mathscr{F}(W), \mathscr{F}(V))$ corresponding to the (linear) charts in (1) are also linear and therefore continuous and smooth in all cases.

2.3.11 Remark (Motivation of Manifold-Enrichment)

The next theorem shows, that we can extend a manifold-enriched functor between $\mathbf{Vect}_{\mathbb{R}}^{\mathbf{f.d.}}$ to a functor between $\mathbf{VB}_M$, the category of smooth vector bundles over a fixed base space M with smooth vector bundle M-morphisms. So far, one wonders why we require in Definition 2.3.9 a smooth manifold structure on the set of morphisms and smoothness of the map assignment by a functor since in our examples, the set of morphisms are finite dimensional vector spaces and the map assignments are linear maps between these vector spaces. The key point is that $\mathrm{Aut}(\mathbb{R}^N)$ is not a vector (sub)space but an open submanifold of $\mathrm{Mor}(\mathbb{R}^N, \mathbb{R}^N)$ such that the restriction of the map

$$\mathscr{F}_{\mathbb{R}^N,\mathbb{R}^N} : \mathrm{Mor}(\mathbb{R}^N, \mathbb{R}^N) \to \mathrm{Mor}(\mathscr{F}(\mathbb{R}^N), \mathscr{F}(\mathbb{R}^N)) \tag{2.50}$$

to $\mathrm{Aut}(\mathbb{R}^N)$ is also a smooth map.

2.3.12 Theorem (Extension of a Manifold-Enriched Functor)

Let $\mathscr{F}$ be a manifold-enriched functor from $\mathbf{Vect}_{\mathbb{R}}^{\mathbf{f.d.}}$ to $\mathbf{Vect}_{\mathbb{R}}^{\mathbf{f.d.}}$.

(a) Let $\xi = (E, \pi_E, M)$ be a smooth vector bundle of rank N. Then $\mathscr{F}(E) = \bigsqcup_{x\in M} \mathscr{F}(E_x)$ has a unique topology and smooth structure such that $\mathscr{F}(\xi) = (\mathscr{F}(E), \pi, M)$ is a smooth vector bundle of rank $\dim(\mathscr{F}(\mathbb{R}^N))$, where π denotes the canonical projection.

(b) Let f be a vector bundle M-morphism from ξ_E to ξ_F.
(1) If $\mathscr{F}$ is covariant, the map $\mathscr{F}(f) : \mathscr{F}(\xi_E) \to \mathscr{F}(\xi_F)$ defined fibrewise by

$$\mathscr{F}(f)_x : \{x\} \times \mathscr{F}(E_x) \to \{x\} \times \mathscr{F}(F_x),\ (x, v) \mapsto (x, \mathscr{F}(f_x)(v)), \tag{2.51}$$

is smooth and a vector bundle M-morphism.
(2) If $\mathscr{F}$ is contravariant, the map $\mathscr{F}(f) : \mathscr{F}(\xi_F) \to \mathscr{F}(\xi_E)$ defined fibrewise by

$$\mathscr{F}(f)_x : \{x\} \times \mathscr{F}(F_x) \to \{x\} \times \mathscr{F}(E_x),\ (x, w) \mapsto (x, \mathscr{F}(f_x)(w)), \tag{2.52}$$

is smooth and a vector bundle M-morphism.

In this way, $\mathscr{F}$ extends to a functor from $\mathbf{VB}_M$ to $\mathbf{VB}_M$ for a fixed base space M, i.e. the assignment preserves composition and identity morphisms.

Proof. (a) The proof is an application of the Vector Bundle Construction Lemma 2.3.2. Let $(U_\alpha)_{\alpha\in\mathscr{A}}$ be a cover of M such that there is a smooth local trivialization (U_α, Φ_α) of E for each $\alpha \in \mathscr{A}$. In particular, the restrictions

$$(\Phi_\alpha)_x : E_x \to \{x\} \times \mathbb{R}^N \cong \mathbb{R}^N \tag{2.53}$$

(treated with target space $\mathbb{R}^N$ in the following) are isomorphisms of vector spaces for all $\alpha \in \mathscr{A}, x \in U_\alpha$. Define

$$\Psi_{\alpha,x} = \begin{cases} \mathscr{F}((\Phi_\alpha)_x), & \text{if } \mathscr{F} \text{ is covariant;} \\ \mathscr{F}((\Phi_\alpha)_x^{-1}), & \text{if } \mathscr{F} \text{ is contravariant.} \end{cases} \tag{2.54}$$

By functoriality, $\Psi_{\alpha,x} : \mathscr{F}(E_x) \to \mathscr{F}(\mathbb{R}^N)$ is also an isomorphism of vector spaces for all $\alpha \in \mathscr{A}$ and $x \in U_\alpha$. Now, we define $\mathscr{F}(\Phi_\alpha) : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathscr{F}(\mathbb{R}^N)$ fibrewise by $\mathscr{F}(\Phi_\alpha)_x : \{x\} \times \mathscr{F}(E_x) \to \{x\} \times \mathscr{F}(\mathbb{R}^N)$, $(x, v_x) \mapsto (x, \Psi_{\alpha,x}(v_x))$. Then $\mathscr{F}(\Phi_\alpha)$ is linear on each fibre and bijective. If $U_{\alpha,\beta} = U_\alpha \cap U_\beta \neq \emptyset$ for some $\alpha, \beta \in \mathscr{A}$, there are the smooth transition functions $\tau_{\alpha,\beta}, \tau_{\beta,\alpha} : U_{\alpha,\beta} \to \mathrm{Aut}(\mathbb{R}^N)$ according to Remark 2.1.8 (2). Since $\mathscr{F}_{\mathbb{R}^N,\mathbb{R}^N}|_{\mathrm{Aut}(\mathbb{R}^N)} : \mathrm{Aut}(\mathbb{R}^N) \to \mathrm{Aut}(\mathscr{F}(\mathbb{R}^N))$ is smooth as a restriction of a smooth map to an open submanifold, we obtain the smooth composition $\tilde{\tau}_{\alpha,\beta} : U_{\alpha,\beta} \to \mathrm{Aut}(\mathscr{F}(\mathbb{R}^N))$,

$$\tilde{\tau}_{\alpha,\beta} = \begin{cases} \mathscr{F}_{\mathbb{R}^N,\mathbb{R}^N}|_{\mathrm{Aut}(\mathbb{R}^N)} \circ \tau_{\alpha,\beta}, & \text{if } \mathscr{F} \text{ is covariant;} \\ \mathscr{F}_{\mathbb{R}^N,\mathbb{R}^N}|_{\mathrm{Aut}(\mathbb{R}^N)} \circ \tau_{\beta,\alpha}, & \text{if } \mathscr{F} \text{ is contravariant.} \end{cases} \tag{2.55}$$

Then, for $x \in U_{\alpha,\beta}$ and $w \in \mathbb{R}^N$, we obtain

$$\mathscr{F}(\Phi_\alpha) \circ \mathscr{F}(\Phi_\beta)^{-1}(x, w) = \mathscr{F}(\Phi_\alpha)_x(x, \Psi_{\beta,x}^{-1}(w)) = (x, \Psi_{\alpha,x}(\Psi_{\beta,x}^{-1})(w))) \tag{2.56}$$

$$= (x, \tilde{\tau}_{\alpha,\beta}(x)(w)). \tag{2.57}$$

(b) Except the smoothness, it is easy to check, that $\mathscr{F}(f)$ is a vector bundle M-morphism. We only show the smoothness if $\mathscr{F}$ is contravariant. Let (U, Φ_E) and (U, Φ_F) be smooth local trivializations of ξ_E and ξ_F, respectively. Under the notations of (a), the composition

$$\mathscr{F}(\Phi_E) \circ \mathscr{F}(f) \circ \mathscr{F}(\Phi_F)^{-1} : U \times \mathscr{F}(\mathbb{R}^L) \to U \times \mathscr{F}(\mathbb{R}^N), \tag{2.58}$$

$$(x, w) \mapsto (x, \mathscr{F}((\Phi_E)_x^{-1}) \circ \mathscr{F}(f_x) \circ \mathscr{F}((\Phi_F)_x)(w)) \tag{2.59}$$

is clearly smooth where the domain and range are considered as product manifolds, such that the smoothness of $\mathscr{F}(f)$ follows.
Let g be another M-morphism between ξ_F and ξ_G. We obtain $\mathscr{F}(g \circ f) = \mathscr{F}(g) \circ \mathscr{F}(f)$ if $\mathscr{F}$ is covariant just by definition as well as $\mathscr{F}(g \circ f) = \mathscr{F}(f) \circ \mathscr{F}(g)$ if $\mathscr{F}$ is contravariant. Further, the identity on ξ_E is mapped to the identity on $\mathscr{F}(\xi_E)$. □

2.3.13 Corollary (Smooth Vector Bundles)

Let $\xi = (E, \pi_E, M)$ be a smooth vector bundle of rank N and $k \in \mathbb{N}_0$. Under the notations of Theorem 2.3.12 and Example 2.3.10, we obtain the following smooth vector bundles

(a) $\mathscr{F}^*(\xi)$ of rank N, called the **dual bundle of** ξ;

(b) $\mathscr{F}_\otimes^k(\xi)$ of rank N^k, called the k**-tensor bundle of** ξ;

(c) $\mathscr{F}^k_\wedge(\xi)$ of rank $\binom{N}{k}$, called the **alternating k-tensor bundle of ξ.**
Moreover, if $f : E \to F$ is a smooth vector bundle M-morphism between smooth vector bundles over M, the pullback of alternating k-tensors $f^* : \Lambda^k(F^*) \to \Lambda^k(E^*)$, defined fibrewise by

$$f_x^*(\omega_x)(v_1, ..., v_k) = \omega_x(f_x(v_1), ..., f_x(v_k)), \quad \text{where } x \in M,\ \omega_x \in \Lambda^k(F_x^*),\ v_i \in E_x, \tag{2.60}$$

is a smooth vector bundle M-morphism and of constant rank, if f is of constant rank.

2.4 Short Exact Sequences of Vector Bundles

2.4.1 Remark (No Abelian Category)

The category $\mathbf{VB}_M$, consisting of smooth vector bundles over the same base space M as objects and smooth vector bundle M-morphisms as morphisms is not an abelian category since kernels and images only exist if the M-morphism has constant rank. Note that the composition of M-morphisms with constant ranks does not have constant rank in general: For example, let $I = (0, 1)$ and $\varphi : I \to \mathbb{R}$ be a smooth function equal to 0 on $\left(0, \frac{1}{3}\right)$ and equal to 1 on $\left(\frac{2}{3}, 1\right)$. Define $A, B : I \to \mathbb{R}^{2\times 2}$ by

$$A = \begin{bmatrix} \varphi & (1-\varphi) \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.61}$$

Then $f_A, f_B : I \times \mathbb{R}^2 \to I \times \mathbb{R}^2$, $f_A(t, x) = (t, A(t)x)$ and $f_B(t, x) = (t, B(t)x))$ define vector bundle morphisms of constant rank 1 between the trivial vector bundle $I \times \mathbb{R}^2$, but $f_A \circ f_B$ does not have constant rank, since $f_A \circ f_B(t, x) = (t, A(t)B(t)x)$ and $A(t)B(t) = \begin{bmatrix} \varphi & 0 \\ 0 & 0 \end{bmatrix}$. Thus, all M-morphisms with constant rank do not serve as morphisms of a category. However, the M-morphisms with constant rank are the *homomorphisms* of $\mathbf{VB}_M$:

2.4.2 Lemma (Vector Bundle Homomorphisms)

Let $f : E \to F$ be a smooth vector bundle M-morphism of constant rank d between two smooth vector bundles (E, π_E, M) and (F, π_F, M). Then, f is a homomorphism in the category $\mathbf{VB}_M$. This means that the induced bijection

$$\tilde{f} : E/\mathrm{Ker}(f) \to \mathrm{Im}(f) \tag{2.62}$$

is an isomorphism of smooth vector bundle M-morphisms, i.e. $(\tilde{f}, \mathrm{id}_M)$ is a smooth vector bundle isomorphism.

Proof. For ease of notation, denote $\tilde{f}$ with g. Then, for each $x \in X$, the linear map $g_x : E_x/\mathrm{Ker}(f_x) \to \mathrm{Im}(f_x)$, $g_x([v_x])) = f_x(v_x)$ is an isomorphism of vector spaces such that there is a linear inverse g_x^{-1}. This defines a map $g^{-1} : \mathrm{Im}(f) \to E/\mathrm{Ker}(f)$ satisfying the inversion equalities and $\pi_E \circ g^{-1} = \pi_F$ since $\pi_E = \pi_F \circ g$. To check smoothness, consider an open subset $U \subseteq M$ such that there are a smooth chart (U, φ) of M and a smooth local trivializations (Φ_E, U) and (Φ_F, U) over both bundles. By Lemma (2.1.6) of local representation, there is a smooth function $A_g : U \to \mathbb{R}^{d\times d}$ satisfying

$$\Phi_F \circ g \circ \Phi_E^{-1}(x, \lambda) = (x, A_g(x) \cdot \lambda) \text{ on } U \times \mathbb{R}^d. \tag{2.63}$$

In particular, A_g maps to $\mathrm{GL}(\mathbb{R}, d)$ because each g_x is an isomorphism of vector spaces. Hence, we obtain by inverting:

$$\Phi_E \circ g^{-1} \circ \Phi_F^{-1}(x, \lambda) = (x, (A_g(x))^{-1} \cdot \lambda) \quad \text{on } U \times \mathbb{R}^d. \tag{2.64}$$

Since the inversion of a matrix is a smooth map from $\mathrm{GL}(\mathbb{R}, d)$ to itself as a consequence of Cramer's Rule, we have a smooth map $A_{g^{-1}} : U \to \mathrm{GL}(\mathbb{R}, d)$, $A_{g^{-1}}(x) = (A_g(x))^{-1}$, such that the coordinate representations of g^{-1} are smooth. □

2.4.3 Definition (Short Exact Sequence of Smooth Vector Bundles over M)

A sequence

$$\cdots \xrightarrow{f^{k-1}} E^{k-1} \xrightarrow{f^k} E^k \xrightarrow{f^{k+1}} E^{k+1} \xrightarrow{f^{k+2}} \cdots \qquad (*)$$

consisting of total spaces of smooth vector bundles over the same base space M together with smooth vector bundle M-morphisms is called a **sequence of smooth vector bundles over M**. Such a sequence is called **exact at E^k**, if f^k and f^{k+1} are of constant rank and satisfy $\mathrm{Im}(f^k) = \mathrm{Ker}(f^{k+1})$, i.e. $\mathrm{Im}(f^k_x) = \mathrm{Ker}(f^{k+1}_x)$, for each $x \in M$. The sequence $(*)$ is called **exact** if it is exact at each total space.
Denote the trivial vector bundle $M \times \{0\}$ simply by 0. A sequence of smooth vector bundles over M $0 \longrightarrow E \xrightarrow{f} F \xrightarrow{g} G \longrightarrow 0$ is called **short**.
Note that a smooth vector bundle M-morphism $f : E \to F$ is injective (resp. surjective)[11] if and only if $0 \longrightarrow E \xrightarrow{f} F$ (resp. $E \xrightarrow{f} F \longrightarrow 0$) is an exact sequence.

By abuse of language, we call the total space of a smooth vector bundle over M a smooth vector bundle.

2.4.4 Splitting Lemma

Consider a short exact sequence of smooth vector bundles over M

$$0 \longrightarrow E \xrightarrow{f} F \xrightarrow{g} G \longrightarrow 0.$$

The following assertions are equivalent:

[11]This means that all f_x are injective (resp. surjective), see Definition 2.1.3.

(a) g has a right inverse in $\mathbf{VB}_M$, i.e. there is a smooth vector bundle M-morphism

$$r : G \to F \text{ with } g \circ r = \mathrm{id}_G \,.$$

(b) f has a left inverse in $\mathbf{VB}_M$, i.e. there is a smooth vector bundle M-morphism

$$\ell : F \to E \text{ with } \ell \circ f = \mathrm{id}_E \,.$$

(c) There is a smooth vector bundle M-morphism $P : F \to F$ (of constant rank), which is a projection, i.e. $P \circ P = P$ such that $\mathrm{Im}(P) = \mathrm{Ker}(g) = \mathrm{Im}(f)$.

(d) There is a complemented smooth vector bundle of $\mathrm{Ker}(g) = \mathrm{Im}(f)$, i.e. a subbundle H of F with $\mathrm{Ker}(g) \cap H = M \times \{0\}$ and $\mathrm{Ker}(g) + H = F$ such that the map

$$S : \mathrm{Ker}(g) \times H \to F,\ (\xi, \eta) \mapsto \xi + \eta, \tag{2.65}$$

is a smooth isomorphism of M-morphisms.

If one of the equivalent conditions is satisfied, we say that the exact sequence of vector bundles over M $0 \longrightarrow E \xrightarrow{f} F \xrightarrow{g} G \longrightarrow 0$ **splits**.

Proof. Throughout this proof, we call a smooth vector bundle M-morphism just an M-morphism and the projection M-morphism in (c) just an M-projection.
(a) $\Rightarrow$ (c) Define $Q = r \circ g$, then Q is an M-morphism of constant rank because g is surjective and further $Q \circ Q = r \circ (g \circ r) \circ g = Q$. Since r is injective as a right inverse of g, we obtain $\mathrm{Ker}(Q) = \mathrm{Ker}(g)$ because it holds fibrewise. Then $P = \mathrm{id}_F - Q$ is again an M-projection with $\mathrm{Im}(P) = \mathrm{Ker}(Q) = \mathrm{Ker}(g)$.
(c) $\Rightarrow$ (a) The restriction $g|_{\mathrm{Ker}(P)} : \mathrm{Ker}(P) \to G$ is bijective:
Let $x \in M$ and $\nu \in \mathrm{Ker}(g_x) \cap \mathrm{Ker}(P_x) = \mathrm{Im}(P_x) \cap \mathrm{Ker}(P_x)$, such that there is $\mu \in F_x$ with $P_x(\mu) = \nu$ and $P_x(\nu) = 0 \in F_x$, hence $\nu = P(\mu) = P \circ P(\mu) = P(\nu) = 0 \in F_x$ yields injectivity. For $\varrho \in G_x$ there is $\sigma \in F_x$ with $g_x(\sigma) = \varrho$. Moreover,

$$g_x(\sigma) = g_x(P_x(\sigma)) + g_x(\sigma - P_x(\sigma)) = g_x(\sigma - P_x(\sigma)) \tag{2.66}$$

and $(\sigma - P_x(\sigma)) \in \mathrm{Ker}(P_x)$, which yields surjectivity. So, $g|_{\mathrm{Ker}(P)}$ is an isomorphism of M-morphisms by Lemma 2.4.2, such that $r = g^{-1}_{\mathrm{Ker}(P)} : G \to \mathrm{Ker}(P)$ is an M-morphism and satisfies $g \circ r = \mathrm{id}_G$.
(b) $\Rightarrow$ (c) Define $P = f \circ \ell$, then P is analogously an M-projection of constant rank and the surjectivity of ℓ as a left inverse, implies $\mathrm{Im}(P) = \mathrm{Im}(f) = \mathrm{Ker}(g)$.
(c) $\Rightarrow$ (b) Since f is injective and of constant rank, the induced bijection $f : E \to \mathrm{Im}(f)$ is an isomorphism of M-morphisms such that $f^{-1} : \mathrm{Im}(f) \to E$ is an M-morphism of constant rank. Define $\ell = f^{-1} \circ P$ such that ℓ is an M-morphism of constant rank. Due to $\mathrm{Im}(P) = \mathrm{Im}(f)$, we obtain $P(f(\mu)) = f(\mu)$ for all $\mu \in E$ and therefore also $\ell \circ f(\nu) = f^{-1}(P(f(\mu))) = f^{-1} \circ f(\mu) = \mu$ on E.
(c) $\Rightarrow$ (d) Define $H = \mathrm{Ker}(P)$, then H is a subbundle of F. By the projection property,

$P \circ P = P$, it follows by fibrewise inspection that $M \times \{0\} = \mathrm{Im}(P) \cap \mathrm{Ker}(P) = \mathrm{Ker}(g) \cap H$ and $\mathrm{Ker}(g) + H = \mathrm{Im}(P) + \mathrm{Ker}(P) = F$ holds, where addition is fibrewise defined. In particular, for every $x \in M$ each $\nu_x \in F_x$ has a unique representation $\nu_x = \xi_x + \eta_x$ with $\xi_x \in \mathrm{Ker}(g_x)$ and $\eta_x \in H_x$ such that $S : \mathrm{Ker}(g) \times H \to F$, $(\xi, \eta) \mapsto \mu + \nu$ is bijective and linear on each fibre. The addition in F, defined fibrewise, is also smooth such that S is a bijective M-morphism and consequently an isomorphism of M-morphisms by Lemma 2.4.2. The inverse is given by $T : F \to \mathrm{Ker}(g) \times H$, $\nu \mapsto (P(\nu), \nu - P(\nu))$.
(d) $\Rightarrow$ (c) Fix $x \in M$. Every $\nu_x \in F_x$ can be uniquely written as $\nu_x = \xi_x + \eta_x$ with $\xi_x \in \mathrm{Ker}(g_x)$ and $\eta_x \in H_x$ since $\mathrm{Ker}(g_x)$ and H_x are algebraically complemented vector spaces. Moreover, the linear map $P_x : F_x \to F_x$, $\nu_x \mapsto \xi_x$ if $\nu_x = \xi_x + \eta_x$, is a projection with $\mathrm{Im}(P_x) = \mathrm{Ker}(g_x)$ such that the induced map $P : F \to F$ is also a projection and has constant rank. On the other hand, P is the first component of the M-morphism $S^{-1} : F \to \mathrm{Ker}(g) \times H$ such that P is also an M-morphism. □

2.4.5 Lemma (Splitting of Short Exact Sequence)

Every short exact sequence $0 \longrightarrow E \xrightarrow{f} F \xrightarrow{g} G \longrightarrow 0$ of smooth vector bundles over M splits.

Proof. Since f is injective and g is surjective, there are natural smooth isomorphisms of M-morphisms by Lemma 2.4.2:

$$\tilde{f} : E \to \mathrm{Im}(f) \ \text{ and } \ \tilde{g} : F/\mathrm{Ker}(g) \to G, \tag{2.67}$$

where $\mathrm{Im}(f) = \mathrm{Ker}(g)$ is a subbundle of F. Therefore it is sufficient to show the splitting of the canonical short exact sequence $0 \longrightarrow E \xrightarrow{i} F \xrightarrow{q} F/E \longrightarrow 0$ for a subbundle E of a smooth vector bundle F with the inclusion morphism i and the quotient morphism q. But this is clear by Lemma 2.3.5 or 2.3.7. □

2.5 Functor of Sections

For a fixed base space M, we want to construct a functor from $\mathbf{VB}_M$, the category of smooth vector bundles over M with smooth vector bundle M-morphisms to **Fréchet**, the category of Fréchet spaces with linear and continuous maps as morphisms. The functor will preserve exact sequences (Lemma 2.5.15). In that case, a short exact sequence of vector bundles that splits, yields a short exact sequence of Fréchet spaces that also splits by functoriality.
First of all, we have to equip the $\mathbb{R}$-vector space of sections with a system of seminorms such that it becomes a Fréchet space. This will be part of the next subsection. By Lemma 2.2.4, we already know that, under a choice of chart and trivialization, smooth sections are locally isomorphic as $\mathbb{R}$-vector spaces to smooth functions of an open domain of some $\mathbb{R}^n$ to some $\mathbb{R}^N$. This enables, at least locally, a way to define seminorms on smooth sections, inherited by seminorms of smooth functions, which induce a Fréchet topology. However, in order to construct a well defined topology on sections over arbitrary open subsets of

a manifold, we have to verify that the induced topology is independent of the choices of charts and trivializations. It will be reasonable to start with the Fréchet space of smooth functions on an open domain of $\mathbb{R}^n$.

The order of a multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ is defined by $|\alpha| = \sum_{j=1}^{n} \alpha_j$.
Further, define for $\alpha \in \mathbb{N}_0^n$ and a smooth function $f \in C^\infty(V, \mathbb{R})$ on some open $V \subseteq \mathbb{R}^n$ the α-partial derivative evaluated at $x \in V$ by

$$f^{(\alpha)}(x) = \left.\frac{\partial^{\alpha_1} \cdots \partial^{\alpha_n}(f)}{\partial x_1 \cdots \partial x_n}\right|_x . \tag{2.68}$$

2.5.1 Remark (Fréchet Space of Smooth Functions)

(1) Let $V \subseteq \mathbb{R}^n$ be open. For a compact set $K \subseteq V$ and a number $\ell \in \mathbb{N}_0$, the maximal order of multi-indices involved, define

$$q_{K,\ell}(f) = \sup \left\{ |f^{(\alpha)}(x)| : x \in K, \alpha \in {\mathbb{N}_0}^n, |\alpha| \leq \ell \right\} \tag{2.69}$$

on $C^\infty(V, \mathbb{R})$. The system $\mathcal{Q}_V = \{q_{K,\ell} : K \subseteq V \text{ compact}, \ell \in \mathbb{N}_0\}$ of seminorms is equivalent to the countable system of seminorms $\mathcal{Q}_V((K_j)_{j \in \mathbb{N}_0}) = \{q_{K_j,\ell} : j, \ell \in \mathbb{N}_0\}$, where $(K_j)_{j \in N_0}$ is a compact exhaustion of V, i.e. $K_j \subseteq K_{j+1} \subseteq V$ are compact such that $V = \bigcup_{j \in \mathbb{N}_0} \overset{\circ}{K}_j$. (Every compact $K \subseteq V$ is contained in some K_j). Therefore, the induced topology does not depend on the choice of exhaustion and is metrizable. It is well known that these seminorms induce a Fréchet topology on $C^\infty(V, \mathbb{R})$.[12]

A countable product of Fréchet spaces is a Fréchet space with respect to the product topology. Further, $C^\infty(V, \mathbb{R}^N)$ is isomorphic to the product $(C^\infty(V, \mathbb{R}))^N$ by $f \mapsto (f_k)_{k=1}^N$ such that we obtain:

(2) Under the same notations as in (1), $C^\infty(V, \mathbb{R}^N)$ equipped with the system of seminorms $\mathcal{Q}_V^N((K_j)_j) = \{q_{K_j,\ell}^N : j, \ell \in \mathbb{N}_0\}$ defined by

$$q_{K_j,\ell}^N(f) = \sup \left\{ \max_{k=1,\dots,N} |f_k^{(\alpha)}(x)| : x \in K_j, \alpha \in {\mathbb{N}_0}^n, |\alpha| \leq \ell \right\}, \tag{2.70}$$

is also a Fréchet space. Further, $\mathcal{Q}_V^N = \{q_{K,\ell}^N : K \subseteq V \text{ compact}, \ell \in \mathbb{N}_0\}$ is an equivalent system of seminorms, such that the induced topology is independent of the choice of exhaustions.

For later applications, we are interested in the continuity of certain linear operators between spaces of smooth functions. Instead of struggling with formulas for (mixed) partial derivatives, we will use a continuity criterion, which is an application of the closed graph theorem.

[12] We refer to [MV97, Examples 5.18 (4)] for instance.

2.5.2 Lemma (Continuity-Criterion)

Let X, Y, Z be Fréchet spaces and $I : Y \to Z$ be an injective continuous map.
Then a linear map $T : X \to Y$ is continuous if and only if $I \circ T : X \to Z$ is continuous.

Proof. Suppose $I \circ T$ is continuous. The map $J : X \times Y \to X \times Y$, $(x, y) \mapsto (x, I(y))$ is continuous (with respect to the product topology) since I is continuous. Because Z is in particular a Hausdorff-space, the graph of the continuous function $I \circ T$, denoted by $\operatorname{graph}(I \circ T)$, is closed. Hence, the following set is also closed:

$$J^{-1}(\operatorname{graph}(I \circ T)) = \{(x, y) \in X \times Y : (x, I(y)) \in \operatorname{graph}(I \circ T)\} \tag{2.71}$$
$$= \{(x, y) \in X \times Y : I(y) = I(T(x))\} \tag{2.72}$$
$$= \{(x, y) \in X \times Y : y = T(x)\} = \operatorname{graph}(T). \tag{2.73}$$

The continuity of T follows now by the closed graph theorem. The other implication is trivially true since compositions of continuous functions are continuous. □

2.5.3 Lemma (Continuity of an Induced Operator)

Let $V \subseteq \mathbb{R}^n$ be open, $N, L \in \mathbb{N}$ and $A : V \to \mathbb{R}^{L \times N}$ a smooth map (all entries are smooth $\mathbb{R}$-valued functions). Then the induced linear map $S_A : C^\infty(V, \mathbb{R}^N) \to C^\infty(V, \mathbb{R}^L)$, $f \mapsto S_A f$ defined by pointwise matrix-vector-multiplication $S_A f(y) = A(y) \cdot f(y)$, is continuous.
In particular, if $A : V \to \mathrm{GL}(\mathbb{R}, N)$ is smooth, the induced map S_A is an isomorphism of Fréchet spaces.

Proof. Endow $C(V, \mathbb{R}^L) = \{f : V \to \mathbb{R}^L : f \text{ continuous}\}$ with the system of seminorms $\{q^L_{K,0} : K \subseteq V \text{ compact}\}$. As in the smooth case, $C(V, \mathbb{R}^L)$ is isomorphic to the product $(C(V, \mathbb{R}))^L$, which is a Fréchet space as a finite product of the Fréchet spaces $C(V, \mathbb{R})$ [13].
The inclusion $I : C^\infty(V, \mathbb{R}^L) \to C(V, \mathbb{R}^L)$, $f \mapsto f$ is clearly continuous such that we only have to verify the continuity of $I \circ S_A : C^\infty(V, \mathbb{R}^N) \to C(V, \mathbb{R}^L)$ due to the Continuity-Criterion, Lemma 2.5.2.
Equip $\mathbb{R}^L$ with the norm $\|\cdot\|_{\max,L}$, defined by $\|y\|_{\max,L} = \max\limits_{k=1,\ldots,L} |y_k|$ and $\mathbb{R}^N$ with $\|\cdot\|_{\max,N}$, respectively. For a matrix $B \in \mathbb{R}^{L \times N}$, denote its operatornorm by

$$\|B\|_{\mathrm{op}} = \sup\{\|B\lambda\|_L : \lambda \in \mathbb{R}^N,\ \|\lambda\|_N \leq 1\}, \tag{2.74}$$

such that

$$\|B\lambda\|_{\max,L} \leq \|B\|_{\mathrm{op}}\, \|\lambda\|_{\max,N} \text{ for all } \lambda \in \mathbb{R}^N. \tag{2.75}$$

Let $K \subseteq V$ be compact, then

$$c_K = \sup\{\|A(y)\|_{\mathrm{op}} : y \in K\} < \infty \tag{2.76}$$

[13] [MV97], Examples 5.18 (2) for instance.

by continuity of $A : V \to \mathbb{R}^{L\times N}$ and we obtain:

$$\begin{aligned} q^L_{K,0}(I \circ S_A(f)) &= \sup\{\|S_A f(y)\|_{\max,L} : y \in K\} && (2.77)\\ &= \sup\{\|A(y) \cdot f(y)\|_{\max,L} : y \in K\} && (2.78)\\ &\leq \sup\{\|A(y)\|_{\mathrm{op}} \, \|f(y)\|_{\max,N} : y \in K\} && (2.79)\\ &\leq c_K \, p^N_{K,0}(f). && (2.80)\end{aligned}$$

Thus, the continuity of S_A has been proven.
Note that if $B : V \to \mathbb{R}^{N\times M}$ is another smooth map, the pointwise product defines a smooth map $AB : V \to \mathbb{R}^{L\times M}$, $AB(x) = A(x) \cdot B(x)$ and $S_A \circ S_B = S_{AB}$ holds. In the special case of a smooth map $A : V \to \mathrm{GL}(\mathbb{R}, N)$, the pointwise inverse matrix map $A^{-1} : V \to \mathrm{GL}(\mathbb{R}, N)$ is also smooth by Cramer's Rule. Their induced linear continuous maps satisfy $S_A \circ S_{A^{-1}} = S_{I_N}$, which is the identity on $C^\infty(V, \mathbb{R}^N)$. □

2.5.4 Lemma (Continuity of the Pullback of Smooth Functions)

Let $\varphi : V \to W$ be a smooth map between open sets $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$.
The pullback $\varphi^* : C^\infty(W, \mathbb{R}^N) \to C^\infty(V, \mathbb{R}^N)$, $\varphi^* f = f \circ \varphi$ is linear and continuous.
For another smooth map $\psi : U \to V$ on $U \subseteq \mathbb{R}^\ell$ open, we have $\psi^* \circ \varphi^* = (\varphi \circ \psi)^*$ and id_V^* is the identity on $C^\infty(V, \mathbb{R}^N)$.
In particular, the pullback of a diffeomorphism is an isomorphism of Fréchet spaces.

Proof. The linearity of φ^* is clear. As in the proof of Lemma 2.5.3, by Lemma 2.5.2 it suffices to show the continuity of $I \circ \varphi^* : C^\infty(W, \mathbb{R}^N) \to C(V, \mathbb{R}^N)$, where I denotes the continuous inclusion from $C^\infty(V, \mathbb{R}^N)$ into $C(V, \mathbb{R}^N)$. Let $K \subseteq V$ be compact, then

$$\begin{aligned} q^N_{K,0}(\varphi^* f) =& \sup\left\{\max_{k=1,\dots,N} |f_k(\varphi(x))| : x \in K\right\} && (2.81)\\ =& q^N_{\varphi(K),0}(f). && (2.82)\end{aligned}$$

Since φ is continuous, $\varphi(K)$ is compact, which shows the continuity of $I \circ \varphi^*$. Just by definition, $\psi^* \circ \varphi^* = (\varphi \circ \psi)^*$ and $\mathrm{id}_V = \mathrm{id}_{C^\infty(V,\mathbb{R}^N)}$ follow and imply for a diffeomorphism φ and its inverse $\psi = \varphi^{-1}$ the inverse identities, such that the pullback becomes an isomorphism of Fréchet spaces. □

We refer to [MV97, Chapter 24] for the concept of projective topologies on locally convex spaces. It can be seen as a generalization of the product topology or the subspace topology and allows to define a topology on a common domain of a family of linear maps, which map to (possible different) locally convex spaces. The projective topology is also called initial, weak or limit topology. We consider only $\mathbb{R}$-vector spaces.

2.5.5 Definition/Remark (Projective Topology)

A vector space X, together with a family of locally convex spaces $(X_i)_{i\in I}$ and linear maps $\pi_i : X \to X_i$ $(i \in I)$, is called a **projective system**, if for each $x \in X \setminus \{0\}$, there is an $i \in I$ with $\pi_i(x) \neq 0$. If $\mathcal{P}_i$ denotes a system of seminorms on X_i, we call

$$\mathcal{PR}((\mathcal{P}_i)_{i\in I}) = \left\{ p = \max_{i\in J} p_i \circ \pi_i : J \subseteq I \text{ finite, } p_i \in \mathcal{P}_i \right\}. \tag{2.83}$$

the **system of projective seminorms** of the projective system $(\pi : X \to X_i)_{i\in I}$ inherited by $(\mathcal{P}_i)_{i\in I}$. It induces a locally convex topology on X, the so called **projective topology**. We call $\Theta : X \to \prod_{\imath\in I} X_i$, $x \mapsto (\pi_i(x))_{i\in I}$ the **evaluation map** of a projective system. It is an isomorphism of topological vector spaces onto its image with respect to the projective topology on X and the subspace topology of the product topology on $\Theta(X)$, respectively. (Θ is clearly linear and injective by the defining property of the projective system. The continuity of Θ and $\Theta^{-1} : \Theta(X) \to X$ follows just by definition of the seminorms.)
The projective topology on X is therefore the coarsest topology such that all maps π_i are continuous.

2.5.6 Definition (Local Choice and Seminorms on Local Sections)

Let $\xi = (E, \pi, M)$ be a smooth vector bundle of rank N over a smooth n-manifold M.

(a) We will call $\mathcal{U} = (U, \varphi, \Phi, (K_j)_{j\in\mathbb{N}_0})$ a **local choice of ξ (over U)**, if

(LC1) $U \subseteq M$ is open;

(LC2) (U, φ) is a smooth chart of M;

(LC3) (U, Φ) is a local trivialization of E over U;

(LC4) $(K_j)_{j\in\mathbb{N}_0}$ is a compact exhaustion of $\varphi(U) \subseteq \mathbb{R}^n$.

(b) For a local choice $\mathcal{U} = (U, \varphi, \Phi, (K_j)_{j\in\mathbb{N}_0})$ of ξ, we define a system of seminorms on $\Gamma(U, E)$ by

$$\mathcal{P}_{\mathcal{U}} = \mathcal{PR}(\mathcal{Q}^N_{\varphi(U)}((K_j)_{j\in\mathbb{N}_0})) = \{p^U_{j,\ell} = q^N_{K_j,\ell} \circ T_U : j, \ell \in \mathbb{N}_0\}, \tag{2.84}$$

where $T_U : \Gamma(U, E) \to C^\infty(\varphi(U), \mathbb{R}^N)$ denotes the isomorphism of vector spaces according to Lemma (2.2.4) induced by (U, φ) and (U, Φ).

2.5.7 Lemma (Fréchet space of Local Sections)

Let $\mathcal{U} = (U, \varphi, \Phi, (K_j)_{j\in\mathbb{N}_0})$ be a local choice of a smooth vector bundle (E, π, M).
Then $(\Gamma(U, E), \mathcal{P}_{\mathcal{U}})$ is a Fréchet space.

Proof. Denote by $T_U : \Gamma(U, E) \to C^\infty(\varphi(U), \mathbb{R}^N)$ the isomorphism of vector spaces according to Lemma (2.2.4) induced by (U, φ) and (U, Φ). Since T_U is in particular injective, it defines a projective system. By definition, $\mathcal{P}_\mathcal{U}$ is the system of projective seminorms of the projective system $T_U : \Gamma(U, E) \to C^\infty(\varphi(U), \mathbb{R}^N)$ inherited by $\mathcal{Q}^N_{\varphi(U)}((K_j)_j)$. Moreover, the evaluation map Θ is just T_U, which is also surjective. Hence, we have an isomorphism of topological vector spaces $T_U : (\Gamma(U, E), \mathcal{P}_\mathcal{U}) \to (C^\infty(\varphi(U), \mathbb{R}^N), \mathcal{Q}^N_{\varphi(U)})$. Since $\mathcal{P}_\mathcal{U}$ is countable and $(C^\infty(\varphi(U), \mathbb{R}^N), \mathcal{Q}^N_{\varphi(U)}((K_j)_j))$ is a Fréchet space, we can follow the assertion. $\square$

The induced topology does not depend on the choice of the compact exhaustion of $\varphi(U)$, since $\mathcal{P}_\mathcal{U}$ is equivalent to $\{p^U_{K,\ell} = q^N_{K,\ell} \circ T_U : K \subseteq \varphi(U) \text{ compact}, \ell \in \mathbb{N}_0\}$. But we also have to check, whether the topology depends on the choice of chart and trivialization, which will be expressed in a lemma:

2.5.8 Lemma (Independence of Local Choices)

Let $\mathcal{U}_\alpha = (U, \varphi_\alpha, \Phi_\alpha, (K^\alpha_j)_{j\in\mathbb{N}_0})$ and $\mathcal{U}_\beta = (U, \varphi_\beta, \Phi_\beta, (K^\beta_j)_{j\in\mathbb{N}_0})$ be local choices of a smooth vector bundle (E, π, M) over an open subset $U \subseteq M$. Then $\mathcal{P}_{\mathcal{U}_\alpha}$ and $\mathcal{P}_{\mathcal{U}_\beta}$ are equivalent systems of seminorms on $\Gamma(U, E)$.

Proof. We already know that the transition map $\varphi^\beta_\alpha = \varphi_\beta \circ \varphi_\alpha^{-1}$ from $\varphi_\alpha(U)$ to $\varphi_\beta(U)$ is a diffeomorphism, such that its pullback is an isomorphism between Fréchet spaces $(\varphi^\beta_\alpha)^* : C^\infty(\varphi_\beta(U), \mathbb{R}^N) \to C^\infty(\varphi_\alpha(U), \mathbb{R}^N)$, $f \mapsto f \circ \varphi^\beta_\alpha$ by Lemma 2.5.4. Thus, $\mathcal{Q}^N_{\varphi_\alpha(U)}(K^\alpha_j)_{j\in\mathbb{N}_0}$ and $\mathcal{Q}^N_{\varphi_\beta(U)}(K^\beta_j)_{j\in\mathbb{N}_0}$ are equivalent systems of seminorms. Therefore the projective topology on $\Gamma(U, E)$ is independent of the compact exhaustion.
It remains to show, that $\mathrm{id} : (\Gamma(U, E), \mathcal{P}_{\mathcal{U}_\alpha}) \to (\Gamma(U, E), \mathcal{P}_{\mathcal{U}_\beta})$ is an isomorphism of Fréchet spaces. Denote by T_α and T_β the isomorphisms of Lemma 2.2.4 corresponding to $\varphi_\alpha, \Phi_\alpha$ and $\varphi_\beta, \Phi_\beta$, respectively. In addition, they are isomorphisms of Fréchet spaces onto $C^\infty(\varphi_\alpha(U), \mathbb{R}^N)$ and $C^\infty(\varphi_\beta(U), \mathbb{R}^N)$, respectively. Hence, we have the following commutative diagram with isomorphisms of Fréchet spaces in the columns:

$$
\begin{array}{ccc}
(\Gamma(U,E), \mathcal{P}_{\mathcal{U}_\alpha}) & \xrightarrow{\ \mathrm{id}\ } & (\Gamma(U,E), \mathcal{P}_{\mathcal{U}_\beta}) \\
\downarrow T_\alpha & & \downarrow T_\beta \\
C^\infty(\varphi_\alpha(U), \mathbb{R}^N) & & C^\infty(\varphi_\beta(U), \mathbb{R}^N) \\
 & \searrow S^\beta_\alpha & \downarrow (\varphi^\beta_\alpha)^* \\
 & & C^\infty(\varphi_\alpha(U), \mathbb{R}^N).
\end{array}
$$

Therefore, it suffices to show that $S^\beta_\alpha = (\varphi^\beta_\alpha)^* \circ T_\beta \circ T_\alpha^{-1}$ is an isomorphism of Fréchet spaces. We compute for $f \in C^\infty(\varphi_\alpha(U), \mathbb{R}^N)$:

$$T_\beta \circ T_\alpha^{-1}(f) = \mathrm{pr}_{II} \circ \Phi_\beta \circ (T_\alpha^{-1}(f)) \circ \varphi_\beta^{-1} \tag{2.85}$$
$$= \mathrm{pr}_{II} \circ \Phi_\beta \circ (\Phi_\alpha^{-1} \circ (\mathrm{id}_U \times (f \circ \varphi_\alpha))) \circ \varphi_\beta^{-1} \tag{2.86}$$
$$= \mathrm{pr}_{II} \circ (\Phi_\beta \circ \Phi_\alpha^{-1}) \circ (\varphi_\beta^{-1} \times (f \circ \varphi_\beta^\alpha)). \tag{2.87}$$

Using the transition function $\tau_\alpha^\beta : U \to \mathrm{GL}(\mathbb{R}, N)$ between Φ_β and Φ_α according to Lemma 2.1.7, we obtain for $y \in \varphi_\alpha(U)$:

$$T_\beta \circ T_\alpha^{-1}(f)(y) = \tau_\alpha^\beta(\varphi_\beta^{-1}(y)) \cdot f(\varphi_\beta^\alpha(y)). \tag{2.88}$$

Applying the pullback $(\varphi_\alpha^\beta)^*$, gives

$$S_\alpha^\beta f(y) = \tau_\alpha^\beta(\varphi_\beta^{-1} \circ \varphi_\alpha^\beta(y)) \cdot f(\varphi_\beta^\alpha \circ \varphi_\alpha^\beta(y)) \tag{2.89}$$
$$= \tau_\alpha^\beta(\varphi_\alpha^{-1}(y)) \cdot f(y). \tag{2.90}$$

For $V = \varphi_\alpha(U)$, the composition $A = \tau_\alpha^\beta \circ \varphi_\alpha^{-1} : V \to \mathrm{GL}(\mathbb{R}, N)$ is smooth. By Lemma 2.5.3, the induced map $S_A = S_\alpha^\beta$ is an isomorphism of Fréchet spaces.
(Note that if $\mathrm{id} : (\Gamma(U, E), \mathcal{P}_{\mathcal{U}_\alpha}) \to (\Gamma(U, E), \mathcal{P}_{\mathcal{U}_\beta})$ is a continuous map, it is already an isomorphism of Fréchet spaces by the open mapping theorem. To achieve this, one can show the continuity of S_α^β. However, by the previous results, this is not much less work than showing that it is already an isomorphism of Fréchet spaces.) □

2.5.9 Definition (Seminorms on the Space of (Global) Sections)

Let $\xi = (E, \pi, M)$ be a smooth vector bundle of rank N over a smooth n-manifold M.

(a) We call $\mathcal{U}_{\mathscr{A}} = (\mathcal{U}_\alpha)_{\alpha \in \mathscr{A}}$ a **cover of local choices of ξ over** M, if

(CLC1) $\mathcal{U}_\alpha = (U_\alpha, \varphi_\alpha, \Phi_\alpha, (K_j^\alpha)_{j \in \mathbb{N}_0})$ is a local choice of ξ over an open subset $U_\alpha \subseteq M$ for each $\alpha \in \mathscr{A}$;

(CLC2) $M = \bigcup_{\alpha \in \mathscr{A}} U_\alpha$.

Another cover of local choices $\mathcal{V}_{\mathscr{B}} = (\mathcal{V}_\beta = (V_\beta, \psi_\beta, \Psi_\beta, (L_j^\beta)_{j \in \mathbb{N}_0}))_{\beta \in \mathscr{B}}$ of ξ over M is called a **local choice refinement of $\mathcal{U}_{\mathscr{A}}$**, if

(LCR) for each $\alpha \in \mathscr{A}$, there is a subset $\mathscr{B}_\alpha \subseteq \mathscr{B}$ such that $\mathcal{V}_{\mathscr{B}_\alpha} = (\mathcal{V}_{\beta_\alpha})_{\beta_\alpha \in \mathscr{B}_\alpha}$ is a cover of local choices of the restricted vector bundle $\xi|_{U_\alpha}$ over U_α and $\mathscr{B} = \bigcup_{\alpha \in \mathscr{A}} \mathscr{B}_\alpha$.

(b) Let $\mathcal{U}_{\mathscr{A}} = (\mathcal{U}_\alpha)_{\alpha \in \mathscr{A}}$ a cover of local choices of ξ over M. For each $\alpha \in \mathscr{A}$ define the restriction map $\varrho^\alpha : \Gamma(M, E) \to \Gamma(U_\alpha, E)$, $\sigma \mapsto \sigma|_{U_\alpha} = \sigma \circ \iota_\alpha$, where ι_α denotes the smooth inclusion of U_α into M. The restriction maps and Fréchet spaces

$(\Gamma(U_\alpha, E), \mathcal{P}_{\mathcal{U}_\alpha})$ make $\Gamma(M,E)$ into a projective system. For $\mathcal{U}_{\mathscr{A}}$, we define $\mathcal{P}_{\mathcal{U}_{\mathscr{A}}}$ to be the system of projective seminorms inherited by $(\mathcal{P}_{\mathcal{U}_\alpha})_{\alpha\in\mathscr{A}}$, i.e.

$$\mathcal{P}_{\mathcal{U}_{\mathscr{A}}} = \mathcal{PR}((\mathcal{P}_{\mathcal{U}_\alpha})_{\alpha\in\mathscr{A}}) \tag{2.91}$$

$$= \left\{ p^I_{j_I,\ell_I} = \max_{\alpha\in I} p^{U_\alpha}_{j_\alpha,\ell_\alpha} \circ \varrho^\alpha : I \subseteq \mathscr{A} \text{ finite, } j_\alpha, \ell_\alpha \in \mathbb{N}_0 \text{ for all } \alpha \in I \right\}. \tag{2.92}$$

2.5.10 Lemma (Topology Invariant Under Refinement)

Let $\mathcal{V}_{\mathscr{B}}$ be a local choice refinement of a cover $\mathcal{U}_{\mathscr{A}}$ of local choices of ξ over M. Then the systems of seminorms $\mathcal{P}_{\mathcal{U}_{\mathscr{A}}}$ and $\mathcal{P}_{\mathcal{V}_{\mathscr{B}}}$ on $\Gamma(M,E)$ are equivalent.

Proof. We will show:

(1) $\mathcal{PR}((\mathcal{P}_{\mathcal{V}_{\mathscr{B}_\alpha}})_{\alpha\in\mathscr{A}}) = \mathcal{P}_{\mathcal{V}_{\mathscr{B}}}$;

(2) If $\mathcal{U} = (U, \varphi, \Phi, (K_j)_{j\in\mathbb{N}_0})$ is a local choice of a vector bundle $\xi = (E, \pi, M)$ over U and $\mathcal{U}_{\mathscr{A}} = (\mathcal{U}_\alpha)_{\alpha\in\mathscr{A}}$ is a cover of local choices of $\xi|_U = (E_U, \pi_{E_U}, U)$ over U, then $\mathcal{P}_{\mathcal{U}}$ and $\mathcal{P}_{U_{\mathscr{A}}}$ are equivalent.

Then we can conclude that $\mathcal{P}_{\mathcal{V}_{\mathscr{B}_\alpha}}$ and $\mathcal{P}_{\mathcal{U}_\alpha}$ are equivalent for each $\alpha \in \mathscr{A}$. It is easy to see, that two systems of projective seminorms inherited by equivalent systems of seminorms are again equivalent. Therefore, $\mathcal{P}_{\mathcal{U}_{\mathscr{A}}}$ and $\mathcal{PR}((\mathcal{P}_{\mathcal{V}_{\mathscr{B}_\alpha}})_{\alpha\in\mathscr{A}}) = \mathcal{P}_{\mathcal{V}_{\mathscr{B}}}$ are equivalent.

(1) After applying the definitions and using $\varrho^{\beta_\alpha}_\alpha \circ \varrho^\alpha = \varrho^{\beta_\alpha}$, a seminorm of $\mathcal{PR}((\mathcal{P}_{\mathcal{V}_{\mathscr{B}_\alpha}})_{\alpha\in\mathscr{A}})$ is given by $p_{(I_\alpha)_{\alpha\in J}} = \max_{\alpha\in J} \max_{\beta_\alpha\in I_\alpha} p^{V_{\beta_\alpha}}_{j_{\beta_\alpha},\ell_{\beta_\alpha}} \circ \varrho^{\beta_\alpha}$ for finite subsets $J \subseteq \mathscr{A}$, $I_\alpha \subseteq \mathscr{B}_\alpha$ and $j_{\beta_\alpha}, \ell_{\beta_\alpha} \in \mathbb{N}_0$ for all $\alpha \in J$ and $\beta_\alpha \in I_\alpha$. Define $I = \bigcup_{\alpha\in J} I_\alpha$. Then $I \subseteq \mathscr{B}$ is finite and $p_{(I_\alpha)_{\alpha\in J}} = \max_{\beta\in I} p^{\mathcal{V}_\beta}_{j_\beta,\ell_\beta} \circ \varrho^\beta \in \mathcal{P}_{\mathcal{V}_{\mathscr{B}}}$.
Othersides, a seminorm of $\mathcal{P}_{\mathcal{V}_{\mathscr{B}}}$ is given by $p^I_{j_I,\ell_I} = \max_{\beta\in I} p^{\mathcal{V}_\beta}_{j_\beta,\ell_\beta} \circ \varrho^\beta$ for a finite subset $I \subseteq \mathscr{B}$ and $j_\beta, \ell_\beta \in \mathbb{N}_0$. Since $I \subseteq \bigcup_{\alpha\in\mathscr{A}} \mathscr{B}_\alpha$ is finite, there is a finite subset $J \subseteq \mathscr{A}$, such that $I \subseteq \bigcup_{\alpha\in J} \mathscr{B}_\alpha$. We divide I into subsets $I_\alpha = I \setminus \left(\bigcup_{\alpha\neq\beta\in J} \mathscr{B}_\beta \right) \subseteq \mathscr{B}_\alpha$, which are also finite. Then, we obtain $p^I_{j_I,\ell_I} = p_{(I_\alpha)_{\alpha\in J}} \in \mathcal{PR}((\mathcal{P}_{\mathcal{V}_{\mathscr{B}_\alpha}})_{\alpha\in\mathscr{A}})$.

(2) Let $I \subseteq \mathscr{A}$ be finite and $j_\alpha, \ell_\alpha \in \mathbb{N}_0$ for all $\alpha \in I$. The finite union $K = \bigcup_{\alpha\in I} K_{j_\alpha}$ is a compact set in U such that there is a $j \in \mathbb{N}_0$ with $K \subseteq K_j$. For $\ell = \max_{\alpha\in I} \ell_\alpha$, we clearly have $p^I_{j_I,\ell_I}(\sigma) = \max_{\alpha\in I} p^{U_\alpha}_{j_\alpha,\ell_\alpha} \circ \varrho^\alpha(\sigma) \leq p_{j,\ell}(\sigma)$ for all $\sigma \in \Gamma(U,E)$.
Otherwise, let $j, \ell \in \mathbb{N}_0$ be arbitrary. The compactness of $K_j \subseteq U = \bigcup_{\alpha\in\mathscr{A}} U_\alpha$ yields a

finite subset $I \subseteq \mathscr{A}$ with $K_j \subseteq \bigcup_{\alpha \in I} U_\alpha$. For $\alpha \in I$, define

$$\tilde{K}_\alpha = K_j \setminus \left(\bigcup_{\beta \in I \setminus \{\alpha\}} U_\beta \right). \tag{2.93}$$

Each $\tilde{K}_\alpha$ is a compact subset of U_α, since it is a closed subset of a compact set in a Hausdorff space. Hence, there is a $j_\alpha \in \mathbb{N}_0$ with $\tilde{K}_\alpha \subseteq K_{j_\alpha}$ for each $\alpha \in I$. Setting $\ell_\alpha = \ell$ for $\alpha \in I$, we now obtain $p_{j,\ell}(\sigma) \leq \max_{\alpha \in I} p^{U_\alpha}_{j_\alpha, \ell_\alpha} \circ \varrho^\alpha(\sigma)$ on $\Gamma(U, E)$.

□

2.5.11 Lemma (Fréchet Space of Global Sections)

Let $\mathcal{U}_\mathscr{A} = (\mathcal{U}_\alpha = (U_\alpha, \varphi_\alpha, \Phi_\alpha, (K^\alpha_j)_{j \in \mathbb{N}_0}))_{\alpha \in \mathscr{A}}$ be a cover of local choices of $\xi = (E, \pi, M)$ over M. Then, $(\Gamma(M, E), \mathcal{P}_{\mathcal{U}_\mathscr{A}})$ is a Fréchet space.
In particular, $(\Gamma(M, E), \mathcal{P}_{\mathcal{U}_\mathscr{A}})$ is isomorphic to the Fréchet space:

$$X_{\mathcal{U}_\mathscr{A}} = \left\{ (\sigma_\alpha)_{\alpha \in \mathscr{A}} \in \prod_{\alpha \in \mathscr{A}} \Gamma(U_\alpha, E) : \sigma_\alpha|_{U_{\alpha,\beta}} = \sigma_\beta|_{U_{\alpha,\beta}} \text{ for all } (\alpha, \beta) \in \mathrm{OD}(\mathscr{A}) \right\}, \tag{2.94}$$

where $\mathrm{OD}(\mathscr{A}) = \{(\alpha, \beta) \in \mathscr{A} \times \mathscr{A} : U_{\alpha,\beta} = U_\alpha \cap U_\beta \neq \emptyset\}$ denotes the set of indices with overlapping domains and the topology on $X_\mathcal{U}$ is the subspace topology of the product topology.

Proof. According to Lemma 2.5.7, each space of local sections $\Gamma(U_\alpha, E)$, equipped with $\mathcal{P}_{\mathcal{U}_\alpha}$, is a Fréchet space. Hence, the countable product $\prod_{\alpha \in \mathscr{A}} (\Gamma(U_\alpha, E), \mathcal{P}_{\mathcal{U}_\alpha})$ of local sections is also a Fréchet space with respect to the product topology, induced by the projections $\pi_\alpha : \prod_{\beta \in \mathscr{A}} \Gamma(U_\beta, E) \to \Gamma(U_\alpha, E)$, $(\sigma_\beta)_{\beta \in \mathscr{A}} \mapsto \sigma_\alpha$.
By definition, $\mathcal{P}_{\mathcal{U}_\mathscr{A}}$ is $\mathcal{PR}((\mathcal{P}_{\mathcal{U}_\alpha})_{\alpha \in \mathscr{A}})$, the system of projective seminorms inherited by $(\mathcal{P}_{\mathcal{U}_\alpha})_{\alpha \in \mathscr{A}}$. Therefore, the evaluation map $\Theta_{\mathcal{U}_\mathscr{A}} : \Gamma(M, E) \to \prod_{\alpha \in \mathscr{A}} \Gamma(U_\alpha, E)$, $\sigma \mapsto (\varrho^\alpha(\sigma))_{\alpha \in \mathscr{A}}$ is a linear homeomorphism onto its image, which is equal to $X_{\mathcal{U}_\mathscr{A}}$.
($\mathrm{Im}(\Theta_{\mathcal{U}_\mathscr{A}})$ is contained in $X_{\mathcal{U}_\mathscr{A}}$ since restrictions commute. If otherwise $(\sigma_\alpha)_{\alpha \in \mathscr{A}}$ is an element of $X_{\mathcal{U}_\mathscr{A}}$, the mapping $x \mapsto \sigma_\alpha(x)$ if $x \in U_\alpha$, yields a well defined smooth section $\sigma \in \Gamma(M, E)$. Indeed, for any other $\beta \in \mathscr{A}$ with $x \in U_\beta$, the local sections σ_α and σ_β are equal on the open intersection $U_{\alpha,\beta} = U_\alpha \cap U_\beta$.)
We will show, that $X_{\mathcal{U}_\mathscr{A}}$ is a closed subspace of the product space. By restricting to a non-empty intersection $U_{\alpha,\beta} = U_\alpha \cap U_\beta$, we receive charts $(U_{\alpha,\beta}, \tilde{\varphi}_\alpha)$, $(U_{\alpha,\beta}, \tilde{\varphi}_\beta)$ and trivializations $(U_{\alpha,\beta}, \tilde{\Phi}_\alpha)$, $(U_{\alpha,\beta}, \tilde{\Phi}_\beta)$. Further, $(\tilde{K}^{\alpha,\beta})_{j \in \mathbb{N}_0} = (\varphi_\alpha^{-1}(K^\alpha_{j+k}) \cap \varphi_\beta^{-1}(K^\beta_{j+k}))_{j \in \mathbb{N}_0}$ defines a compact exhaustion of $U_{\alpha,\beta}$, where $k = \min\{j \in \mathbb{N}_0 : \varphi_\alpha^{-1}(K^\alpha_j) \cap \varphi_\beta^{-1}(K^\beta_j) \neq \emptyset\}$. So

$$\mathcal{U}^\alpha_{\alpha,\beta} = (U_{\alpha,\beta}, \tilde{\varphi}_\alpha, \tilde{\Phi}_\alpha, (\tilde{\varphi}_\alpha(\tilde{K}^{\alpha,\beta}_j))_{j \in \mathbb{N}_0}) \text{ and } \mathcal{U}^\beta_{\alpha,\beta} = (U_{\alpha,\beta}, \tilde{\varphi}_\beta, \tilde{\Phi}_\beta, (\tilde{\varphi}_\beta(\tilde{K}^{\alpha,\beta}_j))_{j \in \mathbb{N}_0}) \tag{2.95}$$

are local choices of ξ over $U_{\alpha,\beta}$. The maps,

$$\varrho_\alpha^{\alpha,\beta} : (\Gamma(U_\alpha, E), \mathcal{P}_{\mathcal{U}_\alpha}) \to (\Gamma(U_{\alpha,\beta}, E), \mathcal{P}_{\mathcal{U}_{\alpha,\beta}^\alpha}),\ \sigma_\alpha \mapsto \sigma_\alpha|_{U_{\alpha,\beta}} = \sigma_\alpha \circ \iota_{\alpha,\beta}^\alpha\,; \tag{2.96}$$

$$\varrho_\beta^{\alpha,\beta} : (\Gamma(U_\beta, E), \mathcal{P}_{\mathcal{U}_\beta}) \to (\Gamma(U_{\alpha,\beta}, E), \mathcal{P}_{\mathcal{U}_{\alpha,\beta}^\beta}),\ \sigma_\beta \mapsto \sigma_\beta|_{U_{\alpha,\beta}} = \sigma_\beta \circ \iota_{\alpha,\beta}^\beta \tag{2.97}$$

are continuous. (Indeed, for $p_{j,\ell}^{U_{\alpha,\beta}} \in \mathcal{P}_{\mathcal{U}_{\alpha,\beta}^\alpha}$ the seminorm $p_{j+k,\ell}^{U_\alpha} \in \mathcal{P}_{\mathcal{U}_\alpha}$ satisfies $p_{j,\ell}^{U_{\alpha,\beta}}(\varrho_\alpha^{\alpha,\beta}(\sigma)) \leq p_{j+k,\ell}^{U_\alpha}(\sigma)$ for all $\sigma \in \Gamma(U_\alpha, E)$ since $\tilde{\varphi}_\alpha(\tilde{K}_j^{\alpha,\beta}) \subseteq K_{j+k}^\alpha$. The continuity of $\varrho_\beta^{\alpha,\beta}$ follows analogously.) But, by Lemma 2.5.8, $\mathcal{P}_{\mathcal{U}_{\alpha,\beta}^\alpha}$ and $\mathcal{P}_{\mathcal{U}_{\alpha,\beta}^\beta}$ are equivalent, such that they induce the same topology on $\Gamma(U_{\alpha,\beta}, E)$. Now, we can write $X_{\mathcal{U}_{\mathscr{A}}}$ as a countable intersection of kernels of continuous and linear functions, which are closed subspaces:

$$X_{\mathcal{U}_{\mathscr{A}}} = \bigcap_{(\alpha,\beta)\in \mathrm{OD}(\mathscr{A})} \mathrm{Ker}(\varrho_\alpha^{\alpha,\beta} \circ \pi_\alpha - \varrho_\beta^{\alpha,\beta} \circ \pi_\beta). \tag{2.98}$$

Hence, $\mathrm{Im}(\Theta_{\mathcal{U}_{\mathscr{A}}})$ is a Fréchet space with respect to the subspace topology of the product topology on $\prod_{\alpha\in\mathscr{A}} \Gamma(U_\alpha, E)$. Since $\Theta_{\mathcal{U}_{\mathscr{A}}}$ is a homeomorphism onto its image, $(\Gamma(U, E), \mathcal{P}_{\mathcal{U}})$ is also a Fréchet space. □

2.5.12 Lemma (Independence of the Topology of Sections)

Let $\xi = (E, \pi, M)$ be a smooth vector bundle over M. Two covers of local choices

$$\mathcal{U}_{\mathscr{A}} = (\mathcal{U}_\alpha = (U_\alpha, \varphi_\alpha, \Phi_\alpha, (K_j^\alpha)_{j\in\mathbb{N}_0}))_{\alpha\in\mathscr{A}}, \mathcal{V}_{\mathscr{B}} = (\mathcal{V}_\beta = (V_\beta, \psi_\beta, \Psi_\beta, (L_j^\beta)_{j\in\mathbb{N}_0}))_{\beta\in\mathscr{B}} \tag{2.99}$$

over M induce equivalent systems of seminorms $\mathcal{P}_{\mathcal{U}_{\mathscr{A}}}$ and $\mathcal{P}_{\mathcal{V}_{\mathscr{B}}}$ on $\Gamma(M, E)$.

Proof. We will construct local choice refinements of $\mathcal{U}_{\mathscr{A}}$ and $\mathcal{V}_{\mathscr{B}}$, respectively. By restricting to a non-empty intersection $W_{\alpha,\beta} = U_\alpha \cap V_\beta$, we receive charts $(W_{\alpha,\beta}, \tilde{\varphi}_\alpha)$, $(W_{\alpha,\beta}, \tilde{\psi}_\beta,)$ and trivializations $(W_{\alpha,\beta}, \tilde{\Phi}_\alpha)$, $(W_{\alpha,\beta}, \tilde{\Psi}_\beta,)$. Moreover, $\tilde{K}_j^{\alpha,\beta} = \varphi_\alpha^{-1}(K_{j+k}^\alpha) \cap \psi_\beta^{-1}(L_{j+k}^\beta)$ defines a compact exhaustion of $W_{\alpha,\beta}$ for $k = \min\{j \in \mathbb{N}_0 : \varphi_\alpha^{-1}(K_j^\alpha) \cap \psi_\beta^{-1}(L_j^\beta) \neq \emptyset\}$. Hence,

$$\mathcal{W}_{\alpha,\beta}^\alpha = (W_{\alpha,\beta}, \tilde{\varphi}_\alpha, \tilde{\Phi}_\alpha, (\tilde{\varphi}_\alpha(\tilde{K}_j^{\alpha,\beta}))_{j\in\mathbb{N}_0}) \ \text{ and } \ \mathcal{W}_{\alpha,\beta}^\beta = (W_{\alpha,\beta}, \tilde{\psi}_\beta, \tilde{\Psi}_\beta, (\tilde{\psi}_\beta(\tilde{K}_j^{\alpha,\beta}))_{j\in\mathbb{N}_0}) \tag{2.100}$$

are local choices of ξ over $W_{\alpha,\beta}$ for all $(\alpha, \beta) \in (\mathscr{A}, \mathscr{B}) = \{(\alpha, \beta) \in \mathscr{A} \times \mathscr{B} : W_{\alpha,\beta} = U_\alpha \cap V_\beta \neq \emptyset\}$. Moreover, for fixed $\alpha \in \mathscr{A}$ define $\mathscr{B}_\alpha = \{\beta \in \mathscr{B} : (\alpha, \beta) \in (\mathscr{A}, \mathscr{B})\}$ and for fixed $\beta \in \mathscr{B}$ define $\mathscr{A}_\beta = \{\alpha \in \mathscr{A} : (\alpha, \beta) \in (\mathscr{A}, \mathscr{B})\}$, respectively. Then, $\mathcal{W}_{\mathscr{A},\mathscr{B}}^{\mathscr{A}} = (\mathcal{W}_{\alpha,\beta}^\alpha)_{(\alpha,\beta)\in(\mathscr{A},\mathscr{B})}$ is a local choice refinement of $\mathcal{U}_{\mathscr{A}}$ such that by Lemma 2.5.10, $\mathcal{P}_{\mathcal{U}_{\mathscr{A}}}$ and $\mathcal{P}_{\mathcal{W}_{\mathscr{A},\mathscr{B}}^{\mathscr{A}}}$ are equivalent systems of seminorms on $\Gamma(M, E)$. Analogously, $\mathcal{P}_{\mathcal{V}_{\mathscr{B}}}$ and $\mathcal{P}_{\mathcal{W}_{\mathscr{A},\mathscr{B}}^{\mathscr{B}}}$ are equivalent. Finally, by Lemma 2.5.8, $\mathcal{P}_{\mathcal{W}_{\alpha,\beta}^\alpha}$ and $\mathcal{P}_{\mathcal{W}_{\alpha,\beta}^\beta}$ are equivalent for each $(\alpha, \beta) \in (\mathscr{A}, \mathscr{B})$, such that their corresponding system of projective seminorms $\mathcal{P}_{\mathcal{W}_{\mathscr{A},\mathscr{B}}^{\mathscr{A}}}$ and $\mathcal{P}_{\mathcal{W}_{\mathscr{A},\mathscr{B}}^{\mathscr{B}}}$ are also equivalent. □

2.5.13 Remark (Comparison to Smooth Functions on Euclidean Space)

As we have seen in the construction of the Fréchet space structure on global sections, all seminorms arise *locally* only from compact exhaustions of *typically small open sets*, on which smooth trivializations and charts exist. This is a difference to the Fréchet space structure of $C^\infty(V)$ for any open $V \subseteq \mathbb{R}^n$. In this case, the seminorms arise from a compact exhaustion of the whole set V, such that one might call this situation *semi-global* in difference to the *local* origin of the Fréchet space of global sections.

2.5.14 Remark (Section Functor)

A smooth vector bundle (E, π, M) of rank N gives rise to the Fréchet-space of sections $\Gamma(M, E)$ by the previous lemmas. For a smooth vector bundle M-morphism $f : E \to F$, define the **pushforward** $f_* : \Gamma(M, E) \to \Gamma(M, F)$ by $f_*(\sigma) = f \circ \sigma$. Since the vector space structure of sections is defined pointwise in each fibre and f restricts to a linear map on each fibre, the linearity of f_* follows. For the continuity, suppose an open cover $(U_\alpha)_{\alpha \in \mathscr{A}}$ of M such that there are a chart $(U_\alpha, \varphi_\alpha)$ of M and smooth local trivializations (U_α, Φ_α) of E and (U_α, Ψ_α) of F, respectively. Locally, we have the following commutative diagram for $V_\alpha = \varphi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$ and $(\tilde{f}_\alpha)_* = T_{F,\alpha} \circ (f_\alpha)_* \circ T_{E,\alpha}^{-1}$, where $T_{E,\alpha}^{-1}$ and $T_{F,\alpha}$ are the isomorphisms of Lemma (2.2.4):

$$\begin{array}{ccc} \Gamma(U_\alpha, E) & \xrightarrow{(f_\alpha)_*} & \Gamma(U_\alpha, F) \\ {\scriptstyle T_{E,\alpha}}\big\downarrow & & \big\downarrow{\scriptstyle T_{F,\alpha}} \\ C^\infty(V_\alpha, \mathbb{R}^N) & \xrightarrow[(\tilde{f}_\alpha)_*]{} & C^\infty(V_\alpha, \mathbb{R}^L) \end{array}$$

By construction of the topology on spaces of sections, we obtain that f_* is continuous if and only if $(\tilde{f}_\alpha)_*$ is continuous from $C^\infty(V_\alpha, \mathbb{R}^N)$ to $C^\infty(V_\alpha, \mathbb{R}^L)$ for all $\alpha \in \mathscr{A}$. To show this, Lemma 2.1.6 gives a smooth map $A_{f_\alpha} : U_\alpha \to \mathbb{R}^{L \times N}$ satisfying $\Psi_\alpha \circ f_\alpha \circ \Phi_\alpha^{-1}(x, \lambda) = (x, A_{f_\alpha}(x) \cdot \lambda)$ on $U_\alpha \times \mathbb{R}^N$. Using all the definitions, we obtain for $g \in C^\infty(V_\alpha, \mathbb{R}^N)$ and $y \in V_\alpha$:

$$(\tilde{f}_\alpha)_*(g)(y) = (T_{F,\alpha} \circ (f_\alpha)_* \circ T_{E,\alpha}^{-1}(g))(y) = (A_{f_\alpha} \circ \varphi^{-1})(y) \cdot g(y). \tag{2.101}$$

Since $A_{f_\alpha} \circ \varphi^{-1} : V_\alpha \to \mathbb{R}^{L \times N}$ is smooth, we obtain the continuity of $(\tilde{f}_\alpha)_*$ by Lemma 2.5.3 for each $\alpha \in \mathscr{A}$ and consequently the continuity of the push-forward f_*
For another smooth vector bundle M-morphism $h : F \to G$, we clearly have $(h \circ f)_* = h_* \circ f_*$ and further, the identity is also preserved. To summarize, we have constructed a covariant functor from $\mathbf{VB}_M$ to **Fréchet**, the category of Fréchet spaces with continuous and linear maps.

2.5.15 Lemma (Section-Functor is Exact)

Let $0 \longrightarrow E \xrightarrow{f} F \xrightarrow{g} G \longrightarrow 0$ be a short exact sequence of smooth vector bundles over M, then the sequence of Fréchet spaces with linear and continuous maps,

$$0 \longrightarrow \Gamma(M,E) \xrightarrow{f_*} \Gamma(M,F) \xrightarrow{g_*} \Gamma(M,G) \longrightarrow 0\ , \tag{2.102}$$

is exact and splits.
In particular, if (E,π_E,M) is a subbundle of (F,π_F,M), we obtain for the canonical sequence, consisting of the inclusion morphism i and the quotient morphism q, a short exact sequence that splits:

$$0 \longrightarrow \Gamma(M,E) \xrightarrow{i_*} \Gamma(M,F) \xrightarrow{q_*} \Gamma(M,{}^{F}\!/_{E}) \longrightarrow 0\ . \tag{2.103}$$

Proof. We show the exactness:
(1) Let $\varrho \in \mathrm{Ker}(f_*)$, i.e. $\varrho \in \Gamma(M,E)$ with $f_*(\varrho) = 0$. Then $0 = f_*(\varrho)(x) = f(\varrho(x))$ for every $x \in M$, such that $\varrho(x) = 0$ on M, by injectivity of f.
(2) Now, let $\sigma \in \mathrm{Im}(f_*)$, i.e. there is $\varrho \in \Gamma(M,E)$ with $\sigma = f_*(\varrho) = f \circ \varrho$. Then, we have $g_*(\sigma) = g \circ f \circ \varrho = 0 \in \Gamma(M,G)$ since $\mathrm{Im}(f) = \mathrm{Ker}(g)$, which yields $\mathrm{Im}(f_*) \subseteq \mathrm{Ker}(g_*)$. For $\sigma \in \mathrm{Ker}(g_*)$, we obtain $0 = g_*(\sigma)(x) = g(\sigma(x))$, such that $\sigma(x) \in \mathrm{Ker}(g_x) = \mathrm{Im}(f_x)$ for every $x \in M$. Pick a frame $\{\varrho_j\}_{j=1}^p$ of E over some open subset $U \subseteq M$. Then $\{f \circ \varrho_j\}$ is a frame of $\mathrm{Im}(f)$ by injectivity of f such that $\sigma|_U$ has a local representation $\sigma|_U = \sum\limits_{j=1}^{p} \lambda_j f \circ \varrho_j$ for some smooth functions $\lambda_j \in C^\infty(U,\mathbb{R})$. Define $\varrho_U = \sum\limits_{j=1}^{p} \lambda_j \varrho_j$, then $\varrho_U \in \Gamma(U,E)$ with $f_*(\varrho) = \sigma|_U$ by fibrewise linearity of f. Moreover, for an open cover $(U_\alpha)_{\alpha\in\mathscr{A}}$ of M such that there is a frame of E over each U_α. Pick a smooth locally finite partition of unity $(\chi_\alpha)_{\alpha\in\mathscr{A}}$ subordinate to $(U_\alpha)_{\alpha\in\mathscr{A}}$. The preceding shows that there are $\varrho_\alpha \in \Gamma(U_\alpha,E)$ satisfying $f_*(\varrho_\alpha) = \sigma|_{U_\alpha}$. Since $\mathrm{supp}(\chi_\alpha) \subseteq U_\alpha$ we can extend each $\chi_\alpha\varrho_\alpha$ smoothly by zero outside of U_α and obtain a smooth section $\varrho = \sum\limits_{\alpha\in\mathscr{A}} \chi_\alpha\varrho_\alpha \in \Gamma(M,E)$ with $f_*(\varrho) = \sigma$.
(3) By an analogously partition of unity argument, it is sufficient to show the surjectivity of g_* locally. So let U be an open subset of M such that $\{\sigma_j\}_{j=1}^N$ is a frame of F over U where $\{\sigma_j\}_{j=1}^p$ is a frame of $\mathrm{Ker}(g)(\overset{(2)}{=} \mathrm{Im}(f))$ over U. Hence, $\{\tau_j = g \circ \sigma_{j+p}\}_{j=1}^{N-p}$ defines a frame of G over U since g is surjective. Every $\tau_U \in \Gamma(U,G)$ has a local representation $\tau_U = \sum\limits_{j=1}^{N-p} \lambda_j \tau_j$ with smooth functions $\lambda_j \in C^\infty(U,\mathbb{R})$ such that $\sigma_U = \sum\limits_{j=1}^{N+p} \lambda_j \sigma_{j+p}$ defines a smooth section in $\Gamma(U,F)$ satisfying $g \circ \sigma_U = \tau_U$ by fibrewise linearity.

Lemma 2.4.5 yields the splitting of $0 \longrightarrow E \xrightarrow{f} F \xrightarrow{g} G \longrightarrow 0$, such that there are a left inverse M-morphism $\ell : F \to E$ of f and a right inverse M-morphism $r : F \to E$ of g. By functoriality, we obtain that ℓ_* is a left inverse of f_* and r_* is a right inverse of g_*, respectively. $\square$

2.5.16 Lemma (Sections of Quotient bundle and Quotient of Sections)

Let (E, π_E, M) be a subbundle of a smooth vector bundle (F, π_F, M).
Then $\Gamma(M, E)$ is a closed subspace of $\Gamma(M, F)$ and $\Gamma(M, F/E)$ is an isomorphic Fréchet space of $\Gamma(M,F)/\Gamma(M,E)$. Moreover, the quotient map $\tilde{q} : \Gamma(M, F) \to \Gamma(M,F)/\Gamma(M,E)$ has a linear and continuous right inverse $\tilde{r}$.

Proof. The short exact sequence (2.103) implies $\Gamma(M, E) = \operatorname{Im}(i_*) = \operatorname{Ker}(q_*)$, which is a closed subspace since q_* is continuous. Therefore, $\Gamma(M,F)/\Gamma(M,E)$ is a Fréchet space and the surjective linear and continuous quotient map $\tilde{q} : \Gamma(M, F) \to \Gamma(M,F)/\Gamma(M,E)$ is a topological homomorphism with $\operatorname{Ker}(\tilde{q}) = \Gamma(M, E) = \operatorname{Ker}(q_*)$. Further, the sequence (2.103) splits, such that there are linear and continuous maps ℓ_*, a left inverse of i_*, and r_*, a right inverse of q_*. This gives the following commutative diagram with short exact sequences of Fréchet-spaces in each row:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(M,E) & \underset{\ell_*}{\overset{i_*}{\rightleftarrows}} & \Gamma(M,F) & \underset{r_*}{\overset{q_*}{\rightleftarrows}} & \Gamma(M, F/E) & \longrightarrow & 0\,. \\ & & \big\updownarrow \text{id} & & \big\updownarrow \text{id} & & & & \\ 0 & \longrightarrow & \Gamma(M,E) & \underset{\ell_*}{\overset{i_*}{\rightleftarrows}} & \Gamma(M,F) & \overset{\tilde{q}}{\longrightarrow} & \Gamma(M,F)/\Gamma(M,E) & \longrightarrow & 0 \end{array} \tag{2.104}$$

Since ℓ_* is also a left inverse of i_* in the lower sequence, the Splitting Lemma (2.4.4) gives a linear and continuous right inverse $\tilde{r} : \Gamma(M,F)/\Gamma(M,E) \to \Gamma(M, F)$ of $\tilde{q}$. The linear and continuous compositions $\tilde{q} \circ r_*$ and $q_* \circ \tilde{r}$ are inverse functions, such that we have an isomorphism of Fréchet spaces between $\Gamma(M, F/E)$ and $\Gamma(M,F)/\Gamma(M,E)$. $\square$

3 Forms and Cohomology of Foliated Manifolds

3.1 Introduction of Smooth Foliations

We refer to [Lee13, Chapter 5, p. 98 ff.] for an introduction of smooth submanifolds and collect some basics. An **immersed submanifold** of a smooth manifold M is the image of an *injective immersion* $f : N \to M$, i.e. a smooth injective map between smooth manifolds such that the differential df_x at each $x \in N$ is injective [14]. The topology of an immersed submanifold may be different from the subspace topology inherited from M. If f is a *topological embedding*, such that its image has the subspace topology of M, the image of f is called an **embedded submanifold**. Locally, an immersed submanifold looks like an embedded one, but globally they can look different.
The idea of a foliation is a decomposition of a manifold into disjoint connected immersed submanifolds of a fixed dimension, called leaves of the foliation. Locally, this decomposition can be seen in charts as (lower dimensional) parallelized surfaces. Roughly speaking, a book as a 3-dimensional object is decomposed by its 2-dimensional pages as leaves. Foliations arise in differential geometry as collections of solutions of (underdetermined) systems of differential equations if they satisfy an integrability condition, known as involutivity. This connection is described by the Frobenius Theorem, named after Ferdinand Georg Frobenius. As John Milnor has recommended,[15] a more appropriate name would be Deahna-Clebsch-Frobenius Theorem. We will not go into details, but refer to [Mil70] or [Lee13, Chapter 19] for an elaboration. However, we include an example of a first-order partial differential equation to illustrate the importance of foliations in that area.

3.1.1 Definition (Smooth Foliation)

Let M be a smooth manifold and $\mathcal{F} = \{\mathcal{L}_\alpha\}_{\alpha\in\mathscr{A}}$ be a partition of M into disjoint, connected, immersed k-dimensional submanifolds of M. A **(smooth) foliated chart** (U, φ) is a diffeomorphism $\varphi : U \to V \times W$ between open subsets $U \subseteq M$, $V \subseteq \mathbb{R}^p$ and $W \subseteq \mathbb{R}^q$ with the following property: For each $\alpha \in \mathscr{A}$ and each connected component $(U \cap \mathcal{L}_\alpha)^\beta$ of $U \cap \mathcal{L}_\alpha$ there exists a constant $c_\alpha^\beta \in W \subseteq \mathbb{R}^q$ such that $\varphi((U \cap \mathcal{L}_\alpha)^\beta) = V \times \{c_\alpha^\beta\}$.
We call $(M, \mathcal{F})$ a **(smooth)** (p, q)**-foliated manifold** if each point has a foliated chart, where p is called the dimension of the foliation and q the codimension. In this case, the partition $\mathcal{F} = \{\mathcal{L}_\alpha\}_{\alpha\in\mathscr{A}}$ is called a **(smooth) foliation** of M and $\mathcal{L}_\alpha$ is called a **leaf** of $\mathcal{F}$. Since $\{\mathcal{L}_\alpha\}_{\alpha\in\mathscr{A}}$ is a partition of M into disjoint sets, each $x \in M$ is contained in exactly one leaf, which will be denoted by $\mathcal{F}_x$. Hence, the mapping $x \mapsto \mathcal{F}_x$ from $M \to \mathcal{F}$ is a function that satisfies $\mathcal{F}_x = \mathcal{F}_y$ if and only if x and y lie in the same leaf. We call this the **leaf function of** $\mathcal{F}$.
A **smooth foliated map** $f : (M, \mathcal{F}) \to (N, \mathcal{G})$ between smooth foliated manifolds is a smooth map $f : M \to N$ such that the image of each leaf of $\mathcal{F}$ is contained in some leaf of $\mathcal{G}$, i.e. for every $F \in \mathcal{F}$ there is some $G \in \mathcal{G}$ such that $f(F) \subseteq G$ or equivalently,

[14]See also Examples 2.1.4 (3) for the introduction of an immersion.
[15]See [Mil70, p. 10].

$f(\mathcal{F}_x) \subseteq \mathcal{G}_{f(x)}$ for all $x \in M$.
Note that every smooth manifold M of dimension m is also a smooth $(m, 0)$-dimensional foliation, given by one leaf M itself. But we could also define a smooth $(0, m)$-foliation by points of M, i.e. $\{\{x\} : x \in M\}$. In order to construct more interesting foliations, we start with transversal maps, which allow to pull back a foliation from the target manifold.

3.1.2 Definition (Transverse Map to a Foliation)

A smooth map $f : M \to N$ between smooth manifolds is called **transverse to a smooth foliation $\mathcal{G}$ of N** if

$$T_{f(x)}N = T_{f(x)}\mathcal{G}_{f(x)} + df_x(T_xM) \text{ for all } x \in M. \tag{3.1}$$

Here, $df : TM \to TN$ is the differential[16] of f, defined fibrewise $df_x : T_xM \to T_{f(x)}N$ by

$$df_x(\nu_x)(g) = \nu_x(g \circ f) \text{ for } \nu_x \in T_xM \text{ and } g \in C^\infty(N). \tag{3.2}$$

Note that the sum does not to be a direct sum in the definition. Moreover, if $M = \dim(M)$, $n = \dim(N)$ and $\mathcal{G}$ is a (p, q)-foliation of N, the existence of a transverse map $f : M \to N$ to $\mathcal{G}$ implies

$$n \le p + m \text{ and therefore } q = n - p \le m. \tag{3.3}$$

3.1.3 Remark (Submersion)

If $f : M \to N$ is a submersion, then df_x is surjective (or equivalently of full rank $\dim(N)$) at each point $x \in M$, such that f is clearly transverse to any foliation of N. If $m \ge n$, the projection map $\mathrm{pr}_{II} : \mathbb{R}^{m-n} \times \mathbb{R}^n \to \mathbb{R}^n$, $(x, y) \mapsto y$ is a submersion. The following result states that each submersion can be transformed locally into a projection by a diffeomorphism, which is a consequence of the *Inverse Function Theorem*[17]. It is called the **Local** (or **Canonical**) **Submersion Theorem** and we use it to proof Proposition 3.1.5. The Local Submersion Theorem is a special case of the so called *Rank Theorem*[18], where $f : M \to N$ needs to be a smooth map such that the differential $df : TM \to TN$ has a constant rank.

3.1.4 Theorem (Local Submersion Theorem)

Let $V \subseteq \mathbb{R}^m$ and $W \subseteq \mathbb{R}^q$ be open sets. If $f : V \to W$ is a submersion (at $x \in V$), then there exist open subsets $\tilde{V} \subseteq V \subseteq \mathbb{R}^m$, $W_1 \subseteq \mathbb{R}^{m-q}$, $W_2 \subseteq \mathbb{R}^q$ (with $x \in \tilde{V}$) and a diffeomorphism $\kappa : W_1 \times W_2 \to \tilde{V}$ such that

$$f \circ \kappa(w_1, w_2) = w_2 \text{ for all } (w_1, w_2) \in W_1 \times W_2. \tag{3.4}$$

[16] See also Examples 2.1.4 (3).
[17] See [Lee13, Theorem C.34, p. 657] for instance.
[18] See [Lee13, Theorem 4.12, p. 81] for instance.

Proof. Since f is a submersion, $q \leq m$ necessarily. The Jacobian matrix $\nabla f(x)$ at $x \in V$ is a $(q \times m)$-matrix of rank q with entries $\left(\frac{f_i}{\partial x_j}(x)\right)$ $(i = 1, ..., q,\ j = 1, ..., m)$. There is a (linear) diffeomorphism $\tau : \mathbb{R}^m \to \mathbb{R}^m$, given by a change of basis such that $\tilde{f} = f \circ \tau : \tau^{-1}(V) \to W$ is a submersion and the left $(q \times q)$-block of its Jacobian matrix at $\tilde{x} = \tau^{-1}(x)$,

$$A = \left(\frac{\tilde{f}_i}{\partial x_j}(\tilde{x})\right)_{1 \leq i,j \leq q}, \tag{3.5}$$

is regular. Now, we define

$$g : \tau^{-1}(V) \to \mathbb{R}^m \text{ by } g(x_1, ..., x_m) = (x_{q+1}, ..., x_m, \tilde{f}(x_1, ..., x_q)). \tag{3.6}$$

For the projection $\mathrm{pr}_{II} : \mathbb{R}^{m-q} \times \mathbb{R}^q \to \mathbb{R}^q$, $\mathrm{pr}_{II}(a, b) = b$, we get $\tilde{f} = \mathrm{pr}_{II} \circ g$. Further, g is smooth and its Jacobian matrix at $\tilde{x}$ is given by

$$\nabla g(\tilde{x}) = \begin{bmatrix} 0 & \mathrm{I}_{m-q} \\ A & 0 \end{bmatrix}, \text{which is invertible with inverse } (\nabla g(\tilde{x}))^{-1} = \begin{bmatrix} 0 & A^{-1} \\ \mathrm{I}_{m-q} & 0 \end{bmatrix}. \tag{3.7}$$

Therefore, the Inverse Function Theorem yields an open neighbourhood $U \subseteq \tau^{-1}(V)$ of $\tilde{x}$ such that $g|_U$ is a diffeomorphism. In particular, $g(U)$ is open. Hence, there are open sets $W_1 \subseteq \mathbb{R}^{m-q}$ and $W_2 \subseteq \mathbb{R}^q$ such that $g(\tilde{x}) \in W_1 \times W_2 \subseteq g(U)$. $\tilde{U} = (g|_U)^{-1}(W_1 \times W_2) \subseteq U$ is open in $\mathbb{R}^m$ with $\tilde{x} \in \tilde{U}$ and $g|_{\tilde{U}}$ is also a diffeomorphism. Moreover, $\kappa = \tau \circ (g|_{\tilde{U}})^{-1}$ is a diffeomorphism between open sets from $W_1 \times W_2$ onto $\tilde{V} = \tau(\tilde{U})$ with $x = \tau(\tilde{x}) \in \tilde{V}$. By $f \circ \tau = \mathrm{pr}_{II} \circ g$, we finally obtain

$$f \circ \kappa(w_1, w_2) = w_2 \text{ for all } (w_1, w_2) \in W_1 \times W_2. \tag{3.8}$$

□

3.1.5 Proposition (Pullback Foliation of a Transverse Map)

Let M be a smooth manifold of dimension m and $(N, \mathcal{G})$ a smooth (p, q)-foliation. Further, let $f : M \to N$ be a smooth map transverse to $\mathcal{G}$.
Then, there is a smooth $(m - q, q)$-foliation $f^*(\mathcal{G})$ of M, called the **pullback foliation of f induced by $\mathcal{G}$**. The leaves are given by the connected components of the preimages $f^{-1}(\mathcal{L}) = \{y \in M : f(y) \in \mathcal{L}\}$, where $\mathcal{L}$ ranges over the leaves of $\mathcal{G}$ which intersect the image $f(M) \subseteq N$.

Proof. Let $x \in M$, $\psi_M : U_M \to V_M$ be a chart of x and $(\psi_N^p, \psi_N^q) : U_N \to W^p \times W^q$ be a foliated chart of $(N, \mathcal{G})$ with $f(x) \in U_N$. By shrinking the domain of ψ_M, we can assume $U_M \subseteq f^{-1}(U_N)$. The second component function of the foliated chart, $\psi_N^q : U_N \to W^q \subseteq \mathbb{R}^q$, is a submersion and since f is transverse to $\mathcal{G}$, the composition $\psi_N^q \circ f|_{f^{-1}(U_N)}$ is also a submersion. Further, $\tilde{f} = \psi_N^q \circ f|_{U_M} \circ \psi_M^{-1} : V_M \subseteq \mathbb{R}^m \to W^q \subseteq \mathbb{R}^q$ is a submersion between open subsets, such that the Local Submersion Theorem 3.1.4 provides open sets

$\tilde{V}_M \subseteq V_M \subseteq \mathbb{R}^m$, $W_1 \times W_2 \subseteq \mathbb{R}^{m-q} \times \mathbb{R}^q$ with $\psi_M(x) \in \tilde{V}_M$ and a smooth diffeomorphism $\kappa : W_1 \times W_2 \to \tilde{V}_M$ satisfying

$$\tilde{f} \circ \kappa(w_1, w_2) = w_2 \text{ for all } (w_1, w_2) \in W_1 \times W_2. \tag{3.9}$$

Set $\tilde{U}_M = \psi_M^{-1}(\tilde{V}_M)$ and $\varphi = \kappa^{-1} \circ \psi_M|_{\tilde{U}_M}$. We will verify, that $(\tilde{U}_M, \varphi)$ is a foliated chart for x of an $(m-q, q)$-foliation of M. φ is a diffeomorphism as composition of diffeomorphisms. Fix a leaf $\mathcal{L} \in G$. If $f^{-1}(\mathcal{L}) \cap \tilde{U}_M = \emptyset$, there is nothing to show. Otherwise, let $y \in \tilde{U}_M$ with $f(y) \in \mathcal{L}$. Then, there is a (unique) pair $(w_1, w_2) = \varphi(y) \in W_1 \times W_2$ satisfying

$$w_2 = \tilde{f} \circ \kappa(w_1, w_2) = \tilde{f} \circ \psi_M(y) = \psi_N^q \circ f(y) \tag{3.10}$$

If $[\tilde{U}_M \cap f^{-1}(\mathcal{L})]^\alpha$ is a connected component, then by $f(\tilde{U}_M \cap f^{-1}(\mathcal{L})) \subseteq U_N \cap \mathcal{L}$ and the continuity of f, the image, $f([\tilde{U}_M \cap f^{-1}(\mathcal{L})]^\alpha)$ is connected in $U_N \cap \mathcal{L}$ and therefore contained in some connected component $[U_N \cap \mathcal{L}]^{\beta(\alpha)}$ on which ψ_N^q is of a constant value $c^{\beta(\alpha)} \in W^q \subseteq \mathbb{R}^q$. Together with (3.10), this gives $\varphi([\tilde{U}_M \cap f^{-1}(\mathcal{L})]^\alpha) = W_1 \times \{c^{\beta(\alpha)}\}$. □

3.1.6 Corollary (Foliations of Submersions)

Let $S : M \to N$ be a submersion between smooth manifolds of dimension m and n, thus $m \geq n$ necessarily. Then, there is a smooth $((m-n), n)$-foliation of M, given by the connected components of the level sets $S^{-1}(\{c\}) = \{x \in M : S(x) = c\}$ as leaves, where c ranges over the image $S(M) \subseteq N$.

Proof. Let $\mathcal{G} = \{\{y\} : y \in N\}$ be the $(0, n)$-foliation by points of N. Since S is a submersion, it is also transverse to $\mathcal{G}$. By proposition 3.1.5, the pullback foliation $S^*(\mathcal{G})$ is the desired foliation on M. □

Note that every p-foliation of an m-dimensional manifold M is locally given by the connected components of the level sets of a submersion. More precisely, define for a foliated chart $\varphi : U \to V \times W \subseteq \mathbb{R}^p \times \mathbb{R}^{m-p}$ the map $S_U = \mathrm{pr}_{II} \circ \varphi : U \to W$, where pr_{II} is the projection of $V \times W$ onto W. Then S_U is a submersion as a composition of submersions and $S_U^{-1}(\{c_\alpha^\beta\}) = \varphi^{-1}(\mathrm{pr}_{II}^{-1}(\{c_\alpha^\beta\})) = \varphi^{-1}(V \times \{c_\alpha^\beta\}) = (U \cap \mathcal{L}_\alpha)^\beta$ for any α and β.

3.1.7 Examples (Foliations)

(1) **Foliation by Points**
Let M and N be smooth manifolds of dimension m and n, respectively.
Then $\mathcal{F}_N(M) = \{M \times \{y\} : y \in N\}$ is a smooth (m, n)-foliation on the product manifold $M \times N$, called the M**-foliation by points of** N. A foliated chart is given by $(V \times W, (\varphi \circ \pi_V, \psi \circ \pi_W))$, where π_V, π_W are the projections from $V \times W$, (V, φ) and (W, ψ) are smooth charts of M and N, respectively.
Note that each smooth (p, q)-foliation $(M, \mathcal{F})$ is locally isomorphic to a V-foliation by points of W with open subsets $V \subseteq \mathbb{R}^p$ and $W \subseteq \mathbb{R}^q$ by a foliated chart $\varphi : U \to V \times W$ of $(M, \mathcal{F})$.

(2) **Foliation induced by Submersion:**
Let $S : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$, $S(x,y) = xy$. Then, S is a submersion since at each point $(x,y) \in \mathbb{R}^2 \setminus \{0\}$ the differential $dS_{(x,y)} : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$dS_{(x,y)}(a,b) = \partial_x S(x,y)a + \partial_y S(x,y)b = ya + xb, \tag{3.11}$$

which has rank 1. Thus, the connected components of the level sets $\{(x,y) : xy = c\}$ $(c \in \mathbb{R})$ of S induce a foliation on $\mathbb{R}^2 \setminus \{0\}$.

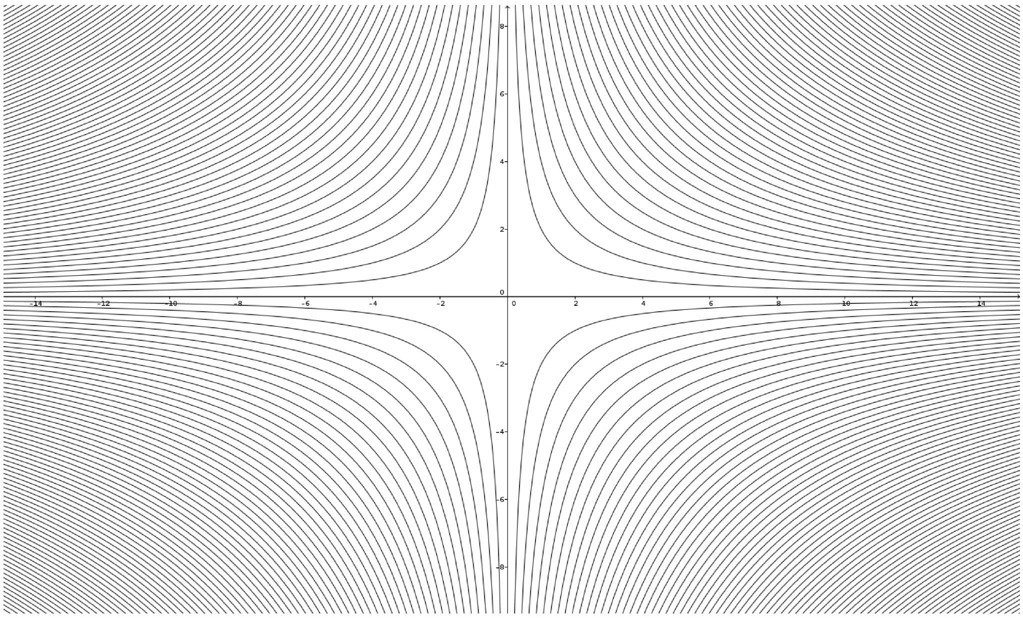


Figure 2: Foliation of the level sets of $(x,y) \mapsto xy$ on $\mathbb{R}^2 \setminus \{0\}$.

(3) **Pullback Foliation**
Let $(M, \mathcal{F})$ be a smooth (p,q)-foliation and $\pi : M \times \mathbb{R} \to M$ the projection map, which is a submersion. Then, the pullback foliation $\pi^*(\mathcal{F})$ on $M \times \mathbb{R}$ is given by $\{\mathcal{L} \times \mathbb{R} : \mathcal{L} \in \mathcal{F}\}$. We will need this specific foliation on the product manifold $M \times \mathbb{R}$ later for a type of homotopy. It satisfies the following property:

(∗) For any $t \in \mathbb{R}$ the smooth injective inclusion map $J_t : M \to M \times \mathbb{R}$, $x \mapsto (x,t)$ is also a foliated map from $(M, \mathcal{F})$ to $(M \times \mathbb{R}, \pi^*(\mathcal{F}))$.

(4) **Restricted Foliation**
Let S be a smooth immersed submanifold of a smooth manifold M. If the inclusion $i_S : S \to M$ is transverse to a (p,q)-foliation $\mathcal{F}$ of M, then $\mathcal{F}|_U = i_S^*(\mathcal{F})$, given by the connected components of all leaf intersections with S, is a $(s-q,q)$-foliation of S, where s denotes the dimension of the manifold S. Note that, if $U \subseteq M$ is an open subset, then $di_U : T_xU \to T_xM$ is an isomorphism for each $x \in U$ (e.g. by [Lee13, Proposition 3.9]). In this case, the inclusion is transversal to any foliation of M.

(5) **Foliation of the Tangent Bundle:**
Let $(M, \mathcal{F})$ be a smooth (p, q)-foliated manifold. Then, $(TM, \tau_M^*(\mathcal{F}))$ is a smooth foliated manifold of dimension $(2p + q, q)$. (By Remark 2.1.2 (4), $\tau_M : TM \to M$ is a submersion as the projection of a smooth vector bundle. The manifold dimension of TM is $2(p + q)$ and Proposition 3.1.5 yields the assertion.)

3.2 Foliated Differential Forms

In this chapter, let $\mathcal{F} = \{\mathcal{L}_\alpha\}_{\alpha \in \mathscr{A}}$ be a smooth (p, q)-foliation on an n-dimensional manifold M and $k \in \mathbb{N}_0$. Recall that a differential form $\omega \in \Omega^k(M)$ is a section of the alternating k-tensor bundle of the tangent bundle of M, i.e. $\omega_x = \omega(x) : (T_xM)^k \to \mathbb{R}$ is a multilinear alternating map for every $x \in M$. A 0-form is just a C^∞ function $M \to \mathbb{R}$. The C^∞-topology provides $\Omega^k(M)$ with the structure of a Fréchet space.
There are two ways to introduce the so called *smooth foliated differential k-forms* on a foliated manifold.[19] We will show that both constructions will lead to isomorphic Fréchet spaces.

3.2.1 Definition (Some Vector Bundles Induced By a Foliation)

(a) **Foliated Tangential Subbundle**
All leaves are immersed submanifolds of dimension p, such that we can identify the tangent space $T_x\mathcal{F}_x$ with a p-dimensional subspace of T_xM for each $x \in M$. Define $T\mathcal{F} = \bigsqcup_{x \in M} T_x\mathcal{F}_x$ and $\tau_{\mathcal{F}} = \tau_M|_{T\mathcal{F}}$. Then $(T\mathcal{F}, \tau_{\mathcal{F}}, M)$ is a p-subbundle of the n-rank tangent bundle (TM, τ_M, M), called the **foliated tangential subbundle** of $(M, \mathcal{F})$.

(b) **Transversal Bundle**
The quotient bundle $(T\mathcal{F}^\perp, \tau_{\mathcal{F}^\perp}, M) = (TM/T\mathcal{F}, \pi_{TM/T\mathcal{F}}, M)$ of the tangent bundle by the foliated tangential p-subbundle is called the **transversal** or **normal bundle** of M and has rank $n - p$.

(c) **Foliated Alternating k-Tensor Bundle**
Applying the alternating k-tensor functor (see Corollary 2.3.13 (c)) to the foliated tangential subbundle, we obtain the **foliated alternating k-tensor bundle** $(\Lambda^k(T^*\mathcal{F}), \Lambda^k(\pi), M)$ of rank $\binom{p}{k}$. The inclusion M-morphism $i : T\mathcal{F} \to TM$ induces a restriction map $i^* : \Lambda^k(T^*M) \to \Lambda^k(T^*\mathcal{F})$ by its pullback, which is a smooth vector bundle M-morphism with constant rank $\binom{p}{k}$. Hence, i^* is surjective and its image vector bundle is the foliated alternating k-tensor bundle. By Lemma 2.4.2, it is isomorphic to the *quotient bundle* of $(\Lambda^k(T^*M), \Lambda^k(\pi), M)$ by the kernel vector bundle of i^*. This motivates the following vector bundle:

[19] Compare to [Ber01] and [Ber11].

(d) k**-Annihilator**
Define the k**-annihilator of** $T\mathcal{F}$ to be the kernel vector bundle of the smooth vector bundle M-morphism $i^* : \Lambda^k(T^*M) \to \Lambda^k(T^*\mathcal{F})$, which is the pullback of the inclusion M-morphism $i : T\mathcal{F} \to TM$. To be more precise, define

$$\Lambda^k(T^*M \perp\!\!\!\perp T\mathcal{F}) = \bigsqcup_{x \in M} \Lambda^k(T_x^*M \perp\!\!\!\perp T_x\mathcal{F}_x), \text{ where}$$
$$\Lambda^k(T_x^*M \perp\!\!\!\perp T_x\mathcal{F}_x) = \{\omega \in \Lambda^k(T_x^*M) : \omega(v_1, ..., v_k) = 0 \ \forall\, v_1, ..., v_k \in T_x\mathcal{F}_x\} \quad (x \in M),$$

For $k = 0$, we set $\Lambda^0(T_x^*M \perp\!\!\!\perp T_x\mathcal{F}_x) = \{0\}$ such that $\Lambda^0(T^*M \perp\!\!\!\perp T\mathcal{F}) = M \times \{0\}$. Together with the canonical projection, it is a smooth $\left(\binom{n}{k} - \binom{p}{k}\right)$-subbundle of the $\binom{n}{k}$-ranked smooth vector bundle $(\Lambda^k(T^*M), \Lambda^k(\pi), M)$.

Foliations arise in the solvability theory of differential equations. We refer to [Lee13, Chapter 19] for a general elaboration and illustrate the connections between foliations and systems of differential equations in some examples.

3.2.2 Examples (Connections of Foliations to Differential Equations)

Consider a vector field X on a smooth manifold M of dimension n. For each point $x \in M$, there is some open interval I and a smooth map $u : I \to M$ such that for all $t \in I$, the equation $X_{u(t)} = u'(t) (\in T_{u(t)}M)$ holds, and $u(t_0) = x$ is satisfied for some $t_0 \in I$ by [Lee13, Proposition 9.2]. This u is called an **integral curve** of X through the point $x \in M$ and the image of u is called an **integral manifold** of X at $x \in M$. If the vector field X vanishes nowhere, it spans a one-dimensional subbundle of TM over M which is equal to the foliated tangential subbundle of an $(1, n-1)$-foliation, where the leaves are given by the *maximal* integral manifolds of X. For instance, to find the integral curve $u : I \to \mathbb{R}^2$, $u(t) = (x(t), y(t))$ of the vector field X on $\mathbb{R}^2$ defined by $X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$, we have to solve

$$X_{(x(t),y(t))} = u'(t) = x'(t)\frac{\partial u}{\partial x} + y'(t)\frac{\partial u}{\partial y}. \tag{3.12}$$

Comparison of coefficients yields the equivalent system of first-order ordinary differential equations

$$x'(t) = y(t), \qquad\qquad y'(t) = -x(t). \tag{3.13}$$

The solutions are $x(t) = a\sin(t) + b\cos(t)$ and $y(t) = a\cos(t) - b\sin(t)$ with constants $a, b \in \mathbb{R}$. If we want to obtain the integral curve through $(x_0, y_0) \in \mathbb{R}^2$, $u(0) = (b, a)$ yields $a = y_0$ and $b = x_0$. The maximal interval of u can be chosen to be $\mathbb{R}$ and the integral manifold through $(x_0, y_0) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ is a circle through that point with center $(0, 0)$, which is a one-dimensional manifold. The integral manifold through $(0, 0)$, where the vector field vanishes, is just the origin, which is a zero-dimensional manifold. So X

induces a $(1,1)$-foliation only on $\mathbb{R}^2 \setminus \{(0,0)\}$ given by circles. For each parameter $t \in \mathbb{R}$, we can define the *flow* $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ of X at time $t \in \mathbb{R}$ by setting

$$\varphi_t(x, y) = (y \sin(t) + x \cos(t), y \cos(t) - x \sin(t)). \tag{3.14}$$

Intuitively, the flow of X tells us that a particle $(x, y) \in \mathbb{R}^2$ at time $t_0 = 0$ is moved to the coordinates $\varphi_t(x, y)$ after time t by the impact of X. For a higher dimensional foliation, we consider $M = \mathbb{R}^3$ and two vector fields X and Y defined by

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \qquad Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}. \tag{3.15}$$

X and Y are pointwise linearly independent vector fields on $\mathbb{R}^3$. Hence, the span of X and Y, denoted by $\text{span}\{X, Y\}$, defines a 2-subbundle of $T\mathbb{R}^3$. The question whether the span of these two vector fields arise as the foliated tangential subbundle of some $(2,1)$-foliation of $\mathbb{R}^3$ is answered by the **Frobenius Theorem**. The answer is affirmative if and only if the subbundle $\text{span}\{X, Y\}$ is *involutive*, which means that the *Lie-bracket* $[A, B]$ of any two vector fields $A, B \in \Gamma(M, \text{span}\{X, Y\})$ in $\text{span}\{X, Y\}$, defined pointwise by $[A, B]_x(f) = A_x(B_x(f)) - B_x(A_x(f))$ for $x \in M$ and $f \in C^\infty(M)$, is again a section of $\text{span}\{X, Y\}$. Equivalently, $[X, Y]$ can be written as a linear combination of X and Y. In our example, we can compute

$$X \circ Y = \frac{\partial^2}{\partial x \partial y} + \frac{\partial}{\partial z} + x\frac{\partial^2}{\partial x \partial z} + y\frac{\partial^2}{\partial y \partial z} + xy\frac{\partial^2}{\partial z^2} = Y \circ X, \tag{3.16}$$

such that X and Y even commute, which is equivalent to $[X, Y] = 0$. To obtain the underlying foliation of $\text{span}\{X, Y\}$, we form the flows $\varphi_t, \psi_s : \mathbb{R}^3 \to \mathbb{R}^3$ of the corresponding vector fields and get a parametrization of the integral manifold through $(x, y, z) \in \mathbb{R}^3$ by the function $\Phi(s, t) = \psi_s \circ \varphi_t(x, y, z)$, defined on $I_X \times I_Y$, where I_X and I_Y are the (maximal) open intervals of the integral curves corresponding to X and Y, respectively. Note that the flows of X and Y commute if and only if the vector fields X and Y commute, see for instance [Lee13, Theorem 9.44]. In the case, $0 \neq [X, Y] \in \Gamma(M, \text{span}\{X, Y\})$, one has to find vector fields $\tilde{X}$ and $\tilde{Y}$ which generate $\text{span}\{X, Y\}$ and commute, in order to compute a parametrization of the integral manifolds[20]. In our example, we can calculate the integral curves and flows of X and Y through $(x, y, z) \in \mathbb{R}^3$ to be:

$$u, v : \mathbb{R} \to \mathbb{R}^3, \qquad u(t) = (t + x, y, yt + z), \qquad v(s) = (x, s + y, xs + z), \tag{3.17}$$

$$\varphi_t, \psi_s : \mathbb{R}^3 \to \mathbb{R}^3, \quad \varphi_t(x, y, z) = (t + x, y, yt + z), \quad \psi_s(x, y, z) = (x, s + y, xs + z) \tag{3.18}$$

for $t, s \in \mathbb{R}$. Hence, $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$, $\Phi(s, t) = \psi_s \circ \varphi_t(x, y, z) = (t + x, s + y, (t + x)s + yt + z)$ yields a parametrization of the integral manifold through $(x, y, z) \in \mathbb{R}^3$.

[20] Compare to [Lee13, Example 19.14].

3.2.3 Definition (Foliated Differential k-Forms)

(a) Define $\Omega^k(M, \mathcal{F})$ to be the Fréchet space of sections of the foliated alternating k-tensor bundle $\Lambda^k(T^*\mathcal{F})$:

$$\Omega^k(M, \mathcal{F}) = \Gamma\left(M, \Lambda^k(T^*\mathcal{F})\right). \tag{3.19}$$

(b) Consider $\Lambda^k(T^*M \amalg T\mathcal{F})$, the k-annihilator of $T\mathcal{F}$ and $i^* : \Lambda^k(T^*M) \to \Lambda^k(T^*\mathcal{F})$, defined in Definition 3.2.1 (d). Applying the section functor (Remark 2.5.14), we obtain the continuous pushforward $(i^*)_* : \Omega^k(M) \to \Omega^k(M, \mathcal{F})$. The space of sections $\Omega^k_{\amalg}(M, \mathcal{F}) = \Gamma(M, \Lambda^k(T^*M \amalg T\mathcal{F}))$ is exactly the kernel of $(i^*)_*$ and therefore a *closed subspace* of the Fréchet space $\Omega^k(M)$. Now, we define the quotient space

$$\Omega^k_{\text{fol}}(M, \mathcal{F}) = \Omega^k(M)/\Omega^k_{\amalg}(M, \mathcal{F}), \tag{3.20}$$

which is also a Fréchet space.

3.2.4 Lemma (Isomorphic Constructions of Foliated Forms)

There is an isomorphism of Fréchet spaces between $\Omega^k_{\text{fol}}(M, \mathcal{F})$ and $\Omega^k(M, \mathcal{F})$ for each $k \in N_0$.

Proof. For $k = 0$ both spaces are $C^\infty(M)$, so assume $k \in \mathbb{N}$. As we have mentioned in Definition 3.2.1 (c), the induced bijection of the surjective smooth vector bundle M-morphism $\varrho = i^* : \Lambda^k(T^*M) \to \Lambda^k(T^*\mathcal{F})$,

$$\tilde{\varrho} : \Lambda^k(T^*M)/\Lambda^k(T^*M \amalg T\mathcal{F}) \to \Lambda^k(T^*\mathcal{F}) \tag{3.21}$$

is an isomorphism of smooth vector bundle M-morphisms by Lemma 2.4.2. By applying the section functor, the pushforward $\tilde{\varrho}_* : \Gamma(M, \Lambda^k(T^*M)/\Lambda^k(T^*M \amalg T\mathcal{F})) \to \Omega^k(M, \mathcal{F})$ is an isomorphism of Fréchet spaces. Further, $\Gamma(M, \Lambda^k(T^*M)/\Lambda^k(T^*M \amalg T\mathcal{F}))$ is an isomorphic Fréchet space of $\Omega^k(M)/\Omega^k_{\amalg}(M, \mathcal{F}) = \Omega^k_{\text{fol}}(M, \mathcal{F})$ by Lemma 2.5.16, which completes the proof. □

3.2.5 Example (Foliated Forms on a Foliation Induced by Submersions)

Consider a submersion $S : U \to \mathbb{R}^n$ of an open set $U \subseteq \mathbb{R}^{n+1}$. We know by Lemma 3.1.6 that a 1-dimensional foliation $\mathcal{F}$ on U is given by the connected components of the level sets as leaves. It is well known, that the gradients of the component functions $S_1, ..., S_n$ are orthogonal to these level sets. For simplification we suppose that the Euclidean norm of the gradients in each point of U is equal 1. Fix any $\xi \in U$. Then there is a unique leaf $\mathcal{F}_\xi$ (a connected component of a level set) which contains ξ. The cross product of $\nu_1(\xi) := \nabla S_1(\xi), ..., \nu_n(\xi) := \nabla S_n(\xi)$ complements these vectors to a basis of the $(n+1)$ dimensional tangent space $T_\xi U$. So $\nu_{n+1}(\xi) = \nu_1(\xi) \times \cdots \times \nu_n(\xi)$ is an orthonormal basis of the 1-dimensional tangent space $T_\xi \mathcal{F}_\xi$.

We obtain a basis of $T^*_\xi U$, the dual space of $T_\xi U$, by building the dual basis $\nu^1(\xi), ..., \nu^{n+1}(\xi)$ of the basis $\nu_1(\xi), ..., \nu_{n+1}(\xi)$ (such that $\nu^i(\nu_j(\xi)) = \delta_{i,j}$).
In this way, we receive for each $\xi \in U$ a basis $\nu^1(\xi), ..., \nu^{n+1}(\xi)$ of $T^*_\xi U$, such that $\nu^{n+1}(\xi)$ is an orthonormal basis of $T^*_\xi \mathcal{F}_\xi$. Therefore a foliated 1 form ω can be represented as the product of some C^∞-function $\lambda : U \to \mathbb{R}$ with the 1-form ν^{n+1}:

$$\Omega^1(U, \mathcal{F}) = \{\omega = \lambda\, \nu^{n+1} : \lambda \in C^\infty(U)\} \tag{3.22}$$

Thus $\Omega^1_{\text{fol}}(U, \mathcal{F})$ is isomorphic to $C^\infty(U)$ by $\lambda\, \nu^{n+1} \mapsto \lambda$. We can compute the quotient map $\Omega^1(U) \to \Omega^1_{\text{fol}}(U, \mathcal{F})$ as a projection $\omega \mapsto \langle \omega, \nu^{n+1} \rangle \nu^{n+1} = \left(\sum_{j=1}^{n+1} \omega_j (\nu^{n+1})_j \right) \nu^{n+1}$.

If $\omega = dg = \sum_{j=1}^{n+1} \partial_j g$ for some $g \in C^\infty(U)$, we obtain by using the projection and isomorphism above a differential operator $\tilde{d} : C^\infty(U) \to C^\infty(U)$, $g \mapsto \sum_{j=1}^{n+1} (\nu^{n+1})_j\, \partial_j g$, such that the following diagram commutes:

$$\begin{array}{ccc} \Omega^0_{\text{fol}}(U, \mathcal{F}) & \xrightarrow{\tilde{d}_{\mathcal{F}}} & \Omega^1_{\text{fol}}(U, \mathcal{F}) \\ {\scriptstyle \text{id}} \updownarrow & & \updownarrow \wr \\ C^\infty(U) & \xrightarrow[\tilde{d}]{} & C^\infty(U). \end{array}$$

Here, the upper map $\tilde{d}_{\mathcal{F}} : \Omega^0_{\text{fol}}(U, \mathcal{F}) \to \Omega^1_{\text{fol}}(U, \mathcal{F})$ is the composition $q^1 \circ d$ of the quotient map $q^1 : \Omega^1(U, \mathcal{F}) \to \Omega^1_{\text{fol}}(U, \mathcal{F})$ and the Cartan-differential $d : C^\infty(U) \to \Omega^1(U)$.
Hence, $\Omega^1_{\text{fol}}(M, \mathcal{F}) / \text{Im}(d_{\mathcal{F}})$ is Hausdorff if and only if the differential operator $\tilde{d}$ (with components of ν^{n+1} as non constant coefficients) has closed range.

3.3 Foliated Cartan-Differential

Next, we want to define an *exterior derivative* between foliated differential forms similar to the Cartan-differential between differential forms. If one uses the spaces $\Omega^k_{\text{fol}}(M,\mathcal{F})$, defined as quotient spaces (Definition 3.2.3 (b)) and q^{k+1} denotes the quotient map from $\Omega^{k+1}(M)$ onto $\Omega^{k+1}_{\text{fol}}(M,\mathcal{F})$, a linear and continuous map $\tilde{d}_{\mathcal{F}} : \Omega : \text{fol}^k(M,\mathcal{F}) \to \Omega^k_{\text{fol}}(M,\mathcal{F})$ is induced by the composition $q^{k+1} \circ d$. This $\tilde{d}_{\mathcal{F}}$ is well-defined since the composition is constant on equivalence classes of $\Omega^k_{\text{fol}}(M,\mathcal{F})$ because the Cartan-differential d commutes with restrictions, which are pullbacks of inclusions of the leaves. It might be reasonable to achieve also a foliated Cartan-differential between the foliated differential forms $\Omega^k(M,\mathcal{F})$ obtained by Definition 3.2.3 (a). We will do this elementary without using the just defined $\tilde{d}_{\mathcal{F}}$ and the isomorphism between the two ways of definition to avoid forming quotient spaces completely. Our approach is based on the introduction of the classical Cartan-differential that can be found in [Lee13, Exterior Derivatives, p. 362 ff.]. As a positive side effect, we obtain the classical Cartan-differential between differential forms by considering the trivial one-leaf foliation $\mathcal{L} = \{M\}$.

3.3.1 Remark (Differential of a Smooth Function)

Recall, the **differential of a smooth function** $f : M \to \mathbb{R}$ on a smooth manifold is a smooth section df of the cotangent bundle T^*M, i.e. $df \in \Omega^1(M)$ defined by

$$(df)_x(\nu_x) = \nu_x(f) \text{ for } x \in M \text{ and a derivation } \nu_x \in T_xM. \tag{3.23}$$

If $(U,(x^1,...,x^n))$ is a chart of M, the coordinate representation of df is:

$$df|_U = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i, \tag{3.24}$$

where dx^i is the differential of the ith-coordinate function x^i. Furthermore, $(dx^1,...,dx^n)$ is a frame of T^*M over U which is dual to the frame $\left(\frac{\partial}{\partial x^1},...,\frac{\partial}{\partial x^n}\right)$ of TM over U in the sense, that $dx^i\left(\frac{\partial}{\partial x^j}\right) = \delta_{i,j}$ on U.[21] The coordinate representation yields therefore also the continuity of $d : C^\infty(M) \to \Omega^1(M)$.
Involving the wedge product, one gets also a frame of the exterior k-form vector bundle $\Lambda^k(T^*M)$ over U[22], such that a 1-forms dx^i can be seen as an elementary building block of differential forms. Besides, the differential of a function enables a definition of the exterior derivative in terms of local representations. Since $T\mathcal{F}$ is a subbundle of TM, it is quite natural to look at the restriction of df and expect similar results for the theory of foliated forms.

[21] We refer to [Lee13, Chapter 6, p.132 ff.] for details.

[22] The frame consists of *elementary alternating tensors* (Remark 2.3.8) built of a frame and its dual frame.

3.3.2 Definition (Foliated Differential of a Smooth Function)

Denote by $i_{T\mathcal{F}} : T\mathcal{F} \to TM$ the inclusion of $T\mathcal{F}$, which is a smooth injective vector bundle M-morphism. The **foliated differential** $d_{\mathcal{F}} f : M \to T^*\mathcal{F}$ of a smooth function $f : M \to \mathbb{R}$ is defined as the pushforward[23] of the pullback of $i_{T\mathcal{F}}$ evaluated at df:

$$(d_{\mathcal{F}} f) = (i^*_{T\mathcal{F}})_*(df) = i^*_{T\mathcal{F}} \circ df, \text{ i.e.} \tag{3.25}$$

$$(d_{\mathcal{F}} f)_x(\nu_x) = \nu_x(f|_{\mathcal{F}_x}) \text{ for } x \in M \text{ and a derivation } \nu_x \in T_x\mathcal{F}_x. \tag{3.26}$$

It follows by Remark 2.5.14, that $d_{\mathcal{F}} : C^\infty(M) \to \Omega^1(M, \mathcal{F})$ is continuous and linear even by definition.

3.3.3 Proposition (Local Representation of the Foliated Differential)

Let $(U, (x^1, ..., x^n))$ be a smooth foliated chart of $(M, \mathcal{F})$ and $f \in C^\infty(M)$, then

$$(d_{\mathcal{F}} f)|_U = \sum_{i=1}^{p} \frac{\partial f}{\partial x^i} d_{\mathcal{F}} x^i. \tag{3.27}$$

Moreover, $(U, d_{\mathcal{F}} x^1, ..., d_{\mathcal{F}} x^p)$ is a frame of $T^*\mathcal{F}$.

Proof. The foliated chart induces the frame $\left(U, \{\frac{\partial}{\partial x^i}\}_{i=1}^p\right)$ of $T\mathcal{F}$. Define sections $\lambda^1, ..., \lambda^p : U \to T^*\mathcal{F}$ by the equations $\lambda^j\left(\frac{\partial}{\partial x^j}\right) = \delta_{i,j}$ for $i, j = 1, ..., p$. To be precise, $\lambda^j_x(\nu_x) = g_i(x)$ for $x \in U$ and $\nu = \sum_{j=1}^{p} g_j \frac{\partial}{\partial x^j} \in \Gamma(U, T\mathcal{F})$, such that λ^j is a smooth section of $T^*\mathcal{F}$. Moreover, $(\lambda^1_x, ..., \lambda^p_x)$ is the dual basis of $\left(\frac{\partial}{\partial x^1}\big|_x, ..., \frac{\partial}{\partial x^p}\big|_x\right)$ for each $x \in U$, such that $(U, \{\lambda^j\}_{j=1}^p)$ is a frame of $T^*\mathcal{F}$. Since $d_{\mathcal{F}} f \in \Gamma(M, T^*\mathcal{F}) = \Omega^1(M, \mathcal{F})$, we obtain $d_{\mathcal{F}} f|_U = \sum_{j=1}^{p} c_j \lambda^j$ for some component functions $c_j \in C^\infty(U)$, which we can compute by $c_j(x) = (d_{\mathcal{F}} f)_x\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^j}\big|_x$ for $x \in U$. Finally, the foliated differential of a local coordinate yields

$$(d_{\mathcal{F}} x^i)|_U = \sum_{j=1}^{p} \frac{\partial x^i}{\partial x^j} \lambda^j = \sum_{j=1}^{p} \delta_{i,j} \lambda^j = \lambda^i. \tag{3.28}$$

Therefore, $(U, d_{\mathcal{F}} x^1, ..., d_{\mathcal{F}} x^p)$ is the dual frame of $(U, \left(U, \{\frac{\partial}{\partial x^i}\}_{i=1}^p\right))$ and in particular a frame of $T^*\mathcal{F}$ over U. $\square$

A frame of $T^*\mathcal{F}$ allows one to construct a frame of $\Lambda^k(T^*\mathcal{F})$ using the wedge-product for foliated forms, which is defined pointwise. This constructed frame will be the same as the elementary alternating k-tensors described in Remark 2.3.8 for frames instead of just a basis.[24] This gives the following corollary.

[23] See Remark 2.5.14 (section functor).

[24] See [Lee13, Proposition 14.11 (d), p. 356] for instance.

3.3.4 Corollary (Local Representation of Foliated Forms)

Let $(U, (x^1, ..., x^n))$ be a smooth foliated chart of $(M, \mathcal{F})$ and denote

$$d_{\mathcal{F}} x^I = (d_{\mathcal{F}} x^{i_1}) \wedge ... \wedge (d_{\mathcal{F}} x^{i_k}) \tag{3.29}$$

for a (not necessarily increasing) multi-index $I = (i_1, ..., i_k) \in \{1, ..., p\}^k$.
Then $\left(U, \{d_{\mathcal{F}} x^I\}_{I \in I^k_{\text{inc}}(p)}\right)$ is a local frame of $\Lambda^k(T^*\mathcal{F})$ and $\omega \in \Omega^k(M, \mathcal{F})$ has the local representation

$$\omega|_U = \sum_{I \in I^k_{\text{inc}}(p)} \omega_I \, d_{\mathcal{F}} x^I, \tag{3.30}$$

where $\omega_I \in C^\infty(U)$ is given by $\omega_I(x) = \omega_x \left(\frac{\partial}{\partial x^1}\Big|_x, ..., \frac{\partial}{\partial x^k}\Big|_x \right)$.

Differential forms can be pulled back by smooth functions. In order to pull back foliated forms, one needs smooth **foliated** maps, which are the morphisms in the category of foliated manifolds. Recall that, under the notation of leaf functions, a foliated map $f : (M, \mathcal{F}) \to (N, \mathcal{G})$ needs to satisfy:

$$\forall x \in M : f(\mathcal{F}_x) \subseteq \mathcal{G}_{f(x)}, \tag{3.31}$$

where $\mathcal{F}_x \in \mathcal{F}$ denotes the leaf with $x \in \mathcal{F}_x$ and $\mathcal{G}_{f(x)} \in \mathcal{G}$ denotes the leaf with $f(x) \in \mathcal{G}_{f(x)}$.

3.3.5 Definition (Pushforward and Pullback of Foliated Maps)

Let $f : (M, \mathcal{F}) \to (N, \mathcal{G})$ be a smooth foliated map between foliated manifolds.

(a) The **pushforward** $f_* : T\mathcal{F} \to T\mathcal{G}$ **of the foliated map** f is defined by

$$(f_*)_x : T_x\mathcal{F}_x \to T_{f(x)}\mathcal{G}_{f(x)}, \; (f_*)_x(\nu_x)(g) = \nu_x(g \circ f|_{\mathcal{F}_x}), \tag{3.32}$$

where $\mathcal{F}_x \in \mathcal{F}$ denotes the leaf containing $x \in M$, $\mathcal{G}_{f(x)} \in \mathcal{G}$ denotes the leaf containing $f(x)$ (and therefore also $f(\mathcal{F}_x)$), $\nu_x \in T_x\mathcal{F}_x$ is a derivation and $g \in C^\infty(\mathcal{G}_{f(x)})$. (Alternatively, f_* could be defined in terms of derivatives of curves through x by $(f_*)_x(\gamma'(0)) = (f \circ \gamma)'(0)$, where γ is a smooth curve in $\mathcal{F}_x$ with $\gamma(0) = x$.)

(b) The **pullback** $f^* : \Omega^k(N, \mathcal{G}) \to \Omega^k(M, \mathcal{F})$ **of the foliated map** f is defined by

$$f^* g = g \circ f, \qquad \text{if } g \in \Omega^0(N, \mathcal{G}) = C^\infty(N), \tag{3.33}$$

$$(f^*\omega)_x = (f_*)^*_x(\omega_{f(x)}), \qquad \text{if } \omega \in \Omega^k(N, \mathcal{G}), \; x \in M, \tag{3.34}$$

$$(f_*)^*(\omega_{f(x)})(\nu^1_x, ..., \nu^k_x) = \omega_{f(x)}((f_*)_x(\nu^1_x), ..., (f_*)_x(\nu^k_x)) \quad \text{for } \nu^j_x \in T_x\mathcal{F}_x. \tag{3.35}$$

Here, $(f_*)^*_x : \Lambda^k \left(T_{f(x)}\mathcal{G}_{f(x)}\right) \to \Lambda^k \left(T_x\mathcal{F}_x\right)$ is the pullback map of the linear map $(f_*)_x : T_f\mathcal{F}_x \to T_{f(x)}\mathcal{G}_{f(x)}$, which we already know from example 2.3.10 (2) (iii).

3.3.6 Lemma (Properties of Foliated Pullback)

Let $f : (M, \mathcal{F}) \to (N, \mathcal{G})$ be a smooth foliated map between foliated manifolds. Then:

(a) $d_{\mathcal{F}}(f^*g) = f^*(d_{\mathcal{G}}g)$ for all $g \in C^\infty(N)$.

(b) If $(V, \psi = (y^1, ..., y^{r+s}))$ is a smooth foliated chart of the (r, s)-foliated manifold $(N, \mathcal{G})$, we have the following local representation of $f^* : \Omega^k(N, \mathcal{G}) \to \Omega^k(M, \mathcal{F})$

$$f^* \left(\sum_{I \in I^k_{\text{inc}}(r)} g_I \, d_{\mathcal{G}} y^{i_1} \wedge ... \wedge d_{\mathcal{G}} y^{i_k} \right) = \sum_{I \in I^k_{\text{inc}}(r)} (g_I \circ f) \, d_{\mathcal{F}}(y^{i_1} \circ f) \wedge ... \wedge d_{\mathcal{F}}(y^{i_k} \circ f). \tag{3.36}$$

(c) If $g : (N, \mathcal{G}) \to (P, \mathcal{H})$ is another smooth foliated map, then

$$(g \circ f)^* = f^* \circ g^* \tag{3.37}$$

and id_M^* is the identity map on $\Omega^k(M, \mathcal{F})$.

Proof. (a) Let $g \in C^\infty(N)$, $x \in M$ and $\nu_x \in T_x\mathcal{F}(x)$. We obtain

$$(d_{\mathcal{F}}(f^*g))_x(\nu_x) = d_{\mathcal{F}}(g \circ f)_x(\nu_x) \tag{3.38}$$
$$= \nu_x((g \circ f)|_{\mathcal{F}_x}), \tag{3.39}$$

and on the other hand

$$(f^*(d_{\mathcal{G}}g))_x(\nu_x) = f^*((d_{\mathcal{G}}g)_{f(x)})(\nu_x) \tag{3.40}$$
$$= (d_{\mathcal{G}}g)_{f(x)}((f_*)_x(\nu_x) \tag{3.41}$$
$$= (f_*)_x(\nu_x)(g|_{\mathcal{G}_{f(x)}}) \tag{3.42}$$
$$= \nu_x(g|_{\mathcal{G}_{f(x)}} \circ f|_{\mathcal{F}_x}), \tag{3.43}$$

which is the same since $f(\mathcal{F}_x) \subseteq \mathcal{G}_{f(x)}$ for a foliated map f.
(b) Since f^* is linear and $f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta)$ for foliated forms on $(N, \mathcal{G})$, the formula follows by (a) using $f^*(d_{\mathcal{G}}y^j) = d_{\mathcal{F}}(f^*y^j) = d_{\mathcal{F}}(y^j \circ f)$.
(c) If $k = 0$, this is clear, hence assume $k > 0$. Note that the pushforward satisfies $((g \circ f)_*)_x = (g_*)_{f(x)} \circ (f_*)_x$ for $x \in M$. The alternating k-tensor functor of example 2.3.10 (2) (iii) is contravariant and yields therefore, $((g \circ f)_*)^*_x = (f_*)^*_x \circ (g_*)^*_{f(x)}$. For $\eta \in \Omega^k(P, \mathcal{H})$, we get

$$((g \circ f)^*\eta)_x = ((g \circ f)_*)^*_x(\eta_{g(f(x))}) = (f_*)^*_x \left((g_*)^*_{f(x)}(\eta_{g(f(x))})\right) \tag{3.44}$$
$$= (f_*)^*_x((g^*\eta)_{f(x)}) = (f^*(g^*\eta))_x, \tag{3.45}$$

which gives $(g \circ f)^* = f^* \circ g^*$. If $\mathrm{id}_M : (M, \mathcal{F}) \to (M, \mathcal{F})$ is the identity, the pushforward $(\mathrm{id}_M)_* : T\mathcal{F} \to T\mathcal{F}$ is the identity on $T\mathcal{F}$ and therefore, id_M^* is the identity map on $\Omega^k(M, \mathcal{F})$. □

In order to define a Cartan-differential for arbitrary foliated forms, we consider first the local case of a foliation and collect some properties.

3.3.7 Definition (Local Version of Foliated Cartan-Differential)

Let $V \subseteq \mathbb{R}^p$, $W \subseteq \mathbb{R}^q$ be open subsets and $(x^1, ..., x^p)$ the Euclidean basis of $\mathbb{R}^p$. Consider on $V \times W$ the V-foliation $\mathcal{F} = \mathcal{F}_W(V) = \{V \times \{w\} : w \in W\}$ by points of W. For $\omega \in \Omega^k(V \times W, \mathcal{F}_W(V))$ we have the representation

$$\omega = \sum_{I \in I^k_{\text{inc}}(p)} \omega_I \, d_{\mathcal{F}} x^I \tag{3.46}$$

and define the **foliated Cartan-differential** $d_{\mathcal{F}} : \Omega^k(V \times W, \mathcal{F}) \to \Omega^{k+1}(V \times W, \mathcal{F})$ by

$$d_{\mathcal{F}}\omega = \sum_{I \in I^k_{\text{inc}}(p)} (d_{\mathcal{F}}\omega_I) \wedge (d_{\mathcal{F}} x^I), \tag{3.47}$$

On $\Omega^0(V \times W, \mathcal{F}) = C^\infty(V \times W)$, $d_{\mathcal{F}}$ is given by the foliated differential of definition 3.3.2.

3.3.8 Proposition (Properties of Local Foliated Cartan-Differential)

In the setting of definition 3.3.7 the following holds:

(a) $d_{\mathcal{F}} : \Omega^k(V \times W, \mathcal{F}) \to \Omega^{k+1}(V \times W, \mathcal{F})$ is $\mathbb{R}$-linear.

(b) If $\omega \in \Omega^k(V \times W, \mathcal{F})$ and $\eta \in \Omega^\ell(V \times W, \mathcal{F})$, then

$$d_{\mathcal{F}}(\omega \wedge \eta) = (d_{\mathcal{F}}\omega) \wedge \eta + (-1)^k \omega \wedge (d_{\mathcal{F}}\eta). \tag{3.48}$$

(c) $d_{\mathcal{F}} \circ d_{\mathcal{F}} = 0$.

(d) The foliated Cartan-differential commutes with pullbacks of foliated maps:
If $\tilde{V} \subseteq \mathbb{R}^{\tilde{p}}$, $\tilde{W} \subseteq \mathbb{R}^{\tilde{q}}$ are open, $\tilde{\mathcal{F}} = \mathcal{F}_{\tilde{W}}(\tilde{V})$ is the $\tilde{V}$-foliation by points of $\tilde{W}$ and $f : (V \times W, \mathcal{F}) \to (\tilde{V} \times \tilde{W}, \tilde{\mathcal{F}})$ is a smooth foliated map, then

$$f^*(d_{\tilde{F}}\mu) = d_{\mathcal{F}}(f^*\mu) \quad \text{for } \mu \in \Omega^k(\tilde{V} \times \tilde{W}, \tilde{F}). \tag{3.49}$$

Proof. (a) The linearity follows by definition and linearity of the foliated differential on smooth functions.
(b) By linearity, it will be sufficient to show (b) for $\omega = \omega_I \, d_{\mathcal{F}} x^I \in \Omega^k(V \times W, \mathcal{F})$ and $\eta = \eta_J \, d_{\mathcal{F}} x^J \in \Omega^\ell(V \times W, \mathcal{F})$. Note that the product rule for functions yields

$$d_{\mathcal{F}}(\omega_I \eta_J) = d_{\mathcal{F}}\omega_I \, \eta_J + \omega_I \, d_{\mathcal{F}}\eta_J. \tag{3.50}$$

Next, we show $d_{\mathcal{F}}(\omega_I \, d_{\mathcal{F}} x^I) = (d_{\mathcal{F}}\omega_I) \wedge (d_{\mathcal{F}} x^I)$ for a (not necessarily increasing) multi-index $I \in \{1, ..., p\}^k$. If I contains repeated indices, both sides are 0. Otherwise, there exists a permutation σ sending I to an increasing multi-index $K \in I^k_{\text{inc}}(p)$ such that $d_{\mathcal{F}} x^I = \text{sgn}(\sigma) d_{\mathcal{F}} x^K$ and we obtain

$$d_{\mathcal{F}}(\omega_I \, d_{\mathcal{F}} x^I) = \text{sgn}(\sigma) d_{\mathcal{F}}(\omega_I \, d_{\mathcal{F}} x^K) = \text{sgn}(\sigma)(d_{\mathcal{F}}\omega_I) \wedge (d_{\mathcal{F}} x^K) = (d_{\mathcal{F}}\omega_I) \wedge (d_{\mathcal{F}} x^I). \tag{3.51}$$

Now, we can compute

$$d_{\mathcal{F}}(\omega \wedge \eta) = d_{\mathcal{F}}((\omega_I\, d_{\mathcal{F}}x^I) \wedge (\eta_J\, d_{\mathcal{F}}x^J)) \tag{3.52}$$
$$= d_{\mathcal{F}}(\omega_I \eta_J\, (d_{\mathcal{F}}x^I) \wedge (d_{\mathcal{F}}x^J)) \tag{3.53}$$
$$= (d_{\mathcal{F}}\omega_I\, \eta_J + \omega_I\, d_{\mathcal{F}}\eta_J) \wedge (d_F x^I) \wedge (d_{\mathcal{F}}x^J) \tag{3.54}$$
$$= (d_{\mathcal{F}}\omega_I) \wedge (d_{\mathcal{F}}x^I) \wedge (\eta_J\, dx^J) + d_{\mathcal{F}}\eta_J \wedge (\omega_I\, d_{\mathcal{F}}x^I) \wedge (d_{\mathcal{F}}x^J) \tag{3.55}$$
$$= (d_{\mathcal{F}}\omega) \wedge \eta + (-1)^k \omega \wedge (d_{\mathcal{F}}\eta_J \wedge d_{\mathcal{F}}x^J) \tag{3.56}$$
$$= (d_{\mathcal{F}}\omega) \wedge \eta + (-1)^k \omega \wedge (d_{\mathcal{F}}\eta). \tag{3.57}$$

(c) For $f \in C^\infty(V \times W) = \Omega^0(V \times W, \mathcal{F})$ we obtain by Schwarz's Theorem,

$$d_{\mathcal{F}}(d_{\mathcal{F}}f) = d_{\mathcal{F}}\left(\sum_{j=1}^{p} \frac{\partial f}{\partial x^j} d_{\mathcal{F}}x^j\right) \tag{3.58}$$
$$= \sum_{j=1}^{p}\sum_{i=1}^{p} \frac{\partial^2 f}{\partial x^i \partial x^j}(d_{\mathcal{F}}x^i) \wedge (d_{\mathcal{F}}x^j) \tag{3.59}$$
$$= \sum_{1 \le i < j \le p} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i}\right)(d_{\mathcal{F}}x^i) \wedge (d_{\mathcal{F}}x^j) = 0. \tag{3.60}$$

In particular, $d_{\mathcal{F}}(d_{\mathcal{F}}x^i)) = 0$. More generally, we show $d_{\mathcal{F}}(d_{\mathcal{F}}x^I) = 0$ for $I \in I^k_{\text{inc}}(p)$ by induction over k. Assume this is the case for $k-1$, then by the anti-derivation property (b), we obtain for $I = (i_1, ..., i_k) \in I^k_{\text{inc}}(p)$ and $J = (i_2, ..., i_k) \in I^{k-1}_{\text{inc}}(p)$,

$$d_{\mathcal{F}}(d_{\mathcal{F}}x^I) = d_{\mathcal{F}}((d_{\mathcal{F}}x^{i_1}) \wedge (d_{\mathcal{F}}x^J)) \tag{3.61}$$
$$= d_{\mathcal{F}}(d_{\mathcal{F}}x^{i_1}) \wedge (d_{\mathcal{F}}x^J) - (d_{\mathcal{F}}x^{i_1}) \wedge d_{\mathcal{F}}(d_{\mathcal{F}}x^J) = 0. \tag{3.62}$$

Together with (b) this yields for $\omega = \omega_I\, dx^I \in \Omega^k(V \times W, \mathcal{F})$,

$$d_{\mathcal{F}}(d_{\mathcal{F}}\omega) = d_{\mathcal{F}}\left((d_{\mathcal{F}}\omega_I) \wedge (d_{\mathcal{F}}x^I)\right) \tag{3.63}$$
$$= d_{\mathcal{F}}(d_{\mathcal{F}}\omega_I) \wedge (d_{\mathcal{F}}x^I) - (d_{\mathcal{F}}\omega_I) \wedge d_{\mathcal{F}}(d_{\mathcal{F}}x^I) = 0. \tag{3.64}$$

By linearity of $d_{\mathcal{F}}$, (c) follows.
(d) Denote the Euclidean basis of $\mathbb{R}^{\tilde{p}}$ by $(y^1, ..., y^{\tilde{p}})$. Again by linearity, it sufficies to consider only forms $\mu = \mu_I\, dy^I \in \Omega^k(\tilde{V} \times \tilde{W}, \tilde{\mathcal{F}})$. We get by Lemma 3.3.6 (a)

$$f^*(d_{\tilde{\mathcal{F}}}\mu) = f^*((d_{\tilde{\mathcal{F}}}\mu_I) \wedge (d_{\tilde{\mathcal{F}}}y^I)) \tag{3.65}$$
$$= d_{\mathcal{F}}(\mu_I \circ f) \wedge d_{\mathcal{F}}(y^{i_1} \circ f) \wedge ... \wedge d_{\mathcal{F}}(y^{i_k} \circ f). \tag{3.66}$$

On the other hand, Lemma 3.3.6 (b) yields

$$d_{\mathcal{F}}(f^*\mu) = d_{\mathcal{F}}\left((\mu_I \circ f)\, d_{\mathcal{F}}(y^{i_1} \circ f) \wedge ... \wedge d_{\mathcal{F}}(y^{i_k} \circ f)\right) \tag{3.67}$$
$$= d_{\mathcal{F}}(\mu_I \circ f) \wedge d_{\mathcal{F}}(y^{i_1} \circ f) \wedge ... \wedge d_{\mathcal{F}}(y^{i_k} \circ f). \tag{3.68}$$

□

3.3.9 Theorem (Existence and Uniqueness of Foliated Cartan-Differential)

Let $(M,\mathcal{F})$ be a smooth (p,q)-foliated manifold. For each $k \in \mathbb{N}_0$ there is a unique map $d_{\mathcal{F}} : \Omega^k(M,\mathcal{F}) \to \Omega^{k+1}(M,\mathcal{F})$, called the **foliated Cartan-differential**, satisfying:

(a) $d_{\mathcal{F}}$ is $\mathbb{R}$-linear.

(b) If $\omega \in \Omega^k(M,\mathcal{F})$ and $\eta \in \Omega^\ell(M,\mathcal{F})$, then

$$d_{\mathcal{F}}(\omega \wedge \eta) = (d_{\mathcal{F}}\omega) \wedge \eta + (-1)^k \omega \wedge (d_{\mathcal{F}}\eta). \tag{3.69}$$

(c) $d_{\mathcal{F}} \circ d_{\mathcal{F}} = 0$.

(d) For $k = 0$, $d_{\mathcal{F}}$ is the foliated differential of definition 3.3.2.

(e) For $k \geq 1$ and a foliated chart $(U,(x^1,...,x^n))$ of $(M,\mathcal{F})$, $d_{\mathcal{F}}$ is locally given by

$$d_{\mathcal{F}}\left(\sum_{I \in I^k_{\mathrm{inc}}(p)} \omega_I \, d_{\mathcal{F}}x^I\right) = \sum_{I \in I^k_{\mathrm{inc}}(p)} (d_{\mathcal{F}}\omega_I) \wedge (d_{\mathcal{F}}x^I). \tag{3.70}$$

(f) $d_{\mathcal{F}}$ is continuous.

(g) The foliated Cartan-differential commutes with pullpacks of foliated maps:
If $f : (M,\mathcal{F}) \to (N,\mathcal{G})$ is a smooth foliated map between smooth foliated manifolds, then

$$f^*(d_{\mathcal{G}}\mu) = d_{\mathcal{F}}(f^*\mu) \quad \text{for } \mu \in \Omega^k(N,\mathcal{G}). \tag{3.71}$$

Proof. We start with existence. If $\omega \in \Omega^k(M,\mathcal{F})$ and $\varphi : U \to V \times W$ is a foliated chart of $(M,\mathcal{F})$, then $(\varphi^{-1})^*\omega|_U \in \Omega^k(V \times W, \mathcal{F}_W(V))$ and we define the foliated Cartan-differential by

$$(d_{\mathcal{F}}\omega)|_U = \varphi^* d_{\mathcal{F}_W(V)}((\varphi^{-1})^*\omega|_U). \tag{3.72}$$

For well-definition, suppose $\psi : U \to \tilde{V} \times \tilde{W}$ is also a foliated chart of $(M,\mathcal{F})$. Then $\psi \circ \varphi^{-1}$ is a foliated diffeomorphism from $(V \times W, \mathcal{F}_W(V))$ to $(\tilde{V} \times \tilde{W}, \mathcal{F}_{\tilde{W}}(\tilde{V}))$. Proposition 3.3.8 (d) and $(\psi \circ \varphi^{-1})^* \circ (\psi^{-1})^* = (\varphi^{-1})^*$ yields

$$(\psi \circ \varphi^{-1})^* d_{\mathcal{F}_{\tilde{W}}(\tilde{V})}((\psi^{-1})^*\omega|_U) = d_{\mathcal{F}_W(V)}((\varphi^{-1})^*\omega|_U). \tag{3.73}$$

Together with $\varphi^* \circ (\psi \circ \varphi^{-1})^* = \psi^*$, we obtain

$$\psi^* d_{\mathcal{F}_{\tilde{W}}(\tilde{V})}((\psi^{-1})^*\omega|_U) = \varphi^* d_{\mathcal{F}_W(V)}((\varphi^{-1})^*\omega|_U). \tag{3.74}$$

Now, properties (a)-(d) follow by definition and proposition 3.3.8 (a)-(c).
Before proving the remaining properties, we show that $d_{\mathcal{F}}$ is already uniquely defined.

Suppose d is any map satisfying properties (a)-(d). Then d is locally determined, i.e. if $\omega_1, \omega_2 \in \Omega^k(M, \mathcal{F})$ satisfy $\omega_1|_U = \omega_2|_U$ on some open $U \subseteq M$, then $(d\omega_1)|_U = (d\omega_2)|_U$. Let $x \in U$ be arbitrary and $\chi \in C^\infty(M)$ be a bump function which is identically 1 on some open neighbourhood of x with $\operatorname{supp}(\chi) \subseteq U$. For $\eta = \omega_1 - \omega_2$ we obtain that $\chi\eta$ is identically 0 everywhere. By properties (b) and (d): $0 = d(\chi\eta) = d_\mathcal{F}\chi \wedge \eta + \chi d\eta$. Using $\chi(x) = 1$, $(d_\mathcal{F}\chi)_x = 0$ and (a), we get $(d\omega_1)|_x - (d\omega_2)_x = (d_\eta)_x = 0$.
Let $\omega \in \Omega^k(M, \mathcal{F})$ and $(U, \varphi = (x^1, ..., x^n))$ be a foliated chart of $(M, \mathcal{F})$ such that

$$\omega|_U = \sum_{I \in I^k_{\text{inc}}(p)} \omega_I \, d_\mathcal{F} x^I. \tag{3.75}$$

For any $x \in U$ and a bump function $\chi \in C^\infty(M)$ which is identically 1 on an open neighbourhood $V \subseteq U$ of x with $\operatorname{supp}(\chi) \subseteq U$, we can extend $\omega_I|_V$ and $x^i|_V$ to global smooth functions $\tilde{\omega}_I, \tilde{x}^i \in C^\infty(M)$. Setting $d_\mathcal{F}\tilde{x}^I = d_\mathcal{F}\tilde{x}^{i_1} \wedge \cdots \wedge d_\mathcal{F}\tilde{x}^{i_k} \in \Omega^k(M, \mathcal{F})$ for a multi-index $I = (i_1, ..., i_k)$, we obtain

$$\tilde{\omega} = \sum_{I \in I^k_{\text{inc}}(p)} \tilde{\omega}_I \, d_\mathcal{F}\tilde{x}^I \in \Omega^k(M, \mathcal{F}) \text{ with } \tilde{\omega}|_V = \omega|_V. \tag{3.76}$$

By (b)-(d), it follows that $d(\tilde{\omega}_I \, d_\mathcal{F}\tilde{x}^I) = (d_\mathcal{F}\tilde{\omega}_I) \wedge (d_\mathcal{F}\tilde{x}^I)$. Since d is locally determined, this gives with (a) for the evaluation in x

$$(d\omega)_x = (d\tilde{\omega})_x = \sum_{I \in I^k_{\text{inc}}(p)} (d_\mathcal{F}\omega_I)_x \wedge (d_\mathcal{F}x^I)_x. \tag{3.77}$$

Since $x \in U$ was arbitrary, we have proven uniqueness and (e).
The continuity of $d_\mathcal{F}$ follows by (e). To see this elementary, use the representation of $d_\mathcal{F}\omega_I$ and compute the component functions of $(d\omega)|_U$ with respect to the frame $\left(U, \{d_\mathcal{F}x^J\}_{J \in I^{k+1}_{\text{inc}}(p)}\right)$. These component functions are less than p many plus and minus combinations of some $\frac{\partial \omega_I}{\partial x^j}$ such that the order of differentiation is only increased by one. Hence, increasing the maximal differentiation order of a seminorm by one and setting the continuity constant equals p will yield continuity locally, which suffices to show for (f).
To prove (g), let $\varphi : U \to V \times W$ and $\psi : \tilde{U} \to \tilde{V} \times \tilde{W}$ be smooth foliated charts of $(M, \mathcal{F})$ and $(N, \mathcal{G})$, respectively. The coordinate representation $\psi \circ f \circ \varphi^{-1}$ is then a smooth foliated map from $(V \times W, \mathcal{F}_W(V))$ to $(\tilde{V} \times \tilde{W}, \mathcal{F}_{\tilde{W}}(\tilde{V}))$ such that we can apply 3.3.8 (d). Together with (3.72) twice, this gives for $\omega \in \Omega^k(N, \mathcal{G})$ on $U \cap f^{-1}(\tilde{U})$:

$$f^*(d_\mathcal{G}\omega) = f^*\psi^* d_{\mathcal{F}_{\tilde{W}}(\tilde{V})}((\psi^{-1})^*\omega) \tag{3.78}$$
$$= \varphi^* \circ (\psi \circ f \circ \varphi^{-1})^* d_{\mathcal{F}_{\tilde{W}}(\tilde{V})}((\psi^{-1})^*\omega) \tag{3.79}$$
$$= \varphi^* d_{\mathcal{F}_W(V)}((\psi \circ f \circ \varphi^{-1})^*(\psi^{-1})^*\omega) \tag{3.80}$$
$$= \varphi^* d_{\mathcal{F}_W(V)}((\varphi^{-1})^* f^*\omega) \tag{3.81}$$
$$= d_\mathcal{F}(f^*\omega). \tag{3.82}$$

□

3.3.10 Corollary (Connection with Cartan-Differential)

Considering the one leaf foliation $\{M\}$ of a smooth manifold M, there exists a unique map $d : \Omega^k(M) \to \Omega^{k+1}(M)$ with likewise properties (a)-(g) of Theorem 3.3.9, called the (classical) **Cartan-differential**.
If $(M, \mathcal{F})$ is a foliated manifold and $\varrho^k = (i^*_{T\mathcal{F}})_* : \Omega^k(M) \to \Omega^k(M, \mathcal{F})$ denotes the push-forward (as in the definition of the section functor) of the pullback of alternating k-tensor bundles, $i^*_{T\mathcal{F}} : \Lambda^k(T^*M) \to \Lambda^k(T^*\mathcal{F})$, the following diagram commutes:

$$\begin{array}{ccc} \Omega^k(M) & \xrightarrow{d} & \Omega^{k+1}(M) \\ \downarrow{\scriptstyle \varrho^k} & & \downarrow{\scriptstyle \varrho^{k+1}} \\ \Omega^k(M, \mathcal{F}) & \xrightarrow[d_{\mathcal{F}}]{} & \Omega^{k+1}(M, \mathcal{F}). \end{array}$$

Proof. The existence and uniqueness of d is a consequence of the considered special case of a one leaf foliation since foliated differential forms are just differential forms. For the second part, let $(M, \mathcal{F})$ be a foliated manifold. Since each leave $\mathcal{L}_\alpha$ of $\mathcal{F}$ is contained in M, the identity map $\mathrm{id}_M : (M, \mathcal{F}) \to (M, \{M\})$ is a smooth foliated map. Its pushforward $(\mathrm{id}_M)_* : T\mathcal{F} \to TM$ is just the inclusion morphism $i_{T\mathcal{F}}$ of vector bundles and its pullback $\mathrm{id}^*_M : \Omega^k(M) \to \Omega^k(M, \mathcal{F})$ agrees with the (section functor) pushforward of the pullback of $i^*_{T\mathcal{F}} : \Lambda^k(T^*M) \to \Lambda^k(T^*\mathcal{F})$, such that $\varrho^k = \mathrm{id}^*_M$ and we obtain the commutativity of the diagram as a special case of Theorem 3.3.9 (g). $\square$

3.4 Foliated De Rham Cohomology

The de Rham cohomology of a smooth manifold is a useful tool to classify smooth manifolds and gives information about their geometry and topology as well. Although, the foliated de Rham cohomology will not be that illuminative for foliated manifolds, we will develop tools to compute the foliated cohomology in some cases. The main results in this chapter are the integrable homotopy invariance (Corollary 3.4.9), the Poincaré Lemma (Lemma 3.4.10), the Mayer-Vietoris Theorem for foliated cohomology (Theorem 3.4.11) and Theorem 3.4.14 to compute the foliated de Rham cohomology of foliations by points.

3.4.1 Definition (Foliated de Rham Cohomology)

Let $(M, \mathcal{F})$ be a smooth (p, q)-foliated manifold. A foliated k-form $\omega \in \Omega^k(M, \mathcal{F})$ with $d_{\mathcal{F}}\omega = 0$ will be called **closed**. A foliated k-form $\eta \in \Omega^k(M, \mathcal{F})$ will be called **exact** if there is a foliated $(k-1)$-form $\mu \in \Omega^{k-1}(M, \mathcal{F})$ with $\eta = d_{\mathcal{F}}\mu$. We define the spaces of closed and exact foliated k-forms by

$$\mathcal{Z}^k(M, \mathcal{F}) = \operatorname{Ker}(d_{\mathcal{F}} : \Omega^k(M, \mathcal{F}) \to \Omega^{k+1}(M, \mathcal{F})) = \{\text{closed } k\text{-forms on } (M, \mathcal{F})\}, \quad (3.83)$$

$$\mathcal{B}^k(M, \mathcal{F}) = \operatorname{Im}(d_{\mathcal{F}} : \Omega^{k-1}(M, \mathcal{F}) \to \Omega^k(M, \mathcal{F})) = \{\text{exact } k\text{-forms on } (M, \mathcal{F})\}. \quad (3.84)$$

Both are $\mathbb{R}$-vector spaces by linearity of $d_{\mathcal{F}}$. If $k < 0$ or $k > p$, $\Omega^k(M, \mathcal{F})$ is just the zero space such that $\mathcal{Z}^p(M, \mathcal{F}) = \Omega^p(M, \mathcal{F})$ and $\mathcal{B}^0(M, \mathcal{F}) = 0$, in particular. Since $d_{\mathcal{F}} \circ d_{\mathcal{F}} = 0$, we have $\mathcal{B}^k(M, \mathcal{F}) \subseteq \mathcal{Z}^k(M, \mathcal{F})$, i.e. every exact form is closed. This allows the definition of the kth **foliated de Rham cohomology group of** $(M, \mathcal{F})$ to be the quotient vector space

$$H^k(M, \mathcal{F}) = {}^{\mathcal{Z}^k(M, \mathcal{F})}\!/\!_{\mathcal{B}^k(M, \mathcal{F})}. \quad (3.85)$$

(It is indeed an $\mathbb{R}$-vector space and in particular a group under its addition. But most cohomology theories produce only groups, such that we use the traditional term of a cohomology group and keep in mind, that we are dealing with vector spaces.)
If $k < 0$ or $k > p$, it follows $H^k(M, \mathcal{F}) = 0$ by $\Omega^k(M, \mathcal{F}) = 0$.
Further, $H^0(M, \mathcal{F}) = \mathcal{Z}^0(M, \mathcal{F}) = \{f \in C^\infty(M) : f|_{\mathcal{L}} \text{ is constant for each } \mathcal{L} \in \mathcal{F}\}$.
(Since $\frac{\partial f}{\partial x^j} = 0$ for $j = 1, ..., p$ in every foliated chart $(U, (x^1, ..., x^n))$ and leaves are connected.)

3.4.2 Example (Foliation by Points)

Let F and T be (connected) smooth manifolds of dimension p and q, respectively. Consider the F-foliation $\mathcal{F}_T(F) = \{F \times \{t\} : t \in T\}$ by points of T on $F \times T$. Then,

$$H^0(F \times T, \mathcal{F}_T(F)) \cong C^\infty(T). \quad (3.86)$$

(The smooth projection $\pi_T : F \times T \to T,\ (x, t) \mapsto t$ between manifolds induces the pullback $\pi_T^* : C^\infty(T) \to C^\infty(F \times T)$, $\pi_T^* g = g \circ \pi_T$. This map is linear and also injective since π_T

is surjective. We show $\mathrm{Im}(\pi_T^*) = H^0(F \times T, \mathcal{F}_T(F))$. If $g \in C^\infty(T)$, then $g \circ \pi_T|_{F\times\{t\}}$ is of constant value $g(t)$ for each $t \in T$. Otherwise, if $f \in C^\infty(F \times T)$ such that $f|_{F\times\{t\}}$ is of constant value c_t for each $t \in T$, we can define a smooth map $g : T \to \mathbb{R}$ by $g(t) = c_t$ satisfying $g \circ \pi_T = f$. Hence, π_T^* is an isomorphism of vector spaces from $C^\infty(T)$ to $H^0(F \times T, \mathcal{F}_T(F))$.)

3.4.3 Proposition (Induced Cohomology Maps and Foliated de Rham Functor)

For a smooth foliated map $f : (M, \mathcal{F}) \to (N, \mathcal{G})$ between smooth foliated manifolds, the pullback $f^* : \Omega^k(N, \mathcal{G}) \to \Omega^k(M, \mathcal{F})$ maps $\mathcal{Z}^k(N, \mathcal{G})$ into $\mathcal{Z}^k(M, \mathcal{F})$ and $\mathcal{B}^k(N, \mathcal{G})$ into $\mathcal{B}^k(M, \mathcal{F})$. This gives rise to a well-defined linear map between cohomologies

$$f^* : H^k(N, \mathcal{G}) \to H^k(M, \mathcal{F}),\ f^*[\omega] = [f^*\omega], \tag{3.87}$$

called **induced cohomology map**. Together with the assignment $(M, \mathcal{F}) \mapsto H^k(M, \mathcal{F})$, this defines a contravariant functor, called **foliated de Rham functor**, from the category **FolMfld** of smooth foliated manifolds with smooth foliated maps to the category $\mathbf{Vect}_{\mathbb{R}}$ of $\mathbb{R}$-vector spaces with linear maps, i.e.

(a) if $g : (N, \mathcal{G}) \to (P, \mathcal{H})$ is another smooth foliated map, then

$$(g \circ f)^* = f^* \circ g^* : H^k(P, \mathcal{H}) \to H^k(M, \mathcal{F}) \tag{3.88}$$

(b) and id_M^* is the identity map on $H^k(M, \mathcal{F})$.

Proof. By naturality with pullbacks (proposition 3.3.9 (g)), we obtain for $\omega \in \mathcal{Z}^k(N, \mathcal{G})$, $d_{\mathcal{F}}(f^*\omega) = f^*(d_{\mathcal{G}}\omega) = 0$ and if $\eta \in \Omega^{k-1}(N, \mathcal{G})$, then $f^*(d_{\mathcal{G}}\eta) = d_{\mathcal{F}}(f^*\eta) \in B^k(M, \mathcal{F})$ since $f^*\eta \in \Omega^{k-1}(M, \mathcal{F})$. Further, if $\tilde{\omega} = \omega + d_{\mathcal{G}}\eta \in \mathcal{Z}^k(N, \mathcal{G})$, then $[f^*\tilde{\omega}] = [f^*\omega + d(f^*\eta)] = [f^*\omega]$. Thus, the induced cohomology map is well defined. (a) and (b) follow by Lemma 3.3.6 (c). □

3.4.4 Corollary (Foliated Diffeomorphism Invariance)

If $f : (M, \mathcal{F}) \to (N, \mathcal{G})$ is a smooth foliated diffeomorphism between smooth foliated manifolds, then

$$H^k(M, \mathcal{F}) \cong H^k(N, \mathcal{G}) \text{ for all } k \in \mathbb{N}_0. \tag{3.89}$$

Proof. For every $k \in \mathbb{N}_0$, the induced cohomology maps of f and its inverse are isomorphisms of vector spaces by proposition 3.4.3 (a) and (b). □

The de Rham groups of homotopy equivalent manifolds (without a foliation) are also isomorphic (as vector spaces). Regarding the next example, even a smooth foliated homotopy map will not yield an invariance. We need a stronger type of homotopy, which we will call **integrable homotopy**. This type of homotopy was introduced by Haefliger, [Hae71] and named *homotopy intégrable* by El Kacimi-Alaoui, [KA83]. We will show, that integrable homotopic maps induce the same cohomology maps. For that purpose, we use a homotopy operator.

3.4.5 Example (Foliated De Rham Cohomology Not a Homotopy Invariance)

Consider $M = S^1 \times \mathbb{R}$ with the S^1-foliation $\mathcal{F}_M = \mathcal{F}_\mathbb{R}(S^1)$ by points of $\mathbb{R}$ and $S = S^1 \times \{0\}$ with the 1-leaf foliation $\mathcal{F}_S = \{S\}$. Then, S is a deformation retract of M but

$$H^k(M, \mathcal{F}_M) \cong C^\infty(\mathbb{R}) \not\cong \mathbb{R} \cong H^k(S, \mathcal{F}_S) \text{ for } k = 0, 1. \tag{3.90}$$

Proof. Denote the inclusion from S into M by i_S and let $r : M \to S$, $r(z, s) = (z, 0)$. Then, $r \circ i_S = \mathrm{id}_S$ and we show, that $i_S \circ r$ is homotopic to id_M. Let be $\psi \in C^\infty(\mathbb{R})$ with $\psi(t) = 0$ if $t \leq 0$ and $\psi(t) = 1$ if $t \geq 1$. Define $H : M \times \mathbb{R} \to M$ by $H((z, s), t) = (z, \psi(t)s)$ such that $H((z, s), t) = (z, 0) = i_S \circ r(z, s)$ for $t \leq 0$ and $H((z, s), t) = (z, s) = \mathrm{id}_M(z, s)$ for $k \geq 1$. Note that H is a smooth map and if one considers the foliation $\mathcal{G} = \{S^1 \times \{(s, t)\} : (s, t) \in \mathbb{R}^2\}$ on $M \times \mathbb{R}$, H is also a foliated map between $(M \times \mathbb{R}, \mathcal{G}) \to (M, \mathcal{F}_M)$, mapping the leaf $S^1 \times \{(s, t)\}$ into the leaf $S^1 \times \{\psi(t)s\}$. In summary, S is a deformation retract of M, where the used homotopy map is also a smooth foliated map.
The map $f : (S, \mathcal{F}_S) \to (S^1, \{S^1\})$, $f(z, 0) = z$ is clearly a smooth foliated diffeomorphism such that $H^k(S, \mathcal{F}_S) = H^k(S^1) \cong \mathbb{R}$ for $k = 0, 1$ because the foliated de Rham cohomology of the 1-leaf foliation is just the usual de Rham cohomology. On the other hand, we know $H^0(M, \mathcal{F}_M) \cong C^\infty(\mathbb{R})$ by example 3.4.2. For $k = 1$, we refer to the later result 3.4.14. □

3.4.6 Definition (Integrable Homotopy)

(a) Let $f, g : (M, \mathcal{F}) \to (N, \mathcal{G})$ be smooth foliated maps between smooth foliated manifolds and let $\pi : M \times \mathbb{R} \to M$ be the projection map. We call f **integrable homotopic to** g, if there is a smooth foliated map $H : (M \times \mathbb{R}, \pi^*(\mathcal{F})) \to (N, \mathcal{G})$ such that

$$H(x, t) = f(x), \quad \text{if } t \leq 0 \text{ and} \tag{3.91}$$

$$H(x, t) = g(x), \quad \text{if } t \geq 1. \tag{3.92}$$

In this case, H is called an **integrable homotopy from** f **to** g.

(b) Two smooth foliated manifolds $(M, \mathcal{F})$ and $(N, \mathcal{G})$ will be called **integrable homotopy equivalent** if there exist two smooth foliated maps $f : (M, \mathcal{F}) \to (N, \mathcal{G})$ and $g : (N, \mathcal{G}) \to (M, \mathcal{F})$ such that $g \circ f$ is integrable homotopic to id_M and $f \circ g$ is integrable homotopic to id_N. Then, f and g are called **integrable homotopy inverse**.

(c) Let $(M, \mathcal{F})$ be a smooth foliated manifold and S a smooth immersed submanifold of M such that the inclusion map $i_S : S \to M$ is transverse to $\mathcal{F}$, i.e. for all $x \in S$, we have $T_x M = d\, i_S(T_x S) + T_x \mathcal{F}_x$. A smooth foliated map $r : (M, \mathcal{F}) \to (S, \mathcal{F}|_S)$ will be called an **integrable deformation retraction** if i_S and r are integrable homotopy inverse, where $\mathcal{F}|_S = i_S^*(\mathcal{F})$ is the considered foliation on S, given by the connected components of all leaf intersections with S. In this case, S is called an **integrable deformation retract** of $(M, \mathcal{F})$.

Recall from example 3.1.7, (3), that $\pi^*(\mathcal{F}) = \{\mathcal{L} \times \mathbb{R} : \mathcal{L} \in \mathcal{F}\}$ satisfies:

(∗) For any $t \in \mathbb{R}$ the smooth injective inclusion map $J_t : M \to M \times \mathbb{R}$, $x \mapsto (x, t)$ is also a foliated map from $(M, \mathcal{F})$ to $(M \times \mathbb{R}, \pi^*(\mathcal{F}))$.

3.4.7 Proposition (Existence of a Homotopy Operator)

Let $(M, \mathcal{F})$ be a smooth (p,q)-foliated manifold and $\pi : M \times \mathbb{R} \to M$ the projection map. Then, for each $k \in \mathbb{N}_0$ there is a linear map $h : \Omega^k(M \times \mathbb{R}, \pi^*(\mathcal{F})) \to \Omega^{k-1}(M, \mathcal{F})$ satisfying

$$d_{\mathcal{F}} \circ h + h \circ d_{\pi^*(\mathcal{F})} = J_1^* - J_0^*, \tag{3.93}$$

where $J_0^*, J_1^* : \Omega^k(M \times \mathbb{R}, \pi^*(\mathcal{F})) \to \Omega^k(M, \mathcal{F})$ are the foliated pullback maps of the inclusion maps mentioned in $(*)$ and $\Omega^{-1}(M, \mathcal{F}) = \{0\}$ by convention. Such a map h is called a homotopy operator between J_0 and J_1.

Proof. (1) We start with the **local version**.
Let $V \subseteq \mathbb{R}^p$, $W \subseteq \mathbb{R}^q$ be open subsets and $(x^1, ..., x^p)$ the Euclidean basis of $\mathbb{R}^p$. Consider on $M = V \times W$ the V-foliation $\mathcal{F} = \mathcal{F}_W(V) = \{V \times \{w\} : w \in W\}$ by points of W. Then, $\pi^*(\mathcal{F}) = \{V \times \{w\} \times \mathbb{R} : w \in W\}$ is the pullback foliation on $M \times \mathbb{R}$. If $f \in C^\infty(M \times \mathbb{R})$, set $h(f) = 0$. For $k \in \mathbb{N}$, any $\omega \in \Omega^k(M \times \mathbb{R}, \pi^*(\mathcal{F}))$ can be written as a linear combination of the following two types of k-forms:

(α) $\alpha = a\, d_{\pi^*(\mathcal{F})}x^{i_1} \wedge \cdots \wedge d_{\pi^*(\mathcal{F})}x^{i_k}$, where $a \in C^\infty(M \times \mathbb{R})$;

(β) $\beta = b\, d_{\pi^*(\mathcal{F})}t \wedge d_{\pi^*(\mathcal{F})}x^{i_1} \wedge \cdots \wedge d_{\pi^*(\mathcal{F})}x^{i_{k-1}}$, where $b \in C^\infty(M \times \mathbb{R})$.

We set $h(\alpha) = 0$ and $h(\beta) = \int_0^1 b(\cdot, t)dt\, d_{\pi^*(\mathcal{F})}x^{i_1} \wedge \cdots \wedge d_{\pi^*(\mathcal{F})}x^{i_{k-1}} \in \Omega^{k-1}(M, \mathcal{F})$. Define $h : \Omega^k(M \times \mathbb{R}, \pi^*(\mathcal{F})) \to \Omega^{k-1}(M, \mathcal{F})$ by linear extension. Then,

$$(d_{\mathcal{F}} \circ h + h \circ d_{\pi^*(\mathcal{F})})(f) = \int_0^1 \frac{\partial f}{\partial t} dt = f(\cdot, 1) - f(\cdot, 0) \tag{3.94}$$

$$= f \circ J_1 - f \circ J_0 = (J_1^* - J_0^*)(f), \quad \text{if } f \in C^\infty(M \times \mathbb{R}). \tag{3.95}$$

Further, $d_{\mathcal{F}} \circ h(\alpha) = 0$ and

$$h \circ d_{\pi^*(\mathcal{F})}(\alpha) = \left(\int_0^1 \frac{\partial a}{\partial t} dt \right) d_{\pi^*(\mathcal{F})}x^{i_1} \wedge \cdots \wedge d_{\pi^*(\mathcal{F})}x^{i_k} = (J_1^* - J_0^*)(\alpha). \tag{3.96}$$

Moreover, $J_1^*(\beta) = J_0^*(\beta) = 0$ and

$$d_{\mathcal{F}} \circ h(\beta) = \sum_{j=1^p} \left(\int_0^1 \frac{\partial b}{\partial x^j} dt \right) d_{\pi^*(\mathcal{F})}x^j \wedge d_{\pi^*(\mathcal{F})}x^{i_1} \wedge \cdots \wedge x^{i_{k-1}}, \tag{3.97}$$

$$h \circ d_{\pi^*(\mathcal{F})}(\beta) = h \left(\sum_{j=1}^p \frac{\partial b}{\partial x^j} d_{\pi^*(\mathcal{F})}x^j \wedge d_{\pi^*(\mathcal{F})}t \wedge d_{\pi^*(\mathcal{F})}x^{i_1} \wedge \cdots \wedge d_{\pi^*(\mathcal{F})}x^{i_{k-1}} \right) \tag{3.98}$$

$$= -\sum_{j=1}^p \left(\int_0^1 \frac{\partial b}{\partial x^j} dt \right) d_{\pi^*(\mathcal{F})}x^j \wedge d_{\pi^*(\mathcal{F})}x^{i_1} \wedge \cdots \wedge d_{\pi^*(\mathcal{F})}x^{i_{k-1}}, \tag{3.99}$$

where the minus sign comes from $d_{\pi^*(\mathcal{F})}x^j \wedge d_{\pi^*(\mathcal{F})}t = -d_{\pi^*(\mathcal{F})}t \wedge d_{\pi^*(\mathcal{F})}x^j$. Hence, (3.93) follows in the considered special case.

(2) Before we treat the global case, we look at **foliated transition functions**:
Let $V, \tilde{V} \subseteq \mathbb{R}^p$, $W, \tilde{W} \subseteq \mathbb{R}^q$ be open sets and $\tau : (V \times W, \mathcal{F}_W(V)) \to (\tilde{V} \times \tilde{W}, \mathcal{F}_{\tilde{W}}(\tilde{V}))$ be a foliated diffeomorphism. Then,

$$\tilde{\tau} : (V \times W \times \mathbb{R}, \pi^*(\mathcal{F}_W(V))) \to (\tilde{V} \times \tilde{W} \times \mathbb{R}, \tilde{\pi}^*(\mathcal{F}_{\tilde{W}}(\tilde{V}))),\ \tilde{\tau}(u, v, t) = (\tau(u, v), t) \tag{3.100}$$

defines a foliated diffeomorphism, where π and $\tilde{\pi}$ denote the projections onto $V \times W$ and $\tilde{V} \times \tilde{W}$, respectively. Moreover, we get on $\Omega^k(\tilde{V} \times \tilde{V} \times \mathbb{R}, \tilde{\pi}(\mathcal{F}_{\tilde{W}}(V)))$ for each $k \in \mathbb{N}_0$:

$$\tau^* \circ h = h \circ \tilde{\tau}^*. \tag{3.101}$$

Indeed, on smooth functions, both sides are 0. $\tilde{\tau}^*$ maps k-forms of type α to k-forms of type α such that in this case also both sides are 0. As well, $\tilde{\tau}^*$ preserves k-forms of type β and it is easy to compute the component function of both sides equal $\int_0^1 b(\varphi(\cdot), t)dt$.

(3) Now, consider the **global case**. Define $h : \Omega^k(M \times \mathbb{R}, \pi^*(\mathcal{F})) \to \Omega^{k-1}(M, \mathcal{F})$ in terms of foliated charts, i.e. if $\varphi : U \to V \times W$ is a foliated chart of $(M, \mathcal{F})$, we set

$$(h(\omega))|_U = \varphi^* \circ h \circ (\tilde{\varphi}^{-1})^*(\omega|_{U \times \mathbb{R}}). \tag{3.102}$$

Here, $\tilde{\varphi} : U \times \mathbb{R} \to V \times W \times \mathbb{R}$, $\tilde{\varphi}(x, t) = (\varphi(x), t)$ and h on the right side is the map from (1). For well-definition, suppose there is another foliated chart $\psi : U \to \tilde{V} \times \tilde{W}$. Since $\tau = \psi \circ \varphi^{-1}$ is a foliated diffeomorphism as desired in (2), (3.101) gives together with $\tilde{\tau}^* \circ (\tilde{\psi}^{-1})^* = (\tilde{\varphi}^{-1})^*$:

$$\tau^* \circ h \circ (\tilde{\psi}^{-1})^*(\omega|_{U \times \mathbb{R}}) = h \circ (\tilde{\varphi}^{-1})^*(\omega|_{U \times \mathbb{R}}). \tag{3.103}$$

We obtain by $\varphi^* \circ (\psi \circ \varphi^{-1})^* = \psi^*$,

$$\psi^* \circ h \circ (\tilde{\psi}^{-1})^*(\omega|_{U \times \mathbb{R}}) = \varphi^* \circ h \circ (\tilde{\varphi}^{-1})^*(\omega|_{U \times \mathbb{R}}). \tag{3.104}$$

To prove (3.93), it sufficies to show the identity on open sets of a foliated atlas. Again, let $\varphi : U \to V \times W$ be a foliated chart of $(M, \mathcal{F})$. Denote the inclusions of the local case by $\tilde{J}_t : U \to U \times \mathbb{R}$, the projection from $V \times W \times \mathbb{R}$ onto $V \times W$ by $\tilde{\pi}$ and set $\tilde{\mathcal{F}} = \mathcal{F}_W(V)$. Then, $\tilde{\varphi}^{-1} \circ \tilde{J}_t \circ \varphi(x) = \tilde{\varphi}^{-1}(\varphi(x), t) = (x, t) = J_t(x)$ for $x \in U$ and $t \in \mathbb{R}$. Since the foliated differential commutes with pullbacks, we obtain for $\omega \in \Omega^k(M \times \mathbb{R}, \pi^*(\mathcal{F}))$,

$$(d_{\mathcal{F}} \circ h(\omega) + h \circ d_{\pi^*(\mathcal{F})}(\omega))|_U = d_{\mathcal{F}}((h(\omega))|_U) + (h \circ d_{\pi^*(\mathcal{F})}(\omega))|_U \tag{3.105}$$

$$= (d_{\mathcal{F}} \circ \varphi^*) \circ h \circ (\tilde{\varphi}^{-1})^*(\omega|_{U \times \mathbb{R}}) + \varphi^* \circ h \circ ((\tilde{\varphi}^{-1})^* \circ d_{\pi^*(\mathcal{F})})(\omega|_{U \times \mathbb{R}}) \tag{3.106}$$

$$= \varphi^* \circ \left(d_{\tilde{\mathcal{F}}} \circ h + h \circ d_{\tilde{\pi}^*(\tilde{\mathcal{F}})}\right) \left((\tilde{\varphi}^{-1})^*(\omega|_{U \times \mathbb{R}})\right) \tag{3.107}$$

$$\overset{(1)}{=} \varphi^* \circ (\tilde{J}_1^* - \tilde{J}_0^*) \circ (\tilde{\varphi}^{-1})^*(\omega|_{U \times \mathbb{R}}) = (J_1^* - J_0^*)(\omega|_{U \times \mathbb{R}}) \tag{3.108}$$

$$= ((J_1^* - J_0^*)(\omega))\,|_U. \tag{3.109}$$

□

3.4.8 Proposition (Cohomology Maps of Integrable Homotopic Maps)

Let $f, g : (M, \mathcal{F}) \to (N, \mathcal{G})$ be smooth foliated maps between smooth foliated manifolds. If f is integrable homotopic to g, then for every $k \in \mathbb{N}_0$, the induced cohomology maps f^* and g^* from $H^k(N, \mathcal{G})$ to $H^k(M, \mathcal{F})$ are equal.

Proof. At first, we show the assertion for the inclusion maps $J_0, J_1 : M \to M \times \mathbb{R}$.
Let $\omega \in \Omega^k(M \times \mathbb{R}, \pi^*(\mathcal{F}))$ be closed, i.e. $d_{\pi^*(\mathcal{F})}(\omega) = 0$. The homotopy operator h of proposition 3.4.7 yields

$$J_1^*(\omega) - J_0^*(\omega) = d_{\mathcal{F}}(h(\omega)) + h(d_{\pi^*(\mathcal{F})}(\omega)) = d_{\mathcal{F}}(h(\omega)) \in \mathcal{B}^k(M, \mathcal{F}), \tag{3.110}$$

because $h(\omega) \in \Omega^k(M, \mathcal{F})$. Hence, the induced cohomology maps are equal.
Now, let $H : (M \times \mathbb{R}, \pi^*(\mathcal{F})) \to (N, \mathcal{G})$ be an integrable homotopy from f to g. Since $f = H \circ J_0$ and $g = H \circ J_1$, we get by funtoriality (proposition (3.4.3),

$$f^* = J_0^* \circ H^* = J_1^* \circ H^* = g^*. \tag{3.111}$$

□

3.4.9 Corollary (Integrable Homotopy Invariance)

If $(M, \mathcal{F})$ and $(N, \mathcal{G})$ are integrable homotopy equivalent smooth foliated manifolds, then

$$H^k(M, \mathcal{F}) \cong H^k(N, \mathcal{G}) \text{ for all } k \in N_0. \tag{3.112}$$

In particular, if S is an integrable deformation retract of $(M, \mathcal{F})$, then

$$H^k(M, \mathcal{F}) \cong H^k(S, \mathcal{F}|_S) \text{ for all } k \in N_0. \tag{3.113}$$

Proof. There exist smooth foliated maps $f : (M, \mathcal{F}) \to (N, \mathcal{G})$ and $g : (N, \mathcal{G}) \to (M, \mathcal{F})$ such that $g \circ f$ is integrable homotopic to id_M and $f \circ g$ is integrable homotopic to id_N. The preceding proposition and proposition 3.4.3 imply

$$f^* \circ g^* = (g \circ f)^* = (\mathrm{id}_M)^* = \mathrm{id}_{H^k(M,\mathcal{F})} \text{ and} \tag{3.114}$$

$$g^* \circ f^* = (f \circ g)^* = (\mathrm{id}_N)^* = \mathrm{id}_{H^k(N,\mathcal{G})}. \tag{3.115}$$

If S is an integrable deformation retract of $(M, \mathcal{F})$, then $(S, \mathcal{F}|_S)$ and $(M, \mathcal{F})$ are in particular integrable homotopy equivalent. □

3.4.10 Lemma (Poincaré Lemma for Star-Shaped Foliation by Points)

Let M be a star-shaped open subset of $\mathbb{R}^p$ with center $c \in M$ and let N be a q-dimensional smooth manifold. Then, $\{c\} \times N$ is an integrable deformation retract of $(M \times N, \mathcal{F}_N(M))$ and

$$H^k(M \times N, \mathcal{F}_N(M)) \cong H^k(\{c\} \times N, \mathcal{F}_N(\{c\})) \cong \begin{cases} C^\infty(N) & \text{if } k = 0; \\ \{0\} & \text{otherwise.} \end{cases} \tag{3.116}$$

Proof. We will verify $S = \{c\} \times N$ as an integrable deformation retract of $(M \times N, \mathcal{F}_N(M))$. The inclusion $i_S : S \to M \times N$ is transversal to $\mathcal{F}_N(M)$ since the differential of i_S is injective and its q-dimensional image of $T_{(c,y)}(\{c\} \times N)$ interesects the p-dimensional foliated tangent space $T_{(c,y)}(M \times \{y\})$ only at zero for each point (c, y) of S. The induced foliation $i_S^*(\mathcal{F}_N(M))$ is just $\mathcal{F}_N(\{c\}) = \{\{(c, y)\} : y \in N\}$. Define

$$r : (M \times N, \mathcal{F}_N(M)) \to (S, \mathcal{F}_N(\{c\})) \text{ by } r(x, y) = (c, y). \tag{3.117}$$

Then, r is a smooth foliated map and satisfies $r \circ i_S(c, y) = (c, y) = \mathrm{id}_S(c, y)$ on S. In order to define an integrable homotopy from $i_S \circ r$ to $\mathrm{id}_{M \times N}$, let $\varphi : \mathbb{R} \to [0, 1]$ be a smooth function which is equal to 0 on $(-\infty, 0]$ and to 1 on $[1, \infty)$. Denote the projection from $(M \times N) \times \mathbb{R}$ to $M \times N$ by π and define

$$H : ((M \times N) \times \mathbb{R}, \pi^*(\mathcal{F}_N(M))) \to (M \times N, \mathcal{F}_N(M)) \text{ by} \tag{3.118}$$
$$H((x, y), t) = (c + \varphi(t)(x - c), y). \tag{3.119}$$

Since M is star-shaped, H is well-defined. It is also a smooth foliated map satisfying

$$H((x, y), t) = (c, y) = i_S \circ r(c, y) \text{ if } t \leq 0 \text{ and} \tag{3.120}$$
$$H((x, y), t) = (x, y) = \mathrm{id}_{M \times N} \text{ if } t \geq 1. \tag{3.121}$$

Thus, S is an integrable deformation retract of $(M \times N, \mathcal{F}_N(M))$ and $H^k(M \times N, \mathcal{F}_N(M))$ is isomorphic to $H^k(S, \mathcal{F}_N(\{c\}))$ for each $k \in \mathbb{N}_0$. Moreover, $(S, \mathcal{F}_N(\{c\}))$ is diffeomorphic to the smooth foliated manifold $(N, \{\{y\} : y \in N\})$ of dimension 0, such that for $k \geq 1$ all foliated cohomology groups are $\{0\}$ and for $k = 0$, we have

$$H^0(N, \{\{y\} : y \in N\}) = \{f \in C^\infty(N) : f|_{\{y\}} \text{ is constant for each } y \in N\} = C^\infty(N). \tag{3.122}$$

□

One powerful tool for computing the de Rham cohomology is the Mayer-Vietoris Theorem. If U and V are submanifolds of some smooth manifold M, the theorem connects the de Rham cohomology of the union $U \cup V$ with the cohomologies of U and V involving also the cohomology of their intersection $U \cap V$, ordered in a long exact sequence. In [KA83] there is a version of a foliated Mayer-Vietoris Theorem for two submanifolds of M which have the same boundary such that this boundary is transverse to the foliation $\mathcal{F}$ on M and admits a *tubular neighbourhood* such that the boundary is an integrable deformation retract of that neighbourhood. Our version needs some assumptions of transversality but is applicable in more general situations. The proof is similar to the unfoliated case with a few replacements. Basically, one needs an induced foliation on the submanifolds such that the inclusion becomes a smooth foliated map. As we clarify in Remark 3.4.12, the assumptions are satisfied for open submanifolds and we can also obtain the classical Mayer-Vietoris Theorem for (unfoliated) smooth manifolds.

3.4.11 Theorem (Mayer-Vietoris for Foliated Cohomology)

Let $(M, \mathcal{F})$ be a smooth foliated manifold. Further, let U and V be immersed submanifolds of M with $U \cup V = M$ such that the inclusion maps satisfy

(1) $i_U : U \to M$ and $i_V : V \to M$ are transverse to $\mathcal{F}$;

(2) $i^U : U \cap V \to U$ is transverse to $\mathcal{F}|_U$;

(3) $i^V : U \cap V \to V$ is transverse to $\mathcal{F}|_V$.

Then, for each $k \in \mathbb{N}_0$, there is a linear map $\delta : H^k(U \cap V, \mathcal{F}|_{U\cap V}) \to H^{k+1}(M, \mathcal{F})$ such that the following sequence of vector spaces, called **foliated Mayer-Vietoris sequence**, is exact:

$$\cdots \xrightarrow{\delta} H^k(M,\mathcal{F}) \xrightarrow{i_U^* \oplus i_V^*} H^k(U,\mathcal{F}|_U) \oplus H^k(V,\mathcal{F}|_V) \xrightarrow{(i^U)^* - (i^V)^*} H^k(U \cap V, \mathcal{F}|_{U\cap V})$$

$$\xrightarrow{\delta} H^{k+1}(M,\mathcal{F}) \xrightarrow{i_U^* \oplus i_V^*} H^{k+1}(U,\mathcal{F}|_U) \oplus H^{k+1}(V,\mathcal{F}|_V) \xrightarrow{(i^U)^* - (i^V)^*} \cdots$$

Proof. We have the following commutative diagram of smooth foliated inclusion maps:

$$\begin{array}{ccccc} & & (U,\mathcal{F}|_U) & & \\ & \overset{i^U}{\nearrow} & & \overset{i_U}{\searrow} & \\ (U \cap V, \mathcal{F}|_{U\cap V}) & & & & (M,\mathcal{F}) \\ & \underset{i^V}{\searrow} & & \underset{i_V}{\nearrow} & \\ & & (V,\mathcal{F}|_V) & & \end{array}$$

This gives for each $k \in \mathbb{N}_0$ the following commutative diagram of foliated pullbacks:

$$\begin{array}{ccccc} & & \Omega^k(U,\mathcal{F}|_U) & & \\ & \overset{i_U^*}{\nearrow} & & \overset{(i^U)^*}{\searrow} & \\ \Omega^k(M,\mathcal{F}) & & & & \Omega^k(U \cap V, \mathcal{F}|_{U\cap V}) \\ & \underset{i_V^*}{\searrow} & & \underset{(i^V)^*}{\nearrow} & \\ & & \Omega^k(V,\mathcal{F}|_V) & & \end{array}$$

Notice that the diagram has been mirrored due to the contravariance. Define for each $k \in \mathbb{N}_0$ the linear maps

$$f = i_U^* \oplus i_V^* : \Omega^k(M,\mathcal{F}) \to \Omega^k(U,\mathcal{F}|_U) \oplus \Omega^k(V,\mathcal{F}|_V), \tag{3.123}$$

$$f(\omega) = (i_U^*(\omega), i_V^*(\omega)) = (\omega|_U, \omega_V) \text{ and} \tag{3.124}$$

$$g = ((i^U)^* - (i^V)^*) : \Omega^k(U,\mathcal{F}|_U) \oplus \Omega^k(V,\mathcal{F}|_V) \to \Omega^k(U \cap V, \mathcal{F}_{U\cap V}) \tag{3.125}$$

$$g(\mu,\nu) = (i^U)^*(\mu) - (i^V)^*(\nu) = \mu|_{U\cap V} - \nu|_{U\cap V}. \tag{3.126}$$

We will show, that the short sequence

$$(*) \quad 0 \longrightarrow \Omega^k(M,\mathcal{F}) \xrightarrow{f} \Omega^k(U,\mathcal{F}|_U) \oplus \Omega^k(V,\mathcal{F}|_V) \xrightarrow{g} \Omega^k(U\cap V,\mathcal{F}|_{U\cap V}) \longrightarrow 0$$

is exact.
f is injective: $f(\omega) = (0,0)$ yields $\omega|_U = 0$ and $\omega|_V = 0$ such that $\omega = 0$ since $U \cap V = M$.
$\mathrm{Im}(f) = \mathrm{Ker}(g)$: $\subseteq$ follows by

$$(g \circ f)(\omega) = (i^U)^* - (i^V)^*(\omega|_U, \omega|_V) = \omega|_{U\cap V} - \omega|_{U\cap V} = 0. \tag{3.127}$$

Now, suppose $(\mu,\nu) \in \Omega^k(U,\mathcal{F}|_U) \oplus \Omega^k(V,\mathcal{F}|_V)$ satisfies $g(\mu,\nu) = 0$. Then $\mu|_{U\cap V} = \nu|_{U\cap V}$ and we can define a foliated k-form on $M = U \cup V$ by

$$\omega = \begin{cases} \mu \text{ on } U, \\ \nu \text{ on } V, \end{cases} \tag{3.128}$$

such that $f(\omega) = (\omega|_U, \omega|_V) = (\mu,\nu)$.
g is surjective: Let be $\eta \in \Omega^k(U\cap V,\mathcal{F}|_{U\cap V})$ and let $\{\varphi_U, \varphi_V\}$ be a smooth partition of unity subordinate to the open cover $\{U, V\}$. Define

$$\mu = \begin{cases} \varphi_V\, \eta & \text{on } U \cap V, \\ 0 & \text{on } U \setminus \mathrm{supp}(\varphi_V), \end{cases} \quad \text{and} \quad \nu = \begin{cases} -\varphi_U\, \eta & \text{on } U \cap V, \\ 0 & \text{on } V \setminus \mathrm{supp}(\varphi_U). \end{cases} \tag{3.129}$$

Note that on $(U \cap V) \setminus \mathrm{supp}(\varphi_V)$, where both definitions of μ overlap, they both are zero. Hence $\mu \in \Omega^k(U,\mathcal{F}|_U)$ and analogous $\nu \in \Omega^k(V,\mathcal{F}|_V)$. Finally, we obtain

$$g(\mu,\nu) = \mu|_{U\cap V} - \nu|_{U\cap V} = \varphi_V\eta - (-\varphi_U\eta) = \eta. \tag{3.130}$$

Therefore, the exactness of the sequence $(*)$ has been proven. Since pullbacks commute with the foliated differential, we obtain the short exact sequence in foliated de Rham cohomology with induced cohomology maps

$$(*) \quad 0 \longrightarrow H^k(M,\mathcal{F}) \xrightarrow{i_U^* \oplus i_V^*} H^k(U,\mathcal{F}|_U) \oplus H^k(V,\mathcal{F}|_V) \xrightarrow{(i^U)^* - (i^V)^*} H^k(U\cap V,\mathcal{F}|_{U\cap V}) \longrightarrow 0.$$

The *Zigzag Lemma*[25] (also known as *Snake Lemma*), which is proved by *diagram chasing*, yields now the linear connecting maps

$$\delta : H^k(U\cap V,\mathcal{F}|_{U\cap V}) \to H^{k+1}(M,\mathcal{F}), \tag{3.131}$$

such that the Mayer-Vietoris sequence of foliated cohomology is exact. □

[25] See [Lee13, Lemma 17.40, p. 461] for instance.

3.4.12 Remark (Foliated Mayer Vietoris Sequence)

(1) If $U, V \subseteq M$ are open subsets, then all required inclusions in (1)-(3) are transversal for any foliation of their target manifold. Indeed, if $i : U \to M$ is the inclusion of an open subset $U \subseteq M$, the map $di_x : T_xU \to T_xM$, $di_x(\nu_x)(g) = \nu_x(g \circ i)$, where $\nu_x \in T_xU$ and $g \in C^\infty(M)$ is an isomorphism for each $x \in U$ by [Lee13, Proposition 3.9, p. 56]. Even if $U = \emptyset$, this holds since otherwise there would be some $x \in \emptyset$ at which point the differential is no isomorphism.

(2) The connecting linear map $\delta : H^k(U \cap V, \mathcal{F}|_{U\cap V}) \to H^{k+1}(M, \mathcal{F})$ in the Mayer-Vietoris sequence of foliated cohomology can be described as follows:
By surjectivity, for each $\eta \in \mathcal{Z}^k(U \cap V, \mathcal{F}|_{U\cap V})$, there are $\mu \in \Omega^k(U, \mathcal{F}|_U)$ and $\nu \in \Omega^k(V, \mathcal{F}|_V)$ such that $\eta = \mu|_{U\cap V} - \nu|_{U\cap V}$. Then $\delta[\eta] = [\sigma]$, where $\sigma \in \Omega^{k+1}(M, \mathcal{F})$ is equal to $d_{\mathcal{F}|_U}\mu$ on U and to $d_{\mathcal{F}|_V}\nu$ on V. Note that on $U \cap V$, both definitions agree since $0 = d_{\mathcal{F}|_{U\cap V}}\mu - d_{\mathcal{F}|_{U\cap V}}\nu$. If $\{\varphi_U, \varphi_V\}$ is a smooth partition of unity subordinate to $\{U, V\}$, it is possible to take $\mu = \varphi_V\eta$ and $\nu = -\varphi_U\eta$, both extended by zero outside of the supports of φ_V and φ_U, respectively. This is a consequence of a characterization of the connecting linear map of the Zigzag Lemma and coincides with the unfoliated case. For instance, see [Lee13, Lemma 17.40 and Corollary 17.42].

(3) If we consider the one leaf foliation $\mathcal{F} = \{M\}$ of M, all inclusions of (1)-(3) are transversal and we obtain the classical Mayer-Vietoris sequence of (unfoliated) de Rham cohomology.

3.4.13 Remark (Good Cover of a Manifold)

A **good cover** $(U_\alpha)_{\alpha\in I}$ of a smooth manifold M of dimension n is a collection of open sets U_α such that all sets and all finite non-empty intersections $U_{\alpha_0} \cap ... \cap U_{\alpha_\ell}$ of the cover are diffeomorphic to $\mathbb{R}^n$. By [BT82, Theorem 5.1, p.42], every smooth manifold has a good cover and if the manifold is compact, the good cover may be chosen to be finite. Moreover, if a smooth manifold has a finite good cover, then its de Rham cohomology is finite dimensional. (This can be proved by an induction on the cardinality of a good cover together with the Mayer-Vietoris sequence and the Poincaré Lemma, see [BT82, Proposition 5.3.1, p.43]. Another proof uses an isomorphism between the de Rham cohomology and the Čech-cohomology of a good cover together with the observation that the Čech-cohomology of a finite cover is finite dimensional. We obtain this result later by Corollary 3.5.15.)

3.4.14 Theorem (Foliated Cohomology of Foliation by Points)

Let M and N be smooth manifolds of dimension p and q, respectively. Assume M has a finite good cover (and consequently a finite dimensional de Rham cohomology), then

$$H^k(M \times N, \mathcal{F}_N(M)) \cong H^k(M) \otimes C^\infty(N) \text{ for each } k \in \mathbb{N}_0, \tag{3.132}$$

where $H^k(M)$ denotes the usual de Rham cohomology group of the smooth manifold M.

Proof. The proof is inspired by the proof of the Künneth Formula for (unfoliated) de Rham cohomology that can be found in [BT82, p.47 ff.].
Let $\mathcal{F}_M = \{M\}$ be the smooth $(p, 0)$-foliation of M and $\mathcal{F}_N = \{\{y\} : y \in N\}$ be the smooth $(0, q)$-foliation of N, such that the projections π_M and π_N from $(M \times N, \mathcal{F}_N(M))$ onto $(M, \mathcal{F}_M)$ and $(N, \mathcal{F}_N)$, respectively, are smooth foliated maps. For $k, \ell \in \mathbb{N}_0$ with $k \geq \ell$, this allows the definition of

$$\varphi^{k,\ell} : \Omega^k(M, \mathcal{F}_M) \times \Omega^{k-\ell}(N, \mathcal{F}_N) \to \Omega^\ell(M \times N, \mathcal{F}_N(M)),\ (\mu, \nu) \mapsto \pi_M^*(\mu) \wedge \pi_N^*(\nu). \tag{3.133}$$

Note that $\Omega^k(M, \mathcal{F}_M) = \Omega^k(M)$ for all $k \in \mathbb{N}_0$, $\Omega^0(N, \mathcal{F}_N) = C^\infty(N)$ and $\Omega^\ell(N, \mathcal{F}_N) = \{0\}$ for all $\ell \geq 1$ because $\mathcal{F}_N$ is of dimension 0. Moreover, $\varphi^{k,0}$ is bilinear such that the universal property of the tensor product induces an $\mathbb{R}$-linear map

$$\psi^k : \Omega^k(M) \otimes C^\infty(N) \to \Omega^k(M \times N, \mathcal{F}_N(M)),\ \mu \otimes f \mapsto \pi_M^*(\mu) \wedge \pi_N^*(f) = \pi_N^*(f)\, \pi_M^*(\mu). \tag{3.134}$$

Next, define $\tilde{d} : \Omega^k(M) \otimes C^\infty(N) \to \Omega^{k+1}(M) \otimes C^\infty(N)$ by $\tilde{d}(\mu \otimes f) = (d\mu) \otimes f$ for each $k \in \mathbb{N}_0$, where $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is the Cartan-differential on k-forms. Hence, $\tilde{d}$ is linear and satisifes $\tilde{d} \circ \tilde{d} = 0$ such that we obtain cohomology groups $H^k(M) \otimes C^\infty(N)$, which are also $\mathbb{R}$-vector spaces. Since the (foliated and unfoliated) Cartan-differential commutes with pullbacks, we can compute for $\mu \otimes f \in \Omega^k(M) \otimes C^\infty(N)$,

$$\psi^{k+1} \circ \tilde{d}(\mu \otimes f) = \pi_N^* f\ \pi_M^*(d\mu) = \pi_N^* f\ d^*_{\mathcal{F}_N(M)}(\pi_M^* \mu) = d_{\mathcal{F}_N(M)} \circ \psi^k(\mu \otimes f). \tag{3.135}$$

Therefore, we receive for each $k \in \mathbb{N}_0$ a linear map in cohomology

$$\Psi^k : H^k(M) \otimes C^\infty(N) \to H^k(M, \times N, \mathcal{F}_M(N)),\ [\mu] \otimes f \mapsto [\psi^k(\mu \otimes f)]. \tag{3.136}$$

We will show, that Ψ^k is an isomorphism. If M is a star-shaped open subset of $\mathbb{R}^p$, this is just the Poincaré Lemma for foliated cohomology. This can be used together with the Mayer-Vietoris sequence of a good cover, requiring a finite good cover for an induction argument. Let $U, V \subseteq M$ be open subsets such that we have the exact Mayer-Vietoris sequence

$$\cdots \to H^k(U \cup V) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V) \to \cdots . \tag{3.137}$$

Tensoring with the vector space $C^\infty(N)$ preserves exactness. Moreover, $U \times N$ and $V \times N$ are open subsets of $M \times N$, such that the Mayer-Vietoris sequence of foliated cohomology is exact. This gives the following diagram with exact rows,

$$\begin{array}{ccccccc}
\cdots \to & H^k(U \cup V) \otimes C^\infty(N) & \to & (H^k(U) \otimes C^\infty(N)) \oplus (H^k(V) \otimes C^\infty(N)) & \to & H^k(U \cap V) \otimes C^\infty(N) & \to \cdots \\
& \downarrow{\scriptstyle \Psi^k} & & \downarrow{\scriptstyle \Psi^k \oplus \Psi^k} & & \downarrow{\scriptstyle \Psi^k} & \\
\cdots \to & H^k((U \cup V) \times N, \mathcal{F}_N(U \cup V)) & \to & H^k(U \times N, \mathcal{F}_N(U)) \oplus H^k(V \times N, \mathcal{F}_N(V)) & \to & H^k((U \cap V) \times N, \mathcal{F}_N(U \cap V)) & \to \cdots
\end{array}$$

and it is also commutative. This follows essentially by the commutativity of the (foliated) Cartan-differential with pullbacks. However, the square containing the connecting linear maps of the Mayer-Vietoris sequences,

$$\begin{array}{ccc} H^k(U\cap V)\otimes C^\infty(N) & \xrightarrow{\tilde{\delta}\otimes \mathrm{id}} & H^{k+1}(U\cup V)\otimes C^\infty(N) \\ \downarrow{\scriptstyle \Psi^k} & & \downarrow{\scriptstyle \Psi^{k+1}} \\ H^k((U\cap V)\times N, \mathcal{F}_N(U\cap V))) & \xrightarrow{\delta} & H^{k+1}((U\cup V)\times N, \mathcal{F}_N(U\cup V)), \end{array}$$

requires a closer look. Let $\{\varphi_U, \varphi_V\}$ be a smooth partition of unity subordinate to the open cover $\{U, V\}$ and let $[\eta]\otimes f \in H^k(U\cap V)\otimes C^\infty(N)$, then by Remark 3.4.12 (2), $\tilde{\delta}[\eta] = [\tilde{\sigma}]$, where $\tilde{\sigma} = d(\varphi_V \eta)$ on U and $\tilde{\sigma} = d(-\varphi_U\eta)$ on V (extended by zero outside the supports of φ_U and φ_V, respectively). Hence,

$$\Psi^{k+1}\circ(\tilde{\delta}\otimes \mathrm{id})([\eta]\otimes f) = \Psi^{k+1}(\tilde{\delta}[\eta]\otimes f) = [\pi_N^*(f)\pi_{U\cup V}^*(\tilde{\sigma})] \tag{3.138}$$

$$= \begin{cases} [\pi_N^*(f)\pi_{U\cup V}^*(d(\varphi_V\eta))] & \text{on } U, \\ [\pi_N^*(f)\pi_{U\cup V}^*(d(-\varphi_U\eta))] & \text{on } V. \end{cases} \tag{3.139}$$

Since $\{\pi_{U\cup V}^*(\varphi_U), \pi_{U\cup V}^*(\varphi_V)\}$ is a smooth partition of unity subordinate to the open cover $\{U\times N, V\times N\}$ of $(U\cup V)\times N$, we obtain again by Remark 3.4.12 (2)

$$\delta\circ\Psi^k([\eta]\otimes f) = \delta[\pi_N^*(f)\pi_{U\cap V}^*(\eta)] = [\sigma], \tag{3.140}$$

where σ is given on U by

$$d_{\mathcal{F}_N(U)}\left(\pi_{U\cup V}^*(\varphi_V)\, \pi_N^*(f)\, \pi_{U\cup V}^*(\eta)\right) = \pi_N^*(f) d_{\mathcal{F}_N(U)}(\pi_{U\cup V}^*(\varphi_V\eta)) \tag{3.141}$$

$$= \pi_N^*(f)\, \pi_{U\cup V}^*(d(\varphi_V\eta)). \tag{3.142}$$

At the first step, we used the antiderivation property together with $d_{\mathcal{F}_N(U)}(\pi_N^*(f)) = 0$ and the commutativity of pullbacks with products. Analogously, σ is given on V by

$$d_{\mathcal{F}_N(U)}\left(\pi_{U\cup V}^*(-\varphi_U)\, \pi_N^*(f)\, \pi_{U\cup V}^*(\eta)\right) = \pi_N^*(f) d_{\mathcal{F}_N(U)}(\pi_{U\cup V}^*(-\varphi_U\eta)) \tag{3.143}$$

$$= \pi_N^*(f)\, \pi_{U\cup V}^*(d(-\varphi_U\eta)). \tag{3.144}$$

Thus, the square diagram commutes and therefore also the huge diagram with the long exact rows is commutative. By the *Five Lemma*, it suffices to prove that Ψ^k is an isomorphism for U, V and $U\cap V$, then it is also true for $U\cup V$. Therefore, by induction on the cardinality L of a finite good cover $\{U_j\}_{j=1}^L$ of M, we can conclude together with the Poincaré Lemma for star shaped foliations by points, that

$$\Psi^k : H^k(M)\otimes C^\infty(N) \to H^k(M, \times N, \mathcal{F}_M(N)) \tag{3.145}$$

is an isomorphism for each $k\in\mathbb{N}_0$, if M has a finite good cover. □

Proposition III.1a of [KA83] states the same formula of Theorem 3.4.14 without assuming a finite good cover. The proof requires some knowledge about spectral sequences which we could avoid.

3.4.15 Example (Torus Foliated by Points of a Circle)

Consider the torus $\mathbb{T}^2 = S^1 \times S^1$. The leaves of a S^1-foliation by points of S^1 are either $\{x\} \times S^1$ for $x \in S^1$ or $S^1 \times \{y\}$ for $y \in S^1$, depending over which factor we parametrize. However, both foliations are isomorphic in the sense that $\Phi : \mathbb{T}^2 \to \mathbb{T}^2$, $\Phi(x, y) = (y, x)$ is a diffeomorphism which is also a foliated map with respect to mentioned foliations. Since S^1 is compact, such that it has a finite good cover and its non-zero de Rham cohomology groups are $H^k(S^1) \cong \mathbb{R}$ $(k = 0, 1)$, we obtain by Theorem 3.4.14

$$H^k(\mathbb{T}^2, \mathcal{F}_{S^1}(S^1)) = H^k(S^1) \otimes C^\infty(S^1) = C^\infty(S^1) \text{ for } k = 0, 1. \tag{3.146}$$

In Example 3.6.2 we consider more interesting foliations of the torus.

3.5 Čech Cohomology and Generalized Mayer-Vietoris

We want to generalize the Mayer-Vietoris sequence of just two open sets to a cover of countable many open sets. This generalization will involve foliated forms restricted to intersections of finitely many open sets of the cover. These spaces arise as special cases of the so called **Čech complex**, which is a differential complex depending on an open cover of a topological space and a presheaf.

3.5.1 Definition (Presheaf, Čech complex and Cover Operator)

(a) In general, a ($\mathbf{Vect}_\mathbb{R}$-valued) **presheaf** $\mathscr{G}$ on a toplogical space X is a contravariant functor from the category $\mathbf{Open}\,(\mathbf{X})$, consisting of open sets in X as objects and inclusions of open sets as morphisms to the category $\mathbf{Vect}_\mathbb{R}$, consisting of $\mathbb{R}$-vector spaces and $\mathbb{R}$-linear maps as morphisms. This means, if $V \subseteq U$ are open sets in X, then $\mathscr{G}$ assigns to the inclusion

$$i_V^U : V \to U \tag{3.147}$$

an $\mathbb{R}$-linear map, called restriction

$$\varrho_U^V = \mathscr{G}(i_V^U) : \mathscr{G}(U) \to \mathscr{G}(V) \tag{3.148}$$

satisfying $\varrho_V^V = \mathscr{G}(i_V^V) = \mathrm{id}_{\mathscr{G}(V)}$ and if $W \subseteq V \subseteq U$ are all open in X, then

$$\varrho_U^W = \mathscr{G}(i_W^U) = \mathscr{G}(i_V^U \circ i_W^V) = \mathscr{G}(i_W^V) \circ \mathscr{G}(i_V^U) = \varrho_V^W \circ \varrho_U^V. \tag{3.149}$$

A **morphism of presheaves**, $f : \mathscr{G}_1 \to \mathscr{G}_2$, is a natural transformation between functors from $\mathbf{Open}\,(\mathbf{X})$ to $\mathbf{Vect}_\mathbb{R}$. To be precise, for every open $U \subseteq X$, there is a linear function $f_U : \mathscr{G}_1(U) \to \mathscr{G}_2(U)$ such that for all inclusions of open sets $i_V^U : V \to U$ the following diagram with restrictions is commutative:

$$\begin{array}{ccc} \mathscr{G}_1(U) & \xrightarrow{f_U} & \mathscr{G}_2(U) \\ {\scriptstyle (\varrho_1)_U^V}\downarrow & & \downarrow{\scriptstyle (\varrho_2)_U^V} \\ \mathscr{G}_1(V) & \xrightarrow{f_V} & \mathscr{G}_2(V) \end{array}$$

A morphism of presheaves, $f : \mathscr{G}_1 \to \mathscr{G}_2$, is called an **isomorphism of presheaves**, if it is a natural isomorphism between functors, i.e. $f_U : \mathscr{G}_1(U) \to \mathscr{G}_2(U)$ is an isomorphism of vector spaces for all open $U \subseteq X$ and we call $\mathscr{G}_1$ and $\mathscr{G}_2$ **isomorphic presheaves** in that case.
For a fixed $\mathbb{R}$-vector space A, the so called **trivial presheaf associated to** A is the presheaf which associates to every open set of X the vector space A and to every inclusion $V \subseteq U$ of open sets the identity map id_A on A. If a presheaf is isomorphic to the trivial presheaf associated to A, we call it a **constant presheaf associated to** A. Moreover, we call a presheaf $\mathscr{G}$ a **locally constant presheaf associated to** A, if every $x \in X$ has an open neighbourhood $U \subseteq X$ such that $\mathscr{G}|_U$, defined on **Open(U)**, is a constant presheaf associated to A.

(b) Let $\mathcal{U} = (U_\alpha)_{\alpha \in J}$ be an open cover of a topological space X, where J is an ordered countable set. For $\ell \in \mathbb{N}_0$ and $\alpha_0, ..., \alpha_\ell \in J$, we write

$$U_{\alpha_0,\dots,\alpha_\ell} = \bigcap_{j=0}^{\ell} U_{\alpha_j} \quad \text{and} \quad U_{\alpha_0,\dots,\hat{\alpha_i},\dots,\alpha_\ell} = \bigcap_{\substack{j=0\\ j\neq i}}^{\ell} U_{\alpha_j}. \tag{3.150}$$

Let $\mathscr{G}$ be a presheaf on X. For $\ell \in \mathbb{N}_0$ define the ℓ-cochains of the cover $\mathcal{U}$ with values in the presheaf $\mathscr{G}$ by

$$\check{C}^\ell(\mathcal{U}, \mathscr{G}) = \prod_{\alpha_0<\dots<\alpha_\ell} \mathscr{G}(U_{\alpha_0,\dots,\alpha_\ell}), \tag{3.151}$$

where all indices of type α and α_j are supposed to be in J without further mentioning. Each inclusion from U_α into X induces a restriction $\varrho_X^{U_\alpha}$. We define the **global restriction map** to be $r : \mathscr{G}(X) \to \check{C}^0(\mathcal{U}, \mathscr{G})$ by $(r\omega)_\alpha = \varrho_X^{U_\alpha}(\omega)$ for all $\alpha \in J$. Further, we denote for $i \in \{0, ..., \ell\}$ the inclusion map from $U_{\alpha_0,\dots,\alpha_\ell}$ into $U_{\alpha_0,\dots,\hat{\alpha_i},\dots,\alpha_\ell}$ by ∂_i^ℓ. This induces a restriction map

$$\delta_i^\ell = \mathscr{G}(\partial_i^\ell) : \mathscr{G}(U_{\alpha_0,\dots,\hat{\alpha_i},\dots,\alpha_\ell}) \to \mathscr{G}(U_{\alpha_0,\dots,\alpha_\ell}). \tag{3.152}$$

Now, we define for $\ell \in \mathbb{N}_0$ the cover operator

$$\delta^\ell : \check{C}^\ell(\mathcal{U}, \mathscr{G}) \to \check{C}^{\ell+1}(\mathcal{U}, \mathscr{G}) \tag{3.153}$$

which maps an ℓ-cochain $\omega \in \check{C}^\ell(\mathcal{U}, \mathscr{G})$ with entries $\omega_{\alpha_0,\dots,\alpha_\ell} \in \mathscr{G}(U_{\alpha_0,\dots,\alpha_\ell})$ to an $\ell+1$-cochain $\delta^\ell\omega \in \check{C}^{\ell+1}(\mathcal{U}, \mathscr{G})$ where the entries are given by the alternating sum of restrictions

$$(\delta^\ell\omega)_{\alpha_0,\dots,\alpha_{\ell+1}} = \sum_{i=0}^{\ell} (-1)^i \delta_i^{\ell+1}(\omega_{\alpha_0,\dots,\hat{\alpha_i},\dots,\alpha_{\ell+1}}), \tag{3.154}$$

where the hat on α_i indicates that this index is omitted. As we will show below, the cover operator is a coboundary operator ($\delta^{\ell+1} \circ \delta^\ell = 0$) of the cochain complex

$\check{C}^*(\mathcal{U},\mathscr{G}) = \bigoplus_{\ell\in\mathbb{N}_0} \check{C}^\ell(\mathcal{U},\mathscr{G})$, which we will call the **Čech complex of the cover $\mathcal{U}$ with values in the presheaf $\mathscr{G}$**. Further, we denote by

$$\check{H}^*(\mathcal{U},\mathscr{G}) = \bigoplus_{\ell\in\mathbb{N}_0} \check{H}^\ell(\mathcal{U},\mathscr{G}) \text{ where } \check{H}^\ell(\mathcal{U},\mathscr{G}) = {}^{\mathrm{Ker}(\delta^{\ell+1})}\!/\!{}_{\mathrm{Im}(\delta^\ell)}, \tag{3.155}$$

the **Čech cohomology of the cover $\mathcal{U}$ with values in the presheaf $\mathscr{G}$**.

3.5.2 Lemma (Cover Operator Is a Coboundary Operator)

In the setting of 3.5.1 (b), the cover operator satisfies $\delta^{\ell+1}\circ\delta^\ell = 0$ for each $\ell\in\mathbb{N}_0$.

Proof. Let $\omega\in\check{C}^\ell(\mathcal{U},\mathscr{G})$. For ease of notation, we denote the following entry of ω with two omitted indices by

$$e_{i,j} = \omega_{\alpha_0,\dots,\hat{\alpha_i},\dots,\hat{\alpha_j},\dots,\alpha_{\ell+2}} \text{ if } 0\le i<j\le\ell+2. \tag{3.156}$$

Using the definition of δ^ℓ, we compute for $j\in\{0,\dots,\ell+2\}$

$$(\delta^\ell\omega)_{\alpha_0,\dots,\hat{\alpha_j},\dots,\alpha_{\ell+2}} = \sum_{i=0}^{j-1}(-1)^i\delta_i^{\ell+1}(e_{i,j}) + \sum_{i=j}^{\ell+1}(-1)^i\delta_i^{\ell+1}(e_{j,i+1}) \tag{3.157}$$

$$= \sum_{i=0}^{j-1}(-1)^i\delta_i^{\ell+1}(e_{i,j}) - \sum_{i=j+1}^{\ell+2}(-1)^i\delta_i^{\ell+1}(e_{j,i}). \tag{3.158}$$

Of course, for $j=0$ the first sum vanishes as well as the second sum vanishes for $j=\ell+2$. Together with the linearity of $\delta_j^{\ell+2}$, we get

$$(\delta^{\ell+1}(\delta^\ell\omega))_{\alpha_0,\dots,\alpha_{\ell+2}} = \sum_{j=0}^{\ell+2}(-1)^j\delta_j^{\ell+2}((\delta^\ell\omega)_{\alpha_0,\dots,\hat{\alpha_j},\dots,\alpha_{\ell+2}}) \tag{3.159}$$

$$= \sum_{j=1}^{\ell+2}\sum_{i=0}^{j-1}(-1)^{i+j}\delta_j^{\ell+2}(\delta_i^{\ell+1}(e_{i,j})) - \sum_{j=0}^{\ell+2}\sum_{i=j+1}^{\ell+1}(-1)^{i+j}\delta_j^{\ell+2}(\delta_i^{\ell+1}(e_{j,i}))$$

$$= \sum_{0\le i<j\le\ell+2}(-1)^{i+j}\delta_j^{\ell+2}(\delta_i^{\ell+1}(e_{i,j})) - \sum_{0\le j<i\le\ell+2}(-1)^{i+j}\delta_j^{\ell+2}(\delta_i^{\ell+1}(e_{j,i}))$$

$$= 0, \tag{3.160}$$

since $\delta_j^{\ell+2}(\delta_i^{\ell+1}(e_{j,i})) = \varrho^{U_{\alpha_0,\dots,\hat{\alpha_j},\dots,\hat{\alpha_i},\dots,\alpha_{\ell+2}}}_{U_{\alpha_0,\dots,\alpha_{\ell+2}}}(e_{j,i}) = \delta_i^{\ell+2}(\delta_j^{\ell+1}(e_{j,i}))$ in the last sum. $\square$

3.5.3 Examples (Presheaves)

(1) **(Presheaf of Sections)**
Consider a smooth vector bundle (E, π, M) over a smooth manifold M. For an open subset $U \subseteq M$, the space of smooth sections $\Gamma(U, E)$ is an $\mathbb{R}$-vector space (even a Fréchet space). If $i_V^U : V \subseteq U$ is an inclusion of open sets, the restriction map given by the pullback, $\varrho_U^V = (i_V^U)^* : \Gamma(U, E) \to \Gamma(V, E)$, $\varrho_U^V(\sigma) = \sigma \circ i_V^U = \sigma|_V$, is a (continuous) linear map. The linking properties of the restrictions also satisfied, such that we have defined a (even a **Fréchet**-valued) presheaf on M, called the **presheaf of sections of** E **over** M and denoted by Γ_E. Each point $x \in M$ has an open neighbourhood $U \subseteq M$ such that $\Gamma(U, E)$ is isomorphic (even as Fréchet spaces) to $C^\infty(\mathbb{R}^n, \mathbb{R}^N)$ by Lemma 2.2.4 using smooth charts diffeomorphic to $\mathbb{R}^n$, where N is the rank of E and n the dimension of the manifold M. Hence, $\Gamma|_E$ is a locally constant presheaf associated to $C^\infty(\mathbb{R}^n, \mathbb{R}^N)$. It is only a constant presheaf, if there is a global smooth chart and the vector bundle is trivial. We want to emphasize two presheaves of sections:

(2) **(Presheaf of Differential Forms)**
Let M be a smooth manifold and $k \in \mathbb{N}_0$. We denote the presheaf of sections of the alternating k-tensor bundle $\Lambda^k(T^*M)$ over M by Ω^k and call it the **presheaf of differential** k**-forms on** M.

(3) **(Presheaf of Foliated Differential Forms)**
This is the most important example for us and we repeat the definition of the presheaf of sections in this situation. Consider a smooth foliated manifold $(M, \mathcal{F})$. Since every open set $U \subseteq M$ is transversal to the foliation $\mathcal{F}$, we can pullback the foliation $\mathcal{F}$ by the inclusion map $i_U : U \to M$, such that $\mathcal{F}|_U = i_U^*(\mathcal{F})$ is a foliation on U and the inclusion becomes a smooth foliated map. For each $k \in \mathbb{N}_0$ we can define a presheaf $\Omega^k_{\mathcal{F}}$ on M by setting $\Omega^k_{\mathcal{F}}(U) = \Omega^k(U, \mathcal{F}|_U)$ for an open set $U \subseteq M$ and if $i_U^V : U \to V$ is an inclusion of open sets, then $\Omega^k_{\mathcal{F}}(i_U^V) = (i_U^V)^* : \Omega^k_{\mathcal{F}}(V) \to \Omega^k_{\mathcal{F}}(U)$ is the restriction of foliated k-forms, given by the pullback of the foliated inclusion map. We call $\Omega^k_{\mathcal{F}}$ the **presheaf of foliated differential** k**-forms**. Note that we obtain for the trivial foliation $\mathcal{F} = \{M\}$ on a manifold M the same presheaf as in (2).

(4) **(Presheaf of (foliated) de Rham Cohomology)**
Let $(M, \mathcal{F})$ be a smooth foliated manifold and $k \in \mathbb{N}_0$. As in (3), for an open subset $U \subseteq M$, $(U, \mathcal{F}|_U)$ is a smooth (p, q)-foliated manifold and we define $H^k_{\mathcal{F}}(U)$ to be the kth foliated de Rham cohomology group $H^k(U, \mathcal{F}|_U)$, which is a vector space as we know. Moreover, we set $H^k_{\mathcal{F}}(i_V^U) = (i_V^U)^* : H^k_{\mathcal{F}}(U) \to H^k_{\mathcal{F}}(V)$, which is the linear induced cohomology map of the inclusion $i_V^U : V \to U$ of open sets. This defines the **presheaf** $H^k_{\mathcal{F}}$ **of the** k**th foliated de Rham cohomology**. Further, if $\mathcal{F} = \{M\}$ is the trivial foliation on M, we omit the sub index $\mathcal{F}$ since we obtain the classical unfoliated de Rham cohomology and call H^k the **presheaf of the** k**th de Rham cohomology**.

3.5.4 Remark (Convention for Arbitrary Indices)

So far, the entries $\omega_{\alpha_0,...,\alpha_\ell} \in \mathscr{G}(U_{\alpha_0,...,\alpha_\ell})$ of an ℓ-cochain $\omega \in \check{C}^\ell(\mathcal{U}, \mathscr{G})$ appear only for increasing indices in J. By the following convention, we can allow more general indices of any order, even with repetitions.
Convention: Whenever two indices are changed, the entry become its negative, hence

$$\omega_{...,\alpha,...,\beta,...} = -\omega_{...,\beta,...,\alpha,...}. \tag{3.161}$$

As a consequence, an entry with repeated indices is 0. Otherwise, there is a permutation σ which puts $\alpha_0, ..., \alpha_\ell$ in an increasing order $\alpha_{\sigma(0)} < ... < \alpha_{\sigma(\ell)}$ such that

$$\omega_{\alpha_0,...,\alpha_\ell} = \operatorname{sgn}(\sigma)\, \omega_{\alpha_{\sigma(0)},...,\alpha_{\sigma(\ell)}}. \tag{3.162}$$

Note that this convention is consistent with the definition of the cover operator δ.

3.5.5 Lemma (The Generalized Mayer-Vietoris Sequence)

Let $(M, \mathcal{F})$ be a smooth foliated manifold and $\mathcal{U} = (U_\alpha)_{\alpha\in J}$ an open cover of M indexed by a countable ordered set J. The sequence of the Čech complex with values in the presheaf of foliated differential k-forms, augmented by the global restriction map r (see 3.5.1, (b) for definition),

$$0 \to \Omega^k_{\mathcal{F}}(M) \overset{r}{\to} \check{C}^0(\mathcal{U}, \Omega^k_{\mathcal{F}}) \overset{\delta^0}{\to} \check{C}^1(\mathcal{U}, \Omega^k_{\mathcal{F}}) \overset{\delta^1}{\to} \check{C}^2(\mathcal{U}, \Omega^k_{\mathcal{F}}) \to \cdots \tag{3.163}$$

is exact. In other words, the Čech cohomology with values in the presheaf of foliated differential k-forms vanishes identically.

Proof. If the restriction of a foliated global form is constantly zero on every set of the open cover, the global form must be constantly zero. Hence, r is injective.
We show $\delta^0 \circ r = 0$. Let $\omega \in \Omega^k_{\mathcal{F}}(M)$ be a global foliated form, then

$$\begin{aligned}(\delta^0(r\omega))_{\alpha_0,\alpha_1} &= \delta^1_0((r\omega)_{\alpha_1}) - \delta^1_1((r\omega)_{\alpha_0}) &(3.164)\\ &= \varrho^{U_{\alpha_0,\alpha_1}}_{U_{\alpha_1}} \circ \varrho^{U_{\alpha_1}}_X(\omega) - \varrho^{U_{\alpha_0,\alpha_1}}_{U_{\alpha_0}} \circ \varrho^{U_{\alpha_0}}_X(\omega) = 0. &(3.165)\end{aligned}$$

Therefore, $\operatorname{Im}(r) \subseteq \operatorname{Ker}(\delta^0)$. If $\mu = (\mu_\alpha)_{\alpha\in J} \in \operatorname{Ker}(\delta^0)$, then $\mu_\alpha|_{U_{\alpha,\beta}} = \mu_\beta|_{U_{\alpha,\beta}}$ for all $\alpha < \beta$ in J with $U_{\alpha,\beta} \neq \emptyset$. Hence, μ defines a global foliated form $\omega \in \Omega^k_{\mathcal{F}}(M)$ with $r(\omega) = \mu$ such that we can conclude $\operatorname{Im}(r) = \operatorname{Ker}(\delta^0)$.
It remains to be shown that $\operatorname{Ker}(\delta^{\ell+1}) \subseteq \operatorname{Im}(\delta^\ell)$ for $\ell \in \mathbb{N}_0$. Let $(\chi_\alpha)_{\alpha\in J}$ be a smooth partition of unity subordinate to the open cover $\{U_\alpha\}_{\alpha\in J}$. Suppose ω is an $(\ell+1)$-cocycle, i.e. $\omega \in \check{C}^{\ell+1}(\mathcal{U}, \Omega^k_{\mathcal{F}})$ with $\delta^{\ell+1}\omega = 0$. We define a cochain $\tau \in \check{C}^\ell(\mathcal{U}, \Omega^k_{\mathcal{F}})$ by

$$\tau_{\alpha_0,...,\alpha_\ell} = \sum_{\alpha\in J} \chi_\alpha\, \omega_{\alpha,\alpha_0,...,\alpha_\ell}\ , \tag{3.166}$$

where we use the convention above for non increasing indices. We compute,

$$(\delta^\ell \tau)_{\alpha_0,\ldots,\alpha_{\ell+1}} = \sum_{i=0}^{\ell+1} (-1)^i \tau_{\alpha_0,\ldots,\hat{\alpha_i},\ldots,\alpha_{\ell+1}} \tag{3.167}$$

$$= \sum_{\alpha\in J} \chi_\alpha \sum_{i=0}^{\ell+1} (-1)^i \omega_{\alpha,\alpha_0,\ldots,\hat{\alpha_i},\ldots,\alpha_{\ell+1}}. \tag{3.168}$$

Since ω is a cocycle,

$$0 = (\delta^{\ell+1}\omega)_{\alpha,\alpha_0,\ldots,\alpha_{\ell+1}} = \omega_{\alpha_0,\ldots,\alpha_{\ell+1}} + \sum_{i=0}^{\ell+1} (-1)^{i+1} \omega_{\alpha,\alpha_0,\ldots,\hat{\alpha_i},\ldots,\alpha_{\ell+1}}. \tag{3.169}$$

Finally,

$$(\delta^\ell \tau)_{\alpha_0,\ldots,\alpha_{\ell+1}} = \sum_{\alpha\in J} \chi_\alpha\, \omega_{\alpha_0,\ldots,\alpha_{\ell+1}} = \omega_{\alpha_0,\ldots,\alpha_{\ell+1}}. \tag{3.170}$$

□

Combining the foliated de Rham complex and the Čech complex with values in the presheaf of foliated forms will lead us to a double complex:

3.5.6 Definition (Foliated Čech-de Rham Complex and Cohomology)

Define the spaces

$$K^{k,\ell} = \check{C}^\ell(\mathcal{U}, \Omega^k_{\mathcal{F}}) = \prod_{\alpha_0<\ldots<\alpha_\ell} \Omega^k(U_{\alpha_0,\ldots,\alpha_\ell}, \mathcal{F}_{\alpha_0,\ldots,\alpha_\ell}) \ \text{ for } k,\ell \in \mathbb{N}_0, \tag{3.171}$$

consisting of the **ℓ-cochains of the cover $\mathcal{U}$ with values in the foliated k-forms**. The cover operator $\delta^\ell : K^{k,\ell} \to K^{k,\ell+1}$ as horizontal operator and the foliated Cartan-differential $d_{\mathcal{F}} : K^{k,\ell} \to K^{k+1,\ell}$ as vertical operator turn $\check{C}^*(\mathcal{U}, \Omega^*_{\mathcal{F}}) = \bigoplus_{k,\ell\in\mathbb{N}_0} K^{k,\ell}$ into a double complex, called **the foliated Čech-de Rham complex**.
By setting

$$D : \bigoplus_{k+\ell=j} K^{k,\ell} \to \bigoplus_{k+\ell=j+1} K^{k,\ell}, D\omega = \delta^\ell\omega + (-1)^k d_{\mathcal{F}}\omega \text{ if } \omega \in K^{k,\ell}, \tag{3.172}$$

the double complex can be made into a single cochain complex consisting of the anti-diagonal lines $A^j = \bigoplus_{k+\ell=j} K^{k,\ell}$ of the double complex together with D as coboundary operator. For $j \in \mathbb{N}_0$ define the jth foliated Čech-de Rham cohomology group by

$$H^j_D\{\check{C}^*(\mathcal{U}, \Omega^*_{\mathcal{F}})\} = {}^{\text{Ker}(D:A^j\to A^{j+1})}\!/\!_{\text{Im}(D:A^{j-1}\to A^j)}. \tag{3.173}$$

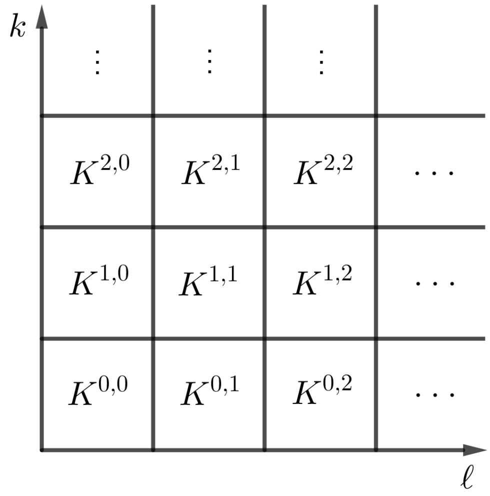


Figure 3: The foliated Čech-de Rham double complex.

A D-**cocycle** $\omega \in A^j$ is a sum $\omega = \sum\limits_{k+\ell=j} \omega^{k,\ell}$ with $\omega^{k,\ell} \in K^{k,\ell}$ and $D\omega = 0$, which is equivalent to

$$d_{\mathcal{F}}\omega^{j,0} = 0, \;\; \delta\omega^{k,\ell} = (-1)^{k+1} d_{\mathcal{F}}\omega^{k-1,\ell+1} \;\text{ and }\; \delta\omega^{0,j} = 0. \tag{3.174}$$

A D-**coboundary** $\eta \in A^j$ is a sum $\eta = \sum\limits_{k+\ell=j} \eta^{k,\ell}$ with $\eta^{k,\ell} \in K^{k,\ell}$ such that there exists $\omega \in A^{j-1}$ with $D\omega = \eta$, which is equivalent to

$$\eta^{j,0} = (-1)^j d_{\mathcal{F}}\omega^{j-1,0}, \;\; \eta^{k,\ell} = \delta\omega^{k,\ell-1} + (-1)^k d_{\mathcal{F}}\omega^{k-1,\ell} \;\text{ and }\; \eta^{0,j} = \delta\omega^{0,j-1}. \tag{3.175}$$

We call $\omega \in A^j$ D-**cohomologous** to $\tilde{\omega} \in A^j$, if ω and $\tilde{\omega}$ differ by a D-coboundary, i.e. there is some $\eta \in A^{j-1}$ with $\omega - \tilde{\omega} = D\eta$.

The exactness of all rows of the double complex (Lemma 3.5.5) yields the following lemma.

3.5.7 Lemma (D-Cohomologous Cocycles and Coboundaries)

Consider the foliated Čech-de Rham complex $\check{C}^*(\mathcal{U}, \Omega^*_{\mathcal{F}}) = \bigoplus\limits_{k,\ell \in \mathbb{N}_0} K^{k,\ell}$, then:

(1) Every D-cocycle $\omega = \sum\limits_{k+\ell=j} \omega^{k,\ell} \in A^j$ is D-cohomologous to a D-cocycle $\tilde{\omega} \in K^{j,0} \subseteq A^j$, consisting only of a top component.

(2) If the D-coboundary $\omega = D\nu \in K^{j,0} \subseteq A^j$ of $\nu \in A^{j-1}$ has only a top component, then ν is D-cohomologous to some $\tilde{\nu} \in K^{j-1,0}$ which has only a top component and satisfies $D\tilde{\nu} = \omega$.

Proof. (1) Set $\omega_0 = \omega$. If $\omega_i = \sum\limits_{\substack{k+\ell=j \\ k \geq i}} \omega_i^{k,\ell} \in A^j$ is a D-cocycle, then since $\delta\omega^{i,j-i} = 0$ there is by δ-exactness a $\mu_i \in K^{i,j-1-i} \subseteq A^{j-1}$ with $\delta\mu_i = w_i^{i,j-1}$. Then $\omega_{i+1} = \omega_i - D\mu_i$ is

D-cohomologous to ω_i and of the form $\omega_{i+1} = \sum_{\substack{k+\ell=j \\ k\geq i+1}} \omega_{i+1}^{k,\ell}$. Iterating this procedure at most j-times, any D-cocycle $\omega \in A^j$ is D-cohomologous to $\tilde{\omega} = \omega - D\left(\sum_{i=0}^{j-1} \mu_i\right) \in K^{j,0}$ where $\mu_i \in K^{i,j-1-i}$.

(2) Set $\nu_0 = \nu$. If $\nu_i = \sum_{\substack{k+\ell=j-1 \\ k\geq i}} \nu^{k,\ell}$ for $0 \leq i < j-1$ satisfies $D\nu_i = \omega$, then

$$0 = (D\nu)^{i,j-i} = \delta\nu^{i,j-i-1} + (-1)^{i-1} d_{\mathcal{F}} \nu^{i-1,j-i} = \delta\nu^{i,j-i-1}. \tag{3.176}$$

Thus, by δ-exactness, there is some $\mu_i \in K^{i,j-i-2}$ with $\delta\mu_i = \nu^{i,j-i-1}$. Now, $\nu_{i+1} = \nu_i - D\mu_i$ satisfies $D\nu_{i+1} = D\nu_i = \omega$. Moreover, ν_{i+1} is D-cohomologous to ν_i and of the form $\nu_{i+1} = \sum_{\substack{k+\ell=j-1 \\ k\geq i+1}} \nu_{i+1}^{k,\ell}$. Iterating this procedure at most $(j-1)$-times, we end up with

$$\tilde{\nu} = \nu_{j-1} = \nu - D\left(\sum_{i=0}^{j-2} \mu_i\right) \in K^{j-1,0} \text{ satisfying } D\tilde{\nu} = D\nu. \tag{3.177}$$

□

In [BT82, Proposition 8.8, p. 96] it is shown that the de Rham cohomology is isomorphic to the Čech-de Rham cohomology. This is also true for the corresponding foliated cohomologies by a similar proof using Lemmas 3.5.5 and 3.5.7.

3.5.8 Theorem (Generalized Mayer-Vietoris Principle)

The restriction map $r : \Omega^*_{\mathcal{F}}(M) \to \check{C}^*(\mathcal{U}, \Omega^*_{\mathcal{F}})$, defined for $k \in \mathbb{N}_0$ by the global restriction map $r : \Omega^k_{\mathcal{F}}(M) \to \check{C}^0(\mathcal{U}, \Omega^k_{\mathcal{F}})$, induces an isomorphism between the foliated de Rham cohomology and the foliated Čech-de Rham cohomology, i.e. for each $j \in \mathbb{N}_0$ the cohomology map $r^* : H^j(M, \mathcal{F}) \to H^j_D\{\check{C}^*(\mathcal{U}, \Omega^*_{\mathcal{F}})\}$ is an isomorphism of vector spaces.

Proof. Since $\delta^0 \circ r = 0$ on each $\Omega^k_{\mathcal{F}}(M)$ and $d_{\mathcal{F}}$ commutes with restrictions, we obtain

$$D \circ r = (-1)^k d_{\mathcal{F}} \circ r = (-1)^k r \circ d_{\mathcal{F}} \text{ on } \Omega^k_{\mathcal{F}}(M). \tag{3.178}$$

Hence, r maps $d_{\mathcal{F}}$-cocycles to D-cocycles and $d_{\mathcal{F}}$-coboundaries to D-coboundaries, respectively such that r induces for each $j \in \mathbb{N}_0$ a linear map between cohomologies $r^* : H^j(M, \mathcal{F}) \to H^j_D\{\check{C}^*(\mathcal{U}, \Omega^*_{\mathcal{F}})\}$. We augment the double complex by an initial column consisting of global foliated forms such that all rows of the augmented double complex are exact by Lemma 3.5.5.

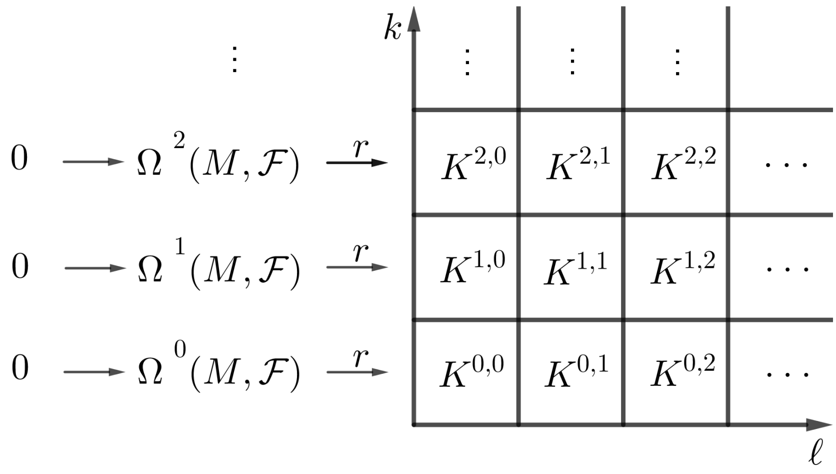


Figure 4: The augmented foliated Čech-de Rham double complex.

(1) r^* is surjective:
Let $\omega \in A^j$ be a D-cocycle. By Lemma 3.5.7 (1) there is some D-cocycle $\tilde{\omega} \in K^{j,0}$ which is in the same D-cohomology class as ω. Since $\delta\tilde{\omega} = 0$, Lemma 3.5.5 yields $\tilde{\omega} \in \Omega^j(M,\mathcal{F})$. Moreover, $d_{\mathcal{F}}\tilde{\omega} = 0$ such that $\tilde{\omega}$ is a $d_{\mathcal{F}}$-cocycle with $r^*([\tilde{\omega}]_{d_{\mathcal{F}}}) = [\tilde{\omega}]_D = [\omega]_D$.

(2) r^* is injective:
Let be $\mu \in \Omega^j(M,\mathcal{F})$ with $d_{\mathcal{F}}\mu = 0$ satisfying $r(\mu) = D\nu$ for some $\nu \in A^{j-1}$. Since the D-coboundary $\eta = D\nu$ is in particular a D-cocycle, Lemma 3.5.7 (1) yields a D-cohomologous coboundary with only a top component $\tilde{\eta} = D\tilde{\nu} \in K^{j,0}$, satisfying $r(\mu) = D\tilde{\nu}$. By Lemma 3.5.7 (2) we can assume $\tilde{\nu} \in K^{j-1,0}$ having only a top component. Now $\delta\tilde{\nu} = 0$ gives a global form $\lambda \in \Omega^{j-1}(M,\mathcal{F})$ with $r(\lambda) = \tilde{\nu}$, such that

$$r(\mu) = D\tilde{\nu} = D \circ r(\lambda) = r((-1)^{j-1} d_{\mathcal{F}}(\lambda)). \tag{3.179}$$

Since r is injective, $\mu = d_{\mathcal{F}}((-1)^{j-1}\lambda)$ is a $d_{\mathcal{F}}$-coboundary.

□

3.5.9 Remark (General Argument for Augmented Double Complex)

Combining Lemma 3.5.7 and Theorem 3.5.8, we have proven:
If all the rows of a double complex, augmented by an initial column, are exact, then the D-cohomology of the double complex is isomorphic to the cohomology of the initial column of the augmented double complex.
By switching the indices of the double complex and changing the horizontal and vertical operator, we get a similar result:
If all the columns of a double complex, augmented by an initial row, are exact, then the

D-cohomology of the double complex is isomorphic to the cohomology of the initial row of the augmented double complex.

We can augment the Čech-de Rham complex by an initial row. Note that the following *presheaf of leafwise constant smooth functions* is just $H^0_{\mathcal{F}}$, the presheaf of the zeroth foliated de Rham cohomology, defined in Examples 3.5.3 (d).

3.5.10 Definition (Čech Complex of Leafwise Constant Functions)

If $(M, \mathcal{F})$ is a smooth foliated manifold, we define a presheaf on M by

$$\mathscr{G}^{\mathrm{lc}}_{\mathcal{F}}(U) = \mathrm{Ker}(d_{\mathcal{F}|_U} : \Omega^0(U, \mathcal{F}|_U) = C^\infty(U) \to \Omega^1(U, \mathcal{F}|_U)) \subseteq C^\infty(U), \tag{3.180}$$

which is the space of leafwise constant smooth functions on U. Further, if $i^U_V : V \to U$ is an inclusion of sets, then $\mathscr{G}^{\mathrm{lc}}_{\mathcal{F}}(i^U_V) = (i^U_V)^* : \mathrm{Ker}(d_{\mathcal{F}|_U}) \to \mathrm{Ker}(d_{\mathcal{F}|_V})$ is the restriction of smooth functions. It is well-defined since $d_{\mathcal{F}}$ commutes with restrictions. We call $\mathscr{G}^{\mathrm{lc}}_{\mathcal{F}}$ the **presheaf of leafwise constant smooth functions**. The Čech complex $\check{C}^*(\mathcal{U}, \mathscr{G}^{\mathrm{lc}}_{\mathcal{F}})$ of an open cover $\mathcal{U} = (U_\alpha)_{\alpha \in J}$ with values in the presheaf of leafwise constant smooth functions augments the Čech-de Rham complex by an initial row such that each inclusion $\check{C}^\ell(\mathcal{U}, \mathscr{G}^{\mathrm{lc}}_{\mathcal{F}}) \to \check{C}^\ell(\mathcal{U}, \Omega^0_{\mathcal{F}})$ is injective.

3.5.11 Lemma (Exact Columns under Good Covers)

If $\mathcal{U} = (U_\alpha)_{\alpha \in J}$ is a good cover, then the augmented columns of the Čech-de Rham complex are exact. To be precise, for each $\ell \in \mathbb{N}_0$, the following sequence is exact:

$$0 \to \check{C}^\ell(\mathcal{U}, \mathscr{G}^{\mathrm{lc}}_{\mathcal{F}}) \to \check{C}^\ell(U, \Omega^0_{\mathcal{F}}) \overset{d_{\mathcal{F}}}{\to} \check{C}^\ell(U, \Omega^1_{\mathcal{F}}) \overset{d_{\mathcal{F}}}{\to} \check{C}^\ell(U, \Omega^2_{\mathcal{F}}) \overset{d_{\mathcal{F}}}{\to} \cdots . \tag{3.181}$$

Proof. The exactness at $\check{C}^\ell(\mathcal{U}, \mathscr{G}^{\mathrm{lc}}_{\mathcal{F}})$ and $\check{C}^\ell(U, \Omega^0_{\mathcal{F}})$ is clear by definiton. For $k \in \mathbb{N}$, the failure of exactness at $\check{C}^\ell(U, \Omega^k_{\mathcal{F}})$ is measured by the cohomology

$$\prod_{\alpha_0 < \ldots < \alpha_\ell} H^k(U_{\alpha_0, \ldots, \alpha_\ell}, \mathcal{F}|_{U_{\alpha_0, \ldots, \alpha_\ell}}). \tag{3.182}$$

Since all sets of the good cover and their finite non-empty intersections are contractible, we obtain by the Poincaré Lemma 3.4.10, that the cohomology vanishes for $k \in \mathbb{N}$. Hence, the sequence above is exact. □

Combining the last results, we get:

3.5.12 Theorem (Foliated Cohomology Isomorphic to Čech Cohomology)

Let $(M, \mathcal{F})$ be a smooth foliated manifold and $\mathcal{U} = (U_\alpha)_{\alpha \in J}$ a good cover of M, indexed by a countable ordered set J. Then, the foliated de Rham cohomology of $(M, \mathcal{F})$ is isomorphic to the Čech cohomology of the good cover with values in the presheaf of leafwise constant smooth functions, i.e. for each $j \in \mathbb{N}_0$ we have

$$H^j(M, \mathcal{F}) \cong H^j_D\{\check{C}^*(\mathcal{U}, \Omega^*_{\mathcal{F}})\} \cong \check{H}^j(\mathcal{U}, \mathscr{G}^{\mathrm{lc}}_{\mathcal{F}}). \tag{3.183}$$

Proof. By Remark 3.5.9, Lemma 3.5.11 yields an isomorphism between each $\check{H}^j(\mathcal{U}, \mathscr{G}^{\mathrm{lc}}_{\mathcal{F}})$ and $H^j_D\{\check{C}^*(\mathcal{U}, \Omega^*_{\mathcal{F}})\}$, which is isomorphic to $H^j(M, \mathcal{F})$ by Proposition 3.5.8. □

3.5.13 Definition/Remark (Direct Limit of Čech-Cohomology)

Recall, an open cover $\mathcal{U} = (U_\alpha)_{\alpha \in J}$ of M is called a **refinement** of another open cover $\mathcal{V} = (V_\beta)_{\beta \in I}$ of M, if for each $\alpha \in J$, there is a $\beta \in I$ with $U_\alpha \subseteq V_\beta$. This relation defines a partial order on $\mathcal{OC}(M)$, the system of all open covers of M. Since two open covers always have a refinement given by the intersections, $\mathcal{OC}(M)$ together with the partial order given by refinements, is a directed set. If $\mathcal{U}$ is a refinement of $\mathcal{V}$, then there is a well-defined map in cohomology, induced by inclusions:

$$\varrho^{\mathcal{U}}_{\mathcal{V}} : \check{H}^*(\mathcal{V}, \mathscr{G}^{\mathrm{lc}}_{\mathcal{F}}) \to \check{H}^*(\mathcal{U}, \mathscr{G}^{\mathrm{lc}}_{\mathcal{F}}). \tag{3.184}$$

This makes $\{H^*(\mathcal{U}, \mathscr{G}^{\mathrm{lc}}_{\mathcal{F}})\}_{\mathcal{U} \in \mathcal{OC}(M)}$ into a direct system of groups and we define $\check{H}^*(M, \mathscr{G}^{\mathrm{lc}}_{\mathcal{F}})$ to be the direct limit of this system:

$$\check{H}^j(M, \mathscr{G}^{\mathrm{lc}}_{\mathcal{F}}) = \lim_{\mathcal{U} \in \mathcal{OC}(M)} \check{H}^j(\mathcal{U}, \mathscr{G}^{\mathrm{lc}}_{\mathcal{F}}). \tag{3.185}$$

Since every open cover has a refinement, which is a good cover (the good covers are cofinal in the set of all covers of a manifold)[26], we receive the following corollary of Theorem 3.5.12:

3.5.14 Corollary (Čech-Cohomology of Space)

Let $(M, \mathcal{F})$ be a smooth foliated manifold. Then, for every good cover $\mathcal{U}$ of M, we have

$$\check{H}^j(M, \mathscr{G}^{\mathrm{lc}}_{\mathcal{F}}) = \check{H}^j(\mathcal{U}, \mathscr{G}^{\mathrm{lc}}_{\mathcal{F}}) \cong H^j(M, \mathcal{F}). \tag{3.186}$$

We can also obtain results for the unfoliated case, which can be found in [BT82, Theorem 8.9 and Corollary 8.9.3, p. 98-99]:

[26] See [BT82, Corollary 5.2, p. 43].

3.5.15 Corollary (De Rham and Čech Cohomology in Unfoliated Case)

If $\mathcal{U}$ is a good cover of a smooth manifold M, then its (unfoliated) de Rham cohomology $H^j(M)$ is isomorphic to $\check{H}^j(\mathcal{U}, \mathbb{R})$, the Čech cohomology of the good cover with values in the presheaf of locally constant functions.
Moreover, if $\mathcal{U}$ is a finite good cover, then $H^*(M)$ is finite dimensional.

Proof. We consider the trivial 1-leaf foliation $\mathcal{F} = \{M\}$ on M. Then the Čech complex of leafwise constant functions $\check{C}(\mathcal{U}, \mathscr{G}_{\mathcal{F}}^{\mathrm{lc}})$ is the Čech complex with values in the presheaf of locally constant functions $\check{C}(\mathcal{U}, \mathbb{R})$ and we receive the first assertion by Theorem 3.5.12. Further, for a finite good cover, it is easy to see, that $\check{H}^*(\mathcal{U}, \mathbb{R})$ is finite dimensional, such that $H^*(M)$ must be also finite dimensional. □

3.6 Künneth Formula For Foliated Cohomology

If we consider two foliated smooth manifolds $(M, \mathcal{F})$ and $(N, \mathcal{G})$, then there is a foliation $\mathcal{F} \times \mathcal{G}$ given by the products of leaves on the product manifold $M \times N$. A statement relating the (co)homology of a product of two objects with the (co)homology of these objects is called a **Künneth formula**. The formula of Theorem 3.4.14 suggestes

$$H^k(M \times N, \mathcal{F} \times \mathcal{G}) \cong \bigoplus_{i+j=k} H^i(M, \mathcal{F}) \otimes H^j(N, \mathcal{G}) \text{ for } k \in \mathbb{N}_0. \tag{3.187}$$

We have to be careful. If we consider the discrete foliation $\mathcal{F} = \{\{x\} : x \in \mathbb{R}\}$ of $\mathbb{R}$, then $H^0(\mathbb{R}, \mathcal{F}) = C^\infty(\mathbb{R})$. Since $\mathcal{F} \times \mathcal{F} = \{\{(x, y)\} : (x, y) \in \mathbb{R}^2\}$ is the discrete foliation on $\mathbb{R}^2$, we have $H^0(\mathbb{R}^2, \mathcal{F} \times \mathcal{F}) = C^\infty(\mathbb{R}^2)$. But the injection $C^\infty(\mathbb{R}) \otimes C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}^2)$ is not surjective. The function $f(x, y) = \exp(xy)$ can not be written as a finite linear combination of functions in $C^\infty(\mathbb{R}) \otimes C^\infty(\mathbb{R})$, for example.[27] Only if we consider a so called topological tensor product $\hat{\otimes}$, we get an isomorphism between $C^\infty(\mathbb{R}) \hat{\otimes} C^\infty(\mathbb{R})$ and $C^\infty(\mathbb{R}^2)$. In general, we have no topology yet on the cohomology to build a topological tensor product. The coboundary operator $d_{\mathcal{F}}$ is a linear and continuous map from $\Omega^k(M, \mathcal{F})$ to $\Omega^{k+1}(M, \mathcal{F})$, which are Fréchet spaces. Since $\mathrm{Ker}(d_{\mathcal{F}})$ is a closed subpsace, it is also a Fréchet space. But the quotient spaces of the cohomology are only Fréchet spaces if $d_{\mathcal{F}}$ has closed image which is by the open mapping theorem equivalent to $d_{\mathcal{F}}$ being a homomorphism. In general, this does not have to be the case as we see in Example 3.6.2 for a slope α which is a Liouville number. A Liouville number is an irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ such that for all $k \in \mathbb{N}_0$ there are relatively prime numbers $m, n \in \mathbb{Z}, n \geq 2$ satisfying

$$\left|\alpha - \frac{m}{n}\right| < \frac{1}{|n|^k}. \tag{3.188}$$

For example Liouville's constant $\sum\limits_{k=1}^{\infty} 10^{-k!}$ is a Liouville number. We need the following characterization of a Liouville number for Example 3.6.2.

[27] See [MSE17] for a contradiction to such a representation.

3.6.1 Lemma (Characterization of a Liouville Number)

An irrational number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is a Liouville number if and only if

$$\forall\, k \in \mathbb{N}\ \exists\, m, n \in \mathbb{Z},\ n \geq 2 : |\alpha n + m| < \frac{1}{(|n| + |m|)^k}. \tag{3.189}$$

Proof. (Let be $k \in \mathbb{N}$. If (3.189) holds, then there are some $m, n \in \mathbb{Z}$, $n \geq 2$ such that we obtain after dividing by n for $\tilde{m} = -m$

$$\left|\alpha - \frac{\tilde{m}}{n}\right| < \frac{1}{n(|n| + |m|)^k} < \frac{1}{n^k}. \tag{3.190}$$

The other implication is more subtle. Let α be a Liouville number and $k \in \mathbb{N}$. Choose $\ell \in \mathbb{N}$ such that $2^\ell > |\alpha|^k$. The Liouville condition yields for $j + 1 \geq 3k + \ell + 1$ some $m, n \in \mathbb{Z}$, $n \geq 2$ satisfying $|\alpha n - m| \leq n^{-j}$. (Note that we can assume α and m to have the same sign for j sufficiently large.) If $\alpha n < m$, then $m - \alpha n < 1/2$, such that $n > \frac{m - 1/2}{\alpha} > 0$ follows. Since $n|m - 1/2| \geq |m|$ for all $m \in \mathbb{Z}$, $n \geq 2$, we can conclude $n^2 > \left|\frac{m}{\alpha}\right|$. Otherwise, $\alpha n > m$ and in particular $n^2 \geq \left|\frac{m}{\alpha}\right|$. Since $n^3 > n + n^2$ for $n \geq 2$, we finally obtain for $\tilde{m} = -m$

$$|\alpha n + \tilde{m}| < \frac{1}{n^j} < \frac{1}{n^\ell(|n| + |m/\alpha|)^k} < \frac{1}{(|n| + |m|)^k}, \tag{3.191}$$

where the last inequality follows by an estimation based on the *binomial theorem* and $n^\ell \geq 2^\ell > |\alpha|^k$ by the choice of ℓ

$$n^\ell(|n| + |m/\alpha|)^k > n^{\ell+k} + |m|^k \frac{n^\ell}{|\alpha|^k} > |n|^k + |m|^k. \tag{3.192}$$

□

3.6.2 Example (Kronecker Foliation of the Torus)

The torus $\mathbb{T}^2 = S^1 \times S^1$ can be obtained as a quotient space $\mathbb{R}^2/\mathbb{Z}^2$ where $x, y \in \mathbb{R}^2$ are equivalent if $x - y \in \mathbb{Z}^2$. More figurative, one can glue the opposite edges of a rectangle $R = [0,1]^2$ together along the same orientations, such that one can imagine $\mathbb{T}^2$ as the surface of a round tire or a donut. Consider the foliation of $\mathbb{R}^2$ (or R) by parallel lines of a fixed slope $\alpha \in \mathbb{R}$. This induces a foliation $\mathcal{F}_\alpha$ on $\mathbb{T}^2$, called **Kronecker foliation** of the torus. (The translation of a line in $\mathbb{R}^2$ by a pair of integers represents the same set in $\mathbb{R}^2/\mathbb{Z}^2$ as the originally line.) The leaves of $\mathcal{F}_\alpha$ are images of curves twisting spirally around the torus, given by $\{\gamma_{\alpha,\theta}(t) = \left(e^{it}, e^{i(\alpha t + \theta)}\right) : t \in \mathbb{R}\}$ as θ ranges over $\mathbb{R}$ (or just $[0, 2\pi)$). If $\alpha \in \mathbb{Q}$, the curves are closed and the leaves of $\mathcal{F}_\alpha$ are circles. (For $\alpha = 0$, we obtain the S^1-foliation of $\mathbb{T}^2$ by points of S^1, see Example 3.4.15.) Otherwise, if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the curves are not closed, such that each leaf is a dense line on the torus.[28] Smooth functions on $\mathbb{T}^2$

[28]See [Lee13, Example 19.18 (Foliations) (e), p. 502].

which are constant on these dense leaves are constant on $\mathbb{T}^2$, such that $H^0(M, \mathcal{F}_\alpha) \cong \mathbb{R}$ for an irrational α. Surprisingly, the first foliated de Rham cohomology $H^1(\mathbb{T}^2, \mathcal{F}_\alpha)$ is not the same for all irrational numbers α. As we see below, if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is not a Liouville number, $H^1(\mathbb{T}^2, \mathcal{F}_\alpha) \cong \mathbb{R}$, i.e. the first foliated cohomology is 1-dimensional and Hausdorff. But if α is a Liouville number, $H^1(\mathbb{T}^2, \mathcal{F}_\alpha)$ is infinite-dimensional and non-Hausdorff. (In that case, the image of $d_\mathcal{F}$ can not be closed or equivalently be a homomorphism between Fréchet spaces.)
The present author was not able to find a proof of the result as stated, by following the references [MS06] and [Hae80] given in [Ber11]. Hence, we include a detailed proof, also based on the work of Greenfield and Wallach, who considered global hypoellipticity of a constant coefficient differential operator P on the torus in [GW72]. P is said to be globally hypoelliptic on $\mathbb{T}^2$ if for any $g \in C^\infty(\mathbb{T}^2)$ and a distribution $f \in \mathcal{D}'(\mathbb{T}^2)$ with $Pf = g$ already follows that $f \in C^\infty(\mathbb{T}^2)$. They proved for an irrational number α that the vector field $P = D_1 - \alpha D_2$ is globally hypoelliptic if and only if α is not a Liouville number. There is also mentioned a connection to the work of Herz on divergence. Herz showed in [Her70, Example 1] that the Lie algebra of derivations induced by constant multiples of vector fields, locally given by $L = \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}$, has a closed divergence if and only if $\frac{\alpha}{\beta}$ is not a Liouville number. Both articles considered the Fourier expansions and deduced a system of equations for the Fourier coefficients, which is solvable depending on whether the slope is a Liouville number or not.

A C^∞-atlas on $\mathbb{T}^2$ is given by smooth charts of the following type, referred to as *standard coordinates*, which are the inverse of the most natural local parametrization of the torus: $(x, y) : U \to I_1 \times I_2$ is a diffeomorphism from an open set $U \subseteq \mathbb{T}^2$ onto a rectangle with open intervals I_i of length shorter than 2π as edges such that $\pi_I \circ x^{-1}(r) = e^{ir}$ and $\pi_{II} \circ y^{-1}(s) = e^{is}$ for the projections $\pi_I, \pi_{II} : \mathbb{T}^2 \to S^1$.
There is a global vector field $X^\alpha \in \Gamma(\mathbb{T}^2, T\mathbb{T}^2)$ with local representation $X^\alpha|_U = \frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial y}$, such that the leaves of $\mathcal{F}_\alpha$ are exactly the images of the integral curves of the vector field X^α, i.e. $\gamma'_{\alpha,\theta}(t) = X^\alpha_{\gamma_{\alpha,\theta}(t)} (\in T_{\gamma_{\alpha,\theta}(t)}\mathbb{T}^2)$ for all $t \in \mathbb{R}$. In particular, X^α is a global frame of the foliated tangent bundle $T\mathcal{F}_\alpha$. Now, there is the global coframe $\xi^\alpha = (X^\alpha)^*$ dual to X^α, which is a global frame of $T^*\mathcal{F}_\alpha$ satisfying $\xi^\alpha(X^\alpha) = 1$ on $\mathbb{T}^2$. In standard coordinates, ξ^α is the foliated differential of the first coordinate $x : U \to \mathbb{R}$, i.e. $\xi^\alpha|_U = d_{\mathcal{F}_\alpha} x$ because for any $z \in U$,

$$(dx)_z(X^\alpha_z) = dx_z\left(\left.\frac{\partial}{\partial x}\right|_z\right) + \alpha\, dx_z\left(\left.\frac{\partial}{\partial y}\right|_z\right) = 1 \tag{3.193}$$

and by definition, we have $d_{\mathcal{F}_\alpha} = i^*_{T\mathcal{F}_\alpha} \circ d$ on $C^\infty(\mathbb{T}^2)$. Hence, for each $\omega \in \Omega^1(\mathbb{T}^2, \mathcal{F}_\alpha)$ there is some $f \in C^\infty(\mathbb{T}^2)$ with $\omega = f \cdot \xi^\alpha$. Since $\mathcal{F}_\alpha$ is a $(1, 1)$-foliation of $\mathbb{T}^2$, every foliated 1-form is already closed. Moreover, $\omega = f \cdot \xi^\alpha$ is exact, if there is some $g \in C^\infty(\mathbb{T}^2)$ with $f \cdot \xi^\alpha = d_{\mathcal{F}_\alpha} g = X^\alpha(g) \cdot \xi^\alpha$, which is equivalent to $f = X^\alpha(g)$. On the other hand, we can consider $f, g \in C^\infty(\mathbb{T}^2)$ as smooth functions from $\mathbb{R}^2$ to $\mathbb{R}$ with $f((u, v) + \kappa) = f((u, v))$,

$g((u,v)+\kappa) = g((u,v))$ for all $(u,v) \in \mathbb{R}^2$ and $\kappa \in \mathbb{Z}^2$. Their Fourier expansions

$$f(u,v) = \sum_{m,n\in\mathbb{Z}} \hat{f}(m,n)\, e^{2\pi i(mu+nv)} \quad \text{and} \quad g(u,v) = \sum_{m,n\in\mathbb{Z}} \hat{g}(m,n)\, e^{2\pi i(mu+nv)} \tag{3.194}$$

yield the characterization of $f = X^\alpha(g)$ by the system of equations

$$\hat{f}(m,n) = 2\pi i(m+\alpha n)\hat{g}(m,n) \text{ for } (m,n) \in \mathbb{Z}^2, \text{ or equivalently by} \tag{3.195}$$

$$\hat{g}(m,n) = \frac{\hat{f}(m,n)}{2\pi i(m+\alpha n)} \text{ for } (m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\} \text{ and } \hat{f}(0,0) = 0. \tag{3.196}$$

Hence, $\omega = f \cdot \xi^\alpha$ is exact if and only if $\hat{f}(0,0) = 0$ and $\hat{g}(m,n) = \hat{f}(m,n)(2\pi i(m+\alpha n))^{-1}$ are the Fourier coefficients for $(0,0) \neq (m,n) \in \mathbb{Z}^2$ of a smooth function $g \in C^\infty(\mathbb{T}^2)$, i.e. for each $j \in \mathbb{N}$ there is a $C > 0$ such that

$$\frac{|\hat{f}(m,n)|}{2\pi|m+\alpha n|} = |\hat{g}(m,n)| \leq \frac{C}{(|m|+|n|)^j} \text{ for sufficiently large } |m|, |n|. \tag{3.197}$$

We refer to [Gra08, Chapter 3] for details about Fourier analysis on the torus.

(1) $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ **is not a Liouville number:**
Let be $f \in C^\infty(\mathbb{T}^2)$ with $f(0,0) = 0$. The negation of (3.189) yields some $k \in \mathbb{N}$ such that $|\alpha n + m| > (|n|+|m|)^{-k}$ for all $m,n \in \mathbb{Z}, n \geq 2$. Let $j \in \mathbb{N}$. Since $f \in C^\infty(\mathbb{T}^2)$, its Fourier coefficients satisfy for $i = j+k$ and some constant $\tilde{C} > 0$ the inequalities $|\hat{f}(m,n)| \leq \tilde{C}(|m|+|n|)^{-i}$ for $|m|, |n|$ sufficiently large. This gives

$$|\hat{g}(m,n)| = \frac{|\hat{f}(m,n)|}{2\pi|m+\alpha n|} < \frac{1}{2\pi}(|n|+|m|)^k|\hat{f}(m,n)| \tag{3.198}$$

$$\leq \frac{\tilde{C}}{2\pi}\frac{(|m|+|n|)^k}{(|m|+|n|)^i} = \frac{C}{(|m|+|n|)^j} \tag{3.199}$$

for $|m|, |n|$ sufficiently large and $C = \frac{\tilde{C}}{2\pi}$. Thus, there is solution $g \in C^\infty(\mathbb{T}^2)$ of $X(g) = f$, such that $f \cdot \xi^\alpha \in \Omega^1(M, \mathcal{F}_\alpha)$ is exact whenever $f_{0,0} = 0$.
If $0 \neq f_{0,0}$, then f must be constant on $\mathbb{T}^2$ since otherwise

$$f_{0,0} = \int_{\mathbb{T}^2} f = \int_0^{2\pi}\int_0^{2\pi} f(e^{ir}, e^{is})dr\, ds = 0 \tag{3.200}$$

by the *fundamental theorem of calculus* and the 2π-periodicity of $t \mapsto e^{it}$. Therefore, $H^1(\mathbb{T}^2, \mathcal{F}_\alpha) \cong \mathbb{R}$ if α is not a Liouville number.

(2) $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ **is a Liouville number:**
According to (3.189), there is a sequence $(m_k, n_k)_{k\in\mathbb{N}} \in (\mathbb{Z}^2 \setminus \{(0,0)\})^{\mathbb{N}}$ such that $|\alpha n_k + m_k| \leq (|n_k|+|m_k|)^{-k}$. For each $\ell \in \mathbb{N}$,

$$f_\ell(u,v) = \sum_{k\in\mathbb{N}} |\alpha n_k + m_k|^{1/\ell}\, e^{2\pi i(m_k u + n_k v)} \tag{3.201}$$

defines a smooth function $f_\ell \in C^\infty(\mathbb{T}^2)$. Indeed, let be $j \in \mathbb{N}$ and $k \geq \ell \cdot j$, then

$$|\hat{f}_\ell(m_k, n_k)| = |\alpha n_k + m_k|^{1/\ell} \leq \frac{1}{(|n_k| + |m_k|)^{k/\ell}} \leq \frac{1}{(|n_k| + |m_k|)^j}. \tag{3.202}$$

But $X(g_\ell) = f_\ell$ has no solution $g_\ell \in C^\infty(\mathbb{T}^2)$ for any $\ell \in \mathbb{N}$, because

$$|\hat{g}_\ell(m_k, n_k)| = \frac{|\hat{f}_\ell(m_k, n_k)|}{2\pi|\alpha n_k + m_k|} = \frac{1}{2\pi|\alpha n_k + m_k|^{1-1/\ell}} \geq \frac{1}{2\pi}(|n_k| + |m_k|)^{k(1-1/\ell)} \tag{3.203}$$

does not even tend to zero. By linearity of the Fourier transformation, it suffices to verify that $(\hat{f}_\ell : \mathbb{Z}^2 \to \mathbb{C})_{\ell \in \mathbb{N}}$ is linearly independent to obtain linearly independence of the family $(f_\ell)_{\ell \in \mathbb{N}}$. Denote $c_k = |\alpha n_k + m_k| > 0$ for each $k \in \mathbb{N}$ and let $J \subseteq \mathbb{N}$ be a finite subset. If $\lambda_\ell \in \mathbb{R}$ for $\ell \in J$ such that $\sum_{\ell \in J} \lambda_\ell \hat{f}_\ell = 0$, we obtain

$$0 = \sum_{\ell \in J} \lambda_\ell \, \hat{f}_\ell(m_k, n_k) = \sum_{\ell \in J} \lambda_\ell \, c_k^{1/\ell} \text{ for each } k \in \mathbb{N}. \tag{3.204}$$

Hence, for each $j \in J$,

$$\lambda_j = -\sum_{\ell \in J \setminus \{j\}} \lambda_\ell \, c_k^{\frac{1}{\ell} - \frac{1}{j}} \text{ for each } k \in \mathbb{N}. \tag{3.205}$$

One can show by induction on the cardinality of J that λ_j is constant for each $k \in \mathbb{N}$ only if all λ_ℓ for $\ell \in J \setminus \{j\}$ (and consequently also λ_j) are zero. We have shown that there are infinitely many linearly independent functions $f_\ell \in C^\infty(\mathbb{T}^2)$ such that the closed 1-forms $f_\ell \cdot \xi^\alpha \in \Omega^1(M, \mathcal{F}_\alpha)$ are not exact. Consequently, $H^1(\mathbb{T}^2, \mathcal{F}_\alpha)$ is infinite-dimensional if α is a Liouville number.

3.6.3 Remark (Topological Tensor Product)

To enable a topological tensor product for the Künneth formula, we need the Hausdorff-property, such that we require $d_\mathcal{F}$ to be a homomorphism. As clarified in [Ber11, p. 259 ff.], $\Omega^k(M, \mathcal{F})$ is a nuclear Fréchet space. Subspaces and quotient spaces by closed subspaces of nuclear spaces are also nuclear.[29] If one factor is a nuclear Fréchet space, the two natural ways to construct a topological tensor product will yield the same topological tensor product, which we denote by $\hat{\otimes}$.[30]

[29] See [Tre67, Proposition 50.1, p. 514].
[30] See [Tre67, Theorem 50.1, p.511].

3.6.4 Definition (Topological Tensor Product Cochain Complex)

Let $E = (E^j, d_E^j)_{j\in\mathbb{N}_0}$ and $F = (F^j, d_F^j)_{j\in\mathbb{N}_0}$ be two cochain complexes consisting of nuclear Fréchet spaces with coboundary operators d_E^j and d_F^j, respectively, which are all homomorphisms. The topological tensor product cochain complex $E\hat{\otimes}F$ is defined by

$$(E\hat{\otimes}F)^j = \bigoplus_{k+\ell=j} E^k\hat{\otimes}F^\ell \tag{3.206}$$

with coboundary operator $D^j = \sum_{k+\ell=j} d_E^k\hat{\otimes}\operatorname{id}_{F^\ell} + (-1)^k \operatorname{id}_{E^k}\hat{\otimes}d_F^\ell$.

In [Ber11] it is shown that there is a Künneth formula for foliated de Rham cohomology in two situations:

3.6.5 Theorem (Künneth Formula for Foliated Cohomology)

Let $(M,\mathcal{F})$ and $(N,\mathcal{G})$ be two smooth foliated manifolds.

(1) If $d_\mathcal{F}$ and $d_\mathcal{G}$ are homomorphisms, then

$$H^j(M\times N,\mathcal{F}\times\mathcal{G}) \cong \bigoplus_{k+\ell=j} H^k(M,\mathcal{F})\hat{\otimes}H^\ell(N,\mathcal{G}). \tag{3.207}$$

(2) If the foliated de Rham cohomology of one factor is finite dimensional and the underlying manifold of that factor is compact, then the Künneth formula with only the (algebraic) tensor product is valid

$$H^j(M\times N,\mathcal{F}\times\mathcal{G}) \cong \bigoplus_{k+\ell=j} H^k(M,\mathcal{F})\otimes H^\ell(N,\mathcal{G}). \tag{3.208}$$

For (2), the proof requires a continuous right inverse of the foliated Cartan differential on the compact manifold, which is obtained by using the following splitting theorem of exact sequences of Fréchet spaces, Theorem 3.6.7. It can be found in [MV97, Theorem 30.1, p. 378] and uses the properties (DN) and (Ω) of Fréchet spaces, which we will now define [31]. We also present the closely related properties $(\mathrm{DN}_{\mathrm{loc}})$ and (Ω_{loc}) which are used in Theorem 3.6.8 and can be found in [Vog04, Definition 2.1 and Definition 2.4, p. 815 f.]

3.6.6 Definition (Properties (DN), (Ω), $(\mathrm{DN})_{\mathrm{loc}}$ and $(\Omega)_{\mathrm{loc}}$)

A Fréchet space X with an increasing fundamental system of seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ is said to have property

(DN) if there is a so called **dominating norm** $\|\cdot\|$ on X, which is a continuous norm such that for every $k\in\mathbb{N}$ there are $K\in\mathbb{N}$ and $C>0$ satisfying $\|\cdot\|_k^2 \le C\|\cdot\|\,\|\cdot\|_K$;

[31] See [MV97, Definition, p. 359 and p. 367]

(Ω) if for every $k \in \mathbb{N}$ there is some $m \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ there are $0 < \theta < 1$ and $C > 0$ such that the dual seminorms satisfy $\|\cdot\|_m^* \le C\,(\|\cdot\|_k^*)^{1-\theta}\,(\|\cdot\|_n^*)^{\theta}$;

(DN)$_{\text{loc}}$ if for every $n \in \mathbb{N}$ the quotient map $q_n : X \to X/\mathrm{Ker}(\|\cdot\|_n)$ factorizes through a Fréchet space Y with property (DN), i.e. there are linear and continuous maps $f : X \to Y$ and $g : Y \to X/\mathrm{Ker}(\|\cdot\|_n)$ with $q_n = g \circ f$;

(Ω)$_{\text{loc}}$ if X has property (Ω) and for every $k \in \mathbb{N}$ there are some $K \in \mathbb{N}$, $K \ge k$ and a Fréchet space Z with property (Ω) such that $\mathrm{Ker}(\|\cdot\|_K) \subseteq Z \subseteq \mathrm{Ker}(\|\cdot\|_k)$ with continuous inclusions.

Note that $(\Omega)_{\text{loc}}$ is a stronger property than (Ω) while $(\text{DN})_{\text{loc}}$ is weaker than (DN).

3.6.7 Theorem (Splitting Theorem)

Let E, F, G be Fréchet-Hilbert spaces and let $0 \to F \to G \to E \to 0$ be a short exact sequence of continuous linear maps. If E has property (DN) and F has property (Ω), then the sequence splits.

We refer to [Ber11, p. 270 ff.] why this theorem is applicable. In short, by the compactness of the manifold, the foliated differential forms are isomorphic, as a topological vector space, to the Schwartz space $s = \{(x_j)_{j\in\mathbb{N}} : \sum_{j=1}^{\infty} |x_j|^2 j^{2k} < \infty \,\forall k \in \mathbb{N}\}$ which satisfies the invariants (DN) and (Ω). If the manifold is not compact, then the foliated differential k-forms are isomorphic as topological vector spaces to $s^{\mathbb{N}}$, which does not any more satisfy the property (DN). But $s^{\mathbb{N}}$ satisfies the properties (DN_{loc}) and (Ω_{loc}).[32] In this situation, another splitting theorem[33] might be applicable, if one can show that the foliated Cartan differential is an SK-homomorphism:

3.6.8 Theorem (Splitting Theorem)

Let $0 \to G \to E \xrightarrow{A} F \to 0$ be an exact sequence of nuclear Fréchet spaces and A an SK-homomorphism. If F has property (DN_{loc}) and G has property (Ω_{loc}), then the sequence splits.

We will have a closer look at the splitting theory and SK-properties in the next chapter.

[32] [Vog04, Lemma 2.2 and Lemma 2.5, p. 815 ff.].
[33] [Vog04, Theorem 3.5, p. 820].

4 Splitting Theory

4.1 SK-Topology

A locally convex space X is a vector space over $\mathbb{R}$ (or $\mathbb{C}$) with a directed system of seminorms $\mathcal{P}$, i.e. for each $p, q \in \mathcal{P}$ there is $r \in \mathcal{P}$ with $r \geq p$ and $r \geq q$, such that the topology $\mathcal{T}$, induced by the seminorm balls $B_p(x, \varepsilon) = \{y \in X : p(x-y) < \varepsilon\}$ $(p \in \mathcal{P},\ x \in X,\ \varepsilon > 0)$, is Hausdorff or equivalently for each $x \in X \setminus \{0\}$ there is $p \in \mathcal{P}$ with $p(x) \neq 0$. Of course, every $p \in \mathcal{P}$ is continuous with respect to this topology and the set $\mathrm{CS}(X)$ of all continuous seminorms generates the same topology such that $\mathcal{P}$ is often assumed to be the system of all continuous seminorms on X, which we will denote by $\mathrm{CS}(X)$. Note that $\mathrm{CS}(X)$ is a directed system since finite sums of continuous seminorms are continuous seminorms. We refer to [Vog04] and [Mei10] for an introduction of the presented SK-theory in this chapter.

4.1.1 Motivation

Let $f : X \to Y$ be a continuous linear map between locally convex spaces X and Y. In general, a (linear) right inverse $R : \mathrm{Im}(f) \to X$ of f will not be continuous. But, if we consider a continuous linear right inverse R, then for each $p \in \mathrm{CS}(X)$ there is $q \in \mathrm{CS}(Y)$ such that $q \geq p \circ R$ on $\mathrm{Im}(f)$. Hence,

$$\mathrm{Ker}(q) \cap \mathrm{Im}(f) \subseteq \{y \in \mathrm{Im}(f) : p(R(y)) = 0\} = f(\{x \in X : p(x) = 0\}) = f(\mathrm{Ker}(p)). \tag{4.209}$$

Therefore, the condition

$$\forall p \in \mathrm{CS}(X)\ \exists\, q \in \mathrm{CS}(Y) : \mathrm{Ker}(q) \cap \mathrm{Im}(f) \subseteq f(\mathrm{Ker}(p)) \tag{4.210}$$

is necessary for the existence of a continuous linear right inverse of f. Moreover, since f is a continuous linear map, we obtain,

$$\forall q \in \mathrm{CS}(Y)\ \exists\, p(= q \circ f) \in \mathrm{CS}(X) : f(\mathrm{Ker}\, p) \subseteq \mathrm{Ker}\, q. \tag{4.211}$$

In both observations, the kernels of seminorms play a central role, such that it is advisable to study the system of seminorm kernels $\mathrm{SK}(X) = \{\mathrm{Ker}(p) : p \in \mathrm{CS}(X)\}$. This collection is stable under finite intersections by adding seminorms but fails to be stable under unions, hence it is not a topology yet. However, we can construct the coarsest topology containing $\mathrm{SK}(X)$, the so called **SK-topology**, such that the seminorm kernels are neighbourhoods of zero. The condition (4.211) is then just the continuity of f with respect to the SK-topologies. In that case, f is a homomorphism with respect to the SK-topologies iff (4.210) holds. We collect these considerations in the following definition:

4.1.2 Definition (SK-Topology)

Let X, Y be locally convex spaces and $f : X \to Y$ be a linear map.

(a) The topology,

$$\mathcal{T}^{\mathrm{SK}} = \{U \subseteq X : \forall x \in U \,\exists\, p \in \mathrm{CS}(X) \text{ with } x + \mathrm{Ker}(p) \subseteq U\}, \tag{4.212}$$

induced by the system of seminorm kernels $\mathrm{SK}(X) = \{\mathrm{Ker}(p) : p \in \mathrm{CS}(X)\}$, is called the **SK-topology** of X.

(b) f is called **SK-continuous**, if

$$\forall q \in \mathrm{CS}(Y) \,\exists\, p \in \mathrm{CS}(X) : f(\mathrm{Ker}\, p) \subseteq \mathrm{Ker}\, q. \tag{4.213}$$

(c) f is called an **SK-homomorphism**, if it is SK-continuous and

$$\forall p \in \mathrm{CS}(X) \,\exists\, q \in \mathrm{CS}(Y) : \mathrm{Ker}(q) \cap \mathrm{Im}(f) \subseteq f(\mathrm{Ker}(p)). \tag{4.214}$$

4.1.3 Lemma (SK-Topology and Additive Hausdorff-Group)

Let X be a locally convex space. Then $\mathcal{T}^{\mathrm{SK}}$ is a topology on X such that $\mathrm{SK}(X)$ is a base of 0-neighbourhoods. Moreover, $(X, \mathcal{T}^{\mathrm{SK}}, +)$ is a topological Hausdorff group, i.e. the maps $+ : X \times X \to X$ $(x, y) \mapsto x + y$ and $- : X \to X$, $x \mapsto -x$ are continuous with respect to $\mathcal{T}^{\mathrm{SK}}$ and $(X, \mathcal{T}^{\mathrm{SK}})$ is a Hausdorff space.
If a system of seminorms $\mathcal{P}$ on X is equivalent to $\mathrm{CS}(X)$, then

$$\mathcal{T}^{sk} = \{U \subseteq X : \forall x \in U \,\exists\, p \in \mathcal{P} \text{ with } x + \mathrm{Ker}(p) \subseteq U\}. \tag{4.215}$$

Proof. $\emptyset, X \in \mathcal{T}^{\mathrm{SK}}$ and the stability under unions follow straight from the definition of $\mathcal{T}^{\mathrm{SK}}$. We have to verify that $U \cap V \in \mathcal{T}^{\mathrm{SK}}$ for $U, V \in \mathcal{T}^{\mathrm{SK}}$. For each $x \in U \cap V$ there are $p, q \in \mathrm{CS}(X)$ such that $x + \mathrm{Ker}(p) \subseteq U$ and $x + \mathrm{Ker}(q) \subseteq V$. Then $r = p + q \in \mathrm{CS}(X)$ and $\mathrm{Ker}(r) = \mathrm{Ker}(p) \cap \mathrm{Ker}(q)$. Hence $x + \mathrm{Ker}(r) = (x + \mathrm{Ker}(p)) \cap (x + \mathrm{Ker}(q)) \subseteq U \cap V$, such that $\mathcal{T}^{\mathrm{SK}}$ is indeed a topology.
Let $x, y \in X$ and $W \in \mathcal{T}^{\mathrm{SK}}$ be a neighbourhood of $x + y$, then there is a $p \in \mathrm{CS}(X)$ such that $(x + y) + \mathrm{Ker}(p) \subseteq W$. Since $\mathrm{Ker}(p) + \mathrm{Ker}(p) \subseteq \mathrm{Ker}(p)$, by the triangle inequality, it follows that $(x + \mathrm{Ker}(p)) + (y + \mathrm{Ker}(p)) \subseteq (x + y) + \mathrm{Ker}(p)$.
Let $\mathcal{P}$ be a system of seminorms which is equivalent to $\mathrm{CS}(X)$. It suffices to show, that $\{\mathrm{Ker}(p) : p \in \mathcal{P}\}$ is equivalent to $\mathrm{SK}(X)$. Let $p \in \mathcal{P}$, then there is a $C > 0$ and a $q \in \mathrm{CS}(X)$ with $p \leq C\, q$. Hence, $\mathrm{Ker}(q) \subseteq \mathrm{Ker}(p)$. On the other hand, if $q \in \mathrm{CS}(X)$, then there is a $C > 0$ and a $p \in \mathcal{P}$ with $q \leq C\, p$. Then $\mathrm{Ker}(p) \subseteq \mathrm{Ker}(p)$. Therefore, the induced topologies coincide. □

4.1.4 Lemma (Compatibility)

(a) Let $(X_i, \mathcal{T}_i)_{i\in I}$ be a family of locally convex spaces, then the SK-topology of the product topology is the product topology of the SK-topologies,

$$\left(\prod_{i\in I} \mathcal{T}_i\right)^{\mathrm{SK}} = \prod_{i\in I} \mathcal{T}_i^{\mathrm{SK}}. \tag{4.216}$$

(b) Let Y be a subspace of a locally convex space $(X, \mathcal{T})$, then the SK-topology of the relative topology is the relative topology of the SK-topology,

$$(\mathcal{T}|_Y)^{\mathrm{SK}} = (\mathcal{T}^{\mathrm{SK}})|_Y. \tag{4.217}$$

(c) Let Y be a closed subspace of a locally convex space $(X, \mathcal{T})$, then the SK-topology of the quotient topology is coarser than the quotient topology of the SK-topology,

$$\left(\mathcal{T}_{X/Y}\right)^{\mathrm{SK}} \subseteq \left(\mathcal{T}^{\mathrm{SK}}\right)_{X/Y}. \tag{4.218}$$

(d) Let $(X, \mathcal{T})$ be a locally convex space, then the SK-topology is finer than the originally topology $\mathcal{T}$,

$$\mathcal{T} \subseteq \mathcal{T}^{\mathrm{SK}}. \tag{4.219}$$

Proof. (a) Let $U \subseteq \prod_{i\in I} X_i$ be a $(\prod_{i\in I} \mathcal{T}_i)^{\mathrm{SK}}$-neighbourhood of zero. Then there is a continuos seminorm p on $\prod_{i\in I} X_i$ with $\mathrm{Ker}(p) \subseteq U$. The continuity yields a finite subset $E \subseteq I$, $C > 0$ and $p_i \in \mathrm{CS}(X_i)$ for each $i \in E$, such that

$$p((x_i)_{i\in I}) \leq C \sup\{p_i(x_i) : i \in E\} \text{ for all } (x_i)_{i\in I} \in \prod_{i\in I} X_i. \tag{4.220}$$

Therefore, we obtain $\prod_{i\in E} \mathrm{Ker}(p_i) \times \prod_{i\in I\setminus E} X_i \subseteq \mathrm{Ker}(p) \subseteq U$, which implies that U is a $\left(\prod_{i\in I} \mathcal{T}_i^{\mathrm{SK}}\right)$-neighbourhood of zero.

Vice versa, a $\left(\prod_{i\in I} \mathcal{T}_i^{\mathrm{SK}}\right)$-neighbourhood of zero U yields a finite set $E \subseteq I$ and $p_i \in \mathrm{CS}(X_i)$ for each $i \in I$ with $\prod_{i\in E} \mathrm{Ker}(p_i) \times \prod_{i\in I\setminus E} X_i \subseteq U$. Then, $p((x_i)_{i\in I}) = \sup\{p_i(x_i) : i \in E\}$ defines a continuous seminorm on $\prod_{i\in I} X_i$ with $\mathrm{Ker}(p) \subseteq U$.

(b) Let $U \subseteq Y$ be a $(\mathcal{T}|_Y)^{\mathrm{SK}}$-neighbourhood of zero. Hence, there is a $p \in \mathrm{CS}(X)$ with $\mathrm{Ker}(p|_Y) \subseteq U$. Since $Y \cap \mathrm{Ker}(p) = \mathrm{Ker}(p|_Y)$, this is equivalent to U being a $(\mathcal{T}^{\mathrm{SK}})|_Y$-neighbourhood of zero.

(c) Let $U \subseteq X/Y$ be a $(\mathcal{T}_{X/Y})^{\mathrm{SK}}$-neighbourhood of zero, hence there is $p \in \mathrm{CS}(X)$ defining $\tilde{p} \in \mathrm{CS}(X/Y)$ by $\tilde{p}([x]) = \inf\{p(x+y) : y \in Y\}$ such that $\mathrm{Ker}(\tilde{p}) \subseteq U$. For the quotient map $\pi : X \to X/Y$, we obtain $\mathrm{Ker}(p) \subseteq \pi^{-1}(U)$ such that $\pi^{-1}(U)$ is a $\mathcal{T}^{\mathrm{SK}}$-neighbourhood of zero and U is a $(\mathcal{T}^{\mathrm{SK}})_{X/Y}$-neighbourhood of zero.
(d) Let $U \subseteq X$ be a $\mathcal{T}$-neighbourhood of zero. Then, there are an $\varepsilon > 0$ and a $p \in \mathrm{CS}(X)$ with $B_p(0,\varepsilon) = \{x \in X : p(x) < \varepsilon\} \subseteq U$. Therefore, $\mathrm{Ker}(p) \subseteq U$ follows, such that U is a $\mathcal{T}^{\mathrm{SK}}$-neighbourhood of zero. □

4.1.5 Remarks/Examples

(1) **(SK-Theory for Normed Spaces)**
If X has a norm $\|\cdot\| \in \mathrm{CS}(X)$, then $\{0\}$ is a $\mathcal{T}^{\mathrm{SK}}$-neighbourhood of zero, since $\mathrm{Ker}(\|\cdot\|) \subseteq \{0\}$. Therefore, $\mathcal{T}^{\mathrm{SK}}$ is the discrete topology on X. On the other hand, if $\mathcal{T}^{\mathrm{SK}}$ is the discrete topology on X, then there is a $p \in \mathrm{CS}(X)$ with $\mathrm{Ker}(p) \subseteq \{0\}$, such that p is indeed a norm on X.
If Y has a norm $\|\cdot\| \in \mathrm{CS}(Y)$, then every $\Phi \in L(X,Y)$ is already an SK-homomorphism because,

$$\{0\} = \mathrm{Ker}\,\|\cdot\| \cap \mathrm{Im}(\Phi) \subseteq \Phi(\mathrm{Ker}(p)) \text{ for every } p \in \mathrm{CS}(X). \tag{4.221}$$

(2) **(Topological homomorphism that is no SK-homomorphism)**
Consider the linear function

$$\Phi : C^\infty([-1,1]) \to \mathbb{R}^{\mathbb{N}_0}, \Phi(f) = (f^{(n)}(0))_{n\in\mathbb{N}_0}. \tag{4.222}$$

By a classical Theorem of E. Borel[34], Φ is a surjective map. Since Φ is a map between Fréchet spaces it is also a topological homomorphism by the open mapping theorem. Since $[-1,1]$ is compact, there is a continuous norm on $C^\infty([-1,1])$, given by

$$\|f\| := \sup\{|f(x)| : x \in [-1,1]\}. \tag{4.223}$$

Assume Φ is an SK-homomorphism, then there is a $q \in \mathrm{CS}(\mathbb{R}^{\mathbb{N}_0})$ satisfying

$$\mathrm{Ker}(q) \subseteq \Phi(\mathrm{Ker}\,\|\cdot\|)) = \{0\}. \tag{4.224}$$

Hence, q would be a continuous norm. But there is no norm on $\mathbb{R}^{\mathbb{N}_0}$:
A fundamentalsystem of seminorms on $\mathbb{R}^{\mathbb{N}_0}$ is given by $(p_k)_{k\in\mathbb{N}_0}$ defined by

$$p_k((x_n)_{n\in\mathbb{N}_0}) = \sup\{|x_i| : i = 0, ..., k\}. \tag{4.225}$$

So the continuity of q yields a $C > 0$ and a $k \in \mathbb{N}_0$ with $q \leq C\,p_k$ on $\mathbb{R}^{\mathbb{N}}_0$. But for $n \geq k+1$ and $(e_n)_i = \delta_{n,i}$ $(i \in \mathbb{N}_0)$, we obtain a contradiction,

$$q(e_n) > 0 = C\,p_k(e_n). \tag{4.226}$$

[34]For instance, see [MV97, Proposition 26.29.]

More generally, we have seen, a surjective map $\Phi \in L(X, Y)$ can not be an SK-homomorphism, if the domain has a norm but the range does not. Further, the quotient space of $C^\infty([-1, 1])$ by its closed subspace $\mathrm{Ker}(\Phi)$ is an example, where the SK-topology of the quotient topology is strictly coarser than the quotient topology of the SK-topology.

4.1.6 Lemma (Characterization of SK-Homomorphism)

Let $f : X \to Y$ be a linear and continuous map between locally convex spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$. The following assertions are equivalent:

(a) f is an SK-homomorphism.

(b) $f : X \to \mathrm{Im}(f)$ is an SK-homomorphism.

(c) For all $p \in \mathrm{CS}(X)$ there is $q \in \mathrm{CS}(Y)$ with $f^{-1}(\mathrm{Ker}(q)) \subseteq \mathrm{Ker}(p) + \mathrm{Ker}(f)$.

Proof.

(a)⇒(b): Let $p \in \mathrm{CS}(X)$, then there is $q \in \mathrm{CS}(Y)$ with $\mathrm{Ker}(q) \cap (\mathrm{Im}(f)) \subseteq f(\mathrm{Ker}(p)$. But then $q|_{\mathrm{Im}(f)} \in \mathrm{CS}(\mathrm{Im}(f))$ satisfies $\mathrm{Ker}(q|_{\mathrm{Im}(f)}) \cap \mathrm{Im}(f) = \mathrm{Ker}(q) \cap \mathrm{Im}(f) \subseteq f(\mathrm{Ker}(p))$.

(b)⇒(c) Let $p \in \mathrm{CS}(X)$, then there is a $q|_{\mathrm{Im}(f)} \in \mathrm{CS}(\mathrm{Im}(f))$ induced by $q \in \mathrm{CS}(Y)$ satisfying $\mathrm{Ker}(q) \cap \mathrm{Im}(f) = \mathrm{Ker}(q|_{\mathrm{Im}(f)}) \cap \mathrm{Im}(f) \subseteq f(\mathrm{Ker}(p))$. For $x \in f^{-1}(\mathrm{Ker}(q))$ we know $f(x) \in \mathrm{Ker}(q) \cap \mathrm{Im}(f) \subseteq f(\mathrm{Ker}(p))$, such that there is a $\tilde{x} \in \mathrm{Ker}(p)$ with $f(x) = f(\tilde{x})$. Hence, $x = \tilde{x} + (x - \tilde{x}) \in \mathrm{Ker}(p) + \mathrm{Ker}(f)$.

(c)⇒(a) Let $p \in \mathrm{CS}(X)$, then there is a $q \in \mathrm{CS}(X)$ with $f^{-1}(\mathrm{Ker}(q)) \subseteq \mathrm{Ker}(p) + \mathrm{Ker}(f)$. Applying f yields already $\mathrm{Ker}(q) \cap \mathrm{Im}(f) \subseteq f(\mathrm{Ker}(p))$.

□

4.1.7 Lemma (Composition of SK-Homomorphisms)

Let $f : X \to Y$ and $g : Y \to Z$ be linear and continuous maps between locally convex spaces. If f and g are SK-homomorphisms and $\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$, then $g \circ f$ is an SK-homomorphism.

Proof.
It follows by $\mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$ that

$$g(V) \cap g(\mathrm{Im}(f)) \subseteq g(V \cap \mathrm{Im}(f)) \text{ for } V \subseteq Y. \tag{4.227}$$

(For $v \in V$ and $x \in X$ with $z = g(v) = g(f(x))$ it follows, that $v - f(x) \in \mathrm{Ker}(g) \subseteq \mathrm{Im}(f)$. So, there is $a \in X$ with $f(a) = v - f(x)$ such that $v = f(a + x) \in \mathrm{Im}(f) \cap V$ and $g(v) = z$.) Now start with $p \in \mathrm{CS}(X)$, then the SK-property of f yields an $r \in \mathrm{CS}(Y)$ satisfying

$$\mathrm{Ker}(r) \cap \mathrm{Im}(f) \subseteq f(\mathrm{Ker}(p)). \tag{4.228}$$

By the SK-property of g, there is $q \in \mathrm{CS}(Z)$ with

$$\mathrm{Ker}(q) \cap \mathrm{Im}(g) \subseteq g(\mathrm{Ker}(r)). \tag{4.229}$$

Further, since $\mathrm{Im}(g \circ f) = g(\mathrm{Im}(f)) \subseteq \mathrm{Im}(g)$, we obtain

$$\mathrm{Ker}(q) \cap \mathrm{Im}(g \circ f) \subseteq g(\mathrm{Ker}(r)) \cap g(\mathrm{Im}(f)) \overset{(4.227)}{\subseteq} g(\mathrm{Ker}(r) \cap \mathrm{Im}(f)) \subseteq g \circ f(\mathrm{Ker}(p)). \tag{4.230}$$

□

4.1.8 Lemma (SK-Homomorphism Criterion)

Consider the following commutative diagram of linear and continuous maps between locally convex spaces:

$$\begin{array}{ccc} X & \xrightarrow{f} & Z \\ {\scriptstyle h}\downarrow & \nearrow{\scriptstyle g} & \\ Y & & \end{array} . \tag{4.231}$$

If f is an SK-homomorphism and h is surjective, then g is an SK-homomorphism.
In particular, if Y is a closed subspace of X, and $f : X \to Z$ is a linear and continuous SK-homomorphism that is constant on equivalence classes of X/Y, then the induced linear and continuous map $\tilde{f} : X/Y \to Z$ is an SK-homomorphism.

Proof. Let be $p \in \mathrm{CS}(Y)$, then there is $r \in \mathrm{CS}(X)$ with

$$h(\mathrm{Ker}(r)) \subseteq \mathrm{Ker}(p) \tag{4.232}$$

since h is SK-continuous. Because f is an SK-homomorphism, there is $q \in \mathrm{CS}(Z)$ satisfying

$$\mathrm{Ker}(q) \cap \mathrm{Im}(f) \subseteq f(\mathrm{Ker}(r)). \tag{4.233}$$

The surjectivity of h and $f = g \circ h$ implies $\mathrm{Im}(f) = \mathrm{Im}(g)$ and we obtain

$$\mathrm{Ker}(q) \cap \mathrm{Im}(g) \subseteq g(h(\mathrm{Ker}(r))) \subseteq g(\mathrm{Ker}(p)), \tag{4.234}$$

such that g is an SK-homomorphism. Finally, the special case follows immediately for g being the surjective quotient map from X to X/Y. □

Next, we want to characterize the closed subspaces Y of a locally convex space $(X, \mathcal{T})$, such that $\left(\mathcal{T}_{X/Y}\right)^{\mathrm{SK}} = (\mathcal{T}^{\mathrm{SK}})_{X/Y}$ holds. We call such a subspace an **SK-subspace**.
For $p \in \mathrm{CS}(X)$ we set

$$\mathcal{T}_p := \{U \subseteq X : \forall\, x \in U\ \exists\, \varepsilon > 0 \text{ such that } B_p(x, \varepsilon) \subseteq U\}. \tag{4.235}$$

$\mathcal{T}_p$ is a topology on X induced by the system of p-balls and for a subset $S \subseteq X$, we denote by $\overline{S}^{\,p}$ the closure of S with respect to the topology $\mathcal{T}_p$. For instance $\overline{\{0\}}^{\,p} = \mathrm{Ker}(p)$.

4.1.9 Lemma (Characterization SK-Subspaces)

Let Y be a closed subspace of a locally convex space $(X, \mathcal{T})$. Then the following are equivalent:

(a) $\left(\mathcal{T}_{X/Y}\right)^{\mathrm{SK}} = (\mathcal{T}^{\mathrm{SK}})_{X/Y}$.

(b) For every $p \in \mathrm{CS}(X)$ there is a $q \in \mathrm{CS}(X)$ with $\overline{Y}^{\,q} \subseteq \mathrm{Ker}(p) + Y$.

(c) The quotient map $\pi : X \to X/Y$ is an SK-homomorphism.

Proof. Every $p \in \mathrm{CS}(X)$ induces a $\tilde{p} \in \mathrm{CS}(X/Y)$ by setting $\tilde{p}([x]) = \inf\{p(x+y) : y \in Y\}$. Then:

$$(*) \qquad \pi^{-1}(\mathrm{Ker}(\tilde{p})) \subseteq \mathrm{Ker}(p) + Y.$$

(Let $x \in X \setminus Y$ with $\tilde{p}(\pi(x)) = 0$ and suppose $x \notin \mathrm{Ker}(p)$. Since p is $\mathcal{T}_p$-continuous, its kernel is $\mathcal{T}_p$-closed ($\mathrm{Ker}(p) = \overline{\mathrm{Ker}(p)}^{\,p}$). Hence there is an $\varepsilon > 0$ with $B_p(x, \varepsilon) \cap \mathrm{Ker}(p) = \emptyset$. But since $\tilde{p}(\pi(x)) = 0$, there is a $y \in Y$ with $p(x-y) < \varepsilon$ and we obtain a contradiction.)

(a)⇒(b) Let $p \in \mathrm{CS}(X)$. Since $\tilde{p} \in \mathrm{CS}(X/Y)$, we get for $\widetilde{\mathrm{Ker}(p)} \in (\mathcal{T})^{\mathrm{SK}}_{X/Y}$ an $U \in (\mathcal{T}^{\mathrm{SK}})_{X/Y}$ with $U \subseteq \mathrm{Ker}(\tilde{p})$ by (a). Then, $\pi^{-1}(U) \in \mathcal{T}^{\mathrm{SK}}$ by definition of the quotient topology, such that there is a $q_0 \in \mathrm{CS}(X)$ with $\mathrm{Ker}(q_0) \subseteq \pi^{-1}(U)$. If we set $q = \tilde{q}_0 \circ \pi$, then $q \in \mathrm{CS}(X)$ and $Y \in \mathrm{Ker}(q)$. Since, $\overline{\mathrm{Ker}(q)}^{\,q} = \mathrm{Ker}(q)$, we obtain

$$\overline{Y}^{\,q} \subseteq \mathrm{Ker}(q) = \mathrm{Ker}(\tilde{q}_0 \circ \pi) = \pi^{-1}(\mathrm{Ker}(\tilde{q}_0)) \tag{4.236}$$

$$\overset{(*)}{\subseteq} \mathrm{Ker}(q_0) + Y \subseteq \pi^{-1}(U) \subseteq \pi^{-1}(\mathrm{Ker}(\tilde{p})) \tag{4.237}$$

$$\overset{(*)}{\subseteq} \mathrm{Ker}(p) + Y. \tag{4.238}$$

(b)⇒ (c) Let $p \in \mathrm{CS}(X)$. By (2), there is a $q \in \mathrm{CS}(X)$ with $\overline{Y}^{\,q} \subseteq \mathrm{Ker}(p) + Y$. Then, $\mathrm{Ker}(\tilde{q}) \subseteq \pi(\overline{Y}^{\,q})$. (Otherwise there is a $x \in X$ with $\tilde{q}(\pi(x)) = 0$ and $x \notin \overline{Y}^{\,q}$. The last condition yields an $\varepsilon > 0$ with $B_q(x, \varepsilon) \cap F = \emptyset$. But this is a contradiction to $\tilde{q}(\pi(x)) = 0$, because there is a $y \in Y$ with $p(x+y) < \varepsilon$.) This yields the SK-condition,

$$\mathrm{Ker}(\tilde{q}) \subseteq \pi(\overline{Y}^{\,q}) \subseteq \pi(\mathrm{Ker}(p)). \tag{4.239}$$

(c)⇒ (a) By 4.1.4 (c), we have only to show $(\mathcal{T}^{\mathrm{SK}})_{X/Y} \subseteq (\mathcal{T}_{X/Y})^{\mathrm{SK}}$. Let U be a $(\mathcal{T}^{\mathrm{SK}})_{X/Y}$-neighbourhood of zero. Then, $\pi^{-1}(U)$ is a $\mathcal{T}^{\mathrm{SK}}$-neighbourhood of zero, such that there is a $p \in \mathrm{CS}(X)$ with $\mathrm{Ker}(p) \subseteq \pi^{-1}(U)$. By (3), there is a $q \in \mathrm{CS}(X/Y)$ with $\mathrm{Ker}(q) \subseteq \pi(\mathrm{Ker}(p)) \subseteq U$. Hence, U is a $(\mathcal{T}_{X/Y})^{\mathrm{SK}}$-neighbourhood of zero.

□

We will verify that the closed subspace $\Omega^k_{\perp\!\!\perp}(M, \mathcal{F}) = \Gamma(M, \Lambda^k(T^*M \perp\!\!\perp T\mathcal{F}))$ of $\Omega^k(M)$ is an SK-subspace. The sections in the k-annihilator vector bundle were used to give an isomorphic construction of the foliated differential k-forms. See 4.1.10 and 3.2.3 for details. At first, we deal with the local situation:

4.1.10 Lemma (k-Annihilator Criterion)

Let $\mathcal{F} = \{\mathcal{L}_\alpha\}_{\alpha\in\mathscr{A}}$ be a p-foliation of an n-manifold M and let $(X_1, ..., X_n)$ be a frame of TM over some open set $U \subseteq M$ such that $(X_{n-p+1}, ..., X_n)$ is a frame of $T\mathcal{F}$ over U.
For any compact $L \subseteq U$ and $p_{L,0} \in \mathrm{CS}(\Omega^k(M))$, defined by

$$p_{L,0}(\omega) = \sup\{|\omega_x(X_{n-p+1}(x), ..., X_n(x))| : x \in L\}, \tag{4.240}$$

we have the following inclusion:

$$\{\omega \in \Omega^k(M) : \mathrm{supp}(\omega) \subseteq \mathrm{Int}(L)\} \cap \overline{\Omega^k_{\perp\!\!\perp}(M,\mathcal{F})}^{\,p_{L,0}} \subseteq \Omega^k_{\perp\!\!\perp}(M,\mathcal{F}), \tag{4.241}$$

where $\overline{\Omega^k_{\perp\!\!\perp}(M,\mathcal{F})}^{\,p_{L,0}}$ denotes the closure of $\Omega^k_{\perp\!\!\perp}(M,\mathcal{F})$ in $\Omega^k(M)$ w.r.t $\mathcal{T}_{p_{L,0}}$.

Proof. For any $\omega \in \Omega^k(M)$, the local representation on U is given by

$$\omega|_U = \sum_{1\le i_1<...<i_k\le n} c_{i_1...i_k}(\omega) dX_{i_1} \wedge ... \wedge dX_{i_k}, \tag{4.242}$$

where $c_{i_1...i_k}(\omega) \in C^\infty(U)$ is defined by $c_{i_1...i_k}(\omega)(x) = \omega_x(X_{i_1}(x), ..., X_{i_k}(x))$.
If $\omega \in \Omega^k_{\perp\!\!\perp}(M,\mathcal{F})$, we have $i^*_\alpha\omega = 0$ for each inclusion $i_\alpha : \mathcal{L}_\alpha \to M$. So the local representation becomes

$$\omega|_U = \sum_{j=1}^{n-p} dX_j \wedge \theta_j \tag{4.243}$$

with $\theta_j = \sum_{j<i_2<...<i_k\le n} c_{ji_2...i_k}(\omega) dX_{i_2} \wedge ... \wedge dX_{i_k} \in \Omega^{k-1}(U)$.
On the other hand, if $V \subseteq U$ is open and $\omega \in \Omega^k(M)$ with $\mathrm{supp}(\omega) \subseteq V$ has a local representation on V of the form (4.243), it is also an element of $\Omega^k_{\perp\!\!\perp}(M,\mathcal{F})$.

Now let $L \subseteq U$ be compact and $\omega \in \overline{\Omega^k_{\perp\!\!\perp}(M,\mathcal{F})}^{\,p_{L,0}}$ with $\mathrm{supp}(\omega) \subseteq \mathrm{Int}(L)$.
So there is a sequence $(\omega_r)_{r\in\mathbb{N}} \in \Omega^k_{\perp\!\!\perp}(M,\mathcal{F})^{\mathbb{N}}$ such that $p_{L,0}(\omega_r - \omega) \to 0$ for $r \to \infty$, i.e. $c_{i_1...i_k}(\omega_r) \to c_{i_1...i_k}(\omega)$ uniformly on L for all ordered indices $1 \le i_1 < ... < i_k \le n$. But since $\omega_r \in \Omega^k_{\perp\!\!\perp}(M,\mathcal{F})$, we obtain $c_{i_1...i_k}(\omega) = \lim_{r\to\infty} c_{i_1...i_k}(\omega_r) = 0$ for all $i_1 > n-p$ by (4.243). Thus,

$$\omega|_{\mathrm{Int}(L)} = \sum_{1\le i_1<\cdots<i_k\le n} c_{i_1\cdots i_k}(\omega) dX_{i_1} \wedge \cdots \wedge dX_{i_k} \tag{4.244}$$

$$= \sum_{j=1}^{n-p} \sum_{j<i_2<\cdots<i_k\le n} c_{j\,i_2\cdots i_k}(\omega) dX_j \wedge dX_{i_2} \cdots \wedge dX_{i_k} \tag{4.245}$$

$$= \sum_{j=1}^{n-p} dX_j \wedge \left(\sum_{j<i_2<\cdots<i_k\le n} c_{j\,i_2\cdots i_k}(\omega) dX_{i_2} \cdots \wedge dX_{i_k} \right). \tag{4.246}$$

□

4.1.11 Proposition (k-Annihilator is an SK-Subspace)

Let $\mathcal{F} = \{\mathcal{L}_\alpha\}_{\alpha \in \mathscr{A}}$ be a p-foliation of an n-manifold M. Then $\Omega^k_{\perp\!\!\perp}(M, \mathcal{F})$ is an SK-subspace of $\Omega^k(M)$ for any $k \in \mathbb{N}_0$.

Proof. For $p \in \mathrm{CS}(\Omega^k(M))$ there is some finite set I such that for every $i \in I$ there are

(1) a frame $\{X^i_j\}_{j=1}^n$ of TM over some open $U_i \subseteq M$, such that $\{X^i_j\}_{j=n-p+1}^n$ is a frame of $T\mathcal{F}$ over U_i,

(2) a compact subset $K_i \subseteq U_i$ and an order of differentiation $\ell_i \in \mathbb{N}_0$,

such that $p = \max\{p_{K_i,\ell_i} : i \in I\}$. Recall, that p_{K_i,ℓ_i} is defined for $\omega \in \Omega^k(M)$ by its local representation $\omega|_{U_i} = \sum\limits_{1 \leq i_1 < \ldots < i_k \leq n} c^i_{i_1 \ldots i_k}(\omega) dX^i_{i_1} \wedge \ldots \wedge dX^i_{i_k}$ with smooth coefficients on U_i, $x \mapsto c^i_{i_1 \ldots i_k}(\omega)(x) = \omega_x(X^i_{i_1}(x), \ldots, X^i_{i_k}(x))$, such that

$$p_{K_i,\ell_i}(\omega) = \sup\{\sup\{|D^\alpha(c^i_{i_1 \cdots i_k}(\omega))(x)| : x \in K_i,\ |\alpha| \leq \ell_i\} : 1 \leq i_1 < \cdots < i_k \leq n\}. \tag{4.247}$$

Let $L_i \subseteq U_i$ be compact with $K_i \subseteq \mathrm{Int}(L_i)$ for each $i \in I$. Hence, $q = \max\{p_{L_i,0} : i \in I\}$ defines a continuous seminorm on $\Omega^k(M)$. The finite union $K = \bigcup\limits_{i \in I} K_i$ is compact and there is a finite partition of unity $\{\varphi_i\}_{i \in I}$ such that $\mathrm{supp}(\varphi_i) \subseteq \mathrm{Int}(L_i)$ and $\sum\limits_{i \in I} \varphi_i = 1$ on K.

We will show $\overline{\Omega^k_{\perp\!\!\perp}(M, \mathcal{F})}^{\,q} \subseteq \Omega^k_{\perp\!\!\perp}(M, \mathcal{F}) + \mathrm{Ker}(p)$: So let $\omega \in \overline{\Omega^k_{\perp\!\!\perp}(M, \mathcal{F})}^{\,q}$, i.e. $\omega \in \Omega^k(M)$ and there is a sequence $(\omega_r)_{r \in \mathbb{N}} \in \Omega^k_{\perp\!\!\perp}(M, \mathcal{F})^{\mathbb{N}}$ with $q(\omega_r - \omega) \to 0$ for $r \to \infty$. This means, $c^i_{i_1 \ldots i_k}(\omega_r) \to c^i_{i_1 \ldots i_k}(\omega)$ uniformly on L_i for each $i \in I$. Because of $\sum\limits_{i \in I} \varphi_i = 1$ on K, we obtain $\left(1 - \sum\limits_{i \in I} \varphi_i\right) \omega \in \mathrm{Ker}(p)$. By the decomposition

$$\omega = \sum_{i \in I} \varphi_i \omega + \left(1 - \sum_{i \in I} \varphi_i\right) \omega, \tag{4.248}$$

it remains to show $\sum\limits_{i \in I} \varphi_i \omega \in \Omega^k_{\perp\!\!\perp}(M, \mathcal{F})$. Since each φ_i is bounded and $p_{L_i,0} \leq q$, it follows that $p_{L_i,0}(\varphi_i \omega_r - \varphi_i \omega) \to 0$, i.e. $\varphi_i \omega \in \overline{\Omega^k_{\perp\!\!\perp}(M, \mathcal{F})}^{\,p_{L_i,0}}$. Moreover, $\mathrm{supp}(\varphi_i \omega) \subseteq \mathrm{Int}(L_i)$ such that Lemma 4.1.10 yields $\varphi_i \omega \in \Omega^k_{\perp\!\!\perp}(M, \mathcal{F})$ for each $i \in I$, hence $\sum\limits_{i \in I} \varphi_i \omega \in \Omega^k_{\perp\!\!\perp}(M, \mathcal{F})$. Now, we have proven (b) of Lemma 4.1.9 and $\Omega^k_{\perp\!\!\perp}(M, \mathcal{F})$ is an SK subspace. □

4.1.12 Remark (Alternative Foliated Cartan-Differential)

Originally, we were interested in the question if the foliated Cartan-differential is an SK-homomorphism. If we use the definition $\Omega^k_{\mathrm{fol}}(M, \mathcal{F}) = \Omega^k(M) / \Omega^k_{\perp\!\!\perp}(M, \mathcal{F})$ of foliated forms as quotient spaces and denote the quotient map by $q^k : \Omega^k(M) \to \Omega^k_{\mathrm{fol}}(M, \mathcal{F})$, then since

$$q^{k+1} \circ d : \Omega^k(M) \to \Omega^{k+1}_{\mathrm{fol}}(M, \mathcal{F}) \tag{4.249}$$

is constantly zero on $\Omega^k_{\perp\!\!\perp}(M,\mathcal{F})$ by $i^*_{\mathcal{L}_\alpha} \circ d = d \circ i^*_{\mathcal{L}_\alpha}$, there is a linear continuous map

$$\tilde{d}_{\mathcal{F}} : \Omega^k_{\text{fol}}(M,\mathcal{F}) \to \Omega^{k+1}_{\text{fol}}(M,\mathcal{F}) \tag{4.250}$$

satisfying $\tilde{d}_{\mathcal{F}} \circ q^k = q^{k+1} \circ d$. Moreover, using the isomorphisms of Fréchet spaces $\Phi^k : \Omega^k_{\text{fol}}(M,\mathcal{F}) \to \Omega^k(M,\mathcal{F})$, which exist by Lemma 3.2.4, $d_{\mathcal{F}} = \Phi^{k+1} \circ \tilde{d}_{\mathcal{F}} \circ (\Phi^k)^{-1}$ holds. Since isomorphisms of Fréchet spaces are in particular sk-homomorphisms, we obtain by Lemma 4.1.7 that $d_{\mathcal{F}}$ is an SK-homomorphism if and only if $\tilde{d}_{\mathcal{F}}$ is one.
By Lemma 4.1.8, $\tilde{d}_{\mathcal{F}}$ is an SK-homomorphism if $q^{k+1} \circ d$ is one. We already know that q^k is an SK-homomorphism for each $k \in \mathbb{N}_0$ by the previous proposition. However, even if d is an SK-homomorphism, Lemma 4.1.7 requires $\Omega^{k+1}_{\perp\!\!\perp}(M,\mathcal{F}) = \operatorname{Ker}(q^{k+1}) \subseteq \operatorname{Im}(d)$ to conclude that the composition $q^{k+1} \circ d$ is also an SK-homomorphism. Even if the inclusion is not satisfied in general, we will investigate whether the (unfoliated) Cartan-differential is an SK-homomorphism.

4.1.13 Remark (SK-Condition for Cartan-Differential)

Denote the set of open and relatively compact subsets of a manifold M by $\mathcal{T}_{\text{rc}}(M)$. An open and relatively compact exhaustion of M is a sequence of sets $M_j \in \mathcal{T}_{\text{rc}}(M)$ such that $\overline{M_j} \subseteq M_{j+1}$ and $M = \bigcup\limits_{j\in\mathbb{N}} M_j$. Every (relatively) compact set $A \subseteq M$ is contained in some M_n since $\overline{A} \subseteq \bigcup\limits_{j\in\mathbb{N}} M_j$ yields an $n \in \mathbb{N}$ with $\overline{A} \subseteq \bigcup\limits_{j=1}^{n} M_j = M_n$. The kernel of a seminorm on spaces of C^∞-sections consists of sections vanishing on some compact set $A \subseteq M$. So, an equivalent system of zero neighbourhoods for the SK-topology is given by sections that vanish on some open relatively compact set and the SK-homomorphism condition for $d : \Omega^k(M) \to \Omega^{k+1}(M)$ becomes

$$\begin{gathered} \forall\, U \in \mathcal{T}_{\text{rc}}(M) \;\; \exists\, V \in \mathcal{T}_{\text{rc}}(M) \;\; \forall\, \omega \in \Omega^k(M) : \\ d\omega|_V = 0 \Rightarrow \;\exists\; \nu \in \Omega^k(M) : d\omega = d\nu \text{ and } \nu|_U = 0. \end{gathered} \tag{4.251}$$

4.1.14 Lemma (Characterization Cartan-Differential SK-Homomorphism)

Let M be a manifold with an open and relatively compact exhaustion $(M_j)_{j\in\mathbb{N}}$ and $k \geq 0$. Then the following are equivalent:

(a) $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is an SK-homomorphism.

(b) $\forall\, U \in \mathcal{T}_{\text{rc}}(M) \;\exists\, V \in \mathcal{T}_{\text{rc}}(M)\; (U \subseteq V)$: $\varrho^U_V(\Omega^k_{\text{cl}}(V)) \subseteq \varrho^U_M(\Omega^k_{\text{cl}}(M))$.

(c) $\forall\, n \in \mathbb{N} \;\exists\, m \geq n : \varrho^n_m\left(\Omega^k_{\text{cl}}(M_m)\right) \subseteq \varrho^n_M\left(\Omega^k_{\text{cl}}(M)\right)$.

Proof. (a) $\Rightarrow$ (b): Let $U \in \mathcal{T}_{\text{rc}}(M)$, then there is $W \in \mathcal{T}_{\text{rc}}(M)$ according to (4.251). Choose $V \in \mathcal{T}_{\text{rc}}(M)$ such that $\overline{W} \subseteq V$ and let $\chi \in C^\infty(M)$ be a bump function with $\chi|_{\overline{W}} = 1$ and $\operatorname{supp}(\chi) \subseteq V$. For $\omega \in \Omega^k_{\text{cl}}(V)$ define $\tilde{\omega} = \chi\omega \in \Omega^k(M)$ such that $d\tilde{\omega}|_W = d\omega|_W = 0$.

By (4.251) there is $\nu \in \Omega^k(M)$ such that $\nu|_U = 0$ and $d\nu = d\tilde{\omega}$. Set $\eta = \tilde{\omega} - \nu$, then $\varrho_M^U(\eta) = \tilde{\omega}|_U - \nu|_U = \omega|_U = \varrho_V^U(\omega)$ and $d\eta = d\tilde{\omega} - d\nu = 0$.
(b) $\Rightarrow$ (c): Let $n \in \mathbb{N}$. For $U = M_n \in \mathcal{T}_{\mathrm{rc}}(M)$ there is $V \in \mathcal{T}_{\mathrm{rc}}(M)$ such that $\varrho_V^n(\Omega_{\mathrm{cl}}^k(V)) \subseteq \varrho_M^n(\Omega_{\mathrm{cl}}^k(M))$. Now there is $m \geq n$ with $\overline{V} \subseteq M_m$ and by $\varrho_m^n = \varrho_V^n \circ \varrho_m^V$ it follows $\varrho_m^n\left(\Omega_{\mathrm{cl}}^k(M_m)\right) \subseteq \varrho_V^n\left(\Omega_{\mathrm{cl}}^k(V)\right) \subseteq \varrho_M^n\left(\Omega_{\mathrm{cl}}^k(M)\right)$.
(c) $\Rightarrow$ (a): Let $U \in \mathcal{T}_{\mathrm{rc}}(M)$, then there is $n \in \mathbb{N}$ such that $\overline{U} \subseteq M_n$. For this n it exists $m \geq n$ such that $V = M_m$ contains U and satisfies $\varrho_n^U \circ \varrho_V^n\left(\Omega_{\mathrm{cl}}^k(V)\right) \subseteq \varrho_n^U \circ \varrho_M^n\left(\Omega_{\mathrm{cl}}^k(M)\right)$.
If $d\omega|_V = 0$ for some $\omega \in \Omega^k(M)$, then $\omega|_V \in \Omega_{\mathrm{cl}}^k(V)$ and it exists $\eta \in \Omega_{\mathrm{cl}}^k(M)$ satisfying $\omega|_U = \eta|_U$. Setting $\nu = \omega - \eta \in \Omega^k(M)$ gives $\nu|_U = \omega|_U - \eta|_U = 0$ and $d\nu = d\omega - d\eta = d\omega$ since η is closed.

□

The last condition (c), is a so called *strictness* condition of a projective spectrum induced by an open and relatively compact exhaustion of the manifold. We will have a closer look at the theory of projective spectra and the strictness condition to finally conclude that the Cartan-differential is indeed an SK-homomorphism.

4.2 Projective Limit

The introduction of projective spectra and projective limits is based on [Wen03, Chapter 3]. As we have seen before, we are interested in a so called strictness condition which can be defined for projective spectra and can be interpreted as a relaxation of the surjectivity of the linking maps.

4.2.1 Definition (Projective Spectrum)

A **projective spectrum** $\mathscr{X} = (X_n, \varrho_m^n)$ is a sequence $(X_n)_{n\in\mathbb{N}}$ of linear spaces (over the same scalar field) together with linear maps $\varrho_m^n : X_m \to X_n$ for $n \leq m$ such that $\varrho_n^n = \mathrm{id}_{X_n}$ and $\varrho_n^k \circ \varrho_m^n = \varrho_m^k$ for $k \leq n \leq m$.
A morphism $f = (f_n)_{n\in\mathbb{N}} : \mathscr{X} \to \mathscr{Y}$ between two projective spectra $\mathscr{X} = (X_n, \varrho_m^n)$ and $\mathscr{Y} = (Y_n, \sigma_m^n)$ is a sequence of linear maps $f_n : X_n \to Y_n$ such that $f_n \circ \varrho_m^n = \sigma_m^n \circ f_m$ for $n \leq m$, i.e. the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle \varrho_m^n}\uparrow & & \uparrow{\scriptstyle \sigma_m^n} \\ X_m & \xrightarrow[f_m]{} & Y_m \end{array}$$

commutes.
The composition of two morphisms $f : (X_n, \varrho_m^n) \to (Y_n, \sigma_m^n)$ and $g : (Y_n, \sigma_m^n) \to (Z_n, \tau_m^n)$ is defined by composing the components $g \circ f = (g_n \circ f_n)_{n\in\mathbb{N}}$.
We call $\mathscr{Y} = (Y_n, \sigma_m^n)$ a **subsequence of a projective spectrum** $\mathscr{X} = (X_n, \sigma_m^n)$ if there is a strictly increasing sequence $(k(n))_{n\in\mathbb{N}}$ of natural numbers such that $Y_n = X_{k(n)}$ and

$\sigma_m^n = \varrho_{k(m)}^{k(n)}$. A projective spectrum $\mathscr{X} = (X_n, \varrho_m^n)$ is called **strict**, if it satisfies

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, \ell \geq m: \quad \varrho_m^n(X_m) \subseteq \varrho_\ell^n(X_\ell). \tag{4.252}$$

Notice that $\varrho_\ell^n(X_\ell) = \varrho_m^n(\varrho_\ell^m(X_\ell)) \subseteq \varrho_m^n(X_m)$ is always satisfied for $\ell \geq m \geq n$.
A short sequence of projective spectra

$$0 \to \mathscr{X} \xrightarrow{f} \mathscr{Y} \xrightarrow{g} \mathscr{Z} \to 0 \tag{4.253}$$

is exact if it consists of short exact sequences of linear spaces such that the following diagram is commutative for every $m \geq n$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X_n & \xrightarrow{f_n} & Y_n & \xrightarrow{g_n} & Z_n & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \varrho_m^n} & & \uparrow{\scriptstyle \sigma_m^n} & & \uparrow{\scriptstyle \tau_m^n} & & \\
0 & \longrightarrow & X_m & \xrightarrow[f_m]{} & Y_m & \xrightarrow[g_m]{} & Z_n & \longrightarrow & 0.
\end{array}$$

4.2.2 Lemma (Strictness-Invariance under Subsequences)

Let $\mathscr{X} = (X_n, \varrho_m^n)$ be a projective spectrum. Then the following are equivalent:

(a) $\mathscr{X}$ is strict;

(b) Every subsequence of $\mathscr{X}$ is strict;

(c) There is a strict subsequence of $\mathscr{X}$.

Proof.
(a) $\Rightarrow$ (b) Let $(k(n))_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers and let $n \in \mathbb{N}$. The strictness of $\mathscr{X}$ provides for $k(n) \in \mathbb{N}$ an $M \in \mathbb{N}$ such that $\varrho_M^{k(n)}(X_M) \subseteq \varrho_\ell^{k(n)}(X_\ell)$ for every $\ell \geq M$ holds. Since the sequence is strictly increasing there is $m \geq n$ such that $k(m) \geq M$ and for every $\ell \geq m$ it follows:

$$\varrho_{k(m)}^{k(n)}\left(X_{k(m)}\right) = \varrho_M^{k(n)}\left(\varrho_{k(m)}^M(X_{k(m)})\right) \subseteq \varrho_M^{k(n)}\left(X_M\right) \subseteq \varrho_{k(\ell)}^{k(n)}\left(X_{k(\ell)}\right). \tag{4.254}$$

(b) $\Rightarrow$ (c) is trivially true.
(c) $\Rightarrow$ (a) Let $(k(n))_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers such that the corresponding subsequence of $\mathscr{X}$ is strict and let $n \in \mathbb{N}$. There is $N \in \mathbb{N}$ with $k(N) \geq n$ and the strictness provides an $M \geq N$ such that for $m = k(M) \geq k(N)$ and every $L \geq M$:

$$\varrho_m^n\left(X_m\right) = \varrho_{k(N)}^n \circ \varrho_m^{k(N)}\left(X_m\right) \subseteq \varrho_{k(N)}^n \circ \varrho_{k(L)}^{k(N)}\left(X_{k(L)}\right) = \varrho_{k(L)}^n\left(X_{k(L)}\right). \tag{4.255}$$

Given $\ell \geq m$ there is some $L \geq M$ with $k(L) \geq \ell$. Hence we obtain

$$\varrho_m^n\left(X_m\right) \subseteq \varrho_{k(L)}^n\left(X_{k(L)}\right) = \varrho_\ell^n\left(\varrho_{k(L)}^\ell\left(X_{k(L)}\right)\right) \subseteq \varrho_\ell^n\left(X_\ell\right). \tag{4.256}$$

$\square$

4.2.3 Definition (Projective Limit)

For a projective spectrum $\mathscr{X} = (X_n, \varrho^n_m)$ we define its **projective limit** by

$$\operatorname{Proj} \mathscr{X} = \varprojlim X_n = \left\{ (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n : \varrho^n_m(x_m) = x_n \text{ for all } m \geq n \right\}. \tag{4.257}$$

We denote by $\varrho^n : \operatorname{Proj} \mathscr{X} \to X_n$ the projection onto the nth-component. For a morphism $f = (f_n)_{n\in\mathbb{N}} : \mathscr{X} \to \mathscr{Y}$ between two projective spectra, we set

$$\operatorname{Proj} f : \operatorname{Proj} \mathscr{X} \to \operatorname{Proj} \mathscr{Y},\ (x_n)_{n\in\mathbb{N}} \mapsto (f_n(x_n))_{n\in\mathbb{N}}. \tag{4.258}$$

4.2.4 Remark (Proj-functor)

The linking property of the restrictions implies that $\operatorname{Proj} \mathscr{X}$ is the kernel of the linear map

$$\Psi = \Psi_{\mathscr{X}} : \prod_{n\in\mathbb{N}} X_n \to \prod_{n\in\mathbb{N}} X_n,\ (x_n)_{n\in\mathbb{N}} \mapsto \left(x_n - \varrho^n_{n+1}(x_{n+1})\right)_{n\in\mathbb{N}}.$$

Furthermore, $\operatorname{Proj} f$ is a well-defined linear map since f is a morphism of projective spectra and for another morphism $g = (g_n)_{n\in\mathbb{N}} : \mathscr{Y} \to \mathscr{Z}$ the composition $\operatorname{Proj} g \circ \operatorname{Proj} f$ equals $\operatorname{Proj}(g \circ f)$. In addition, $\operatorname{Proj} \mathrm{id}_{\mathscr{X}} = \mathrm{id}_{\operatorname{Proj} \mathscr{X}}$ such that Proj is a covariant functor acting on the category of projective spectra with values in the category of linear spaces.
If $\mathscr{Y} = \left(X_{k(n)}, \varrho^{k(n)}_{k(m)}\right)$ is a subsequence of $\mathscr{X}$, there is a morphism $f = (f_n)_{n\in\mathbb{N}} : \mathscr{Y} \to \mathscr{X}$ defined by $f_n = \varrho^n_{k(n)} : X_{k(n)} \to X_n$ such that the induced map $\operatorname{Proj} f : \operatorname{Proj} \mathscr{Y} \to \operatorname{Proj} \mathscr{X}$, $(x_{k(n)})_{n\in\mathbb{N}} \mapsto \left(\varrho^n_{k(n)}(x_{k(n)})\right)_{n\in\mathbb{N}}$ is bijective where its inverse is given by $\varphi : \operatorname{Proj} \mathscr{X} \to \operatorname{Proj} \mathscr{Y}$, $(x_n)_{n\in\mathbb{N}} \mapsto (x_{k(n)})_{n\in\mathbb{N}}$. We can use this in the following:

4.2.5 Lemma (Mittag-Leffler Lemma of Algebra)

Let $\mathscr{X} = (X_n, \varrho^n_m)$ be a projective spectrum. Then $\mathscr{X}$ is strict if and only if

$$\forall\, n \in \mathbb{N}\ \ \exists\, m \geq n :\ \ \varrho^n_m\left(X_m\right) \subseteq \varrho^n\left(\operatorname{Proj} \mathscr{X}\right). \tag{4.259}$$

Proof. Let $n \in \mathbb{N}$. Assume that (4.259) holds, then there is some $m \geq n$ such that $\varrho^n_m(X_m) \subseteq \varrho^n\left(\operatorname{Proj} \mathscr{X}\right)$. So for every $x_m \in X_m$, there is $y \in \operatorname{Proj} \mathscr{X}$ with $\varrho^n_m(x_m) = y_n = \varrho^n_\ell(y_\ell)$ for every $\ell \geq n$ since $y \in \operatorname{Proj} \mathscr{X}$. Hence $\mathscr{X}$ is strict.

(1) At first, we show that the condition (4.259) is satisfied if it holds for some subsequence of $\mathscr{X}$. So let $(k(n))_{n\in\mathbb{N}}$ be a strictly increasing sequence of natural numbers and $\mathscr{Y} = \left(X_{k(n)}, \varrho^{k(n)}_{k(m)}\right)$ such that:

$$\forall\, N \in \mathbb{N}\ \exists\, M \in \mathbb{N} : \varrho^{k(N)}_{k(M)}\left(X_{k(M)}\right) \subseteq \varrho^{k(N)}\left(\operatorname{Proj} \mathscr{Y}\right). \tag{4.260}$$

Let $n \in \mathbb{N}$. For $N \in \mathbb{N}$ with $k(N) \geq n$ there is $M \in \mathbb{N}$ such that for $m = k(M) \geq k(N)$:

$$\varrho_m^n(X_m) = \varrho_{k(N)}^n\left(\varrho_m^{k(N)}(X_m)\right) \subseteq \varrho_{k(N)}^n\left(\varrho^{k(N)}(\operatorname{Proj}\mathscr{Y})\right). \tag{4.261}$$

As mentioned before, there is a bijection $\operatorname{Proj} f : \operatorname{Proj}\mathscr{Y} \to \operatorname{Proj}\mathscr{X}$, defined by $(x_{k(n)})_{n\in\mathbb{N}} \mapsto \left(\varrho_{k(n)}^n(x_{k(n)})\right)_{n\in\mathbb{N}}$. Moreover, $\varrho^{k(n)}(\operatorname{Proj}\mathscr{Y}) = \varrho^{k(n)}(\operatorname{Proj} f(\operatorname{Proj}\mathscr{Y})) = \varrho^{k(n)}(\operatorname{Proj}\mathscr{X})$ for each $n \in \mathbb{N}$ such that (4.259) follows.

Now let $\mathscr{X}$ be strict. By passing to a subsequence, we can assume the strictness condition in the following way:

$$\forall\, n \in \mathbb{N},\ \ell \geq n+1: \ \ \varrho_{n+1}^n(X_{n+1}) \subseteq \varrho_\ell^n(X_\ell). \tag{4.262}$$

Let $n \in \mathbb{N}$ and put $m = n+1$. For $x_m \in X_m$ and $j \in \{1, ..., n\}$ define $y_j = \varrho_m^j(x_m)$.
According to (4.262), we can construct inductively $(x_j)_{j\geq n+2} \in \prod_{j\geq n+2} X_j$ such that $\varrho_{j+1}^j(x_{j+1}) = \varrho_{j+2}^j(x_{j+2})$ holds for every $j \geq n$. Setting $y_j = \varrho_{j+1}^j(x_{j+1})$ for $j \geq n+1$ defines an element $y = (y_j)_{j\in\mathbb{N}} \in \operatorname{Proj}\mathscr{X}$ since $\varrho_{j+1}^j(y_{j+1}) = y_j$ for each $j \in \mathbb{N}$ and it satisfies $\varrho^n(y) = \varrho_m^n(x_m)$. □

4.2.6 Lemma (Strictness of the Image of a Strict Projective Spectrum)

Let $\mathscr{X} \xrightarrow{f} \mathscr{Y} \to 0$ be an exact sequence of projective spectra. If $\mathscr{X}$ is strict, then $\mathscr{Y}$ is also strict.

Proof. The strictness of $\mathscr{X} = (X_n, \varrho_m^n)$ implies for $n \in \mathbb{N}$

$$(1) \qquad \exists\, m \geq n\ \forall\, k \geq m: \ \ \varrho_m^n(X_m) \subseteq \varrho_k^n(X_k).$$

For $y \in Y_m$ there is $x \in X_m$ with $y = f_m(x)$ because of the exactness. Consider the following commutative diagram with exact rows for $k \geq m$:

$$\begin{array}{ccccc}
X_n & \xrightarrow{f_n} & Y_n & \longrightarrow & 0 \\
\uparrow{\scriptstyle \varrho_m^n} & & \uparrow{\scriptstyle \sigma_m^n} & & \\
X_m & \xrightarrow{f_m} & Y_m & \longrightarrow & 0 \\
\uparrow{\scriptstyle \varrho_k^m} & & \uparrow{\scriptstyle \sigma_k^m} & & \\
X_k & \xrightarrow{f_k} & Y_k & \longrightarrow & 0.
\end{array}$$

So $\sigma_m^n(y) = \sigma_m^n(f_m(x)) = f_n(\varrho_m^n(x)) \in f_n(\varrho_m^n(X_m)) \overset{(1)}{\subseteq} f_n(\varrho_k^n(X_k)) = \sigma_k^n(f_k(X_k)) = \sigma_k^n(Y_k)$. □

4.2.7 Lemma (Strictness-Criterion for the Middle)

Let $\mathscr{X} \overset{f}{\to} \mathscr{Y} \overset{g}{\to} \mathscr{Z} \to 0$ be an exact sequence of projective spectra. If $\mathscr{X}$ and $\mathscr{Z}$ are strict, then $\mathscr{Y}$ is also strict.

Proof. We have to show the strictness condition for $\mathscr{Y} = (Y_n, \sigma_m^n)$. So let $n \in \mathbb{N}$. $\mathscr{X} = (X_n, \varrho_m^n)$ being strict implies

$$(1) \qquad \exists\, \ell \geq n \ \forall\, k \geq \ell: \ \varrho_\ell^n(X_\ell) \subseteq \varrho_k^n(X_k)$$

and for this ℓ the strictness of $\mathscr{Z} = (Z_n, \tau_m^n)$ provides

$$(2) \qquad \exists\, m \geq \ell \ \forall\, k \geq m: \ \tau_m^\ell(Z_m) \subseteq \tau_k^\ell(Z_k).$$

It suffices to show for $k \geq m$ and $y \in Y_m$ that $\sigma_m^n(y) \in \sigma_k^n(Y_k)$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
X_n & \xrightarrow{f_n} & Y_n & \xrightarrow{g_n} & Z_n & \longrightarrow & 0 \\
\uparrow{\scriptstyle \varrho_\ell^n} & & \uparrow{\scriptstyle \sigma_\ell^n} & & \uparrow{\scriptstyle \tau_\ell^n} & & \\
X_\ell & \xrightarrow{f_\ell} & Y_\ell & \xrightarrow{g_\ell} & Z_\ell & \longrightarrow & 0 \\
\uparrow{\scriptstyle \varrho_m^\ell} & & \uparrow{\scriptstyle \sigma_m^\ell} & & \uparrow{\scriptstyle \tau_m^\ell} & & \\
X_m & \xrightarrow{f_m} & Y_m & \xrightarrow{g_m} & Z_m & \longrightarrow & 0 \\
\uparrow{\scriptstyle \varrho_k^m} & & \uparrow{\scriptstyle \sigma_k^m} & & \uparrow{\scriptstyle \tau_k^m} & & \\
X_k & \xrightarrow{f_k} & Y_k & \xrightarrow{g_k} & Z_k & \longrightarrow & 0.
\end{array}$$

Then $g_\ell(\sigma_m^\ell(y)) = \tau_m^\ell(g_m(y)) \in \tau_m^\ell(g_m(Y_m)) = \tau_m^\ell(Z_m) \overset{(2)}{\subseteq} \tau_k^\ell(Z_k) = \tau_k^\ell(g_k(Y_k))$ yields $\tilde{y} \in Y_k$ such that $g_\ell(\sigma_m^\ell(y)) = \tau_k^\ell(g_k(\tilde{y})) = g_\ell(\sigma_k^\ell(\tilde{y}))$. Therefore $\sigma_m^\ell(y) - \sigma_k^\ell(\tilde{y}) \in \operatorname{Ker}(g_\ell) = \operatorname{Im}(f_\ell)$. So there is $x \in X_\ell$ satisfying $\sigma_m^\ell(y) - \sigma_k^\ell(\tilde{y}) = f_\ell(x)$.

Hence $\sigma_m^n(y) - \sigma_k^n(\tilde{y}) = \sigma_\ell^n(\sigma_m^\ell(y) - \sigma_k^\ell(\tilde{y})) = \sigma_\ell^n(f_\ell(x)) = f_n(\varrho_\ell^n(x)) \in f_n(\varrho_\ell^n(X_\ell)) \overset{(1)}{\subseteq} f_n(\varrho_k^n(X_k))$ and it exists $\tilde{x} \in X_k$ with $\sigma_m^n(y) - \sigma_k^n(\tilde{y}) = f_n(\varrho_k^n(\tilde{x})) = \sigma_k^n(f_k(\tilde{x}))$. Finally $\sigma_m^n(y) = \sigma_k^n(\tilde{y} + f_k(\tilde{x})) \in \sigma_k^n(Y_k)$. □

4.2.8 Lemma (Linking Maps With Finite Dimensional Images)

Let $\mathscr{X} = (X_n, \varrho_m^n)$ be a projective spectrum with finite dimensional images $\varrho_m^n(X_m)$ for all $m \geq n$. Then $\mathscr{X}$ is strict.

Proof. Let $n \in \mathbb{N}$. Define $x_k = \dim(\varrho_k^n(X_k))$ for $k \geq n$ then $(x_k)_{k \geq n}$ is a monotone decreasing sequence of natural numbers since $\varrho_k^n(X_k) = \varrho_m^n(\varrho_k^m(X_k)) \subseteq \varrho_m^n(X_m)$ for $k \geq m$. Hence there is $m \geq n$ such that $x_k = x_m$ for all $k \geq m$, i.e. $\varrho_m^n(X_m) = \varrho_k^m(X_k)$ because their dimensions agree. □

Recall that a **good cover** $(U_\alpha)_{\alpha\in I}$ of a smooth manifold M of dimension n is a collection of open sets U_α such that all sets and all finite non-empty intersections $U_{\alpha_0}\cap\ldots\cap U_{\alpha_\ell}$ of the cover are diffeomorphic to $\mathbb{R}^n$. As mentioned in Remark 3.4.13, every smooth manifold has a good cover and if the manifold is compact, the good cover may be chosen to be finite. Moreover, if a smooth manifold has a finite good cover, then its de Rham cohomology is finite dimensional by Corollary 3.5.15.

4.2.9 Lemma (Restrictions in Cohomology Got Finite Dimensional Image)

Let M be a smooth manifold and $N\subseteq M$ be open and relatively compact. Then the restriction $\varrho: H^k(M)\to H^k(N)$ has finite dimensional image for each $k\in\mathbb{N}_0$.

Proof. Let $(U_\alpha)_{\alpha\in I}$ be a good cover of M. Since $\overline{N}$ is compact in M, there is a finite subset $E\subseteq I$ such that $N\subseteq\bigcup\limits_{\alpha\in E}U_\alpha$. Define $V=\bigcup\limits_{\alpha\in E}U_\alpha$. Then V is a submanifold of M with a finite good cover, namely $(U_\alpha)_{\alpha\in E}$ and consequently its de Rahm cohomologies $H^k(V)$ are finite dimensional. Now $\varrho: H^k(M)\to H^k(N)$ factorizes through the finite dimensional space $H^k(V)$ such that ϱ has finite dimensional image. □

For $k\in\mathbb{N}_0$ and a smooth manifold M, we denote the closed and exact k-forms by

$$\Omega^k_{\mathrm{cl}}(M)=\mathrm{Ker}(d:\Omega^k(M)\to\Omega^{k+1}(M))\text{ and}\tag{4.263}$$

$$\Omega^k_{\mathrm{ex}}(M)=\mathrm{Im}(d:\Omega^{k-1}(M)\to\Omega^k(M)).\tag{4.264}$$

4.2.10 Examples (Strict Projective Spectra)

Let M be a smooth manifold and $(M_n)_{n\in\mathbb{N}}$ be an open and relatively compact exhaustion of M, i.e. $M_n\subseteq M$ open and realtively compact, $\overline{M_n}\subseteq M_{n+1}$ for each $n\in\mathbb{N}$ and $M=\bigcup\limits_{n\in\mathbb{N}}M_n$. Then the following projective spectra are strict for each $k\in\mathbb{N}_0$:

(1) $(\Omega^k(M_n),\varrho^n_m)$; (2) $(\Omega^k_{\mathrm{ex}}(M_n),\varrho^n_m)$; (3) $(H^k(M_n),\varrho^n_m)$; (4) $(\Omega^k_{\mathrm{cl}}(M_n),\varrho^n_m)$,

where ϱ^n_m denotes the restriction between the corresponding spaces, induced by the pullback of the inclusion $i^m_n: M_n\hookrightarrow M_m$ for $m\geq n$.

Proof. (1) Let $n\in\mathbb{N}$, $\ell\geq n+1$ and $\omega\in\Omega^k(M_{n+1})$. For a bump function $\varphi\in C^\infty(M)$ with $\mathrm{supp}(\varphi)\subseteq M_{n+1}$ and $\varphi|_{M_n}\equiv 1$ define $\eta=(\varphi\,\omega)|_{M_\ell}$. Then, $\eta\in\Omega^\ell(M_\ell)$ and $\varrho^n_{n+1}(\omega)=\omega|_{M_n}=(\varphi\,\omega)|_{M_n}=\varrho^n_\ell(\eta)$.
(2) The exterior derivatives $d_n:\Omega^{k-1}(M_n)\to\Omega^k_{\mathrm{ex}}(M_n)$ give rise to a morphism $d=(d_n)_{n\in\mathbb{N}}$ of projective spectra such that $(\Omega^{k-1}(M_n))_{n\in\mathbb{N}}\overset{d}{\to}(\Omega^k_{\mathrm{ex}}(M_n))_{n\in\mathbb{N}}\to 0$ is exact. Now (1) implies (2) by Lemma 4.2.6.
(3) The restriction $\varrho^n_m: H^k(M_m)\to H^k(M_n)$ has finite dimensional image for each $m\geq n$ by Lemma 4.2.9. Hence the assertion follows by Lemma 4.2.8.
(4) For each $n\in\mathbb{N}$, the sequence $0\to\Omega^k_{\mathrm{ex}}(M_n)\hookrightarrow\Omega^k_{\mathrm{cl}}(M_n)\to H^k(M_n)\to 0$ with the

inclusion and quotientmap is exact, such that there is an exact sequence of projective spectra:

$$0 \to (\Omega^k_{\mathrm{ex}}(M_n))_{n\in\mathbb{N}} \hookrightarrow (\Omega^k_{\mathrm{cl}}(M_n))_{n\in\mathbb{N}} \to (H^k(M_n))_{n\in\mathbb{N}} \to 0.$$

Thus, $(\Omega^k_{\mathrm{cl}}(M_n))_{n\in\mathbb{N}}$ is strict because of (2), (3) and Lemma 4.2.7.

□

4.2.11 Theorem (Exterior Derivative is an Sk-Homomorphism)

Let M be a manifold and $k \in \mathbb{N}_0$. Then $d : \Omega^k(M) \to \Omega^{k+1}(M)$ is an SK-homomorphism.

Proof. Let $(M_n)_{n\in\mathbb{N}}$ be an open and relatively compact exhaustion of M. By example 4.2.10 and Lemma 4.2.5 we obtain:

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n:\ \ \varrho^n_m(\Omega^k_{\mathrm{cl}}(M_m)) \subseteq \varrho^n\left(\mathrm{Proj}(\Omega^k_{\mathrm{cl}}(M_n), \varrho^n_m)\right).$$

The map $\varphi : \Omega^k_{\mathrm{cl}}(M) \to \mathrm{Proj}(\Omega^k_{\mathrm{cl}}(M_n), \varrho^n_m)$, $\omega \mapsto (\varrho^n_M(\omega))_{n\in\mathbb{N}}$ is a bijection. Injectivity follows by locality, i.e. if a form coincides on open sets, it coincides on their union and surjectivity follows by glueing, using a partition of unity subordinate to $(M_n)_{n\in\mathbb{N}}$. Since $\varrho^n \circ \varphi = \varrho^n_M$ we conclude

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n:\ \ \varrho^n_m\left(\Omega^k_{\mathrm{cl}}(M_m)\right) \subseteq \varrho^n_M\left(\Omega^k_{\mathrm{cl}}(M)\right). \tag{4.265}$$

Finally, d is an SK-homomorphism by Lemma 4.1.14. □

4.2.12 Corollary (Foliated Cartan-Differential SK-Homomorphism Criterion)

Let $(M, \mathcal{F})$ be a smooth foliated manifold and $k \in \mathbb{N}_0$. If $\Omega^{k+1}_{\perp\!\!\!\perp}(M, \mathcal{F}) \subseteq d(\Omega^k(M))$ is satisfied, then $d_{\mathcal{F}} : \Omega^k(M, \mathcal{F}) \to \Omega^{k+1}(M, \mathcal{F})$ is an SK-homomorphism.

Proof. This follows by Remark 4.1.12 and the previous theorem. □

4.2.13 Remark (Čech Cohomology of Leafwise Constant Functions)

As in 4.1.14, the foliated Cartan differential will be also an SK-homomorphism if the kernel spectrum is strict. In the unfoliated case, we used the strictness of the projective spectrum of the de Rham cohomology. Therefore, we showed that the linking/restriction maps got finite dimensional images, which was basically a consequence of the fact, that the de Rham cohomology is finite dimensional if there exists a finite good cover. The foliated de Rham cohomology of a finite good cover is not finite dimensional in general and we need a replacement. We would like to show:
If $(M, \mathcal{F})$ is a smooth foliated manifold with finite dimensional foliated de Rham cohomology and $P \subseteq N \subseteq M$ are relatively open and compact, then the restriction map $\varrho : H^k(N, \mathcal{F}|_N) \to H^k(P, \mathcal{F}|_P)$ has finite dimensional image.
If we consider the Kronecker foliation (Example 3.6.2) on the torus corresponding to an irrational slope α which is not a Liouville number, then the foliated de Rham cohomology is finite dimensional. But if we take a zylinder on $\mathbb{T}^2$, $Z = \{(e^{i\alpha}, e^{i\beta}) : \alpha \in (0, \pi), \beta \in [0, 2\pi]\}$, then it is an open and relatively compact subset of $\mathbb{T}^2$. A dense leaf on the torus is cut into uncountable many lines by restricting to Z. $(Z, \mathcal{F}_\alpha|_Z)$ is foliated isomorphic to $(S^1 \times \mathbb{R}, \mathcal{F}_{S^1}(\mathbb{R}))$ such that we obtain by Theorem 3.4.14,

$$H^k(Z, \mathcal{F}_\alpha|_Z) \cong H^k(\mathbb{R}) \otimes C^\infty(S^1) = \begin{cases} C^\infty(S^1), & \text{if } k = 0 \\ 0, & \text{else.} \end{cases} \tag{4.266}$$

Hence, the restriction map $\varrho : H^k(\mathbb{T}^2, \mathcal{F}_\alpha) \to H^k(Z, \mathcal{F}_\alpha|_Z)$ can not be surjective for $k = 0$. One wants also to know, what happens if we take a (finite) subset $I \subseteq J$ and set $\mathcal{V} = (U_\alpha)_{\alpha \in I}$. Then for each $\ell \in \mathbb{N}_0$ there is the projection map

$$\varrho^\ell : \check{C}^\ell(\mathcal{U}, \mathscr{G}^{\mathrm{lc}}_{\mathcal{F}}) \to \check{C}^\ell(\mathcal{V}, \mathscr{G}^{\mathrm{lc}}_{\mathcal{F}}),\ \varrho^\ell(\omega)_{\alpha_0,\dots,\alpha_\ell} = \omega_{\alpha_0,\dots,\alpha_\ell}, \tag{4.267}$$

which satisfies $\varrho^{\ell+1} \circ \delta^\ell = \delta^\ell \circ \varrho^\ell$. Hence it is a δ-cochain map. The map

$$\tau^\ell : \check{C}^\ell(\mathcal{V}, \mathscr{G}^{\mathrm{lc}}_{\mathcal{F}}) \to \check{C}^\ell(\mathcal{U}, \mathscr{G}^{\mathrm{lc}}_{\mathcal{F}}),\ \tau^\ell(\mu) = \begin{cases} \mu_{\alpha_0,\dots,\alpha_\ell}, & \text{if } \alpha_0, \dots, \alpha_0 \in I \\ 0, & \text{else .} \end{cases} \tag{4.268}$$

is a right inverse of ϱ^ℓ, but it is not a δ-cochain map: If exactly one index α_i is in $J \setminus I$ and $\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_{\ell+1} \in I$, then $(\delta^\ell \circ \tau^\ell(\nu))_{\alpha_0,\dots,\alpha_\ell} = \delta^{\ell+1}_i(\nu_{\alpha_0,\dots,\hat{\alpha}_i,\dots,\alpha_{\ell+1}})$ which is in general not $0 = (\tau^{\ell+1} \circ \delta^\ell(\nu))_{\alpha_0,\dots,\alpha_{\ell+1}}$. Therefore the induced map of ϱ^ℓ in Čech-cohomology needs not to be surjective.

References

[Ber01] Mélanie Bertelson. Foliations associated to regular poisson structures. *Communications in Contemporary Mathematics*, 3(03):441–456, 2001.

[Ber11] Mélanie Bertelson. Remarks on a künneth formula for foliated de rham cohomology. *Pacific Journal of Mathematics*, 252(2):257–274, 2011.

[BT82] Raoul Bott and Loring W. Tu. Differential forms in algebraic topology, 1982.

[Die72] Jean Dieudonné. *Treatise on analysis. Vol. III, Translated from the French by IG MacDonald*. Academic Press, 1972.

[Die76] Jean Dieudonné. *Treatise on analysis. Vol. II, Translated from the French by IG MacDonald*. Academic Press, 1976.

[Eur] Chris Eur. On constructions of new vector bundles, (https://math.berkeley.edu/~ceur/notes_pdf/Eur_OnBuildingNewVectorBundles.pdf).

[Gra08] Loukas Grafakos. *Classical fourier analysis*, volume 2. Springer, 2008.

[GW72] Stephen J. Greenfield and Nolan R. Wallach. Global hypoellipticity and liouville numbers. *Proceedings of the American Mathematical Society*, 31(1):112–114, 1972.

[Hae71] André Haefliger. Homotopy and integrability. In *Manifolds—Amsterdam 1970*, pages 133–163. Springer, 1971.

[Hae80] André Haefliger. Some remarks on foliations with minimal leaves. *Journal of Differential Geometry*, 15(2):269–284, 1980.

[Her70] Carl S. Herz. Functions which are divergences. *American Journal of Mathematics*, 92(3):641–656, 1970.

[KA83] Aziz El Kacimi-Alaoui. Sur la cohomologie feuilletée. *Compositio Mathematica*, 49(2):195–215, 1983.

[Lee13] John M. Lee. *Introduction to Smooth Manifolds*. Springer, 2013.

[Mei10] Patrick Meier. Diplomarbeit, *Splittingverhalten von C^∞-Komplexen mit nichttrivialer Kohomologie*. Master's thesis, Bergische Universität Wuppertal, 2010.

[Mil70] John Milnor. Foliations and foliated vector bundles, mimeographed notes. *Institute for Advanced Study, Princeton*, 1970.

[MS06] Calvin C. Moore and Claude L. Schochet. *Global analysis on foliated spaces*, volume 9. Cambridge University Press, 2006.

[MSE17] User: levaplevap of Mathematics Stack Exchange. What is an example of a smooth function in $C^\infty(\mathbb{R}^2)$ which is not contained in $C^\infty(\mathbb{R}) \otimes C^\infty(\mathbb{R})$, (https://math.stackexchange.com/questions/2244402/what-is-an-example-of-a-smooth-function-in-c-infty-mathbbr2-which-is-not/2244646#2244646), 2017.

[MV97] Reinhold Meise and Dietmar Vogt. *Introduction to functional analysis*. Clarendon Press, 1997.

[Tan14] Hiro Lee Tanaka. Fall 2014 Math 230a, (http://www.hiroleetanaka.com/index.php?pageid=2014-230a), 2014.

[Tre67] François Treves. *Topological Vector Spaces, Distributions and Kernels*. Academic Press, 1967.

[Vog04] Dietmar Vogt. Splitting of exact sequences of fréchet spaces in the absence of continuous norms. *Journal of Mathematical Analysis and Applications*, 297(2):812–832, 2004.

[Wen03] Jochen Wengenroth. *Derived functors in functional analysis*. Springer, 2003.

[Whi36] Hassler Whitney. Differentiable manifolds. *Annals of Mathematics*, pages 645–680, 1936.

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