Dissertationsschrift

Implementing Efficient Outcomes In Combinatorial Allocation Problems

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Deutsche Zusammenfassung

Der Entwurf von Regeln, welche in Märkten mit mehreren unteilbaren Gütern effiziente Allokation induzieren, ist eine der grundlegensten Fragestellungen in der Mechanismus-Design-Theorie. In der vorliegenden Dissertation wird ebendieses Problem unter verschiedenen Gesichtspunkten betrachtet. Die Dissertation leistet vorallem drei Forschungsbeiträge, die sich alle mit der Implementierung von effizienten Allokationen befassen. Die zentrale Gemeinsamkeit der drei Forschungsarbeiten ist die Gegebenheit, dass die den Allokationsproblemen zu Grunde liegenden kombinatorischen Strukturen im engen Zusammenhang zu Matroiden stehen.

Im ersten Kapitel werden die notwendigen mathematischen Definitionen und Notationen bereitgestellt, insbesondere aus der Matroid Theorie.

Das zweite Kapitel betrachtet das Problem der Implementierung wohlfahrtsmaximierender Allokationen durch aufsteigende Auktionen im Falle, dass ein einzelner Verkäufer eine Menge von Gütern an eine endliche Anzahl von Bietern auktioniert. Dabei sind die Bewertungsfunktionen der Bieter separable nichtfallende diskrete konkave Funktionen und die Menge der zulässigen Allokationen ist durch die Basen eines ganzzahligen Polymatroids gegeben. In diesem Kontext entwickeln wir eine aufsteigende Auktion, welche eine wohlfahrtsmaximierende Allokation implementiert. Unsere Auktion ist so gestaltet, dass sie in Polynomialzeit abläuft und nur von einem wachsenden Preis abhängt, sowie den Bietern den Anreiz setzt, gemäß ihrer wahren Bewertungen zu bieten.

Das dritte Kapitel befasst sich mit Verallgemeinerungen des Problems der Approximationsgarantie des klassichen Greedy Algorithmus im Verhältnis zur optimalen Lösung beim Finden einer gewichtsmaximalen Basis eines gewichteten Unabhängigkeitssystems. Dieses eigentlich rein kombinatorisch-mathematische Problem ist unter anderem motiviert durch die ökonomisch relevante Fragestellung, warum bei tatsächlich stattfindenden Auktionen der heuristische Greedy Algorithmus oftmals sehr gute Lösungen liefert, obwohl die zu Grunde liegende Menge zulässiger Allokationen nicht die Struktur eines Matroids, sondern nur eines Unabhängigkeitssystems hat. Wir führen das Konzept der inneren Unabhängigkeitssysteme ein und zeigen, dass die ebenso von uns eingeführte Verallgemeinerung des Rangquotienten für Unabhängigkeitssysteme eine strenge Abschätzung für die schlechtest mögliche Approximationsgarantie des Greedy Algorithmus im Verhältniss zur optimalen Lösung liefert. Weiterhin beweisen wir das überraschende Resultat, dass ebendiese Gütegarantie auf inneren Unabhängigkeitssystemen simultan besser sein kann als jene des Greedy Algorithmus angewandt auf das ursprüngliche Unabhängigkeitssystem und jene des Greedy Algorithmus angewandt auf das beste innere Matroid. Im Anschluss erweitern wir unsere Resultate auf die Maximierung von nichtfallenden separablen diskret konkaven Zielfunktionen über ganzzahligen Packungsinstanzen und beweisen hier analoge Resultate zum Problem des Findens einer gewichtsmaximalen Basis eines gewichteten Unabhängigkeitssystems.

Das vierte Kapitel befasst sich mit der Implementierung von wohlfahrtsmaximierenden Allokationen durch so genannte dynamische Preise. In dem von uns betrachteten Kontext verkauft ein einzelner Verkäufer eine endliche Menge von unteilbaren Gütern an eine endliche Anzahl von Käufern, welche öffentlich bekannte Bewertungsfunktionen haben. Die Käufer erscheinen einmalig, sequentiell und in vorher unbekannter Reihenfolge, um von dem Verkäufer Güter zu erwerben. Bevor der nächste Käufer erscheint, hat der Verkäufer die Möglichkeit die Preise der verbliebenen Güter neu fest zu setzen. Das Ziel des Verkäufers ist es, die Preise so zu wählen, dass der nächste jeweils erscheinende Käufer notwendigerweise eine Menge von Gütern kauft, die ihm auch in einer wohlfahrtsmaximierenden Allokation zugeteilt werden könnte. In diesem Rahmen zeigen wir, dass immer eine Folge von Preisen existiert, die eine wohlfahrtsmaximierenden Allokation induziert, wenn die Bewertungsfunktion eines jeden Käufers eine gewichtete uniforme Matroidrangfunktion ist und die Käufer rational handeln, folglich Mengen kaufen, die ihren individuellen Nutzen maximieren.

Chapter 1.

Introduction

Allocating scarce resources efficiently is a major task in mechanism design. One of the most fundamental problems in mechanism design theory is the problem of selling a single indivisible item to bidders with private valuations for the item. In this setting, the classic Vickrey auction of Vickrey [1961] describes a simple mechanism to implement a social welfare maximizing allocation. The Vickrey auction for a single item asks every buyer to report its valuation and allocates the item to the highest bidder for a price of the second highest bid. This auction features some desirable properties, e.g., buyers cannot benefit from misreporting their true value for the item (incentive compatibility) and the auction can be executed in polynomial time. However, when there is more than one item for sale and buyers' valuations for sets of items are not additive or the set of feasible allocations is constrained, then constructing mechanisms that implement efficient allocations and have polynomial runtime might be very challenging. Consider a single seller selling $n \in \mathbb{N}$ heterogeneous indivisible items to several bidders. The Vickrey-Clarke-Groves auction generalizes the idea of the Vickrey auction to this multi-item setting. Naturally, every bidder has an intrinsic value for every subset of items. As in in the Vickrey auction, bidders report their valuations (Now, for every subset of items!). Then, the auctioneer computes a social welfare maximizing allocation according to the submitted bids and charges buyers the social cost of their winning that is incurred by the rest of the buyers. (This is the analogue to charging the second highest bid to the winning bidder in the single item Vickrey auction.) It turns out that the Vickrey-Clarke-Groves auction is also incentive compatible but it poses some problems: In fact, say for $n = 40$, bidders would have to submit $2^{40} - 1$ values (one value for each nonempty subset of the ground set) in total. Thus, asking every bidder for its valuation might be impossible due to time complexity issues. Therefore, even though the Vickrey-Clarke-Groves auction implements a social welfare maximizing allocation in this multi-item setting it might be impractical and there is need for alternative approaches to implement social welfare maximizing allocations.

This dissertation represents the results of three independent research papers all of them tackling the problem of implementing efficient allocations in different combinatorial settings. All these settings are related to the combinatorial structure of matroids. Matroids, independently introduced by Whitney [1935] and by Nakasawa [1935], Nakasawa [1936a] and Nakasawa [1936b], are some of the most fundamental objects studied in combinatorial optimization since they abstract and generalize the notion of linear independence in vector spaces. Several important combinatorial optimization problems can be formulated as matroid optimization problems, for example:

a) Given a matrix A with column weights, find a maximum weight linearly independent set of columns of A.

b) Given a graph with edge weights, find a subgraph that is a maximum weight forest.

c) Given a graph with nonnegative node weights, find the maximum weighted set of nodes that can be covered by a matching of the graph.

However, even more important, matroids are exactly the family of sets that, given nonnegative item weights, allows for a simple greedy algorithm to find a maximum weighted set of the family. Greedy algorithms can be considered as the discrete version of the method of steepest ascent making the locally optimal choice at each stage. Note that that the greedy algorithm applied to the problems a)-c) at each stage chooses the most valuable feasible item among the remaining items, but there obviously also exist matroid minimization problems. On the one hand, the greedy algorithm solves matroid optimization problems exactly. On the other hand, it is one of the most common heuristics to approximate optimal solutions of combinatorial optimization problems. Several important nonmatroidal combinatorial optimization problems can be approximated by the greedy algorithm with performance guarantees, e.g.,

- Finding a maximum weight matching in a graph.
- The knapsack problem. (Taking the maximum of the greedy solution and the most valuable item.)
- The k-center clustering problem.

One of the most fundamental and general problems that can be approximated by the greedy algorithm is the problem of finding a maximum weight basis of an independence system. This problem is $\mathbb{NP}$-hard since it includes e.g., the knapsack problem, the problem of finding a maximum independence set in a graph and the traveling salesman problem (define the feasible sets as the subgraphs of Hamiltonian cycles). We consider the problem of finding a maximum weight basis of an independence system in detail in Chapter 3 and propose a heuristic to improve the approximation guarantee of the greedy algorithm.

Consider a setting when there are many heterogeneous items to be sold by an auctioneer. Even though the valuations of the bidders might be expressed compactly e.g., if they are additive, there exist more reasons to avoid the direct application of Vickrey-Clarke-Groves auctions. However, for reasons of privacy, bidders may not be willing to report their entire valuation to the seller. Therefore, in multi-item settings, a reasonable alternative to the Vickrey-Clarke-Groves mechanism are ascending auctions, which preserve more privacy than the Vickrey-Clarke-Groves mechanism and additionally are group strategyproof (Milgrom and Segal [2014]), hence no group of buyers can collude to misreport such that it makes every member of the group better off. We consider ascending auctions in Chapter 2 under the assumption that the seller is constrained to sell bases of a polymatroid and bidders' valuations are the sum of concave and piecewise linear functions. In this context, we provide an ascending auction that implements an efficient allocation.

Besides (ascending) auctions, posted prices are the fundamental real life mechanisms for selling goods. Typical examples of auctions include sales at eBay, Google adwords auctions and the classic English auction for selling works of art while the canonical example of posted prices are goods sold in supermarkets or cars sold at car dealers for fixed prices. A main drawback of auctions is that they need a central authority coordinating the market, in particular, if ties occur, the market coordinator, e.g., the auctioneer, has to break ties according to some rule and assign items to buyers. In contrast, in a posted price mechanism buyers arrive sequentially and purchase some of their utility maximizing sets, given the price, from the seller. In Chapter 4 we consider posted price mechanisms in a uniform matroid setting when the seller is allowed to update the prices of the remaining items after each buyer leaves and show how to implement efficient allocations in this context.

1.1. Our Contribution

In the second chapter we consider a single auctioneer who sells indivisible items to buyers with private valuations. Even though every item to be auctioned has possibly multiple identical copies, the set of feasible allocations is constrained to be an integer base polyhedron and the buyers' valuations are the sum of nondecreasing discrete concave functions. This seemingly abstract setting is of theoretical relevance and has various interesting applications. In this context, Bikhchandani et al. [2011] provide an ascending auction that returns the Vickrey-Clarke-Groves outcome. Moreover, their auction is incentive compatible. Since their original auction was designed for matroids, the applicability to integer polymatroids is achieved by a transformation. Although it is the fastest known auction in this setting it still runs in pseudo-polynomial time on integer polymatroids in the quantity of the item with the most copies, using a pseudo-polynomial transformation from integer polymatroids to matroids. In this context, we develop an ascending auction that implements a social welfare maximizing allocation and charges the buyers Vickrey-Clarke-Groves prices. Our ascending auction is ex post incentive-compatible and has polynomial runtime in the number of buyers, items and copies and relies only on a single increasing price and directly generalizes the auction of Bikhchandani et al. [2011].

The third chapter provides a purely mathematical results regarding the approximation guarantee of the well-known greedy algorithm. A classic result of Korte and Hausmann [1978] and Jenkyns [1976] bounds the quality of the greedy solution to the problem of finding a maximum weight basis of an independence system $(E, \mathcal{I})$ in terms of the rank quotient. We extend this result in two ways. First, we apply the greedy algorithm to an *inner independence system* contained in $\mathcal{I}$. Additionally, following an idea of Milgrom [2017], we incorporate exogenously given prior information about the set of likely candidates for an optimal basis in terms of a set $\mathcal{O} \subseteq \mathcal{I}$. We provide a generalization of the rank quotient that yields a tight bound on the worst case performance of the greedy algorithm applied to the inner independence system relative to the optimal solution in $\mathcal{O}$. Furthermore, we show that for a worst-case objective the inner independence system approximation may outperform not only the standard greedy algorithm but also the inner matroid approximation proposed by Milgrom [2017]. Second, we generalize the inner approximation framework to inner approximation of packing instances in $\mathbb{Z}_+^n$ by inner polymatroids and inner packing instances. We consider the problem of maximizing a separable discrete concave function and show that our inner approximation can be better than the greedy algorithm applied to the original packing instance. Our result shows that the generalized rank-quotient lower bounds the worst case approximation guarantee of the greedy algorithm to the optimal solution in this more general setting and subsumes Malinov and Kovalyov [1980]. We remark that even though our result seems primarily as a theoretic result in discrete mathematics, it is closely related to mechanism design theory and the task of implementing efficient allocations in markets. It is motivated by the question which circumstances makes greedy algorithms perform well on (nonmatroidal) independence structures in real world auctions and was raised by Milgrom [2017].

The fourth chapter considers the problem of supporting optimal allocations in combinatorial markets by means of dynamic pricings. Consider one seller selling a finite number of heterogeneous indivisible items to a finite set of buyers, each of them having a publicly known valuation function. Then, the set of social welfare maximizing allocations regarding the valuations can be determined. The seller sets item prices and buyers arrive one at a time and sequentially to purchase one of their utility maximizing sets among the remaining items from the seller. The order of the arriving buyers is unknown to the seller but he is allowed to update the prices of the remaining items before the next buyer arrives. The goal of the seller is to induce a social welfare maximizing allocation by setting the item prices such that it is in the best interest of every arriving buyer to choose a set of items she gets allocated in some social welfare maximizing allocation. Any price such that every possibly next arriving buyer necessarily has to chose a set of items she might get allocated in some social welfare maximizing allocation is called a dynamic price and a sequence of dynamic prices is called a dynamic pricing. Cohen-Addad et al. [2016] and Berger et al. [2020] show that dynamic pricings exist in matching markets and in weighted uniform matroid markets with at most three buyers. We substantially generalize their work by proving the existence of dynamic pricings in general weighted uniform matroid markets. En route, we show that every interior Walrasian price already is a dynamic price in matching markets, thereby providing an independent proof of Cohen-Addad et al. [2016].

The research question of the second chapter extends a result of Bikhchandani et al. [2011] by my advisor. Similarly, the research question of the third chapter was motivated by de Vries and Vohra [2020] of my advisor. In contrast to the chapters two and three, the research question of chapter four arose during my own studies. In the fourth chapter, I completely solve an open problem in the theory of posted price mechanisms for which several recent research publications only provide partial solutions for special cases.

All three papers which form the basis of the respective chapters are or will be submitted for publication in mathematical journals. Chapter 4 is written in sole authorship while Chapters 2 and 3 are written with co-authors.

1.2. Fundamentals and Notations

We fix some basic notations:

Definition 1.1. *For a finite set E and a subset $S \subseteq E$ we define the* ***complement*** $\overline{S} := E \setminus S$ *and denote by χ^S the* ***characteristic vector*** *of S in $\mathbb{R}^E$. Furthermore, we denote by $\mathbf{0}|_S$ the zero vector in $\mathbb{R}^S$ and for a set T that is disjoint to S we denote the* ***disjoint union*** *of S and T by $S \sqcup T$. We denote the* ***power set*** *of E by 2^E. For any pair of sets $A, B \subseteq 2^E$ we define $A \bigoplus B := \{s \colon s = a \cup b, a \in A, b \in B\}$. We denote $\mathbb{R}_+ := \{x \in \mathbb{R} \colon x \geq 0\}$.*

Next, we recall basic definitions from matroid theory, which are relevant for several chapters of this dissertation.

Definition 1.2. *An* ***independence system*** *is an ordered pair $(E, \mathcal{I})$ where the finite set E is the* ***ground set*** *and the* ***independent sets*** *$\mathcal{I} \subseteq 2^E$ is a collection of subsets of E satisfying:*

(I1) $\emptyset \in \mathcal{I}$,

(I2) If $T \in \mathcal{I}$ and $S \subset T$ then $S \in \mathcal{I}$.

Given an independence system $(E, \mathcal{I})$ *every* $I \in \mathcal{I}$ *is called* ***independent*** *and every* $I \notin \mathcal{I}$ *is called* ***dependent****. A set* $B \in \mathcal{I}$ *is called a* ***basis of*** $S \subseteq E$ *iff* $B \subseteq S$ *and* $B \cup \{s\} \notin \mathcal{I}$ *for all* $s \in S \setminus B$*. A basis of* E *is simply called a* ***basis*** *and the* ***set of bases*** *of* E *is also called the set of bases of* $(E, \mathcal{I})$ *and denoted* $\mathcal{B}_{\mathcal{I}}$*. A set* $C \subseteq E$ *that is minimal dependent, hence* $C \notin \mathcal{I}$ *and* $C \setminus \{c\} \in \mathcal{I}$ *for all* $c \in C$ *is called a* ***circuit****. A subset* $\mathcal{I}$ *of* 2^E *for which holds* $\{i\} \in \mathcal{I}$ *for all* $i \in E$ *is called* ***normal****, thus an independence system* $(E, \mathcal{I})$ *is called* ***normal*** *if it holds* $\{i\} \in \mathcal{I}$ *for all* $i \in E$*. The cardinality of the largest basis of a set* $S \subseteq E$ *is called the* ***rank*** *of* S*, hence* $r(S) := \max\{|I| \colon I \in \mathcal{I}, I \subseteq S\}$ *and the size of the smallest basis of a set* $S \subseteq E$ *is called the* ***lower rank*** *of* S*, i.e.,* $l(S) := \min\{|I| \colon I \in \mathcal{I}, I \subseteq S, I \cup \{i\} \notin \mathcal{I} \text{ for all } i \in S \setminus I\}$*. The* ***rank quotient*** *of the independence system* $(E, \mathcal{I})$ *is denoted by* $q(\mathcal{I}) := \min_{S \subseteq E \colon r(S) \neq 0} \frac{l(S)}{r(S)}$*. An independence systems* $(E, \mathcal{I})$ *can also be characterized via a* ***rank function****: Let* $r \colon 2^E \to \mathbb{N}_0$ *such that*

(R1) $r(\emptyset) = 0$ *(normalized),*

(R2) $r(S) \leq r(S \cup \{x\}) \leq r(S) + 1$ *for all* $S \subset E, x \in E \setminus S$*,*

then, the pair $(E, \{I \subseteq E \colon r(I) = |I|\})$ *is an independence system. If not clear from context, we denote the rank function of the independence system* $(E, \mathcal{I})$ *by* $r_{\mathcal{I}}$*.*

Now that we have the basic definitions of independence systems we characterize matroids.

Definition 1.3. *An independence system* $(E, \mathcal{I})$ *is called a* ***matroid*** M *if* $q(\mathcal{I}) = 1$*. For a matroid* $M = (E, \mathcal{I})$ *we also denote its ground set by* $E(M)$*, its independent sets by* $\mathcal{I}(M)$ *and its set of bases by* $\mathcal{B}(M)$*. There exist several equivalent characterizations of matroids of which we make use of throughout this thesis. A matroid* M *is an independence system* $(E, \mathcal{I})$ *for which one of the following holds:*

(Submodularity) The rank function r *satisfies* $r(S) + r(S \cup \{x, y\}) \leq r(S \cup \{x\}) + r(S \cup \{y\})$ *for all* $x, y \in E$*,* $S \subseteq E \setminus \{x, y\}$*.*

(Basis Exchange) For every pair of bases $B_1, B_2 \in \mathcal{B}(M)$ *and* $x \in B_1 \setminus B_2$ *there exists a* $y \in B_2 \setminus B_1$ *such that* $(B_2 \cup \{x\}) \setminus \{y\}$ *is a basis.*

(Augmention property) For all $S, T \in \mathcal{I}$ *such that* $|S| < |T|$ *there exists* $e \in T \setminus S$ *with* $S \cup \{e\} \in \mathcal{I}$*.*

The ***dual*** *of a matroid* $M = (E, \mathcal{I})$ *is another matroid* $M^* := (E, \{I \subseteq E \colon$ *there exists* $B \in \mathcal{B}(M)$ *such that* $I \cap B = \emptyset\}$*. A set* $C^* \subseteq E$ *is a* ***cocircuit*** *of* $M = (E, \mathcal{I})$ *if it holds* $C^* \cap B \neq \emptyset$ *for all bases* $B \in \mathcal{B}(M)$ *and for all* $S \subset C^*$ *there exists a basis* $B \in \mathcal{B}(M)$ *such that* $S \cap B = \emptyset$*. We denote* $\mathcal{C}^*(M) := \{C^* \subseteq E \colon C^* \text{ is cocircuit}\}$*. Note that it holds* $\mathcal{C}^*(M) = \mathcal{C}(M^*)$*, hence the set of cocircuits of* M *is the set of circuits of the dual matroid*

*M^* of M. A **loop** is a circuit that is singleton and a **coloop** is a cocircuit that is singleton. Notice that normal matroids have no loops, and vice versa. For a matroid $M = (E, \mathcal{I})$ the **deletion** of $X \subseteq E$ from M is the matroid $M \setminus X := (E \setminus X, \{I \in \mathcal{I} \colon I \subseteq E \setminus X\})$, the **restriction** to $X \subseteq E$ is the matroid $M|_X := M \setminus (E \setminus X)$ and the **contraction** of $X \subseteq E$ from M is the matroid $M/X := (M^* \setminus X)^*$. Note that for $X \in \mathcal{I}$ it holds that $\mathcal{I}(M/X) := \{I \in \mathcal{I}(M) \colon I \sqcup X \in \mathcal{I}(M)\}$ and the rank function of M/X is given by $r_{M/X}(S) := r(S \sqcup X) - r(S)$.*

We provide a standard example of a matroid.

Example 1.4. *Let E a finite set and $n \in \mathbb{Z}_+$, $n \leq |E|$, then, the **uniform matroid** on E is given by $U_E^n := (E, \{S \subseteq E \colon |S| \leq n\})$.*

Matroid rank functions are special cases of nondecreasing submodular functions.

Definition 1.5. *For the finite set E recall that $f \colon 2^E \to \mathbb{R}$ is called **submodular** iff*

$$f(S) + f(S \cup \{x, y\}) \leq f(S \cup \{x\}) + f(S \cup \{y\}) \text{ for all } x, y \in E, S \subseteq E \setminus \{x, y\}.$$

*Further, we call $f \colon 2^E \to \mathbb{R}$ **nondecreasing** if for each $S \subseteq T \subseteq E$ holds $f(S) \leq f(T)$ and **normalized** if it holds $f(\emptyset) = 0$. Every normalized nondecreasing submodular function f can be identified* bijective *with the **polymatroid** $P_f \subset \mathbb{R}_+^E$, that is the polyhedron described by the inequalities*

$$\sum_{e \in S} x_e \leq f(S) \quad \text{for all } S \subseteq E,$$
$$x_e \geq 0 \quad \text{for all } e \in E.$$

*We call a polymatroid **normal** if $\chi^{\{e\}} \in P_f$ for all $e \in E$, hence if it holds $f(\{e\}) > 0$ for all $e \in E$. A **basis** of a polymatroid P_f is a vector $x \in P_f$ with $\sum_{e \in E} x_e = f(E)$ and the **base polyhedron** B_f is the set of bases of the polymatroid P_f.*

We are interested in maximizing the integer restriction of nondecreasing concave functions $f \colon \mathbb{R}^n \to \mathbb{R}_+$ over polymatroids.

Definition 1.6. *Call a function $f \colon \mathbb{Z}_+ \to \mathbb{R}_+$ **nondecreasing discrete concave** if it holds $f(x+1) - f(x) \geq f(y+1) - f(y)$ for $x, y \in \mathbb{Z}_+$, $x \leq y$. For $f \colon \mathbb{Z}_+ \to \mathbb{R}_+$ nondecreasing discrete concave define the **extension to the nonnegative real numbers** of f by $\bar{f} \colon \mathbb{R}_+ \to \mathbb{R}_+$, $\bar{f}(x) := f(x)$ for $x \in \mathbb{Z}_+$ and $\bar{f}(x) := f(\lfloor x \rfloor) + (f(\lceil x \rceil) - f(\lfloor x \rfloor)) \cdot (x - \lfloor x \rfloor)$. For $n \in \mathbb{N}$ and $i \in \{1, ..., n\}$ let $f_i \colon \mathbb{Z}_+ \to \mathbb{R}_+$ be a nondecreasing discrete concave function, then call the function $f \colon \mathbb{Z}_+^n \to \mathbb{R}_+, f := \sum_{i=1}^n f_i$ **separable nondecreasing discrete concave**.*

Note that for a nondecreasing discrete concave function f the extension $\bar{f}$ is nondecreasing, concave and piecewise linear with breakpoints exclusively integral (hence, there are at most countable many breakpoints).

Chapter 2.

An Ascending Vickrey Auction for Selling Bases of an Integer Polymatroid

2.1. Introduction

In this chapter we consider a single auctioneer who wants to assign a set of indivisible goods, each having possibly multiple copies, to bidders with private valuations. This problem has been addressed frequently in the mechanism design literature in various contexts. In this paper, we study the situation when the set of feasible allocations is constrained to be an integer base polyhedron, hence the set of bases of an integer polymatroid, and bidders' valuations are the sum of nondecreasing, concave functions. Our interest in this seemingly abstract problem is twofold: From a practical perspective, it turns out to contain several interesting applications and theoretically it is the most general multi-bidder, multi-item and multi-unit structure in which a single price ascending auction can be used to determine the optimal outcome.

Our primary goal is to allocate the goods efficiently, hence to determine an optimal element of the base polyhedron regarding the valuations of the bidders. If the bidders' valuations are public, an optimal allocation can be calculated trivially by a polymatroid greedy algorithm. However, this does not apply in our setting. Therefore, we need to create incentives for the bidders to reveal their true valuations. One method that implements an optimal outcome and meets the incentive objective is the *VCG auction*, a special case of a *sealed-bid auction*. It asks the bidders to report their entire (true) valuations, computes an optimal allocation according to them and charges the bidders *VCG payments*. For completeness, we briefly recap the concept of a VCG auction: Let A be a set of abstract outcomes and each bidder $j \in N$ reports a valuation $v_j \colon A \to \mathbb{R}_+$. Then, an *efficient allocation* according to the reported valuations chooses $a^* \in \arg\max_{a \in A} \sum_{j \in N} v_j(a)$. The VCG payments for bidder j are defined as the loss in attainable welfare suffered by the rest of the bidders, hence $p_j := \max_{a \in A} \sum_{k \in N \setminus \{j\}} v_k(a) - \sum_{k \in N \setminus \{j\}} v_k(a^*)$. This induces the bidders to report their valuation truthfully since the utility bidder j obtains from outcome

$b \in A$ is

$$\sum_{j \in N} v_j(b) - \max_{a \in A} \sum_{k \in N \setminus \{j\}} v_k(a).$$

Hence, the utility of any bidder is clearly maximized if for $b \in A$ an efficient allocation is chosen and the VCG auction is *dominant strategy incentive compatible*, hence it provides the incentive that every bidder reports truthfully, independent of other bidders' reported valuations.

2.1.1. Ascending Auctions

A common alternative to a sealed bid auction is an *ascending auction*, which we also propose in this setting. Roughly speaking, an ascending auction works in the following way: The seller (auctioneer) announces prices and bidders report their demand according to the prices. If the demand of the bidders equals the supply the auction ends and bidders get their demanded items and pay the prices. If the demand exceeds the supply the prices are adjusted upwards. As the auction progresses, bidders are only allowed to reduce their demand. Our ascending auction implements an optimal solution if bidders behave truthfully and truthful behavior is an ex post equilibrium.

There are several advantages of ascending auctions over sealed-bid auctions. According to Ausubel [2004] and Cramton [1998] advantages of ascending auctions include that

- they are more transparent,
- bidders reveal less private information,
- communication and computation costs for bidders may be lower,
- they may be better even with mild interdependencies among bidders.

Further, ascending auctions are easier to handle for buyers since they only require any bidder to answer a yes or no question instead of specifying the entire valuation in advance.

There exist several ex post incentive compatible ascending auctions in the literature that include our setting and implement the optimal outcome. However, they either make use of exponentially many (in the number of items) prices (see Parkes and Ungar [2000], de Vries et al. [2007]) or require superpolynomial time in the number of copies of an item (see Bikhchandani et al. [2011]). In contrast, we provide a fully polynomial (in the number of items, copies and bidders) incentive compatible ascending auction that implements the optimal outcome and charges VCG prices.

2.1.2. Related Literature

This paper continues the line of research on ascending price auctions on multiple objects. Fundamental contributions to ascending auctions with gross substitutes valuations are Kelso and Crawford [1982] and Gul and Stacchetti [2000]. Although these auctions implement the efficient outcome when bidders behave truthfully they are not incentive compatible. The work of Demange et al. [1986] which considers an ascending auction in a matching market can be seen as the starting point for incentive compatible multi-object ascending auctions. Subsequently, Ausubel [2004] and Ausubel [2006] analyze a multi-item and multi-bidder environment for homogeneous respectively heterogeneous items without any combinatorial constraints. A primal-dual ascending auction for heterogeneous items and bidders with arbitrary valuations is provided by de Vries et al. [2007]. Finally, Bikhchandani et al. [2011] generalizes the clinching auction of Ausubel [2004] to the setting that the set of feasible allocations is constrained by a matroid. Their auction can be modified to be applied in our setting, but has only pseudo-polynomial runtime in the highest quantity of any item.

Note that when auctioning only a single indivisible item to a set of bidders even the famous English auction has complexity issues and only pseudo-polynomial runtime. This follows easily since in the English auction the price increases by unit steps as long as at least two players announce their willingness to pay the price but the bidders' values for the item are encoded logarithmically. However, by a binary search approach, Grigorieva et al. [2007] provide an iterative auction that runs in polynomial time and implements the VCG outcome in this setting.

The class of all ascending auctions for unit-demand bidders that terminate in an optimal allocation and charge Walrasian prices was characterized by Andersson et al. [2013].

A possible approach to overcome computational issues in more complex auctions is to restrict the valuation classes of the bidders: Candogan et al. [2015] prove that an iterative auction yields an efficient allocation when bidders have tree valuations, a strict subset of graphical valuations. Their auction has polynomial runtime in the number of items, bidders and the highest unknown but fixed value of any node and edge of any agent's underlying graphical valuation but it is neither an ascending auction since it increases and reduces prices iteratively by solving a certain integer program nor does it rely on a single price.

An important subset of ascending auctions is the class of deferred-acceptance auctions introduced by Milgrom and Segal [2014], which are remarkable since they are obviously strategyproof and group-strategyproof. Deferred-acceptance auctions were introduced in binary settings (a bidder gets served or not) and subsequently generalized by Gkatzelis et al. [2017] to bidders with multiple levels of services and various feasibility constraints on the set of feasible outcomes. Further, Dütting et al. [2017] study deferred-acceptance auctions from an approximation standpoint and measure the performance guarantee by comparing the outcome of deferred-acceptance auctions with the social welfare maximizing allocation in various settings.

For quite some time there is also increasing interest in ascending auctions where bidders are budget constrained. This additional constraint rules out the existence of an ascending auction that is incentive compatible and simultaneously implements a *Pareto* optimal allocation even in the simple case of a single-item and multi-unit auction and two bidders with linear valuation functions, see Dobzinski et al. [2012]. It turns out that our mechanism is similar to the one in Goel et al. [2015], which applies to a polymatroid environment where bidders are constrained by publicly known budgets. However, they consider a market that consists only of multiple copies of a single divisible good and bidders are restricted to have linear valuations, which is in two ways less general than our setting. Further, the analysis of our mechanism is closely related to Hirai and Sato [2022] which generalizes Goel et al. [2015] and deals with auctions in two-sided markets with polymatroid constraints on the set of edges.

2.1.3. Chapter Structure

We briefly outline the organization of this chapter: In Section 2.2 we recap the ascending auction of Bikhchandani et al. [2011], which implements a VCG outcome when the set of feasible allocations is constrained by a matroid. This auction is ex post incentive compatible and has polynomial runtime in its long step version. Recall that the matroid environment is a special case of our setting. We provide an alternative ascending auction for selling bases of a matroid that is extendable to integer base polyhedrons. The advantages of our auction over the one of Bikhchandani et al. [2011] are twofold. First, we are able to allocate bundles of items in a single step instead of awarding items sequentially. Second, we avoid the cocircuit terminology used by Bikhchandani et al. [2011], which is rather unwieldy in terms of polymatroids and makes a direct application of their auction to our setting impossible. We show that our auction implements the same outcome as the aforementioned and inherits its incentive properties. Section 2.3 is dedicated to our main result. We elucidate that our ascending auction for selling bases of a matroid, after a pseudo-polynomial transformation, applies to polymatroids and yields a VCG outcome. Subsequently, we adapt our auction so that it is directly applicable on polymatroids and show that the outcome of both auctions coincide. The new auction performs directly on the integer polymatroid and has strongly polynomial runtime in the number of items, bidders and time it takes to evaluate an oracle call and therefore provides the first polynomial ascending auction in this setting. Afterwards, we illustrate our algorithm by an example and in Section 2.4 we provide some applications. We finish with a conclusion in Section 2.5.

2.2. Selling Bases of a Matroid

In this section, we provide an ascending auction that implements a social welfare maximizing allocation if the auctioneer is constrained to sell bases of a matroid.

2.2.1. Economic Model

We define the economic model in the matroid setting. Consider a seller and a finite set of bidders N that are interested in purchasing indivisible goods of a finite set E of cardinality m from the seller. The ground set E is partitioned into sets $(E_j)_{j\in N}$, such that bidder j is exclusively interested in the items E_j, hence there are no two bidders that demand the same good. This assumption is without loss of generality (see the more general polymatroid model in Section 2.3 for a proof of this). Furthermore, there is a normal matroid $M = (E, \mathcal{I})$ that constrains the feasible allocations. The seller is only allowed to sell combinations of the elements of E that form a basis of the matroid. As usual, we assume that the matroid M is given by an **independence oracle**, hence for any set $S \subseteq E$ one can determine by an oracle call if S is independent.

If bidder j acquires element $e \in E_j$ it provides him with value $v_e \in \mathbb{R}_+$ and a set $S \subseteq E_j$ provides him value $v_j(S) := \sum_{e\in S} v_e$. The goal of the seller is to identify a maximum weight basis of M regarding the unknown values v_e for $e \in E$, hence determine an element of $\arg\max_{B\in\mathcal{B}(M)} \sum_{e\in B} v_e$. Obviously, the values v_e are private information; otherwise, a simple greedy algorithm would find an optimal solution.

We assume that a basis of M can be formed without allocating any elements to any fixed bidder, hence for all $j \in N$ and $E_{-j} := E \setminus E_j$ it holds $\text{rank}(E) = \text{rank}(E_{-j})$. We call this assumption **no-monopoly**. If this assumption is not fulfilled for some bidder $j \in N$ she would be able to block any feasible outcome. However, the idea of our ascending auction is as follows: By price increments, we force bidders to give up interest in some items, hence we delete these items from the matroid. Because of the deletion some bidders might achieve for some of their items monopoly power. These uncontested elements are awarded to the bidder at current price.

2.2.2. The Ascending Auction of Bikhchandani et al. [2011]

In the following we present the ascending auction of Bikhchandani et al. [2011] that finds an optimal basis of the matroid M and implements VCG payments. Note that p is a price clock that gradually ascends.

Algorithm 1:

Require: A matroid $M = (E, \mathcal{I})$ with a no-monopoly partition $E = \bigsqcup_{j \in N} E_j$ and nonnegative integer valuations

1 $a \leftarrow 0$, $p \leftarrow 0$, $r \leftarrow r(E)$, $B = \emptyset$

2 Choose an arbitrary ordering of N (for tie-breaking)

3 **while** $a < r$ **do**

4 Ask bidders to determine $F = \{f_1, ..., f_k\} \leftarrow \{f \in \mathcal{I} \colon v_f = p\}$ and label the elements in increasing order according to the tie-breaking

5 **for** $l \leftarrow 1$ **to** k **do**

6 **if** *f_l is contained in the groundset of M* **then**

7 **while** *there exists a bidder $j \in N$ and a cocircuit C^*_j of $M \setminus \{f_l\}$ with $C^*_j \subseteq E_j$* **do**

8 Ask bidder j to determine $\arg\max_{e \in C^*_j} v_e$ and let x be the most valuable element from this set according to tie-breaking.

9 Award x to j and charge him p,

10 $M \leftarrow M/\{x\}$, $B \leftarrow B \cup \{x\}$, $a \leftarrow a + 1$

11 $M = M \setminus \{f_l\}$

12 $p \leftarrow p + 1$

Output : The optimal basis B.

We have added the if condition Line 6 to the original auction of Bikhchandani et al. [2011] for additional clarity, otherwise for $f_l \notin E(M)$ we can define $M \setminus \{f_l\} := M$.

We remark that Bikhchandani et al. [2011] assumes the existence of a cocircuit oracle. However, the existence of an independence oracle is the more natural and common assumption. A cocircuit oracle can be implemented by polynomial many calls to an independence oracle: It is well known for matroids that an independence oracle is polynomial equivalent to a rank oracle, see e.g., Hausmann and Korte [1981]. To see that a rank oracle is polynomial reducible to a cocircuit oracle observe that for any set $C^* \subseteq E$ it holds that C^* is a cocircuit iff it holds $r(E \setminus C^*) = r(E) - 1$ and $r(E \setminus (C^* \setminus \{c\})) = r(E)$ for all $c \in C^*$. Therefore, for any $S \subseteq E$ it can be checked by at most $|S| + 1$ rank oracle calls if S is a cocircuit.

The auction in Auction 1 runs pseudo-polynomial in the highest unknown but fixed value $\max_{e \in E} v_e$ and can be adapted to a polynomial auction by modifying the unit step increase in the price p to a so called *long step* version (see [Bikhchandani et al., 2011]).

Theorem 2.1. *[Bikhchandani et al., 2011] Auction 1 outputs an optimal basis and yields VCG payments, independently of the ordering of F in Line 4 and the possible tie breaking used.*

2.2.3. An Alternative Algorithm for Matroidal Settings

Our goal is to provide an ascending auction that efficiently allocates items when the set of feasible allocations is constrained by an integer base polyhedron. To this end, we first provide an alternative algorithm for selling bases of a matroid that generalizes well to polymatroids. Unlike Auction 1, our algorithm avoids the cocircuit terminology which is rather unwieldy in terms of polymatroids. However, to prove correctness of our auction we will frequently use characterizations of cocircuits of matroid minors:

Lemma 2.2. *[Oxley, 2006, Proposition 3.1.16, 3.1.17] Let $M = (E,\mathcal{I})$ a matroid, $\mathcal{C}^*(M)$ its set of cocircuits and $T \subseteq E$, then $\mathcal{C}^*(M/T) := \{C^* \in \mathcal{C}^*(M) \colon C^* \subseteq E \backslash T\}$ and $\mathcal{C}^*(M \backslash T) := \{C^* \setminus T \colon C^* \in \mathcal{C}^*(M), \nexists D^* \in \mathcal{C}^*(M)$ such that $D^* \setminus T \subsetneq C^* \setminus T\}$.*

In some sense, Auction 1 is based on the cocircuit-greedy algorithm introduced by Dawson [1980]. Its correctness depends on the following Lemma.

Lemma 2.3. *[Dawson, 1980]. Let $M = (E,\mathcal{I})$ a matroid, $I \in \mathcal{I}$ and $a \in E \setminus I$. Then, it holds $I \cup \{a\} \in \mathcal{I}$ iff there exists a cocircuit $C^* \in \mathcal{C}^*(M)$ with $a \in C^*$ and $C^* \cap I = \emptyset$.*

A direct consequence of Lemmata 2.2 and 2.3 is:

Corollary 2.4. *Let $M = (E,\mathcal{I})$ a matroid, $I \in \mathcal{I}$ and $a \in E \setminus I$. Then, it holds $I \cup \{a\} \in \mathcal{I}$ iff there exists a cocircuit $C^* \in \mathcal{C}^*(M/I)$ with $a \in C^*$.*

We want to show that one can formulate Lines 6 and 7 of Auction 1 in an equivalent alternative way.

Definition 2.5. *For a price p denote $F^p := \{e \in E : v_e = p\}$ and $F^p_{-j} := F^p \cap E_{-j}$ and omit the p if clear by context.*

If Auction 1 chooses an item x to allocate it to bidder j, then the other bidders are not negatively affected by this. Therefore, for each independent set I that consists exclusively of items that provides strict positive utility to the agents interested in the items at the current price and that is demanded by any set of bidders excluded j, there has to exist a basis containing I and x, hence for all $I \in \mathcal{I}(M|_{E_{-j} \setminus F_{-j}})$ it holds $I \sqcup \{x\} \in \mathcal{I}(M)$. Conceptually, this condition necessarily has to be implied by the cocircuit condition in Line 7 of Auction 1 and it provides a good intuition why Auction 1 works correctly and implements a social welfare maximizing allocation. Before proving the equivalence of both conditions we first reformulate our new condition in terms of matroid minors:

Lemma 2.6. *Let $M = (E,\mathcal{I})$ a matroid, $S \subseteq E$, $x \in E \setminus S$. Then, it holds $\{x\} \in \mathcal{I}(M/S)$ iff for all $I \in \mathcal{I}(M|_S)$ holds $I \sqcup \{x\} \in \mathcal{I}(M)$.*

Proof. Let $\{x\} \in \mathcal{I}(M/S)$, then it holds by definition of the rank function $r_{M/S}$ that $r_M(S \sqcup \{x\}) - r_M(S) = 1$. By the submodularity of r_M it holds for all $I \in \mathcal{I}(M|_S)$ that $r_M(I \sqcup \{x\}) - r_M(I) \geq r_M(S \sqcup \{x\}) - r_M(S)$, hence $r_M(I \sqcup \{x\}) - r_M(I) \geq 1$ and therefore it holds for all $I \in I(M|_S)$ that $r_M(I \sqcup \{x\}) = |I \sqcup \{x\}|$, hence $I \cup \{x\} \in \mathcal{I}(M)$.

Conversely, let for all $I \in \mathcal{I}(M|_S)$ hold $I \sqcup \{x\} \in \mathcal{I}(M)$, then for $B \in \mathcal{B}(M|_S)$ it clearly holds $B \sqcup \{x\} \in \mathcal{I}(M)$, hence $r(S \cup \{x\}) - r(S) \geq r(B \cup \{x\}) - r(B) = 1$ and therefore $x \in \mathcal{I}(M/S)$ by definition. □

Now, we provide an equivalent condition to the cocircuit condition in Line 7 of Auction 1.

Lemma 2.7. *Let $M = (E, \mathcal{I})$ a matroid, $E = E' \sqcup E''$, $F'' \subseteq E''$. Then, for $x \in E'$ it holds $\{x\} \in \mathcal{I}(M/(E'' \setminus F''))$ iff there exists a cocircuit $C^* \in \mathcal{C}^*(M \setminus F'')$ with $x \in C^* \subseteq E'$.*

Proof. Let $x \in E^1$ such that it holds $\{x\} \in \mathcal{I}(M/(E'' \setminus F''))$ and $I \in \mathcal{B}(M \setminus F''|_{E''}) \subseteq \mathcal{I}(M \setminus F'')$, then, it holds that $I \cup \{x\} \in \mathcal{I}(M \setminus F')$ by assumption and by Corollary 2.4 there has to exists a cocircuit $C^* \in \mathcal{C}^*(M \setminus F''/I)$ such that $x \in C^*$. Furthermore, it has to hold $C^* \subseteq E'$ by Lemma 2.3 since for all $y \in E'' \setminus (F'' \cup I)$ it holds $I \cup \{y\} \notin \mathcal{I}(M \setminus F'')$ by the basis-assumption. Then, it follows directly by Lemma 2.2 that $C^* \in \mathcal{C}^*(M \setminus F'')$.

Conversely, let $C^* \in \mathcal{C}^*(M \setminus F'')$ with $x \in C^* \subseteq E'$ and $I \in \mathcal{I}(M \setminus F''|_{E''})$. It holds $C^* \cap I = \emptyset$ and therefore $I \cup \{x\} \in \mathcal{I}(M \setminus F^2)$ by Lemma 2.3. Then, it holds $x \in \mathcal{I}(M \setminus F''/(E'' \setminus F'')) \subseteq \mathcal{I}(M/(E'' \setminus F''))$ by Lemma 2.6. □

As a direct consequence of Lemma 2.7 we can reformulate Lines 7 and 8 of Auction 1: Ask bidder j to determine

$$\arg\max\{v_e \colon \{e\} \in \mathcal{I}(M|_{E_j}) \text{ and } \{e\} \in \mathcal{I}(M/(E_{-j} \setminus F_{-j}))\}$$

and let x be the most valuable element from this set according to tie-breaking.

Consequently, assuming that the set of items $F_{-j} \subseteq E_{-j}$ drops out then every set of items $T_j \in \mathcal{I}(M)|_{E_j}$ that fulfills the condition $T_j \in \mathcal{I}(M/(E_{-j} \setminus F_{-j}))$ might be awarded to bidder j without impairing the bidders in $N \setminus \{j\}$, hence bidder j has a monopoly for the set T_j.

Definition 2.8. *For a matroid $M = (E_j \sqcup E_{-j}, \mathcal{I})$ and $F_{-j} \subseteq E_{-j}$ we denote by $\boldsymbol{Monopoly}_{M,E_{-j},F_{-j},T_j}$ the property that $T_j \in \mathcal{I}(M|_{E_j})$ fulfills the condition $T_j \in \mathcal{I}(M/(E_{-j} \setminus F_{-j}))$.*

Note that in our economic model initially for all $j \in N$ it holds that the only set for which holds $\mathbf{Monopoly}_{M,E_{-j},F_{-j},T_j}$ is $T_j = \emptyset$ due to the no-monopoly condition. We modify Auction 1 making use of the $\textbf{Monopoly}_{M,E_{-j},F_{-j},T_j}$ condition.

Algorithm 2:

Require: No-monopoly matroid $M = (E,\mathcal{I})$ with nonnegative integer valuation and a partition $E = \bigsqcup_{j\in N} E_j$

1 $a \leftarrow 0$, $p \leftarrow 0$, $r \leftarrow r(E)$
2 Determine an (arbitrary) ordering of E (for tie-breaking)
3 **while** $a < r$ **do**
4   Ask bidders to determine $F = \{f \in \mathcal{I} \colon v_f = p\}$
5   **for** $j \in N$ **do**
6   **if** $a < r$ **then**
7   find $\arg\max\{\sum_{e\in T_j} v_e \colon T_j \in \mathcal{I}(M|_{E_j})$ with $\textbf{Monopoly}_{M,E_{-j},F_{-j},T_j}\}$ and let S_j be a maximal cardinality element of this set. Award S_j to j and charge him $|S_j| \cdot p$
8             $M \leftarrow M/S_j$, $a \leftarrow a + |S_j|$, $B \leftarrow B \cup S_j$, $F \leftarrow F \setminus S_j$
9     $M \leftarrow M \setminus F$
10     $p \leftarrow p + 1$

Output : The optimal basis B.

We remark that the maximum cardinality condition in Line 7 is necessary (see Example 2.26) and defer the explanation how to implement this condition to the analogous statement in Section 2.3, in which bases of a polymatroid are auctioned. Notice that Auction 2, in contrast to Auction 1, allocates *all elements awardable* to bidder j at price p in a single step. In Theorem 2.10 we will show that a slight variation of Auction 2 has polynomial runtime. Before that, we prove that Auction 2 outputs an optimal basis and yields VCG payments.

Theorem 2.9. *Auction 2 finds an optimal basis and implements VCG payments.*

Proof. Notice that at price p there may exist cocircuits which are contained in F. For any item contained in such a cocircuit it holds that the later it is ordered in F in Line 4 of Auction 1, the more likely this item gets allocated before it gets deleted. Similarly, the earlier the bidder that is interested in this item is ordered in Line 5 of Auction 2, the more likely this item gets allocated before it gets deleted. We want the set F in Line 4 of Auction 1 and the set N in Line 5 of Auction 2 to be ordered such that any item that is more likely to be allocated in Auction 1 is also more likely to be allocated in Auction 2. Therefore, let $<$ a total order on N and define $\sigma \colon E \to N$ such that for $e \in E$ by $\sigma(e)$ is denoted the unique bidder that initially is interested in item e. Let $\prec$ a partial order on $E \times E$ such that $e \prec f$ if $\sigma(e) < \sigma(f)$. Assume $F \subseteq E$ in Line 4 of Auction 1 to be partially ordered by $\prec$ and the for-loop in Line 5 of Auction 2 to be carried out in reverse to the order $<$ on $N \times N$.

Let p be the current price and $M = (E, \mathcal{I})$ denote the matroid that coincides in both algorithms before any element got deleted or awarded at p. Call a *sequence* $((b_1, C_1^*), \dots, (b_l, C_l^*))$ of element-cocircuit pairs *suitable* for Auction 1, if it can be constructed during an execution of Auction 1 at price p and a *set* of elements $\{x_1, \dots, x_l\}$ *suitable* for Auction 2 if it can be awarded during an execution of Auction 1 at price p. Let $(((b_1, C_1^*), \dots, (b_l, C_l^*)), \{x_1, \dots, x_l\})$ a pair of a suitable sequence for Auction 1 and a suitable set for Auction 2 at price p such that the index h of the first element b_h for which holds $b_h \notin \{x_1, \dots, x_l\}$ is maximal among all pairs of suitable sequences and suitable sets at price p. Assume that the item b_h gets awarded at the $j-$th iteration of the for-loop in Line 6 of Auction 1 by choosing b_h as a maximal valued element of the circuit $C_h^* \in \mathcal{C}^*(M \setminus F^j / B^{h-1})$, $C_h^* \subseteq E_{\sigma(b_h)}$ where $F^j := \{f_1, \dots, f_j\}$ and $B^{h-1} := \{b_1, \dots, b_{h-1}\}$ and $B_k^{h-1} := \{b \in B^{h-1} \colon \sigma(b) \geq k\}$ for $k \in N$.

If it holds $\sigma(f_j) < \sigma(b_h)$, then it is $F^j \subseteq F_{-\sigma(b_h)}$ and by Lemma 2.2 there exists a cocircuit $D_h^* \subseteq C_h^*$ with $D_h^* \in \mathcal{C}(M \setminus F_{-\sigma(b_h)}/B^{h-1})$ and $b_h \in D_h^*$, hence $D_h^* \in \mathcal{C}(M \setminus F_{-\sigma(b_h)}/B_{\sigma(b_h)}^{h-1})$ by Lemma 2.2 . Then, by Lemma 2.7 there exists a value maximal element $e_h \in D_h^*$ that gets allocated to bidder $\sigma(b_h)$ by Auction 2, hence it has to hold $v(e_h) = v(b_h)$ and therefore Auction 1 could equivalently award item e_h instead of b_h and $((b_1, C_1^*), \dots, (e_h, C_h^*))$ is a suitable subsequence and there obviously has to exist a suitable sequence
$((b_1, C_1^*), \dots, (b_{h-1}, C_{h-1}^*), (e_h, C_h^*), (b'_{h+1}, C'^*_{h+1}), \dots, (b'_l, C'^*_l))$ that could be awarded by Auction 1 in contrast to the maximality of the pair $(((b_1, C_1^*), \dots, (b_l, C_l^*)), \{x_1, \dots, x_l\})$.

If it holds $\sigma(f_j) > \sigma(b_h)$ it has to hold $v_{\sigma(b_h)}(b_h) - p > 0$ since otherwise b_h already got deleted in Line 10 in a previous iteration at a lower price than p. Then, by Lemma 2.2 there exists $D_h^* \in \mathcal{C}^*(M \setminus (F_{-\sigma(b_h)} \cup F^j)/B^{h-1})$ such that $b_h \in D_h^* \subseteq C_h^*$ and again by Lemma 2.2 there exists $G_h^* \in C^*(M \setminus F_{-\sigma(b_h)}/B^{h-1})$ such that $G_h^* \setminus (F^j \cap F_{\sigma(b_h)}) = D_h^*$, hence $b_h \in G_h^*$. Note that it may not hold $G_h^* \subseteq C_h^*$ since it is possible that it holds $F^j \cap F_{\sigma(b_h)} \neq \emptyset$ but it holds $\{e \in G_h^* \colon v_{\sigma(b_h)}(e) - p > 0\} = \{e \in D_h^* \colon v_{\sigma(b_h)}(e) - p > 0\}$ by construction and therefore $\{e \in G_h^* \colon v_{\sigma(b_h)}(e) - p > 0\} \subseteq \{e \in C_h^* \colon v_{\sigma(b_h)}(e) - p > 0\}$. However, by Lemma 2.7 there exists a value maximal element $e_h \in G_h^*$ that gets allocated to bidder $\sigma(b_h)$ by Auction 2, hence it has to hold $v(e_h) = v(b_h)$ and therefore Auction 1 could equivalently award item e_h instead of b_h and $((b_1, C_1^*), \dots, (e_h, C_h^*))$ is a suitable subsequence and there obviously has to exist a suitable sequence due to the no-monopoly condition, $((b_1, C_1^*), \dots, (b_{h-1}, C_{h-1}^*), (e_h, C_h^*), (b'_{h+1}, C'^*_{h+1}), \dots, (b'_l, C'^*_l))$ that could be awarded by Auction 1 in contrast to the maximality of the pair $(((b_1, C_1^*), \dots, (b_l, C_l^*)), \{x_1, \dots, x_l\})$.

Therefore, Auction 1 and 2 award the same elements if appropriate tie-breaking is used in Auction 1 and therefore they also delete the same elements and the claim follows directly by Theorem 2.1.

□

We remark that the runtime of Auction 2 is only pseudo-polynomial in the highest encoded value $\max_{e \in E} v_e$ since it uses unit step price increments to find the largest integer value. However, it can be easily adapted to a polynomial auction, if we modify it to a *long step* version by changing Line 10 to

Ask each bidder $j \in N$ to determine $p_j := \min\{v_e \colon e \in E_j\}$ and set $p \leftarrow \min\{p_j \colon j \in N\}$.

This bounds the number of price increments by $|E|$ and makes Auction 2 a polynomial auction.

Theorem 2.10. *The long step version of Auction 2 runs in polynomial time in the number of items $|E|$ and the number of bidders $|N|$, if agents are truthful.*

Proof. Note that the price increment rule forces at least one element to be deleted after every price increase. Therefore, the While-Loop in Line 3 is carried out at most $|E|$ times. Clearly, the inner for-loop in Line 5 is carried out $|N|$ times. The optimization problem Line 7 can be solved easily by applying the greedy algorithm to $M/(E_{-j} \setminus F_{-j})$ with weights $v_e - p$ for $e \in E_j$, which clearly has runtime polynomial in the number of items and the number of bidders. Furthermore, by the Lines 8 and 9, the matroid M is a minor of the initial matroid, hence its rank function can easily be determined from the rank function of the initial matroid and therefore the optimization problem Line 7 can also be solved in polynomial oracle time and the claim follows. $\square$

Note that in the single item setting the long step version of Auction 2 is the Vickrey-Clarke-Groves auction and has a runtime of $\mathcal{O}(n-1)$.

2.3. Selling Elements of an Integer Base Polyhedron

In this chapter we present our main result, a polynomial runtime ascending auction for selling bases of an integer polymatroid.

2.3.1. Economic Setting in the Polymatroid Environment

Similar to the matroid setting there is one seller and a set of bidders $N = \{1, \dots n\}$, $n \in \mathbb{N}$ that are interested in purchasing indivisible goods of the finite set $E := \{1, \dots, m\}$, $m \in \mathbb{N}$ from the seller. For each good $e \in E$ there are $f(\{e\})$ identical copies available. Furthermore, there is a nondecreasing, normalized submodular function $f \colon 2^E \to \mathbb{Z}_+$ determining the set of feasible allocations given by the integer base polyhedron $B_f \subset \mathbb{Z}_+^E$ and P_f is assumed to be normal. The submodular function f is given by a (polynomial time) *value giving oracle* and we denote by EO_f the time it takes to evaluate $f(S)$ for any given $S \subseteq E$.

The ground set E is partitioned into sets $(E_j)_{j\in N}$ such that bidder $j \in N$ is exclusively interested in the items E_j, hence there are no two bidders interested in the same good. This assumption is w.l.o.g. because, if necessary, we can split one item into two items with a joint capacity constraint. For that, assume there are two bidders 1 and 2 interested in $e \in E$. Consider the set $(E \setminus \{e\}) \sqcup \{e_1, e_2\}$ and $\bar{f}\colon 2^{(E\setminus\{e\})\sqcup\{e_1,e_2\}} \to \mathbb{Z}_+$ with

$$\bar{f}(S) = \begin{cases} f(S) & \text{for } S \cap \{e_1, e_2\} = \emptyset, \\ f((S \cup \{e\}) \setminus \{e_1, e_2\}) & \text{for } S \cap \{e_1, e_2\} \neq \emptyset. \end{cases}$$

Lemma 2.11. *If f is submodular then the function $\bar{f}$ is submodular.*

Then, since $\bar{f}$ obviously is nondecreasing and normal it holds that $P_{\bar{f}}$ is a polymatroid and for every $\bar{x} \in P_{\bar{f}}$ it also holds $x \in P_f$ if we define $x_i := \bar{x}_i$ for $i \neq e$ and $x_e = \bar{x}_{e_1} + \bar{x}_{e_2}$.

For every element $e \in E_j$ that bidder j is interested in the bidder has a function v^j_e mapping the quantity x_e of e to $\mathbb{R}_+$ expressing her value. We assume v^j_e to be nondecreasing concave. Then, bidder $j's$ value for $x \in P_f$ is given by $v^j(x) := \sum_{e\in E_j} v^j_e(x_e)$ and $v^j\colon \mathbb{R}^{E_j}_+ \to \mathbb{R}_+$ is a separable concave function. Analogous to the matroid setting, initially, the polymatroid P_f has to fulfill the following condition, which we also denote by **no-monopoly**: No bidder is initially able to prevent that a basis of the polymatroid is sold, therefore, for all $j \in N$ it holds $f(E_{-j}) = f(E)$.

2.3.2. Applying Auction 2 to Integer Polymatroids

First, we show how to directly apply Auction 2 to the integer polymatroid setting. We convert the integer polymatroid to a matroid via the following transformation:

Definition 2.12. *Let P_f be an integer polymatroid on the ground set E. For $e \in E$ let $X_e := \{e\} \times \{1, ..., f(\{e\})\}$ and $X := \bigcup_{e\in E} X_e$ and for $j \in N$ let $X^j := \bigcup_{e\in E_j} X_e$. Define $\mathcal{I} := \{I \subseteq X\colon \sum_{(e,b)\in I} \chi^{\{e\}} \in P_f\}$ and the* ***underlying matroid*** *to P_f as $M(P_f) := (X, \mathcal{I})$.*

Observe that $(X^j)_{j\in N}$ is the partition of the set X such that X^j is the set of items that bidder j is exclusively interested in $M(P_f)$ and define $X^{-j} := X \setminus X^j$ for $j \in N$.

Lemma 2.13. *[Helgason, 1974] Let P_f be an integer polymatroid on the ground set E, then, $M(P_f)$ is a matroid.*

Proof. Let $x, y \in E$ and $S \subseteq E \setminus \{x, y\}$ and

Case 1: If $(S \cup \{x, y\}) \cap \{e_1, e_2\} = \emptyset$ it holds $\bar{f}(S \cup \{x, y\}) - \bar{f}(S \cup \{x\}) = f(S \cup \{x, y\}) - f(S \cup x) \leq f(S \cup \{y\}) - f(S) = \bar{f}(S \cup \{y\}) - \bar{f}(S)$.

Case 2: If w.l.o.g. $e_1 \in S$ it holds $\bar{f}(S \cup \{x, y\}) - \bar{f}(S \cup \{x\}) = f((S \cup \{x, y, e\}) \setminus \{e_1, e_2\}) - f((S \cup \{x, e\}) \setminus \{e_1, e_2\}) \leq f((S \setminus \{e_1, e_2\}) \cup \{y, e\}) - f((S \cup \{e\}) \setminus \{e_1, e_2\}) = \bar{f}(S \cup \{y\}) - \bar{f}(S)$.

Case 3: If $e_1 = x$, $e_2 \notin S \cup \{y\}$, it holds $\bar{f}(S \cup \{x, y\}) - \bar{f}(S \cup \{x\}) = f(S \cup \{y, e\}) - f(S \cup \{e\}) \geq f(S \cup \{y\}) - f(S) = \bar{f}(S \cup y) - \bar{f}(S)$.

Case 4: If $e_1 = x$, $e_2 = y$, it holds $\bar{f}(S \cup \{x, y\}) - \bar{f}(S \cup \{x\}) = f(S \cup \{e\}) - f((S \cup \{e\})) \leq f(S \cup \{e\}) - f(S) = \bar{f}(S \cup \{y\}) - \bar{f}(S)$. □

Definition 2.14. *For a polymatroid P_f with underlying matroid $M(P_f) := (X, \mathcal{I})$ and $a \in \mathbb{Z}^n_+$, $a_e \leq f(\{e\})$ we denote $A^l := \{(e, b) \in X : b \leq a_e, e \in E\}$ and $A^r := \{(e, b) \in X : b > f(\{e\}) - a_e, e \in E\}$ and, if necessary, we identify the point a with the modular function $a : 2^E \to \mathbb{R}_+, a(S) = \sum_{e \in S} a_e$ without explicitly stating it.*

Set the values of $(e, 1) \in E(X)$ as $v^j_{(e,1)} := v^j_e(1)$ and $v^j_{(e,b)} := v^j_e(b) - v^j_e(b - 1)$ for $e \in E_j$ and $1 < b \leq f(\{e\})$. Let B an optimal basis of $M(P_f)$. Define $B_e := \{(e, k) \in B\}$ for $e \in E$ and $B^l := \{(e, k) : e \in E, k \leq |B_e|\}$. Then, B^l is obviously also a basis of $M(P_f)$. Furthermore, B^l is optimal regarding $\sum_{j \in N} v^j$ since for all $e \in E$ it holds $\sum_{(e,b) \in B} \chi^{\{e\}} = \sum_{(e,b) \in B^l} \chi^{\{e\}}$ and $v_{(e,b)} \geq v_{(e,b+1)}$ and for $(e, k) \in B$ it either holds $(e, k) \in B^l$ or there exists $(e, k') \in B^l$ with $k' < k$. Therefore, for every optimal basis $B \in \mathcal{B}(M(P_f))$ it holds that $x := \sum_{(e,b) \in B} \chi^{\{e\}}$ is an optimal basis for P_f with the same value as B and for every optimal basis $x \in B_f$ it holds that $B^l := \{(e, b) \in X : x_e \leq b, e \in E\}$ is an optimal basis for $M(P_f)$ with the same value as x.

Now, it is easy to see that applying Auction 2 on the underlying matroid has pseudo-polynomial runtime in the highest encoded value $\max_{e \in E} f(\{e\})$ since a pseudo-polynomial transformation is required: Storing $\max_{e \in E} f(\{e\})$ just requires $\log \max_{e \in E} f(\{e\})$ bits for which $\max_{e \in E} f(\{e\})$ elements are necessary in the matroid $M(P_f)$.

2.3.3. Polynomial Auction for Selling Bases of an Integer Polymatroid

We present our main result, a mechanism for selling bases of an integer polymatroid. In contrast to Auction 2 our auction performs directly on the integer polymatroid. The main idea of our algorithm is to generalize our nonimpairing condition from matroids to polymatroids. To show the correctness of our mechanism, we will prove that its output is equivalent to the output of Auction 2, performed on the underlying matroid.

We start with some properties of submodular functions and polymatroids.

Lemma 2.15. *Let $f : 2^E \to \mathbb{R}_+$ be a submodular function and $d : 2^E \to \mathbb{R}$, $d(S) := \sum_{e \in S} d_e$ a modular function. Then, it holds that $g : 2^E \to \mathbb{R}_+, g(S) := f(S) + d(S)$ is submodular.*

Proof. Let $X \subset E$ and $S \subseteq T \subseteq E \setminus X$, it holds

$$\begin{aligned} g(S \cup X) - g(S) =& f(S \cup X) - f(S) + d(S \cup X) - d(S) \\ \geq& f(T \cup X) - f(T) + d(T \cup X) - d(T) = g(T \cup X) - g(T). \end{aligned}$$ □

Lemma 2.16. *[Lovász, 1983, Theorem 2.5]. Let $f\colon 2^E \to \mathbb{R}_+$ be a submodular and $d\colon 2^E \to \mathbb{R}$, $d(S) := \sum_{e\in S} d_e$ a modular function. Then, it holds that $g\colon 2^E \to \mathbb{R}_+, g(S) := \min_{K\subseteq S} f(K) + d(S \setminus K)$ is submodular.*

We define analogously to polymatroids.

Definition 2.17. *Let $f : 2^E \to \mathbb{R}_+$, then $P_f := \{x \in \mathbb{R}^E_+ \colon \sum_{e\in S} x_e \leq f(S)$ for all $S \subseteq E\}$.*

Lemma 2.18. *[Schrijver, 2003, Subsection 44.4]. Let $f : 2^E \to \mathbb{R}_+$ be submodular and normalized, then the function $f'\colon 2^E \to \mathbb{R}_+, f'(S) := \min_{T\supseteq S} f(T)$ is submodular, normalized and* nondecreasing *and it holds $P_f = P_{f'}$.*

Therefore, to every *non* nondecreasing normalized submodular function there can be associated a polymatroid. However, in contrast to nondecreasing normalized submodular functions there clearly exists no bijection between the set of polymatroids and the set of normalized submodular functions. Note that the nondecreasing property for submodular functions f is desirable since it allows for a simple polymatroid greedy algorithm to optimize linear functions over P_f efficiently while for nonsubmodular functions a simple polymatroid greedy algorithm may not work correctly in this setting.

Lemma 2.19. *Let $f : 2^E \to \mathbb{R}_+$ be submodular, $d\colon 2^E \to \mathbb{R}_+$ modular and $g\colon 2^E \to \mathbb{R}_+, g(S) := \min_{K\subseteq S} \sum_{e\in K} d(e) + f(S \setminus K)$. Then, it holds $P_{\min\{f,d\}} = P_g$, hence $P_{\min\{f,d\}}$ is a polymatroid.*

Proof. Clearly, g is nondecreasing, submodular and integer valued and it holds $P_g = \{x \in \mathbb{R}^E_+ \colon \sum_{e\in S} x_e \leq \min_{K\subseteq S} \sum_{e\in K} d(e) + f(S \setminus K)$ for $S \subseteq E\}$ by Lemmata 2.16 and 2.18. Let $y \in P_g$, then it holds $y \leq f(S)$ for all $S \subseteq E$ by choosing $K = \emptyset$ and it holds $y_e \leq d(e)$ for all $e \in E$ by choosing $K = \{e\} = S$. Therefore, $\sum_{e\in S} y_e \leq \min\{f(S), d(S)\}$ for all $S \subseteq E$.

Conversely, let $y \in P_{\min\{f,d\}}$, then for all $K \subseteq S \subseteq E$ it holds $\sum_{e\in K} y_e \leq \sum_{e\in K} d_e$ and $\sum_{e\in S\setminus K} y_e \leq f(S \setminus K)$, hence $y \in P_g$. □

Assume one executes Auction 2 on a matroid $M(P_f)$. We show that awarding sets to bidders (matroid contraction) and elements dropping out due to price increments (matroid deletion) can also be expressed directly as operations on the polymatroid P_f.

Shrinking the polymatroid turns out to be equivalent to matroid contraction.

Lemma 2.20. *Let P_f be an integer polymatroid on the ground set E and $M(P_f) := (X, \mathcal{I})$ its underlying matroid. Then, the matroid $M(P_f)/A^r$ is the underlying matroid to the integer polymatroid $P_{f-a} := \{x \in \mathbb{R}_+^E \colon \sum_{e \in S} x_e \leq f(S) - a(S)$ for all $S \subseteq E\}$.*

Proof. It holds by Lemma 2.15 that $f - a$ is normalized submodular and by Lemma 2.18 that P_{f-a} is a well-defined integer polymatroid. It follows

$$I \in \mathcal{I}(M(P_{f-a})) \Leftrightarrow \sum_{(e,b) \in I} \chi^{\{e\}} \in P_{f-a} \Leftrightarrow \sum_{e \in S} \sum_{(e,b) \in I} \chi^{\{e\}} \leq f(S) - a(S) \text{ for all } S \subseteq E$$

$$\Leftrightarrow \sum_{e \in S} \sum_{(e,b) \in I} \chi^{\{e\}} + \sum_{e \in S} a_e \leq f(S) \text{ for all } S \subseteq E$$

$$\Leftrightarrow \sum_{e \in S} \sum_{(e,b) \in I} \chi^{\{e\}} + \sum_{e \in S} \sum_{(e,b) \in A^r} \chi^{\{e\}} \leq f(S) \text{ for all } S \subseteq E$$

$$\Leftrightarrow I \sqcup A^r \in M(P_f) \Leftrightarrow I \in M(P_f)/A^r. \qquad \square$$

Clearly, for a polymatroid P_f with underlying matroid $M(P_f)$ for $e \in E_j$ and $1 < b < f(\{e\})$ it holds $v^j_{(e,b)} \geq v^j_{(e,b+1)}$, hence agents prefer the set A^l over the contracted set A^r in Lemma 2.20. A direct consequence of the definition of $M(P_f)$ is:

Lemma 2.21. *Let P_f be an integer polymatroid on the ground set E and $M(P_f) := (X, \mathcal{I})$ its underlying matroid. Then, for $a \in P_f$ it holds $M(P_f)/A^l \simeq M(P_f)/A^r$.*

Analogously to Lemma 2.20, truncating a polymatroid turns out to be equivalent to deleting elements in the underlying matroid:

Lemma 2.22. *Let P_f be an integer polymatroid on the ground set E and $M(P_f) := (X, \mathcal{I})$ its underlying matroid, $a \in P_f$ and and $d_e := f(\{e\}) - a_e$ for $e \in E$. Then, $M(P_f) \setminus A^r$ is the underlying matroid to $P_{\min\{f,d\}}$.*

Proof. By Lemma 2.19, it holds that $P_{\min\{f,d\}}$ is an integer polymatroid. It follows

$$I \in \mathcal{I}(M(P_f) \setminus A^r) \Leftrightarrow I \in \mathcal{I}(M(P_f)) \text{ and } I \cap A^r = \emptyset$$

$$\Leftrightarrow \sum_{e \in S} \sum_{(e,b) \in I} \chi^{\{e\}} \leq f(S) \text{ for all } S \subseteq E \text{ and } \sum_{(e,b) \in X} \chi^{\{e\}} \leq f(\{e\}) - \sum_{(e,b) \in A^r} \chi^{\{e\}} \text{ for all } e \in E$$

$$\Leftrightarrow \sum_{e \in S} \sum_{(e,b) \in I} \chi^{\{e\}} \leq f(S) \text{ for all } S \subseteq E \text{ and } \sum_{(e,b) \in X} \chi^{\{e\}} \leq d_e \text{ for all } e \in E$$

which, by Lemma 2.19, is equivalent to $\sum_{(e,b) \in I} \chi^{\{e\}} \in P_{\min\{f,d\}} \Leftrightarrow I \in \mathcal{I}(M(P_{\min\{f,d\}}))$. $\square$

The correspondences between integer polymatroids and matroids can be summarized by the following commutative diagram.

$$
\begin{array}{ccc}
P_{f-a} & \longrightarrow & M(P_f)/A^r \\
\uparrow {\scriptstyle -} & & \uparrow {\scriptstyle /} \\
P_f & \longrightarrow & M(P_f) \\
\downarrow {\scriptstyle \min} & & \downarrow {\scriptstyle \setminus} \\
P_{\min\{f, \sum_{e\in E}(f(\{e\})-a_e)\cdot\chi^{\{e\}}\}} & \longrightarrow & M(P_f)\setminus A^r
\end{array}
$$

Next, we adapt Auction 2 to the polymatroid setting and state our mechanism for selling bases of a polymatroid. Let P_f the polymatroid that constrains the feasible allocations at current price p and $a \in \mathbb{Z}_+^E$ the already awarded quantities. All bidders $j \in N$ announce for each item $e \in E_j$ the minimum quantity

$$d_e := \min\{y_e \colon y_e \in \arg\max_{x_e \le f(\{e\})} v_e^j(x_e + a_e) - x_e \cdot p\}$$

they would accept. Any additional unit of item e has no strict positive marginal gain for bidder j at the current price p. Clearly, for $F = \{x \in X : v_x = p\}$ and $(X, \mathcal{I}) = M(P_f)$ it holds $d_e = f(\{e\}) - |\{k \colon (e,k) \in F\}|$. We remark that, in contrast to the matroidal setting, the demanded amount of e explicitly depends on the quantity a_e because of the possible nonlinearity of v_e^j.

We define analogous to the **Monopoly** condition:

Definition 2.23. *For a polymatroid P_f given on a groundset $E = E_j \sqcup E_{-j}$ and $d|_{E_{-j}} \in \mathbb{Z}_+^{E_{-j}}$ we denote by **P-Monopoly**$_{P_f, d|_{E_{-j}}, x^j}$ the property that $x^j \in P_f|_{E_j}$ fulfills the condition $(x^j, \mathbf{0}|_{E_{-j}}) \in P_{f-(\mathbf{0}, d|_{E_{-j}})}|_{E_j}$.*

Then, by the Lemmata 2.20 and 2.22 we can express the **P-Monopoly** condition equivalently by the **Monopoly** condition and vice versa.

Corollary 2.24. *For a polymatroid P_f with underlying matroid $M(P_f)$ it holds that $x^j \in P_f|_{E_j}$ and **P-Monopoly**$_{P_f, d|_{E_{-j}}, x^j}$ iff $T_j \in M(P_f)|_{X^j}$ and **Monopoly**$_{M(P_f), X^{-j}, F_{-j}, T_j}$ with $T_j = \{(e,b) \colon e \in E_j, b \le x_e^j\}$ and $F_{-j} = \{(e,b) \in X^{-j} \colon (d|_{E_{-j}})_e < b \le f(\{e\}), e \in E_{-j}\}$.*

We adapt Auction 2 to the polymatroid setting making use of the **P-Monopoly** condition.

Algorithm 3:

Require: No-monopoly submodular function $f\colon 2^E \to \mathbb{Z}_+$ with a partition $E = \bigsqcup_{j\in N} E_j$ of the ground set, valuations that are the sum of nondecreasing, concave functions.

1 $p \leftarrow 0$, $a_e \leftarrow 0$, $c_j \leftarrow 0$, $r \leftarrow f(E)$

2 **while** $\sum_{e\in E} a_e \neq r$ **do**

3   Ask bidders to determine $d_e := \min \arg\max_{x_e \leq f(\{e\})}\{v_e^j(x_e + a_e) - p \cdot x_e\}$

4   **for** $j \in N$ **do**

5   Ask bidder j to determine $\arg\max\{\sum_{e\in E_j} v_e(x_e^j + a_e)\colon x^j \in \mathbb{Z}_+^{N_j}$ such that **P-Monopoly**$_{P_f, d|_{E_{-j}}, x^j}\}$ and let $q^j \in \mathbb{N}^{E_j}$ a norm-maximal element of that set.

6   Award q^j to j and charge him $k_j = \sum_{e\in E_j} p \cdot q_e^j$, $c_j \leftarrow c_j + k_j$.

7         $f \leftarrow f - (q^j, \mathbf{0}|_{E_{-j}})$

8   **for** $e \in E_j$ **do**

9             $a_e \leftarrow a_e + q_e^j$

10     $f \leftarrow \min\{f, d\}$

11     $p \leftarrow p + 1$

Output : a is the optimal allocation, c are the VCG payments

We prove the correctness of Auction 3.

Theorem 2.25. *Auction 3 outputs a welfare maximizing allocation and charges VCG prices.*

Proof. We can adapt Auction 2 such that for any S_j which is chosen in Line 7 it holds for every $e \in E_j$ the set $\{k\colon (e,k) \in S_j\}$ is either the emptyset or a set of descending consecutive integers starting with $f(\{e\})$. Then, it follows directly by Lemmata 2.21 and 2.22 and Corollary 2.24 that Auctions 2 and 3 produce the same welfare maximizing allocation, after transforming the allocation obtained by Auction 2 onto the original space, and charge the same prices. Then, the claim follows by Theorem 2.9. □

We mention that the maximal value element in Line 5 that fulfills the **P-Monopoly** condition has not to be unique. In order to obtain a basis of the polymatroid P_f, Auction 3 chooses a quantity such that its norm is maximal among all quantities that are contained in the arg max. Otherwise, we might obtain a suboptimal allocation and the auction may not terminate. To see this, consider the following example, which also applies to matroids:

Example 2.26. *Let $P_f := \{x \in \mathbb{R}_+^2 : x_1 + x_2 \leq 1\}$ with bidder 1 interested in x_1 and bidder 2 interested in x_2 and both bidders $j \in \{1,2\}$ have the valuation $v_{x_j}^j(a) = a$. Then, at price $p = 0$ both bidders report the demand $d_{x_j}^0 = 1$ and nothing will be awarded. When the price* *gets increased to* 2 *bidder* j *is indifferent about getting awarded a unit of* j *or not. If there is no condition to take the maximum in Line 5 of Auction 3, it might happen that both bidders* $j \in \{1, 2\}$ *choose* $q_{x_j} = 0$. *They have to drop out of the auction since* $P_{\min\{f,d^2\}} = \emptyset$ *and no item was awarded and additionally Auction 3 does not terminate.*

However, the auctioneer can ensure that norm-maximal elements are awarded in Line 6 of Auction 3.

Lemma 2.27. *Bidders can not sabotage the auction by choosing non norm-maximal elements in Line 5 of Auction 3.*

Proof. Let P_f be the polymatroid at Line 2 of Auction 3 and $P_{\tilde{f}}$ the polymatroid in Line 10 when bidders tried to manipulate the auction by choosing possibly non norm-maximal elements q_e for $e \in E$ at price p. Then, the auctioneer can check if it holds $f(E) = \tilde{f}(E) + \sum_{e \in E} q_e$ and if this is not the case he can award $\tilde{q}$ such that $f(E) = \tilde{f}(E) + \sum_{e \in E} q_e + \tilde{q}_e$ and $P_{\tilde{f}} = P_{\min\{f-(q+\tilde{q}),d\}}$ and charge the current price p for the additionally awarded quantities $\tilde{q}$.

Note that in Theorem 2.30 we show that the quantities q can be computed efficiently and obviously, the additional quantities $\tilde{q}$ can be calculated equivalently to the quantities q.

Long Step Version of Auction 3

We have seen that Auction 3 executed on an integer polymatroid yields the same allocation as Auction 2 executed on the underlying matroid. It is rather obvious that the runtime of Auction 3 is not polynomial in the highest encoded value $\max_{e \in E} v_e^j(1)$, since we have eventually to determine d_e for every smaller integer than $\max_{e \in E} v_e^j(1)$. To overcome this issue, we modify Auction 3 to a faster long step version. We can change Line 11 to ask each bidder $j \in N$ to determine

$$p_j := \underset{e \in E_j, f(\{e\}) \neq 0}{\arg\min} \; v_e^j(f(\{e\}) - v_e^j(f(\{e\}) - 1) \text{ and set } p \leftarrow \min_{j \in N} p_j.$$

Therefore, we ask every bidder for the smallest price change such that the demand for any of the items she is interested in changes and set the new price as the minimum. No bidder can profit from underreporting since it only delays the auction without allocating anything.

However, there is a drawback of the long step version in contrast to the single step version of Auction 3. In a single item setting the long step version turns out to be the Vickrey-Clarke-Groves auction, while the single step version remains an ascending auction. Therefore, the long step version should preferably be applied in settings with many items and a moderate number of breakpoints of the separated valuation functions v_e^j where it is unlikely that any winning bidder has to reveal the values of its smallest quantities to preserve the bidder's privacy.

We want to show that the long step version of Auction 3 has polynomial runtime in the number of items, bidders and breakpoints of buyers' valuation functions. The main difficulty is to verify that Line 5 can be computed efficiently. To this end, we need some algorithmic results regarding submodular functions.

Proposition 2.28. *The minimum of a submodular function* $f\colon 2^E \to \mathbb{R}_+$ *can be calculated in polynomial oracle time. Orlin [2009] showed that there exists an algorithm with runtime* $\mathcal{O}(m^6 + m^5 \cdot \mathrm{EO}_f)$ *and Lee et al. [2015] provides a strongly polynomial algorithm of runtime* $\mathcal{O}(m^4 \cdot \log m + m^3 \cdot \log^2 m \cdot \mathrm{EO}_f)$.

In order to compute Line 5 of Auction 3, every bidder has to maximize a separable concave function over a polymatroid that is given by value oracle access to its associated monotone nondecreasing submodular function. We outline a procedure, implicitly described in Groenevelt [1991] and Nagano [2007], that is specifically tailored to this task.

Algorithm 4:

Require: Polymatroid P_f on the ground set E and a separable concave function $v\colon \mathbb{R}^E_+ \to \mathbb{R}_+$.

1 Determine a solution y to the problem $\max\{\sum_{e\in E} v_e(x_e)\colon\ \sum_{e\in E} x_e \le f(E),\ x \in \mathbb{R}^E_+\}$.
2 Determine the unique inclusionwise maximal set $E' \in \arg\min\{f(S) - \sum_{e\in S} y_e\colon S \subseteq E\}$.
3 **return** E'

Algorithm 5: Decomposition Algorithm

Require: Polymatroid P_f on the ground set E

```
Call Algorithm 4 on P_f
if E' = E then
    z|_E = y
else
    Call Algorithm 5 on P_{f|_{E'}} and v|_{E'}.
    Set E'' := E \ E' and g: E'' → R_+, g(S) = f(S ⊔ E') − f(E').
    Call Algorithm 5 on P_g and v|_{E''}.
return z
```

The Decomposition Algorithm 5 is a recursion of Algorithm 4 that either states that the solution y determined in Line 1 of Algorithm 4 is optimal and *norm-maximal* or it decomposes the ground set nontrivially into two disjoint subsets to call Algorithm 5 on these subsets again. Then, the solution to the original problem is given by joining the solutions to the subproblems.

Lemma 2.29. *[Groenevelt, 1991], [Nagano, 2007] Maximizing a separable concave function over a polymatroid given by its associated submodular function can be done by Algorithm 5 in polynomial oracle time.*

Proof. Let $f\colon 2^E \to \mathbb{R}_+$ the function that describes the polymatroid. We refer to Groenevelt [1991] or Nagano [2007] for a detailed proof of the correctness of the Decomposition Algorithm. Here, we outline that Algorithm 5 has polynomial runtime in m and makes just polynomial many value oracle calls to the initial function f.

The single constraint problem in Line 1 can be solved in $\mathcal{O}(m \cdot \log(\frac{|f(E)|}{m})$ by Frederickson and Johnson [1982] to determine $y \in \arg\max\{\sum_{j\in E} v_e^j(x_e)\colon \sum_{e\in E} x_e \leq f(E)\}$.

In Line 2 of Algorithm 4, one needs to find an inclusionwise maximal minimizer of $g\colon 2^E \to \mathbb{R}_+, g(S) := f(S) - \sum_{e\in S} y_e$. The submodular function minimizing algorithms in Orlin [2009] and Iwata et al. [2001] output a solution $T \in \arg\min\{g(S)\colon S \subseteq E)\}$ and also a vector $z \in B_g$ and a presentation $z = \sum_{e\in E} \lambda_e b_e$, $\lambda_e \geq 0, \sum_{e\in E} \lambda_e = 1$, b_e extreme point of P_g, for which holds $\sum_{e\in E} \min\{z_e, 0\} = g(T)$. The vectors z and b can be used to compute a maximal minimizer of g in $\mathcal{O}(m^3 \cdot \mathrm{EO}_f)$ additional time, see Murota [2003, Note 10.11]. Therefore, one can solve the problem in Line 2 of Algorithm 4 in $\mathcal{O}(m^6 + (m^5 + m^3) \cdot \mathrm{EO}_f)$ by the Algorithm in Orlin [2009] by Proposition 2.28. In contrast, the faster submodular function minimization algorithm of Lee et al. [2015] does not output a vector $z \in B_g$ from which an inclusionwise maximal minimizer of g can be computed.

The Decomposition Algorithm is a recursion of Algorithm 4 that either states optimality or decomposes the ground set nontrivially into two disjoint subsets to call Algorithm 4 again on these subsets. The ground set E can be decomposed nontrivially at most $n-1$ times, hence $n-1$ is an upper bound for the number of executions of Algorithm 4. In total, a separable concave function f can be maximized over a polymatroid in $\mathcal{O}(m \cdot (m \cdot \log(\frac{|f(E)|}{m}) + m^6 + (m^5 + m^3) \cdot \mathrm{EO}_f) = \mathcal{O}(m^2 \cdot \log(\frac{|f(E)|}{m}) + m^7 + m^6 \cdot \mathrm{EO}_f)$.

□

We need that Line 5 of Auction 3 can be computed efficiently.

Theorem 2.30. *Line 5 of Auction 3 can be calculated in polynomial time.*

Proof. Proof. Recall that the initial function f is given via a value giving oracle and denote $p_0 := 0$, $f^0 = f$ and $d_e^0 = f(\{e\})$ for $e \in E$ and let $(p_i)_{1\leq i\leq k}$ be a sequence of prices computed during a run of (the long step version) of Auction 3. For $1 \leq i \leq k$ denote by $d^{p_i} \in \mathbb{Z}_+^E$ the demand and by $q^{p_i} \in \mathbb{Z}_+^E$ the awarded quantities at price p_i and by $a^{p_i} \in \mathbb{Z}_+^E$ the already awarded quantities $\sum_{j=0}^{i-1} q^{p_j}$. It holds by the Lines 7 and 10 for $1 \leq i \leq k$ that $f^{p_i} := \min\{f^{p_{i-1}} - q^{p_{i-1}}, d^{p_{i-1}}\}$ describes the feasible allocations at price p_i, hence the polymatroid when the While-Loop in Line 2 is evaluated at price p^i. Note that for $2 \leq i \leq k$ it holds $f^{p_i} = \min\{\min\{f^{p_{i-2}} - q^{p_{i-2}}, d^{p_{i-2}}\} - q^{p_{i-1}}, d^{p_{i-1}}\} = \min\{\min\{f^{p_{i-2}} -$

$q^{p_{i-2}} - q^{p_{i-1}}, d^{p_{i-2}} - q^{p_{i-1}}\}, d^{p_{i-1}}\} = \{\min\{f^{p_{i-2}} - q^{p_{i-2}} - q^{p_{i-1}}, d^{p_{i-2}} - q^{p_{i-1}}, d^{p_{i-1}}\}$ and therefore by induction

$$f^{p_i} = \min\{f - \sum_{k=0}^{i-1} q^{p_k}, \min\{d^{p_j} - \sum_{k=j+1}^{i-1} q^{p_k}, 0 \leq j \leq i-1\}\} \text{ for } 1 \leq i \leq k.$$

We assume w.l.o.g. that the for-loop in Line 4 is executed in lexicographic order and define $q^{p_i}_{<l} := (q^{p_i}|_{\bigcup_{1\leq k<l} E_k}, \mathbf{0}|_{\bigcup_{l\leq k\leq n} E_k})$ for $l \in N$, hence $q^{p_i}_{<l}$ equals the awarded quantities at price p^i after $l-1$ iterations of the for loop in Line 4. Then, it holds that the polymatroid that constrains the feasible allocations at the l-th iteration of the for loop in Line 4 at price p_i is given by

$$f^{p_i}_l := f^{p_i} - q^{p_i}_{<l} = \min\{f - \sum_{k=0}^{i-1} q^{p_k} - q^{p_i}_{<l}, \min\{d^{p_j} - \sum_{k=j+1}^{i-1} q^{p_k} - q^{p_i}_{<l}, 0 \leq j \leq i-1\}\} \text{ for } 1 \leq i \leq k$$

and in Line 5 of Auction 3 agent $l \in N$ has to compute

$$\arg\max\{\sum_{e\in E_l} v_e(x^l_e + a^{p_{i-1}}_e) \colon x^l \in P_{f^{p_i}_l - (\mathbf{0}|_{E_l}, d^{p_i}|E_{-l})}|_{E_l}\}.$$

It holds that $f^{p_i}_l - (\mathbf{0}|_{E_l}, d^{p_i}|E_{-l}) = \min\{f - \sum_{k=0}^{i-1} q^{p_k} - q^{p_i}_{<l} - (\mathbf{0}|_{E_l}, d^{p_i}|E_{-l}), \min\{d^{p_j} - \sum_{k=j+1}^{i-1} q^{p_k} - q^{p_i}_{<l} - (\mathbf{0}|_{E_l}, d^{p_i}|E_{-l}), 0 \leq j \leq i-1\}\}$. We denote $\tilde{q}^{p_i}_{<l} := \sum_{k=0}^{i-1} q^{p_k} + q^{p_i}_{<l} + (\mathbf{0}|_{E_l}, d^{p_i}|E_{-l})$ and $\tilde{d}^{p_i}_{<l} := \min\{d^{p_j} - \sum_{k=j+1}^{i-1} q^{p_k} - q^{p_i}_{<l} - (\mathbf{0}|_{E_l}, d^{p_i}|E_{-l}), 0 \leq j \leq i-1\}$, observe that $\tilde{q}^{p_i}_{<l}$ and $\tilde{d}^{p_i}_{<l}$ can be computed easily out of the previously submitted demands and awarded quantities and define $\bar{f}^{p_i}_l \colon 2^E \to \mathbb{Z}_+$, $\bar{f}^{p_i}_l(S) := \min_{K\subseteq S}\{(f - \tilde{q}^{p_i}_{<l})(K)\} + \tilde{d}^{p_i}_{<l}(S \setminus K)$ and recall that $P_{f^{p_i}_l - (\mathbf{0}|_{E_l}, d^{p_i}|E_{-l})} = P_{\bar{f}^{p_i}_l}$ by Lemma 2.19.

Then, to calculate the quantity that bidder l is awarded at price p^i she has to maximize a separable concave function over the polymatroid $P_{\bar{f}^{p_i}_l}|_{E_l}$ which can be done in $\mathcal{O}(m_l^2 \cdot \log(\frac{|f(E_l)|}{m_l}) + m_l^7 + m_l^6 \cdot \mathrm{EO}_{\bar{f}^{p_i}_l}) \subseteq \mathcal{O}(m^2 \cdot \log(\frac{|f(E)|}{m}) + m^7 + m^6 \cdot \mathrm{EO}_{\bar{f}^{p_i}_l})$ where $|E_l| = m_l$, by Lemma 2.29. However, evaluating $\bar{f}^{p_i}_l(S)$ for $S \subseteq E_l$ can be done in $\mathcal{O}(m^4 \cdot \log m + m^3 \cdot \log^2 m \cdot \mathrm{EO}_f)$ by Proposition 2.28. Therefore, Line 5 of Auction 3 can be calculated in $\mathcal{O}(m^2 \cdot \log(\frac{|f(E)|}{m}) + m^7 + m^6 \cdot \mathrm{EO}_{\bar{f}^{p_i}_l}) = \mathcal{O}(m^2 \cdot \log(\frac{|f(E)|}{m}) + m^7 + m^6 \cdot (m^4 \cdot \log m + m^3 \cdot \log^2 m \cdot \mathrm{EO}_f) = \mathcal{O}(m^2 \cdot \log(\frac{|f(E)|}{m}) + m^{10} \cdot \log m + m^9 \cdot \log^2 m \cdot \mathrm{EO}_f)$. □

Now, the polynomial runtime of Auction 3 follows easily.

Theorem 2.31. *The long step version of Auction 3 runs in polynomial time in $m, n, f(E)$ and $|\{v^j_e(x) - v^j_e(x-1) \colon x$ is breakpoint of $\bar{v}^j_e, e \in E_j, j \in N\}|$ if bidders behave truthfully.*

Proof. The outer While-loop in Line 2 is carried out at most the number of price increments times, which is at most the number of prices at which any function $\bar{v}^j_e, e \in E$ has a breakpoint, hence $|\{v_e^j(x) - v_e^j(x-1)\colon x \text{ is breakpoint of } \bar{v}_e^j, e \in E_j, j \in N\}|$. (Notice that there might exist $j, k \in N$, $j \neq k$ and $e \in E_j$ and $f \in E_k$ and breakpoints x of v_e^j and y of v_f^k such that $v_e^j(x) - v_e^j(x-1) = v_f^k(y) - v_f^k(y-1)$ and thus $|\{v_e^j(x) - v_e^j(x-1)\colon x$ is breakpoint of $\bar{v}_e^j, e \in E\}| \leq \sum_{e \in E} \#(\text{of integral breakpoints of } \bar{v}_e^j)$.) The inner For-loop in Line 4 is carried out n times. To show polynomial runtime it remains to show that we are able to check the While-condition in Lines 2 and calculate Line 5 of Auction 3 in polynomial time. Clearly, the While-Condition in Lines 2 can be checked easily e.g., by a greedy algorithm and the most time consuming computation in Line 5 can be executed in $\mathcal{O}(m^2 \cdot \log(\frac{|f(E)|}{m}) + m^{10} \cdot \log m + m^9 \cdot \log^2 m \cdot \mathrm{EO}_f)$ by Theorem 2.30. In total, an optimal allocation can be computed in $\mathcal{O}((m^2 \cdot \log(\frac{|f(E)|}{m}) + m^{10} \cdot \log m + m^9 \cdot \log^2 m \cdot \mathrm{EO}_f) \cdot n \cdot \sum_{e \in E} \#(\text{of integral breakpoints of } v_e^j))$. □

We remark that the runtime of Auction 3 primarily depends on the time consuming repeated application of submodular function minimization and that the bound obtained in Theorem 2.31 holds for arbitrary submodular functions. Furthermore, in contrast to the Auctions 1 and 2 applied to integer polymatroids, the runtime of Auction 3 does *not* depend linearly on the highest encoded value $\max_{e \in E} f(\{e\})$.

2.3.4. Incentives

It was shown by Bikhchandani et al. [2011] that in Auction 1 truthful bidding forms an ex post equilibrium. That is, no bidder can benefit by bidding untruthfully provided that all other bidders bid truthfully. This is a stronger equilibrium property than Bayesian-Nash equilibrium. Therefore, it follows directly by the observation made in Theorem 2.9 that the awarded and deleted elements in Auction 1 and Auction 2 coincide at every price p and by Corollary 2.24 that in Auction 3 truthful bidding forms an ex post equilibrium. However, truthful bidding is not dominant strategy incentive compatible in Auction 3 since Auction 3 is vulnerable to biding in a way that is inconsistent with any valuation function, as shown in the following example

Example 2.32. *Let $E := \{E_1 \sqcup E_2\}$ with $E_1 := \{a, b\}$ and $E_2 := \{c, d\}$, $N := \{1, 2\}$ and $P_f := \{x \in \mathbb{R}_+^E\colon x_a + x_c \leq 1, x_a + x_c \leq 1\}$. Suppose now that the true valuations of the bidders are $v_a^1(1) = 3, v_b^1(1) = 10, v_c^2(1) = 2, v_a^2(1) = 5$. Then, the optimal bases is $x_a = x_b = 1, x_c = x_d = 0$. If both bidders bid truthfully then a gets allocated at a price of 2 and b gets allocated at a price of 5 by Auction 3. However, if bidder 2 bids inconsistent with his valuation, e.g. he pretends to have a higher value for item d because of being mad to have been forced to drop out of the competition for item c at price 2 it might occur that bidder 1 had been better of by give up his interest in item a and secure item b for a price of 5 and a higher utility then he gets by only getting awarded item a or getting awarded item b for a price higher than 6.*

2.3.5. Impossibly Beyond Polymatroids

We briefly mention that the polymatroid structure is essential so that any single price ascending auction implements an efficient allocation, even if bidders valuations are separable linear functions. Bikhchandani et al. [2011, Theorem 18] show, using a variant of Korte and Lovász [1984, Theorem 4.1], that in every multi-item, single-unit (hence, every item is unique) and multi bidder setting some truthful single price ascending auction is able to find the optimal basis for all valuations iff the inclusionwise maximal feasible subsets are the set of basis of some matroid. Then, it follows directly by the transformation from integer polymatroids to matroids in Subsection 2.3.2 that there cannot exists an environment beyond integer polymatroids for which a truthful single price ascending auction implements an optimal allocation in a multi-item, multi-copy and multi-bidder setting.

2.3.6. An Example

We provide an example of the long step version of Auction 3.

Example 2.33. *Let* $E := \{E_1 \sqcup E_2\}$ *with* $E_1 := \{a, b\}$ *and* $E_2 := \{c, d\}$, $N := \{1, 2\}$, *the polymatroid that constraints the feasible allocations is given by*

$$P_f := \{x \in \mathbb{R}_+^E \colon x_e \leq 60 \text{ for } e \in E, x_a + x_b + x_c + x_d \leq 100, x_a + x_c \leq 80\}$$

and the concave valuation functions are depicted in the following table.

	$[0,10]$	$(10,20]$	$(20,30]$	$(30,40]$	$(40,50]$	$(50,60]$
$v_a^1(x)$	$\mathbf{50} \cdot x$	$50 \cdot x$	$50 \cdot x$	$50 \cdot x$	$\mathbf{25} \cdot x$	$25 \cdot x$
$v_b^1(x)$	$\mathbf{30} \cdot x$	$\mathbf{15} \cdot x$	$15 \cdot x$	$15 \cdot x$	$15 \cdot x$	$\mathbf{10} \cdot x$
$v_c^2(x)$	$\mathbf{40} \cdot x$	$40 \cdot x$	$40 \cdot x$	$40 \cdot x$	$40 \cdot x$	$\mathbf{10} \cdot x$
$v_d^2(x)$	$\mathbf{25} \cdot x$	$25 \cdot x$	$25 \cdot x$	$\mathbf{10} \cdot x$	$10 \cdot x$	$10 \cdot x$

Clearly, for $j \in \{1, 2\}$ *holds* $100 = f(E_{-j}) = f(E)$, *hence no bidder can prevent that a basis is sold.*

Bidders announce the smallest price change such that the demand of any item changes and the price is set to 10 *and the updated demands are* $d_a^{10} = 60$, $d_b^{10} = 50$, $d_c^{10} = 50$ *and* $d_d^{10} = 30$. *Then, bidder* 1 *can clinch* 20 *units from item* a *or item* b *since* $d_c^{10} + d_d^{10} = 80 = 100 - 20$ *and* $d_c^{10} = 50 = 80 - 30$. *He gets awarded* 20 *units of item* a *because it maximizes the valuation function of bidder 1. Bidder* 2 *gets awarded nothing since* $d_a^{10} + d_b^{10} \geq 100$. *Then, the polymatroid gets updated to*

$$\{x \in \mathbb{R}_+^E \colon x_a \leq 40, x_b \leq 50, x_c \leq 50, x_d \leq 30, x_a + x_b + x_c + x_d \leq 80, x_a + x_c \leq 60\}$$

and the relevant parts of the concave valuation functions at price 10 *are depicted in the following table:*

	$[0,10]$	$(10,20]$	$(20,30]$	$(30,40]$	$(40,50]$	Allocated	Charged
$v_a^1(x+20)$	$\mathbf{50} \cdot x$	$50 \cdot x$	$\mathbf{25} \cdot x$	$25 \cdot x$		20	200
$v_b^1(x)$	$\mathbf{30} \cdot x$	$\mathbf{15} \cdot x$	$15 \cdot x$	$15 \cdot x$			
$v_c^2(x)$	$\mathbf{40} \cdot x$	$40 \cdot x$	$40 \cdot x$	$40 \cdot x$	$40 \cdot x$		
$v_d^2(x)$	$\mathbf{25} \cdot x$	$25 \cdot x$	$25 \cdot x$				

The price increases to 15 *and bidder* 1 *updates the demand* $d_b^{20} = 10$*, hence* $d_a^{20} + d_b^{20} = 50$ *and bidder* 2 *gets allocated* 20 *units of item c and* 10 *units of item d (He cannot get* 30 *units of item c since* $d_a^{20} = 40$ *and* $x_a + x_c \leq 60$*). Then, the polymatroid gets updated to*

$$\{x_a \leq 40, x_b \leq 10, x_c \leq 30, x_d \leq 20, x_a + x_b + x_c + x_d \leq 50, x_a + x_c \leq 40\}$$

and the relevant parts of the concave valuation functions at price 15 *are depicted in the following table:*

	$[0,10]$	$(10,20]$	$(20,30]$	$(30,40]$	Allocated	Charged
$v_a^1(x+20)$	$\mathbf{50} \cdot x$	$50 \cdot x$	$\mathbf{25} \cdot x$	$25 \cdot x$	20	200
$v_b^1(x)$	$\mathbf{30} \cdot x$					
$v_c^2(x+20)$	$\mathbf{40} \cdot x$	$40 \cdot x$	$40 \cdot x$		20	400
$v_d^2(x+10)$	$\mathbf{25} \cdot x$	$25 \cdot x$			10	200

Next, the price increases to 25 *and bidder* 1 *updates the demand* $d_a^{25} = 20$ *and bidder* 2 *updates* $d_d^{25} = 0$ *and bidder* 1 *gets allocated* 10 *items of a and* 10 *items of b (he cannot get allocated* 20 *items of a since* $d_c^{25} = 30$ *and* $x_a + x_c \leq 40$*) and bidder* 2 *gets allocated* 20 *items of c. Then, the polymatroid gets updated to*

$$\{x_a \leq 10, x_b = 0, x_c \leq 10, x_d = 0, x_a + x_c \leq 10\}$$

and the relevant parts of the concave valuation functions at price 25 *are depicted in the following table:*

	$[0,10]$	Allocated	Charged
$v_a^1(x+30)$	$\mathbf{50} \cdot x$	30	450
$v_b^1(x+10)$		10	250
$v_c^2(x+40)$	$\mathbf{40} \cdot x$	40	900
$v_d^2(x+10)$		10	200

Next, the price increases to 40 *and bidder* 2 *updates the demand* $d_c^{40} = 0$ *and bidder* 1 *gets allocated* 10 *items of a for a cost of* 400.

The auction terminates with the allocation $(x_a = 40, x_b = 10, x_c = 40, x_d = 10)$ *for a social welfare of* $v_a^1(40) + v_b^1(10) + v_c^2(40) + v_d^2(10) = 2300 + 1850 = 4150$ *and payments* $c_1 = 1100$ *and* $c_2 = 1085$*. This are exactly the VCG payments, since* $\max_{x_a+x_b=100} v_a^1(x_a) + v_b^1(x_b) = 40 \cdot 50 + 10 \cdot 30 + 20 \cdot 25 + 30 \cdot 15 = 3385$ *and* $3385 - 2300 = 1085$ *what equals the loss suffered by bidder* 1 *and* $\max_{x_c+x_d=100} v_c^2(x_c) + v_d^2(x_d) = 50 \cdot 40 + 30 \cdot 25 + 20 \cdot 10 = 2950$ *and* $2950 - 1850 = 1100$ *what equals the loss suffered by bidder* 2*.*

2.4. Applications

Clearly, Auction 3 applies to various matroidal settings considered in the literature, e.g., scheduling matroids (see Demange et al. [1986]), transversal matroids for kidney exchange (see Roth et al. [2005]) or graphical matroids (see Bikhchandani et al. [2011]). However, in all of these examples even the simpler Auctions 1 and 2 apply and to highlight the impact of Auction 3 we provide some genuine polymatroid applications.

Auction 3 applies to submodular functions defined as the cut functions of a directed graph: Consider the setting of a directed graph $G = (V \sqcup \{s\}, A)$ where the bidders can be identified with disjoint subsets of V and the seller is presented by the node s. The utility of bidder $j \in N$ is the sum of concave functions of the incoming flow x_e into each of the agents' nodes. Every edge has a capacity and an allocation $x \in \mathbb{R}_+^V$ is feasible iff for all $S \subseteq V$ it holds that $\sum_{v \in S} x_v \leq f(S)$ where $f(S)$ denotes the minimal cut from s to S. It is well known that f is submodular and therefore the set of feasible allocations is a polymatroid. Thus, Auction 3 applies to this setting, and following Bikhchandani et al. [2011], as a practical application one can interpret s as a company that provides video streams and every bidder represents a group of friends that shares an account and wants to stream movies provided by the company. In this setting, the allocation x represents the transmitted data, hence for $v \in V$ the incoming data is x_v.

A more sophisticated polymatroid application arises from electricity markets. Following the increase of the global gas price since March 2022 there are ongoing discussions about the inefficiency of the German electricity *day-ahead spot market prices.* A daily day-ahead auction to optimize energy production is held, hence, to determine which power plants produce which amount at which price.

One of the main issues of the current auction format is that all producers get payed the same price although in Northern Germany there is more energy production than consumption while in Southern Germany there is more consumption than production and the constrained capacity of the high voltage grid to transport the northern surplus to the south is not taken into account. Simplified, one can represent this particular situation as a network flow problem on the digraph $(V \sqcup \{n, s\}, A)$ of which the set of feasible flows turns out to be a polymatroid: There is one node $n \in V$ for consumption in the north and one node

$s \in V$ for consumption in the south and for every energy producer (bidder) there is a node $v \in V$. Bidders $V = V_n \sqcup V_s$ are partitioned regarding their location, in which V_n is the set of producers that are located in the north and V_s is the set of producers that are located in the south. There is an arc (v, n) if producer $v \in V_n$ and an arc (v, s) if $v \in V_s$ Further, there is an arc (n, s) with capacity $c_{(n,s)}$, which is the physical transportation capacity of the high voltage cables connecting Northern Germany and Southern Germany. For each time interval consumers in the north and south announce their demand, which is aggregated to the respective consumption and corresponds to the node capacities $c(n)$ and $c(s)$ of the nodes n and s. Then, the set of feasible allocations is the set of feasible flows from the producers to s and n and from n to s that meet the node capacities $c(n)$ and $c(s)$. It turns out that the set of feasible flows is a polymatroid. To see this, notice that the set of feasible flows is constrained by the normalized nondecreasing submodular function f, given by

$$f\colon A \to \mathbb{Z}_+, f(S) = \begin{cases} 0 \text{ if } S = \emptyset, \\ c(n) + c_{(n,s)} \text{ if } \emptyset \neq S \subseteq V_n, \\ c(n) + c(s) \text{ if } S \cap V_n \neq \emptyset, S \cap V_s \neq \emptyset, \\ c(s) \text{ if } \emptyset \neq S \subseteq V_s. \end{cases}$$

To determine an efficient allocation one can apply our auction as follows: One starts with a negative price at which buyers submit their potential production. Notice that in this setting buyers purchase the duty to feed in energy for a in general negative price, hence they get payed. Then, if the starting price is low enough, potential production exceeds demand and increasing the price step by step yields reduction of the potential production and results in an efficient production.

2.5. Conclusion

We presented a stronger polynomial time ascending auction for selling elements of an integer base polyhedron. Our auction is incentive compatible and implements a social welfare maximizing allocation if bidders behave truthfully.

Future research may extend from integer polymatroids to integer packing instances: Generally, applying our algorithm to packing instances may come at the cost of loosing the ex post incentive compatibility and truthful bidding also may result in a non optimal allocation.

However, if the rank quotient of the integer packing instance is close to one then our auction may be incentive compatible for basically all valuations and the output by the auction is approximately optimal. In particular, consider a more elaborated model of the electricity spot market. Then the set of feasible flows of the underlying network should be an integer packing instance that is nearly a polymatroid. Thus, it seems worthy to investigate this model from a theoretical and practical perspective.

Chapter 3.

On Inner Independence Systems

3.1. Introduction

In the first part of this chapter we examine the problem of determining a maximum weight basis of an independence system. The problem is of course $\mathbb{NP}$-hard and it is common to resort to heuristics to solve it (or any of its special cases). The most well known heuristic is the greedy algorithm, in which elements of the ground set with nonnegative weight are selected in order of declining weight as long they form an independent set. A classic result of Korte and Hausmann [1978] and Jenkyns [1976] bounds the value of the greedy solution relative to the optimal value in terms of the rank quotient of the underlying independence system.

We examine an alternative approximation approach. The idea is to find an 'inner approximation' of the feasible region of the underlying optimization problem and optimize over that. Milgrom [2017] and de Vries and Vohra [2020] examine this approach in the context of independence systems. Specifically, find an inner matroid that approximates the underlying independence system well. An inner matroid is a matroid with the property every one of its basis sets is contained in a basis of the original independence system (but not conversely). Then, apply the greedy algorithm to the inner matroid. It is well known that this will recover the maximum weight basis of the inner matroid.

In this chapter, we propose to approximate the given independence system by an inner independence system (defined in a similar way to inner matroid). Then, apply the greedy algorithm to the inner independence system. At first blush, this appears silly. Surely, the best inner independence system would be the given independence system itself? Second, the greedy algorithm applied to the inner independence system is itself suboptimal. Hence, one is introducing two sources of approximation error. First from the inner independence system, and second, from the greedy solution itself. Surprisingly, this is not the case. We show that an approximation by inner independence systems may simultaneously outperform the approximation by inner matroids as well as direct application of the greedy algorithm regarding a worst-case objective.

The basic intuition is this. The greedy algorithm is suboptimal because it gets ‘stuck’ in a low rank basis. An inner independence system can be obtained from the given independence system by deleting some of the low rank basis, i.e., removing potential *local* optima.

The second part of this chapter extends the inner approximation idea to more general combinatorial objects than independence systems, namely *packing instances*. To motivate this generalization represent the independent sets of the independence system by their characteristic vector. The set of these binary vectors forms an independence system if and only they are downward closed. Specifically, if x is in the set, then any binary vector $y \leq x$ is also in the set. The maximal vectors in this set correspond to the basis sets of the corresponding independence system. Packing problems correspond to collection of vectors in $\mathbb{Z}_+^E$ that are downward closed. We consider packing problems where the objective function is separable discrete concave and show that the worst-case approximation guarantee of the greedy algorithm can be outperformed by an approximation with inner polymatroids and by inner integer packing instances.

In the next section we describe the greedy algorithm for independence systems and present the approximation guarantee of inner approximation by independence systems. The subsequent section covers packing problems.

3.2. Greedy Algorithm and Inner Approximation by Independence Systems

Let $(E, \mathcal{I})$ a normal independence system with weights $v_i \in \mathbb{R}_+$ for $i \in E$. Then, the problem of finding a maximum weight basis of $(E, \mathcal{I})$ is $\max_{I \in \mathcal{I}} v(I) := \max_{I \in \mathcal{I}} \sum_{i \in I} v_i$. A standard approach to solve the problem $V^*(\mathcal{I}, v) := \max_{I \in \mathcal{I}} v(I)$ is the greedy algorithm:

Algorithm 6: Greedy algorithm for independence systems

Input : An independence system $(E, \mathcal{I})$ given by an independence oracle. Weights $v_i \in \mathbb{R}_+$

Output: A basis $I_g \in \mathcal{I}$

Order the elements of E in nondecreasing order such that $v_1 \geq v_2 \geq \cdots \geq v_E$

Set $I = \emptyset$

for $i \leftarrow 1$ **to** E **do**

  if $I \cup \{i\} \in \mathcal{I}$ *(oracle call)* **then**

    $I = I \cup \{i\}$

Set $I_g = I$ and return

Let $V^{greedy}(\mathcal{I}, v)$ denote the value of the *worst* possible (if ties occurs) greedy solution. A common approach to evaluate the quality of an algorithm is to compare the value of its output with the optimal value in the worst case, here $\min_{v \in \mathbb{R}_+^E \setminus \{\mathbf{0}\}} \frac{V^{greedy}(\mathcal{I}, v)}{V^*(\mathcal{I}, v)}$, if $V^*(\mathcal{I}, v) >$

0.

Korte and Hausmann [1978] and Jenkyns [1976] showed

$$\min_{v \in \mathbb{R}_+^E \setminus \{\mathbf{0}\}} \frac{V^{greedy}(\mathcal{I}, v)}{V^*(\mathcal{I}, v)} = q(\mathcal{I}) \text{ for normal independence systems } (E, \mathcal{I}).$$

Motivated by Milgrom [2017], we incorporate exogenously given prior information about the set of likely candidates for an optimal basis, denoted $\mathcal{O} \subseteq \mathcal{I}$.

Definition 3.1. *We call $\mathcal{O}$ the* ***acceptable set****. If $\mathcal{O} = \mathcal{I}$ we call this the* ***zero prior knowledge*** *case.*

In contrast to Milgrom [2017] we do not require $\mathcal{O}$ to be downward closed but will point out where it becomes necessary.

For any acceptable set $\mathcal{O}$ we define $\mathcal{O}^{\subseteq} := \{I \subseteq O \colon O \in \mathcal{O}\}$. The optimal value with respect to the acceptable set $\mathcal{O}$ is denoted by $V^*(\mathcal{O}, v) := \max_{I \in \mathcal{O}} v(I)$. We consider an independence system $(E, \mathcal{J})$ inside the independence system $(E, \mathcal{I})$ (but not necessarily containing all elements of $\mathcal{O}$) and apply the greedy algorithm to find an optimal weight basis in $\mathcal{J}$. We call an independence system $(E, \mathcal{J})$ an **inner independence system** of $(E, \mathcal{I})$, if $\mathcal{J} \subseteq \mathcal{I}$ and propose that the greedy solution to $(E, \mathcal{J})$ be used as a solution to the problem of finding a maximum weight basis in $\mathcal{O}$.

We want to evaluate the quality of the greedy solution of the inner independence system to $V^*(\mathcal{O}, v)$.

Definition 3.2. *In the following, we abbreviate the condition ($S \subseteq E, S \neq \emptyset$) by $S \sqsubseteq E$.*

Assumption 3.3. *For an independence system $(E, \mathcal{I})$, an inner independence system $(E, \mathcal{J})$ and an acceptable set $\mathcal{O}$ we assume that $\mathcal{O}^{\subseteq}$ is normal, hence for every $i \in E$ there exists $O \in \mathcal{O}$ with $i \in O$.*

Definition 3.4. *We define the* ***generalized rank-quotient*** *as*

$$\omega(\mathcal{I}, \mathcal{O}, \mathcal{J}) := \min_{S \sqsubseteq E} \frac{l_{\mathcal{J}}(S)}{\max\{|O| \colon O \subseteq S, O \in \mathcal{O}^{\subseteq}\}} = \min_{S \sqsubseteq E} \frac{l_{\mathcal{J}}(S)}{r_{\mathcal{O}^{\subseteq}}(S)}.$$

The normality assumption of $\mathcal{O}^{\subseteq}$ is needed to ensure that $\omega(\mathcal{I}, \mathcal{O}, \mathcal{J})$ is well defined. Otherwise, there exists $i \in E$ with $\{i\} \notin \mathcal{O}^{\subseteq}$ such that $\frac{l_{\mathcal{J}}(\{i\})}{\max\{|O| \colon O \subseteq \{i\}, O \in \mathcal{O}^{\subseteq}\}}$ is undefined.

The definition of the generalized rank-quotient is similar to the rank-quotient and depends only indirectly on $\mathcal{I}$. We provide a bound on the quality of the solution $V^{greedy}(\mathcal{J}, v)$ in terms of $\omega(\mathcal{I}, \mathcal{O}, \mathcal{J})$

Theorem 3.5. *Let $(E,\mathcal{I})$ be an independence system, $(E,\mathcal{J})$ an inner independence system of $(E,\mathcal{I})$ and $\mathcal{O}$ the acceptable set. Then,*

$$\min_{v\in\mathbb{R}_+^E\setminus\{\mathbf{0}\}} \frac{V^{greedy}(\mathcal{J},v)}{V^*(\mathcal{O},v)} = \omega(\mathcal{I},\mathcal{O},\mathcal{J}).$$

Proof. Let $S^* \in \arg\min_{S\subseteq E} \frac{l_{\mathcal{J}}(S)}{r_{\mathcal{O}^{\subseteq}}(S)} = \omega(\mathcal{I},\mathcal{O},\mathcal{J})$. Then, with $v = \mathbb{1}_{S^*}$ holds $l_{\mathcal{J}}(S^*) = V^{\text{greedy}}(\mathcal{J},\mathbb{1}_{S^*})$ and $r_{\mathcal{O}^{\subseteq}}(S^*) = V^*(\mathcal{O},\mathbb{1}_{S^*})$ and $\min_{v\in\mathbb{R}_+^E\setminus\{\mathbf{0}\}} \frac{V^{greedy}(\mathcal{J},v)}{V^*(\mathcal{O},v)} \leq \omega(\mathcal{I},\mathcal{O},\mathcal{J})$.

To prove $\min_{v\in\mathbb{R}_+^E\setminus\{0\}} \frac{V^{greedy}(\mathcal{J},v)}{V^*(\mathcal{O},v)} \geq \min_{S\subseteq E} \frac{l_{\mathcal{J}}(S)}{r_{\mathcal{O}^{\subseteq}}(S)}$ we argue similar to Korte and Hausmann [1978]. Let $v \in \mathbb{R}_+^E \setminus \{\mathbf{0}\}$ and X_g a worst possible greedy solution for $(E,\mathcal{J})$ and O the optimal solution for $\mathcal{O}$. Let E be ordered such that $v_1 \geq v_2 \geq \cdots \geq v_n \geq v_{n+1} := 0$. Define $E_i := \{1,\ldots,i\}$. Then, for $F \in \mathcal{I}$ we can rewrite

$$v(F) = \sum_{i=1}^{n} |F \cap E_i|(v_i - v_{i+1}). \tag{3.1}$$

Note that $O \cap E_i \subseteq O$ and therefore $O \cap E_i \in \mathcal{O}^{\subseteq}$, hence $r_{\mathcal{O}^{\subseteq}}(E_i) \geq |O \cap E_i|$ for all $i \in E$. Furthermore, $X_g \cap E_i$ is a basis of E_i for $\mathcal{J}$ for all $i \in E$ since it is generated by the greedy algorithm. That implies $|X_g \cap E_i| \geq l_{\mathcal{J}}(E_i)$ for all $i \in E$. It follows

$$\begin{aligned}
\frac{v(X_g)}{v(O)} &= \frac{\sum_{i=1}^n |X_g \cap E_i| \cdot (v_i - v_{i+1})}{\sum_{i=1}^n |O \cap E_i| \cdot (v_i - v_{i+1})} \\
&\geq \frac{\sum_{i=1}^n l_{\mathcal{J}}(E_i) \cdot \frac{|O\cap E_i|}{r_{\mathcal{O}^{\subseteq}}(E_i)} \cdot (v_i - v_{i+1})}{\sum_{i=1}^n |O \cap E_i| \cdot (v_i - v_{i+1})} = \frac{\sum_{i=1}^n |O \cap E_i| \cdot (v_i - v_{i+1}) \cdot \frac{l_{\mathcal{J}}(E_i)}{r_{\mathcal{O}^{\subseteq}}(E_i)}}{\sum_{i=1}^n |O \cap E_i| \cdot (v_i - v_{i+1})} \\
&\geq \frac{\sum_{i=1}^n |O \cap E_i| \cdot (v_i - v_{i+1}) \cdot \min_{S\subseteq E} \frac{l_{\mathcal{J}}(S)}{r_{\mathcal{O}^{\subseteq}}(S)}}{\sum_{i=1}^n |O \cap E_i| \cdot (v_i - v_{i+1})} \\
&= \frac{\sum_{i=1}^n |O \cap E_i| \cdot (v_i - v_{i+1})}{\sum_{i=1}^n |O \cap E_i| \cdot (v_i - v_{i+1})} \cdot \min_{S\subseteq E} \frac{l_{\mathcal{J}}(S)}{r_{\mathcal{O}^{\subseteq}}(S)} \\
&= \min_{S\subseteq E} \frac{l_{\mathcal{J}}(S)}{r_{\mathcal{O}^{\subseteq}}(S)}.
\end{aligned}$$

□

The classic result of Hausmann et al. [1980] is a special case of Theorem 3.5 for $\mathcal{O} = \mathcal{I}$:

Corollary 3.6. *For an independence system $(E,\mathcal{I})$ it holds $\omega(\mathcal{I},\mathcal{I},\mathcal{I}) = q(\mathcal{I})$.*

Proof. It holds $\omega(\mathcal{I},\mathcal{I},\mathcal{I}) = \min_{S\subseteq E} \frac{l_{\mathcal{I}}(S)}{\max\{|O|:\, O\subseteq S, O\in\mathcal{I}\}} = \min_{S\subseteq E} \frac{l_{\mathcal{I}}(S)}{r_{\mathcal{I}}(S)} = q(\mathcal{I})$. □

Definition 3.7. *Call $(E,\mathcal{J}^*)$ a **best inner independence system** of an independence system $(E,\mathcal{I})$ and acceptable set $\mathcal{O}$ if*

$$(E,\mathcal{J}^*) \in \underset{(E,\mathcal{J}) \text{ is independence system}, \mathcal{J}\subseteq\mathcal{I}}{\arg\max} \omega(\mathcal{I},\mathcal{O},\mathcal{J}).$$

We give an example which shows that an inner independence system approximation can outperform the direct application of the greedy algorithm regarding a worst-case objective.

Example 3.8. *Consider the independence system* $(E, \mathcal{I})$ *with* $E = \{a, b\} \cup C$ *with* $C := \{c_1, ..., c_{10}\}$ *defined via its set of bases* $\mathcal{B}_{\mathcal{I}} = \{\{a, b\}, C\} \cup (\{\{a\}\} \bigoplus \binom{C}{5}) \cup (\{\{b\}\} \bigoplus \binom{C}{5})$ *whilst* $\binom{C}{5} := \{S \subseteq C \colon |S| = 5\}$*. Let* $\mathcal{O} = \mathcal{I}$*, so that the acceptable set coincides with the set of all independent sets. The ground set has a basis* C *of cardinality* 10 *and* $\{a, b\}$ *is a low rank basis of cardinality* 2*, therefore,* $q(I) \leq \frac{1}{5}$*. We show for* $\mathcal{J} := \mathcal{I} \setminus \{\{a, b\}\}$ *that* $\omega(\mathcal{I}, \mathcal{O}, \mathcal{J}) = \frac{1}{2}$ which yields a better worst case approximation guarantee than $q(\mathcal{I})$*. It is*

$$\mathcal{B}_{\mathcal{J}} = \{C\} \cup (\{\{a\}\} \bigoplus \binom{C}{5}) \cup (\{\{b\}\} \bigoplus \binom{C}{5}).$$

For $\{a, b\}$ *it is* $\frac{l_{\mathcal{J}}(\{a,b\})}{r_{\mathcal{I}}(\{a,b\})} = \frac{1}{2}$ *since* $\{a\}, \{b\} \in \mathcal{J} \not\ni \{a, b\}$*. Now, let* $S \subseteq E$*,* $S_1 = S \cap \{a, b\}$ *and* $S_2 = S \cap C$*. If* $S_1 = \emptyset$ *then* $\mathcal{B}_{\mathcal{J}}(S) = \{S_2\}$ *and therefore* $\frac{l_{\mathcal{J}}(S)}{r_{\mathcal{I}}(S)} = 1$*. If* $S_1 = \{a\}$ *(note that this is equivalent to* $S_1 = \{b\}$*) and* $|S_2| \leq 5$ *it is* $\mathcal{B}_{\mathcal{J}}(S) = \{\{a\}\} \bigoplus \{S_2\}$ *and therefore* $\frac{l_{\mathcal{J}}(S)}{r_{\mathcal{I}}(S)} = 1$ *and if* $S_1 = \{a\}$ *and* $|S_2| > 5$ *it is* $\mathcal{B}_{\mathcal{J}}(S) = (\{\{a\}\} \bigoplus \binom{C}{5}) \cup \{S_2\}$ *and therefore* $\frac{l_{\mathcal{J}}(S)}{r_{\mathcal{I}}(S)} \geq \frac{5}{r_{\mathcal{I}}(S)} \geq \frac{5}{10} = \frac{1}{2}$*. It remains the case that* $S_1 = \{a, b\}$*. Then, if* $|S_2| < 5$ *it is* $\mathcal{B}_{\mathcal{J}}(S) = (\{\{a\}\} \bigoplus \{S_2\}) \cup (\{\{b\}\} \bigoplus \{S_2\})$ *and therefore* $\frac{l_{\mathcal{J}}(S)}{r_{\mathcal{I}}(S)} \geq \frac{1+|S_2|}{2+|S_2|} \geq \frac{1}{2}$*. If* $|S_2| \geq 5$ *it is* $B_{\mathcal{J}}(S) = (\{\{a\}\} \bigoplus \binom{C}{5}) \cup (\{\{b\}\} \bigoplus \binom{C}{5}) \cup \{S_2\}$ *and therefore* $\frac{l_{\mathcal{J}}(S)}{r_{\mathcal{I}}(S)} \geq \frac{5}{10} = \frac{1}{2}$*. In total, it follows*

$$\omega(\mathcal{I}, \mathcal{I}, \mathcal{J}) = \frac{1}{2} > q(\mathcal{I}) = \frac{1}{5}.$$

It is obvious that $(E, \mathcal{J})$ *is a best inner independence system: For any inner independence system* $(E, \mathcal{J}')$ *with* $\{a, b\} \in \mathcal{J}'$ *holds* $\omega(\mathcal{I}, \mathcal{I}, \mathcal{J}') \leq \frac{l_{\mathcal{J}'}(E)}{r_{\mathcal{I}}(E)} = \frac{2}{10}$ *and any independence system* $\mathcal{J}' \subset \mathcal{J}$ *cannot improve the approximation guarantee since* $\omega(\mathcal{I}, \mathcal{I}, \mathcal{J}') \leq \frac{l_{\mathcal{J}'}(\{a,b\})}{r_{\mathcal{I}}(\{a,b\})} \leq \frac{1}{2}$.

One can extend Example 3.8 to show that there exist cases where the inner independence system approximation yields an arbitrary better worst-case approximation guarantee than the original independence system:

Example 3.9. *For* $i \in \mathbb{N}$ *define the independence system* $(E^{2i}, \mathcal{I}^{2i})$ *with* $E^{2i} := \{a, b\} \cup C^{2i}$ *and* $C^{2i} := \{c_1, ..., c_{2i}\}$ *via its set of bases* $\mathcal{B}_{\mathcal{I}} = \{\{a, b\}, C^{2i}\} \cup (\{\{a\}\} \bigoplus \binom{C^{2i}}{i}) \cup (\{\{b\}\} \bigoplus \binom{C^{2i}}{i})$*. Let* $\mathcal{O} = \mathcal{I}$ *and* $\mathcal{J}^{2i} := \mathcal{I}^{2i} \setminus \{\{a, b\}\}$*. An analogous argument as in Example 3.8 shows that* $q(\mathcal{I}^{2i}) = \frac{2}{2i}$ *and* $\omega(\mathcal{I}^{2i}, \mathcal{I}^{2i}, \mathcal{J}^{2i}) = \frac{1}{2}$ *and therefore* $\lim_{i \to \infty} \frac{\omega(\mathcal{I}^{2i}, \mathcal{I}^{2i}, \mathcal{J}^{2i})}{q(\mathcal{I}^{2i})} = \lim_{i \to \infty} \frac{i}{2} \to \infty$.

3.3. Comparison to Inner Matroid Approximation

We compare our result with the inner matroid approximation proposed by Milgrom [2017] and de Vries and Vohra [2020]. Milgrom [2017] considers an **inner matroid** $(E, \mathcal{M})$ of the independence system $(E, \mathcal{I})$ with $\mathcal{M} \subseteq \mathcal{I}$ and proposes the greedy solution to $(E, \mathcal{M})$ be used as a solution to the problem of finding a maximum weight basis in the downward closed acceptable set $\mathcal{O}^{\subseteq} \subseteq \mathcal{I}$. He calls the term $\min_{S \in \mathcal{O}^{\subseteq} \setminus \{\emptyset\}} \frac{r_{\mathcal{M}}(S)}{|S|} =: \rho(\mathcal{I}, \mathcal{O}^{\subseteq}, \mathcal{M})$ the **substitutability index** of $\mathcal{M}$ for $\mathcal{I}$ with respect to the acceptable set $\mathcal{O}^{\subseteq} \subseteq \mathcal{I}$. Note that $\min_{S \in \mathcal{O}^{\subseteq} \setminus \{\emptyset\}} \frac{r_{\mathcal{M}}(S)}{|S|} = \min_{S \in \mathcal{O} \setminus \{\emptyset\}} \max_{M \in \mathcal{M}, M \subseteq S} \frac{|M|}{|S|}$. Consequently, he defines the **best approximating matroid** by

$$(E, \mathcal{M}^*) \in \underset{(E,\mathcal{M}) \text{ is matroid, } \mathcal{M} \subseteq \mathcal{I}}{\arg\max} \; \min_{S \in \mathcal{O}^{\subseteq} \setminus \{\emptyset\}} \; \max_{M \in \mathcal{M}, M \subseteq S} \frac{|M|}{|S|}.$$

We show that the definition of substitutability index is unsuitable when applied to inner independence system instead of inner matroids.

Example 3.10. *Let $N := \{1, 2, 3\}$ be an independence system $(E, \mathcal{I})$ whose bases are $\mathcal{B}_{\mathcal{I}} := \{\{1, 2\}, \{3\}\}$, thus $q(\mathcal{I}) = \frac{1}{2}$. Let the inner independence system $(E, \mathcal{J})$ coincide with $(E, \mathcal{I})$ and the acceptable set is $\mathcal{O} = \mathcal{B}_{\mathcal{I}}$. Clearly, $\rho(\mathcal{I}, \mathcal{O}, \mathcal{J}) = 1$. The greedy algorithm, however, may fail to find an optimal basis. Assume $v_1 = v_2 = 1, v_3 = 1 + \varepsilon$. Clearly, $V^*(\mathcal{J}, v) = 2 = V^*(\mathcal{O}, v)$. The greedy algorithm carried out on $\mathcal{J}$ gets stuck with the low rank basis $\{3\}$ with $V^{greedy}(E, \mathcal{J}) = 1 + \varepsilon$ and therefore $\frac{V^{greedy}(E,\mathcal{J})}{V^*(\mathcal{O},v)} = \frac{1+\varepsilon}{2}$. Hence, the greedy algorithm performs worse than $\rho(\mathcal{I}, \mathcal{O}, \mathcal{J})$ suggests.*

However, the substitutability index of $\mathcal{M}$ coincides with the generalized rank-quotient of $\mathcal{M}$ in the case of $(E, \mathcal{M})$ being an inner matroid and $(E, \mathcal{O})$ a normal independence system.

Theorem 3.11. *Let $(E, \mathcal{M})$ an inner matroid of the independence system $(E, \mathcal{I})$ and $\mathcal{O} \subseteq \mathcal{I}$ an acceptable set such that $(E, \mathcal{O})$ is a normal independence system. Then, it holds $\omega(\mathcal{I}, \mathcal{O}, \mathcal{M}) = \rho(\mathcal{I}, \mathcal{O}, \mathcal{M})$.*

Proof. Clearly, for $S \in \mathcal{O} \setminus \{\emptyset\}$ it holds $\frac{l_{\mathcal{M}}(S)}{|r_{\mathcal{O}^{\subseteq}}(S)|} = \frac{r_{\mathcal{M}}(S)}{|S|}$ and therefore $\omega(\mathcal{I}, \mathcal{O}, \mathcal{M}) \geq \rho(\mathcal{I}, \mathcal{O}, \mathcal{M})$. Suppose $S' \in \arg\min_{S \sqsubseteq E} \frac{l_{\mathcal{M}}(S)}{r_{\mathcal{O}^{\subseteq}}(S')}$ but $S' \notin \mathcal{O}^{\subseteq}$. Let F be upper rank basis of S' for $\mathcal{O}^{\subseteq}$, then $F \subset S'$ and $r_{\mathcal{O}^{\subseteq}}(S') = r_{\mathcal{O}^{\subseteq}}(F)$. But since $(E, \mathcal{M})$ is a matroid it holds $l_{\mathcal{M}}(F) = r_{\mathcal{M}}(F) \leq r_{\mathcal{M}}(S') = l_{\mathcal{M}}(S')$ and therefore $\frac{l_{\mathcal{M}}(F)}{r_{\mathcal{O}^{\subseteq}}(F)} \leq \frac{l_{\mathcal{M}}(S')}{r_{\mathcal{O}^{\subseteq}}(S')}$, hence $F \in \arg\min_{S \sqsubseteq E} \frac{l_{\mathcal{M}}(S)}{r_{\mathcal{O}^{\subseteq}}(S)}$ and it holds $\omega(\mathcal{I}, \mathcal{O}, \mathcal{M}) \leq \rho(\mathcal{I}, \mathcal{O}, \mathcal{M})$ and the claim follows. □

As reported in Milgrom [2017] and de Vries and Vohra [2020], the substitutability index can be used to bound the approximation quality of an inner matroid $(E, \mathcal{M})$ if $\mathcal{O}$ is a normal independence system.

Corollary 3.12. *[de Vries and Vohra, 2020, Milgrom, 2017] Let $(E, \mathcal{M})$ an inner matroid of the independence system $(E, \mathcal{I})$. If $\mathcal{O}$ is a normal independence system then*

$$\min_{v \in \mathbb{R}_+^E \setminus \{\mathbf{0}\}} \frac{V^*(\mathcal{M}, v)}{V^*(\mathcal{O}, v)} = \min_{S \in \mathcal{O} \setminus \{\emptyset\}} \frac{r_{\mathcal{M}}(S)}{|S|}.$$

Proof. Follows directly from Theorems 3.5 and 3.11. □

Remark 3.13. *Note that in de Vries and Vohra [2020], the condition that $\mathcal{O}$ is an independence system is omitted. The following explains, why the condition is necessary.*

Let $E = \{1,2,3,4,5\}$ and $\mathcal{I} = 2^{\{1,2,3\}} \cup 2^{\{1,4,5\}}$ and $\mathcal{B}_{\mathcal{M}} = \{\{1,2\}\{1,3\},\{1,4\},\{1,5\}\}$ and $\mathcal{O} = \{\{1,2,3\},\{1,4,5\}\}$. It is easy to see that $\min_{S \in \mathcal{O} \setminus \{\emptyset\}} \frac{r_{\mathcal{M}}(S)}{|S|} = \frac{2}{3}$. However, setting $v_1 = v_2 = v_3 = 0$ and $v_4 = v_5 = \frac{1}{2}$ yields $V^*(\mathcal{M}, v) = \frac{1}{2}$ and $V^*(\mathcal{O}, v) = 1$ and therefore $\min_{v \in \in \mathbb{R}_+^E \setminus \{0\}} \frac{V^*(\mathcal{M}, v)}{V^*(\mathcal{O}, v)} \leq \frac{1}{2} \stackrel{!}{<} \frac{2}{3}$.

It is natural to ask if the inner independence system approximation outperforms the best inner matroid approximation. In Example 3.8, this is not the case, as the following shows.

Example 3.14. *Let $(E, \mathcal{I})$ defined as in Example 3.8 and $(E, \mathcal{M})$ defined by its bases $\mathcal{B}_{\mathcal{M}} := \binom{C}{6} \cup (\{\{a\}\} \bigoplus \binom{C}{5}) \cup (\{\{b\}\} \bigoplus \binom{C}{5})$. Verification of the basis exchange axiom (see Appendix) would confirm that $(E, \mathcal{M})$ is a matroid. An analogous argument as in Example 3.8 yields*

$$\omega(\mathcal{I}, \mathcal{I}, \mathcal{M}) = \frac{1}{2}.$$

In spite of this, there exist examples, even in the zero prior knowledge case $\mathcal{O} = \mathcal{I}$, where the approximation guarantee of the *best* inner independence system strictly dominates the approximation guarantee of the best inner matroid. First, we provide some intuition for why the later inequality might occur before giving an example in Theorem 3.15.

Suppose for some independence system $(E, \mathcal{I})$ we know a nonmatroidal best approximating independence system $(E, \mathcal{J})$. Assume now that we want to construct an inner matroid $(E, \mathcal{M})$ contained in $(E, \mathcal{J})$ with the same approximation guarantee. Recall that for any pair of rank function $r_{\mathcal{M}}$ and lower rank function $l_{\mathcal{M}}$ of a matroid it has to be that $l_{\mathcal{M}}(S) = r_{\mathcal{M}}(S)$ for every $S \subseteq E$. As $(E, \mathcal{I})$ is nonmatroidal there exists $S^* \subseteq E$ such that $l_{\mathcal{J}}(S^*) < r_{\mathcal{J}}(S^*)$ with F lower rank basis of S^* and G upper rank basis of S^*. Therefore, to construct an inner matroid from $(E, \mathcal{J})$ one either has to reduce the rank of F to eliminate the low rank basis F and make the newly independent subsets of F contained in a higher rank basis of S^*, or one has to reduce the rank of G to obtain $r_{\mathcal{M}}(S^*) = r_{\mathcal{M}}(F) = r_{\mathcal{M}}(G)$. Note that both ideas can be incorporated by adding circuits to $\mathcal{J}$. If we follow the idea to reduce the rank of F it might occur that $\frac{r'_{\mathcal{M}}(F')}{r_{\mathcal{O}\subseteq}(F')} < \min_{S \subseteq E} \frac{l_{\mathcal{J}}(S)}{r_{\mathcal{O}\subseteq}(S)}$ for some $F' \subseteq F$. Conversely, in the case that we reduce the rank of G it could happen that we implicitly reduce the rank of some $A \supset G$ such that $\frac{r'_{\mathcal{M}}(A)}{r_{\mathcal{O}\subseteq}(A)} < \min_{S \subseteq E} \frac{l_{\mathcal{J}}(S)}{r_{\mathcal{O}\subseteq}(S)}$.

This explanation tacitly assumes that the best inner matroid is contained in the best inner independence system. This need not be true because the best inner matroid could be a *superset* of *some* best inner independence system. To see this, consider the independence system $(E, \mathcal{I})$ and the best inner matroid $(E, \mathcal{M})$ defined as in Example 3.14. The inner independence system $(E, \mathcal{J})$ given via its bases $\mathcal{B}_{\mathcal{J}} := \mathcal{B}_{\mathcal{M}} \setminus \{\{c_1, c_2, c_3, c_4, c_5, c_6\}\}$ is a best inner independence system but $\mathcal{M} \supset \mathcal{J}$.

In the next theorem we exhibit an independence system $\mathcal{I}$ which is a best inner independence system which contains a best inner matroid.

Theorem 3.15. *There exists an independence system $(E, \mathcal{I})$ and an inner independence system $(E, \mathcal{J})$ such that $\omega(\mathcal{I}, \mathcal{I}, \mathcal{J}) > \omega(\mathcal{I}, \mathcal{I}, \mathcal{M})$ for every matroid $(E, \mathcal{M})$ contained in $(E, \mathcal{I})$.*

Proof. Let $N := \{1, 2, 3, 4, 5\}$ and the independence system $(E, \mathcal{I})$ defined via its bases $\mathcal{B}_{\mathcal{I}} := \{\{1, 2, 3\}, \{1, 4\}, \{2, 5\}\}$.

Claim 1: $q(\mathcal{I}) = \frac{1}{2}$.
Proof of claim: Since $\{l_{\mathcal{I}}(S) \colon S \sqsubseteq E\} = \{r_{\mathcal{I}}(S) \colon S \sqsubseteq E\} = \{1, 2, 3\}$ it has to hold $q(\mathcal{I}) \in \{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}\}$. It is $q(\mathcal{I}) \leq \frac{l(\{1,3,5\})}{r(\{1,3,5\})} = \frac{1}{2}$. If there exists $S \subseteq E$ with $r_{\mathcal{I}}(E) = 3$ it has to hold $E \supseteq \{1, 2, 3\}$. It is $l_{\mathcal{I}}(\{1, 2, 3\}) = 3$ and $l_{\mathcal{I}}(\{1, 2, 3, 4\}) = l_{\mathcal{I}}(\{1, 2, 3, 5\}) = l_{\mathcal{I}}(\{1, 2, 3, 4, 5\}) = 2$. Therefore, there can not exist $S \subseteq E$ with $\frac{l(E)}{r(E)} = \frac{1}{3}$ and it follows $q(\mathcal{I}) = \frac{1}{2}$. ∎

Claim 2: $\omega(\mathcal{I}, \mathcal{I}, \mathcal{J}) = \frac{1}{2}$ for a best inner independence system $(E, \mathcal{J})$.
Proof of claim: Since $\mathcal{I}$ is an independence system $\mathcal{J} := \mathcal{I}$ itself is an inner independence system with $\omega(\mathcal{I}, \mathcal{I}, \mathcal{J}) = \frac{1}{2}$. Assume there exists a better approximating inner independent system $(N, \mathcal{J}')$. Obviously, it either has to hold that the only basis of $\mathcal{J}'$ containing 4 is $\{4\}$ or $\{1, 4\}$. Either way, it is $\frac{l_{\mathcal{J}'}(\{2,4,5\})}{\max\{|O| \colon O \subseteq \{2,4,5\}, O \in \mathcal{I}\}} \leq \frac{1}{2}$ and therefore $\omega(\mathcal{I}, \mathcal{I}, \mathcal{J}') \leq \frac{1}{2}$. ∎

Claim 3: $\omega(\mathcal{I}, \mathcal{I}, \mathcal{M}) = \frac{1}{3}$ for a best inner matroid $(E, \mathcal{M})$.
Proof of claim: We argue that the uniform matroid U_5^1 is the best inner matroid. Clearly, a best inner matroid $(E, \mathcal{M})$ has to be contained in $(E, \mathcal{I})$. Consider $E = \{1, 4, 5\}$ and note that $\{5\}$ is a low rank basis of E in $(E, \mathcal{I})$. In order to construct $r_{\mathcal{M}}$ out of $r_{\mathcal{J}}$ we cannot reduce the rank of $\{5\}$ since this would make $\{5\}$ inapproximable. Therefore, $r_{\mathcal{M}}(\{1, 4, 5\}) = 1$, hence $r_{\mathcal{M}}(\{1, 4\}) = 1$. Since $\{1, 4\} \notin \mathcal{M}$ it must be that $r_{\mathcal{M}}(\{4\}) = 1$ because setting $r_{\mathcal{M}}(\{4\}) = 0$ would would make $\{4\}$ inapproximable. Now, since $\{2, 4\} \notin \mathcal{I}$ and $\{3, 4\} \notin \mathcal{I}$ it follows that $r_{\mathcal{M}}(\{1, 2, 3, 4\}) = r_{\mathcal{M}}(\{4\}) = 1$. Then, as $\{4, 5\} \notin \mathcal{I}$ it has to hold that $\{4\}$ is a basis of $\{1, 2, 3, 4, 5\}$ from which follows $\mathcal{M} = U_5^1$. This yields $\omega(\mathcal{I}, \mathcal{I}, \mathcal{M}) = \frac{l_{\mathcal{M}}(\{1,2,3,4,5\})}{r_{\mathcal{I}}\{1,2,3,4,5\},} = \frac{l_{\mathcal{M}}(\{1\})}{r_{\mathcal{I}}(\{1,2,3\})} = \frac{1}{3}$. ∎

In sum, we can conclude

$$\max_{\mathcal{M}\subseteq\mathcal{I},(E,\mathcal{M})\text{ matroid}} \omega(\mathcal{I},\mathcal{I},\mathcal{M}) = \frac{1}{3} \overset{!}{<} \frac{1}{2} = \max_{\mathcal{J}\subseteq\mathcal{I},(E,\mathcal{J})\text{ indpendence system}} \omega(\mathcal{I},\mathcal{I},\mathcal{J}). \qquad \square$$

Definition 3.16. *An independence system $(E,\mathcal{I})$ arises from a knapsack problem if there exists $W \in \mathbb{R}_+$ and $w\colon E \to \mathbb{R}_+$ such that $\mathcal{I} = \{I \subseteq E\colon \sum_{x\in I} w(x) \leq W\}$.*

We remark that the independence system in Theorem 3.15, unlike that in Example 3.8, can not arise from a knapsack problem.[1] In a knapsack instance, either $\{1,5\} \in \mathcal{I}$ or $\{2,4\} \in \mathcal{I}$ would be necessary, since for the weight function w it has to be that $w(1) \leq w(2)$ or $w(2) \leq w(1)$. Therefore, we conjecture a connection between knapsack like independence systems and the equivalence in the generalized rank-quotient of inner matroids and inner independence systems.

Conjecture 3.17. *For every knapsack instance the underlying independence system $(E,\mathcal{I})$ fulfills*

$$\max_{\mathcal{J}\subseteq\mathcal{I},(E,\mathcal{J})\text{ independence system}} \omega(\mathcal{I},\mathcal{I},\mathcal{J}) = \max_{\mathcal{M}\subseteq\mathcal{I},(E,\mathcal{M})\text{ matroid}} \omega(\mathcal{I},\mathcal{I},\mathcal{M}).$$

However, the converse of Conjecture 3.17 is false.

Example 3.18. *Consider the independence system $(E,\mathcal{I})$ on the ground set $E = \{1,2,3,4\}$ given by its bases $\mathcal{B}_\mathcal{I} := \{\{1,2\},\{3,4\}\}$. No knapsack instance can induce the independence system $(E,\mathcal{I})$: Assume w.l.o.g. that $w_1 \leq w_i$ for $i \in \{2,3,4\}$. Then, it has to hold that $\{1,3\},\{1,4\} \in \mathcal{B}_\mathcal{I}$ since $w_1+w_3 \leq w_3+w_4$ and $w_1+w_4 \leq w_3+w_4$. Nevertheless, it holds $\max_{\mathcal{J}\subseteq\mathcal{I},(E,\mathcal{J})\text{ is independence system}} \omega(\mathcal{I},\mathcal{I},\mathcal{J}) = q(\mathcal{I}) = \frac{1}{2} = \max_{\mathcal{M}\subseteq\mathcal{I},(E,\mathcal{M})\text{is matroid}} \omega(\mathcal{I},\mathcal{I},\mathcal{M})$ $= \omega(\mathcal{I},\mathcal{I},U_4^1)$.*

We have seen in Theorem 3.15 that the best inner independence system may outperform the best inner matroid, but the approximation guarantee of the best inner independence system does not improve over the standard greedy algorithm. Example 3.8 and Remark 3.14 show that the best inner independence system may have the same approximation guarantee as the best inner matroid but outperforms the standard greedy algorithm. Our goal is to provide an example where the best inner independence system outperforms not only the best inner matroid but the standard greedy algorithm as well.

Lemma 3.19. *Let $a,b,c,d \in \mathbb{R}^+$, then, $\frac{a}{b} \leq \frac{c}{d}$ if and only if $\frac{a}{b} \leq \frac{a+c}{b+d}$.*

Proof. $\frac{a}{b} \leq \frac{c}{d} \Leftrightarrow \frac{ad}{b} \leq c \Leftrightarrow \frac{ad}{b} + \frac{ab}{b} \leq c + a \Leftrightarrow \frac{a(d+b)}{b} \leq c + a \Leftrightarrow \frac{a}{b} \leq \frac{a+c}{b+d}$. $\square$

[1] To see that the independence system in Example 3.8 is a knapsack problem set $v_{c_i} = 1$ for all $i \in \{1,...,10\}$, $v_a = v_b = 5$ and the capacity to 10.

Lemma 3.20. *Let* $(E_1, \mathcal{I}_1)$ *and* $(E_2, \mathcal{I}_2)$ *be independence systems with* $E_1 \cap E_2 = \emptyset$ *and* $(E, \mathcal{I}) := (E_1 \dot\cup E_2, \mathcal{I}_1 \bigoplus \mathcal{I}_2)$*, where* $E_1 \dot\cup E_2$ *denotes the disjoint union of* E_1 *and* E_2*. Then, it holds* $q(\mathcal{I}) = \min\{q(\mathcal{I}_1), q(\mathcal{I}_2)\}$*. Furthermore, let* $(E_1, \mathcal{J}_1)$ *and* $(E_2, \mathcal{J}_2)$ *be inner independence systems (matroids) and* $\mathcal{O}_1, \mathcal{O}_2$ *acceptable sets. Then, for the inner independence system (matroid)* $(E, \mathcal{J}) := (E_1 \dot\cup E_2, \mathcal{J}_1 \bigoplus \mathcal{J}_2\}$ *and acceptable set* $\mathcal{O} := \mathcal{O}_1 \dot\cup \mathcal{O}_2$ *we have* $\omega(\mathcal{I}, \mathcal{O}, \mathcal{J}) = \min\{\omega(\mathcal{I}_1, \mathcal{O}_1, \mathcal{J}_1), \omega(\mathcal{I}_2, \mathcal{O}_2, \mathcal{J}_2)\}$*.*

Proof. Let $S \in \arg\min_{T \subseteq E} \frac{l_{\mathcal{I}}(T)}{r_{\mathcal{I}}(T)}$ with $S_1 = S \cap E_1, S_2 = S \cap E_2$ and w.l.o.g. let $q(\mathcal{I}_1) \leq q(\mathcal{I}_2)$. It follows by Lemma 3.19 that

$$q(\mathcal{I}) = \frac{l_{\mathcal{I}}(S)}{r_{\mathcal{I}}(S)} = \frac{l_{\mathcal{I}_1}(S_1) + l_{\mathcal{I}_2}(S_2)}{r_{\mathcal{I}_1}(S_1) + r_{\mathcal{I}_2}(S_2)} \geq \min\left\{\frac{l_{\mathcal{I}_1}(S_1)}{r_{\mathcal{I}_1}(S_1)}, \frac{l_{\mathcal{I}_2}(S_2)}{r_{\mathcal{I}_2}(S_2)}\right\} \geq \min\{q(\mathcal{I}_1), q(\mathcal{I}_2)\}$$

and since clearly $q(\mathcal{I}) \leq \min\{q(\mathcal{I}_1), q(\mathcal{I}_2)\}$ it follows that $q(\mathcal{I}) = \min\{q(\mathcal{I}_1), q(\mathcal{I}_2)\}$.

Analogously, let $F \in \arg\min_{S \subseteq E} \frac{l_{\mathcal{J}}(S)}{r_{\mathcal{O}^{\subseteq}}(S)}$ and $T \in \arg\max\{|O| \colon O \subseteq F, O \in \mathcal{O}^{\subseteq}\}$ with $F_1 = F \cap E_1$, $F_2 = F \cap E_2$, $T_1 = T \cap E_1$ and $T_2 = T \cap E_2$ and w.l.o.g. $\omega(\mathcal{I}_1, \mathcal{O}_1, \mathcal{J}_1) \leq \omega(\mathcal{I}_2, \mathcal{O}_2, \mathcal{J}_2)$. It holds by Lemma 3.19 that

$$\begin{aligned}\omega(\mathcal{I}, \mathcal{O}, \mathcal{J}) =& \frac{l_{\mathcal{J}}(F)}{|T|} = \frac{l_{\mathcal{J}_1}(F_1) + l_{\mathcal{J}_2}(F_2)}{|T_1| + |T_2|} \geq \min\left\{\frac{l_{\mathcal{J}_1}(F_1)}{|T_1|}, \frac{l_{\mathcal{J}_2}(F_2)}{|T_2|}\right\} \\ \geq& \min\{\omega(\mathcal{I}_1, \mathcal{O}_1, \mathcal{J}_1), \omega(\mathcal{I}_2, \mathcal{O}_2, \mathcal{J}_2)\}\end{aligned}$$

and since trivially holds $\omega(\mathcal{I}, \mathcal{O}, \mathcal{J}) \leq \min\{\omega(\mathcal{I}_1, \mathcal{O}_1, \mathcal{J}_1), \omega(\mathcal{I}_2, \mathcal{O}_2, \mathcal{J}_2)\}$ the claim follows. □

We provide an independence system $(E, \mathcal{I})$, a best inner matroid $(E, \mathcal{M})$ and a best inner independence system $(E, \mathcal{J})$ for which hold $q(\mathcal{I}) < \omega(\mathcal{I}, \mathcal{I}, \mathcal{M}\} < \omega(\mathcal{I}, \mathcal{I}, \mathcal{J})$ by joining Example 3.8 and Theorem 3.15.

Theorem 3.21. *Let* $(E, \mathcal{I})$ *the direct sum of the independence systems in Example 3.8 and Theorem 3.15, hence* $E_1 = \{a, b, c_1, .., c_{10}\}$ *with* $(E_1, \mathcal{I}_1)$ *given by the bases* $\mathcal{B}_{\mathcal{I}_1} := \{\{a, b\}, C\} \cup (\{\{a\}\} \bigoplus \binom{C}{5}) \cup (\{\{b\}\} \bigoplus \binom{C}{5})$ *whilst* $C := \{c_1, ..., c_{10}\}$ *and* $\binom{C}{5} := \{S \subset C \colon |S| = 5\}$ *and* $E_2 = \{1, 2, 3, 4, 5\}$ *with* $(E_2, \mathcal{I}_2)$ *defined by its bases* $\mathcal{B}_{\mathcal{I}_2} := \{\{1, 2, 3\}, \{1, 4\}, \{2, 5\}\}$ *and* $(E, \mathcal{I}) := (E_1 \dot\cup E_2, \{I \colon I = I_1 \dot\cup I_2, I_1 \in I_1, I_2 \in I_2\}$*, hence*

$$\mathcal{B}_{\mathcal{I}} := (\{\{a, b\}, C\} \cup (\{\{a\}\} \bigoplus \binom{C}{5}) \cup (\{\{b\}\} \bigoplus \binom{C}{5})) \bigoplus \{\{1, 2, 3\}, \{1, 4\}, \{2, 5\}\}.$$

Then, $q(\mathcal{I}) < \omega(\mathcal{I}, \mathcal{I}, \mathcal{M}) < \omega(\mathcal{I}, \mathcal{I}, \mathcal{J})$ *where* $(E, \mathcal{J})$ *and* $(E, \mathcal{M})$ *are the best inner independence system and best inner matroid, respectively.*

Proof. Let $\mathcal{B}_{\mathcal{J}_1} := \{C\} \cup (\{\{a\}\} \bigoplus \binom{C}{5}) \cup (\{\{b\}\} \bigoplus \binom{C}{5})$ and $\mathcal{B}_{\mathcal{J}_2} := \{\{1, 2, 3\}, \{1, 4\}, \{2, 5\}\}$

be the set of bases of best inner independence systems regarding $(E, \mathcal{I}_1)$ and $(E, \mathcal{I}_2)$ (due to Example 3.8 and Theorem 3.15). Then, since E_1 and E_2 are disjoint, it holds that $\mathcal{B}_{\mathcal{J}} := \Big(\{C\} \cup (\{\{a\}\} \bigoplus \binom{C}{5}) \cup (\{\{b\}\} \bigoplus \binom{C}{5})\Big) \bigoplus \{\{1,2,3\},\{1,4\},\{2,5\}\}$ is the set of bases of a best inner independence system regarding $(E, \mathcal{I})$ by Lemma 3.20. Analogously, $\mathcal{B}_{\mathcal{M}_1} := \binom{C}{6} \cup (\{\{a\}\} \bigoplus \binom{C}{5}) \cup (\{\{b\}\} \bigoplus \binom{C}{5})$ and $\mathcal{B}_{\mathcal{M}_2} := \mathcal{B}_{U_5^1}$ are best inner matroids regarding $(E, \mathcal{I}_1)$ respectively $(E, \mathcal{I}_2)$ due to Remark 3.14 and Theorem 3.15. Then, again by Lemma 3.20, it holds that $\mathcal{B}_{\mathcal{M}} := ((\binom{C}{6} \cup (\{\{a\}\} \bigoplus \binom{C}{5}\}) \cup (\{\{b\}\} \bigoplus \binom{C}{5})) \bigoplus \mathcal{B}_{U_5^1}$ is the set of bases of a best inner matroid regarding $(E, \mathcal{I})$. It follows directly from Lemma 3.20 that $q(\mathcal{I}) = \frac{1}{5} < \frac{1}{3} = \omega(\mathcal{I}, \mathcal{I}, \mathcal{M}\} < \frac{1}{2} = \omega(\mathcal{I}, \mathcal{I}, \mathcal{J})$. □

We have demonstrated that one can improve the performance guarantee of the greedy algorithm by using an inner independence system. Next, we extend our findings to a more general structure than independence systems.

3.4. Inner Approximations to Packing Instances

In the previous section we approximated independence systems by inner independence systems. Here we consider inner approximations of more general structures. Every independence system $(E, \mathcal{I})$ can be interpreted as a subset of $\{0,1\}^n$ by identifying independent sets $I \in \mathcal{I}$ with their characteristic vectors χ^I. We allow for multiplicities and generalize independence systems:

Definition 3.22. *A set $D \subset \mathbb{Z}_+^n$ is called a* ***packing instance*** *if $x \in D$ implies $y \in D$ for all $y \in \mathbb{Z}_+^n$ such that $y \leq x$ (component wise). Consequently, $G \subseteq D$ is an* ***inner packing instance*** *of D if G is a packing instance. Call a packing instance* ***normal****, if it contains all unit vectors.*

Although, one can define packing instances over arbitrary posets, here we focus on the special case of the poset $(\mathbb{Z}_+^n, \leq)$.

For each $i \in E := \{1, ..., n\}$ let there be a nondecreasing discrete concave function $f_i \colon \mathbb{Z}_+ \to \mathbb{R}_+$ and $f \colon \mathbb{Z}_+^n \to \mathbb{R}_+, f := \sum_{i=1}^n f_i$, hence f is separable nondecreasing discrete concave. We are interested in the problem $\max_{x \in D} f(x)$. Note, that this is more general than $\max_{x \in D} \sum_{i=1}^n x_i \cdot v_i$ with weights v_i associated to $i \in E$.

Definition 3.23. *For any packing instance $D \subset \mathbb{Z}_+^n$, an element $x \in D$ is called* ***maximal*** *if $x + e^i \notin D$ for all $i \in \{1, \ldots, n\}$ where e^i denotes the i-th unit vector. Let D^+ denote the* ***set of maximal elements*** *of D and $r(D) := \max\{\sum_{i \in E} x_i \colon x \in D^+\}$ the* ***maximal height*** *and $l(D) := \min\{\sum_{i \in E} x_i \colon x \in D^+\}$ the* ***minimal height*** *of D.*

*In the following, assume D is normal and for $\alpha \in \mathbb{Z}_+^n$ define the α-**truncated** packing instance $D_\alpha := \{x \in D \colon x_i \leq \alpha_i \text{ for } i \in E\}$ and the **height-quotient of** D as*

$$\rho(D) := \min_{\alpha \in \mathbb{Z}_+^n \setminus \{\mathbf{0}\}} \frac{l(D_\alpha)}{r(D_\alpha)}.$$

*Clearly, $\rho(D) = \min_{\alpha \neq \mathbf{0}, \alpha \leq \beta} \frac{l(D_\alpha)}{r(D_\alpha)}$ with $\beta_i := \max_{x \in D} x_i$, hence $\rho(D)$ is well defined. The packing instance D is called a **polymatroid** (see e.g., Dunstan and Welsh [1973]) iff $\rho(D) = 1$ and an **inner polymatroid** of a packing instance D is a polymatroid $P \subseteq D$.*

This polymatroid definition coincides with the standard definition of a polymatroid (hence, Definition 1.5), as the function $g \colon 2^E \to \mathbb{R}_+, g(S) := \max_{x \in D}\{\sum_{i \in S} x_i\}$ turns out to be submodular and $D = \{x \in \mathbb{R}_+^n \colon \sum_{i \in S} x_i \leq g(S)$ fo all $S \subseteq E\}$ (see e.g., [Schrijver, 2003, Theorem 44.5]).

The solution of $\max_{x \in D} f(x)$ of a packing instance D can be approached by the following greedy algorithm generalizing Algorithm 6 for independence systems, which yields an exact solution if D is a polymatroid:

Algorithm 7: Greedy algorithm for integer packing instances

Input : A packing instance $D \subset \mathbb{Z}_+^n$ given by a membership oracle. Separable nondecreasing discrete concave function $f \colon \mathbb{Z}_+^n \to \mathbb{R}_+$

Output: A maximal element $x_g \in D^+$

```
1 Set x = 0
2 while x is not maximal (oracle call) do
3     choose j ∈ arg max_{i∈E: x+e^i∈D} (oracle call) f(x + e^i)
4     x = x + e^j
5 Set x_g = x and return
```

Let $V^*(D, f) := \max_{x \in D} \sum_{i=1}^n f(x)$ and $V^{greedy}(D, f)$ the worst possible (if tie breaking occurs) solution obtained by Algorithm 7. Malinov and Kovalyov [1980] show that the height-quotient $\rho(D)$ to be the worst case approximation guarantee.

Theorem 3.24. *[Malinov and Kovalyov, 1980] Let $D \subset \mathbb{Z}_+^n$ be a normal packing instance and $f(x) := \sum_{i=1}^n f_i(x_i)$ with $f_i \colon \mathbb{Z}_+ \to \mathbb{R}_+$ being nondecreasing and discrete concave. Then, $\frac{V^{greedy}(D,f)}{V^*(D,f)} \geq \rho(D)$. Furthermore, for any normal packing instance $D \subseteq \mathbb{Z}_+^n$ there exists a separable nondecreasing discrete concave function f such that*

$$\frac{V^{greedy}(D, f)}{V^*(D, f)} = \rho(D). \tag{3.2}$$

We want to approximate packing instances by inner packing instances respectively inner polymatroids. For economy of exposition we assume that the whole of the packing instance is acceptable, so $D = \mathcal{O}$. Our results easily generalize to other acceptable sets that are not necessarily downward closed subsets of the packing instance D. So as to achieve a better performance than Equation (3.2) we first search for an inner packing instance G contained in D and then apply 7 to G. Our goal is to find G such that the worst case performance of 7 on G is better than the worst case performance on D.

Definition 3.25. *For a normal packing instance $D \subset \mathbb{Z}^n_+$ and a packing instance $G \subseteq D$ we define the* ***generalized height-quotient*** *as*

$$\min_{\alpha \in \mathbb{Z}^n_+ \setminus \{\mathbf{0}\}} \frac{l(G_\alpha)}{r(D_\alpha)} =: \omega(D, G).$$

The next result generalizes Theorem 3.5 and Equation (3.2).

Theorem 3.26. *Let $D, G \subset \mathbb{Z}^n_+$ packing instances such that D is normal, $G \subseteq D$ and*

$$\mathcal{C} := \{f \colon \mathbb{Z}^n_+ \to \mathbb{R}_+ \colon f \text{ is separable nondecreasing discrete concave}\}.$$

Then, for $f \in \mathcal{C}$ it holds

$$\min_{f \in \mathcal{C}} \frac{V^{greedy}(G, f)}{V^*(D, f)} = \omega(D, G).$$

Proof. Let $\beta \in \arg\min_{\alpha \in \mathbb{Z}^n_+ \setminus \{\mathbf{0}\}} \frac{l(G_\alpha)}{r(D_\alpha)}$ and $u \in \arg\max\{\|x\|_1 \colon x \in D^+_\beta\}$, hence $\|u\|_1 = r(D_\beta)$ and $v \in \arg\min\{\|x\|_1 \colon x \in G^+_\beta\}$, hence $\|v\|_1 = l(G_\beta)$.

Define $g_i(x_i) := \frac{1}{\|u\|_1} \cdot x_i$ for $x_i \le \beta_i$ and $g_i(x_i) := \frac{\beta_i}{\|u\|_1}$ for $x_i > \beta_i$ and $g(x) := \sum_i^n g_i(x_i)$. Hence, g_i is a piecewise linear function in x_i that has slope $\frac{1}{\|u\|_1}$ between 0 and β_i in x_i and constant value $\frac{\beta_i}{\|u\|}$ starting from β_i. Observe that for $y \in D \setminus D_\beta$ holds $g(y) = \max_{x \le y, x \in D^+_\beta} g(x)$, hence $\max_{x \in D}\{g(x)\} = \max_{x \in D_\beta}\{g(x)\}$. For every $x \in D_\beta$ it holds $g(x) = \frac{\sum_{i=1}^n x_i}{\|u\|_1} \le \frac{\sum_{i=1}^n u_i}{\|u\|_1}$. Hence, $V^*(D, g) = \sum_{i=1}^n g_i(u_i) = \sum_{i=1}^n \frac{u_i}{\|u\|_1} = 1$. The worst case greedy solution that Algorithm 7 may produce on G is $V^{\text{greedy}}(G, g) = g(v) = \sum_{i=1}^n \frac{v_i}{\|u\|} = \frac{\|v\|_1}{\|u\|_1}$. In total, we get $\frac{V^{\text{greedy}}(G,g)}{V^*(D,g)} = \frac{\|v\|_1}{\|u\|_1}$ and therefore $\min_{f \in \mathcal{C}} \frac{V^{\text{greedy}}(G,f)}{V^*(D,f)} \le \frac{V^{\text{greedy}}(G,g)}{V^*(D,g)} = \frac{\|v\|_1}{\|u\|_1} = \min_{\alpha \in \mathbb{Z}^n_+ \setminus \{\mathbf{0}\}} \frac{l(G_\alpha)}{r(D_\alpha)}$.

Conversely, let $f \in \mathcal{C}$. Denote the marginal value of $f_i(\cdot)$ of j by $\Delta f_i(j) := f_i(j) - f_i(j-1)$ for $i \in E, 1 \le j \le k$ and totally order $F := \{(i,j) \colon 1 \le i \le n, 1 \le j \le k)\}$ such that (i,j) is ordered before (u,w) if $\Delta f_i(j) > \Delta f_u(w)$ and (i,j) is ordered before $(i, j+1)$ if $\Delta f_i(j) = \Delta f_i(j+1)$. Hence, F is ordered such that the higher the marginal value of an additional unit of i added to $(j-1)$ the earlier (i,j) occurs. Let $p \colon F \to \{1, \ldots, n \cdot m\}$ be such that $p((i,j))$ is the position of (i,j) according to the ordering. For $1 \le t \le n \cdot m$ and $i \in E$ define $q_i(t) := |\{(i,j) \colon p((i,j)) \le t\}|$, which can be interpreted as the total number of units of i among the t most valuable units, and set $q(t) := (q_1(t), \ldots, q_n(t))$. Furthermore, for $x \in \mathbb{Z}_+^n$ we set $(q(t) \wedge x)_i := \min\{x_i, q_i(t)\}$, hence $(q(t) \wedge x)_i$ counts how many of the t best elements of f are among the x_i best of $i \in \{1, \ldots, n\}$. Consequently, $q(t) \wedge x = (q^1(t) \wedge x, \ldots, q^n(t) \wedge x)$ and $\|q(t) \wedge x\|_1 = \sum_{i=1}^n (q(t) \wedge x)_i$. For convenience, we define $p^{-1}(m \cdot n + 1) := 0$, $h\colon F \to \mathbb{R}_+, h((i,j)) := \Delta f_i(j)$ and thus for $x \in \mathbb{Z}_+^n$ we can rewrite analogous to Equation 3.1

$$f(x) = \sum_{t=1}^{m \cdot n} (h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot \|q(t) \wedge x\|_1.$$

Now, let x^g be a worst case greedy solution obtained by Algorithm 7 on G and x^* an optimal solution on D. Note that for $x \in D$ or $x \in G$ we can interpret $q(t) \wedge x$ as the quantity of i in x which also is contained in $D_{q(t)}$ respectively $G_{q(t)}$. Then, it holds $q(t) \wedge x^g \in G_{q(t)}$ and $q(t) \wedge x^g + e^i \notin G_{q(t)}$ due to the definition of $q_i^g(t)$. Hence, $q(t) \wedge x^g \in G_{q(t)}^+$ what yields the simple inequality $l(G_{q(t)}) \leq \|q(t) \wedge x^g\|_1$. Analogously, it holds that $q(t) \wedge x^* \in D_{q(t)}$ and $q(t) \wedge x^* + e^i \notin D_{q(t)}$, therefore $q(t) \wedge x^* \in D_{q(t)}^+$ what yields $r(D_{q(t)}) \geq \|q(t) \wedge x^*\|_1$. We use these two inequalities to show

$$\begin{aligned}
\frac{f(x^g)}{f(x^*)} &= \frac{\sum_{t=1}^{m \cdot n} (h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot \|q(t) \wedge x^g\|_1}{\sum_{t=1}^{m \cdot n} (h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot \|q(t) \wedge x^*\|_1} \\
&\geq \frac{\sum_{t=1}^{m \cdot n} (h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot l(G_{q(t)}) \cdot \frac{\|q(t) \wedge x^*\|_1}{r(D_{q(t)})}}{\sum_{t=1}^{m \cdot n} (h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot \|q(t) \wedge x^*\|_1} \\
&= \frac{\sum_{t=1}^{m \cdot n} (h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot \|q(t) \wedge x^*\|_1 \cdot \frac{l(G_{q(t)})}{r(D_{q(t)})}}{\sum_{t=1}^{m \cdot n} (p^{-1}(t)' - p^{-1}(t+1)' \cdot \|q(t) \wedge x^*\|_1} \\
&\geq \frac{\sum_{t=1}^{m \cdot n} (h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot \|q(t) \wedge x^*\|_1 \cdot \min_{1 \leq s \leq m \cdot n} \frac{l(G_{q(s)})}{r(D_{q(s)})}}{\sum_{t=1}^{m \cdot n} (h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot \|q(t) \wedge x^*\|_1} \\
&= \frac{\sum_{t=1}^{m \cdot n} (h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot \|q(t) \wedge x^*\|_1}{\sum_{t=1}^{m \cdot n} (h(p^{-1}(t)) - h(p^{-1}(t+1))) \cdot \|q(t) \wedge x^*\|_1} \cdot \min_{1 \leq s \leq m \cdot n} \frac{l(G_{q(s)})}{r(D_{q(s)})} \\
&= \min_{1 \leq s \leq m \cdot n} \frac{l(G_{q(s)})}{r(D_{q(s)})} = \frac{l(G_\beta)}{r(D_\beta)}
\end{aligned}$$

and the claim follows. □

Note that Theorem 3.26 also implies the result of Theorem 3.24 and Theorem 3.5, hence it also implies the classic result of Hausmann et al. [1980].

Next, we provide an example beyond independence systems in which the worst case approximation guarantee of the best inner polymatroid approximation improves over Algorithm 7, carried out on the original packing instance.

Example 3.27. *Let $D \subset \mathbb{Z}_+^2$ be defined by its set of maximal elements $D^+ := \{(10,2),(0,3)\}$. Then, $\rho(D) \leq \frac{l(D_{(10,3)})}{r(D_{(10,3)})} = \frac{3}{12} = \frac{1}{4}$.*

Consider $P \subset D$ given by its set of maximal element $P^+ := \{(10,2)\}$. Then, P is a box and therefore a polymatroid.

To prove $\omega(D,P) > \rho(D)$ observe that the unique maximal basis of P_α is given by $(\min\{\alpha_1,10\},\min\{\alpha_2,2\})$ for $\alpha = (\alpha_1,\alpha_2) \leq (10,10)$. For $\alpha_1 \geq 1$ it also holds that $(\min\{\alpha_1,10\},\min\{\alpha_2,2\})$ are elements of maximal height of D_α, hence $\frac{l(D_\alpha)}{r(D_\alpha)} = 1$. In the remaining case that $\alpha_1 = 0$, $1 \leq \alpha_2 \leq 10$ it holds that $(0,\min\{\alpha_2,3\})$ is the unique maximal height basis of D_α and hence $\frac{l(P_\alpha)}{r(D_\alpha)} = \frac{\min\{\alpha_2,2\}}{\min\{\alpha_2,3\}}$. Therefore, $\omega(D,P) = \frac{l(P_{(0,3)})}{r(D_{(0,3)})} = \frac{2}{3}$, which is better than $\rho(D)$.

Note that P is a best inner polymatroid since any better polymatroid P' must contain $(0,3)$ and $(0,3)$ has to be a maximal element of P' (see also Figure 4.1.), hence $\omega(D,P') \leq \frac{l(P'_{(10,10)})}{r(D_{(10,10)})} = \frac{3}{12} = \frac{1}{4}$.

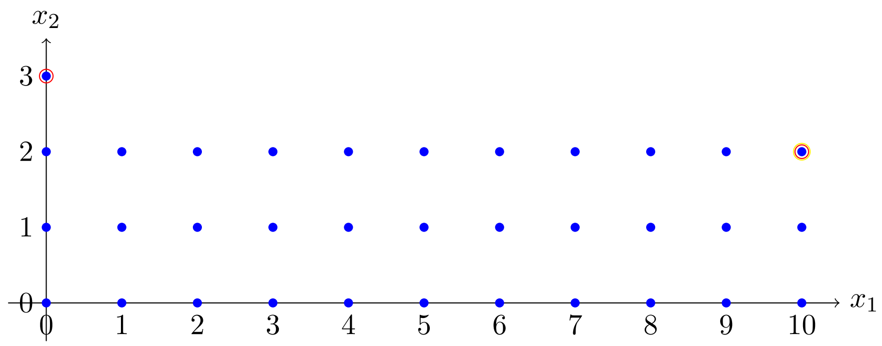

Figure 3.1.: Packing instance in Example 3.27 with $D^+ = \{(0,3),(10,2)\}$ and $P^+ = \{(10,2)\}$.

As seen in Theorem 3.21, there exist cases in which the inner independence system approximation simultaneously outperforms the inner matroid approximation and the greedy algorithm performed on the original independence system. For completeness, by combining the ideas of Example 3.27 and of Theorem 3.15, we provide an additional, simultaneously nonpolymatroidal nonindependent system example, in which the inner packing instance approximation outperforms simultaneously the direct application of Algorithm 7 to the original packing instance and the inner polymatroid approximation. We make use of an analogue to Lemma 3.20.

Lemma 3.28. *Let $E \subset \mathbb{Z}_+^n$ and $F \subset \mathbb{Z}_+^m$ be packing instances. and $H \subseteq E$ and $I \subseteq F$ inner packing instances and $D := E \bigoplus F$ and $G := H \bigoplus I$. Then, it holds $\omega(D,G) = \min\{\omega(E,H),\omega(F,I)\}$.*

Proof. Let $\alpha^n \in \arg\min \omega(E,H)$, $\alpha^m \in \arg\min \omega(F,I)$ and w.l.o.g. $\omega(E,H) \leq \omega(F,I)$.

For any $\beta := (\beta^n, \beta^m) \in \mathbb{Z}^n_+ \bigoplus \mathbb{Z}^m_+$ it is $\frac{l_{(\beta^n,\beta^m)}(G)}{r_{(\beta^n,\beta^m)}(D)} = \frac{l_{\beta^n}(H)+l_{\beta^m}(I)}{r_{\beta^n}(E)+r_{\beta^m}(F)} \geq \frac{l_{\alpha^n}(H)+l_{\beta^m}(I)}{r_{\alpha^n}(E)+r_{\beta^m}(F)} \geq \frac{l_{\alpha^n}(H)+l_{\alpha^m}(I)}{r_{\alpha^n}(E)+r_{\alpha^m}(F)} \geq \frac{l_{\alpha^n}(H)}{r_{\alpha^n}(E)} = \omega(E,H)$. Obviously $\omega(E,H) \geq \omega(D,G)$, therefore $\omega(E,H) = \omega(D,G)$ and this concludes the proof. □

Note that Lemma 3.28 also implies $\rho(D) = \min\{\rho(E), \rho(F)\}$ since $\rho(X) = \omega(X,X)$ for any packing instance X.

Example 3.29. *Let $D \subset \mathbb{Z}^7_+$ given by its set of maximal elements*

$$D^+ := \{(10,2),(0,3)\} \bigoplus \{(1,1,1,0,0),(1,0,0,1,0),(0,1,0,0,1)\}.$$

Then, D is the direct sum of the packing instance in Example 3.27 and Theorem 3.15. It follows by Lemma 3.28, Example 3.27 and Theorem 3.15 that $P^+ := \{(10,2)\} \bigoplus \{x \in \mathbb{R}^5 \colon x = e^i \text{ for } 1 \leq i \leq 5\}$ is the set of maximal elements of a best inner polymatroid, $G^+ := \{(10,2)\} \bigoplus \{(1,1,1,0,0),(1,0,0,1,0),(0,1,0,0,1)\}$ is the set of maximal elements of a best inner packing instance and

$$\rho(D) \leq \frac{1}{4} < \max_{P \subseteq D, P \text{ is polmatroid}} \omega(D,P) = \frac{1}{3} < \max_{G \subseteq D, G \text{ is packing instance}} \omega(D,G) = \frac{1}{2}.$$

3.5. Conclusion

We studied the problem of maximizing a separable nondecreasing discrete concave function over a packing instance and in particular the important special case of finding a maximum weighted basis of an independence system.

The concept of inner independence systems respectively inner packing instances was introduced and a generalization of the rank quotient of independence systems respectively the height-quotient of packing instances was given to provide a bound on the approximation quality of the greedy algorithm applied to inner independence systems respectively inner packing instances to the optimal solution. We showed that the generalized rank-quotient and the height-quotient provide tight bounds on the worst case performance of our algorithms. Furthermore, we provided examples in which our approach concurrently outperforms the standard greedy algorithm for independence systems and the approach of Milgrom [2017] and de Vries and Vohra [2020] of approximating independence systems by inner matroids in terms of worst case approximation guarantee. Analogously, we showed that our algorithm may outperform the greedy algorithm applied to maximize a separable nondecreasing discrete concave function over a packing instance.

Chapter 4.

Dynamic Pricing in Weighted Uniform Matroid Markets

4.1. Introduction

In this chapter, we examine the problem of implementing social welfare maximizing allocations in combinatorial markets through prices. Consider a finite set of buyers N who want to acquire a finite subset of heterogeneous indivisible goods M from a single seller. Each buyer $j \in N$ has a publicly known valuation function that represents the buyer's value for every subset of M. The goal is to allocate items efficiently, hence in such a way that the social welfare, the sum of the buyers' values for the set of items they get allocated, is maximized. Clearly, with known valuation functions, a social welfare maximizing allocation, in the following also called optimal allocation, can be easily determined. Moreover, this can be done in polynomial oracle time, if the buyers' valuations are gross substitutes[1] (see e.g., [Nisan and Segal, 2006]), a set of valuation functions that contains weighted matroid rank functions, among others. In order to implement an efficient allocation, the seller sets item prices. Buyers arrive one at a time and successively to purchase items from the seller, taking any of their *utility* maximizing sets given the price (hence, their valuation for the set minus the sum of the item prices of the items contained in the set) among the remaining items. The order of the arriving buyers is unknown to the seller, but he is allowed to update the prices of the remaining items before the next buyer arrives. The goal of the seller is to induce a social welfare maximizing allocation by setting the item prices such that it is in the best interest of the next arriving buyer to choose a set of items she gets allocated in some social welfare maximizing allocation. Then, any price such that every possibly next arriving buyer necessarily has to chose a set of items she might get allocated in some social welfare maximizing allocation is called a *dynamic price* and a sequence of dynamic prices is called a *dynamic pricing*.

[1] A function $v\colon 2^M \to \mathbb{R}_+$ is *gross substitutes* if it holds for all $S, T \subseteq M$, $x \in S \setminus T$ that $v(S) + v(T) \leq \max\{v(S \setminus \{x\}) + v(T \cup \{x\}), \max_{y \in T \setminus S} v((S \setminus \{x\}) \cup \{y\}) + v((T \setminus \{y\}) \cup \{x\})\}$.

It is well known, and also shown in the next example, that static prices (hence, the seller sets item prices once at the beginning and is *not* allowed to update the prices before the next buyer arrives) cannot induce an efficient allocation when buyers arrive successively, even in markets composed of buyers with unweighted rank-1 matroids valuations.

Example 4.1. *Consider a market composed of three items x, y, z and three buyers $1, 2, 3$ with valuations $v_1(\emptyset) = v_2(\emptyset) = v_3(\emptyset) = v_1(\{x\}) = v_2(\{y\}) = v_3(\{z\}) = 0$ and $v_j(S) = 1$ for $j \in \{1, 2, 3\}$ and all other $S \subseteq \{x, y, z\}$. Then, it is easy to see that the set of optimal allocations is $\{(S_1 = \{y\}, S_2 = \{z\}, S_3 = \{x\}), (S_1 = \{z\}, S_2 = \{x\}, S_3 = \{y\})\}$ and every optimal allocation provides a social welfare of 3. However, no static price can guarantee more than $\frac{2}{3}$ of the optimal welfare if buyers arrive sequentially. To see this, let w.l.o.g. $0 \leq p(x) \leq p(y) \leq p(z) < 1$. Assume buyer 1 arrives first and he purchases item y from the seller and buyer 2 arrives next and purchases item x from the seller. Then, the remaining buyer 3 will purchase nothing and the allocation $(S_1 = \{y\}, S_2 = \{x\}, S_3 = \emptyset)$ provides a social welfare of 2.*

Therefore, it is an important open question for which valuation classes the additional ability of the seller to sequentially update the prices results in the power to induce optimal allocations.

4.1.1. Markets with Dynamic Pricings

The only markets with an arbitrary number of items and bidders for which the existence of dynamic pricings is known are markets composed of buyers with weighted rank-1 matroids, also denoted as matching markets (Cohen-Addad et al. [2016]). For completeness, we present a dynamic pricing for the matching market in Example 4.1.

Example 4.2 (continued)**.** *Consider the initial price $0 \leq p(x) = p(y) = p(z) < 1$ and assume w.l.o.g. that buyer 1 arrives first to the market and purchases w.l.o.g. the item y. Then, the seller updates the prices of the remaining items to $0 \leq p(z) < p(x) < 1$. Then, if buyer 2 arrives next to the market he acquires z and if buyer 3 arrives next to the market he acquires x. In either case, if the seller leaves the price of the remaining item constant, the final buyer will purchase the remaining item and the final allocation will be $(S_1 = \{y\}, S_2 = \{z\}, S_3 = \{x\})$ which is optimal.*

There are weak results regarding the existence of dynamic pricings in weighted uniform matroid markets. Berger et al. [2020] proved the existence of dynamic pricings for weighted uniform matroid markets with up to three three buyers and Bérczi et al. [2021b] showed the existence of dynamic pricings for balanced (when the number of items equals the sum of the rank functions of the buyers) weighted uniform matroid markets when the rank of each buyer's matroid is at most two under the assumption that every optimal allocation gives each buyer exactly its rank many items. Recently, Pashkovich and Xie [2022] extended both results for balanced weighted uniform matroid markets under the same assumption as Bérczi et al. [2021b] to four buyers and to buyers with rank at most three.

Further, Bérczi et al. [2021a] consider the dynamic pricing problem in weighted matroid markets with 2 buyers under the assumption that there exists a pair of disjoint sets which consists of one basis of each buyer's underlying matroid. They prove the existence of dynamic pricings under this assumption if one of the underlying matroids is a partition matroid or both underlying matroids are strongly base orderable.

4.1.2. Our Contribution

We prove that dynamic pricings always exist for weighted uniform matroid markets. In this setting, we provide an algorithm to compute a dynamic price which runs in polynomial time in the number of items and buyers. En route, we show that relative interior Walrasian prices in weighted uniform matroid markets induce no unnecessary demand and that in matching markets, every relative interior Walrasian price already is a dynamic price. In the following table the results we generalize and our results are depicted:

Paper	Result	Conditions
Cohen-Addad et al. [2016]	Existence of dynamic prices for matching markets.	Matroids have rank 1.
This chapter	Interior Walrasian prices are dynamic prices for matching markets.	Matroids have rank 1.
Berger et al. [2020]	Existence of dynamic prices for weighted uniform matroid markets.	Matroids have arbitrary rank, $\lvert N\rvert \leq 3$.
Bérczi et al. [2021b]	”	Matroids have rank ≤ 2, $\sum_{j\in N} r_j = \lvert M\rvert$, every optimal allocation allocates all items.
Pashkovich and Xie [2022]	”	Matroids have rank ≤ 3, $\sum_{j\in N} r_j = \lvert M\rvert$, every optimal allocation allocates all items.
Pashkovich and Xie [2022]	”	Matroids have arbitrary rank, $\lvert N\rvert \leq 4$, $\sum_{j\in N} r_j = \lvert M\rvert$, every optimal allocation allocates all items.
This chapter	Existence of dynamic prices for weighted uniform matroid markets.	Matroids have arbitrary rank.

4.1.3. Walrasian Prices are not Dynamic Prices

Maximizing social welfare through prices is based on the concept of *Walrasian equilibrium* and *Walrasian prices*. A Walrasian equilibrium is a pair of allocation and item prices (called Walrasian prices) such that every buyer gets allocated a bundle that maximizes her utility given the price, and every unallocated item has price zero. Then, it can be shown easily that the allocation in a Walrasian equilibrium maximizes social welfare. Clearly, a Walrasian equilibrium implies the desirable characteristic that no buyer strictly prefers any other buyer's allocated bundle at the Walrasian price. However, the main drawback of Walrasian prices is that a central coordinator might be required to allocate the bundles to the buyers since Walrasian prices might induce ties. Therefore, as also shown in the next example, which turns out to be a weighted uniform matroid market, Walrasian prices are not necessarily dynamic prices.

Example 4.3. *Consider a market composed of three items $M = \{x, y, z\}$ and two buyers $1, 2$ with valuations $v_1(S) = 2 \cdot |S|$ for $|S| \leq 2$ and $v_1(\{x, y, z\}) = 4$ and $v_2(S) = |S|$ for $|S| \leq 2$ and $v_2(\{x, y, z\}) = 2$. Then, the set of optimal allocations is $\{(S_1, S_2)\colon |S_1| = 2, S_2 = M \setminus S_1\}$ and it is easy to see that the only Walrasian price for this market is $p \equiv 1$. However, if buyers arrive successively and buyer 2 arrives first at price p, she might purchase nothing or a set of cardinality two, which results in either way in a nonoptimal allocation.*

Kelso and Crawford [1982] showed that every market with buyers with gross substitutes valuations admits a Walrasian equilibrium. The maximal domain theorem of Gul and Stacchetti [1999] states that for some buyer 1 with a non-gross substitutes valuation v_1 on a set of items M there exists $n \in \mathbb{N}$ and a set of buyers $\{2, \ldots, n\}$ with unit-demand valuations on M such that the resulting market composed of the buyers $1, \ldots, n$ does not have a Walrasian equilibrium. Inspired by this maximal domain theorem, it was shown by Berger et al. [2020] that for some buyer 1 with a non-gross substitutes valuation v_1 on M there exists $n \in \mathbb{N}$ and a set of buyers $\{2, \ldots, n\}$ with unit-demand valuations on M such that the resulting market composed of the buyers $1, \ldots, n$ does not admit a dynamic price. It is an open question whether the latter domain theorem is strict and there is little known about the existence of dynamic prices even in (weighted) (uniform) matroid markets, a small subset of gross substitutes markets.

4.1.4. Further Related Literature

The notion of Walrasian equilibria is due to Walras [1896]. Note that gross substitutes valuations are also known as $M^\natural$-concave functions in discrete convex analysis (e.g., [Murota, 2003]) and valuated matroids (e.g., [Dress and Wenzel, 1990]).

Hsu et al. [2016] observed that minimal Walrasian prices promote indifferences and proposed a genericity condition on valuations to reduce overdemand and Leme and Wong [2020] showed that in markets where optimal allocations are unique and a Walrasian equilibrium exists there also exist Walrasian prices that induce no overdemand and therefore serve as a static instance of a dynamic pricing. However, these observations require strong conditions and it is desirable to establish the existence of prices that induce social welfare maximizing allocations without a central tie-breaking authority in more general settings.

To this end, Cohen-Addad et al. [2016] proposed dynamic pricings as an alternative to static posted price mechanism (e.g., Walrasian prices). As already mentioned in Subsection 4.1.1, there are few existence results on markets that allow for social welfare maximizing allocations induced by dynamic pricings. For completeness, we mention that to achieve their result, Berger et al. [2020] also make use of auxiliary graphs but their techniques are very different from ours and mainly inspired by the cycle-canceling algorithm of Murota [1996].

For completeness, we mention some work that is related to our results in a broader sense: Applications of dynamic prices can be found in Cohen et al. [2014], which considers the k-server problem and provide an optimal k-competitive dynamic pricing scheme for the k-server problem on the line and Cohen et al. [2019] extend this result to tree metric spaces. Feldman et al. [2017] apply dynamic prices to makespan minimization in job scheduling on multiple machines and provide tight results on the approximation guarantee of dynamic pricing to the optimal solution.

Static pricing for social welfare maximization was considered in several papers. Ezra et al. [2020] study the power and limits of static posted prices in terms of approximating the maximizing social welfare in different markets. Feldman et al. [2014] show that there exist posted prices that capture, in expectation, at least half of the optimal social welfare in fractionally subadditive markets and Dutting et al. [2020] extend this work. Bundle pricing that approximately maximizes social welfare in substitutes markets is considered in Feldman et al. [2016] and Chawla et al. [2019] focus on bundle pricing in markets with complements.

There is also plenty of work on revenue maximizing static posted price mechanisms, e.g., [Guruswami et al., 2005], [Chawla et al., 2007], [Chawla et al., 2010], [Babaioff et al., 2015] and [Anshelevich and Sekar, 2017].

4.1.5. Chapter Structure

We briefly outline the organization of this chapter: We start with the economic setting and introduce the dynamic pricing problem and previous results regarding dynamic pricing in Section 4.2. In Section 4.3 we focus on Walrasian equilibria and show that in weighted uniform matroid markets the prices in the relative interior of the polyhedron of Walrasian prices avoid unnecessary demand and thereby avoid one, and in this setting the only, source of ties. On the one hand, this result leads us to conclude that interior Walrasian prices are already dynamic prices in matching markets. On the other hand, in Section 4.4, we interpret weighted uniform matroid markets via interior Walrasian prices as node capacitated bipartite graphs in which the set of optimal allocations is encoded as saturating capacitated matchings respectively perfect capacitated matchings. We use a generalization of Hall's marriage theorem and establish combinatorial auxiliary results that apply to these graphs. Subsequently, in Section 4.5, we show our main results, hence, prove the existence of dynamic pricings in *all* weighted uniform matroid markets and provide an algorithm to compute them in polynomial time in the number of items and buyers.

4.2. Preliminaries

4.2.1. Economic Setting

We present the economic setting. For convenience, we provide a list of the most frequently used symbols and results of this chapter at the end of the chapter.

Definition 4.4. *In the following, let $m, n \in \mathbb{N}$ and $M = \{1, \dots, m\}$ a finite set of items $N = \{1, \dots, n\}$ a finite set of buyers. (Note that throughout this chapter we denote the set of items by M, since following standard notation we use the letter E for the set of edges in graphs.) Each buyer $j \in N$ has a* ***valuation function*** *$v_j \colon 2^M \to \mathbb{R}_+$ that is nondecreasing and normalized and maps every bundle $S \subseteq M$ to j's value for the bundle. Given a matroid $(M, \mathcal{I})$ and weights $\omega_x \in \mathbb{R}$ for $x \in M$ we define the* ***weighted matroid valuation*** *with underlying matroid $(M, \mathcal{I})$ by $v \colon 2^M \to \mathbb{R}_+$, $v(S) := \max_{T \subseteq S, T \in \mathcal{I}} \sum_{x \in T} \omega_x$, hence for the uniform matroid U^r_M the* ***weighted uniform matroid valuation*** *is given by $v \colon 2^M \to \mathbb{R}_+$, $v(S) = \max_{T \subseteq S, |T| \leq r} \sum_{x \in T} \omega_x$. We define a* ***price*** *$p \colon M \to \mathbb{R}_+$ and the price of a bundle $S \subseteq M$ as $p(S) := \sum_{i \in S} p(i)$. Clearly, the set of prices can be identified with $\mathbb{R}^M_+$. For a buyer $j \in N$ and her valuation $v_j \colon 2^M \to \mathbb{R}_+$, a set of items $S \subseteq M$ and a price p her quasilinear* ***utility*** *of S at price p is denoted by $u_j(S, p) := v_j(S) - p(S)$ and for ease of notation for $x \in M$ we denote $u_j(x, p) := u_j(\{x\}, p)$. Then, buyer j's* ***utility maximizing sets*** *at price p, also called the* ***demanded sets****, are given by $D_j(p) := \arg\max_{S \subseteq M} v_j(S) - p(S)$ and the by j* ***demanded items*** *at price p are defined as $\bigcup D_j(p) := \bigcup_{S \in D_j(p)} S$.*

A ***market*** *will be denoted by the 3-tuple $\mathcal{U} = (M, N, \mathbf{v})$ with $\mathbf{v} := (v_1, \dots, v_n)$ and $v_j \colon 2^M \to \mathbb{R}_+$ is the valuation of buyer $j \in N$. A market $\mathcal{U} = (M, N, \mathbf{v})$ is called a* ***weighted uniform matroid market*** *if each buyer's valuation is a weighted uniform matroid valuation with domain M. For $j \in N$ and a weighted uniform matroid valuation v_j let r_j denote the* ***rank*** *of the underlying uniform matroid $U^{r_j}_M$ and $\omega_{j,x}$ denotes the weight of $x \in M$ for j. Then, call a weighted uniform matroid market $\mathcal{U} = (M, N, \mathbf{v})$* ***balanced*** *if it holds $\sum_{j \in N} r_j = |M|$ and* ***with excess demand*** *if it holds $\sum_{j \in N} r_j > |M|$.*

In a market $\mathcal{U} = (M, N, \mathbf{v})$ any $\mathbf{A} := (A_1, \ldots, A_n) \in 2^{M^N}$ such that for $i, j \in N$, $i \neq j$ it holds that $A_i \cap A_j = \emptyset$ is called an ***allocation****. Notice that for $j \in N$ any $A_j \subseteq M$ could be allocated, but its value is going to be the value of the $\max\{r_i, |A_i|\}$ best elements. Note that it may hold $\bigsqcup_{j=1}^{n} A_j \subsetneq M$. The* ***social welfare*** *of an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ is defined as $\sum_{j\in N} v_j(A_j)$ and $\mathrm{Opt}^{\mathcal{U}} := \{\mathbf{A} \colon \mathbf{A} \in \arg\max_{\mathbf{B}=(B_1,\ldots,B_n)\colon \mathbf{B} \text{ is allocation}} \sum_{j\in N} v_j(B_j)\}$ is called the* ***set of optimal allocations****. For any buyer $j \in N$, the* ***optimal sets for*** *j are given by $\mathrm{Opt}_j^{\mathcal{U}} := \{A \subseteq M \colon A = A_j,\ \mathbf{A} = (A_1, \ldots, A_n) \in \mathrm{Opt}^{\mathcal{U}}\}$ and the* ***optimal items for*** *j are $\bigcup \mathrm{Opt}_j^{\mathcal{U}} := \bigcup_{A\in \mathrm{Opt}_j^{\mathcal{U}}} A$. For any buyer $j \in N$ we say that any T such that there exists $S \supseteq T, S \in \mathrm{Opt}_j^{\mathcal{U}}$ is an* ***optimal subset for*** *j and S is an* ***strict optimal subset for*** *j if S is strict subset of an optimal set for j. For any $x \in M$ we denote the set of buyers that get allocated x in some optimal allocation by $\alpha(x) := \{j \in N \colon\ x \in \bigcup \mathrm{Opt}_j^{\mathcal{U}}\}$ and for some allocation $\mathbf{A}$ we denote by $\alpha^{\mathbf{A}}(x)$ the buyer that gets allocated the item x in the allocation $\mathbf{A}$. We say that some price p induces* ***unnecessary demand*** *if there exists $j \in N$ such that it holds $\bigcup D_j(p) \neq \bigcup \mathrm{Opt}_j^{\mathcal{U}}$ or it holds $\bigcap_{S\in D_j(p)} S \neq \{x \in M \colon \{j\} = \alpha(x)\}$. (Hence, in the first case buyer j demands a set of items that contains some item that she does not get allocated in any optimal allocation and in the second case j demands some set of items which does not contain a certain item that she gets allocated in every optimal allocation.)*

In a market $\mathcal{U} = (M, N, \mathbf{v})$ a ***Walrasian equilibrium*** *is a pair $(p, \mathbf{A})$ which consist of a price p and an allocation $\mathbf{A} = (A_1, \ldots, A_n)$ such that it holds $A_j \in D_j(p)$ for all $j \in N$ and it holds $p(x) = 0$ for all $x \in M \setminus \bigcup_{j\in N} A_j$. Every price p which is part of some Walrasian equilibrium $(p, \mathbf{A})$ is called a* ***Walrasian price****.*

Note that Walrasian equilibria exist in weighted uniform matroid markets. This follows e.g., by the equivalence of gross substitutes valuations and M$^\natural$-concave functions shown by Fujishige and Yang [2003] and the facts that matroid rank functions are M$^\natural$-concave functions ([Fujishige, 2005, Theorem 17.15]) and Walrasian equilibria are guaranteed to exist in gross substitutes markets ([Gul and Stacchetti, 1999]).

We recall the two fundamental theorems of welfare economics.

Theorem 4.5. *In any market, it holds:*

i) The allocation in a Walrasian equilibrium is optimal.

ii) Every pair $(p, \mathbf{A})$ of some Walrasian price p and some optimal allocation $\mathbf{A}$ forms a Walrasian equilibrium.

Proof. i) Let $\mathcal{U} = (M, N, \mathbf{v})$ a market and $(p, \mathbf{A} = (A_1, \ldots, A_n))$ a Walrasian equilibrium. Clearly, for every alternative allocation $\mathbf{B} = (B_1, \ldots, B_n)$ that allocates all items, it holds for $j \in N$ that $v_j(A_j) - p(A_j) \geq v_j(B_j) - p(B_j)$ and therefore $\sum_{j\in N} v_j(A_j) - \sum_{x\in M} p(x) = \sum_{j\in N} v_j(A_j) - p(A_j) \geq \sum_{j\in N} v_j(B_j) - p(A_j) =$

$\sum_{j\in N} v_j(B_j) - \sum_{x\in M} p(x)$, hence $\sum_{j\in N} v_j(A_j) \geq \sum_{j\in N} v_j(B_j)$ and $\mathbf{A}$ is an optimal allocation.

ii) Now, let $(p', \mathbf{A}' = (A'_1, \ldots, A'_n))$ an alternative Walrasian equilibrium. Then, it is $\sum_{j\in N} v_j(A_j) - \sum_{x\in M} p(x) = \sum_{j\in N} v_j(A'_j) - \sum_{x\in M} p(x)$ and there cannot exist $j \in N$ such that $v_j(A_j) - p(A_j) > v_j(A'_j) - p(A'_j)$ since then there would have to exist $k \in N\backslash\{j\}$ such that $v_j(A_k) - p(A_k) < v_j(A'_k) - p(A'_k)$ in contrast to $(p, \mathbf{A})$ is Walrasian equilibrium, hence it holds for all $j \in N$ that $v_j(A_j) - p(A_j) = v_j(A'_j) - p(A'_j)$ and therefore $(p', \mathbf{A})$ is a Walrasian equilibrium. □

4.2.2. Dynamic Prices

We formalize the notion of dynamic prices.

Definition 4.6. *In a weighted uniform matroid market $\mathcal{U} = (M, N, \mathbf{v})$ we call a price p* ***dynamic for buyer*** *$j \in N$ if it holds $\emptyset \neq D_j(p) \subseteq \mathrm{Opt}_j^{\mathcal{U}}$ and* ***dynamic*** *if it holds $\emptyset \neq D_j(p) \subseteq \mathrm{Opt}_j^{\mathcal{U}}$ for all $j \in N$.*

Given the existence of a dynamic price for every weighted uniform matroid market and assuming that buyers arrive once and *sequentially* to the market to acquire any of their utility maximizing set of the current inventory at the current price, then there exists a sequence of dynamic prices, called a **dynamic pricing**, such that every buyer j purchases a set A_j of the optimal sets for j and therefore $(A_1, \ldots, A_n)$ is an optimal allocation. This follows inductively by the fact that every residual market of a weighted uniform matroid market is again a weighted uniform matroid market. Therefore, since it suffices to prove the existence of dynamic prices for weighted uniform matroid markets to guarantee the existence of dynamic pricings, we focus on proving the existence of dynamic prices from here on.

Note that the seller does not care about its revenue and given a dynamic price it might be lower than in any minimal Walrasian price.

A Naïve Approach

The existence of a dynamic price in matching markets was shown by Cohen-Addad et al. [2016]. A naïve approach to generalize this result to weighted uniform matroid markets composed of matroids of arbitrary rank is as follows: Reduce the weighted uniform matroid market to a matching market by artificially splitting every buyer $j \in N$ into r_j unit-demand buyers and compute a dynamic price for the resulting matching market, which is candidate to serve as a dynamic price for the original market. It is obvious that a dynamic price for the original market also serve for dynamic price for the matching market. However, the converse is not necessarily true, as the following example shows.

Example 4.7. *Consider a weighted uniform matroid market with the items $M = \{w, x, y, z\}$ and buyers $N = \{1, 2, 3\}$ with valuations given by the table below on the left.*

	w	x	y	z	rank
v_1	1	1	1	1	2
v_2	1	1	0	0	1
v_3	0	0	1	1	1

	w	x	y	z	rank
v_{1_a}	1	1	1	1	1
v_{1_b}	1	1	1	1	1
v_2	1	1	0	0	1
v_3	0	0	1	1	1

Figure 4.1.: On the left the original market $\mathcal{U}$ and on the right the split market $\tilde{\mathcal{U}}$.

Then, the set of optimal allocations is given by $\{(A_1, A_2, A_3)\colon A_i \cap A_j = \emptyset$ for $1 \leq i < j \leq 3, |A_1 \cap \{w, x\}| = 1, |A_1 \cap \{y, z\}| = 1, |A_2 \cap \{w, x\}| = 1, |A_3 \cap \{y, z\}| = 1\}$. For completeness, we observe that $p \equiv 1$ is the only Walrasian price for $\mathcal{U}$, but p clearly does not serve as a dynamic price since $\{y, z\} \in D_1(p) \setminus \mathrm{Opt}_j^{\mathcal{U}}$. Now, assume that buyer 1 is split into the buyers 1_a and 1_b, each with unit-demand, then this results in the split market $\tilde{\mathcal{U}}$, given in the table above on the right. It is easy to see that $q \equiv 1 - \varepsilon$ for $0 < \varepsilon < 1$ is a dynamic price for $\tilde{\mathcal{U}}$ since $D_{1_a}(q) = \{\{w\}, \{x\}, \{y\}, \{z\}\} = \mathrm{Opt}_{1_a}^{\tilde{\mathcal{U}}}$. However, q is not a dynamic price for the original market $\mathcal{U}$ since it holds $\{w, x\} \in D_1(q) \setminus \mathrm{Opt}_1^{\mathcal{U}}$.

Notice, however, that for $0 < \varepsilon < 1$ prices $p^1(w) = p^1(y) = 1 - \varepsilon$, $p^1(x) = p^1(z) = 1$ do form a dynamic price for the original market. Clearly, it holds $D_1(p^1) = \{\{w, y\}\} \subseteq \mathrm{Opt}_1^{\mathcal{U}}$ since $(\{w, y\}, \{x\}, \{z\})$ is an optimal allocation and $D_2(p^1) = \{w\} \subseteq \mathrm{Opt}_2^{\mathcal{U}}$ since e.g., $(\{x, y\}, \{w\}, \{z\})$ and $D_3(p^1) = \{y\} \subseteq \mathrm{Opt}_3^{\mathcal{U}}$ since e.g., $(\{x, z\}, \{w\}, \{y\})$ is an optimal allocation.

Therefore, this naïve approach may not succeed and a more sophisticated approach is needed.

Sketch of our Approach

We briefly sketch how to construct a dynamic price in a weighted uniform matroid market: Our algorithm starts with an arbitrary interior Walrasian price. As it will turn out, interior Walrasian prices have the characteristic that for any buyer every demanded set contains only items that the buyer gets allocated in some optimal allocation. Further, the set of all items, given any interior Walrasian price, can be partitioned such that one part of the partition contains every item that is demanded by exactly one buyer and the item is contained in every demanded set of this buyer (hence, the item gets allocated to this buyer in every optimal allocation) and the other part of the partition contains the items that are contained in some demanded set of several buyers but are not contained in every demanded set of any buyer. We successively adjust the interior Walrasian start price by either decreasing the price of a well-chosen item slightly or increasing the price of all items slightly.

The price decrease of a fixed item does not induce new demand for this item but makes this item contained in *every* demanded set of every buyer who demanded this item before the price increase, hence reduces the number of items that are *not* contained in every demanded set for some buyer given the price. Therefore, the goal is to shrink sequentially the cardinality of the set of items that are contained in some demanded set of several buyers but not contained in every demanded set of some buyer given the price, until it becomes the empty set. The main difficulty in the choice of the next item x, of which the price decreases, is that if x is contained at the current price in the demanded items of e.g., the buyers i and j and buyer i will acquire a set of items containing x from the seller, (the price decrease will make x contained in every demanded set for i and j, hence the buyer necessarily will acquire x) then buyer j can be compensated with another item of equal utility for j (given the initial interior Walrasian price!) as x and vice versa.

Further, increasing the price of all items slightly forces any buyer, for whom the set of items that is contained in every of its demanded sets already is a set she gets allocated in some optimal allocation, to drop out of the competition.

4.2.3. Relevant and Trivial Markets

Definition 4.8. *We call a weighted uniform matroid market $\mathcal{U} = (M, N, \mathbf{v})$ **relevant** if:*

i) The market consists of at most $\sum_{j\in N} r_j$ items.

ii) It holds $\omega_{j,x} > 0$ for all $j \in N$ and $x \in \bigcup \mathrm{Opt}_j^{\mathcal{U}}$.

Note that a weighted uniform matroid market is relevant iff every optimal allocation allocates all items.

For ease of presentation, we make the following assumption for weighted uniform matroid markets:

Assumption 4.9. *All markets are relevant (if not otherwise stated).*

We briefly show that this assumption is w.l.o.g..

Lemma 4.10. *If there exist a dynamic price in every relevant weighted uniform matroid market, then there exists a dynamic price in every weighted uniform matroid market.*

Proof. To see that i) of definition 4.8 is w.l.o.g. assume that $\mathcal{U} = (M, N, \mathbf{v})$ is a weighted uniform matroid market with $|M| > \sum_{j\in N} r_j$ items. Clearly, there exists an optimal allocation $\mathbf{A} = (A_1, \ldots, A_n)$ for $\mathcal{U}$ in which $|A_j| = r_j$ for all $j \in N$. Let $\tilde{M} := \bigcup_{j\in N} A_j$ and consider the market $\tilde{\mathcal{U}} = (\tilde{M}, N, \mathbf{v}|_{2^{\tilde{M}}})$ and assume there exists a dynamic price $p|_{\tilde{M}}$ for $\tilde{\mathcal{U}}$, then setting $p(x) := \infty$ for $x \in M \setminus \tilde{M}$ yields a dynamic price p for $\mathcal{U}$.

To see that ii) of definition 4.8 is also w.l.o.g. assume there exists a weighted uniform matroid market $\mathcal{U} = (M, N, \mathbf{v})$ and an optimal allocation $\mathbf{A} = (A_1, \ldots, A_n)$ and $j \in N$ and $x \in A_j$ such that $\omega_{j,x} = 0$. Then, it holds $p(x) = 0$ and for $\tilde{M} := M \setminus \{x\}$ and $\tilde{\mathcal{U}} = (\tilde{M}, N, \mathbf{v}|_{2^{\tilde{M}}})$ it follows analogously to i) that if there exists a dynamic price $p|_{\tilde{M}}$ for $\tilde{\mathcal{U}}$, then setting $p(x) := \infty$ yields a dynamic price for $\mathcal{U}$. □

Next, we focus on trivial weighted uniform matroid markets.

Definition 4.11. *For a relevant weighted uniform matroid market with Walrasian price p we define*

$$\Delta(p) := \frac{1}{2} \cdot \min\{|a - b| \colon a \neq b, a, b \in \{0\} \cup \{u_j(x, p) \colon x \in M, j \in N\}\}$$

*which is one half of the minimum difference between different utility amounts of single items at a given price and the utility of the empty set. We call a relevant weighted uniform matroid market $\mathcal{U} = (M, N, \mathbf{v})$ **trivial**, if there exists a Walrasian price p such that $2 \cdot \Delta(p) = \min \emptyset := \infty$, hence $\omega_{j,x} = p(x)$ for all $j \in N$ and $x \in M$.*

Clearly, in any trivial weighted uniform matroid market there exists Walrasian price and the existence of dynamic prices is guaranteed, as the following theorem shows.

Theorem 4.12. *There exists a dynamic price for every trivial relevant weighted uniform matroid market.*

Proof. Let $\mathcal{U} = (M, N, \mathbf{v})$ a trivial relevant weighted uniform matroid market with Walrasian price p. Clearly, every allocation that allocates all items in an arbitrary way is optimal. Therefore, $p - \varepsilon$ for $0 < \varepsilon < \min_{x \in M} p(x)$ is a dynamic price. □

Assumption 4.13. *All Markets are nontrivial.*

4.2.4. Definitions from Graph Theory

For the exposition of our results we need some definitions and notations from graph theory.

Definition 4.14. *For either a simple graph or a digraph $G = (V, E)$ and a subset of edges (arcs) $F \subseteq E$ we denote by $G[F]$ the graph whose vertices are the vertices of G that are contained in some $e \in F$ and whose edges (arcs) are given by F and call it the by F **induced subgraph** and for $W \subseteq V$ we denote by $G - W := (V \setminus W, \{\{a, b\} \in E \colon a, b \notin W\})$ the **deletion** of W from V. In a graph $G = (V, E)$ the **degree** of a vertex $v \in V$ is denoted by $\deg_G(v)$ and in a digraph $G = (V, E)$ the **indegree** of $v \in V$ is denoted by $\operatorname{indeg}_G(v)$ and its **outdegree** is denoted by $\operatorname{outdeg}_G(v)$. For ease of notation, in a digraph we interpret a sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that $v_i \neq v_j$ and $(v_i, v_{i+1}) \in E$ for $1 \leq i < j \leq k$*

as the path $\{(v_i, v_{i+1})\colon 1 \leq i < k\}$ *and call a (di)graph* ***acyclic*** *if it contains no cycles. We call a bipartite graph* $G = (U, W, E)$ ***balanced****, if it holds* $|U| = |W|$ *and otherwise* ***unbalanced****. In a bipartite graph* $G = (U, W, E)$ *any subset of edges* $F \subseteq E$ *such that* $\deg_{G[F]}(x) \leq 1$ *for all* $x \in U \cup W$ *is called a* ***matching****. For* $S \subseteq U \cup W$ *a matching* F *is called* S***-saturating*** *if it holds* $\deg_{G[F]}(x) = 1$ *for all* $x \in S$*. In any balanced graph any* U*-saturating matching is also* W*-saturating and is called a* ***perfect matching****. We extend the notion of matchings to node capacitated bipartite graphs: Given a bipartite graph* $G = (U, W, E)$ *equipped with node capacities* $r_x \in \mathbb{Z}_+$ *for* $x \in U \cup W$*, any subset of edges* $F \subseteq E$ *such that* $\deg_{G[F]}(x) \leq r_x$ *for all* $x \in U \cup W$ *is called a* ***capacitated matching*** *and for* $S \subseteq U \cup W$ *a capacitated matching* F *is called* S***-saturating*** *if it holds* $\deg_{G[F]}(x) = r_x$ *for all* $x \in S$*. Call a node capacitated bipartite graph* $G = (U, W, E)$ *equipped with capacities* r_x *for* $x \in U \cup W$ ***capacitated balanced*** *if it holds* $\sum_{x \in U} r_x = \sum_{x \in W} r_x$ *and note that in a capacitated balanced graph* $G = (U, W, E)$ *it holds that any capacitated* U*-saturating matching is also capacitated* W*-saturating and called* ***perfect****.*

4.3. Interior Walrasian Prices

Recall that Walrasian prices exist in weighted uniform matroid markets. It is well known that in every market the set of Walrasian prices is polyhedral. (Recall that the empty set is also a polyhedron.) To see this, observe that, given an optimal allocation $\mathbf{A} = (A_1, \ldots, A_n)$ for the market $\mathcal{U} = (M, N, \mathbf{v})$, the set of Walrasian prices can, by Theorem 4.5, be described by

$$\{p \in \mathbb{R}_+^M : u_j(A_j, p) \geq u_j(T, p), \text{ for all } T \subseteq M, j \in N\}$$

and therefore is the intersection of finitely many closed half-spaces. In this section, we focus on Walrasian prices in the relative interior of this polyhedron.

Definition 4.15. *For a market* $\mathcal{U} = (M, N, \mathbf{v})$ *denote the* ***set of Walrasian prices*** *by* $\mathcal{W}$ *and call the relative interior of* $\mathcal{W}$ *defined by* $\operatorname{relint}(\mathcal{W}) := \{p \in \mathcal{W}\colon$ *there exists* $\varepsilon > 0$ *such that* $B_\varepsilon(p) \cap \operatorname{aff}(\mathcal{W}) \subseteq \mathcal{W}\}$*, where* $\operatorname{aff}(\mathcal{W})$ *is the affine hull of* $\mathcal{W}$ *and* $B_\varepsilon(p)$ *is the open ball of radius* ε *centered on* p*, the* ***set of interior Walrasian prices****. Then, any price* $p \in \operatorname{relint}(\mathcal{W})$ *is called an* ***interior Walrasian price****.*

In this section, we show that in any weighted uniform matroid market any interior Walrasian price induces no unnecessary demand, which clearly is a necessary condition for a price to be dynamic. Not only does this result nearly directly imply that interior Walrasian prices in relevant matching markets are already dynamic prices but it will be useful since interior Walrasian prices share several important characteristics (e.g., inducing no unnecessary demand) with dynamic prices and makes any interior Walrasian price suited to serve as the starting price of our dynamic price algorithm in Section 4.5. However, even though inducing no unnecessary demand is necessary for a price to be dynamic, it is not sufficient, as it is demonstrated in the Examples 4.7 and 4.27.

We put the definition of dynamic prices in a bit more context to Walrasian prices: On the one hand the notion of a Walrasian price is stronger in the sense that it supports allocations in which no buyer envies the bundle that any other buyer gets allocated. On the other hand the notion of a Walrasian price is weaker since for a given buyer it allows that the set of demanded sets given the price contains nonoptimal sets while a dynamic price requires the demanded sets to be a subset of the optimal sets.

4.3.1. Structure of Demand for Weighted Uniform Matroid Valuations

We explore the structure of the demanded sets and the demanded items of weighted uniform matroid valuations.

Definition 4.16. *Let $j \in N$ and v_j a (weighted uniform matroid) valuation and p a price. We define $X_j(p) := \bigcap_{S \in D_j(p)} S$ as the **strongly demanded items for** j and $Y_j(p) := \{x \in \bigcup D_j(p) \setminus X_j(p) \colon$ there exists $S \in D_j(p)$ such that $x \in S$ and $|S| \leq r_j\}$ as the **weakly demanded items for** j. (Note that $(X_j(p), Y_j(p))$ is not necessarily a partition of $\bigcup D_j(p)$ since there might exist items with a price of zero which could be contained in $\bigcup D_j(p)$ but there might already exist at least r_j other items in $\bigcup D_j(p)$ which provide a strict higher utility to the buyer.) In a (weighted uniform matroid) market $\mathcal{U} = (M, N, \mathbf{v})$ we define $X(p) := \bigcup_{j \in N} X_j(p)$ as the set of **strongly demanded items** and $Y(p) := \bigcup_{j \in N} Y_j(p)$ as the set of **weakly demanded items**.*

Clearly, in a weighted uniform matroid market with an arbitrary Walrasian price p it may hold $X(p) \cap Y(p) \neq \emptyset$ as it already can be seen by a simple one item and two buyers example. Clearly, if both buyers have different values for the item, then setting the price p as the lower of the buyers' values is a Walrasian price but it holds $X(p) \cap Y(p) \neq \emptyset$. However, it will turn out that every interior Walrasian price p of a weighted uniform matroid market induces no unnecessary demand, hence in particular it holds $X(p) \cap Y(p) = \emptyset$.

Based on the distinction of some buyer's strongly and weakly demanded items, given a price, we state the rather simple but important property of buyers with weighted uniform matroid valuations that any buyer's weakly demanded items all provide the same utility to her.

Lemma 4.17. *Let $j \in N$ and $v_j \colon 2^M \to \mathbb{R}_+$ be a weighted uniform matroid valuation with underlying matroid $U_M^{r_j}$ and p a price. Then, it holds*

$$u_j(x, p) = u_j(y, p) \text{ for all } x, y \in Y_j(p).$$

Proof. Assume there exists $x, y \in Y_j(p)$ such that $u_j(x,p) > u_j(y,p) \geq 0$. Since $x \in Y_j(p)$ there has to exist $A \in D_j(p)$ such that $x \notin A$ and $|A| = r_j$. (If $|A| < r_j$, by $u_j(x,p) > 0$, one could add x to A and improve j's utility and if $|A| > r_j$ then for $S \in \arg\max_{T \subseteq A, |T|=r_j} u_j(T,p)$ it holds $u_j(S,p) \geq u_j(A,p)$) and in particular it holds $y \notin A$ (otherwise, exchanging y with x yields a set with higher utility). Then, it has to hold $u_j(z,p) \geq u_j(x,p)$ for all $z \in A$ since if there exists $z \in A$ with $u_j(z,p) < u_j(x,p)$, then it holds $u_j(A,p) < u_j((A \setminus \{z\}) \cup \{x\}, p)$ in contrast to $A \in D_j(p)$. Then, for $B \in D_j(p)$ such that $y \in B$ and $|B| \leq r_j$ there exists $z \in A \setminus B$ and it holds $u_j(z,p) \geq u_j(x,p) > u_j(y,p)$, hence exchanging y with z yields a set with higher utility contradicting $B \in D_j(p)$. Therefore, the assumption can not hold and the claim follows. □

Note that it follows easily by the definition of the demanded sets and Lemma 4.17 that for any weighted uniform matroid valuation v_j and any price p and $y \in Y_j(p)$ and $z \in M$ such that $u_j(z,p) = u_j(y,p)$ it clearly holds $z \in Y_j(p)$.

We give an example of the demanded sets of a weighted uniform matroid valuation to develop some intuition for its specific structure.

Example 4.18. *Let $v_j \colon 2^{\{w,x,y,z\}} \to \mathbb{R}_+$ a weighted uniform matroid valuation with $r_j = 2$ with its item weights and item prices given by the following table.*

	w	x	y	z
v_j	10	10	8	4
p	1	2	0	0

Figure 4.2.: Item weights of v_j and item prices.

Then, it holds $D_j(p) = \{\{w,x\}, \{w,x,y\} \{w,x,z\}, \{w,x,y,z\}, \{w,y\}, \{w,y,z\}\}$ and thus $X_j(p) = \{w\}$ and $Y_j(p) = \{x,y\}$.

We define the direct sum of two families of sets to describe the demanded sets in a concise way.

Definition 4.19. *For any pair of sets A, B of sets we define $A \bigoplus B := \{s \colon s = a \cup b, a \in A, b \in B\}$.*

It follows directly by Lemma 4.17:

Corollary 4.20. *Let $j \in N$ and $v_j \colon 2^M \to \mathbb{R}_+$ a weighted uniform matroid valuation with underlying matroid $U^M_{r_j}$ and p a price.*

i) For all $x \in Y_j(p)$ it holds $u_j(x,p) > 0$ iff $D_j(p) = \{A \colon A = X_j(p) \sqcup B, B \subseteq Y_j(p), |B| = r_j - |X_j(p)|\} \bigoplus \{K \colon K \subseteq \{x \in M \setminus X_j(p) \colon p(x) = 0\}\}$.

ii) For all $x \in Y_j(p)$ *it holds* $u_j(x,p) = 0$ *iff* $D_j(p) = \{A\colon A = (X_j(p) \sqcup B), B \subseteq Y_j(p), |B| \leq r_j - |X_j(p)|\} \bigoplus \{K\colon K \subseteq \{x \in M \setminus X_j(p)\colon p(x) = 0\}\}$

Let v_j a weighted uniform matroid valuation and p a price such that $u_j(x,p) > 0$ for all $x \in Y_j(p)$, then it necessarily holds $|Y_j(p)| > r_j - |X_j(p)|$ by Corollary 4.20. Observe that for v_j it may hold $Y_j(p) \cap \{x \in M\colon p(x) = 0\} \neq \emptyset$. Furthermore, if p is strict positive, then the second $\bigoplus$-summand in Corollary 4.20 is empty.

As we consider only relevant markets it follows directly by Corollary 4.20, Theorem 4.5 and Assumption 4.9:

Corollary 4.21. *Let* $\mathcal{U} = (M, N, \mathbf{v})$ *a weighted uniform matroid market and* p *a Walrasian price. Then, it holds:*

i) For $j \in N$ *such that* $u_j(x,p) > 0$ *for* $x \in Y_j(p)$ *it holds that* $\mathrm{Opt}_j^{\mathcal{U}} \subseteq \{A\colon A = (X_j(p) \sqcup B), B \subseteq Y_j(p),\ |B| = r_j\} \subseteq D_j(p)$.

ii) For $j \in N$ *such that* $u_j(x,p) = 0$ *for* $x \in Y_j(p)$ *it holds that* $\mathrm{Opt}_j^{\mathcal{U}} \subseteq \{A\colon A = (X_j(p) \sqcup B), B \subseteq Y_j(p),\ |B| \leq r_j\} \subseteq D_j(p)$.

4.3.2. Equivalent Item Graph

For a weighted uniform matroid market $\mathcal{U} = (M, N, \mathbf{v})$ with Walrasian price p we fix some optimal allocation $\mathbf{A} = (A_1, \dots, A_n) \in \mathrm{Opt}^{\mathcal{U}}$ and introduce a directed auxiliary graph $G_{\mathbf{A}}(p)$ to represent transforms of $\mathbf{A}$ into other optimal allocations. The set of vertices of $G_{\mathbf{A}}(p)$ is $M \sqcup N$ and the set of arcs is defined as follows: From every item x that, given p, is weakly demanded by the buyer $\alpha^{\mathbf{A}}(x)$ there is an outgoing arc to every by $\alpha^{\mathbf{A}}(x)$ weakly demanded item $y \notin A_{\alpha^{\mathbf{A}}(x)}$, hence the utility of $\alpha^{\mathbf{A}}(x)$ does not change by adding y and giving up x (to someone yet undetermined; in fact one who chooses an arc into x). Furthermore, if any item $x \in A_j$ provides zero utility at price p to the buyer j she can give up x without changing her utility. This will be represented by an arc going from x to the vertex j. Conversely, for any buyer j such that $|A_j| < r_j$ and $x \in Y_j(p) \setminus A_j$ it necessarily holds $u_j(x,p) = 0$ by Theorem 4.5, hence buyer j can add x to A_j without changing her utility and this is indicated by an arc going from j to x.

Definition 4.22. *Let* $\mathcal{U} = (M, N, \mathbf{v})$ *a weighted uniform matroid market and* $(p, \mathbf{A} = (A_1, \dots, A_n))$ *a Walrasian equilibrium. We define the directed* ***equivalent item graph of*** $\mathbf{A}$ *by* $G_{\mathbf{A}}(p)$ *with* $V(G_{\mathbf{A}}(p)) := M \sqcup N$ *and*

$$\begin{aligned} E(G_{\mathbf{A}}(p)) := \quad & \{(x,y)\colon x \in A_j \cap Y_j(p), y \in Y_j(p) \setminus A_j, j \in N\} \\ & \sqcup \{(x,j)\colon x \in A_j \cap Y_j(p), u_j(x,p) = 0, j \in N\} \\ & \sqcup \{(j,y)\colon |A_j| < r_j, y \in Y_j(p) \setminus A_j, j \in N\}. \end{aligned}$$

*We say that we **reallocate along an arc** $(x, y) \in E(G_{\mathbf{A}}(p))$ iff for $x, y \in M$ we reallocate item y to buyer $\alpha^{\mathbf{A}}(x)$ or for $y \in M$ and $x \in N$ we additem y to buyer x or for $x \in M$ and $y \in N$ buyer y gives item x up.*

Next, we show that *all* optimal allocations are encoded in $G_{\mathbf{A}}(p)$.

Lemma 4.23. *Let $\mathcal{U} = (M, N, \mathbf{v})$ a weighted uniform matroid market and $(p, \mathbf{A})$ a Walrasian equilibrium. Given $F \subseteq E(G_{\mathbf{A}}(p))$ such that $\operatorname{indeg}_{G_{\mathbf{A}}(p)[F]}(x) = \operatorname{outdeg}_{G_{\mathbf{A}}(p)[F]}(x) \leq 1$ for all $x \in M$ and $0 \leq |A_j| - \operatorname{indeg}_{G_{\mathbf{A}}(p)[F]}(j) + \operatorname{outdeg}_{G_{\mathbf{A}}(p)[F]}(j) \leq r_j$ for all $j \in N$ a reallocation along the arcs of F transforms $\mathbf{A} = (A_1, \ldots, A_n)$ into another optimal allocation. Conversely, for every different optimal allocation $\mathbf{B} \in \mathrm{Opt}^{\mathcal{U}}$ there exist a set $F \subseteq E(G_{\mathbf{A}}(p))$ such that $\operatorname{indeg}_{G_{\mathbf{A}}(p)[F]}(x) = \operatorname{outdeg}_{G_{\mathbf{A}}(p)[F]}(x) \leq 1$ for all $x \in M$ and $0 \leq |A_j| - \operatorname{indeg}_{G_{\mathbf{A}}(p)[F]}(j) + \operatorname{outdeg}_{G_{\mathbf{A}}(p)[F]}(j) \leq r_j$ for all $j \in N$ and reallocating along the arcs of F transforms $\mathbf{A}$ to $\mathbf{B}$.*

Proof. Clearly, every $F \subseteq E(G_{\mathbf{A}}(p))$ such that $\operatorname{indeg}_{G_{\mathbf{A}}(p)[F]}(x) = \operatorname{outdeg}_{G_{\mathbf{A}}(p)[F]}(x) \leq 1$ for all $x \in M$ and $0 \leq |A_j| - \operatorname{indeg}_{G_{\mathbf{A}}(p)[F]}(j) + \operatorname{outdeg}_{G_{\mathbf{A}}(p)[F]}(j) \leq r_j$ allocates all items and therefore transforms $\mathbf{A}$ into another optimal allocation due to Lemma 4.17 and Corollary 4.21.

Conversely, for an alternative optimal allocation $\mathbf{B} = (B_1, \ldots, B_n)$ let F be constructed as follows: For every $j \in N$ such that $|A_j| = |B_j|$ choose any bijection $f_j \colon A_j \setminus B_j \to B_j \setminus A_j$ and let $(x, y) \in F$ iff $f_j(x) = y$. For every $j \in N$ such that $|A_j| < |B_j|$ choose any injective $f_j \colon A_j \setminus B_j \to B_j \setminus A_j$ and let $(x, y) \in F$ iff $x \in A_j \setminus B_j$ and $y = f_j(x)$ or $x = j$ and $y \in (B_j \setminus A_j) \setminus f(A_j)$. For every $j \in N$ such that $|A_j| > |B_j|$ choose any injective $f_j \colon B_j \setminus A_j \to A_j \setminus B_j$ and let $(x, y) \in F$ iff $y \in B_j \setminus A_j$ and $f_j(y) = x$ or $y = j$ and $x \in (A_j \setminus B_j) \setminus f(B_j)$. Then, reallocating along the arcs of F transforms $\mathbf{A}$ to $\mathbf{B}$ by construction and F fulfills $\operatorname{indeg}_{G_{\mathbf{A}}(p)[F]}(x) = \operatorname{outdeg}_{G_{\mathbf{A}}(p)[F]}(x) \leq 1$ for all $x \in M$ and $0 \leq |A_j| - \operatorname{indeg}_{G_{\mathbf{A}}(p)[F]}(j) + \operatorname{outdeg}_{G_{\mathbf{A}}(p)[F]}(j) \leq r_j$. □

We remark that every set F in Lemma 4.23 is the disjoint union of cycles and (u, v)-paths with $u, v \in N$. To see this, let $\mathbf{B}$ an alternative optimal allocation and $F \subseteq E(G_{\mathbf{A}}(p))$ such that reallocating along the arcs of F transforms $\mathbf{A}$ into $\mathbf{B}$. Let $x \in M$ such that $\alpha^{\mathbf{A}}(x) \neq \alpha^{\mathbf{B}}(x)$, hence $\operatorname{indeg}_{G_{\mathbf{A}}(p)[F]}(x) = \operatorname{outdeg}_{G_{\mathbf{A}}(p)[F]}(x) = 1$ and assume x not to be contained in any cycle. Then, there exists $y, z \in M$ such that $\{(y, x), (x, z)\} \subseteq F$ and since M is finite and it holds $\operatorname{indeg}_{G_{\mathbf{A}}(p)[F]}(w) = \operatorname{outdeg}_{G_{\mathbf{A}}(p)[F]}(w) \leq 1$ for all $w \in M$ it follows inductively that $\{(y, x), (x, z)\}$ is contained in some (u, v)-path with $u, v \in N$.

For a directed graph, we formally define the set of vertices that are reachable by a path from a given vertex.

Definition 4.24. *For a digraph $G = (V, E)$ we say an item $y \in V$ is a **successor of** $x \in V$ in G if it holds $x = y$ or there exist a path $P = (x, \ldots, y) \subseteq E$. For $x \in V$ we define*

$R_G(x) := \{y \in V : y$ is a successor from x in $G\}$ and $L_G(x) := \{y \in V : x$ is a successor from y in $G\}$.

The following corollary puts the concept of successor in relation with some fixed buyer's weakly demanded items in weighted uniform matroid markets.

Corollary 4.25. *Let $\mathcal{U} = (M, N, \mathbf{v})$ a weighted uniform matroid market, $(p, \mathbf{A})$ a Walrasian equilibrium and $x \in Y(p)$.*

i) If it holds $R_{G_{\mathbf{A}}(p)}(x) \cap N = \emptyset$ then for $j \in \alpha(R_{G_{\mathbf{A}}(p)}(x))$ it holds $Y_j(p) \setminus \bigcap_{S \in \mathrm{Opt}_j^{\mathcal{U}}} S \subseteq R_{G_{\mathbf{A}}(p)}(x)$.

ii) If it holds $L_{G_{\mathbf{A}}(p)}(x) \cap N = \emptyset$ then for $j \in \alpha(L_{G_{\mathbf{A}}(p)}(x))$ it holds $Y_j(p) \subseteq L_{G_{\mathbf{A}}(p)}(x)$.

Proof. i) For every $y \in R_{G_{\mathbf{A}}(p)}(x)$ and $j \in \alpha(y)$ it either holds $\alpha^{\mathbf{A}}(y) = j$ or by Lemma 4.23 there exists a cycle $C \subseteq E(G_{\mathbf{A}}(p))$ without N and reallocating along the arcs of C yields an optimal allocation $\mathbf{B}$ with $j = \alpha^{\mathbf{B}}(y)$. In either case, it has to exist $z \in R_{G_{\mathbf{A}}(p)}(x)$ with $\alpha^{\mathbf{A}}(z) = j$. Then, it clearly holds $R_{G_{\mathbf{A}}(p)}(z) \cap N = \emptyset$ and therefore $R_{G_{\mathbf{A}}(p)}(t) \cap N = \emptyset$ for all $t \in Y_j(p)$ by the definition of $G_{\mathbf{A}}(p)$. Thus, by Lemma 4.23, every $t \in Y_j(p) \setminus \bigcap_{S \in \mathrm{Opt}_j^{\mathcal{U}}} S$ has to be contained in a cycle $C = (t = t_1, t_2, \dots, t_k, t)$ and there exists a $z - t$ path $P = (z, t_2, \dots, t_k, t)$ and it follows $Y_j(p) \setminus \bigcap_{S \in \mathrm{Opt}_j^{\mathcal{U}}} S \subseteq R_{G_{\mathbf{A}}(p)}(x)$.

ii) For every $y \in L_{G_{\mathbf{A}}(p)}(x)$ and $\alpha^{\mathbf{A}}(y) \neq j$ it clearly holds $(z, y) \in G_{\mathbf{A}}(p)(x)$ for all $z \in Y_j(p)$ by the definition of $G_{\mathbf{A}}(p)(x)$, hence $z \in L_{G_{\mathbf{A}}(p)}(x)$. For every $y \in L_{G_{\mathbf{A}}(p)}(x)$ and $j \in \alpha(y)$ and $\alpha^{\mathbf{A}}(y) = j$ there exists a path $P = (y = y_1, y_2, \dots, x)$ and it holds $(z, y_2) \in E(G_{\mathbf{A}}(p)(x))$ for every $z \in Y_j(p)$, hence $z \in L_{G_{\mathbf{A}}(p)}(x)$. □

4.3.3. Interior Walrasian Prices Induce no Unnecessary Demand

Next, we make use of the equivalent item graph to show the main result of this section, namely that in every weighted uniform matroid market all interior Walrasian prices induce no unnecessary demand.

Theorem 4.26. *Let $\mathcal{U} = (M, N, \mathbf{v})$ a weighted uniform matroid market and p an interior Walrasian price. Then, for all $j \in N$ it holds $\bigcup D_j(p) = \bigcup \mathrm{Opt}_j^{\mathcal{U}}$ and $X_j(p) = \{x \in M : \alpha(x) = \{j\}\}$.*

Proof. Let $\mathbf{A} = (A_1, \dots, A_n)$ an arbitrary optimal allocation, thus $\bigsqcup_{j \in N} A_j = M$.

<u>Claim 4:</u> It holds $p > 0$.
<u>Proof of claim:</u> Suppose there exists $x \in M$ such that $p(x) = 0$. Then, it holds $R_{G_{\mathbf{A}}(p)}(x) \cap N = \emptyset$ since if there exists $v \in R_{G_{\mathbf{A}}(p)}(x) \cap N$ then there has to exist a (x, v)-path P and

reallocating along the edges of P yields an alternative optimal allocation in which x remains unallocated in contrast to $\mathcal{U}$ being relevant. Therefore, for $j \in \bigcup_{y \in R_{G_{\mathbf{A}}(p)}(x)} \alpha(y)$ it holds $v_j(z) - p(z) > 0$ for all $z \in Y_j(p)$. Let $p' := p + \Delta(p) \cdot \chi^{R_{G_{\mathbf{A}}(p)}(x)}$, then it holds for all $j \in \bigcup_{y \in R_{G_{\mathbf{A}}(p)}(x)} \alpha(y)$ that $Y_j(p') = (Y_j(p) \setminus \bigcap_{S \in \mathrm{Opt}_j^{\mathcal{U}}} S) \cup \{x\}$ and $X_j(p') = \bigcap_{S \in \mathrm{Opt}_j^{\mathcal{U}}} S \setminus \{x\}$ and by Corollary 4.25 and the definition of $\Delta(p)$ and therefore it holds $\mathrm{Opt}_j^{\mathcal{U}} = \mathrm{Opt}_j^{\mathcal{U}} \cap D_j(p) = \mathrm{Opt}_j^{\mathcal{U}} \cap D_j(p')$ by Corollary 4.21. Clearly, for all $j \in N \setminus \bigcup_{y \in R_{G_{\mathbf{A}}(p)}(x)} \alpha(y)$ it holds $\bigcup \mathrm{Opt}_j^{\mathcal{U}} \subseteq M \setminus R_{G_{\mathbf{A}}(p)}(x)$ and since $p' \geq p$ it trivially holds $\mathrm{Opt}_j^{\mathcal{U}} = \mathrm{Opt}_j^{\mathcal{U}} \cap D_j(p) = \mathrm{Opt}_j^{\mathcal{U}} \cap D_j(p')$, thus p' is a Walrasian price. However, for all $\varepsilon > 0$ it holds that $p'' := p - \varepsilon \cdot \chi^{R_{G_{\mathbf{A}}(p)}(x)} \notin \mathcal{W}$ since $p''(x) < 0$. This contradicts $p \in \mathrm{relint}(\mathcal{W})$. ■

<u>Claim 5:</u> For every $x \in M$ it holds $\mathrm{indeg}_{G_{\mathbf{A}}(p)}(x) > 0$ iff it holds $\mathrm{outdeg}_{G_{\mathbf{A}}(p)}(x) > 0$.

<u>Proof of claim:</u> Assume there exists $x \in M$ such that $(y, x) \in E(G_{\mathbf{A}}(p))$ and it holds $\mathrm{outdeg}_{G_{\mathbf{A}}(p)}(x) = 0$. Then, it holds $\alpha(x) = \{\alpha^{\mathbf{A}}(x)\}$ and $x \in X_{\alpha^{\mathbf{A}}(x)}(p)$ and therefore $p + \Delta(p) \cdot \chi^{\{x\}} \in \mathcal{W}$ and for $\Delta(p) > \varepsilon > 0$ and $p' := p - \varepsilon \cdot \chi^{\{x\}}$ it holds $p' > 0$ by Claim 4. As the price of x is lowered it holds $x \in X_y(p')$ if $y \in N$ respectively $x \in X_{\alpha^{\mathbf{A}}(y)}(p')$ if $y \in M$ by Lemma 4.17 but $x \notin \bigcup \mathrm{Opt}_y^{\mathcal{U}}$ respectively $x \notin \bigcup \mathrm{Opt}_{\alpha^{\mathbf{A}}(y)}^{\mathcal{U}}$ and therefore $p' \notin \mathcal{W}$ by Theorem 4.5, hence $p \notin \mathrm{relint}(\mathcal{W})$.

Analogously, assume there exists $y \in M$ such that $(y, x) \in E(G_{\mathbf{A}}(p))$ and it holds $\mathrm{indeg}_{G_{\mathbf{A}}(p)}(y) = 0$. Then, it holds $\alpha(y) = \{\alpha^{\mathbf{A}}(y)\}$ but $y \in Y_{\alpha^{\mathbf{A}}(y)}(p)$ and therefore $p - \Delta(p) \cdot \chi^{\{y\}} \in \mathcal{W}$ but for $\varepsilon > 0$ and $p' := p + \varepsilon \cdot \chi^{\{y\}}$ it holds $y \notin \bigcup D_j(p')$ by Lemma 4.17 and therefore $p' \notin \mathcal{W}$ by Theorem 4.5, hence $p \notin \mathrm{relint}(\mathcal{W})$. ■

<u>Claim 6:</u> Every $(x, y) \in E(G_{\mathbf{A}}(p))$ is contained in a cycle or a (u, v)-path with $u, v \in N$.

<u>Proof of claim:</u> Let $F := \{(x, y) \in E(G_{\mathbf{A}}(p)) \colon (x, y)$ is not contained in a cycle or a (u, v)-path with $u, v \in N\}$ and $F_1 := \{(x, y) \in F \colon L_{G_{\mathbf{A}}(p)}(x) \cap N = \emptyset\}$ and $F_2 := \{(x, y) \in F \setminus F_1 \colon R_{G_{\mathbf{A}}(p)}(x) \cap N = \emptyset\}$. If $F_1 \neq \emptyset$ let $(x, y) \in F_1$ such that $\bigcup_{(w,z) \in F_1} \{z\} \cap L_{G_{\mathbf{A}}(p)}(x) = \emptyset$. (Clearly, such a (x, y) has to exist due to Claim 5 and the finiteness of M since otherwise there would exist a cycle containing a subset of F_1 contradicting the definition of F.) Notice that for every $z \in L_{G_{\mathbf{A}}(p)}(x) \setminus \{x\}$ and $(z, w) \in E(G_{\mathbf{A}}(p))$ it holds $(z, w) \notin F$ since clearly $L_{G_{\mathbf{A}}(p)}(z) \cap N = \emptyset$. Then, for $p' := p - \Delta(p) \cdot \chi^{L_{G_{\mathbf{A}}(p)}(x)}$ it holds $p' > 0$ by Claim 4 and for $j \in \bigcup_{z \in L_{G_{\mathbf{A}}(p)}(x)} \alpha(z)$ it holds $X_j(p) = X_j(p')$ and $Y_j(p) = Y_j(p')$ by Corollary 4.25 and therefore $\mathrm{Opt}_j^{\mathcal{U}} = \mathrm{Opt}_j^{\mathcal{U}} \cap D_j(p) = \mathrm{Opt}_j^{\mathcal{U}} \cap D_j(p')$ by Corollary 4.21 and for $j \in N \setminus \bigcup_{z \in L_{G_{\mathbf{A}}(p)}(x)} \alpha(z)$ it trivially holds $D_j(p) = D_j(p')$ and therefore holds $p' \in \mathcal{W}$. However, for $\varepsilon > 0$ and $p'' := p + \varepsilon \cdot \chi^{L_{G_{\mathbf{A}}(p)}(x)}$ it holds $u_{\alpha^{\mathbf{A}}(x)}(x, p'') < u_{\alpha^{\mathbf{A}}(x)}(y, p'')$ and by Theorem 4.5 and the Corollaries 4.20 and 4.21 this contradicts that p'' is Walrasian. Therefore, $p \notin \mathrm{relint}(\mathcal{W})$ contrary to the assumption and $F_1 = \emptyset$.

Analogously, if it holds $F_2 \neq \emptyset$ let $(x, y) \in F_2$ such that $\bigcup_{(w,z) \in F_2} \{w\} \cap R_{G_{\mathbf{A}}(p)}(y) = \emptyset$.

Then, for $p' := p + \Delta(p) \cdot \chi^{R_{G_{\mathbf{A}}(p)}(y)}$ it holds that $p' \in \mathcal{W}$. However, for $\Delta(p) > \varepsilon > 0$ and $p'' := p - \varepsilon \cdot \chi^{R_{G_{\mathbf{A}}(p)}(y)}$ it has to hold $y \in X_{\alpha^{\mathbf{A}}(x)}(p'')$ if $x \in M$ respectively $y \in X_x(p'')$ if $x \in N$ by Lemma 4.17 and therefore $p'' \notin \mathcal{W}$ by Theorem 4.5 since the allocation $\mathbf{A}$ is not supported by p''. Hence, $F_2 \neq \emptyset$ contradicts $p \in \operatorname{relint}(\mathcal{W})$ and therefore holds $F_2 = \emptyset$, hence $F = \emptyset$ what yields the claim. ■

Now, the theorem follows directly by Claim 6 and Lemma 4.23. □

It follows directly by Theorem 4.26 that for every weighted uniform matroid market and every interior Walrasian price p it holds that $X(p) \cap Y(p) = \emptyset$.

For completeness, we note that the assumption in Theorem 4.26 is not sufficient to guarantee Walrasian prices to be contained in the relative interior, as the following example shows:

Example 4.27. *Consider the matching market $\mathcal{U}$ with item set $M = \{x, y\}$ and buyers $N = \{1, 2\}$ given by the following table:*

	x	y	rank
v_1	2	2	1
v_2	1	1	1

Figure 4.3.: The market $\mathcal{U}$.

Then, $\mathcal{W} = \{p \in \mathbb{R}^2 \colon 0 \leq p(x) = p(y) \leq 1\}$ and e.g., for $p = (1, 1)$ it holds $\bigcup D_j(p) = \bigcup \operatorname{Opt}_j^{\mathcal{U}} = \{x, y\}$ and $X_j(p) = \emptyset = \{x \in M \colon \alpha(x) = \{j\}\}$ for $j \in \{1, 2\}$ but $p \in \mathcal{W} \setminus \operatorname{relint}(\mathcal{W})$.

We use the nice characterization of interior Walrasian prices given by Theorem 4.26 to strengthen the main result of Cohen-Addad et al. [2016], which shows that dynamic prices are guaranteed to exist in *matching* markets. To this end, we partition the set of buyers into those that secure in every optimal allocation a bundle of their rank and those that do not.

Definition 4.28. *For a weighted uniform matroid market $\mathcal{U} = (M, N, \mathbf{v})$ we denote $N_= := \{j \in N \colon |S| = r_j$ for all $S \in \operatorname{Opt}_j^{\mathcal{U}}\}$ and $N_< := N \setminus N_= = \{j \in N \colon |S| < r_j$ for some $S \in \operatorname{Opt}_j^{\mathcal{U}}\}$.*

By Assumption 4.9 of relevance there cannot exists $j \in N$ with $S \in \operatorname{Opt}_j^{\mathcal{U}}$ such that $|S| > r_j$.

Lemma 4.29. *In a weighted uniform matroid market $\mathcal{U} = (M, N, \mathbf{v})$ for every $p \in \operatorname{relint}(\mathcal{W})$ it holds:*

i) $u_j(x, p) = 0$ for $j \in N_<$ and $x \in Y_j(p)$, *ii)* $u_j(x,p) > 0$ *for* $j \in N_=$ *and* $x \in Y_j(p)$.

Proof. Trivially, i) follows by Lemma 4.17. To see that ii) holds for true assume there exists a weighted uniform matroid market $\mathcal{U} = (M, N, \mathbf{v})$ such that $p \in \text{relint}(\mathcal{W})$ and $j \in N_=$ and $y \in Y_j(p)$ such that $u_j(y,p) = 0$. Let $\mathbf{A} = (A_1, \ldots, A_n)$ an optimal allocation such that $y \in A_j$ and recall that it holds $(y,j) \in E(G_\mathbf{A}(p))$. If it holds $L_{(G_\mathbf{A}(p)}(y) \cap N = \emptyset$, then it holds $p - \Delta(p) \cdot \chi^{L_{G_\mathbf{A}(p)}(y)} \in \mathcal{W}$ by Corollary 4.25 but for all $\varepsilon > 0$ it holds $p' := p + \varepsilon \cdot \chi^{L_{G_\mathbf{A}(p)}(y)} \notin \mathcal{W}$ since it holds $u_j(y,p') < 0$ what contradicts $p \in \text{relint}(\mathcal{W})$. If it holds $u \in L_{G_\mathbf{A}(p)}(y) \cap N$ then it holds $u \neq y$ by definition of $G_\mathbf{A}(p)$ and there exists a path $P = (u, \ldots, y, \ldots, j)$, thus by Lemma 4.23 reallocating along the edges P yields an optimal allocation $\mathbf{B} = (B_1, \ldots, B_n)$ such that $|B_j| = r_j - 1$ contradicting $j \in N_=$. □

Now, a stronger version of the main result of Cohen-Addad et al. [2016] follows easily by Theorem 4.26 and Lemma 4.29:

Corollary 4.30. *Every interior Walrasian price of a weighted matching market is a dynamic price.*

Proof. Let $\mathcal{U} = (M, N, \mathbf{v})$ a matching market with $p \in \text{relint}(\mathcal{W})$, then it holds for $j \in N_=$ such that $X_j(p) \neq \emptyset$ that that there exists exactly one item x that j gets allocated in every optimal allocation, hence $D_j(p) = \{X_j(p)\} = \{\{x\}\} = \text{Opt}_j^\mathcal{U}$. For $j \in N_=$ such that $X_j(p) = \emptyset$ it holds that $\{\{x\} \colon x \in Y_j(p)\} = \text{Opt}_j^\mathcal{U}$ but clearly also $D_j(p) = \{\{x\} \colon x \in Y_j(p)\}$. For $j \in N_<$ it holds that in any optimal allocation j either gets allocated nothing or some item of $Y_j(p)$, hence $\{\emptyset\} \cup \{\{x\} \colon x \in Y_j(p)\} = \text{Opt}_j^\mathcal{U}$ but clearly also $D_j(p) = \{\emptyset\} \cup \{\{x\} \colon x \in Y_j(p)\}$. □

We briefly rephrase the result of this section: Interior Walrasian prices in weighted uniform matroid markets induce for any fixed buyer no demand for items that the buyer does not get allocated in some optimal allocation and thus serve as dynamic prices for matching markets. However, by Example 4.7, interior Walrasian prices in more general markets may induce some buyers demand for *sets* that the buyer does not get allocated in optimal allocation and thus generally do not serve as dynamic prices in weighted uniform matroid markets.

4.4. Interpreting Weighted Uniform Matroid Markets as Bipartite Graphs

In this section we use interior Walrasian prices to interpret weighted uniform matroid markets as node capacitated bipartite graphs, which encode all optimal allocations through their saturating capacitated matchings.

Motivated by Lemma 4.17 and Theorem 4.26 we partition the set of items of a weighted uniform matroid market depending on the set of buyers that receive an item in some optimal allocation:

Definition 4.31. *For a weighted uniform matroid market* $\mathcal{U} = (M, N, \mathbf{v})$ *and* $S \subseteq N$ *we define* $[S] := \{x \in M : S = \alpha(x)\}$.

Clearly, $M = \bigsqcup_{S \subseteq N}[S]$ by Assumption 4.9. Furthermore, for every interior Walrasian price p it holds $X(p) = \bigsqcup_{j \in N}[\{j\}]$ and $Y(p) = M \setminus X(p)$ by Theorem 4.26. Then, for every optimal allocation $\mathbf{A} = (A_1, \ldots, A_n)$ and every $S \subseteq N$ such that $|S| \geq 2$ and $|[S]| \geq 2$ and $x, y \in [S]$, $x \neq y$ it holds that exchanging the items x and y between $\alpha^{\mathbf{A}}(x)$ and $\alpha^{\mathbf{A}}(y)$ still yields an optimal allocation by Corollary 4.21 and Theorem 4.12. (Note that this clearly does not hold for general weighted matroid markets! See Appendix A.1 for such an example.) Therefore, for $S \subseteq N$ the set $[S]$ forms an equivalence class of these items which are in the set of optimal items of every buyer in S and *not* in the set of optimal items of any buyer in $N \setminus S$. Then, in order to implement an arbitrary optimal allocation it suffices to consider the equivalence class of an item without distinguishing the members of the class. In the following, we identify the set of items M with $\{(S, k) : S \subseteq N, [S] \neq \emptyset, 1 \leq k \leq |[S]|\}$ where $(S, 1), \ldots, (S, |[S]|)$ are the equivalent items of the class $[S]$. Clearly, for any interior Walrasian price p and $S \subseteq N$ it holds by Theorem 4.26 that $[S] = \{x \in M : x \in \bigcap_{j \in S} \bigcup D_j(p), x \notin \bigcup_{j \in N \setminus S} \bigcup D_j(p)\}$.

Notice that the partition of M into the equivalence classes $[S]$, $S \subseteq N$ can be induced by any interior Walrasian price and is independent of the concrete choice of the interior Walrasian price.

Next, we associate every weighted uniform matroid market $\mathcal{U} = (M, N, \mathbf{v})$ a unique bipartite node capacitated graph in which the vertices are the buyers and the items of the market and there is an edge going from some buyer j to an item x iff it holds $j \in \alpha(x)$. We denote

$$G(\mathcal{U}) := (M, N, \{\{(S, k), j\} : (S, k) \in M, j \in S \subseteq N\})$$

and call $G(\mathcal{U})$ **market graph** associated to $\mathcal{U}$. In the following, let the vertices $V(G(\mathcal{U}))$ be equipped with node capacities $r_x = 1$ for $x \in M$ and r_j for $j \in N$ (recall that r_j is the rank oft the v_j underlying matroid $U_M^{r_j}$). For $x \in M$ and $j \in N$ interpret the edge $\{x, j\} \in E(G(\mathcal{U}))$ as the allocation of x to j. This allows to identify any allocation $\mathbf{A} = (A_1, \ldots, A_n)$ such that $A_j \subseteq \bigcup \mathrm{Opt}_j^{\mathcal{U}}$, $|A_j| \leq r_j$ for all $j \in N$ with a capacitated matching of $G(\mathcal{U})$ by setting $F := \{\{x, j\} \in E(G(\mathcal{U})) : x \in A_j\}$. Therefore, it follows directly by Corollary 4.21 and the Theorems 4.5 and 4.26 that *all* optimal allocations are encoded by $G(\mathcal{U})$ simultaneously:

Lemma 4.32. *For any weighted uniform matroid market $\mathcal{U} = (M, N, \mathbf{v})$ and its associated market graph $G(\mathcal{U})$ the set of optimal allocations can be identified with the set of $M \sqcup N_=$-saturating capacitated matchings of $G(\mathcal{U})$.*

Clearly, in every optimal allocation of every balanced weighted uniform matroid market every buyer j gets allocated a set of exactly rank r_j by Assumption 4.9. Therefore, it follows directly by Lemma 4.32:

Corollary 4.33. *In any balanced weighted uniform matroid market $\mathcal{U}$ the set of optimal allocations can be identified with the set of perfect capacitated matchings of $G(\mathcal{U})$.*

It follows directly by Lemma 4.32 that a buyer's optimal subsets are encoded in $G(\mathcal{U})$.

Corollary 4.34. *Let $\mathcal{U} = (M, N, \mathbf{v})$ a weighted uniform matroid market. Then, for all $F \subseteq G(\mathcal{U})\}$ it holds that $\mathbf{A} = (A_1, \dots, A_n)$ with $A_j := \{x \in M : \{x, j\} \in F\}$ for $j \in N$ is an allocation that can be completed to an optimal allocation iff there exists $L \subseteq E(G(\mathcal{U}))$ such that $F \subseteq L$ and L is a $M \sqcup N_=$-saturating capacitated matching.*

In particular, Corollary 4.34 implies for the weighted uniform matroid market $\mathcal{U} = (M, N, \mathbf{v})$ and $j \in N$ that A_j is an optimal subset for j iff $\{\{x, j\} : x \in A_j\}$ is subset of some $M \sqcup N_=$-saturating capacitated matching of $G(\mathcal{U})$.

In weighted uniform matroid markets, we want to establish a combinatorial condition that guarantees a set to be an optimal subset for a fixed buyer. To this end, we recall the marriage theorem of Hall.

Theorem 4.35. *[Hall, 1935] Let $G = (U, W, E)$ a bipartite graph. Then, there exists a W-saturating matching iff for all $S \subseteq W$ holds $|N_G(S)| \geq |S|$.*

As we need a statement in the spirit of Hall's theorem that applies to the market graph, we generalize Hall's theorem to node capacitated bipartite graphs.

Corollary 4.36. *Let $G = (U, W, E)$ a node-capacitated bipartite graph with capacities $r \in \mathbb{Z}_+^{U \sqcup W}$. Then, there exists a W-saturating capacitated matching iff for all $S \subseteq W$ holds $\sum_{i \in N_G(S)} r_i \geq \sum_{i \in S} r_i$.*

Proof. Let $G' = (U', W', E')$ with $U' := \{(u, k) : u \in U, 1 \leq j \leq r(u)\}$, $W' := \{(w, k) : w \in W, 1 \leq k \leq r(p)\}$ and $E' := \{\{(u, j), (w, k)\} : (u, j) \in U', (w, k) \in W', \{u, w\} \in E\}$, then, clearly, every W'-saturating matching $F \subseteq E$ can be easily transferred into a capacitated W-saturating matching $F' \subseteq E'$ and vice versa.

Now, observe that for $A' \subseteq W'$ and $T' := \bigcup_{(w,l) \in A'} \{(w, k) : 1 \leq k \leq r_w\}$ it holds $|N_{G'}(A')| = |N_{G'}(T')|$ and $|T'| \geq |A'|$ and therefore by Theorem 4.35 there exists a W'-saturating matching in G' iff for all $A' \subseteq W'$ such that for all $w \in W$ holds $|A' \cap \{(w, k) : 1 \leq$

$k \le r_w\}| \in \{0, r_w\}$ holds $|A'| \le |N_{G'}(A')|$. Then, the claim follows directly since there exists a bijection from 2^W to $\{A' \subseteq W' \colon A' \cap \{(w,k) \colon 1 \le k \le r_w\}| \in \{0, r_w\}$ for all $w \in W\}$ and for $S \subseteq W$ it clearly holds $\sum_{i \in S} r_i = |\{(w,k) \colon w \in S, 1 \le k \le r_w\}|$ and $\sum_{i \in N_G(S)} r_i = |N_{G'}(\{(w,k) \colon w \in S, 1 \le k \le r_w\})|$. □

4.4.1. Hall's Condition for Balanced Weighted Uniform Matroid Markets

Before we use Corollary 4.36 to provide a necessary and sufficient condition that a given set of items is an optimal subset for a given buyer in a balanced weighted uniform matroid market, we define for ease of notation:

Definition 4.37. *Let $\mathcal{U} = (M, N, \mathbf{v})$ a balanced weighted uniform matroid market. For $S \subseteq N$ we denote $\rho(S) := \bigcup_{T \subseteq S}[T]$ and $\sigma(S) := \bigcup_{T \subseteq N \colon T \cap S \neq \emptyset}[T]$.*

Therefore, for every subset of buyer S the set $\rho(S)$ are those items which in every perfect capacitated matching of the market graph get matched with some buyers of S and $\sigma(S)$ are those items which in at least one perfect capacitated matching of the market graph get matched with some buyer of S. Notice that for $j \in N$ it holds $\sigma(\{j\}) = \bigcup \mathrm{Opt}_j^{\mathcal{U}}$, thus σ generalizes the notion of any buyer's optimal items from a single buyer to a set of buyers.

Observe that ρ and σ are nondecreasing and for $S, T \subseteq N$ it holds:

- $\sigma(S) = M \setminus \rho(\overline{S})$,
- $\sigma(\{j\}) = \bigcup_j^{\mathcal{U}}$ for $j \in N$,
- $\rho(S) \cap \rho(T) = \{x \in M \colon \alpha(x) \subseteq S \text{ and } \alpha(x) \subseteq T\} = \{x \in M \colon \alpha(x) \subseteq S \cap T\} = \rho(S \cap T)$,
- $\sigma(S) \cup \sigma(T) = \{x \in M \colon \alpha(x) \cap S \neq \emptyset \text{ or } \alpha(x) \cap T \neq \emptyset\} = \{x \in M \colon \alpha(x) \cap S \cup T \neq \emptyset\} = \sigma(S \cup T)$,
- $\rho(S) \cup \rho(T) = \{x \in M \colon \alpha(x) \subseteq S \text{ or } \alpha(x) \subseteq T\} \subseteq \{x \in M \colon \alpha(x) \subseteq S \cup T\} = \rho(S \cup T)$.

Lemma 4.38. *Let $\mathcal{U} := (M, N, \mathbf{v})$ a balanced weighted uniform matroid market and $j \in N$ and $I \subseteq \sigma(\{j\})$. Then, it holds that I is an optimal subset for j (hence, the allocation $\mathbf{A} = (A_1, \dots, A_n)$ with $A_j = I, A_i = \emptyset$ for $i \in N \setminus \{j\}$ can be completed to an optimal allocation) iff it holds*

$$|\sigma(S) \setminus I| \ge \sum_{i \in S} r_i \quad \text{for all } S \subseteq N \setminus \{j\}. \tag{4.1}$$

Proof. Let $\bar{r}_j = r_j - |I|$ and $\bar{r}_i = r_i$ for $i \in N \setminus \{j\}$. Clearly, by Corollary 4.36, there exists a perfect capacitated matching with the capacities $\bar{r}$ in $G(\mathcal{U}) - I$ if it holds $|\sigma(S) \setminus I| \ge \sum_{i \in S} \bar{r}_i$ for all $S \subseteq N$. Since $G(\mathcal{U})$ contains a perfect capacitated matching, it holds for all $S \subseteq N$ that $0 \leq |\sigma(S)| - \sum_{i \in S} r_i$ by Corollary 4.36. For $S \ni j$ the last term equals $|\sigma(S) \setminus I| - \sum_{i \in S} \bar{r}_i$ and therefore it trivially holds $|\sigma(S) \setminus I| \geq \sum_{i \in S} \bar{r}_i$ for all $S \ni j$. Therefore, iff Inequality (4.1) holds true for all $S \subseteq N \setminus \{j\}$ then there exists a perfect capacitated matching with the capacities $\bar{r}$ in $G(\mathcal{U}) - I$ by Corollary 4.36. However, for every perfect capacitated matching F for $G(\mathcal{U}) - I$ with capacities $\bar{r}$ it holds that $F \sqcup I$ is a perfect capacitated matching of $G(\mathcal{U})$ with capacities r and vice versa and the claim follows directly. □

Note that for any balanced weighted uniform matroid market $\mathcal{U} := (M, N, \mathbf{v})$ and $j \in N$ it follows directly by Lemma 4.38 that for $I \subseteq \sigma(\{j\})$ it holds $I \in \mathrm{Opt}_j^{\mathcal{U}}$ iff Inequality (4.1) is fulfilled for all $S \subseteq N \setminus \{j\}$ and it holds $|I| = r_j$.

Next, we observe that under the assumption of Lemma 4.38 the set of sets for which Inequality 4.1 is tight has a specific structure.

Lemma 4.39. *Let $\mathcal{U} = (M, N, \mathbf{v})$ a balanced weighted uniform matroid market, $j \in N$ and I an optimal subset for j. Then,*

$$\arg\max\{|S| \colon \sum_{i \in S} r_i = |\sigma(S) \setminus I|, S \subseteq N \setminus \{j\}\} =: T \text{ is unique}$$

and for every $S \subseteq N \setminus (\{j\} \sqcup T)$, $S \neq \emptyset$ holds $\sum_{i \in S} r_i < |\sigma(S) \setminus I|$.

Proof. Recall that it holds $\sum_{i \in S} r_i \leq |\sigma(S) \setminus I|$ for all $S \subseteq N \setminus \{j\}$ by Lemma 4.38 and assume that $A_1, A_2 \in \{S \subseteq N \setminus \{j\} \colon \sum_{i \in S} r_i = |\sigma(S) \setminus I|\}$. Then, by the inclusion-exclusion principle it holds that

$$\begin{aligned}
&|\sigma(A_1 \cup A_2) \setminus I| = |\sigma(A_1) \setminus I| + |\sigma(A_2) \setminus I| - |(\sigma(A_1) \cap \sigma(A_2)) \setminus I| \\
&= \sum_{i \in A_1} r_i + \sum_{i \in A_2} r_i - |(\sigma(A_1) \cap \sigma(A_2)) \setminus I| \leq \sum_{i \in A_1} r_i + \sum_{i \in A_2} r_i - \sum_{i \in A_1 \cap A_2} r_i = \sum_{i \in A_1 \cup A_2} r_i.
\end{aligned}$$

Therefore, it holds that $A_1 \cup A_2 \in \{S \subseteq N \setminus \{j\} \colon \sum_{i \in S} r_i = |\sigma(S) \setminus I|\}$ and the claim follows directly. □

4.5. Dynamic Prices for Weighted Uniform Matroid Markets

In this section, we apply the auxiliary results of Section 4.4 to establish our main result, the existence of dynamic prices in weighted uniform matroid markets and a simple algorithm to compute them in polynomial time in the number of items and buyers.

For any given price we partition the set of buyers into those that have a unique utility maximizing set and those that have not.

Definition 4.40. *For a weighted uniform matroid market* $\mathcal{U} = (M, N, \mathbf{v})$ *and a price* p *call buyer* $j \in N$ ***satisfied*** *if it holds* $D_j(p) = \{X_j(p)\}$ *and* ***unsatisfied*** *if* j *is not satisfied. We denote by* $N^s(p)$ *the set of satisfied buyers at feasible price* p *and by* $N^u(p) = N \setminus N^s(p)$ *the set of unsatisfied buyers at price* p*.*

4.5.1. Feasible Prices

We sequentially calculate prices (non-Walrasian!), that preserve the following important characteristics with *interior* Walrasian prices. We mention per property where it was shown for interior Walrasian prices.

Definition 4.41. *In a weighted uniform matroid market* $\mathcal{U} = (M, N, \mathbf{v})$ *call a price* p ***feasible*** *if it holds*

- $p > 0$, *(Claim 4)*
- $X(p) \cap Y(p) = \emptyset$, *(After Theorem 4.26)*
- $[\{j\}] \subseteq X_j(p)$ *and* $X_j(p)$ *is an optimal subset for* j *for all* $j \in N$ *(After Definition 4.31),*
- $X_j(p) \in \mathrm{Opt}_j^{\mathcal{U}}$ *for all* $j \in N^s(p)$, *(Definition of Walrasian price and Definition 4.40)*
- $X_j(p) = X(p) \cap \sigma(\{j\})$ *and* $Y_j(p) = \sigma(\{j\}) \setminus X_j(p)$ *for all* $j \in N^u(p)$, *(Definition of interior Walrasian price)*
- $u_j(x, p) > 0$ *for all* $j \in N_=$ *and* $x \in Y_j(p)$ *and*
  $u_j(x, p) = 0$ *for all* $j \in N_<$ *and* $x \in Y_j(p)$. *(Lemma 4.29)*

Recall that in any balanced weighted uniform matroid market $\mathcal{U} = (M, N, \mathbf{v})$ it holds $N_= = N$ by Assumption 4.9, hence given any feasible price p it holds $u_j(x, p) > 0$ for all $j \in N$ and $x \in Y_j(p)$. By Theorem 4.26 and Lemma 4.29 it holds that every interior Walrasian price is feasible, hence the set of feasible prices is nonempty for every weighted uniform matroid market. In contrast to Walrasian prices, for any feasible price p and buyers $i, j \in N$, $i \neq j$ it might hold (and frequently occurs!) that $X_i(p) \cap X_j(p) \neq \emptyset$ and furthermore $X(p) \sqcup Y(p) \subsetneq M$. Note that for any weighted uniform matroid market $\mathcal{U} = (M, N, \mathbf{v})$ with feasible price p it holds

a) that p is a dynamic price for all $j \in N^s(p)$ (since $D_j(p) = \{X_j(p)\}$ by the definition of satisfied buyers and $X_j(p) \in \mathrm{Opt}_j^{\mathcal{U}}$ by the definition of feasible price) and thus p is a dynamic price if it holds $N^s(p) = N$.

b) $D_j(p) = \begin{cases} \{S\colon S = X_j(p) \sqcup T, T \subseteq Y_j(p), |T| = r_j - |X_j(p)|\} \text{ if } j \in N_= \cap N^u(p), \\ \{S\colon S = X_j(p) \sqcup T, T \subseteq Y_j(p), |T| \leq r_j - |X_j(p)|\} \text{ if } j \in N_< \cap N^u(p) \end{cases}$

by $p > 0$ and Corollary 4.21.

c) $\mathrm{Opt}_j^{\mathcal{U}} \cap D_j(p) \neq \emptyset$ for all $j \in N$ since for all $j \in N^s(p)$ trivially holds $\mathrm{Opt}_j^{\mathcal{U}} \cap D_j(p) \neq \emptyset$ by a) and for $j \in N^u(p)$ clearly holds $\emptyset \neq \{S \in \mathrm{Opt}_j^{\mathcal{U}} \colon X_j(p) \subseteq S\} \subseteq D_j(p)$ by b) Corollary 4.21 and the fact that $X_j(p)$ is an optimal subset.

We briefly outline the importance of feasible prices for our algorithm to compute dynamic prices: Assume in a weighted uniform matroid market $\mathcal{U} = (M, N, \mathbf{v})$ for every feasible price p that is not already a dynamic price (hence, $N^u(p) \neq \emptyset$) it holds that there exists $x \in Y(p)$ and a feasible price p^1 (that can be generated by taking p and decreasing $p(x)$ slightly after possibly increasing p slightly) such that it holds $X(p^1) = X(p) \sqcup \{x\}$. Then, it follows directly that $Y(p^1) \subsetneq Y(p)$ and by the finiteness of M one can construct inductively a feasible price q such that $Y(q) = \emptyset$. However, this implies that the market allows for a dynamic price since it has to hold $X_j(q) \in \mathrm{Opt}_j^{\mathcal{U}}$ for all $j \in N$ by definition.

4.5.2. Balanced Markets

We show that the for any balanced weighted uniform matroid market $\mathcal{U}$ and for every feasible but non dynamic price p there exists an item x such that decreasing the item price of x slightly yields a feasible price p^1 for which holds $X(p^1) = X(p) \sqcup \{x\}$. To this end, we need a definition in the spirit of Lemma 4.39.

Definition 4.42. *Given a balanced weighted uniform matroid market $\mathcal{U} = (M, N, \mathbf{v})$ we define the **conflict function** at feasible price p by*

$$g_p\colon N^u(p) \to 2^N \setminus \{N\},\ g_p(j) := \operatorname*{arg\,max}_{S \subseteq N\setminus\{j\}\colon \sum_{i \in S} r_i = |\sigma(S) \setminus X_j(p)|} |S|.$$

The concept of the conflict function is crucial for the rest of this chapter and in particular for the proof of our main statement. Clearly, for every feasible price p the function g_p is well defined by Lemma 4.39 as for $j \in N^u(p)$ the set $X_j(p)$ is a (strict!) optimal subset for $j \in N^u(p)$ by the definition of $N^u(p)$ and the fact that $\mathcal{U}$ is balanced. Therefore, given any feasible price p and any unsatisfied buyer $j \in N^u(p)$ there exists a strict superset T of $X_j(p)$ such that $T \in \mathrm{Opt}_j^{\mathcal{U}}$. Then, since $g_p(j)$ is the unique inclusionwise maximal set for which Inequality 4.1 is tight, it is by Corollary 4.33 exactly the maximal set of buyers for which every optimal allocation that allocates $X_j(p)$ to j has to allocate every item in $\sigma(g_p(j)) \setminus X_j(p)$ to some buyer of $g_p(j)$. Therefore, $\sigma(g_p(j)) \cap Y(p)$ is the set of items from which buyer j is not allowed to get allocated any item *on top* of $X_j(p)$ and $g_p(j)$ implicitly describes this set. Before we use the conflict function to describe the set of items from which buyer j is allowed to get allocated any item *on top* of $X_j(p)$, we give an example of the mechanics of the conflict function.

Example 4.43. *Recall the market of Example 4.7 and observe that* $p(x) = p(y) = p(z) = 1$, $p(w) = 0.5$ *is a feasible price for* $\mathcal{U}$ *since* $D_1(p) = \{\{w, x\}, \{w, y\}, \{w, z\}, \{w\}\}$ *and* $\mathrm{Opt}_1^{\mathcal{U}} = \{\{w, y\}, \{w, z\}, \{x, y\}, \{x, z\}\}$*, thus* $D_1(p) \cap \mathrm{Opt}_1^{\mathcal{U}} \neq \emptyset$ *and* $D_2(p) = \{\{w\}\} \subset \mathrm{Opt}_2^{\mathcal{U}} = \{\{w\}, \{x\}\}$ *and* $D_3(p) = \{\{y\}, \{z\}, \emptyset\}$ *and* $\mathrm{Opt}_3^{\mathcal{U}} = \{\{y\}, \{z\}\}$*, thus* $D_3(p) \cap \mathrm{Opt}_3^{\mathcal{U}} \neq \emptyset$ *and it holds that* $N^u(p) = \{1, 3\}$ *and* $N^s(p) = \{2\}$*. It is* $X_3(p) = \{\emptyset\}$ *and* $g_p(3) = N$*, hence buyer* 3 *can get allocated any of her demanded items without hurting any other buyer. In contrast, it is* $X_1(p) = \{w\}$ *and* $g_p(1) = \{2\}$ *since* $|\sigma(\{2\}) \setminus X_1(p)| = |\{x\}| = r_2$*.*

The following corollary describes the set of items from which buyer j *might* get allocated some item *on top* of $X_j(p)$ and turns out to be essential to prove the guaranteed existence of dynamic prices in weighted uniform matroid markets.

In the following, when clear by context, we drop the subscript p and abbreviate the conflict function g_p at price p by g.

Corollary 4.44. *Let* $\mathcal{U} = (M, N, \mathbf{v})$ *a balanced weighted uniform matroid market,* p *a feasible price and* $j \in N^u(p)$*. Then, it holds that* $X_j(p) \sqcup \{x\}$ *is an optimal subset for* j *iff*

$$x \in \sigma(\{j\}) \cap \rho(\overline{g(j)}) \cap Y(p).$$

Proof. It holds by the definition of the conflict function and Corollary 4.33 that $X_j(p) \sqcup \{x\}$ is not an optimal subset for j iff $x \in \sigma(g(j)) \cap \rho(\overline{g(j)}) \cap Y(p)$. Note that $X_j(p)$ is a strict optimal subset for j: Clearly for $j \in N^u(p)$ it trivially cannot hold $|X_j(p)| > r_j$ and it cannot hold $|X_j(p)| > r_j$ by the definition of an unsatisfied buyer and $p > 0$, hence $|X_j(p)| < r_j$. However, since $\mathcal{U}$ is balanced it holds for all $S \in \mathrm{Opt}_j^{\mathcal{U}}$ that $|S| = r_j$. Then, it has to hold by the Corollaries 4.33 and 4.34 that $(\sigma(\{j\}) \setminus \sigma(g(j)) \cap Y(p) \neq \emptyset$ and $X_j(p) \sqcup \{x\}$ is an optimal subset for j iff $x \in (\sigma(\{j\}) \setminus \sigma(g(j))) \cap Y(p)$. However, it holds by definition that $\sigma(\{j\}) \setminus \sigma(g(j)) = \sigma(\{j\}) \cap \rho(\overline{g(j)})$ and the statement follows directly. □

We want to show:

Theorem 4.45. *Let* $\mathcal{U} = (M, N, \mathbf{v})$ *a balanced weighted uniform matroid market with feasible price* p *such that* $N^u(p) \neq \emptyset$*. Then, there exists* $x \in \sigma(N^u(p)) \cap Y(p)$ *such that for all* $j \in N^u(p) \cap \alpha(x)$ *holds that* $X_j(p) \sqcup \{x\}$ *is an optimal subset for* j*.*

Before we get ready to prove Theorem 4.45 we mention a direct implication:

Corollary 4.46. *There exists a dynamic price in every balanced weighted uniform matroid market.*

Proof. Let $\mathcal{U} = (M, N, \mathbf{v})$ a balanced weighted uniform matroid market with feasible price p. Then, it holds by Theorem 4.45 that there exists $x \in \sigma(N^u(p)) \cap Y(p)$ such that for all $j \in N^u(p) \cap \alpha(x)$ holds that $X_j(p) \sqcup \{x\}$ is an optimal subset for j. It follows by Lemma 4.17 and the definition of $\Delta(p)$ that $p^1 := p - \Delta(p) \cdot \chi^{\{x\}}$ is a feasible price such that $X(p^1) = X(p) \sqcup \{x\}$ and $Y(p^1) \subsetneq Y(p)$ and the statement follows successively by the finiteness of M and the definition of feasible prices. □

Two Unsatisfied Buyers

In order to develop some intuition for the proof of Theorem 4.45 we start with the special case of a balanced weighted uniform matroid market with a feasible price such that there are only two unsatisfied buyers given the price.

Theorem 4.47. *Let $\mathcal{U} = (M, N, \mathbf{v})$ a balanced weighted uniform matroid market and p a feasible price and $\{1, 2\} = N^u(p)$. Then, there exists $x \in Y(p) \cap \sigma(\{1, 2\})$ such that for all $j \in \{1, 2\} \cap \alpha(x)$ holds $X_j(p) \sqcup \{x\}$ is an optimal subset for j.*

Proof. Recall that since $\mathcal{U}$ is balanced it holds for $j \in N^u(p)$ that $X_j(p)$ is a strict optimal subset for j and therefore $\sigma(\{j\}) \cap \rho(\overline{g(j)}) \cap Y(p) \neq \emptyset$ by Corollary 4.44.

We distinguish the cases that there exists $j \in \{1, 2\}$ such that $i \in g(j)$ for $i = 3 - j$ and that it holds $1 \notin g(2)$ and $2 \notin g(1)$. We will show that there exists $x \in Y(p) \cap \sigma(N^u(p))$ either with

$$N^u(p) \cap \alpha(x) = \{j\} \text{ and } X_j(p) \sqcup \{x\} \text{ is an optimal subset for } j \text{ in the first case,}$$
$$N^u(p) \cap \alpha(x) = \{1, 2\} \text{ and } X_j(p) \sqcup \{x\} \text{ is an optimal subset for } j \in \{1, 2\} \text{ in the second case.}$$

Case1: W.l.o.g. let $2 \in g(1)$. Then it holds $2 \notin \overline{g(1)}$ and therefore $\rho(\overline{g(1)}) \cap \sigma(\{2\}) = \emptyset$. Obviously, for every $x \in \sigma(\{1\}) \cap \rho(\overline{g(1)}) \cap Y(p) \neq \emptyset$ it holds that $x \in \rho(\overline{g(1)})$, hence $x \notin \sigma(\{2\})$. Therefore, it holds $N^u(p) \cap \alpha(x) = \{1\}$ and since $X_1(p) \sqcup \{x\}$ is an optimal subset for buyer 1 by Corollary 4.44 it follows directly that $X_j(p) \sqcup \{x\}$ is an optimal subset for all $j \in \{1, 2\} \cap \alpha(x)$.

Case2: Now, assume $1 \notin g(2)$ and $2 \notin g(1)$, that is $\{1, 2\} \subseteq \overline{g(1)} \cap \overline{g(2)}$. Recall that for $i \in \{1, 2\}$ it holds $\sum_{j \in g(i)} r_j = |\sigma(g(i)) \setminus X_i(p)| = |\sigma(g(i))| - |X_i(p) \cap \sigma(g(i))|$ and therefore

$$\begin{aligned}\sum_{j \in g(1)} r_j + \sum_{j \in g(2)} r_j &= |\sigma(g(1))| + |\sigma(g(2))| - |X_1(p) \cap \sigma(g(1))| - |X_2(p) \cap \sigma(g(2))| \\ &= |M| - |\rho(\overline{g(1)})| + |M| - |\rho(\overline{g(2)})| - \sum_{j \in \{1,2\}} |X_j(p) \cap \sigma(g(j))|\end{aligned}$$

$$\Leftrightarrow \sum_{j \in \overline{g(1)}} r_j + \sum_{j \in \overline{g(2)}} r_j = |\rho(\overline{g(1)})| + |\rho(\overline{g(2)})| + \sum_{j \in \{1,2\}} |X_j(p) \cap \sigma(g(j))|$$
$$\Leftrightarrow \sum_{j \in \overline{g(1)} \cup \overline{g(2)}} r_j + \sum_{j \in \overline{g(1)} \cap \overline{g(2)}} r_j = |\rho(\overline{g(1)}) \cup \rho(\overline{g(2)})| + |\rho(\overline{g(1)}) \cap \rho(\overline{g(2)})| + \sum_{j \in \{1,2\}} |X_j(p) \cap \sigma(g(j))|. \tag{4.2}$$

Observe that by the Corollaries 4.33 and 4.36 it trivially holds $\sum_{j \in \overline{g(1)} \cup \overline{g(2)}} r_j \geq |\rho(\overline{g(1)} \cup \overline{g(2)}| \geq |\rho(\overline{g(1)}) \cup \rho(\overline{g(2)})|$ and therefore it is

$$\sum_{j \in \overline{g(1)} \cap \overline{g(2)}} r_j \leq |\rho(\overline{g(1)}) \cap \rho(\overline{g(2)})| + \sum_{j \in \{1,2\}} |X_j(p) \cap \sigma(g(j))| \tag{4.3}$$

and again by the Corollaries 4.33 and 4.36 it trivially holds

$$\sum_{j \in (\overline{g(1)} \cap \overline{g(2)}) \setminus \{1,2\}} r_j \geq |(\rho(\overline{g(1)}) \cup \rho(\overline{g(2)}) \setminus \sigma(\{1,2\})|. \tag{4.4}$$

Therefore, summing the Equations (4.3) and (4.4) yields

$$\begin{aligned}\sum_{j \in \{1,2\}} r_j \leq& |\rho(\overline{g(1)}) \cap \rho(\overline{g(2)}) \cap \sigma(\{1,2\})| + \sum_{j \in \{1,2\}} |X_j(p) \cap \sigma(g(j))| \\ =& |\rho(\overline{g(1)}) \cap \rho(\overline{g(2)}) \cap \sigma(\{1,2\}) \cap X(p)| + |\rho(\overline{g(1)}) \cap \rho(\overline{g(2)}) \cap \sigma(\{1,2\}) \cap Y(p)| \\ &+ \sum_{j \in \{1,2\}} |X_j(p) \cap \sigma(g(j))|,\end{aligned} \tag{4.5}$$

where the last equation is due to $\sigma(\{1,2\}) \subseteq X(p) \sqcup Y(p)$ by the definition of $N^u(p)$. Now, observe that for $j \in \{1,2\}$ it holds that $\rho(\overline{g(j)}) \cap (X_j(p) \cap \sigma(g(j))) = \emptyset$ since $\rho(\overline{g(j)})$ and $\sigma(g(j))$ are disjoint by definition. Then, it follows

$$\begin{aligned}&|\rho(\overline{g(1)}) \cap \rho(\overline{g(2)}) \cap \sigma(\{1,2\}) \cap X_j(p)| + |X_j(p) \cap \sigma(g(j))| \\ =&|(\rho(\overline{g(1)}) \cap \rho(\overline{g(2)}) \cap \sigma(\{1,2\}) \cap X_j(p)) \sqcup (X_j(p) \cap \sigma(g(j)))| \leq |X_j(p)| < r_j\end{aligned}$$

and summing for $j = 1, 2$ yields

$$\sum_{j \in \{1,2\}} |\rho(\overline{g(1)}) \cap \rho(\overline{g(2)}) \cap \sigma(\{1,2\}) \cap X_j(p)| + |X_j(p) \cap \sigma(g(j))| < \sum_{j \in \{1,2\}} r_j. \tag{4.6}$$

Furthermore, it obviously holds by the definition of feasible prices that $|\sigma(\{1,2\}) \cap X(p)| =$

$|\sigma(\{1,2\}) \cap (X_1(p) \cup X_2(p))| \leq |\sigma(\{1,2\}) \cap X_1(p)| + |\sigma(\{1,2\}) \cap X_2(p)|$ and therefore

$$|(\rho(\overline{g(1)}) \cap \rho(\overline{g(2)})) \ \cap \ \sigma(\{1,2\}) \cap X(p)| \leq \sum_{j \in \{1,2\}} |(\rho(\overline{g(1)}) \cap \rho(\overline{g(2)})) \ \cap \ \sigma(\{1,2\}) \cap X_j(p)|. \tag{4.7}$$

Then, summing the Inequalities (4.5), (4.6) and (4.7) yields $0 < |\rho(\overline{g(1)}) \cap \rho(\overline{g(2)}) \cap \sigma(\{1,2\}) \cap Y(p)|$, hence $\rho(\overline{g(1)}) \cap \rho(\overline{g(2)}) \cap \sigma(\{1,2\}) \cap Y(p) \neq \emptyset$, and it holds by Corollary 4.44 for every $x \in \rho(\overline{g(1)}) \cap \rho(\overline{g(2)}) \cap \sigma(\{1,2\}) \cap Y(p)$ and $j \in \{1,2\}$ that $X_j(p) \sqcup \{x\}$ is an optimal subset for j. □

Existence of Dynamic Prices in Balanced Markets

Now, we extend the result of Theorem 4.47 from two unsatisfied buyers to an arbitrary number of unsatisfied buyers and prove Theorem 4.45.

We investigate the case $2 \in g(1)$ from the proof of Theorem 4.47 in the general situation in more detail to obtain an auxiliary results that will be necessary for Theorem 4.45.

Lemma 4.48. *Let $\mathcal{U} = (M, N, \mathbf{v})$ a weighted uniform matroid market, p a feasible price and $i, j \in N^u(p)$, $i \neq j$ such that it holds $i \in g(j)$. Then, $\rho(\overline{g(i)}) \cup \rho(\overline{g(j)})$ and $X_i(p) \cap \sigma(g(i))$ are disjoint, i.e.:*

$$|\rho(\overline{g(i)}) \cup \rho(\overline{g(j)})| + |X_i(p) \cap \sigma(g(i))| = |(\rho(\overline{g(i)}) \cup \rho(\overline{g(j)})) \sqcup (X_i(p) \cap \sigma(g(i)))|.$$

Proof. By definition, it holds that $\rho(\overline{g(i)}) \cap \sigma(g(i)) = \emptyset$. It holds $\rho(\overline{g(j)}) \cap X_i(p) = \emptyset$ by the assumption $i \in g(j) \Leftrightarrow i \notin \overline{g(j)}$. □

In order to develop some intuition for the general case we provide some consequences of Lemma 4.48.

Corollary 4.49. *Let $\mathcal{U} = (M, N, \mathbf{v})$ a weighted uniform matroid market, p a feasible price and $i, j \in N^u(p)$, $i \neq j$ such that it holds $i \in g(j)$. Then*

$$\sum_{k \in \overline{g(i)} \cup \overline{g(j)}} r_k \geq |\rho(\overline{g(i)}) \cup \rho(\overline{g(j)})| + |X_i(p) \cap \sigma(g(i))|.$$

Proof. Recall that $X_i(p)$ is a strict optimal subset for i by definition and it obviously holds $X_i(p) \cap \sigma(g(i)) \subseteq X_i(p)$. Clearly, $\rho(\overline{g(i)}) \cup \rho(\overline{g(j)}) \subseteq \rho(\overline{g(i)} \cup \overline{g(j)})$ and in *every* optimal allocation the set $\rho(\overline{g(i)} \cup \overline{g(j)})$ gets allocated to $\overline{g(i)} \cup \overline{g(j)}$ by Lemma 4.32. Then, by the Corollaries 4.33 and 4.36 and $i \in \overline{g(i)}$ (by definition), it holds that $|(\rho(\overline{g(i)}) \cup \overline{g(j)})) \cup (X_i(p) \cap \sigma(g(i)))| \leq \sum_{k \in \overline{g(i)} \cup \overline{g(j)}} r_k$ and the claim follows by Lemma 4.48. □

Under the assumption of Theorem 4.47 and the additional conditions $2 \in g(1)$ and $1 \in \overline{g(2)}$ we strengthen the obvious fact that there exist some by buyer 1 weakly demanded item that in every optimal allocation that allocates $X_1(p)$ to buyer 1 gets *not* allocated to some buyer of buyer 1's conflict set (hence, $\sigma(\{1\}) \cap \rho(\overline{g(1)}) \cap Y(p) \neq \emptyset$).

Corollary 4.50. *Let $\mathcal{U} = (M, N, \mathbf{v})$ a weighted uniform matroid market, p a feasible price and $i, j \in N^u(p)$, $i \neq j$ such that it holds $i \in g(j)$ and $j \in \overline{g(i)}$. Then, it holds*

$$\sigma(\{j\}) \cap \rho(\overline{g(i)}) \cap \rho(\overline{g(j)}) \cap Y(p) \neq \emptyset.$$

Proof. It holds by Corollary 4.49, Equation (4.2) and the fact that p is feasible that

$$\begin{aligned}\sum_{k \in \overline{g(i)} \cap \overline{g(j)}} r_k \leq & |\rho(\overline{g(i)}) \cap \rho(\overline{g(j)})| + |X_j(p) \cap \sigma(g(j)| \\ = & |\rho(\overline{g(i)}) \cap \rho(\overline{g(j)}) \cap \sigma(\{j\})| + |(\rho(\overline{g(i)}) \cap \rho(\overline{g(j)})) \setminus \sigma(\{j\})| + |X_j(p) \cap \sigma(g(j)| \\ = & |\rho(\overline{g(i)}) \cap \rho(\overline{g(j)}) \cap X_j(p)| + |\rho(\overline{g(i)}) \cap \rho(\overline{g(j)})) \cap Y_j(p)| \\ & + |(\rho(\overline{g(i)}) \cap \rho(\overline{g(j)})) \setminus \sigma(\{j\})| + |X_j(p) \cap \sigma(g(j)|. \qquad (4.8)\end{aligned}$$

Recall that for $S, T \subseteq N$, $j \in S \cap T$ it trivially holds $(\rho(S) \cap \rho(T)) \setminus \sigma(\{j\}) = \rho((S \cap T) \setminus \{j\})$ and by the Corollaries 4.33 and 4.36 follows

$$|(\rho(\overline{g(i)}) \cap \rho(\overline{g(j)})) \setminus \sigma(\{j\})| = |(\rho(\overline{g(i)} \cap \overline{g(j)}) \setminus \{j\})| \leq \sum_{k \in (\overline{g(i)} \cap \overline{g(j)}) \setminus \{j\}} r_k. \qquad (4.9)$$

Then, summing the Inequalities (4.8) and (4.9) yields

$$r_j \leq |\rho(\overline{g(i)}) \cap \rho(\overline{g(j)}) \cap X_j(p)| + |\rho(\overline{g(i)}) \cap \rho(\overline{g(j)}) \cap Y_j(p)| + |X_j(p) \cap \sigma(g(j)|. \qquad (4.10)$$

However, it is $\rho(\overline{g(j)}) \cap \sigma(g(j)) = \emptyset$ by definition and therefore

$$\begin{aligned}& |\rho(\overline{g(i)}) \cap \rho(\overline{g(j)}) \cap X_j(p)| + |X_j(p) \cap \sigma(g(j)| = |X_j(p) \cap (\rho(\overline{g(i)} \cap \overline{g(j)}) \sqcup \sigma(g(j)))| \\ \leq & |X_j(p)| < r_j. \qquad (4.11)\end{aligned}$$

Then, summing the Inequalities (4.10) and (4.11) yields $|\rho(\overline{g(i)}) \cap \rho(\overline{g(j)}) \cap \sigma(\{j\}) \cap Y(p)| > 0$ concluding the proof. ☐

Recall that in the proof of Theorem 4.47, instead of searching for a single item $x \in Y(p)$ such that for all $j \in \{1, 2\} \cap \alpha(x)$ it holds $X_j(p) \sqcup \{x\}$ is an optimal subset for j, we have constructed a *set of items* $H \neq \emptyset$ such that it holds $H \subseteq \sigma(\{1, 2\}) \cap Y(p)$ and for all $x \in H$ and $j \in \{1, 2\} \cap \alpha(x)$ it holds $X_j(p) \sqcup \{x\}$ that is an optimal subset for j. Now, given a balanced weighted uniform matroid market $\mathcal{U} = (M, N, \mathbf{v})$ with feasible price p

with $N^u(p) > 2$, we want to extend these results. To this end, we construct a sequence of sequences $(y_i^k)_{1\leq k\leq \bar{k}, 1\leq i\leq l^k}$ of buyers of $N^u(p)$ with $y_i^k \neq y_j^l$ for $k \neq l$ or $i \neq j$ such that $\rho(\bigcap_{k=1}^{\bar{k}} \bigcap_{j=1}^{l^k} \overline{g(y_j^k)}) \cap \sigma(\bigcup_{k=1}^{\bar{k}} \{y_{l^k}^k\}) \cap Y(p) \neq \emptyset$ and for all $x \in \rho(\bigcap_{k=1}^{\bar{k}} \bigcap_{j=1}^{l^k} \overline{g(y_j^k)}) \cap \sigma(\bigcup_{k=1}^{\bar{k}} \{y_{l^k}^k\}) \cap Y(p)$ and $j \in \alpha(x) \cap N^u(p)$ it holds that $X_j(p) \sqcup \{x\}$ is a feasible subset, hence x fulfills the claim of Theorem 4.45. The sequence $(y_i^k)_{1\leq k\leq \bar{k}, 1\leq i\leq l^k}$ fulfills the following properties:

- For $1 < k \leq \bar{k}$ and $1 < i \leq l^k$ it holds that $y_{i-1}^k \in g(y_i^k)$ and $y_i^k \in \bigcap_{j=1}^{i-1} \overline{g(y_j^k)} \subseteq \overline{g(y_{i-1}^k)}$. (Two consecutive elements fulfill a stronger assumption then in Corollary 4.50.)

- For $1 \leq k < k' \leq \bar{k}$ and $1 \leq i \leq l^{k'}$ it holds that the pair $y_{l^k}^k, y_i^{k'}$ fulfills $\{y_{l^k}^k, y_i^{k'}\} \subseteq \overline{g(y_{l^k}^k)} \cap \overline{g(y_{l^{k'}}^{k'})}$. (The pair $y_{l^k}^k, y_i^{k'}$ fulfills the assumption of Case 2 in the proof of Theorem 4.47.)

- For $j \in N^u(p) \setminus \bigcup_{k=1}^{\bar{k}} \bigcup_{j=1}^{l^k} \{y_j^k\}$ it holds $j \in \bigcup_{k=1}^{\bar{k}} \bigcup_{i=1}^{l^k} g(y_i^k)$. (Every unsatisfied buyer that is not contained in the sequence is contained in the conflict set of at least one buyer that is contained in the sequence.)

Then, it clearly holds for every $x \in \rho(\bigcap_{k=1}^{\bar{k}} \bigcap_{j=1}^{l^k} \overline{g(y_j^k)}) \cap \sigma(\bigcup_{k=1}^{\bar{k}} \{y_{l^k}^k\}) \cap Y(p)$ and $j \in \alpha(x) \cap N^u(p)$ that $x \in \rho(\overline{g(j)})$, hence it follows directly by Corollary 4.44 that $X_j(p) \sqcup \{x\}$ is a feasible subset for j. However, even though the sequence $(y_i^k)_{1\leq k\leq \bar{k}, 1\leq i\leq l^k}$ can be constructed easily, as stated in the following Algorithm 8, the main difficulty is to prove that it holds

$\rho(\bigcap_{k=1}^{\bar{k}} \bigcap_{j=1}^{l^k} \overline{g(y_j^k)}) \cap \sigma(\bigcup_{k=1}^{\bar{k}} \{y_{l^k}^k\}) \cap Y(p) \neq \emptyset$.

Algorithm 8:

Input : Balanced weighted uniform matroid market $\mathcal{U} = (M, N, \mathbf{v})$ and a feasible but not dynamic price p.

Output: A nonempty set of items $H \subseteq Y(p) \cap \sigma(N^u(p))$ so that for all $x \in H$ and for all $j \in N^u(p) \cap \alpha(x)$ it holds $X_j(p) \sqcup \{x\}$ is an optimal subset for j.

1 Set $C \leftarrow N^u(p)$, $k \leftarrow 1$, $i \leftarrow 1$, $I \leftarrow \emptyset$.
2 **while** $C \neq \emptyset$ **do**
3   **if** $i = 1$ **then**
4   Choose an arbitary $j \in C$.
5         $y_i^k \leftarrow j$, $I \leftarrow I \sqcup \{j\}$, $C \leftarrow C \setminus \{j\}$, $i \leftarrow i + 1$
6   **if** $C = \emptyset$ **then**
7             $\bar{k} = k$
8   **else if** *there exists* $j \in C$ *such that* $y_{i-1}^k \in g(j)$, $j \in \overline{g(y_1^k)}, \ldots, \overline{g(y_{i-1}^k)}$ **then**
9   Choose such $j \in C$.
10         $y_i^k \leftarrow j$, $I \leftarrow I \sqcup \{j\}$, $C \leftarrow C \setminus \{j\}$, $i \leftarrow i + 1$
11   **else**
12         $I^k := I$, $T^k := \bigcap_{j \in I^k} \overline{g(y_j^k)}$, $l^k := i$,
13         $C \leftarrow \{y \in C \colon y_{l^k}^k \in \overline{g(y)}, y \in \overline{g(y_1^k)}, \ldots, \overline{g(y_{l^k}^k)}\}$
14   **if** $C \neq \emptyset$ **then**
15             $k \leftarrow k + 1$, $i \leftarrow 1$, $I \leftarrow \emptyset$
16   **else**
17             $\bar{k} = k$
18 **return**, $I := \bigsqcup_{k=1}^{\bar{k}} I^k$, $S := \bigcup_{k=1}^{\bar{k}} \{y_{l^k}^k\}$, $T := \bigcap_{k=1}^{\bar{k}} T^k$, $H := \rho(T) \cap \sigma(S) \cap Y(p)$.

Note that the output of Algorithm 8 is not necessarily unique and depends on the choice of j in the Lines 4 and 9.

Clearly, for the output H of Algorithm 8 it holds for all $x \in H$ and $j \in \alpha(x) \cap N^u(p)$ that $X_j(p) \sqcup \{x\}$ is feasible subset for j. To show that $H \neq \emptyset$ we need a simple counting argument, which follows from the inclusion-exclusion principle for two finite sets.

Lemma 4.51. *For finite subsets $A_1, \ldots A_n$ of a set E and $r_x \in \mathbb{Z}_+$ for $x \in E$ it holds that*

$$\sum_{i=1}^{n} \sum_{x \in A_i} r_x = \sum_{i=1}^{n-1} \sum_{x \in \bigcap_{j=1}^{i} A_j \cup A_{i+1}} r_x + \sum_{x \in \bigcap_{i=1}^{n} A_i} r_x.$$

Proof. We prove by induction. Obviously, the statement holds for true for $n = 1$. Next, we assume that the statement holds for true for some $n \in \mathbb{N}$. Let $A_1, \ldots, A_{n+1}$ finite subsets of E, then it holds

$$\sum_{i=1}^{n+1}\sum_{x\in A_i} r_x = \sum_{i=1}^{n}\sum_{x\in A_i} r_x + \sum_{x\in A_{n+1}} r_x = \sum_{i=1}^{n-1} \sum_{x\in \bigcap_{j=1}^{i} A_j \cup A_{i+1}} r_x + \sum_{x\in \bigcap_{i=1}^{n} A_i} r_x + \sum_{x\in A_{n+1}} r_x$$
$$= \sum_{i=1}^{n-1} \sum_{x\in \bigcap_{j=1}^{i} A_j \cup A_{i+1}} r_x + \sum_{x\in \bigcap_{i=1}^{n} A_i \cup A_{n+1}} r_x + \sum_{x\in \bigcap_{i=1}^{n+1} A_i} r_x$$
$$= \sum_{i=1}^{n} \sum_{x\in \bigcap_{j=1}^{i} A_j \cup A_{i+1}} r_x + \sum_{x\in \bigcap_{i=1}^{n+1} A_i} r_x.$$

Therefore, the statement also holds for true for $n+1$, hence for all $m \in \mathbb{N}$. □

Now, we are ready to prove Theorem 4.45.

Proof of Theorem 4.45. It remains to show that $H \neq \emptyset$ to prove the claim. To this end, analogously to derivation of Equation (4.2), we reformulate

$$\begin{aligned}
\sum_{i\in I}\sum_{j\in g(i)} r_j &= \sum_{i\in I} |\sigma(g(i))| - |X_i(p)\cap\sigma(g(i))| \\
&= \sum_{i\in I} |M| - |\rho(\overline{g(i)})| - |X_i(p)\cap\sigma(g(i))| \\
\Leftrightarrow \sum_{i\in I}\sum_{j\in \overline{g(i)}} r_j &= \sum_{i\in I} |\rho(\overline{g(i)})| + |X_i(p)\cap\sigma(g(i))| \\
\underset{I=\bigcup_{k=1}^{\bar{k}}\bigcup_{i=1}^{l^k} y_i^k}{\Leftrightarrow} \sum_{k=1}^{\bar{k}}\sum_{i=1}^{l^k}\sum_{j\in\overline{g(y_i^k)}} r_j &= \sum_{l=1}^{\bar{k}}\sum_{i=1}^{l^k} |\rho(\overline{g(y_i^k)}| + |X_{y_i^k}(p)\cap\sigma(g(y_i^k))|.
\end{aligned} \tag{4.12}$$

For $1 \leq k \leq \bar{k}$ define $\overline{g(y^k_{l^k+1})} = \emptyset$ and use that it holds $\rho(S)\cap\rho(T) = \rho(S\cap T)$ for arbitrary $S, T \subseteq N$. Then, Equation (4.12) can be formulated by Lemma 4.51 as

$$\begin{aligned}
\sum_{k=1}^{\bar{k}}\sum_{i=1}^{l^k}\sum_{h\in(\bigcap_{j=1}^{i}\overline{g(y_j^k)})\cup\overline{g(y_{i+1}^k)}} r_h &= \sum_{k=1}^{\bar{k}}\sum_{i=1}^{l^k} |\rho(\bigcap_{j=1}^{i}\overline{g(y_j^k)})\cup\rho(\overline{g(y_{i+1}^k)})| + |X_{y_i^k}(p)\cap\sigma(g(y_i^k))| \\
&= \sum_{k=1}^{\bar{k}}\sum_{i=1}^{l^k-1} |\rho(\bigcap_{j=1}^{i}\overline{g(y_j^k)}\cup\rho(\overline{g(y_{i+1}^k)})| + |X_{y_i^k}(p)\cap\sigma(g(y_{i+1}^k))| \\
&\quad + \sum_{k=1}^{\bar{k}}\rho(\bigcap_{j=1}^{l^k}\overline{g(y_j^k)})| + |X_{y_{l^k}^k}(p)\cap\sigma(g(y_{l^k}^k))|
\end{aligned}$$

which, by rearranging the terms and defining $s^k := y^k_{l^k}$ for $1 \leq k \leq \bar{k}$ can be formulated as

$$\sum_{k=1}^{\bar{k}} \sum_{i=1}^{l^k-1} \left(\sum_{h \in \bigcap_{j=1}^{i} \overline{g(y_j^k)} \cup \overline{g(y_{i+1}^k)}} r_h - |\rho(\bigcap_{j=1}^{i} \overline{g(y_j^k)}) \cup \rho(\overline{g(y_{i+1}^k)})| - |X_{y_i^k}(p) \cap \sigma(g(y_i^k))| \right)$$

$$= \sum_{k=1}^{\bar{k}} \left(|\rho(\bigcap_{j=1}^{l^k} \overline{g(y_j^k)})| + |X_{s^k}(p) \cap \sigma(g(s^k))| - \sum_{h \in \bigcap_{j=1}^{l^k} \overline{g(y_j^k)}} r_h \right). \tag{4.13}$$

It holds by Lemma 4.48 that $\rho(\overline{g(y_i^k)}) \cup \rho(\overline{g(y_{i+1}^k)})$ and $X_{y_i^k}(p) \cap \sigma(g(y_i^k))$ are disjoint and since it trivially holds $\bigcap_{j=1}^{i} \overline{g(y_j^k)} \subseteq \overline{g(y_i^k)}$ it follows $|\rho(\bigcap_{j=1}^{i} \overline{g(y_j^k)}) \cup \rho(\overline{g(y_{i+1}^k)})| + |X_{y_i^k}(p) \cap \sigma(g(y_i^k))| = |(\rho(\bigcap_{j=1}^{i} \overline{g(y_j^k)}) \cup \rho(\overline{g(y_{i+1}^k)})) \sqcup (X_{y_i^k}(p) \cap \sigma(g(y_i^k)))|$. It follows analogously to Corollary 4.49 that $\sum_{h \in \bigcap_{j=1}^{i} \overline{g(y_j^k)} \cup \overline{g(y_{i+1}^k)}} r_h - |\rho(\bigcap_{j=1}^{i} \overline{g(y_j^k)} \cup \rho(\overline{g(y_{i+1}^k)})| - |X_{y_i^k}(p) \cap \sigma(g(y_i^k))| \geq 0$, hence the left hand side of Equation (4.13) is nonnegative. Then, the right hand side of Equation (4.13) also has to be nonnegative, hence it holds

$$\sum_{k=1}^{\bar{k}} \sum_{h \in T^k} r_h - |X_{s^k}(p) \cap \sigma(g(s^k))| \leq \sum_{k=1}^{\bar{k}} |\rho(T^k)|. \tag{4.14}$$

From here on, the proof is a generalization of the proof of Case 2 of Theorem 4.47 from 2 to $\bar{k}$ buyers. We define $T^{\bar{k}+1} = \emptyset$ and again apply Lemma 4.51 to reformulate Inequality (4.14) as

$$\sum_{k=1}^{\bar{k}} \sum_{f \in \bigcap_{h=1}^{k} T^h \cup T^{k+1}} r_f - |X_{s^k}(p) \cap \sigma(g(s^k))| \leq \sum_{k=1}^{\bar{k}} |\bigcap_{h=1}^{k} \rho(T^h) \cup \rho(T^{h+1})|$$

$$\Leftrightarrow \sum_{k=1}^{\bar{k}-1} \sum_{f \in \bigcap_{h=1}^{k} T^h \cup T^{k+1}} r_f - |\rho(\bigcap_{h=1}^{k} T^h) \cup \rho(T^{k+1})| + \sum_{f \in T} r_f \leq |\rho(T)| + \sum_{k=1}^{\bar{k}} |X_{s^k}(p) \cap \sigma(g(s^k))|. \tag{4.15}$$

Observe that by the Corollaries 4.33 and 4.36 and the trivial inequality $\rho(A \cup B) \supseteq \rho(A) \cup \rho(B)$ holds

$$0 \leq \sum_{k=1}^{\bar{k}-1} \sum_{f \in \bigcap_{h=1}^{k} T^h \cup T^{k+1}} r_f - |\rho(\bigcap_{h=1}^{k} T^h) \cup \rho(T^{k+1})|. \tag{4.16}$$

Then, summing the Inequalities (4.15) and (4.16) yields

$$\begin{aligned}
\sum_{f\in T} r_f \leq & |\rho(T)| + \sum_{k=1}^{\bar{k}} |X_{s^k}(p) \cap \sigma(g(s^k))| \\
\Leftrightarrow \sum_{f\in T\setminus S} r_f + \sum_{f\in S} r_f \leq & |\rho(T) \cap \sigma(S)| + |\rho(T) \cap \overline{\sigma(S)}| + \sum_{f\in S} |X_f(p) \cap \sigma(g(f))|.
\end{aligned}$$

Again, it trivially holds $|\rho(T) \cap \overline{\sigma(S)}| = |\rho(T) \cap \rho(\overline{S})| = |\rho(T \setminus S)| \leq \sum_{f\in T\setminus S} r_f$, hence it has to hold $\sum_{f\in S} r_f \leq |\rho(T) \cap \sigma(S)| + \sum_{f\in S} |X_f(p) \cap \sigma(g(f))|$ and since it holds $\sigma(S) \subseteq X(p) \sqcup Y(p)$ it follows

$$\sum_{f\in S} r_f \leq |\rho(T) \cap \sigma(S) \cap X(p)| + |\rho(T) \cap \sigma(S) \cap Y(p)| + \sum_{f\in S} |X_f(p) \cap \sigma(g(f))|. \tag{4.17}$$

It holds $\sigma(S) \cap X(p) = \bigcup_{f\in S} X_f(p)$ since p is feasible and $S \subseteq N^u(p)$. Further, by definition, it also holds that $\rho(T)$ and $\sigma(g(S))$ are disjoint, hence it follows

$$\begin{aligned}
|\rho(T) \cap \sigma(S) \cap X(p)| + \sum_{f\in S} |X_f(p) \cap \sigma(g(f))| \leq \sum_{f\in S} |\rho(T) \cap X_f(p)| + |\sum_{f\in S} |X_f(p) \cap \sigma(g(f))| \\
= \sum_{f\in S} |X_f(p) \cap (\sigma(g(f)) \sqcup \rho(T))| \leq \sum_{f\in S} |X_f(p)| < \sum_{f\in S} r_f. \qquad (4.18)
\end{aligned}$$

Summing the Inequality (4.17) and the l.h.s. and the r.h.s. of the strict Inequality (4.18) yields

$$0 < |\rho(T) \cap \sigma(S) \cap Y(p)| = |H|$$

and therefore it holds $H \neq \emptyset$ what concludes the proof. □

4.5.3. Dynamic Prices in Markets with Excess Demand

We have proven that dynamic prices exist in balanced weighted uniform matroid markets. In this subsection, we prove the existence of dynamic prices in weighted uniform matroid markets with excess demand.

We can restate Lemma 4.38 for any weighted uniform matroid market with excess demand $\mathcal{U} = (M, N, \mathbf{v})$: For $j \in N$ it holds that $I \subseteq \bigcup \mathrm{Opt}_j^{\mathcal{U}}$ is an optimal subset for j iff it holds

$$\begin{aligned}
&|\{x \in M \colon \alpha(x) \cap S \neq \emptyset\} \setminus I| \geq \sum_{i\in S} r_i \quad \text{for all } S \subseteq N_= \setminus \{j\}, \\
&\sum_{i\in S} r_i - |I| \geq |\{x \in M \colon \alpha(x) \subseteq S\} \setminus I| \text{ for all } S \supseteq \{j\}.
\end{aligned}$$

However, direct search for a $M \sqcup N_=$-saturating capacitated matching in $G(\mathcal{U})$ in order to implement an optimal allocation seems complicated since there does not exist a direct analogous to Theorem 4.45, as the following example shows.

Example 4.52. *Consider the weighted uniform matroid market* $\mathcal{U} = (M, N, \mathbf{v})$ *of Example 4.3 given by the following table:*

	x	y	z	rank
v_1	2	2	2	2
v_2	1	1	1	2

Figure 4.4.: The market $\mathcal{U}$.

Recall that the only Walrasian price for $\mathcal{U}$ *is* $p \equiv 1$. *It is easy to see that, in the spirit of Theorem 4.45 and the proof of Corollary 4.46, we can construct e.g., the feasible price* p^1 *defined by* $p^1(x) := 0.5$ *and* $p^1(y) = p^1(z) := 1$ *starting from* p. *However, given* p^1, *the statement of Theorem 4.45 does not hold since* $2 \in N^u(p^1) = \{1, 2\}$ *and* $Y(p^1) = \{y, z\}$ *and* $2 \in \alpha(y) = \alpha(z) = \{1, 2\}$ *but* $\{x, y\}, \{x, z\} \notin \mathrm{Opt}_2^{\mathcal{U}}$.

Further, Example 4.52 provides the insight that, starting with an arbitrary interior Walrasian price and trying to compute a dynamic price for a market with excess demand iteratively, one necessarily has to increase some item prices at some point. Otherwise, e.g., in Example 4.52, buyer 2 will definitely demand a set of cardinality two which is not optimal.

Therefore, in order to apply an analogous of Theorem 4.45 to a weighted uniform matroid market with excess demand $\mathcal{U} = (M, N, \mathbf{v})$ we extend $\mathcal{U}$ by introducing $\sum_{j \in N} r_j - m$ additional virtual items which we assume to be *exclusively* demanded by the buyers in $N_<$.

Definition 4.53. *Let* $\mathcal{U} := (M, N, \mathbf{v})$ *a weighted uniform matroid market (with excess demand). Denote the* ***set of virtual items*** *by* $\tilde{N}_< := \{(N_<, k) \colon |[N_<]| < k \leq |[N_<]| + \sum_{j \in N} r_j - m\}$, $M' := M \sqcup \tilde{N}_<$. *For* $j \in N_=$ *and* $x \in \tilde{N}_<$ *set* $\omega_{j,x} = -\infty$ *and for* $j \in N_<$ *and* $x \in \tilde{N}_<$ *set* $\omega_{j,x} = 0$. *Then, for* $j \in N$ *define* v'_j *as the weighted uniform matroid valuation with ground set* $2^{M'}$, *weights* $\omega_{j,x}$ *and rank* r_j *and call* $\mathcal{U}' := (M, N, \mathbf{v}')$ *the* ***extended market of*** $\mathcal{U}$. *Consequently, the bipartite* ***extended market graph*** $G(\mathcal{U}') = (M', N, E')$ *is constructed by adding the node set* $[\tilde{N}_<]$ *and an edge* $\{j, x\}$ *for every* $j \in N_<$ *and* $x \in \tilde{N}_<$ *to* $G(\mathcal{U})$. *For ease of presentation for* $S \subseteq N$ *denote* $[S]' := \{x \in M' \colon S = \alpha(x)\}$ *and* $\rho(S) := \bigcup_{T \subseteq N \colon T \subseteq S} [T]'$ *and* $\sigma(S) := \bigcup_{T \subseteq N \colon T \cap S \neq \emptyset} [T]'$. *For a set* $L \subseteq M$ *and an allocation* $\mathbf{A} = (A_1, \ldots, A_n)$, *define the intersection\difference of* $\mathbf{A}$ *and* L *as the componentwise intersection\difference of* A_j, $j \in N$ *with* L *and for another allocation* $\mathbf{B} = (B_1, \ldots, B_n)$ *such that* $\bigcup_{j \in N} A_j \cap \bigcup_{j \in N} B_j = \emptyset$ *define the disjoint union* $\mathbf{A} \sqcup \mathbf{B}$ *as the componentwise disjoint union.*

Note that every balanced weighted uniform matroid market coincides with its extended market.

It follows directly by the notion of the extended market and the extended market graph:

Lemma 4.54. *Let $\mathcal{U} := (M, N, \mathbf{v})$ a weighted uniform matroid market and $\mathcal{U}' = (M', N, \mathbf{v}')$ its extended market, then it holds:*

i) For every optimal allocation $\mathbf{A} = (A_1, \ldots, A_n)$ for $\mathcal{U}$ there exists an optimal allocation $\mathbf{A}' = (A'_1, \ldots, A'_n)$ for $\mathcal{U}'$ (in fact usually several different allocations) that allocates all items M' with $\mathbf{A}' \cap M = \mathbf{A}$. Conversely, for every optimal allocation $\mathbf{A}' = (A'_1, \ldots, A'_n)$ for $\mathcal{U}'$ it holds that $\mathbf{A} \cap M$ is an optimal allocation for $\mathcal{U}$.

ii) The set of optimal allocations of $\mathcal{U}'$ that allocate all items and give every buyer a set of cardinality of its rank can be identified with the set of perfect capacitated matching of $G(\mathcal{U}')$.

Note that in weighted uniform matroid markets with excess demand even though every perfect capacitated matchings of the extended market graph encodes an optimal allocation of the original market there generally does not exist a bijection between the optimal allocations of the original market and the perfect capacitated matchings of the extended market graph, as the allocation of the virtual items is nonunique.

Observe that for a weighted uniform matroid market with excess demand $\mathcal{U}$ its extended market $\mathcal{U}'$ is not relevant, therefore we extend important notations to *non*relevant markets.

Definition 4.55. *Let $\mathcal{U} = (M, N, \mathbf{v})$ a nonrelevant weighted uniform matroid market with price p. Define the **strongly demanded items** and **weakly demanded items** given p analogously to relevant markets and denote them by $X(p)$ and $Y(p)$ respectively. Define the residual market $\hat{\mathcal{U}} := (M|_{\{x\colon p(x) \neq 0\}}, N, \mathbf{v}|_{\{x\colon p(x)\neq 0\}})$ and the price $\hat{p} := p|_{\{x\colon p(x)\neq 0\}}$. Then, call a price p **feasible** for $\mathcal{U}$ if $\hat{\mathcal{U}}$ is relevant, the social welfare of an optimal allocation of $\hat{\mathcal{U}}$ coincides with the social welfare of an optimal allocation of $\mathcal{U}$ and $\hat{p}$ is a feasible price for $\hat{\mathcal{U}}$. Further, given a feasible price p for $\mathcal{U}$, call a buyer $j \in N$polymatroid **satisfied** if it holds $j \in N^s(\hat{p})$ (for the residual market $\hat{\mathcal{U}}$!) and **unsatisfied** if $j \in N^u(\hat{p})$ and denote the set of satisfied buyers of $\mathcal{U}$ by $N^s(p)$ and the set of unsatisfied buyers of $\mathcal{U}$ by $N^u(p)$.*

Therefore, for any weighted uniform matroid market with excess demand $\mathcal{U} = (M, N, \mathbf{v})$ with feasible price p it holds that $p' := (p, \mathbf{0}_{\tilde{N}_<})$ is a feasible price for its extended market $\mathcal{U}' = (M', N, \mathbf{v}')$ and it holds $N^u(p) = N^u(p')$, hence $N^s(p) = N^s(p')$ and $X(p) = X(p')$ and $Y(p') = \begin{cases} Y(p) \sqcup \tilde{N}_< \text{ if } N_< \cap N^u(p) \neq \emptyset, \\ Y(p) \text{ if } N_< \cap N^u(p) = \emptyset \end{cases}$.

Definition 4.56. *For a weighted uniform matroid market with excess demand* $\mathcal{U} = (M, N, \mathbf{v})$ *define the **conflict function** at feasible price* p *by*

$$g_p \colon N^u(p) \to 2^N \setminus \{N\}, g_p(j) := \operatorname*{arg\,max}_{S \subseteq N \setminus \{j\} \colon \sum_{i \in S} r_i = |\sigma(S) \setminus X_j(p)|} |S|.$$

Recall that σ is defined with respect to the extended market (graph)!

Existence of Dynamic Prices in Markets with Excess Demand

We observe that it follows completely analogous to Theorem 4.45:

Theorem 4.57. *Let* $\mathcal{U} = (M, N, \mathbf{v})$ *a weighted uniform matroid market with excess demand and feasible price* p *such that* $N^u(p) \neq \emptyset$*. Then, for the feasible price* $p' := (p, \mathbf{0}|_{\tilde{N}_<})$ *of the extended market* $\mathcal{U}' = (M', N, \mathbf{v}')$ *there exists* $x \in \sigma(N^u(p')) \cap Y(p')$ *such that for all* $j \in N^u(p') \cap \alpha(x)$ *holds that* $X_j(p') \sqcup \{x\}$ *is an optimal subset for* j *(in* $\mathcal{U}'$*).*

Consider any weighted uniform matroid market with excess demand $\mathcal{U} = (M, N, \mathbf{v})$ with feasible price p and its extended market $\mathcal{U}' = (M', N, \mathbf{v}')$ with feasible price $p' := (p, \mathbf{0}|_{\tilde{N}_<})$ and assume that there exists some $x \in \sigma(N^u(p')) \cap Y(p') \cap M$ such that for all $j \in N^u(p') \cap \alpha(x)$ holds $X_j(p') \sqcup \{x\}$ is an optimal subset for j (in $\mathcal{U}'$). Then, it follows directly by Corollary 4.60 that for all $j \in N^u(p) \cap \alpha(x)$ holds $X_j(p) \sqcup \{x\}$ is an optimal subset for j (in $\mathcal{U}$) and therefore analogously to the proof of Corollary 4.46 that $p^1 := p - \Delta(p) \cdot \chi^{\{x\}}$ is a feasible price for $\mathcal{U}$ such that $X(p^1) = X(p) \sqcup \{x\}$ and $Y(p^1) \subsetneq Y(p)$.

However, as also shown in the following example, it may occur that for all $x \in \sigma(N^u(p')) \cap Y(p')$ such that for all $j \in N^u(p') \cap \alpha(x)$ holds that $X_j(p') \sqcup \{x\}$ is an optimal subset for j (in $\mathcal{U}'$) it already holds $x \in \tilde{N}_< = M' \setminus M$. Then, we cannot argue like in the proof of Corollary 4.46 since we cannot reduce the price of a virtual item and it might be impossible to construct a feasible price p^1 starting from p by simply decreasing the item price of some item $x \in M$ slightly.

Example 4.58 (continued)**.** *Recall the market of Example 4.52 with its extended market* $\mathcal{U}'$*, given by the following table:*

	x	y	z	$(\{2\}, 1) =: a$	rank
v_1	2	2	2	$-\infty$	2
v_2	1	1	1	0	2

Figure 4.5.: The extended market $\mathcal{U}'$.

Given the feasible price $p(x) = 0{,}5$*,* $p(y) = 1$*,* $p(z) = 1$ *for* $\mathcal{U}$ *for* $j \in \{1, 2\}$ *it holds* $X_j(p) = \{x\}$ *and* $Y_j(p) = \{y, z\}$*. Then, for the feasible price* $p' := (p, \mathbf{0}_{\{a\}})$ *for* $\mathcal{U}'$ *it holds* *for $j \in \{1,2\}$ that $X_j(p) = X_j(p')$ and $Y_1(p') = Y_1(p)$ and $Y_2(p) = \{a,y,z\}$. Further, the only item $w \in \{a,y,z\}$ for which $X_j(p') \sqcup \{w\}$ is an optimal subset for j (in $\mathcal{U}'$) for all $j \in N^u(p') \cap \alpha(w)$ (here, in fact also an optimal set) is the virtual item a.*

This issue faced in Example 4.58 yields to the idea to increase the prices of all items of M slightly as soon as the only items that fulfill the condition of Theorem 4.57 are virtual items.

Theorem 4.59. *Let $\mathcal{U} = (M,N,\mathbf{v})$ a weighted uniform matroid market with excess demand and p a feasible price such that for the feasible price $p' := (p, \mathbf{0}|_{\tilde{N}_<})$ for the extended market $\mathcal{U}'$ it holds $\emptyset \neq \{x \in \sigma(N^u(p')) \cap Y(p')\colon$ For all $j \in N^u(p') \cap \alpha(x)$ it holds $X_j(p') \sqcup \{x\}$ is an optimal subset for j (in $\mathcal{U}'$)$\} \subseteq \tilde{N}_<$. Then, it holds that $p^1 := p + \Delta(p)$ is a feasible price (for $\mathcal{U}$) such that it holds $X(p) = X(p^1)$ and $N^u(p^1) = N^u(p) \cap N_=$.*

Before we prove Theorem 4.59 we observe that it directly implies:

Corollary 4.60. *There exists a dynamic price in every weighted uniform matroid market.*

Proof. Clearly, by Corollary 4.46, it remains to show the statement for markets with excess demand. However, for any weighted uniform matroid market with excess demand $\mathcal{U} = (M,N,\mathbf{v})$ it follows by Theorem 4.57, the proof of Corollary 4.46 and Theorem 4.59 that there exists a feasible price p such that $N^u(p) \subseteq N_=$ (Recall that any dynamic price q fulfills $N^u(q) = \emptyset \subseteq N_=$.). Then, since $N^u(p) = N^u(p')$, Theorem 4.57 guarantees the existence of some $x \in \sigma(N^u(p)) \cap Y(p)$, hence $x \in M$, such that for all $j \in N^u(p) \cap \alpha(x)$ holds $X_j(p') \sqcup \{x\}$ is an optimal subset for j (in $\mathcal{U}'$), thus it holds by Lemma 4.54 and Corollary 4.36 that $X_j(p) \sqcup \{x\}$ is an optimal subset for all $j \in N^u(p) \cap \alpha(x)$ (in $\mathcal{U}$!). Then, analogously to the proof of Corollary 4.46, the claim follows inductively by Theorem 4.57. □

Now, we prove Theorem 4.59.

Proof of Theorem 4.59. Clearly, for $j \in N_=$ it holds $D_j(p) = D_j(p^1)$ and for $j \in N_<$ it holds $Y_j(p^1) = \emptyset$, hence $X(p) = X(p^1)$ and $D_j(p^1) = \{X_j(p)\}$. Therefore it remains to show that for $j \in N^u(p) \cap N_<$ it holds $X_j(p) \in \mathrm{Opt}_j^{\mathcal{U}}$ to prove the statement.

Let $(y_i^k)_{1 \le k \le \bar{k}, 1 \le i \le l^k}$ be any sequence computed in some run of Algorithm 8 applied to the extended market $\mathcal{U}'$ with feasible price p' with output I^k, I, S, T^k, T and H defined as in Algorithm 8 and $s^k := y_{l^k}^k$ for $1 \le k \le \bar{k}$.

Claim 7: For every $s \in S$ there exists an allocation $\mathbf{A} = (A_1, \dots, A_n)$ such that $A_s = X_s(p)$, $\bigsqcup_{j \in N} A_j = M$ and $A_j \subseteq \bigcup \mathrm{Opt}_j^{\mathcal{U}}$ and $|A_j| \le r_j$ for all $j \in N$.
Proof of claim: It holds by the assumption $\emptyset \neq \{x \in \sigma(N^u(p)) \cap Y(p')\colon$ For all $j \in N^u(p) \cap \alpha(x)$ it holds $X_j(p) \sqcup \{x\}$ is an optimal subset for $j\} \subseteq \tilde{N}_<$ that $H \cap M = \emptyset$ and $\overline{g(i)} \supseteq N_<$ for all $i \in I$, hence $\sigma(g(I)) \subseteq M$.

For $1 \leq k \leq \bar{k}$ and $1 \leq i \leq l^k$ there exists an optimal allocation $\mathbf{A}^{y_i^k} = (A_1^{y_i^k}, \dots A_n^{y_i^k})$ for $\mathcal{U}$ with $A_{y_i^k}^{y_i^k} \supseteq X_{y_i^k}(p)$, $\bigsqcup_{j \in N} A_j^{y_i^k} = M$ and $|A_j^{y_i^k}| \leq r_j$ for all $j \in N$ such that for all $x \in (\sigma(g(y_i^k)) \setminus X_{y_i^k}(p)) \cap M$ holds $\alpha^{\mathbf{A}^{y_i^k}}(x) \in g(y_i^k)$ by Lemma 4.54 and Corollary 4.36. (Note that it might hold $A_{y_i^k} \supsetneq X_{y_i^k}(p)$!) Define $\mathbf{B}^{y_i^k} = (B_1^{y_i^k}, \dots, B_n^{y_i^k}) := \mathbf{A}^{y_i^k} \cap (\sigma(g(y_i^k)) \setminus X_{y_i^k}(p))$ and $\mathbf{C}^{y_i^k} := \mathbf{B}^{y_i^k} \setminus \{x \colon x \in \bigcup_{l \in N} \bigcup_{j=1}^{i-1} B_l^{y_j^k}, 1 \leq j \leq i-1\}$ and $\mathbf{C}^k = (C_1^k, \dots, C_n^k) := \bigsqcup_{1 \leq i \leq l^k} \mathbf{C}^{y_i^k}$. Clearly, for $1 \leq k \leq \bar{k}$ and $j \in N$ it holds that $C_j^k \leq r_j$ and for $j \in N \setminus g(I^k)$ it holds $C_j^k = \emptyset$, hence, in particular $C_{s^k}^k = \emptyset$. However, for $1 \leq k \leq \bar{k}$ we define $\mathbf{Q}^k = (Q_1^k, \dots, Q_n^k) := (C_1^k, \dots, X_{s^k}(p) \setminus \bigcup_{j \in g(I^k)} C_j^k, \dots, C_n^k)$ and show that $\bigsqcup_{j \in N} Q_j^k = \sigma(g(I^k)) \cup X_{s^k}(p))$.

Assume there exists $x \in \sigma(g(I^k))$ such that x gets not allocated in $\mathbf{Q}^k$, then there has to exist $1 \leq i < l^k$ such that $x \in X_{y_i^k}(p)$. However, if $i < l^k$, then it also has to hold $x \in X_{y_{i+1}^k(p)}$ (since otherwise x would get allocated in $\mathbf{C}^{y_{i+1}^k}$, hence in $\mathbf{Q}^k$) and inductively, by finiteness of l^k it holds that $x \in X_{s^k}(p)$. Therefore, x gets allocated to s^k in $\mathbf{Q}^k$ and $\mathbf{Q}^k$ allocates all items of $\sigma(g(I^k)) \cup X_{s^k}(p)$.

Now, for $1 \leq k \leq \bar{k}$ define $\mathbf{L}^k := \mathbf{Q}^k \setminus \{x \in M \colon x$ gets allocated in $\bigcup_{i=1}^{k-1} \bigcup_{j \in N} Q_j^i\}$ and $\mathbf{L} = (L_1, \dots, L_n) := \bigsqcup_{k=1}^{\bar{k}} \mathbf{L}^k$, then it holds for $\mathbf{L}$ that:

- $|L_j| \leq r_j$ for all $j \in N$,
- $\bigsqcup_{j \in N} L_j = \sigma(g(I)) \cup \bigcup_{s \in S} X_s(p) \subseteq M$,
- $L_s \subseteq X_s(p)$ for $s \in S$,
- $L_j = \emptyset$ for $j \in \overline{g(I)} \setminus I$.

Notice that it holds $g(I) = \overline{T}$ and recall that $H \cap M = \emptyset$ by assumption. Then, since it trivially holds $\sigma(S) = (\sigma(S) \cap \sigma(\overline{T})) \sqcup (\sigma(S) \cap \rho(T)) \supseteq \sigma(g(I))$ and $\sigma(S) \cap \rho(T) \cap Y(p') = H$, it holds that all the items of $\sigma(S) \cap M$ get allocated by $\mathbf{L}$ and therefore $\bigsqcup_{j \in N} L_j = \sigma(S \sqcup \overline{T}) \cap M$, hence $M \setminus \bigsqcup_{j \in N} L_j = \rho(\overline{S} \cap T) = \rho(T \setminus S) \subseteq M$.

Then, it holds that $\overline{g(I)} \setminus I = T \setminus S$ and $M \setminus \mathbf{L} = M \setminus \sigma(S \sqcup \overline{T}) = \rho(T \setminus S)$ and by Corollary 4.36 and Lemma 4.32 the allocation $\mathbf{L}$ can be extended to an allocation $\mathbf{F} = (F_1, \dots, F_n)$ such that for every $j \in N$ holds $|F_j| \leq r_j$, $\bigcup_{j \in} F_j = M$ and for $s \in S$ holds $F_s \subseteq X_s(p)$. Then, for any fixed $s \in S$ the allocation $\mathbf{F}$ obviously can be transformed into an allocation $\mathbf{A} = (A_1, \dots, A_n)$ such that for every $j \in N$ it holds $|A_j| \leq r_j$, it holds $\bigcup_{j \in} A_j = M$ and $A_s = X_s(p)$. ■

<u>Claim 8:</u> For $s \in S$ there exists an optimal allocation $\mathbf{A} = (A_1, \dots, A_n)$ such that $A_s = X_s(p)$

<u>Proof of claim:</u> Let $s \in S$ and among all optimal allocations $\mathbf{A} = (A_1, \dots, A_n)$ such that $A_s \supseteq X_s(p)$ let $\mathbf{A}$ be chosen such that $|A_s|$ is minimal. Furthermore, among all allocations

$\mathbf{B} = (B_1, \ldots, B_n)$ such that $B_s = X_s(p)$, $\bigsqcup_{j \in N} B_j = M$, $B_j \subseteq \mathrm{Opt}_j^{\mathcal{U}}$ and $|B_j| \leq r_j$ for all $j \in N$ (such an allocation is guaranteed to exist by Claim 7) let $\mathbf{B}$ be chosen such that $\sum_{j \in N_=} |A_j \setminus B_j|$ is minimal. Clearly, if it holds $\sum_{j \in N_=} |B_j| = \sum_{j \in N_=} r_j$ then $\mathbf{B}$ is optimal, hence assume $\sum_{j \in N_=} |B_j| < \sum_{j \in N_=} r_j$ and thus it holds $\sum_{j \in N_=} |A_j \setminus B_j| > 0$. We will show a contradiction to the later strict inequality what implies $\sum_{j \in N_=} |B_j| = \sum_{j \in N_=} r_j$ and yields the claim.

Let $i \in N_=$ such that $|B_i| < r_i$. Create a digraph $G = (V, E)$ as follows: Define $V := (\bigcup_{j \in N_=} A_j \setminus B_j) \sqcup \{i\}$ and for $j \in N_=$ let $f_j \colon B_j \setminus A_j \to A_j \setminus B_j$ any injective function (it holds $|B_j \setminus A_j| \leq |A_j \setminus B_j|$) and $y \in A_i \setminus f_i(B_i \setminus A_i)$. Clearly, y and f_j for all $j \in N$ are well-defined. Define $E := \{(x, f_j(x)) \colon j \in N_=, x \in B_j \setminus A_j\} \cup \{(i, y)\}$, then it holds $E \neq \emptyset$. Then, if G has a cycle C we construct another allocation starting from $\mathbf{B}$: For $x \in V(C)$ buyer $\alpha^{\mathbf{B}}(x)$ gives up the item x and for every $(x, y) \in C$ buyer $\alpha^{\mathbf{B}}(x)$ adds the item y. This yields an allocation $\mathbf{B}' = (B_1', \ldots, B_n')$ such that $B_s' \supseteq X_s(p)$, $\bigsqcup_{j \in N} B_j' = M$ and $|B_j'| \leq r_j$ and $\sum_{j \in N_=} |A_j \setminus B_j'| < \sum_{j \in N_=} |A_j \setminus B_j|$ contradicting the minimality of $\sum_{j \in N_=} |A_j \setminus B_j|$. Therefore, by the finiteness of V, there exists a path $P = (i = t_0, z = t_1, \ldots, t_h)$ in G such that $\mathrm{outdeg}_G(t_h) = 0$, hence $\alpha^{\mathbf{B}}(t_h) \in N_<$. Then, we can construct another allocation starting from $\mathbf{B}$ by reallocating along the edges of P, hence for every $y \in V(G[P]) \cap M$ and $\{x, y\} \in E(G[p])$, $x \in M$ we take the item y from $\alpha^{\mathbf{B}}(y)$ and add it to $\alpha^{\mathbf{B}}(x)$ and we take the item z from $\alpha^{\mathbf{B}}(z)$ and add it to i. This yields an allocation $\mathbf{B}' = (B_1', \ldots, B_n')$ such that $B_s' \supseteq X_s(p)$, $\bigsqcup_{j \in N} B_j' = M$ and $|B_j'| \leq r_j$ and $\sum_{j \in N_=} |A_j \setminus B_j'| < \sum_{j \in N_=} |A_j \setminus B_j|$ contradicting the minimality of $\sum_{j \in N_=} |A_j \setminus B_j|$. Therefore, it has to hold $\sum_{j \in N_=} |A_j \setminus B_j| = 0$ what proofs the claim. ■

Now, the statement follows directly by Claim 8. □

4.5.4. The Polynomial Runtime Dynamic Price Algorithm

Now, it is easy to provide an algorithm that computes a dynamic price in a weighted uniform matroid market.

Algorithm 9: Dynamic price algorithm for weighted uniform matroid markets

Input : A weighted uniform matroid market $\mathcal{U} := (M, N, \mathbf{v})$ and an interior Walrasian price p.

Output: A dynamic price for $\mathcal{U}$.

$X \leftarrow X(p)$, $Y \leftarrow Y(p)$, $\Delta \leftarrow \Delta(p)$, $N^u \leftarrow N^u(p)$, $N^s \leftarrow N^s(p)$,
while $N^s \neq N$ **do**
  if *there exists* $x \in Y$ *such that* $(X \cap \sigma(\{j\})) \cup \{x\} \in \mathcal{I}(\mathrm{Opt}_j^{\mathcal{U}})$ *for all* $j \in N^u \cap \alpha(x)$ **then**
    Choose such $x \in Y$.
    Set $p \leftarrow p - \Delta \cdot \chi^{\{x\}}$, $X \leftarrow X \cup \{x\}$, $Y \leftarrow Y(p)$, $N^u \leftarrow N^u(p)$, $N^s \leftarrow N^s(p)$, $\Delta \leftarrow \frac{\Delta}{2}$
  else
    $p \leftarrow p + \Delta$, $N^u \leftarrow N^u(p)$, $N^s \leftarrow N^s(p)$, $\Delta \leftarrow \frac{\Delta}{2}$
return p

Clearly, it follows directly by the Theorems 4.47 and 4.59 and the Corollaries 4.46 and 4.60 that Algorithm 9 computes a dynamic price.

For completeness, we calculate a dynamic price in Example 4.52 by Algorithm 9.

Example 4.61 (continued)**.** *Given the feasible price* $p(x) = 0,5.p(y) = p(z) = 1$ *we increase the price of all items by* $\Delta(p) = 0.25$ *to obtain the feasible price* $p^1(x) = 0,5.p^1(y) = p^1(z) = 1.25$*, which turns out to be already a dynamic price. However, for completeness we note that* $\Delta(p^1) = 0.125$ *and compute the feasible price* $p^2(x) = 0.5, p^2(y) = 1.125, p^2(z) = 1,25$ *for which holds* $|D_j(p^2)| = 1$ *for* $j \in \{1, 2\}$.

Furthermore, we remark that it is easy to see that Algorithm 9 is suited to implement a social welfare maximizing allocation through a dynamic pricing even if buyers arrive sequentially and every buyer arrives *at least once*, hence the assumption that buyers exactly arrive is stricter than necessary. Even though this observation has no impact on the pure mathematical result it allows for an even more realistic real life interpretation.

For completeness, we briefly show that the runtime of Algorithm 9 is polynomial in the number of buyers and items: First of all, a Walrasian price and a social welfare maximizing allocation for $\mathcal{U}$ can be computed in polynomial time in m, n and oracle time in the value oracle model in gross substitutes markets (e.g., Murota [1996] or Nisan and Segal [2006]) and even by adapting any algorithm for finding a perfect matching for weighted uniform matroid markets. However, evaluating a weighted matroid valuation for an arbitrary set

$S \subseteq M$ can be done by a simple greedy algorithm and encoding the valuation requires only the buyer's weights for the items and the rank of the underlying matroid, hence $m+1$ different values. Furthermore, an interior Walrasian price can be obtained from an arbitrary Walrasian price in polynomial time in a simple constructive way analogous to the proof of Theorem 4.26.

For $x \in M$ and $j \in N$ we can check if it holds $x \in \bigcup \mathrm{Opt}_j^{\mathcal{U}}$ by simply assigning x to j and considering the residual market $\tilde{\mathcal{U}} = (M \setminus \{x\}, N, \tilde{\mathbf{v}})$ with $\tilde{v}_j(S) := v_j(S \cup \{x\})$ and $\tilde{v}_i(S) := v_i(S)$ for $i \in N \setminus \{j\}$ and for all $S \subseteq M \setminus \{x\}$. Then, it clearly holds that a social welfare maximizing allocation of $\mathcal{U}$ equals the value of a social welfare maximizing allocation of $\tilde{\mathcal{U}}$ iff it holds $x \in \bigcup \mathrm{Opt}_j^{\mathcal{U}}$. Therefore, for $j \in N$ the set $\bigcup \mathrm{Opt}_j^{\mathcal{U}}$ can be determined by computing $n \cdot m + 1$ social welfare maximizing allocations for weighted uniform matroid markets, hence in polynomial time in n and m. Furthermore, the set $\bigcup \mathrm{Opt}_j^{\mathcal{U}}, j \in N$ can be used to determine the sets $X_j(p), Y_j(p)$ for the interior Walrasian start price p and the sets $X_j(p), Y_j(p)$ can be used to compute $N_=, N_<$ easily by Lemma 4.29.

At every price p that is computed during an execution of Algorithm 9 the sets $X_j(p)$ and $Y_j(p)$ for every $j \in N$ can be determined easily either from the sets $N_=$ and $N_<$ (If the If-loop in Line 3 evaluated to false at the previous price) or the buyers strongly and weakly demanded items at the previous price and the item chosen in Line 4 (If the If-loop evaluated to true at the previous price). Clearly, the sets $N^u(p)$ and $N^s(p)$ can easily be determined from the sets $X_j(p)$ and $Y_j(p)$ for every $j \in N$.

There are at most $m+1$ iterations of the While-loop in Line 2 of Algorithm 9, since the If-condition in Line 3 evaluates to false at most once and whenever the If-condition in Line 2 evaluates to true some item gets added to the set X and items will never be eliminated from X.

To show that a dynamic price can be computed in polynomial time it only remains to show that it can be checked efficiently if Line 3 evaluates to false and if the if condition evaluates to true that some item that fulfills the condition in Line 3 can be determined in polynomial time. We remark that Algorithm 8 was needed for technical purposes and it can be shown that it also has polynomial runtime. However, given a feasible price p, an easier approach to check if Line 3 of Algorithm 9 evaluates to false is simply choosing the items in $Y(p) \cap \sigma(N^u(p)) \cap M$ sequentially and check if one of them can be chosen in Line 4. Analogously to determining $\bigcup \mathrm{Opt}_j^{\mathcal{U}}$, it can be checked if the item $x \in Y(p) \cap \sigma(N^u(p)) \cap M$ can be allocated to the buyer $j \in N^u(p) \cap \alpha(x)$ in addition to $X_j(p)$: Temporarily assign $X_j(p) \cup \{x\}$ to buyer j and check if the value of a social welfare maximizing allocation that assigns $X_j(p) \cup \{x\}$ to buyer j coincides with the value of a social welfare maximizing allocation of the original market. Therefore, it can be checked if Line 3 evaluates to false in Algorithm 9 and, if existent, some item that fulfills the condition in Line 4 can be found by solving the social welfare maximizing allocation problem of not more than $m \cdot n$ many residual markets of $\mathcal{U}$.

In total, a dynamic price for weighted uniform matroid markets can be computed by Algorithm 9 in polynomial time in m and n since the most time consuming steps involved are solving polynomial many in m and n social welfare maximizing allocation problems for weighted uniform matroid markets.

4.6. Conclusion

We showed that dynamic prices are guaranteed to exist in weighted uniform matroid markets. En route, we showed that interior Walrasian prices do not induce unnecessary demand in weighted uniform matroid markets and already suffice for dynamic prices in matching markets.

The main open problem regarding dynamic prices remains to establish the maximal domain for which a dynamic price is guaranteed to exist. We remark that the techniques established in Section 4.5 are not directly applicable to general weighted matroid markets since Lemma 4.32 is essential for the existence of dynamic prices in weighted uniform matroid markets but generally does not hold for true beyond weighted uniform matroid markets.

As a first step, it seems worthwhile to prove that interior Walrasian prices in gross substitutes markets do not induce unnecessary demand. This may turn out to be useful in tackling existence proofs of dynamic prices in more complex weighted matroid markets extending the techniques of Section 4.5.

Frequently Used Notation Throughout This Chapter

- $\mathrm{Opt}^{\mathcal{U}} := \{\mathbf{A}\colon \mathbf{A} \in \arg\max_{\mathbf{B}=(B_1,\dots,B_n)\colon \mathbf{B} \text{ is allocation}} \sum_{j\in N} v_j(B_j)\}$: Set of optimal allocations.

- $\mathrm{Opt}_j^{\mathcal{U}} := \{A \subseteq M\colon A = A_j,\ \mathbf{A} = (A_1,\dots,A_n) \in \mathrm{Opt}^{\mathcal{U}}\}$: Sets buyer j gets as part of an optimal allocation.

- $\bigcup \mathrm{Opt}_j^{\mathcal{U}} := \bigcup_{A\in \mathrm{Opt}_j^{\mathcal{U}}} A$.

- $\alpha(x) := \{j \in N\colon\ x \in \bigcup \mathrm{Opt}_j^{\mathcal{U}}\}$: Bidders who get allocated x as part of some optimal allocation.

- $X_j(p) := \bigcap_{S\in D_j(p)} S$: Strongly demanded items for j at price p.

- $Y_j(p) := \{x \in \bigcup D_j(p) \setminus X_j(p)\colon$ there exists $S \in D_j(p)$ such that $x \in S$ and $|S| \leq r_j\}$: Weakly demanded items for j at price p.

- $N_= := \{j \in N\colon |S| = r_j$ for all $S \in \mathrm{Opt}_j^{\mathcal{U}}\}$.

- $N_< := N \setminus N_= = \{j \in N \colon |S| < r_j \text{ for some } S \in \mathrm{Opt}_j^{\mathcal{U}}\}$.
- $S \subseteq N$ then $[S] := \{x \in M \colon \alpha(x) = S\}$: Items that can get allocated to any buyer in S but not allocated to some buyer of $\overline{S}$.
- For balanced markets:
  - $S \subseteq N \colon [S] := \{x \in M \colon \alpha(x) = S\}$: Items that can get allocated to any buyer in S but not allocated to some buyer of $\overline{S}$ to implement an optimal allocation.
  - $\rho(S) := \bigcup_{T \subseteq S}[T]$: Set of items that in very optimal allocation get allocated to a subset of S.
  - $\sigma(S) := \bigcup_{T \subseteq N \colon T \cap S \neq \emptyset}[T]$: Set of items that in at least one optimal allocation get allocated to a buyer of S but never to a buyer of $\overline{S}$.
- For markets with excess demand:
  - $S \subseteq N \colon [S] := \{x \in M' \colon \alpha(x) = S\}$.
  - $\rho(S) := \bigcup_{T \subseteq S}[T]'$.
  - $\sigma(S) := \bigcup_{T \subseteq N \colon T \cap S \neq \emptyset}[T]'$.
- $N^s(p) = \{j \in N \colon D_j(p) = \{X_j(p)\}\}$: Set of satisfied buyers.
- $N^u(p) = N \setminus N^s(p)$: Set of unsatisfied buyers.
- $g_p \colon N^u(p) \to 2^N \setminus \{N\}, g_p(j) := \arg\max_{S \subseteq N \setminus \{j\} \colon \sum_{i \in S} r_i = |\sigma(S) \setminus X_j(p)|} |S|$: $g_p(j)$ is the maximal set $S \subseteq N \setminus \{j\}$ of buyers so that every optimal allocation that gives $X_j(p)$ to j gives $\sigma(S) \setminus X_j(p)$ to buyers of S $\Rightarrow$ There is no optimal allocation giving $X_j(p) \sqcup \{x\}$ to j for all $x \in \sigma(g_p(j)) \setminus X_j(p)$ but there is an optimal allocation giving $X_j(p) \sqcup \{x\}$ to j for all $x \in \rho(\overline{g_p(j)}) \setminus X_j(p)$.

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Appendix A.

Omitted Proofs and Examples

Proof that $(E, \mathcal{M})$ is a matroid in Example 3.14.

Proof. Clearly, each of $\binom{C}{6}, \{\{a\}\} \bigoplus \binom{C}{5}$ and $\{\{b\}\} \bigoplus \binom{C}{5}$ is the set of bases of a (partition) matroid with ground set $C \cup \{a, b\}$ and the bases have cardinality six. Therefore, to prove the basis exchange axiom for matroids it suffices to consider any pair of basis Y, Z which is not contained in the same matroid. Note that $\{\{a\}\} \bigoplus \binom{C}{5}$ and $\{\{b\}\} \bigoplus \binom{C}{5}$ are isomorphic. Let $Y \in \binom{C}{6}$ and $Z \in \{\{a\}\} \bigoplus \binom{C}{5}$. Then, it holds that $a \in Z \setminus Y$ and for each $y \in Y \setminus Z$ it is $(Y \setminus \{y\}) \cup \{a\} \in \{\{a\}\} \bigoplus \binom{C}{5}$. Conversely, for every $y \in Y \setminus Z$ it is $(Y \setminus \{a\}) \cup \{y\} \in \binom{C}{6}$ and for $z \in Z \setminus y$ and $y \in Y \setminus Z$ it is $(Z \setminus \{z\}) \cup \{y\} \in \{\{a\}\} \bigoplus \binom{C}{5}$. It remains $Y \in \{\{a\}\} \bigoplus \binom{C}{5}$ and $Z \in \{\{b\}\} \bigoplus \binom{C}{5}$. Clearly, it holds $(Y \setminus \{a\}) \cup \{b\} \in \{\{b\}\} \bigoplus \binom{C}{5}$ and for $y \in Y \setminus Z$, $y \neq a$ there has to exist $z \in (Z \setminus \{b\}) \setminus (Y \setminus \{a\}) \neq \emptyset$ and it holds $(Y \setminus \{y\}) \cup \{z\} \in \{\{a\}\} \bigoplus \binom{C}{6}$. □

Exchanging items of the same equivalence class in nonuniform weighted matroid markets may result in a suboptimal allocation

Example A.1. *Consider a four items and two buyers weighted matroid market with $M = \{a, b, c, d\}$ and $N = \{1, 2\}$ and $\omega_{j,k} = 1$ for $j \in N, k \in M$. The underlying matroid of the valuation v_1 is the uniform matroid U^2_M and the set of bases of the underlying matroid of v_2 is given by $\{S \subseteq M : |S| = 2\} \setminus \{\{a, b\}\}$. Then, $(A_1 = \{b, d\}, A_2 = \{a, c\})$ and $(B_1 = \{a, c\}, B_2 = \{b, d\})$ are optimal allocations of welfare 4 and thus it holds $M \simeq [\{1, 2\}]$. However, it clearly holds that exchanging b with c in the allocation (A_1, A_2) yields the suboptimal allocation $(\{c, d\}, \{a, b\})$ of welfare 3.*