Strictly and Γ-robust counterparts of electricity market models: Perfect competition and Nash–Cournot equilibria

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ABSTRACT

This paper mainly studies two topics: linear complementarity problems for modeling electricity market equilibria and optimization under uncertainty. We consider both perfectly competitive and Nash–Cournot models of electricity markets and study their robustifications using strict robustness and the Γ-approach. For three out of the four combinations of economic competition and robustification, we derive algorithmically tractable convex optimization counterparts that have a clear-cut economic interpretation. In the case of perfect competition, this result corresponds to the two classic welfare theorems, which also apply in both considered robust cases that again yield convex robustified problems. Using the mentioned counterparts, we can also prove the existence and, in some cases, uniqueness of robust equilibria. Surprisingly, it turns out that there is no such economic sensible counterpart for the case of Γ-robustifications of Nash–Cournot models. Thus, an analog of the welfare theorems does not hold in this case. Finally, we provide a computational case study that illustrates the different effects of the combination of economic competition and uncertainty modeling.

1. Introduction

Modeling of equilibria of liberalized electricity markets and solving these models is of great practical relevance. One of the most important mathematical tools for formulating these equilibrium models are linear complementarity problems (LCPs). These problems are typically obtained by (i) modeling the optimization problem of every player in the market, by (ii) characterizing the optimal actions of these players using their optimality conditions, and by (iii) equilibrating these actions using tailored market-clearing conditions. For a general survey of the theory, applications, and algorithms for LCPs see the book [1]. For a very recent discussion on the importance of complementarity problems in the electricity sector, we refer to [2]. Moreover, the very many examples in [3] also highlight the importance of complementarity modeling in this field of application. One important property of LCPs for modeling electricity market equilibria is that they allow to capture the underlying transmission network if the physics are modeled in a suitable way. Hence, there is a large branch of literature that studies different topics of electricity markets in combination with the transmission system infrastructure; see, e.g., [4–6], to name only a few. Another seminal paper in this field is [7], where a very comprehensive literature survey is given as well. A general tutorial for modeling (electricity)

market equilibria using complementarity problems can be found in [8] and in the book [3]. Based on these LCP papers many authors developed and discussed more complicated techniques for modeling electricity market equilibria using, e.g., mathematical programs with equilibrium constraints (MPECs), generalized Nash equilibria, or (quasi-)variational inequalities. As a primer see, e.g., [9–13] as well as the references therein.

Using LCPs for modeling market equilibria has another important feature that is especially appealing for practice. In many situations, by means of the integrability or symmetry principle of variational inequalities (see, e.g., [14]) a solution of an LCP can be obtained by solving a convex quadratic optimization problem; see, e.g., [1,3,8]. Thus, the approach scales well for large markets and large transmission systems. Moreover, this equivalent optimization model often has a clear-cut economic interpretation, e.g., it corresponds to welfare maximization in the case of suitable LCP models for perfectly competitive markets.

In order to state the above discussed LCPs for modeling electricity markets one has to consider many parameters like consumer's demand in dependence of prices, production and investment costs of suppliers, or technical parameters of the transmission network. Certainly, many of these parameters are uncertain. For example, the willingness to pay of consumers in the future is not known today and many of the technical parameters of the network are more based on educated

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guesses than on certain knowledge. We refer to [15], where uncertain LCPs for electricity markets are modeled as well. In the latter article, the consumers' demand functions are considered to be uncertain, which is the same setting as discussed in this paper.

Mathematical optimization provides many techniques for tackling optimization problems under uncertainty out of which two are very prominent: stochastic and robust optimization. In stochastic optimization, the strategy is to resort to assumptions about the distribution of uncertain model parameters and one then solves for, e.g., minimal expected costs or maximal expected profits. Stochastic optimization has a long tradition due to the practical relevance of the solutions that it provides and because this approach of modeling is quite natural for many practitioners. The literature is by far too broad to be reviewed here. Readers interested in the general topic of stochastic optimization are referred to the textbook [16]. Compared to stochastic optimization, robust optimization is a much younger field. The first robust optimization approach goes back to [17]. One of the most comprehensive treatments is given in the book [18], which mainly considers continuous robust optimization as we do in this paper. A survey about tractability of robust counterparts for linear programming (LP), quadratic programming (QP), and even more general but still convex problems is given in [19], which also provides many more references to the literature. Robust discrete optimization dates back to [20] and has been further investigated in, e.g., [21] and the PhD thesis [22]. The main criticism on robust optimization is that it leads to highly conservative solutions. There have been made many attempts to resolve this criticism. One of the earliest is the so-called flexible Γ-approach in [23], where the authors presented a first approach to decrease what they call the "price of robustness". There have also been other attempts for mitigating the conservatism of robust solutions; e.g., recoverable robustness [24], light robustness [25], or adjustable robustness [26]. Finally, a recent survey on robust optimization with additional references is given in [27].

Due to the strong connection between LCPs and optimization it seems natural that there are also many papers that consider LCPs under uncertainty. As it was the case in optimization, the first LCPs under uncertainty have been studied using stochastic optimization, i.e., the LCP data are considered to be random vectors and matrices; see, e.g., [28]. Since the existence of solutions can typically not be guaranteed in these settings, many authors resort to the residual or gap function formulation of an LCP and then consider what is called "expected residual minimization" in the literature. This or other alternative formulations are studied in, e.g., [29–31].

Although addressing the worst-case is often important in applications, much less literature focuses on robust equilibrium problems or robust LCPs. For instance, to the best of the authors' knowledge, the earliest paper on robust LCPs is [32], where the authors point out that equilibria in the classic robust sense do not need to exist (even if the nominal problem has an equilibrium). Hence, they propose the notion of ρ-robust counterparts and ρ-robust solutions of uncertain LCPs, which is a relaxation of complementarity. Moreover, the authors discuss different types of uncertainty sets like intervals, ellipsoids, or intersections of ellipsoids. Besides the introduction of robust LCPs, the main contribution of [32] is that the authors prove necessary and sufficient conditions for ρ-robust solutions for bounded uncertainty sets, that they characterize ρ-robust solutions for ellipsoidal uncertainty sets, and that they give sufficient conditions for ρ-robust solutions in the case of uncertainty sets represented by intersections of ellipsoids. Very recent papers on robust LCPs are [33–35] and [36,37]. In the latter two papers, the authors also consider the gap function formulation of LCPs and study robust solutions of LCPs as the ones that minimize the worst-case of the gap function. The authors also note an example of Nash–Cournot games in networked power markets and briefly consider the related robustified setting by discussing an uncertain Nash–Cournot LCP. For this case, we show the existence of robust equilibria, discuss uniqueness properties, and provide tractable counterparts from convex quadratic programming that possess a clear-cut economic interpretation in this paper.

The field of robust equilibrium models and LCPs is also related to the young field of robust game theory, which has been introduced in [38]. In this paper, the authors present distribution-free models of incomplete-information games, where the players' payoff uncertainty is modeled by robust optimization. Other papers dealing with robust games or robust Nash equilibria include [39–43]. The concept of robustness has also been applied to Wardrop equilibria; see [44] for a primer.

In contrast to the many papers on stochastic electricity market equilibrium models (see, e.g., [45–50]), only a few publications deal with robust optimization in the electricity sector. Moreover, all existing literature is very recent. For example, [51] considers a two-stage adaptive robust optimization approach for electricity transmission network expansion. A related study of robust transmission network expansion planning with uncertain renewable generation and loads is given in [52]. The most prominent application of robust optimization in the power sector seems to be the unit commitment problem; cf., e.g., [53–56]. Many of these studies consider the net load to be uncertain, which is comparable to what we do in this paper. Adjustable robust optimization for capacity planning is discussed in [57]. Finally, [58] considers robust optimal offering strategies for a price-taking power producer, where uncertainty is taken into account using the flexible Γ-approach for MIPs that has been proposed in [21,23] and that we also consider in this paper. In contrast to what we aim for in this paper, the mentioned papers all consider "standard" robust optimization (albeit with different uncertainty sets and different types of robustifications such as strict or adjustable robustness) but do not consider robust equilibria in the sense of complementarity solutions. In the context of robust equilibrium modeling for electricity market models we are only aware of [15] and the paper [59]. In the latter paper, the authors consider a networked electricity market model of Cournot–Bertrand type and robustify the inverse demand functions of consumers as we do in this paper. They exploit the general insights developed in [37] and solve a convex optimization problem to obtain robustified equilibria. Finally, let us also mention the paper [60], in which possibility theory is used to account for demand function uncertainty in the context of Cournot equilibrium models for electricity markets. In summary, the discussion of the literature both on complementarity problems as well as on robustness shows that these aspects are of great importance for the electricity sector and the combination of these aspects thus leads to relevant models.

In this paper, we combine and extend many of the contributions of the literature discussed so far. Our contribution is the following:

(1) We state robust counterparts of electricity market equilibrium models on a transport network that is modeled using the classic DC power flow approximation; cf., e.g., [61,62].
(2) We consider two different types of economic competition models. First, we study perfectly competitive markets and afterward discuss Nash–Cournot equilibria. Thus, our framework also allows to address issues of market power under uncertainty.
(3) We study two different types of robustness: strict robustness and the more flexible Γ-approach. The strictly robust framework serves as a primer and can also be found, in a related setting, in [59]. In contrast, the case of flexible Γ-robust equilibrium models is more involved and has—to the best of our knowledge—not yet been studied in the literature.

For all four considered combinations of economic competition and uncertainty modeling the following question arises. In the nominal, i.e., deterministic, setting it is folklore knowledge that both the perfectly competitive as well as the Nash–Cournot competition model can be solved by solving a properly chosen convex optimization problem. Consider now the market equilibrium problem consisting of all the optimization problems of the players of the market. Our main research question is now as follows:

Does the robustification of these player's models allow for an LCP formulation that is equivalent to the robustified convex optimization counterpart of the nominal case?

Note that this does not only correspond to the question whether an equilibrium problem possesses an equivalent optimization problem but that the order of robustification and the transition from the equilibrium to the optimization problem has no effect. For the case of perfect competition, the question above is the same as whether the two classic welfare theorems also hold in robustified settings. We prove that this is the case for three out of the four combinations of considered robustifications and economic competition models. By doing so, we also show the tractability of these robust equilibrium models. More specifically, we prove that the tractable counterparts of the strictly robust equilibrium models are convex quadratic problems, whereas those of the Γ-robust equilibrium models are convex quadratically constrained quadratic problems (QCQPs). This is of special importance for practice because it allows to solve large-scale robust equilibrium models by exploiting the equivalent robust convex optimization counterparts. Interestingly, all these results cannot be obtained for the case of Γ-robustification of Nash–Cournot models. In this setting, the complementarity system modeling the robust equilibrium problem is *not* the same as the robustified optimization counterpart. Moreover, it is not possible to derive an optimization counterpart that is equivalent to the robustified equilibrium model, i.e., the above mentioned principle of symmetric Jacobians is not applicable. Finally, we also provide a computational case study on an academic test case that illustrates the effects of the combination of different robustifications with different types of market models.

The remainder of the paper is structured as follows. In Section 2 we present the networked market equilibrium model under perfect competition and review the well-known relation to welfare maximization problems. In Section 3 we then develop both strict and Γ-robust counterparts of the perfectly competitive setting, derive the LCPs modeling the robust equilibrium problem, and prove the equivalence of robustified equilibria and robustified welfare maximal solutions. Afterward, in Section 4 we study the same topics for the case of Nash–Cournot competition among the producers of the market. Surprisingly, the above mentioned equivalence only holds in the strictly robust case but fails to hold for the Γ-approach. Section 5 then discusses the effects of the combination of different robustness concepts and different competition models applied to an academic network before we close the paper with some concluding remarks and some ideas for future research in Section 6.

2. Equilibrium modeling of perfectly competitive markets

We consider electricity networks that we model by using a connected digraph $G := (N, A)$ with node set N and arc set A. Subsequently, all player problems of our market model are stated. Since we first consider perfectly competitive markets, all players are price takers and their optimization problems are formulated using exogenously given market prices $\pi_{u,t}$ at every node $u \in N$ in each time period $t \in [T] := \{1, \dots, T\}$, $T \in \mathbb{N}$. The model is based on standard electricity market models as discussed in, e.g., [3,8].

The first type of players are electricity producers. Without loss of generality, we assume that there exists exactly one producer at each node $u \in N$, which we model by fixed variable production costs $w_u^{\mathrm{var}} > 0$ and investment costs $w_u^{\mathrm{inv}} > 0$. The assumption of a single producer per node is only used to simplify the presentation. In practice, multiple producers at one node can simply be split by introducing artificial nodes that are connected to the original node by lines with "infinite" capacity. Moreover, we restrict ourselves here to the case in which each firm is only acting at a single node. In practice, firms can produce at different locations in the network (e.g., in different countries). Studying such settings would complicate the analysis (see,

e.g., [63]) and is, thus, out of scope of this paper. Production at node u in each time period $t \in [T]$ is denoted by $y_{u,t} \geq 0$ and is bounded from above by the time-independent generation capacity variable $\bar{y}_u$. The objective of the producer at node u is to maximize its profit and, thus, its linear optimization problem reads

$$\max_{y_u, \bar{y}_u} \quad \sum_{t \in [T]} \pi_{u,t} y_{u,t} - \sum_{t \in [T]} w_u^{\mathrm{var}} y_{u,t} - w_u^{\mathrm{inv}} \bar{y}_u \quad \text{s.t.} \quad 0 \leq y_{u,t} \leq \bar{y}_u, \quad t \in [T].$$
(1)

Note that we consider purely continuous problems here that, in particular, do not contain any integrality restrictions regarding production and investment decisions. Thus, the solutions of (1) are characterized by the corresponding Karush–Kuhn–Tucker (KKT) conditions

$$-w_u^{\mathrm{inv}} + \sum_{t \in [T]} \beta_{u,t}^+ = 0, \quad \pi_{u,t} - w_u^{\mathrm{var}} + \beta_{u,t}^- - \beta_{u,t}^+ = 0, \quad t \in [T],$$
(2a)

$$0 \leq y_{u,t} \perp \beta_{u,t}^- \geq 0, \quad 0 \leq \bar{y}_u - y_{u,t} \perp \beta_{u,t}^+ \geq 0, \quad t \in [T],$$
(2b)

where $\beta_{u,t}^{\pm}$ are the dual variables of the production constraints. Here and in what follows, we use the standard $\perp$-notation, where $0 \leq a \perp b \geq 0$ abbreviates $0 \leq a$, $b \geq 0$, $a^{\top} b = 0$ for $a, b \in \mathbb{R}^n$. Moreover, a variable without time index denotes the vector containing all corresponding node or arc variables, e.g., $y_u := (y_{u,t})_{t \in [T]}$.

Consumers, as our second players, are also located at the nodes $u \in N$ and decide on their demand $d_{u,t} \geq 0$. Their demand elasticity is modeled by inverse demand functions $p_{u,t} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, for which we make the following assumption.

Assumption 1. All inverse demand functions $p_{u,t}$, $u \in N$, $t \in [T]$, are strictly decreasing and continuous.

Under Assumption 1, the continuous and concave problem of a surplus maximizing consumer at node u is given by

$$\max_{d_u} \quad \sum_{t \in [T]} \int_0^{d_{u,t}} p_{u,t}(x) \, \mathrm{d}x - \sum_{t \in [T]} \pi_{u,t} d_{u,t} \quad \text{s.t.} \quad 0 \leq d_{u,t}, \quad t \in [T], \quad (3)$$

and its again necessary and sufficient first-order optimality conditions comprise

$$p_{u,t}(d_{u,t}) - \pi_{u,t} + \alpha_{u,t} = 0, \quad 0 \leq d_{u,t} \perp \alpha_{u,t} \geq 0, \quad t \in [T], \quad (4)$$

where $\alpha_{u,t}$ is the dual variable of the lower demand bound.

The third player in our market model is the transmission system operator (TSO). He operates the transmission network, in which every arc $a \in A$ is described by its susceptance B_a and its transmission capacity $f_a^+ > 0$. The latter bounds the flows $f_{a,t}$, $t \in [T]$, by $|f_{a,t}| \leq f_a^+$. The goal of the TSO is to transport electricity from low- to high-price regions and the earnings to be maximized result from the corresponding price differences; cf., e.g., [8]. Power flow in the network is modeled using the standard linear lossless DC approximation—cf., e.g., [61, 62]—which is often used for economic analysis, e.g., in [11,64]. Thus, we obtain additional phase angle variables $\Theta_{u,t}$ for all nodes $u \in N$ and time periods $t \in [T]$. With this notation the continuous and linear problem of the TSO reads

$$\max_{f, \Theta} \quad \sum_{t \in [T]} \sum_{a=(u,v) \in A} (\pi_{v,t} - \pi_{u,t}) f_{a,t}$$
(5a)

$$\text{s.t.} \quad -f_a^+ \leq f_{a,t} \leq f_a^+, \quad f_{a,t} = B_a(\Theta_{u,t} - \Theta_{v,t}), \quad a = (u,v) \in A, \ t \in [T].$$
(5b)

The last constraints in (5b) model the linear lossless DC flow approximation and $\varepsilon_{a,t}$ are the corresponding dual variables. The first constraints in (5b) reflect the network's transmission capacities and have the dual variables $\delta_{a,t}^{\pm}$. The optimality conditions of (5) are given

by

$$f_{a,t} = B_a(\Theta_{u,t} - \Theta_{v,t}), \quad \pi_{v,t} - \pi_{u,t} + \delta_{a,t}^- - \delta_{a,t}^+ + \varepsilon_{a,t} = 0, \quad a = (u,v) \in A, \ t \in [T],$$

$$\sum_{a \in \delta^{\text{out}}(u)} B_a \varepsilon_{a,t} - \sum_{a \in \delta^{\text{in}}(u)} B_a \varepsilon_{a,t} = 0, \quad u \in N, \ t \in [T],$$

$$0 \le f_{a,t} + f_a^+ \perp \delta_{a,t}^- \ge 0, \quad 0 \le f_a^+ - f_{a,t} \perp \delta_{a,t}^+ \ge 0, \quad a \in A, \ t \in [T].$$

$$(6)$$

Here we use the standard δ-notation for the in- and outgoing arcs of a node $u \in N$, i.e., $\delta^{\text{in}}(u) := \{(v,u) \in A\}$ and $\delta^{\text{out}}(u) := \{(u,v) \in A\}$. Putting all first-order optimality conditions as well as the flow balance conditions

$$0 = d_{u,t} - y_{u,t} + \sum_{a \in \delta^{\text{out}}(u)} f_{a,t} - \sum_{a \in \delta^{\text{in}}(u)} f_{a,t}, \quad u \in N, \ t \in [T], \quad (7)$$

together, we obtain the mixed complementarity problem

$$\text{Producers: (2),} \quad \text{Consumers: (4),} \quad \text{TSO: (6),} \quad \text{Market Clearing: (7),} \quad (8)$$

which models the wholesale electricity market under consideration for the case of perfect competition. Hence, solutions of (8) are market equilibria. It can be easily seen that this complementarity system is equivalent to the welfare maximization problem

$$\max_{d,y,\bar{y},f,\Theta} \quad \sum_{t \in [T]} \sum_{u \in N} \int_0^{d_{u,t}} p_{u,t}(x)\,dx - \sum_{t \in [T]} \sum_{u \in N} w_u^{\text{var}} y_{u,t} - \sum_{u \in N} w_u^{\text{inv}} \bar{y}_u \quad (9a)$$

$$\text{s.t.} \quad 0 \le y_{u,t} \le \bar{y}_u, \quad 0 \le d_{u,t}, \quad u \in N, \ t \in [T], \quad (9b)$$

$$- f_a^+ \le f_{a,t} \le f_a^+, \quad a \in A, \ t \in [T], \quad (9c)$$

$$0 = d_{u,t} - y_{u,t} + \sum_{a \in \delta^{\text{out}}(u)} f_{a,t} - \sum_{a \in \delta^{\text{in}}(u)} f_{a,t}, \quad u \in N, \ t \in [T], \quad (9d)$$

$$f_{a,t} = B_a(\Theta_{u,t} - \Theta_{v,t}), \quad a = (u,v) \in A, \ t \in [T]. \quad (9e)$$

The equivalence can be shown by comparing the first-order optimality conditions of Problem (9) with the mixed complementarity system (8) and by identifying the dual variables of the flow balance constraints (9d) with the equilibrium prices $\pi_{u,t}$ of the complementarity problem. Further, we need that the KKT conditions are again necessary and sufficient optimality conditions of Problem (9) under Assumption 1.

Let us briefly comment on why we choose this model setup. First, we consider a long-run model, i.e., we take capacity expansion for producers explicitly into account, because uncertainty is much more important when it comes to long-run investment decisions compared to short-run production decisions. Second, we study a networked setup for being as close as possible to the related literature [37,59]. Third, and finally, we consider static investment problems. This means that the investment decisions are taken once at the very beginning of the time horizon. We are aware of that in the current context of the rapid development of renewable energy technologies, investment costs may change on a yearly basis, which is not covered by our model. We make this assumption here, however, to keep both the robustification of the models as well as their analysis tractable and postpone the study of more dynamic investment aspects to our future work.

3. Robust counterparts under perfect competition

In this section, we study robustified equilibrium models under perfect competition. As it is shown in the last section, the solutions of the corresponding LCP can be computed by solving the welfare maximization problem (9). In other words: Solving the LCP can be replaced by solving a convex optimization problem, which is mainly based on the symmetry principle for variational inequalities; cf. [14]. The main question that we will answer is whether this is also possible in the case of robust equilibrium models, in which the uncertain consumers' demand is robustified using the concepts of strict and Γ-robustness. Second, we also answer the question on how to obtain the
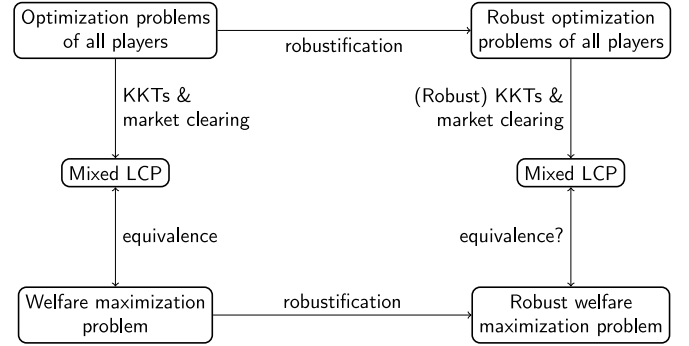

Fig. 1. Overview of the considered models.

robust optimization problem that can be solved as a surrogate for the robust equilibrium model. The overall situation is depicted in Fig. 1. There are two different possibilities on how to obtain the mentioned surrogate model:

(1) One first replaces the nominal LCP (8) by the corresponding welfare maximization problem (9) and then robustifies this problem.
(2) One first robustifies the separate optimal optimization problems of the players with uncertain data and then derives an LCP and an equivalent surrogate model, if possible at all.

For the case of perfect competition, we prove in this section that both approaches yield the same robustified welfare maximization problem, i.e., the diagram in Fig. 1 "commutes".

3.1. Strict robustness

Before we study the robustification of the models discussed in Section 2, let us briefly introduce the basic notions of robust optimization. To this end, we consider a *convex quadratic program (QP)* of the form

$$\max_{x \in \mathbb{R}^n} \quad \frac{1}{2} x^\top Q x + c^\top x \quad \text{s.t.} \quad Ax \le b, \quad Cx = d, \quad (10)$$

where $Q \in \mathbb{R}^{n \times n}$ is a symmetric and positive semi-definite matrix and $A \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{k \times n}$ as well as $b \in \mathbb{R}^m$, $d \in \mathbb{R}^k$, $c \in \mathbb{R}^n$ are matrices and vectors of suitable dimension. In robust optimization, one takes into account that the QP data (Q, A, C, c, b, d) is uncertain, i.e., the matrices and vectors are not known exactly. In particular, one assumes that they are contained in a given uncertainty set $\mathcal{U}$. This yields the so-called *uncertain convex QP*

$$\left\{ \max_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} x^\top Q x + c^\top x : Ax \le b, \ Cx = d \right\} \right\}_{(Q,A,C,c,b,d) \in \mathcal{U}}, \quad (11)$$

which is a family of optimization problems of type (10). Using the abbreviation $u := (Q, A, C, c, b, d)$, the problem

$$\max_{x \in \mathbb{R}^n} \left\{ \inf_{u \in \mathcal{U}} \left\{ \frac{1}{2} x^\top Q x + c^\top x : Ax \le b, \ Cx = d \ \forall u \in \mathcal{U} \right\} \right\}$$

is called the *robust counterpart* of (10).

Our goal in the following is to apply this concept to the models given in Section 2, where we introduced the models of all players that act in the market. In particular, we want to study the case in which the demand functions $p_{u,t}$ of the consumers are uncertain. This corresponds to the special case of (11), in which only objective function data is subject to uncertainty. Thus, we study the case in which Q and c are uncertain and obtain

$$\max_{x \in \mathbb{R}^n} \left\{ \inf_{(Q,c) \in \mathcal{U}} \left\{ \frac{1}{2} x^\top Q x + c^\top x : Ax \le b, \ Cx = d \ \forall (Q,c) \in \mathcal{U} \right\} \right\}.$$

From now on we assume that the inverse demand functions are linear, i.e., $p_{u,t}(x) := \bar{a}_{u,t} + \bar{b}_{u,t} x$ holds for all consumers u in every

time period t. Thus, $\bar{a}_{u,t} \geq 0$ is the price-intercept and $\bar{b}_{u,t} < 0$ is the slope of the function. Here we already used the notation $\bar{a}, \bar{b}$ for so-called *nominal* values. Since we consider uncertain demands, the true values a, b for price-intercepts and slopes are not known explicitly but contained in a given uncertainty set. We consider the case of box-uncertainties, i.e., for all consumers $u \in N$ and every time period $t \in [T]$, we have

$$(a_{u,t}, b_{u,t}) \in \mathcal{U}_{u,t} := \left[a_{u,t}^-, a_{u,t}^+ \right] \times \left[b_{u,t}^-, b_{u,t}^+ \right]. \tag{12}$$

In order to obtain a well-defined economic setting, we make the following assumption.

Assumption 2. For all consumers $u \in N$ and every time period $t \in [T]$ it holds $(a_{u,t}, b_{u,t}) \in \mathcal{U}_{u,t}$ and both $a_{u,t}^- \geq 0$ and $b_{u,t}^+ < 0$ is satisfied.

With these notations and definitions, we can state the robust counterpart of the consumer's model (3):

$$\max_{d_u} \left\{ \inf_{(a_{u,t}, b_{u,t}) \in \mathcal{U}_{u,t}} \left\{ \sum_{t \in [T]} \int_0^{d_{u,t}} (a_{u,t} + b_{u,t} x) \, dx \right. \right.$$
$$\left. \left. - \sum_{t \in [T]} \pi_{u,t} d_{u,t} \ : \ d_{u,t} \geq 0 \ \forall t \in [T], \ \forall (a_{u,t}, b_{u,t}) \in \mathcal{U}_{u,t} \right\} \right\}. \tag{13}$$

This robust counterpart is an optimization problem that is not tractable as it is stated in (13). Fortunately, there is an equivalent and tractable robust counterpart.

Theorem 3.1. *Let $u \in N$ be a consumer with uncertainty set (12) for all $t \in [T]$ and suppose Assumption 2 holds. Then, the robust counterpart (13) is equivalent to the convex optimization problem*

$$\max_{d_u} \quad \sum_{t \in [T]} \int_0^{d_{u,t}} (a_{u,t}^- + b_{u,t}^- x) \, dx - \sum_{t \in [T]} \pi_{u,t} d_{u,t} \quad s.t. \quad 0 \leq d_{u,t}, \quad t \in [T]. \tag{14}$$

The proof is given in Appendix A. The theorem reveals that we obtain the tractable robust counterpart by simply using the lower bounds of the uncertain sets. Since the robust counterpart of a consumer $u \in N$ is convex, its necessary first-order optimality conditions

$$a_{u,t}^- + b_{u,t}^- d_{u,t} - \pi_{u,t} + \alpha_{u,t} = 0, \quad 0 \leq d_{u,t} \perp \alpha_{u,t} \geq 0, \quad t \in [T], \tag{15}$$

are sufficient. In (15), $\alpha_{u,t}$ denotes the dual variable of the consumer's lower demand bound.

Note that the optimization problems of all other market participants are subject to certain data. Putting all first-order optimality conditions as well as the flow balance conditions (7) together, we obtain the *robustified market equilibrium problem (RMEP)*

Producers: (2),   Robustified consumers: (15),

   TSO: (6),   Market clearing: (7), $\tag{16}$

which models the wholesale electricity market for the case of perfect competition and consumers that face demand uncertainties. In general, the techniques discussed in this paper can be used to study uncertainties of other data such as, e.g., future production or investment costs as well. We focus on uncertain demand functions for two reasons. First, this parameter is very important in equilibrium models (with elastic demand) since demand influences prices, which themselves influence generation and thus investment. Second, considering future demand parameters as uncertain is also of great importance for practice; see, e.g., [53,56,59].

In the nominal case discussed in Section 2, it holds that the nominal mixed LCP is equivalent to the nominal welfare maximization problem (9). We now answer the question whether this is also true in the robustified setting. To this end, we robustify the welfare maximization problem (9) and afterward show that it is equivalent to the RMEP (16).

The robust counterpart of the welfare maximization problem (9) reads

$$\max_{d, y, \bar{y}, f, \Theta, \lambda} \quad \lambda \tag{17a}$$

$$\text{s.t.} \quad \lambda \leq \sum_{t \in [T]} \sum_{u \in N} \int_0^{d_{u,t}} (a_{u,t} + b_{u,t} x) \, dx - \sum_{t \in [T]} \sum_{u \in N} w_u^{\text{var}} y_{u,t} \tag{17b}$$

$$- \sum_{u \in N} w_u^{\text{inv}} \bar{y}_u, \quad \text{for all } (a_{u,t}, b_{u,t}) \in \mathcal{U}_{u,t}, \ u \in N, \ t \in [T],$$

$$0 \leq y_{u,t} \leq \bar{y}_u, \quad 0 \leq d_{u,t}, \quad u \in N, \ t \in [T], \tag{17c}$$

$$- f_a^+ \leq f_{a,t} \leq f_a^+, \quad f_{a,t} = B_a(\Theta_{u,t} - \Theta_{v,t}),$$

$$a = (u, v) \in A, \ t \in [T], \tag{17d}$$

$$0 = d_{u,t} - y_{u,t} + \sum_{a \in \delta^{\text{out}}(u)} f_{a,t} - \sum_{a \in \delta^{\text{in}}(u)} f_{a,t}, \quad u \in N, \ t \in [T]. \tag{17e}$$

As it is the case for the consumers, this robust counterpart can be reformulated in a tractable way.

Theorem 3.2. *Let an uncertainty set (12) be given for every consumer $u \in N$ and every time period $t \in [T]$. Suppose further Assumption 2 holds. Then, the robust counterpart (17) of the welfare maximization problem is equivalent to*

$$\max_{d, y, \bar{y}, f, \Theta} \quad \sum_{t \in [T]} \sum_{u \in N} \int_0^{d_{u,t}} (a_{u,t}^- + b_{u,t}^- x) \, dx - \sum_{t \in [T]} \sum_{u \in N} w_u^{\text{var}} y_{u,t} - \sum_{u \in N} w_u^{\text{inv}} \bar{y}_u \tag{18a}$$

$$s.t. \quad (17c)-(17e) \tag{18b}$$

The proof is given in Appendix B. The final question now is whether the robustified welfare maximization problem of Theorem 3.2 is equivalent to the RMEP (16). The answer is positive and can be easily obtained by comparing the KKT conditions of (18) with the RMEP.

Corollary 3.3. *The robust market equilibrium problem (16) and the robust welfare maximization problem (18) are equivalent.*

Note that the latter corollary is in line with the two welfare theorems, which hold for convex player problems (if robustified or not) under perfect competition. We close this section with a final remark on the uniqueness of robust market equilibria.

Remark 3.4. A typical question in the context of market equilibrium models is whether the resulting equilibrium is unique or not. For the nominal case of a market model upon an underlying network structure—as discussed in Section 2—this question has been studied in [65–67]. In these papers that do not consider uncertainties, the uniqueness of the market equilibrium depends on the used model of the transport network. In [65] it is shown that the long-run equilibrium is unique on a capacitated network with a standard linear flow model. For short-run models, i.e., for producer models without endogenous capacity investments, this also holds if the transport model additionally captures some special type of transport costs; see [67]. In contrast, in [66] it is shown that short-run models incorporating a DC network flow model (like we do in this paper) possess multiple solutions.

Since the structure of the robust counterparts do not differ from the structure of the nominal models, the uniqueness and multiplicity results from the nominal settings directly carry over to the strictly robust case discussed in this section.

(1) The RMEP (16) has a unique solution if we neglect the DC constraint in (5b). In this case, both the short- and long-run equilibria are unique under suitable assumptions.

(2) The DC-constrained RMEP (16) has a unique solution on electricity networks that are trees and in the short-run. On general

graphs, it has multiple equilibria. Whether the long-run equilibrium is unique on networks that are tree-shaped is, to the best of our knowledge, not known. However, our hypothesis is that uniqueness can be shown by using the techniques from [65].

Finally, in [68,69] uniqueness is shown for electricity spot markets with transmission losses. The results can be directly applied to the robust setting of this section.

3.2. The Γ-approach

The main criticism of the concept of strict robustness is that it leads, due to its worst-case character, to very conservative solutions. One remedy is the so-called Γ-approach, which has been proposed in [21–23]. This approach does not assume that all parameters are subject to uncertainty but that only a subset of these parameters are to be considered using a worst-case approach. The cardinality of this subset is typically called $\Gamma \in \mathbb{N}$, which gives the approach its name.

Let us briefly discuss this notion of robustness for a general convex QP, as we did in Section 3.1. To this end, we consider an arbitrary convex QP as given in (10) and assume that only objective function data (Q, c) is not exactly known. In this setting, the Γ-approach hedges against Γ_Q many uncertain entries in Q and against Γ_c many uncertain entries in c. The main point now is that we only know these maximum numbers of uncertain parameters but do not know which parameters are subject to change. Thus, using the notation $N := \{1, \dots, n\}$, the Γ-robust counterpart is given by

$$\max_{x \in \mathbb{R}^n} \quad \frac{1}{2} x^\top Q x + c^\top x - \max_{\{S_c \subseteq N \,:\, |S_c| \leq \Gamma_c\}} \sum_{j \in S_c} \Delta c_j x_j$$
$$- \max_{\{S_Q \subseteq N \times N \,:\, |S_Q| \leq \Gamma_Q\}} \sum_{(i,j) \in S_Q} \frac{1}{2} \Delta q_{ij} x_i x_j$$

$$\text{s.t.} \quad Ax \leq b, \; Cx = d,$$

where Δc_j is the deviation of the nominal parameter c_j—and analogous for Δq_{ij}.

As in Section 3.1 we consider consumers that face uncertain demand, which is assumed to lie in a two-dimensional box-uncertainty set as given in (12). Here, we additionally assume that this box is centered around the nominal values, i.e.,

$$a_{u,t}^- = \bar{a}_{u,t} - \Delta a_{u,t}, \qquad a_{u,t}^+ = \bar{a}_{u,t} + \Delta a_{u,t}, \qquad \Delta a_{u,t} \geq 0, \qquad (19a)$$
$$b_{u,t}^- = \bar{b}_{u,t} - \Delta b_{u,t}, \qquad b_{u,t}^+ = \bar{b}_{u,t} + \Delta b_{u,t}, \qquad \Delta b_{u,t} \geq 0, \qquad (19b)$$

and that Assumption 2 still holds. In order to state the Γ-robust counterpart of the consumer model (3), we need some more notation. For every consumer $u \in N$ let $\Lambda_u \in \mathbb{N}$ with $\Lambda_u \leq T$ be the number of scenarios in which the actual price-intercepts of the inverse demand functions may deviate from their nominal value. In analogy, $\Gamma_u \in N$ with $\Gamma_u \leq T$ denotes the number of time periods in which the actual slopes of the inverse demand functions may deviate from their nominal value. With these notations at hand, we can state the Γ-robust counterpart of the model of a consumer located at node $u \in N$:

$$\max_{d_u \geq 0} \left\{ \sum_{t \in [T]} \bar{a}_{u,t} d_{u,t} - \max_{\{I \subseteq [T] \,:\, |I| \leq \Lambda_u\}} \left\{ \sum_{t \in I} \Delta a_{u,t} d_{u,t} \right\} \right.$$
$$\left. + \sum_{t \in [T]} \frac{1}{2} \bar{b}_{u,t} d_{u,t}^2 - \max_{\{S \subseteq [T] \,:\, |S| \leq \Gamma_u\}} \left\{ \sum_{t \in S} \frac{1}{2} \Delta b_{u,t} d_{u,t}^2 \right\} - \sum_{t \in [T]} \pi_{u,t} d_{u,t} \right\}. \tag{20}$$

This robust counterpart can be reformulated in a tractable way.

Theorem 3.5. *For the consumer at node $u \in N$ let an uncertainty set be given as in (12) and (19). Then, the robust counterpart (20) is equivalent*

to

$$\max_{d_u, \mu_u, \eta_u, \nu_u, \tau_u} \quad \sum_{t \in [T]} \int_0^{d_{u,t}} (\bar{a}_{u,t} + \bar{b}_{u,t} x) \, \mathrm{d}x - \sum_{t \in [T]} \pi_{u,t} d_{u,t}$$
$$- \sum_{t \in [T]} (\mu_{u,t} + \nu_{u,t}) - \eta_u \Lambda_u - \tau_u \Gamma_u,$$

$$\text{s.t.} \quad 0 \leq \mu_{u,t} + \eta_u - \Delta a_{u,t} d_{u,t}, \quad t \in [T], \tag{21a}$$
$$0 \leq d_{u,t}, \quad 0 \leq \eta_u, \quad 0 \leq \tau_u, \quad 0 \leq \mu_{u,t}, \quad 0 \leq \nu_{u,t}, \quad t \in [T] \tag{21b}$$
$$0 \leq \nu_{u,t} + \tau_u - \frac{1}{2} \Delta b_{u,t} d_{u,t}^2, \quad t \in [T]. \tag{21c}$$

Proof. The proof mainly follows the technique developed in [22]. First, we rewrite the robust counterpart (20) as

$$\max_{d_u \geq 0, \lambda} \quad \lambda \tag{22a}$$

$$\text{s.t.} \quad \lambda \leq \sum_{t \in [T]} \bar{a}_{u,t} d_{u,t} - \max_{\{I \subseteq [T] \,:\, |I| \leq \Lambda_u\}} \left\{ \sum_{t \in I} \Delta a_{u,t} d_{u,t} \right\} \tag{22b}$$
$$+ \sum_{t \in [T]} \frac{1}{2} \bar{b}_{u,t} d_{u,t}^2 - \max_{\{S \subseteq [T] \,:\, |S| \leq \Gamma_u\}} \left\{ \sum_{t \in S} \frac{1}{2} \Delta b_{u,t} d_{u,t}^2 \right\}$$
$$- \sum_{t \in [T]} \pi_{u,t} d_{u,t}.$$

Next, we reformulate the inner maximization problems that are part of Constraint (22b), state their dual problems, and use strong duality to replace the inner maximization problems with the dual minimization problems. An equivalent formulation of

$$\max_{\{I \subseteq [T] \,:\, |I| \leq \Lambda_u\}} \left\{ \sum_{t \in I} \Delta a_{u,t} d_{u,t} \right\}$$

is given by

$$\max_{z_u} \quad \sum_{t \in [T]} \Delta a_{u,t} d_{u,t} z_{u,t} \quad \text{s.t.} \quad \sum_{t \in [T]} z_{u,t} \leq \Lambda_u, \quad 0 \leq z_{u,t} \leq 1, \quad t \in [T]. \tag{23}$$

This is a linear optimization problem in z_u and its dual problem reads

$$\min_{\mu_u, \eta_u} \quad \sum_{t \in [T]} \mu_{u,t} + \eta_u \Lambda_u$$
$$\text{s.t.} \quad \mu_{u,t} + \eta_u \geq \Delta a_{u,t} d_{u,t}, \quad \mu_{u,t} \geq 0 \quad t \in [T], \quad \eta_u \geq 0,$$

where η_u is the dual variable of the first constraint in (23) and $\mu_{u,t}$ are the dual variables of the second group of constraints. Using the equivalent formulation

$$\max_{z_u} \quad \sum_{t \in [T]} \frac{1}{2} \Delta b_{u,t} d_{u,t}^2 z_{u,t} \quad \text{s.t.} \quad \sum_{t \in [T]} z_{u,t} \leq \Gamma_u, \quad 0 \leq z_{u,t} \leq 1, \quad t \in [T], \tag{24}$$

of

$$\max_{\{S \subseteq [T] \,:\, |S| \leq \Gamma_u\}} \left\{ \sum_{t \in S} \frac{1}{2} \Delta b_{u,t} d_{u,t}^2 \right\} - \sum_{t \in [T]} \pi_{u,t} d_{u,t},$$

we obtain the second dual optimization problem

$$\min_{\nu_u, \tau_u} \quad \sum_{t \in [T]} \nu_{u,t} + \tau_u \Gamma_u \quad \text{s.t.} \quad \nu_{u,t} + \tau_u \geq \frac{1}{2} \Delta b_{u,t} d_{u,t}^2, \; \nu_{u,t} \geq 0, \; t \in [T], \; \tau_u \geq 0. \tag{25}$$

Here, τ_u and $\nu_{u,t}$ are the dual variables of the constraints of Problem (24). We can now apply the strong duality theorem and replace the inner maximization problems in (22b) by the corresponding dual minimization problems. Furthermore, it is easy to see that we can replace these minimization problems by simply stating feasibility, which

is given by

$$\lambda \leq \sum_{t\in[T]} \int_0^{d_{u,t}} (\bar{a}_{u,t} + \bar{b}_{u,t}x)\,\mathrm{d}x - \sum_{t\in[T]} \pi_{u,t}d_{u,t}$$
$$- \sum_{t\in[T]} \mu_{u,t} - \eta_u \Lambda_u - \sum_{t\in[T]} \nu_{u,t} - \tau_u \Gamma_u,$$

$$0 \leq d_{u,t}, \quad 0 \leq \mu_{u,t}, \quad 0 \leq \nu_{u,t}, \quad t \in [T], \quad 0 \leq \eta_u, \quad 0 \leq \tau_u,$$

$$0 \leq \mu_{u,t} + \eta_u - \Delta a_{u,t}d_{u,t}, \quad 0 \leq \nu_{u,t} + \tau_u - \frac{1}{2}\Delta b_{u,t}d_{u,t}^2, \quad t \in [T],$$

instead of Constraint (22b) and the claim follows. □

If we take a closer look at the robust counterpart (21) we see that the structure, compared with the nominal problem, has changed. We obtain new variables, constraints, and objective function terms. This is always the case when the Γ-approach is used; cf. [22]. However, (21) is not a QP anymore but a quadratically constraint quadratic program (QCQP). The nonlinear constraint is (21c). Fortunately, the right-hand side of this constraint is a concave function, yielding the following corollary.

Corollary 3.6. *Suppose that Assumption 1 holds. Then, the Γ-robust counterpart (21) is a convex optimization problem for every consumer $u \in N$.*

Due to this corollary, the KKT conditions of (21) are necessary and sufficient first-order optimality conditions. They are given by

$$\bar{a}_{u,t} + \bar{b}_{u,t}d_{u,t} - \pi_{u,t} + \alpha_{u,t} - \Delta a_{u,t}\sigma_{u,t} - \Delta b_{u,t}d_{u,t}\zeta_{u,t} = 0, \quad t \in [T],$$
$$-1 + \sigma_{u,t} + \xi_{u,t} = 0, \quad -1 + \zeta_{u,t} + \rho_{u,t} = 0, \quad t \in [T],$$
$$-\Lambda_u + \sum_{t\in[T]} \sigma_{u,t} + \chi_u = 0, \quad -\Gamma_u + \psi_u + \sum_{t\in[T]} \zeta_{u,t} = 0,$$
$$0 \leq d_{u,t} \perp \alpha_{u,t} \geq 0, \quad 0 \leq \mu_{u,t} + \eta_u - \Delta a_{u,t}d_{u,t} \perp \sigma_{u,t} \geq 0, \quad t \in [T], \quad (26)$$
$$0 \leq \eta_u \perp \chi_u \geq 0, \quad 0 \leq \tau_u \perp \psi_u \geq 0,$$
$$0 \leq \nu_{u,t} + \tau_u - \frac{1}{2}\Delta b_{u,t}d_{u,t}^2 \perp \zeta_{u,t} \geq 0, \quad t \in [T],$$
$$0 \leq \mu_{u,t} \perp \xi_{u,t} \geq 0, \quad 0 \leq \nu_{u,t} \perp \rho_{u,t} \geq 0, \quad t \in [T],$$

where ζ_u, α_u, σ_u, ξ_u, ρ_u, χ_u, and ψ_u are the dual variables of constraints in Problem (21). We thus obtain the Γ-robustified market equilibrium problem (Γ-RMEP)

Producers: (2), Robustified consumers: (26),

TSO: (6), Market clearing: (7),   (27)

which models the wholesale electricity market for the case of perfect competition and consumers that hedge themselves against demand uncertainty using the Γ-approach. Note that this complementarity problem is not linear due to the quadratic complementarity constraints.

The main question again is whether this Γ-RMEP is equivalent to a Γ-robustified welfare maximization problem. To answer this question, we now consider the Γ-robustification of the welfare maximization problem (9) and prove that the answer is positive. The Γ-robust counterpart of Problem (9) reads

$$\max_{d,y,\bar{y},f,\Theta,\lambda} \quad \lambda \tag{28a}$$

$$\text{s.t.} \quad \lambda \leq \sum_{t\in[T]}\sum_{u\in N} \int_0^{d_{u,t}} (\bar{a}_{u,t} + \bar{b}_{u,t}x)\,\mathrm{d}x - \sum_{t\in[T]}\sum_{u\in N} w_u^{\mathrm{var}}y_{u,t}$$
$$- \sum_{u\in N} w_u^{\mathrm{inv}}\bar{y}_u - \sum_{u\in N} \max_{\{I\subseteq[T]:\,|I|\leq\Lambda_u\}} \left\{\sum_{t\in I} \Delta a_{u,t}d_{u,t}\right\} \tag{28b}$$

$$- \sum_{u\in N} \max_{\{S\subseteq[T]:\,|S|\leq\Gamma_u\}} \left\{\sum_{t\in S} \frac{1}{2}\Delta b_{u,t}d_{u,t}^2\right\},$$

$$(9b)\text{–}(9e). \tag{28c}$$

Again, we state a reformulation of this robust counterpart.

Theorem 3.7. *For all consumers at the nodes $u \in N$ let an uncertainty set be given as in (12) and (19). Then, the Γ-robust counterpart (28) of the welfare maximization problem (9) is equivalent to*

$$\max \quad \sum_{t\in[T]}\sum_{u\in N} \int_0^{d_{u,t}} (\bar{a}_{u,t} + \bar{b}_{u,t}x)\,\mathrm{d}x - \sum_{t\in[T]}\sum_{u\in N} w_u^{\mathrm{var}}y_{u,t} - \sum_{u\in N} w_u^{\mathrm{inv}}\bar{y}_u$$
$$- \sum_{t\in[T]}\sum_{u\in N} \mu_{u,t} - \sum_{u\in N} \eta_u\Lambda_u - \sum_{t\in[T]}\sum_{u\in N} \nu_{u,t} - \sum_{u\in N} \tau_u\Gamma_u, \tag{29a}$$

$$\text{s.t.} \quad (9b)\text{–}(9e), \tag{29b}$$

$$0 \leq \mu_{u,t} + \eta_u - \Delta a_{u,t}d_{u,t}, \quad 0 \leq \mu_{u,t}, \quad 0 \leq \nu_{u,t}, \quad u \in N,\ t \in [T], \tag{29c}$$

$$0 \leq \eta_u, \quad 0 \leq \tau_u, \quad u \in N, \quad 0 \leq \nu_{u,t} + \tau_u - \frac{1}{2}\Delta b_{u,t}d_{u,t}^2,$$
$$u \in N,\ t \in [T]. \tag{29d}$$

Suppose further that Assumption 1 holds. Then, (29) is a convex optimization problem.

Proof. We use the reformulations and dual problems (23)–(25) for every consumer with the same variable names as in the proof of Theorem 3.5. Replacing the corresponding maximization problems in Constraint (28b) of the robust counterpart yields the claim. Finally, we observe that the quadratic constraint again yields a convex feasible set and that we thus obtain a convex problem. □

As a consequence, necessary and sufficient first-order optimality conditions of the robust welfare maximization problem with robust consumers in the flexible Γ-setting (29) are given by the KKT conditions (6) as well as (2), (7), (26) for all $u \in N$, if we use the same names for the dual variables as for the single player problems and if we identify the nodal prices $\pi_{u,t}$ with the dual variable of the corresponding nodal market-clearing conditions. As a result, we obtain the positive answer for our main question.

Corollary 3.8. *The Γ-RMEP (27) and the Γ-robust welfare maximization problem (29) are equivalent.*

We close this section with a brief discussion about the uniqueness of Γ-robust equilibria. Unfortunately, no results from the literature that we are aware of can be used to prove uniqueness. The problem is that the papers [65–67] discussed in Remark 3.4 cannot be used in this setting because the quadratic constraints $0 \leq \nu_{u,t} + \tau_u - 1/2\Delta b_{u,t}d_{u,t}^2$ for all $u \in N$, $t \in [T]$, yield a completely different problem structure, which prevents the application of the results in the cited papers. Nevertheless, existence of robust market equilibria is guaranteed by the existence of solutions of (29).

4. Robust counterparts under Nash–Cournot competition

In this section, we study the same questions as in Section 3 but replace the assumption of a perfectly competitive market by a Nash–Cournot assumption. The nominal Nash–Cournot model is derived in Section 4.1. The case of strictly robust Nash–Cournot equilibria is afterward discussed in Section 4.2, whereas the Γ-approach is topic of Section 4.3.

4.1. The nominal model with Nash–Cournot competition

In this section we derive a model in which we drop the economic assumption of perfect competition but assume Nash–Cournot competition for the producers in our model. Thus, we assume that a producer (i) anticipates the reactions of the consumers to changes in the price but that the producer (ii) naively assumes that the other producers and the TSO do not modify their actions depending on the producer's decision.

For a tutorial of modeling Nash–Cournot competition for electricity markets see [3,8] and the references therein. The TSO, however, is still assumed to be a price taker; cf., e.g., [8].

As it is typically the case for Nash–Cournot models we assume that all consumers have a strictly positive demand and refrain from explicitly stating this demand constraint; cf., e.g., Section 3.4.2.5 in [3]. Hence, the consumer model (3) becomes the unconstrained optimization model

$$\max_{d_u} \quad \sum_{t \in [T]} \int_0^{d_{u,t}} p_{u,t}(x)\,\mathrm{d}x - \sum_{t \in [T]} \pi_{u,t} d_{u,t}.$$

Its optimal solution is characterized by the first-order conditions $p_{u,t}(d_{u,t}) = \pi_{u,t}$ for all $t \in [T]$. Hence, the consumer's demand is unique in dependence of the given prices $\pi_{u,t}$. As usual, we will later substitute these first-order conditions into the optimization problems of the producers.

Next, we state the Nash–Cournot model of a producer. To this end, we need to modify the usage of exogenously given prices $\pi_{u,t}$ in (1) and replace these prices by the inverse demand function of the consumer at that node. This yields the problem

$$\max_{y_u, \bar{y}_u} \quad \sum_{t \in [T]} p_{u,t}\left(y_{u,t} + \sum_{a \in \delta^{\mathrm{in}}(u)} f_{a,t} - \sum_{a \in \delta^{\mathrm{out}}(u)} f_{a,t} \right) y_{u,t} - \sum_{t \in [T]} w_u^{\mathrm{var}} y_{u,t} - w_u^{\mathrm{inv}} \bar{y}_u$$

(30a)

$$\text{s.t.} \quad 0 \le y_{u,t} \le \bar{y}_u, \quad t \in [T],$$ (30b)

in which we explicitly used the market-clearing condition (7) to substitute the nodal demand $d_{u,t}$ into the inverse demand function of the objective. By doing so, the producer anticipates that the price depends on his own supply level and on the supply levels of the other producers. The latter are implicitly given by the in- and outflows at the node at which the producer is located.

We still use the assumption that the inverse demand functions are given as linear functions $p_{u,t}(x) = a_{u,t} + b_{u,t}x$ and thus obtain the producer's KKT conditions

$$p_{u,t}\left(y_{u,t} + \sum_{a \in \delta^{\mathrm{in}}(u)} f_{a,t} - \sum_{a \in \delta^{\mathrm{out}}(u)} f_{a,t} \right) + b_{u,t} y_{u,t} - w_u^{\mathrm{var}} + \beta_{u,t}^- - \beta_{u,t}^+ = 0,$$

$$t \in [T],$$ (31a)

$$-w_u^{\mathrm{inv}} + \sum_{t \in [T]} \beta_{u,t}^+ = 0, \quad 0 \le y_{u,t} \perp \beta_{u,t}^- \ge 0, \quad 0 \le \bar{y}_u - y_{u,t} \perp \beta_{u,t}^+ \ge 0,$$

$$t \in [T].$$ (31b)

Finally, the TSO's model, who still acts as a price taker, is given by (5) and the necessary and sufficient conditions are given by System (6).

Since both the optimal reactions of the consumers and the market clearing are already substituted into the two discussed models, we obtain the LCP

Nash–Cournot Producers: (31), Price-taking TSO: (6) (32)

for modeling Nash–Cournot equilibria.

Note again that we assumed that all demands $d_{u,t}$ are positive. In this situation, it is well-known that the Nash–Cournot equilibrium model (32) can be equivalently replaced by a single optimization problem—see [70] as well as [8]—which is given by

$$\max_{d,y,\bar{y},f,\Theta} \quad \sum_{t \in [T]} \sum_{u \in N} \int_0^{d_{u,t}} p_{u,t}(x)\,\mathrm{d}x - \sum_{t \in [T]} \sum_{u \in N} w_u^{\mathrm{var}} y_{u,t} - \sum_{u \in N} w_u^{\mathrm{inv}} \bar{y}_u \quad (33a)$$

$$+ \sum_{t \in [T]} \sum_{u \in N} \frac{1}{2} b_{u,t} y_{u,t}^2$$

$$\text{s.t.} \quad 0 \le y_{u,t} \le \bar{y}_u, \quad u \in N, \ t \in [T],$$ (33b)

$$-f_a^+ \le f_{a,t} \le f_a^+, \quad f_{a,t} = B_a(\Theta_{u,t} - \Theta_{v,t}),$$

$$a = (u,v) \in A, \ t \in [T],$$ (33c)

$$0 = d_{u,t} - y_{u,t} + \sum_{a \in \delta^{\mathrm{out}}(u)} f_{a,t} - \sum_{a \in \delta^{\mathrm{in}}(u)} f_{a,t}, \quad u \in N, \ t \in [T].$$

(33d)

There are two differences w.r.t. the welfare maximization problem (9). By assumption, the demand constraints in (9b) are neglected and we obtain the additional production-depending quadratic objective terms $1/2 b_{u,t} y_{u,t}^2$ for every producer and every time period. We finally again remark that the equivalence between (32) and (33) only holds under the positivity assumption for the demands. The reason is that, otherwise, the inverse demand functions $p_{u,t}(\cdot)$ do not necessarily equal nodal prices; cf. (4).

Before we start the discussion of robust Nash–Cournot equilibria, we state and prove uniqueness of nominal Nash–Cournot equilibria.

Theorem 4.1. *Suppose that Assumption 1 holds and that the phase angles $\Theta_{r,t}$ are fixed for all $t \in [T]$ at an arbitrary reference node $r \in N$. Then, the Nash–Cournot equilibrium of (32) is unique.*

Proof. Since the LCP (32) and the QP (33) are equivalent, we can consider the latter to prove uniqueness of the equilibrium of (32). The objective function of (33) is strictly concave in the demands $d_{u,t}$ and the productions $y_{u,t}$ for all $u \in N$ and $t \in [T]$. In this case, the uniqueness of these quantities follows from [71]. Thus, it remains to show the uniqueness of the capacities $\bar{y}_u$. For every node $u \in N$, it holds $\bar{y}_u = \max\{y_{u,t} : t \in [T]\}$ by optimality. This also proves the uniqueness of $\bar{y}_u$. The uniqueness of the flows and phase angles follows from Theorem 3.1 in [66]. □

4.2. Strict robustness

In this section, we again use the concept of strictly robust optimization for studying the case in which the demand functions $p_{u,t}$ of the consumers are uncertain. Thus, we follow the same road as in Section 3.1 but replace the economic assumption of perfect competition by Nash–Cournot competition. That is, we robustify the nominal models developed in the last section. We use the same notation as in Section 3.1.

In contrast to Section 3.1, the Nash–Cournot models (30) of the producers now depend on the uncertain inverse demand functions. This is why we now have to consider the corresponding robust counterparts, which are given by

$$\max_{y_u, \bar{y}_u, \lambda} \quad \lambda \quad (34a)$$

$$\text{s.t.} \quad \lambda \le -\sum_{t \in [T]} w_u^{\mathrm{var}} y_{u,t} - w_u^{\mathrm{inv}} \bar{y}_u + \min_{(a_{u,t}, b_{u,t}) \in \mathcal{U}_{u,t}} \left\{ \sum_{t \in [T]} a_{u,t} y_{u,t} \right. \quad (34b)$$

$$\left. + \sum_{t \in [T]} b_{u,t} \left(y_{u,t} + \sum_{a \in \delta^{\mathrm{in}}(u)} f_{a,t} - \sum_{a \in \delta^{\mathrm{out}}(u)} f_{a,t} \right) y_{u,t} \right\}$$

$$0 \le y_{u,t} \le \bar{y}_u, \quad t \in [T].$$ (34c)

As before, we can rewrite this problem in a tractable way.

Theorem 4.2. *Let (12) be the uncertainty set for consumer $u \in N$ for all $t \in [T]$ and suppose Assumption 2 holds. Then, the robust counterpart (34) is equivalent to the convex problem*

$$\max_{y_u, \bar{y}_u} \quad \sum_{t \in [T]} a_{u,t}^- y_{u,t} + \sum_{t \in [T]} b_{u,t}^- \left(y_{u,t} + \sum_{a \in \delta^{\mathrm{in}}(u)} f_{a,t} - \sum_{a \in \delta^{\mathrm{out}}(u)} f_{a,t} \right) y_{u,t} \quad (35a)$$

$$- \sum_{t \in [T]} w_u^{\mathrm{var}} y_{u,t} - w_u^{\mathrm{inv}} \bar{y}_u$$

$$\text{s.t.} \quad 0 \le y_{u,t} \le \bar{y}_u, \quad t \in [T].$$ (35b)

The proof is given in Appendix C. As a consequence, necessary and sufficient first-order optimality conditions of a robust Nash–Cournot producer are given by the KKT conditions

$$a_{u,t}^- + b_{u,t}^- \left(y_{u,t} + \sum_{a \in \delta^{\mathrm{in}}(u)} f_{a,t} - \sum_{a \in \delta^{\mathrm{out}}(u)} f_{a,t} \right) + b_{u,t}^- y_{u,t}$$
$$-w_u^{\mathrm{var}} + \beta_{u,t}^- - \beta_{u,t}^+ = 0, \quad t \in [T], \quad (36a)$$

$$-w_u^{\mathrm{inv}} + \sum_{t \in [T]} \beta_{u,t}^+ = 0, \quad (36b)$$

$$0 \le y_{u,t} \perp \beta_{u,t}^- \ge 0, \quad 0 \le \bar{y}_u - y_{u,t} \perp \beta_{u,t}^+ \ge 0, \quad t \in [T]. \quad (36c)$$

As the optimization problem (5) of the TSO does not depend on the uncertain inverse demand functions of the consumers, no robustification needs to be done for the TSO model. In total, we obtain the robust Nash–Cournot-LCP

Robustified Cournot producers: (36), Price-taking TSO: (6), (37)

which models the wholesale electricity market for the case of Nash–Cournot competition as well as consumers that face demand uncertainties. As before, we now answer the question whether the strictly robust counterpart of the Nash–Cournot QP (33) is equivalent to the RMEP (37) or not.

The robust counterpart of the Nash–Cournot QP (33) reads

$$\max_{d,y,\bar{y},f,\lambda} \quad \lambda \quad (38a)$$

$$\mathrm{s.t.} \quad \lambda \le \min_{(a_{u,t}, b_{u,t}) \in \mathcal{U}_{u,t}} \left\{ \sum_{t \in [T]} \sum_{u \in N} \left(a_{u,t} d_{u,t} + \frac{1}{2} b_{u,t} d_{u,t}^2 \right) \right. \quad (38b)$$

$$\left. + \sum_{t \in [T]} \sum_{u \in N} \frac{1}{2} b_{u,t} y_{u,t}^2 \right\} - \sum_{t \in [T]} \sum_{u \in N} w_u^{\mathrm{var}} y_{u,t} - \sum_{u \in N} w_u^{\mathrm{inv}} \bar{y}_u,$$

$$(33b)\text{–}(33d). \quad (38c)$$

Theorem 4.3. *Let* (12) *be the uncertainty set for every consumer* $u \in N$ *for all* $t \in [T]$ *and suppose Assumption 2 holds. Then, the robust counterpart* (38) *is equivalent to the convex problem*

$$\max_{d,y,\bar{y},\Theta} \quad \sum_{t \in [T]} \sum_{u \in N} \left(a_{u,t}^- d_{u,t} + \frac{1}{2} b_{u,t}^- d_{u,t}^2 \right) + \sum_{t \in [T]} \sum_{u \in N} \frac{1}{2} b_{u,t}^- y_{u,t}^2$$
$$- \sum_{t \in [T]} \sum_{u \in N} w_u^{\mathrm{var}} y_{u,t} - \sum_{u \in N} w_u^{\mathrm{inv}} \bar{y}_u \quad (39)$$

$$\mathrm{s.t.} \quad (33b)\text{–}(33d).$$

The proof can be found in Appendix D. Thus, the first-order conditions are necessary and sufficient for the robust counterpart (39) of the Nash–Cournot QP and are given by

$$a_{u,t}^- + b_{u,t}^- d_{u,t} - \lambda_{u,t} = 0, \quad u \in N, \ t \in [T], \quad (40a)$$

$$b_{u,t}^- y_{u,t} - w_u^{\mathrm{var}} + \beta_{u,t}^- - \beta_{u,t}^+ + \lambda_{u,t} = 0, \quad u \in N, \ t \in [T], \quad (40b)$$

$$-w_u^{\mathrm{inv}} + \sum_{t \in [T]} \beta_{u,t}^+ = 0, \quad u \in N, \quad (40c)$$

$$\delta_{a,t}^- - \delta_{a,t}^+ - \lambda_{u,t} + \lambda_{v,t} + \varepsilon_{a,t} = 0, \quad a = (u,v) \in A, \ t \in [T], \quad (40d)$$

$$\sum_{a \in \delta^{\mathrm{out}}(u)} B_a \varepsilon_{a,t} - \sum_{a \in \delta^{\mathrm{in}}(u)} B_a \varepsilon_{a,t} = 0, \quad u \in N, \ t \in [T], \quad (40e)$$

$$0 \le y_{u,t} \perp \beta_{u,t}^- \ge 0, \quad 0 \le \bar{y}_u - y_{u,t} \perp \beta_{u,t}^+ \ge 0, \quad u \in N, \ t \in [T], \quad (40f)$$

$$0 \le f_{a,t} + f_a^+ \perp \delta_{a,t}^- \ge 0, \quad 0 \le f_a^+ - f_{a,t} \perp \delta_{a,t}^+ \ge 0, \quad a \in A, \ t \in [T], \quad (40g)$$

$$0 = d_{u,t} - y_{u,t} + \sum_{a \in \delta^{\mathrm{out}}(u)} f_{a,t} - \sum_{a \in \delta^{\mathrm{in}}(u)} f_{a,t}, \quad u \in N, \ t \in [T], \quad (40h)$$

$$f_{a,t} = B_a(\Theta_{u,t} - \Theta_{v,t}), \quad a = (u,v) \in A, \ t \in [T]. \quad (40i)$$

Comparing these KKT conditions with the RMEP (37), we see that these problems are equivalent as well.

Theorem 4.4. *The RMEP* (37) *and the strictly robust Nash–Cournot QP* (39) *are equivalent.*

Proof. We identify all dual variables with the same names in both systems (37) and (40) and further identify the dual variables $\lambda_{u,t}$ in (40) with $\pi_{u,t}$ in System (37). Finally, using the market-clearing condition yields the claim. □

Since the structure of the robustified Nash–Cournot QP is the same as for the nominal QP, Theorem 4.1 also holds for (39) and we obtain the uniqueness of the robust Nash–Cournot equilibria. Note that this is a stronger statement compared to what we can say about the uniqueness of robust equilibria in the perfectly competitive case since the uniqueness does not depend on the network model for Nash–Cournot models.

4.3. The Γ-approach

In this section, we again consider the Γ-approach as in Section 3.2 to robustify the Nash–Cournot setting introduced in Section 4.1. Both for the modeling of uncertainty and Nash–Cournot competition we use the same notation as before.

The robust counterpart of the Nash–Cournot producer (30) reads

$$\max_{y_u, \bar{y}_u, \lambda} \quad \lambda \quad (41a)$$

$$\mathrm{s.t.} \quad \lambda \le \sum_{t \in [T]} \left(\bar{a}_{u,t} y_{u,t} + \bar{b}_{u,t} \left(y_{u,t} + \sum_{a \in \delta^{\mathrm{in}}(u)} f_{a,t} - \sum_{a \in \delta^{\mathrm{out}}(u)} f_{a,t} \right) y_{u,t} \right) \quad (41b)$$

$$- \sum_{t \in [T]} w_u^{\mathrm{var}} y_{u,t} - w_u^{\mathrm{inv}} \bar{y}_u - \max_{\{I \subseteq [T] : |I| \le \Lambda_u\}} \left\{ \sum_{t \in I} \Delta a_{u,t} y_{u,t} \right\}$$

$$- \max_{\{S \subseteq [T] : |S| \le \Gamma_u\}} \left\{ \sum_{t \in S} \Delta b_{u,t} \left(y_{u,t} + \sum_{a \in \delta^{\mathrm{in}}(u)} f_{a,t} - \sum_{a \in \delta^{\mathrm{out}}(u)} f_{a,t} \right) y_{u,t} \right\},$$

$$0 \le y_{u,t} \le \bar{y}_u, \quad t \in [T]. \quad (41c)$$

As usual, we rewrite this problem to obtain a tractable counterpart.

Theorem 4.5. *For the consumer at node* $u \in N$ *let an uncertainty set be given as in* (12) *and* (19). *Then, the Γ-robust counterpart* (41) *of the Nash–Cournot producer* (30) *is equivalent to*

$$\max_z \quad \sum_{t \in [T]} \bar{a}_{u,t} y_{u,t} + \sum_{t \in [T]} \bar{b}_{u,t} \left(y_{u,t} + \sum_{a \in \delta^{\mathrm{in}}(u)} f_{a,t} - \sum_{a \in \delta^{\mathrm{out}}(u)} f_{a,t} \right) y_{u,t} \quad (42a)$$

$$- \sum_{t \in [T]} w_u^{\mathrm{var}} y_{u,t} - w_u^{\mathrm{inv}} \bar{y}_u - \sum_{t \in [T]} \mu_{u,t} - \eta_u \Lambda_u - \sum_{t \in [T]} \nu_{u,t} - \tau_u \Gamma_u$$

$$\mathrm{s.t.} \quad 0 \le y_{u,t} \le \bar{y}_u, \quad 0 \le \mu_{u,t} + \eta_u - \Delta a_{u,t} y_{u,t}, \quad 0 \le \mu_{u,t}, \quad 0 \le \nu_{u,t},$$
$$t \in [T], \quad (42b)$$

$$0 \le \eta_u, \quad 0 \le \tau_u, \quad (42c)$$

$$0 \le \nu_{u,t} + \tau_u - \Delta b_{u,t} \left(y_{u,t} + \sum_{a \in \delta^{\mathrm{in}}(u)} f_{a,t} - \sum_{a \in \delta^{\mathrm{out}}(u)} f_{a,t} \right) y_{u,t}, \quad t \in [T], \quad (42d)$$

where the variable vector is abbreviated as $z := (y_u, \bar{y}_u, \mu_u, \eta_u, \nu_u, \tau_u)$. *Suppose further that Assumption 1 holds for consumer* $u \in N$. *Then,* (42) *is a convex QCQP.*

Proof. As in Section 3.2 we reformulate the inner maximization problems that are part of Constraint (41b), state their dual problems, and use strong duality to replace the inner maximization problems with

the dual minimization problems. The dual problem of

$$\max_{\{I \subseteq [T]:\, |I| \le \Lambda_u\}} \left\{ \sum_{t \in I} \Delta a_{u,t} y_{u,t} \right\}$$

can be written as

$$\min_{\mu_u, \eta_u} \quad \sum_{t \in [T]} \mu_{u,t} + \eta_u \Lambda_u$$

$$\text{s.t.} \quad \mu_{u,t} + \eta_u \ge \Delta a_{u,t} y_{u,t}, \quad \mu_{u,t} \ge 0, \quad t \in [T], \quad \eta_u \ge 0,$$

cf., e.g., the proof of Theorem 3.5. To obtain the dual of the second inner maximization problem

$$\max_{\{S \subseteq [T]:\, |S| \le \Gamma_u\}} \left\{ \sum_{t \in S} \Delta b_{u,t} y_{u,t} \left(y_{u,t} + \sum_{a \in \delta^{\text{in}}(u)} f_{a,t} - \sum_{a \in \delta^{\text{out}}(u)} f_{a,t} \right) y_{u,t} \right\}$$

we rewrite it as an equivalent linear program

$$\max_{z_u} \quad \sum_{t \in [T]} \Delta b_{u,t} y_{u,t} \left(y_{u,t} + \sum_{a \in \delta^{\text{in}}(u)} f_{a,t} - \sum_{a \in \delta^{\text{out}}(u)} f_{a,t} \right) z_{u,t}$$

$$\text{s.t.} \quad \sum_{t \in [T]} z_{u,t} \le \Gamma_u, \quad 0 \le z_{u,t} \le 1, \quad t \in [T],$$

for which the dual reads

$$\min_{\tau_u, \nu_u} \quad \Gamma_u \tau_u + \sum_{t \in [T]} \nu_{u,t}$$

$$\text{s.t.} \quad 0 \le \tau_u + \nu_{u,t} - \Delta b_{u,t} y_{u,t} \left(y_{u,t} + \sum_{a \in \delta^{\text{in}}(u)} f_{a,t} - \sum_{a \in \delta^{\text{out}}(u)} f_{a,t} \right), \quad t \in [T],$$

$$0 \le \tau_u, \quad 0 \le \nu_{u,t}, \quad t \in [T].$$

We can now apply the strong duality theorem and replace the inner maximization problems in (41) by the corresponding dual maximization problems and the claim of the theorem follows. □

Due to this result, the KKT conditions of (42) are necessary and sufficient first-order optimality conditions. They are given by

$$\bar{a}_{u,t} + \bar{b}_{u,t} y_{u,t} + \bar{b}_{u,t} \left(y_{u,t} + \sum_{a \in \delta^{\text{in}}(u)} f_{a,t} - \sum_{a \in \delta^{\text{out}}(u)} f_{a,t} \right) - \Delta a_{u,t} \sigma_{u,t}$$

$$- \Delta b_{u,t} \left(2 y_{u,t} + \sum_{a \in \delta^{\text{in}}(u)} f_{a,t} - \sum_{a \in \delta^{\text{out}}(u)} f_{a,t} \right) \zeta_{u,t}$$

$$- w_u^{\text{var}} + \beta_{u,t}^- - \beta_{u,t}^+ = 0, \quad t \in [T], \tag{43a}$$

$$-w_u^{\text{inv}} + \sum_{t \in [T]} \beta_{u,t}^+ = 0, \quad -1 + \sigma_{u,t} + \xi_{u,t} = 0, \quad -1 + \zeta_{u,t} + \rho_{u,t} = 0, \quad t \in [T], \tag{43b}$$

$$-\Lambda_u + \sum_{t \in [T]} \sigma_{u,t} + \chi_u = 0, \quad -\Gamma_u + \psi_u + \sum_{t \in [T]} \zeta_{u,t} = 0, \tag{43c}$$

$$0 \le y_{u,t} \perp \beta_{u,t}^- \ge 0, \quad 0 \le \bar{y}_u - y_{u,t} \perp \beta_{u,t}^+ \ge 0, \quad t \in [T], \tag{43d}$$

$$0 \le \mu_{u,t} + \eta_u - \Delta a_{u,t} y_{u,t} \perp \sigma_{u,t} \ge 0, \quad t \in [T], \tag{43e}$$

$$0 \le \eta_u \perp \chi_u \ge 0, \quad 0 \le \tau_u \perp \psi_u \ge 0, \tag{43f}$$

$$0 \le \nu_{u,t} + \tau_u - \Delta b_{u,t} \left(y_{u,t} + \sum_{a \in \delta^{\text{in}}(u)} f_{a,t} - \sum_{a \in \delta^{\text{out}}(u)} f_{a,t} \right) y_{u,t} \perp \zeta_{u,t} \ge 0, \quad t \in [T], \tag{43g}$$

$$0 \le \mu_{u,t} \perp \xi_{u,t} \ge 0, \quad 0 \le \nu_{u,t} \perp \rho_{u,t} \ge 0, \quad t \in [T]. \tag{43h}$$

As in Section 3.2 we now obtain the Γ-RMEP

Robustified Nash–Cournot Producers: (43), Price-taking TSO: (6),

$$\tag{44}$$

which models the wholesale electricity market for the case of Nash–Cournot competition and the Γ-approach. Again, our main goal is to answer the question whether the Γ-robust counterpart of the Nash–Cournot QP (33) is equivalent to the Γ-RMEP (44) or not. Thus, we consider the Γ-robustification of the Nash–Cournot QP (33), which reads

$$\max_{d, y, \bar{y}, f, \Theta, \lambda} \quad \lambda \tag{45a}$$

$$\text{s.t.} \quad \lambda \le \sum_{t \in [T]} \sum_{u \in N} \int_0^{d_{u,t}} (\bar{a}_{u,t} + \bar{b}_{u,t} x)\, \mathrm{d}x - \sum_{t \in [T]} \sum_{u \in N} w_u^{\text{var}} y_{u,t} \tag{45b}$$

$$- \sum_{u \in N} w_u^{\text{inv}} \bar{y}_u + \sum_{t \in [T]} \sum_{u \in N} \frac{1}{2} \bar{b}_{u,t} y_{u,t}^2$$

$$- \sum_{u \in N} \max_{\{I \subseteq [T]:\, |I| \le \Lambda_u\}} \left\{ \sum_{t \in I} \Delta a_{u,t} d_{u,t} \right\}$$

$$- \sum_{u \in N} \max_{\{S \subseteq [T]:\, |S| \le \Gamma_u\}} \left\{ \sum_{t \in S} \frac{1}{2} \Delta b_{u,t} \left(d_{u,t}^2 + y_{u,t}^2 \right) \right\},$$

$$(33\text{b})\text{–}(33\text{d}). \tag{45c}$$

First of all, we write the latter robust counterpart in a tractable way.

Theorem 4.6. *For all consumers $u \in N$ let an uncertainty set be given as in* (12) *and* (19). *Then, the Γ-robust counterpart* (45) *is equivalent to*

$$\max_{d, y, \bar{y}, f, \Theta, \mu, \eta, \nu, \tau} \quad \sum_{t \in [T]} \sum_{u \in N} \int_0^{d_{u,t}} (\bar{a}_{u,t} + \bar{b}_{u,t} x)\, \mathrm{d}x - \sum_{t \in [T]} \sum_{u \in N} w_u^{\text{var}} y_{u,t}$$

$$- \sum_{u \in N} w_u^{\text{inv}} \bar{y}_u + \sum_{t \in [T]} \sum_{u \in N} \frac{1}{2} \bar{b}_{u,t} y_{u,t}^2$$

$$- \sum_{t \in [T]} \sum_{u \in N} \mu_{u,t} - \sum_{u \in N} \eta_u \Lambda_u - \sum_{t \in [T]} \sum_{u \in N} \nu_{u,t} - \sum_{u \in N} \tau_u \Gamma_u \tag{46a}$$

s.t. $(33\text{b})\text{–}(33\text{d}),$ \hfill (46b)

$$0 \le \mu_{u,t}, \nu_{u,t}, \quad u \in N,\ t \in [T], \tag{46c}$$

$$0 \le \eta_u, \tau_u, \quad u \in N, \tag{46d}$$

$$0 \le \mu_{u,t} + \eta_u - \Delta a_{u,t} d_{u,t}, \quad u \in N,\ t \in [T], \tag{46e}$$

$$0 \le \nu_{u,t} + \tau_u - \frac{1}{2} \Delta b_{u,t} \left(d_{u,t}^2 + y_{u,t}^2 \right), \quad u \in N,\ t \in [T]. \tag{46f}$$

Suppose further that Assumption 2 holds. Then, (46) *is a convex optimization problem.*

Proof. The proof can be carried out in analogy to the one of Theorem 4.5. □

Hence, the KKT conditions of (46) are necessary and sufficient conditions and read

$$\bar{a}_{u,t} + \bar{b}_{u,t} d_{u,t} - \gamma_{u,t} - \Delta a_{u,t} \sigma_{u,t} - \Delta b_{u,t} d_{u,t} \zeta_{u,t} = 0, \quad u \in N,\ t \in [T], \tag{47a}$$

$$-w_u^{\text{var}} + \beta_{u,t}^- - \beta_{u,t}^+ + \gamma_{u,t} + \bar{b}_{u,t} y_{u,t} - \Delta b_{u,t} y_{u,t} \zeta_{u,t} = 0, \quad u \in N,\ t \in [T], \tag{47b}$$

$$-w_u^{\text{inv}} + \sum_{t \in [T]} \beta_{u,t}^+ = 0, \quad u \in N, \tag{47c}$$

$$-1 + \sigma_{u,t} + \xi_{u,t} = 0, \quad -1 + \zeta_{u,t} + \rho_{u,t} = 0, \quad u \in N,\ t \in [T], \tag{47d}$$

$$-\Lambda_u + \sum_{t \in [T]} \sigma_{u,t} + \chi_u = 0, \quad -\Gamma_u + \psi_u + \sum_{t \in [T]} \zeta_{u,t} = 0, \quad u \in N, \tag{47e}$$

$$\delta_{a,t}^- - \delta_{a,t}^+ + \varepsilon_{a,t} - \gamma_{u,t} + \gamma_{v,t} = 0, \quad a = (u,v) \in A,\ t \in [T], \tag{47f}$$

$$\sum_{a \in \delta^{\text{out}}(u)} B_a \varepsilon_{a,t} - \sum_{a \in \delta^{\text{in}}(u)} B_a \varepsilon_{a,t} = 0, \quad u \in N,\ t \in [T], \tag{47g}$$

$$f_{a,t} = B_a(\Theta_{u,t} - \Theta_{v,t}), \quad a = (u,v) \in A, \ t \in [T], \tag{47h}$$

$$0 = d_{u,t} - y_{u,t} + \sum_{a \in \delta^{\text{out}}(u)} f_{a,t} - \sum_{a \in \delta^{\text{in}}(u)} f_{a,t}, \quad u \in N, \ t \in [T], \tag{47i}$$

$$0 \le f_{a,t} + f_a^+ \perp \delta_{a,t}^- \ge 0, \quad 0 \le f_a^- - f_{a,t} \perp \delta_{a,t}^+ \ge 0, \quad a \in A, \ t \in [T], \tag{47j}$$

$$0 \le y_{u,t} \perp \beta_{u,t}^- \ge 0, \quad 0 \le \bar{y}_u - y_{u,t} \perp \beta_{u,t}^+ \ge 0, \quad u \in N, \ t \in [T], \tag{47k}$$

$$0 \le \mu_{u,t} + \eta_u - \Delta a_{u,t} d_{u,t} \perp \sigma_{u,t} \ge 0, \quad u \in N, \ t \in [T], \tag{47l}$$

$$0 \le \eta_u \perp \chi_u \ge 0, \quad 0 \le \tau_u \perp \psi_u \ge 0, \quad u \in N, \tag{47m}$$

$$0 \le \nu_{u,t} + \tau_u - \frac{1}{2}\Delta b_{u,t}\left(y_{u,t}^2 + d_{u,t}^2\right) \perp \zeta_{u,t} \ge 0, \quad u \in N, \ t \in [T], \tag{47n}$$

$$0 \le \mu_{u,t} \perp \xi_{u,t} \ge 0, \quad 0 \le \nu_{u,t} \perp \rho_{u,t} \ge 0, \quad u \in N, \ t \in [T]. \tag{47o}$$

In order to prove or disprove the equivalence between the Γ-robust LCP (44) and the KKT conditions (47) of the Γ-robustified Nash–Cournot QP, we again compare these systems. First, the KKT conditions (47a) and (47b) of the Nash–Cournot QP together are equivalent to the KKT condition (43a) of the Nash–Cournot producer. To this end, one needs to eliminate the demands again by using the market-clearing condition (47i). Furthermore, we again have to identify the dual variables $\gamma_{u,t}$ of this market-clearing condition with the exogenously given prices that the TSO sees in his optimization problem; cf. Problem (5). Note that for obtaining a 1:1 correspondence between the Γ-RMEP equilibria and the robust QCQP solutions, the above discussed identifications are necessary.

We are then left with the pairs of conditions (47l) and (43e) as well as (47n) and (43g). We start with the first pair of conditions, which are equivalent if and only if $\Delta a_{u,t}(y_{u,t} - d_{u,t}) = 0$ holds. This is either the case if $\Delta a_{u,t} = 0$ or $y_{u,t} = d_{u,t}$ holds. The former corresponds to certain price-intercepts whereas the latter contradicts the presence of the transport network. The second pair of conditions are equivalent if and only if $-\frac{1}{2}\Delta b_{u,t}(y_{u,t}^2 + d_{u,t}^2) = -\Delta b_{u,t} d_{u,t} y_{u,t}$ holds. This is the case if either $\Delta b_{u,t} = 0$ or $y_{u,t} = d_{u,t}$ holds. Again, both is false and so the systems are not equivalent due to the same reasons as above.

Summarizing, we observe a very interesting phenomenon. Both robustifications in the case of perfect competition are well-posed in the sense that the RMEPs are equivalent to the robustified welfare optimization counterparts. This is also the case for Nash–Cournot competition and a strictly robust approach for handling data uncertainties. However, and somehow surprisingly, this does not hold anymore for the case of Γ-robustified Nash–Cournot equilibrium models. To be more specific, this shows that solving the Γ-robustification of the nominal optimization counterpart does *not* give a solution of the Γ-robustified market equilibrium problem. Moreover, it is not possible to state an optimization model that possesses KKT conditions that are equivalent to the Γ-RMEP (44). Mathematically speaking, this means that the symmetry principle (cf. Theorem 1.3.1 in [14]) cannot be applied. This implies, that there is no tractable optimization counterpart of the entire Γ-RMEP that also has clear-cut economic interpretation. Let us finally note that the observed failure does not depend on the presence of the transport network but that it is inherent to Γ-robustifications of general Nash–Cournot models; cf. Appendix E where the same failure is observed for a network-free model of Nash–Cournot competition.

5. Computational case study

In this section we study the effects of different robustification techniques and economic competition models using a stylized example on a 3-node network, which is given—together with all relevant nominal data—in Fig. 2. We consider four time periods that can be roughly interpreted as the four seasons. The demand data given in Fig. 2 corresponds to the consumers' willingness to pay in spring and autumn. Demand data for winter and summer are obtained by multiplying the price-intercept values with 1.5 and 0.5, respectively. Production and line data stays the same over all time periods. The uncertainty sets
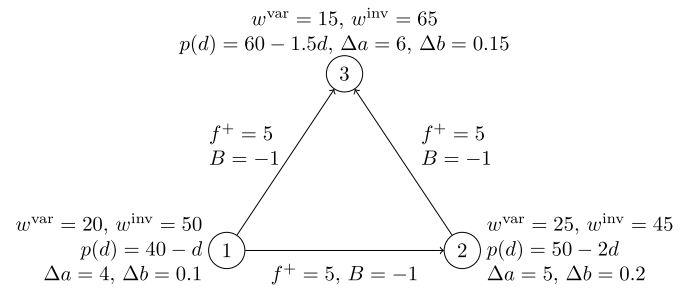

Fig. 2. 3-node network with the technical, economic, and uncertainty data used in the computational study.

Table 1
Welfare values.

Competition model	Uncertainty model	Welfare
Perfect competition	nominal	3137.87
Perfect competition	strictly robust	1778.68
Perfect competition	Γ-robust	2105.71
Nash–Cournot	nominal	1722.19
Nash–Cournot	strictly robust	1023.35

are modeled such that both price- and quantity-intercepts may vary between $\pm 10\,\%$ of the nominal values.

All models have been implemented in Python 2.7.10 and have been solved using Gurobi 7.0.1 on a MacBook Pro with 3.1 GHz Inter Core i7 processor and 16 GB RAM. The LP files of all models are publicly available on GitHub at https://github.com/m-schmidt-math-opt/robust-electricity-market-equilibria.

We start by discussing the results of the three different models for the case of perfect competition, i.e., the nominal, the strictly robust, and the Γ-robust model. The corresponding welfare values are given in Table 1. The nominal model obtains the largest welfare of 3137.87, the strictly robust model yields the smallest value (1778.68), and the Γ-approach with $\Gamma = 2$ yields a value in between (2105.71). This result was to be expected. The strictly robust model captures the worst-case in terms of the consumers' willingness to pay. Since less willingness to pay yields less demand, this decreases the overall welfare. For $\Gamma = 2$, the Γ-approach represents a level of uncertainty between the nominal and the worst-case setting and thus delivers a welfare value in between. Hence, the Γ-approach serves as a suitable equilibrium model that allows for moderate and Γ-parameterized uncertainty effects as it is the case for classic optimization models.

We now consider the demand and production values as well as the resulting prices in more detail. Fig. 3 (top left) displays the demands over all four time periods at all three nodes for all uncertainty models. Roughly speaking, demand is highest where the willingness to pay is highest and the demand pattern over the four seasons is reasonable. Demand in spring and autumn are equal. In summer, demand is lowest and highest in the winter period. Moreover and as expected, nominal demand is higher than Γ-robust demand, which is higher than the strictly robust demand. Interestingly, demand in winter at node 1 is less than in spring and autumn. We think that the reason is that, for the given capacity and production levels, demand is shifted from node 1 to node 3 since the willingness to pay is much higher at this node and thus allows for a larger welfare gain. Production levels mainly follows the ordering of variable production costs; cf. Fig. 2. Most interestingly, all uncertainty models deliver almost the same prices at the nodes. This is a result of the uncertainty level of 10 %. The nodal price differences between the different uncertainty models increase for increasing uncertainty sets. However, the prices are almost the same across the nodes of the network for fixed a uncertainty model. This is surprising since we use a capacitated DC power flow model that might yield different nodal
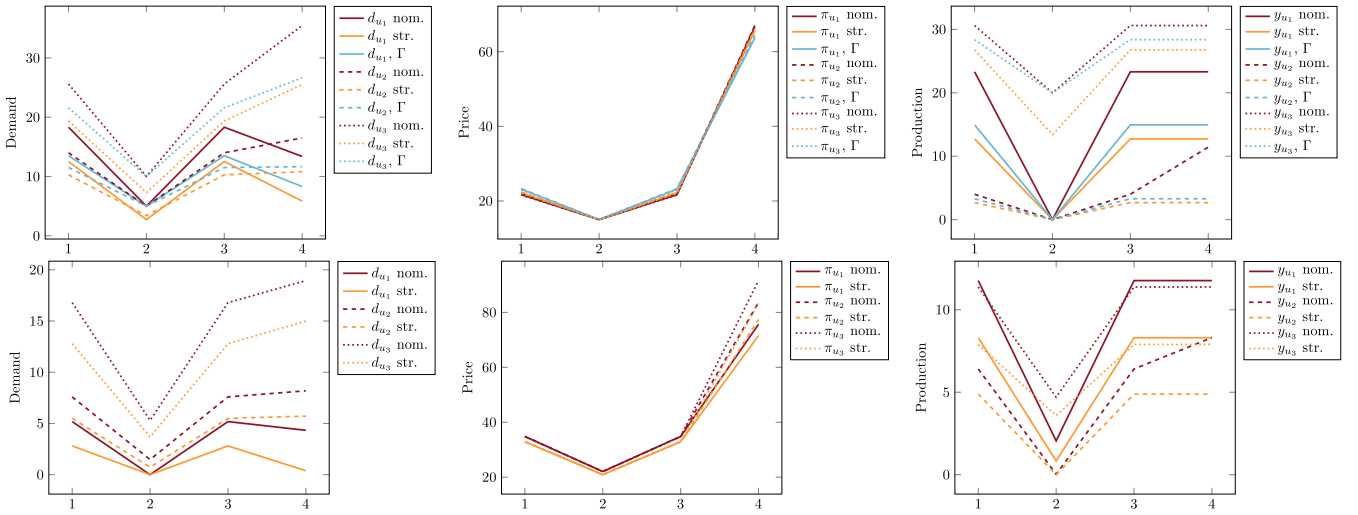
Fig. 3. Nodal demands, prices, and productions for the case of perfect competition (top) and Nash–Cournot competition (bottom).

prices. Thus, it seems to be the case that uncertainty reduces nodal price differences that occur due to the network.

The corresponding results for the Nash–Cournot models are given in Table 1 and Fig. 3 (bottom). Note that we only compare the nominal and strictly robust model since we do not have a tractable counterpart for the Γ-robust case; see Section 4.3. We do not discuss the results in the same detail as for the perfectly competitive cases because the results are rather comparable. Moreover, the classic effects of Cournot competition can be seen. First, welfare values decrease because Nash–Cournot producers do not act in a socially optimal way. Second, Cournot producers tend to produce less in order to obtain higher market prices, which finally yields less demand as compared to the perfectly competitive setting. All these effects can be seen in Fig. 3 (bottom). Finally, one sees that uncertain Nash–Cournot models tend to yield larger price differences than uncertain and perfectly competitive equilibrium models.

6. Conclusion

Although uncertainty plays an increasingly important role in electricity markets, the concept of robust optimization has only been applied to equilibrium models—at least to the best of our knowledge—in the two papers [15,59]. Thus, we investigated basic properties of robustified equilibrium models of electricity markets. To this end, we studied strictly as well as Γ-robust counterparts in the contexts of perfectly competitive as well as Nash–Cournot models of the market. For all but the combination of Nash–Cournot competition and Γ-robustness we derived tractable counterparts that also have a clear-cut economic interpretation. In particular, this corresponds to both classic welfare theorems applied to the robustified but still convex problems in the perfectly competitive setting and an analogous result is established for strictly robust Nash–Cournot equilibria. Finally, we also obtained several existence and uniqueness results by applying already existing results from the literature. Interestingly, the only case in which a tractable counterpart cannot be derived is the one of Γ-robustified Nash–Cournot models. In this case, the questions of existence and uniqueness of equilibria is open.

The field of robust equilibria is very young and, thus, many interesting questions still need to be answered. Examples include the consideration of other robustification techniques to equilibrium models as well as the application of robust concepts to large-scale real-world equilibrium models of electricity markets. Moreover, the extension of our results to uncertain nonlinear inverse demand functions as well as to other uncertain data of the model such as investment or production

costs are interesting topics of future research. Finally, in this paper we considered the case of a static investment problem, i.e., a problem in which the investment decisions are taken once at the very beginning of the time horizon. However, in the current context of the rapid development of renewable energy technologies, investment costs may change on a yearly basis, which is not covered by our model. Thus, the study of an extension of the model where more dynamic investment aspects are reflected is an important topic for future research as well.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgments

Appendix A. Proof of Theorem 3.1

First, we rewrite the robust counterpart (13) as

$$\max_{d_u \geq 0, \lambda} \quad \lambda \tag{48a}$$

$$\text{s.t.} \quad \lambda \leq \sum_{t \in [T]} \int_0^{d_{u,t}} (a_{u,t} + b_{u,t} x)\, dx - \sum_{t \in [T]} \pi_{u,t} d_{u,t} \quad \forall (a_{u,t}, b_{u,t}) \in \mathcal{U}_{u,t},$$

$$t \in [T]. \tag{48b}$$

The uncertainty only appears in Constraint (48b), which is equivalent to

$$\lambda \leq \min_{(a_{u,t}, b_{u,t}) \in \mathcal{U}_{u,t}, t \in [T]} \left\{ \sum_{t \in [T]} \int_0^{d_{u,t}} (a_{u,t} + b_{u,t} x)\, dx - \sum_{t \in [T]} \pi_{u,t} d_{u,t} \right\},$$

which, again, is the same as

$$\lambda \leq \sum_{t \in [T]} \min_{(a_{u,t}, b_{u,t}) \in \mathcal{U}_{u,t}} \left\{ \int_0^{d_{u,t}} (a_{u,t} + b_{u,t} x)\, dx \right\} - \sum_{t \in [T]} \pi_{u,t} d_{u,t}.$$

Using the structure of the uncertainty set (12), we obtain

$$\min_{\substack{(a_{u,t}, b_{u,t}) \in \mathcal{U}_{u,t}}} \left\{ \int_0^{d_{u,t}} (a_{u,t} + b_{u,t} x) \, dx \right\} = \min_{\substack{a_{u,t} \in [a_{u,t}^-, a_{u,t}^+], \\ b_{u,t} \in [b_{u,t}^-, b_{u,t}^+]}} \left\{ a_{u,t} d_{u,t} + \frac{1}{2} b_{u,t} d_{u,t}^2 \right\}$$

and, as we only consider feasible demands $d_{u,t} \geq 0$, this yields

$$\min_{\substack{a_{u,t} \in [a_{u,t}^-, a_{u,t}^+], \\ b_{u,t} \in [b_{u,t}^-, b_{u,t}^+]}} \left\{ a_{u,t} d_{u,t} + \frac{1}{2} b_{u,t} d_{u,t}^2 \right\} = a_{u,t}^- d_{u,t} + \frac{1}{2} b_{u,t}^- d_{u,t}^2 = \int_0^{d_{u,t}} (a_{u,t}^- + b_{u,t}^- x) \, dx$$

for all time periods $t \in [T]$. Thus, the robust counterpart (48) is equivalent to

$$\max_{d_u \geq 0, \lambda} \quad \lambda \quad \text{s.t.} \quad \lambda \leq \sum_{t \in [T]} \int_0^{d_{u,t}} (a_{u,t}^- + b_{u,t}^- x) \, dx - \sum_{t \in [T]} \pi_{u,t} d_{u,t},$$

which is the same as

$$\max_{d_u} \quad \sum_{t \in [T]} \int_0^{d_{u,t}} (a_{u,t}^- + b_{u,t}^- x) \, dx - \sum_{t \in [T]} \pi_{u,t} d_{u,t} \quad \text{s.t.} \quad 0 \leq d_{u,t}, \quad t \in [T].$$

Appendix B. Proof of Theorem 3.2

As the uncertainty only appears in (17b) we only consider these constraints and rewrite them as

$$\lambda \leq \min_{\substack{(a_{u,t}, b_{u,t}) \in \mathcal{U}_{u,t}, \\ u \in N, \ t \in [T]}} \left\{ \sum_{t \in [T]} \sum_{u \in N} \int_0^{d_{u,t}} (a_{u,t} + b_{u,t} x) \, dx - \sum_{t \in [T]} \sum_{u \in N} w_u^{\text{var}} y_{u,t} - \sum_{u \in N} w_u^{\text{inv}} \bar{y}_u \right\}.$$

This can be simplified, yielding

$$\lambda \leq \sum_{t \in [T]} \sum_{u \in N} \min_{(a_{u,t}, b_{u,t}) \in \mathcal{U}_{u,t}} \left\{ \int_0^{d_{u,t}} (a_{u,t} + b_{u,t} x) \, dx \right\} - \sum_{t \in [T]} \sum_{u \in N} w_u^{\text{var}} y_{u,t} - \sum_{u \in N} w_u^{\text{inv}} \bar{y}_u.$$

Using the structure of the uncertainty set $\mathcal{U}_{u,t}$ for all $u \in N$ and $t \in [T]$ as well as the fact that $d_{u,t} \geq 0$, $u \in N$, $t \in [T]$, we obtain

$$\min_{(a_{u,t}, b_{u,t}) \in \mathcal{U}_{u,t}} \left\{ \int_0^{d_{u,t}} (a_{u,t} + b_{u,t} x) \, dx \right\} = \int_0^{d_{u,t}} (a_{u,t}^- + b_{u,t}^- x) \, dx.$$

Thus, the robust counterpart of the welfare model reads

$$\max_{d, y, \bar{y}, f, \Theta, \lambda} \quad \lambda$$

$$\text{s.t.} \quad \lambda \leq \sum_{t \in [T]} \sum_{u \in N} \int_0^{d_{u,t}} (a_{u,t}^- + b_{u,t}^- x) \, dx$$

$$- \sum_{t \in [T]} \sum_{u \in N} w_u^{\text{var}} y_{u,t} - \sum_{u \in N} w_u^{\text{inv}} \bar{y}_u,$$

$$(17c)\text{--}(17e),$$

which is equivalent to (18).

Appendix C. Proof of Theorem 4.2

We only have to consider Constraint (34b). Using the definition of the uncertainty set (12), its right-hand side reads

$$- \sum_{t \in [T]} w_u^{\text{var}} y_{u,t} - w_u^{\text{inv}} \bar{y}_u + \min_{a_{u,t} \in [a_{u,t}^-, a_{u,t}^+]} \left\{ \sum_{t \in [T]} a_{u,t} y_{u,t} \right\}$$

$$+ \min_{b_{u,t} \in [b_{u,t}^-, b_{u,t}^+]} \left\{ \sum_{t \in [T]} b_{u,t} \left(y_{u,t} + \sum_{a \in \delta^{\text{in}}(u)} f_{a,t} - \sum_{a \in \delta^{\text{out}}(u)} f_{a,t} \right) y_{u,t} \right\}. \quad (49)$$

Exploiting Assumption 2, the market-clearing condition (7), the assumption $d_{u,t} > 0$ for all $t \in [T]$, and the constraint $y_{u,t} \geq 0$, we can explicitly state the minima in (49) and obtain

$$\min_{a_{u,t} \in [a_{u,t}^-, a_{u,t}^+]} \left\{ \sum_{t \in [T]} a_{u,t} y_{u,t} \right\} = \sum_{t \in [T]} a_{u,t}^- y_{u,t}$$

as well as

$$\min_{b_{u,t} \in [b_{u,t}^-, b_{u,t}^+]} \left\{ \sum_{t \in [T]} b_{u,t} \left(y_{u,t} + \sum_{a \in \delta^{\text{in}}(u)} f_{a,t} - \sum_{a \in \delta^{\text{out}}(u)} f_{a,t} \right) y_{u,t} \right\}$$

$$= \sum_{t \in [T]} b_{u,t}^- \left(y_{u,t} + \sum_{a \in \delta^{\text{in}}(u)} f_{a,t} - \sum_{a \in \delta^{\text{out}}(u)} f_{a,t} \right) y_{u,t}.$$

Thus, we can write

$$\lambda \leq \sum_{t \in [T]} a_{u,t}^- y_{u,t} + \sum_{t \in [T]} b_{u,t}^- \left(y_{u,t} + \sum_{a \in \delta^{\text{in}}(u)} f_{a,t} - \sum_{a \in \delta^{\text{out}}(u)} f_{a,t} \right) y_{u,t}$$

$$- \sum_{t \in [T]} w_u^{\text{var}} y_{u,t} - w_u^{\text{inv}} \bar{y}_u$$

instead of (34b) and the claim follows.

Appendix D. Proof of Theorem 4.3

As in the other proofs, we only have to consider and reformulate the minimum in Constraint (38b). By the definition of the uncertainty set (12) of every consumer, we have

$$\min_{(a_{u,t}, b_{u,t}) \in \mathcal{U}_{u,t}} \left\{ \sum_{t \in [T]} \sum_{u \in N} \left(a_{u,t} d_{u,t} + \frac{1}{2} b_{u,t} d_{u,t}^2 \right) + \sum_{t \in [T]} \sum_{u \in N} \frac{1}{2} b_{u,t} y_{u,t}^2 \right\}$$

$$= \sum_{t \in [T]} \sum_{u \in N} \left(\min_{a_{u,t} \in [a_{u,t}^-, a_{u,t}^+]} \left\{ a_{u,t} d_{u,t} \right\} + \frac{1}{2} \min_{b_{u,t} \in [b_{u,t}^-, b_{u,t}^+]} \left\{ b_{u,t} (d_{u,t}^2 + y_{u,t}^2) \right\} \right).$$

Then, using Assumption 2 and $d_{u,t} > 0$ for all $t \in [T]$, we obtain

$$\sum_{t \in [T]} \sum_{u \in N} \left(\min_{a_{u,t} \in [a_{u,t}^-, a_{u,t}^+]} \left\{ a_{u,t} d_{u,t} \right\} + \frac{1}{2} \min_{b_{u,t} \in [b_{u,t}^-, b_{u,t}^+]} \left\{ b_{u,t} (d_{u,t}^2 + y_{u,t}^2) \right\} \right)$$

$$= \sum_{t \in [T]} \sum_{u \in N} \left(a_{u,t}^- d_{u,t} + \frac{1}{2} b_{u,t}^- (d_{u,t}^2 + y_{u,t}^2) \right).$$

This yields the first claim. Because all slopes of the inverse demand functions are negative, the robustified Nash–Cournot QP (39) is a convex optimization problem.

Appendix E. The Γ approach for Nash–Cournot competition without a network

In Section 4.3 we observed that the equivalence between the robustified equilibrium model (stated as a robust LCP) and the robustified Nash–Cournot QP does not hold in the case of the Γ-approach. In this section, we briefly show that the reason for this failure is not the transport network by discussing the network-free case of Nash–Cournot competition and the Γ-approach. We omit all proofs in this appendix. All results can be obtained using the same techniques as in the other parts of this paper.

For the network-free case, we consider an aggregated consumer with an aggregated inverse market demand function $P_t(D_t) = A_t + B_t D_t$ with $A_t \geq 0$ and $B_t < 0$ for all $t \in [T]$ and still assume that the aggregated demand D_t is positive for all time periods $t \in [T]$. Following the same principles as in Section 4.1 we then obtain the optimization problem

$$\max_{y_u, \bar{y}_u} \quad \sum_{t \in [T]} P_t \left(\sum_{v \in N} y_{v,t} \right) y_{u,t} - \sum_{t \in [T]} w_u^{\text{var}} y_{u,t} - w_u^{\text{inv}} \bar{y}_u \quad \text{s.t.} \quad 0 \leq y_{u,t} \leq \bar{y}_u,$$

$$t \in [T], \quad (50)$$

of the producer located at node $u \in N$. Note that the price function can be stated explicitly via

$$P_t \left(\sum_{v \in N} y_{v,t} \right) = A_t + B_t \left(\sum_{v \in N} y_{v,t} \right).$$

As usual, the Nash–Cournot-LCP is made up of the necessary and sufficient KKT conditions

$$A_t + B_t y_{u,t} + B_t \left(\sum_{v \in N} y_{v,t} \right) - w_u^{\text{var}} + \beta_{u,t}^- - \beta_{u,t}^+ = 0, \quad u \in N, \ t \in [T],$$

$$-w_u^{\text{inv}} + \sum_{t \in [T]} \beta_{u,t}^+ = 0, \quad u \in N,$$

$$0 \le y_{u,t} \perp \beta_{u,t}^- \ge 0, \quad 0 \le \bar{y}_u - y_{u,t} \perp \beta_{u,t}^+ \ge 0, \quad u \in N, \ t \in [T]$$

of Problem (50) for every producer, i.e., for every $u \in N$. As before, we also obtain an equivalent Nash–Cournot QP that now reads

$$\max_{y, \bar{y}} \quad \sum_{t \in [T]} \int_0^{Y_t} P_t(x) \, dx + \sum_{t \in [T]} \sum_{u \in N} \frac{1}{2} B_t y_{u,t}^2$$

$$- \sum_{t \in [T]} \sum_{u \in N} w_u^{\text{var}} y_{u,t} - \sum_{u \in N} w_u^{\text{inv}} \bar{y}_u \tag{51a}$$

$$\text{s.t.} \quad 0 \le y_{u,t} \le \bar{y}_u, \quad u \in N, \ t \in [T], \tag{51b}$$

where $Y_t := \sum_{u \in N} y_{u,t}$ abbreviates total production (and thus total demand) in time period $t \in [T]$. Next, we consider the robustification of the producer model (50),

$$\max_{y_u, \bar{y}_u, \lambda} \quad \lambda \tag{52a}$$

$$\text{s.t.} \quad \lambda \le \sum_{t \in [T]} \bar{A}_t y_{u,t} + \bar{B}_t \left(\sum_{v \in N} y_{v,t} \right) y_{u,t} - \sum_{t \in [T]} w_u^{\text{var}} y_{u,t} - w_u^{\text{inv}} \bar{y}_u \tag{52b}$$

$$- \max_{\{I \subseteq [T]: \ |I| \le A\}} \left\{ \sum_{t \in I} \Delta A_t y_{u,t} \right\}$$

$$- \max_{\{S \subseteq [T]: \ |S| \le \Gamma\}} \left\{ \sum_{t \in S} \Delta B_t \left(\sum_{v \in N} y_{v,t} \right) y_{u,t} \right\},$$

$$0 \le y_{u,t} \le \bar{y}_u, \quad t \in [T], \tag{52c}$$

for which we can also state a tractable version.

Theorem E.1. *For the aggregated consumer let an uncertainty set be given as in (12) and (19). Then, the Γ-robust counterpart (52) of the Nash–Cournot producer (50) is equivalent to*

$$\max_{y_u, \bar{y}_u, \tau_u, \eta_u, \nu_u, \mu_u} \quad \sum_{t \in [T]} \bar{A}_t y_{u,t} + \bar{B}_t \left(\sum_{v \in N} y_{v,t} \right) y_{u,t} - \sum_{t \in [T]} w_u^{\text{var}} y_{u,t} - w_u^{\text{inv}} \bar{y}_u$$

$$- \tau_u \Gamma - \sum_{t \in [T]} \nu_{u,t} - \eta_u A - \sum_{t \in [T]} \mu_{u,t} \tag{53a}$$

$$\text{s.t.} \quad 0 \le y_{u,t} \le \bar{y}_u, \quad t \in [T], \tag{53b}$$

$$0 \le \tau_u, \eta_u, \tag{53c}$$

$$0 \le \nu_{u,t}, \ \mu_{u,t}, \quad t \in [T], \tag{53d}$$

$$0 \le \eta_u + \mu_{u,t} - \Delta A_t y_{u,t}, \quad t \in [T], \tag{53e}$$

$$0 \le \tau_u + \nu_{u,t} - \Delta B_t \left(\sum_{v \in N} y_{v,t} \right) y_{u,t}, \quad t \in [T]. \tag{53f}$$

Furthermore, (53) is a convex optimization problem if Assumption 2 holds.

Hence, the KKT conditions

$$\bar{A}_t + \bar{B}_t \left(\sum_{v \in N} y_{v,t} \right) + \bar{B}_t y_{u,t} - w_u^{\text{var}} + \beta_{u,t}^- - \beta_{u,t}^+ - \Delta A_t \sigma_{u,t}$$

$$- \Delta B_t \left(\sum_{v \in N} y_{v,t} \right) \zeta_{u,t} - \Delta B_t y_{u,t} \zeta_{u,t} = 0, \quad t \in [T], \tag{54a}$$

$$-w_u^{\text{inv}} + \sum_{t \in [T]} \beta_{u,t}^+ = 0, \tag{54b}$$

$$-1 + \sigma_{u,t} + \xi_{u,t} = 0, \quad -1 + \zeta_{u,t} + \rho_{u,t} = 0, \quad t \in [T], \tag{54c}$$

$$-A + \sum_{t \in [T]} \sigma_{u,t} + \chi_u = 0, \quad -\Gamma + \psi_u + \sum_{t \in [T]} \zeta_{u,t} = 0, \tag{54d}$$

$$0 \le y_{u,t} \perp \beta_{u,t}^- \ge 0, \quad 0 \le \bar{y}_u - y_{u,t} \perp \beta_{u,t}^+ \ge 0, \quad t \in [T], \tag{54e}$$

$$0 \le \mu_{u,t} + \eta_u - \Delta A_t y_{u,t} \perp \sigma_{u,t} \ge 0, \quad t \in [T], \tag{54f}$$

$$0 \le \eta_u \perp \chi_u \ge 0, \quad 0 \le \tau_u \perp \psi_u \ge 0, \tag{54g}$$

$$0 \le \nu_{u,t} + \tau_u - \Delta B_t \left(\sum_{v \in N} y_{v,t} \right) y_{u,t} \perp \zeta_{u,t} \ge 0, \quad t \in [T], \tag{54h}$$

$$0 \le \mu_{u,t} \perp \xi_{u,t} \ge 0, \quad 0 \le \nu_{u,t} \perp \rho_{u,t} \ge 0, \quad t \in [T], \tag{54i}$$

of (53) are necessary and sufficient conditions. Thus, the robust Nash–Cournot-LCP is obtained by collecting the Conditions (54) for every producer, i.e., for every $u \in N$. As before, we now also derive the Γ-robust counterpart of the Nash–Cournot QP (51) and then analyze whether there is a 1:1 correspondence between the latter model and the corresponding robust LCP. The Γ-robust counterpart of the Nash–Cournot QP (51) reads

$$\max_{y, \bar{y}, \lambda} \quad \lambda \tag{55a}$$

$$\text{s.t.} \quad \lambda \le \sum_{t \in [T]} \bar{A}_t \left(\sum_{u \in N} y_{u,t} \right) + \frac{1}{2} \sum_{t \in [T]} \bar{B}_t \left(\sum_{u \in N} y_{u,t} \right)^2 + \sum_{t \in [T]} \sum_{u \in N} \frac{1}{2} \bar{B}_t y_{u,t}^2 \tag{55b}$$

$$- \sum_{t \in [T]} \sum_{u \in N} w_u^{\text{var}} y_{u,t} - \sum_{u \in N} w_u^{\text{inv}} \bar{y}_u$$

$$- \max_{\{I \subseteq [T]: \ |I| \le A\}} \left\{ \sum_{t \in I} \Delta A_t \left(\sum_{u \in N} y_{u,t} \right) \right\}$$

$$- \max_{\{S \subseteq [T]: \ |S| \le \Gamma\}} \left\{ \sum_{t \in S} \frac{1}{2} \Delta B_t \left(\left(\sum_{u \in N} y_{u,t} \right)^2 + \sum_{u \in N} y_{u,t}^2 \right) \right\},$$

$$0 \le y_{u,t} \le \bar{y}_u, \quad t \in [T]. \tag{55c}$$

A tractable version of this robustified model can be derived as before.

Theorem E.2. *For the aggregated consumer let an uncertainty set be given as in (12) and (19). Then, the Γ-robust counterpart (55) is equivalent to*

$$\max_{y, \bar{y}, \tau, \eta, \nu, \mu} \quad \sum_{t \in [T]} \bar{A}_t \left(\sum_{u \in N} y_{u,t} \right) + \frac{1}{2} \sum_{t \in [T]} \bar{B}_t \left(\sum_{u \in N} y_{u,t} \right)^2 + \sum_{t \in [T]} \sum_{u \in N} \frac{1}{2} \bar{B}_t y_{u,t}^2$$

$$- \sum_{t \in [T]} \sum_{u \in N} w_u^{\text{var}} y_{u,t} - \sum_{u \in N} w_u^{\text{inv}} \bar{y}_u$$

$$- \tau \Gamma - \sum_{t \in [T]} \nu_t - \eta A - \sum_{t \in [T]} \mu_t \tag{56a}$$

$$\text{s.t.} \quad 0 \le y_{u,t} \le \bar{y}_u, \quad u \in N, \ t \in [T], \tag{56b}$$

$$0 \le \tau, \eta, \tag{56c}$$

$$0 \le \nu_t, \ \mu_t, \quad t \in [T], \tag{56d}$$

$$0 \le \eta + \mu_t - \Delta A_t \sum_{u \in N} y_{u,t}, \quad t \in [T], \tag{56e}$$

$$0 \le \tau + \nu_t - \frac{1}{2} \Delta B_t \left(\left(\sum_{u \in N} y_{u,t} \right)^2 + \sum_{u \in N} y_{u,t}^2 \right), \quad t \in [T]. \tag{56f}$$

Furthermore, (56) is a convex optimization problem if Assumption 2 holds.

Thus, the KKT conditions of (56) are necessary and sufficient first-order optimality conditions. They are given by

$$\bar{A}_t + \bar{B}_t \left(\sum_{v \in N} y_{v,t} \right) + \bar{B}_t y_{u,t} - w_u^{\text{var}} + \beta_{u,t}^- - \beta_{u,t}^+ - \Delta A_t \sigma_t$$

$$- \Delta B_t \left(\sum_{v \in N} y_{v,t} \right) \zeta_t - \Delta B_t y_{u,t} \zeta_t = 0, \quad u \in N, \ t \in [T],$$

$$- w_u^{\text{inv}} + \sum_{t \in [T]} \beta_{u,t}^+ = 0, \quad u \in N,$$

$$-1 + \sigma_t + \xi_t = 0, \quad -1 + \zeta_t + \rho_t = 0, \quad t \in [T],$$

$$-\Lambda + \sum_{t \in [T]} \sigma_t + \chi = 0, \quad -\Gamma + \psi + \sum_{t \in [T]} \zeta_t = 0,$$

$$0 \le y_{u,t} \perp \beta_{u,t}^- \ge 0, \quad 0 \le \bar{y}_u - y_{u,t} \perp \beta_{u,t}^+ \ge 0, \quad u \in N, \ t \in [T],$$

$$0 \le \mu_t + \eta - \Delta A_t \sum_{v \in N} y_{v,t} \perp \sigma_t \ge 0, \quad t \in [T],$$

$$0 \le \eta \perp \chi \ge 0, \quad 0 \le \tau \perp \psi \ge 0,$$

$$0 \le \nu_t + \tau - \frac{1}{2} \Delta B_t \left(\left(\sum_{v \in N} y_{v,t} \right)^2 + \sum_{v \in N} y_{v,t}^2 \right) \perp \zeta_{u,t} \ge 0, \quad t \in [T],$$

$$0 \le \mu_t \perp \xi_t \ge 0, \quad 0 \le \nu_t \perp \rho_t \ge 0, \quad t \in [T].$$

It is again easy to see that these KKT conditions are not the same as those in (54) for every producer. The main reason is the Nash–Cournot markup terms $\frac{1}{2} B_t y_{u,t}^2$ in the objective of the Nash–Cournot QP (51). These are not present in the single producer problems (50) and are thus not robustified in the robust LCP. This is, however, the case in the robustified Nash–Cournot QP, which clearly leads to different solutions. Thus, the failure of the desired equivalence is not due to network effects but is inherent to Γ-robustifications of general Nash–Cournot models.

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