Hypercyclic, mixing, and chaotic C_0-semigroups

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Introduction

A continuous linear operator T on a topological vector space E is called hypercyclic if it has a hypercyclic vector $x \in E$, i.e. there is a vector x in E such that $\mathrm{orb}(T,x) := \{T^n x;\, n \in \mathbb{N}_0\}$ is dense in E. Using Baire's Category theorem, it can be shown that an operator T on a separable Banach space is hypercyclic if and only if it is topologically transitive, i.e. if and only if for every two open, non-empty subsets U, V of E there is a natural number n such that $U \cap T^n(V) \neq \emptyset$ (see remark 1.10 iii) below).

The first example of a hypercyclic operator on a Banach space was given by Rolewicz [52] in 1969, whereas the very first examples of hypercyclic operators are due to Birkhoff [9] and MacLane [38], who showed that the translation operator and the differentiation operator are hypercyclic on the space of entire functions endowed with the compact open topology.

During the last two decades there has been an extensive study of hypercyclic operators, not only on Banach spaces but also on F-spaces, see for example [2], [8], [10], [11], [26], [27], [29], [47], [55], or [60]. As a survey on this subject we recommend [28], [30].

The notion of hypercyclic operator appeared for the first time in Read's famous paper dealing with the invariant subspace problem [51], although he called an operator T hypercyclic if for *every* non-zero vector x in E the set $\mathrm{orb}(T,x)$ is dense. Obviously, this is a much stronger property than is meant by hypercyclicity nowadays. The terminology stems from the notion of cyclicity in operator theory. Recall that x is a cyclic vector for T if the span of $\mathrm{orb}(T,x)$ is dense. Clearly, if every non-zero vector of E is a cyclic vector of T then T has no non-trivial invariant subspace, and if every non-zero vector of E has a dense orbit under T, then clearly T has no non-trivial invariant closed subset.

Analogously to the single operator case one defines hypercyclicity for C_0-semigroups T on Banach spaces, i.e. T is called hypercyclic if there is a vector x in E such that $\mathrm{orb}(T,x) := \{T(t)x;\, t \geq 0\}$ is dense in E.

As in the single operator case, the first example of a hypercyclic C_0-semigroup was given by Rolewicz [52] in 1969. A systematic study of hypercyclic C_0-semigroups was initiated by Desch et al. [19] in 1997 and various articles dealing with this subject followed, see e.g. [5], [6], [13], [15], [21], or [23].

The present work is mainly devoted to the study of hypercyclic C_0-semigroups on Banach spaces. In the first chapter we introduce some notations and terminology and state some basic properties of hypercyclic C_0-semigroups which will be used in the chapters 2-7.

The most commonly used condition for proving hypercyclicity of a C_0-semigroup T on a separable Banach space E is the so called Hypercyclicity Criterion. We present it and some new equivalent forms in chapter 2 and tackle the problem, if every hypercyclic C_0-semigroup has to satisfy the Criterion. In the single operator case, this question was asked by León, Montes [34], Bès, Peris [8] in 1999 and still remains the great open problem in hypercyclicity (cf. [30]). This question might seem artificial at first sight, but it was shown by Bès and Peris [8] that an operator T satisfies the Criterion if and only if T is weakly mixing, so that the great open problem is in fact equivalent to the question posed by Herrero in 1992 (cf. [31, Problem 1]), if every hypercyclic operator is weakly mixing. This equivalence is also true for C_0-semigroups. Although we do not solve the problem in general, we show that every hypercyclic C_0-semigroup for which the orbit of sufficiently many vectors is "nice" satisfies the Criterion.

If the generator A of the C_0-semigroup T is an unbounded operator, its domain $D(A)$ is of first category in the sense of Baire. So, one cannot use the usual Baire argument to show the existence of a hypercyclic vector x in $D(A)$. Nevertheless, we prove in chapter 3 that for a hypercyclic C_0-semigroup T there are even hypercyclic vectors belonging to $D(A^\infty)$ and that the set of this special hypercyclic vectors is dense in E.

Chapter 4 is devoted to the study of chaotic C_0-semigroups. Despite the deep result that every operator $T(t), t > 0$, of a hypercyclic C_0-semigroup is hypercyclic itself, which was only very recently shown by Conejero, Müller and Peris [14], it remains an open problem if the analogue statement is true for chaotic C_0-semigroups, i.e. if every (or even only one) operator $T(t), t > 0$, of a chaotic C_0-semigroup is chaotic itself. We show that under the chaoticity condition introduced by Desch et al. [19] which is the most commonly used condition for showing chaoticity of concrete C_0-semigroups (see e.g. [41], [40], [58]) it in fact holds true that every $T(t), t > 0$, is a chaotic operator. Furthermore, we prove that under some weak conditions chaoticity of T is equivalent to the existence of a single periodic point of T. Moreover, we show in chapter 4 that a C_0-semigroup T cannot be hypercyclic if the imbedding of $(D(A), \|\cdot\|_A)$ into E is compact and the spectrum of its generator is not empty. We use this result to show that C_0-semigroups on $L^p(\Omega)$ generated by strongly elliptic differential operators are not chaotic if the boundary of the bounded set Ω is smooth.

In chapter 5 we thoroughly investigate transitivity and mixingness of families of weighted composition operators on spaces of integrable functions and spaces continuous functions, respectively. We completely characterise this properties for a large class of families of weighted composition operators and show that transitivity is equivalent to weak mixing. In particular, this implies that a weighted composition operator is hypercyclic if and only if it satisfies the Hypercyclicity Criterion.

In chapter 6 we use the results of the previous chapter to characterise when C_0-semigroups on spaces of integrable and continuous functions generated by first order partial differential operators are hypercyclic and mixing, respectively, and we show that in this framework hypercyclicity is again equivalent to weak mixing, i.e. in this setting a C_0-semigroup is hypercyclic if and only if it satisfies the Hypercyclicity Criterion. The results of this chapter and chapter 7, which deals with chaoticity of the same kind of C_0-semigroups, generalise to a large extend the results about semigroups generated by certain ordinary first order differential operators obtained by Desch et al. [19], Bermúdez et al. [6], Matsui et al. [41, 42], Matsui and Takeo [40], Myjak and Rudnicki [43] and Takeo [58]. In the last part of chapter 6 we also characterise hypercyclicity of evolution families generated by non-autonomous first order partial differential operators. We present an example which shows that, contrary to the autonomous case, in the non-autonomous case not every operator of the hypercyclic evolution family has to be hypercyclic itself.

In appendix A we collect, for the reader's convenience, some of the well-known results about C_0-semigroups which are used throughout this thesis, while appendix B contains four theorems concerning weighted composition operators which are of importance in chapter 5. Finally, in appendix C we proof two propositions which are of interest in the context of C_0-semigroups generated by first order partial differential operators.

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1 Terminology, notions, and some basic facts

In this first chapter we give an introduction to hypercyclic C_0-semigroups and state some of their properties. We use standard notation from functional analysis and semigroup theory, as may be found in [24], [46], [54], or [62]. Our main reference for C_0-semigroups is [24]. For topics related to Baire category we refer to [45]. In appendix A we have collected some well known results about C_0-semigroups which will be used throughout the text.

The vector spaces we consider are always spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If not otherwise specified, by the term "operator" we mean a bounded linear operator. We denote the space of all operators on a Banach space E by $L(E)$.

Analogously to the single operator case one defines hypercyclicity, topological transitivity, weak mixing, and mixing for C_0-semigroups.

Definition 1.1 Let T be a C_0-semigroup on a Banach space E.

i) A vector x in E is called *hypercyclic (for T)* if its orbit under T, i.e. the set $\operatorname{orb}(T,x) := \{T(t)x;\, t \geq 0\}$, is dense in E. T is called *hypercyclic* if it exhibits a hypercyclic vector. We denote the set of all hypercyclic vectors of T by $HC(T)$.

ii) T is called *topologically transitive*, or simply *transitive* if for every pair of open, non-empty subsets U, V of E there is $t > 0$ such that $T(t)(U) \cap V \neq \emptyset$.

iii) T is called *weakly mixing* if the C_0-semigroup $T \oplus T$ is transitive on $E \times E$, where $T \oplus T\,(t)(x,y) := (T(t)x, T(t)y)$.

iv) T is called *mixing* if for every pair of open, non-empty subsets U, V of E there is $t > 0$ such that $T(s)(U) \cap V \neq \emptyset$ for all $s \geq t$.

Remark 1.2 i) Clearly the above notions make perfect sense for arbitrary families $(T_\iota)_{\iota \in I}$ of operators, and we will use them in the sequel.

ii) If T is a C_0-semigroup on E and x belongs to E, it follows from the strong continuity of the mapping $t \mapsto T(t)$ that $\operatorname{orb}(T,x)$ is dense in E if and only if $\{T(t)x;\, t \geq 0, t \in \mathbb{Q}\}$ is dense in E, so that E has to be separable in order to support a hypercyclic C_0-semigroup.

Using the strong continuity of T again one easily sees that $\{T(t)x;\, t \geq t_0\}$ is dense in E for every $t_0 > 0$ if $\operatorname{orb}(T,x)$ is dense in E. From the semigroup property it therefore follows that $\operatorname{orb}(T,x)$ is a subset of $HC(T)$ whenever $x \in HC(T)$.

iii) As in the single operator case, a C_0-semigroup T on a separable Banach space E is hypercyclic if and only if it is transitive. The proof uses the same arguments as in the single operator case (cf. [26, theorem 1.2]). Note that every separable Banach space E is second countable as topological space. Let $(U_k)_{k\in\mathbb{N}}$ be an open base of its topology such that $U_k \neq \emptyset$ for every $k \in \mathbb{N}$. Since obviously x is a hypercyclic vector for T if and only if for each $k \in \mathbb{N}$ there is $t \geq 0$ such that $T(t)x \in U_k$, i.e. $x \in T(t)^{-1}(U_k)$ (here and in what follows $T(t)^{-1}(U_k)$ denotes the pre-image of U_k under $T(t)$) we have $\bigcap_{k\in\mathbb{N}} \bigcup_{t\geq 0} T(t)^{-1}(U_k) = HC(T)$. Thus, $HC(T)$ is always a G_δ-set.

If T is hypercyclic and $x \in HC(T)$, we have already seen that $\text{orb}(T, x) \subset HC(T)$, so that $HC(T)$ is a dense G_δ-subset of E. Now, if U, V are open non-empty subsets of E there is $y \in HC(T) \cap U$ and $t > 0$ such that $T(t)y \in V$, i.e. T is transitive.

If on the other hand T is transitive, it follows that $\bigcup_{t\geq 0} T(t)^{-1}(U_k)$ is an open and dense subset of E for every $k \in \mathbb{N}$. Using Baire's Category theorem we see that $\bigcap_{k\in\mathbb{N}} \bigcup_{t\geq 0} T(t)^{-1}(U_k)$ is a dense G_δ-set, in particular it is not empty, so that T is hypercyclic.

Note that for a C_0-semigroup T we have shown that $HC(T)$ is either empty or a dense G_δ-subset of E.

Additionally, observe that the same proof is valid to show that for arbitrary families of commuting operators $(T_\iota)_{\iota\in I}$ with dense images hypercyclicity is equivalent to transitivity and that the same arguments work not only for separable Banach spaces but also for separable F-spaces.

iv) No C_0-semigroup on a finite dimensional Banach space E can be hypercyclic, because if E is finite dimensional, the generator A of T is a bounded operator. Let $e_1, \dots, e_n$ be a basis of E with respect to which A has Jordan normal form and let $f_1, \dots, f_n$ be the corresponding dual basis. Then, there is $\lambda \in \mathbb{K}$ such that for every $x \in E$ and $t \geq 0$ we have $\langle f_n, T(t)x\rangle = e^{t\lambda}\langle f_n, x\rangle$. This shows that for all x in E the closure of the set $A_x := \{\langle f_n, T(t)x\rangle;\, t \geq 0\}$ is either bounded or does not contain 0. Since f_n is continuous and surjective, it follows that $\text{orb}(T, x)$ cannot be dense in E for any x.

On the other hand, Bermúdez et al. showed in [5] that on every separable infinite dimensional Banach space E there is a hypercyclic C_0-semigroup. In fact they even showed the existence of a uniformly continuous semigroup which is hypercyclic. Their result was later strengthened by A. Conejero in [12].

v) Obviously, every mixing C_0-semigroup on a separable Banach space is transitive, hence hypercyclic.

Example 1.3 The first example of a hypercyclic C_0-semigroup on a Banach space was given by Rolewicz [52] in 1969. Let m_a be the Borel measure on $\mathbb{R}$ with Lebesgue density $\rho_a(x) = a^{-|x|}$ where $a > 1$. It is not hard to see that via $T(t)f := f(\cdot - t)$ one obtains a C_0-semigroup on $L^p(m_a)$ for all $1 \leq p < \infty$. Rolewicz showed that this C_0-semigroup is hypercyclic on $L^p(m_a)$ for every $1 \leq p < \infty$.

This semigroup is a very special case of a composition semigroup which we will thoroughly investigate in chapter 6 and 7. Further examples of hypercyclic C_0-semigroups will be given there.

A useful tool to get new hypercyclic, respectively mixing, C_0-semigroups from old ones is the following so-called Comparison Principle. It is a direct adaption from the single operator case, which is due to Shapiro [56]. We include the short proof for the reader's convenience.

Lemma 1.4 *Let T and S be C_0-semigroups on E resp. F and $\Phi : E \to F$ be a continuous mapping with dense range such that $S(t) \circ \Phi = \Phi \circ T(t)$ for all $t \geq 0$. If T is hypercyclic, respectively weakly mixing, respectively mixing, so is S. Moreover, if $x \in HC(T)$ then $\Phi(x) \in HC(S)$.*

Proof: Let x be a hypercyclic vector for T. From the continuity of Φ, the denseness of $\{T(t)x;\, t \geq 0\}$ in E and the fact that $S(t)\circ\Phi = \Phi\circ T(t)$ for all $t \geq 0$, it follows that $\Phi(\{T(t)x;\, t \geq 0\}) = \{\Phi(T(t)x);\, t \geq 0\} = \{S(t)(\Phi(x));\, t \geq 0\}$ is dense in F, i.e. $\Phi(x) \in HC(S)$.

Now assume that T is weakly mixing. Let $U_i, V_i, i = 1, 2$, be non-empty, open subsets of F. Because Φ is continuous and has dense range, there are non-empty, open subsets $\tilde{U}_i, \tilde{V}_i$ of E such that $\Phi(\tilde{U}_i) \subset U_i$ and $\Phi(\tilde{V}_i) \subset V_i$. Since T is weakly mixing, there is $t > 0$ such that $T(t)(\tilde{U}_i) \cap \tilde{V}_i \neq \emptyset$ for $i = 1, 2$, so that

$$\emptyset \neq \Phi(T(t)(\tilde{U}_i) \cap \tilde{V}_i) \subset S(t)(\Phi(\tilde{U}_i)) \cap \Phi(\tilde{V}_i) \subset S(t)(U_i) \cap V_i$$

for $i = 1, 2$ showing that S is weakly mixing.

Now, let T be mixing and U, V be a pair of non-empty, open subsets of F. Because Φ is continuous and has dense range, there are non-empty, open subsets $\tilde{U}, \tilde{V}$ of E such that $\Phi(\tilde{U}) \subset U$ and $\Phi(\tilde{V}) \subset V$. Since T is mixing, there is $t > 0$ such that $T(s)(\tilde{U}) \cap \tilde{V} \neq \emptyset$ for every $s \geq t$, so that

$$\emptyset \neq \Phi(T(s)(\tilde{U}) \cap \tilde{V}) \subset S(s)(\Phi(\tilde{U})) \cap \Phi(\tilde{V}) \subset S(s)(U) \cap V$$

for all $s \geq t$. □

Before we give another example of a hypercyclic C_0-semigroup we make the following definition.

Definition 1.5 Let E be a Banach space and $B \in L(E)$. B is called a *generalised backward shift* if there is a sequence $(e_n)_{n\in\mathbb{N}}$ in E whose span is dense in E such that $Be_1 = 0$ and $Be_{n+1} = e_n$ for every $n \in \mathbb{N}$.

A different proof of the following theorem can be found in [6, theorem 4.4].

Theorem 1.6 *Let E be a separable Banach space and $B \in L(E)$ a generalised backward shift. Then, the C_0-semigroup generated by B is mixing, in particular hypercyclic.*

PROOF: Since B is an operator on E the C_0-semigroup T generated by B is given by $T(t) = \exp(tB), t \geq 0$, where $\exp(tB) := \sum_{k=0}^{\infty} \frac{(tB)^k}{k!}$. We first prove that $\exp(B) - id = \sum_{k=1}^{\infty} \frac{B^k}{k!}$ again is a generalised backward shift.

Let $(e_n)_{n\in\mathbb{N}}$ be such that its span is dense in E, $Be_1 = 0$ and $Be_{n+1} = e_n$ for every $n \in \mathbb{N}$. Obviously $\sum_{k=1}^{\infty} \frac{B^k}{k!} e_1 = 0$ and $\sum_{k=1}^{\infty} \frac{B^k}{k!} e_2 = e_1$. For $n \geq 3$ let $\alpha_l, 2 \leq l \leq n-1$, be arbitrary scalars. It follows that

$$\sum_{k=1}^{\infty} \frac{B^k}{k!}(e_n + \sum_{l=2}^{n-1} \alpha_l e_l) = e_{n-1} + \sum_{l=1}^{n-2} \Big(\frac{1}{(n-l)!} + (0, \ldots, 0, \underset{l}{1}, 0, \ldots, 0) \begin{pmatrix} 1 & \frac{1}{2!} & \cdots\cdots & \frac{1}{(n-2)!} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots \quad \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots\cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} \Big) e_l.$$

Since the above matrix is invertible for $n \geq 3$, we therefore find inductively $\alpha_{n,l}, 1 \leq l \leq n-1$ such that $\sum_{k=1}^{\infty} \frac{B^k}{k!}(e_n + \sum_{l=1}^{n-1} \alpha_{n,l} e_l) = e_{n-1} + \sum_{l=1}^{n-2} \alpha_{n-1,l} e_l$.

Defining $\tilde{e}_1 := e_1$, $\tilde{e}_2 := e_2$ and $\tilde{e}_n := e_n + \sum_{l=1}^{n-1} \alpha_{n,l} e_l$ for $n \geq 3$, we get a sequence $(\tilde{e}_n)_{n\in\mathbb{N}}$ such that $\exp(B)\tilde{e}_1 - \tilde{e}_1 = 0$ and $\exp(B)\tilde{e}_n - \tilde{e}_n = \tilde{e}_{n-1}$ for all $n \geq 2$. Since obviously $\text{span}\{e_n;\, n \in \mathbb{N}\} = \text{span}\{\tilde{e}_n;\, n \in \mathbb{N}\}$ we see that $\exp(B) - id$ indeed is a generalised backward shift.

If we can show that $id + C$ is mixing whenever C is a generalised backward shift, it follows that $T(1) = \exp(B) = id + (\exp(B) - id)$ is mixing. This was shown in [6, theorem 4.5]. However, we include the proof for the reader's convenience.

So, let C be an arbitrary generalised backward shift on E with sequence $(f_n)_{n\in\mathbb{N}}$. Without loss of generality we can assume that $f_n \neq 0$ for all $n \in \mathbb{N}$. We define

$$\Phi : \ell^1 \to E, (x_n)_{n\in\mathbb{N}} \mapsto \sum_{n=1}^{\infty} \frac{x_n}{\|f_n\|} f_n.$$

Then Φ is a well-defined operator with dense range.

Since C is continuous it follows that $\sup_{n\in\mathbb{N}} \|f_n\|/\|f_{n+1}\| \leq \|C\| < \infty$ so that the weighted backward shift with weights $w_n := \|f_n\|/\|f_{n+1}\|$

$$C_w : \ell^1 \to \ell^1, (x_n)_{n\in\mathbb{N}} \mapsto \left(\frac{\|f_n\|}{\|f_{n+1}\|} x_{n+1}\right)_{n\in\mathbb{N}}$$

is continuous.

For $(x_n)_{n\in\mathbb{N}}$ in ℓ^1 we have

$$\Phi \circ C_w((x_n)_{n\in\mathbb{N}}) = \sum_{n=1}^{\infty} \frac{\|f_n\| x_{n+1}}{\|f_{n+1}\|\|f_n\|} f_n = \sum_{n=1}^{\infty} \frac{x_{n+1}}{\|f_{n+1}\|} f_n = C \circ \Phi((x_n)_{n\in\mathbb{N}}),$$

so that $\Phi \circ (id + C_w) = (id + C) \circ \Phi$.

Since $id + C_w$ is a mixing operator on ℓ^1 (cf. [27, lemma 2.3]), it now follows from lemma 1.4, that $id + C$ is a mixing operator on E.

In particular it follows that $T(1) = \exp(B) = id + (\exp(B) - id)$ is a mixing operator on E. To show that the C_0-semigroup T is mixing, let U_1, U_2 be two non-empty open subsets of E. There are non-empty open subsets V_j of $U_j, j = 1, 2$, and $\varepsilon > 0$ such that $V_j + B(0, \varepsilon) \subset U_j$, where $B(0, \varepsilon)$ denotes the open ball with center 0 and radius ε. From the local equicontinuity of T (see corollary A.2) it follows that there is $\delta > 0$ such that $T(t)(B(0, \delta)) \subset B(0, \varepsilon)$ for all $0 \leq t \leq 1$.

Since $T(1)$ is mixing, there is $N \in \mathbb{N}$ such that for all $m \geq N$ we have $T(m)(V_1) \cap B(0, \delta) \neq \emptyset$ and $T(m)(B(0, \delta)) \cap V_2 \neq \emptyset$. Now, let $t > N$. There are $n \geq N$ and $s_1, s_2 \in [0, 1)$ such that $t = n + s_1 = n + 1 - s_2$. So, we can find $v_1 \in V_1$ and $w \in B(0, \delta)$ such that $T(n)v_1 \in B(0, \delta)$ and $T(n + 1)w \in V_2$. From this we obtain

$$\begin{aligned} T(t)(v_1 + T(s_2)w) &= T(s_1)T(n)v_1 + T(n + 1 - s_2)(T(s_2)w) \\ &\in T(s_1)(B(0, \delta)) + V_2 \subset B(0, \varepsilon) + V_2 \subset U_2. \end{aligned}$$

Since $v_1 + T(s_2)w \in V_1 + B(0, \varepsilon) \subset U_1$ this gives $T(t)(U_1) \cap U_2 \neq \emptyset$. Because $t > N$ was arbitrary the theorem follows. □

Remark 1.7 Note that the last part of the above proof shows that the C_0-semigroup T is mixing if $T(1)$ is a mixing operator. Obviously, every operator

$T(t), t > 0$, of a mixing C_0-semigroup T is mixing as is the rescaled C_0-semigroup $S = (T(at))_{t\geq 0}$, where $a > 0$. With these observations we obtain the equivalence of

i) The C_0-semigroup T is mixing.

ii) For every $t > 0$ the operator $T(t)$ is mixing.

iii) There is $t > 0$ such that $T(t)$ is mixing.

Example 1.8 For $1 \leq p < \infty$ and a sequence of strictly positive numbers $a = (a_n)_{n\in\mathbb{N}}$ such that $\sup_{n\in\mathbb{N}} a_n/a_{n+1} < \infty$ we consider $\ell^p(a) := \{(x_n)_{n\in\mathbb{N}} \in \mathbb{K}^{\mathbb{N}}; \sum_{n\in\mathbb{N}} |x_n|^p a_n < \infty\}$. Then, the backward shift

$$B : \ell^p(a) \to \ell^p(a), (x_n)_{n\in\mathbb{N}} \to (x_{n+1})_{n\in\mathbb{N}}$$

is a well-defined operator. Clearly, B is a generalised backward shift, so that the C_0-semigroup generated by B is mixing on $\ell^p(a)$ by theorem 1.6.

It was already shown by Desch et al. in [19, theorem 5.2] that the C_0-semigroup generated by B is hypercyclic on $\ell^1(a)$. Actually, a careful inspection of their proof shows that by their arguments the semigroup is mixing.

Example 1.9 Let Ω be a bounded open subset of $\mathbb{C}$. We consider the Bergman space $A^2(\Omega)$, i.e. the space of holomorphic functions on Ω which are square integrable with respect to two-dimensional Lebesgue measure. It is well known that $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$, so that it is a Hilbert space when equipped with the L^2-norm.

If ϕ is a bounded holomorphic function on Ω the mapping

$$M_\phi : A^2(\Omega) \to A^2(\Omega), f \mapsto \phi f$$

defines an operator on $A^2(\Omega)$. It was shown by Godefroy and Shapiro [26, proposition 3.1] that in the case of $\phi(z) = z - \alpha$ with $\alpha \in \Omega$ the Hilbert space adjoint of M_ϕ is a generalised backward shift. So, by theorem 1.6 it generates a hypercyclic C_0-semigroup on $A^2(\Omega)$. (Note that our definition of generalised backward shift is slightly more general than the one used by Godefroy and Shapiro, see [26, Section 3, proposition 3.3].)

Little is known about the properties of the generator of a hypercyclic C_0-semigroup. The following theorem gives a necessary condition on the residual spectrum of the generator of a hypercyclic C_0-semigroup. It is due to Desch et al. [19, theorem 3.3].

Theorem 1.10 (cf. [19, theorem 3.3]) *Let T be a hypercyclic C_0-semigroup on a separable Banach space E with generator $(A, D(A))$. Then the residual spectrum $\sigma_r(A)$ of A is empty.*

Proof: Assume that there is λ in $\sigma_r(A)$. It follows from the spectral mapping theorem (see appendix A.7) for the residual spectrum that there is $\phi \in E'\backslash\{0\}$ such that $\langle\phi, T(t)x\rangle = e^{t\lambda}\langle\phi, x\rangle$ for all x in E and $t \geq 0$. From this it follows that $\{\langle\phi, T(t)x\rangle; t \geq 0\}$ is either a bounded subset of $\mathbb{K}$ (in case of $\operatorname{Re}\lambda \leq 0$) or its closure does not contain 0 (in case of $\operatorname{Re}\lambda > 0$) for all $x \in E$.

Since ϕ is different from 0 it must be surjective. Now, if x was a hypercyclic vector for T we could conclude that $\mathbb{K}$ coincides with the image of the closure of $\text{orb}(T,x)$ under ϕ, contradicting the above. So $HC(T)$ must be empty which is a contradiction to the hypothesis. □

As a corollary of this we obtain the following result.

Corollary 1.11 *Let T be a hypercyclic C_0-semigroup on E, $0 \leq t_1 < t_2$ and $(\alpha_1, \alpha_2) \in \mathbb{K}^2 \backslash \{(0,0)\}$.*

i) $\alpha_1 T(t_1) + \alpha_2 T(t_2)$ *has dense range.*

ii) If $x \in HC(T)$ *then* $\alpha_1 T(t_1)x + \alpha_2 T(t_2)x \in HC(T)$.

Proof: It is not hard to see that $T(t)$ has dense image for all $t \geq 0$ due to the hypercyclicity of T. By the spectral mapping theorem (see appendix A.7) and theorem 1.10 it follows that $\sigma_r(T(t)) = \emptyset$ for all $t \geq 0$. If $\alpha_2 \neq 0$ we have $\alpha_1 T(t_1) + \alpha_2 T(t_2) = -\alpha_2 T(t_2)[-\frac{\alpha_1}{\alpha_2}\, id - T(t_2 - t_1)]$, so that $\alpha_1 T(t_1) + \alpha_2 T(t_2)$ is the composition of two operators with dense ranges and therefore has dense range itself. If $\alpha_2 = 0$ we have $\alpha_1 \neq 0$ and a similar decomposition shows that $\alpha_1 T(t_1) + \alpha_2 T(t_2)$ has dense range, which gives i).

Now, if $\{T(t)x;\, t \geq 0\}$ is dense in E it follows from i) that the same is true for $[\alpha_1 T(t_1) + \alpha_2 T(t_2)](\{T(t)x;\, t \geq 0\}) = \{T(t)[\alpha_1 T(t_1) + \alpha_2 T(t_2)]x;\, t \geq 0\}$, which proves ii). □

2 The Hypercyclicity Criterion and some of its consequences

The most commonly used condition for proving hypercyclicity of a C_0-semigroup T on a separable Banach space E is the following theorem which is essentially due to Desch, Schappacher and Webb (cf. [19, theorem 2.3]) and which is a direct modification of the so-called Hypercyclicity Criterion for a single operator. In the single operator case, it was first formulated by Kitai [33] and its hypotheses were later considerably weakened by Gethner and Shapiro [25] in order to give a much more general criterion.

Theorem 2.1 (Hypercyclicity Criterion for C_0-semigroups) *Let E be a separable Banach space and T a C_0-semigroup on E. Let $Y, Z \subset E$ be dense subspaces of E, $(t_k)_{k\in\mathbb{N}}$ a sequence of positive numbers, and $S_t : Z \to E, t \geq 0$, a family of (not necessarily continuous) linear mappings such that*

a) $\lim_{k\to\infty} T(t_k)y = 0$ *for all* $y \in Y$

b) $\lim_{k\to\infty} S_{t_k}z = 0$ *for all* $z \in Z$

c) $T(t)S_t z = z$ *for every* $t \geq 0$ *and all* $z \in Z$.

Then, T is hypercyclic.

We include the short proof for the sake of completeness.

Proof: We will show that under the conditions of the theorem the C_0-semigroup T is transitive, hence hypercyclic by remark 1.2 iii). So, let U, V be a pair of non-empty open subsets of E. We find $y \in U \cap Y$ and $z \in V \cap Z$ such that $\lim_{k\to\infty} T(t_k)y = 0$ and $T(t)S_t z = z$ for all $t \geq 0$. By setting $y_k := y + S_{t_k}z$ we obtain a sequence $(y_k)_{k\in\mathbb{N}}$ which converges to y and for which $\lim_{k\to\infty} T(t_k)y_k = z$, proving that $T(t_k)(U) \cap V \neq \emptyset$ for sufficiently large k. □

If for a C_0-semigroup T on a Banach space E we can find dense subspaces Y and Z of E, a sequence of positive numbers $(t_k)_{k\in\mathbb{N}}$, and a family of linear mappings $S_t : Z \to E, t \geq 0$, satisfying conditions a), b), and c) of theorem 2.1, we say that T *satisfies the Hypercyclicity Criterion*.

Remark 2.2 The proof shows that if T satisfies the Hypercyclicity Criterion there is a sequence $(t_k)_{k\in\mathbb{N}}$ such that for all open and non-empty subsets U and V of E we can find $n \in \mathbb{N}$ such that $T(t_k)(U) \cap V \neq \emptyset$ for all $k \geq n$.

The Hypercyclicity Criterion for the single operator case reads as follows. Let us note that it is valid not only for separable Banach spaces but for arbitrary separable F-spaces (see [48]).

Theorem 2.3 (Hypercyclicity Criterion for single operators) *Let E be a separable Banach space and T an operator on E. Let $Y, Z \subset E$ be dense subspaces of E, $(n_k)_{k\in\mathbb{N}}$ a sequence of positive integers, and $S : Z \to Z$ a (not necessarily continuous) linear mapping such that* *a)* $\lim_{k\to\infty} T^{n_k} y = 0$ *for every* $y \in Y$

b) $\lim_{k\to\infty} S^k z = 0$ *for all* $z \in Z$

c) $TSz = z$ *for all* $z \in Z$.

Then, T is hypercyclic.

It is an open problem if every hypercyclic C_0-semigroup T satisfies the Hypercyclicity Criterion. This question is more interesting as one may think, since the Criterion is not only an easy to check condition for proving hypercyclicity but it is equivalent to $T \oplus T$ being hypercyclic.

The following theorem can be found in Conejero, Peris [13, theorem 2.1] or in El-Mourchid [21, theorem 2.5]. Again, it is a direct adaption of the result for the single operator case, which is due to Bès and Peris [8, theorem 2.3].

Theorem 2.4 (cf. [13, theorem 2.1], [21, theorem 2.5]) *Let E be a separable Banach space and T be a C_0-semigroup on E. Then, the following are equivalent.*

i) T satisfies the Hypercyclicity Criterion.

ii) $T \oplus T$ is hypercyclic.

There are many equivalent formulations of the Hypercyclicity Criterion. Some of them are given below. Apart from the equivalence of i) and ii), which is due to Conejero and Peris [13, theorem 2.1], these results are new for C_0-semigroups. For the single operator case the corresponding equivalence of i) and iii) was independently shown by Bernal and Grosse-Erdmann [7, remark 3.5] and León [36], while the corresponding equivalences of i), iv) and v) are due to Grivaux [27, theorem 3.2]. The single operator analogue of the equivalence of i) and vi) was shown by Peris and Saldivia [49, theorem 2.3].

Theorem 2.5 *Let T be a C_0-semigroup on a separable Banach space E. Then, the following are equivalent.*

i) T satisfies the Hypercyclicity Criterion.

ii) There are dense subsets Y, Z of E, a sequence $(t_k)_{k\in\mathbb{N}}$ of positive real numbers such that

  a) $\lim_{k\to\infty} T(t_k)y = 0$ *for every* $y \in Y$

  b) For every $z \in Z$ there is a sequence $(w_k)_{k\in\mathbb{N}}$ in E converging to 0 such that $\lim_{k\to\infty} T(t_k)w_k = z$.

iii) For every pair of non-empty open subsets U, V of E and every zeroneighbourhood W in E there is $t > 0$ such that $T(t)(U) \cap W \neq \emptyset$ and $T(t)(W) \cap V \neq \emptyset$.

iv) For every pair of non-empty open subsets U, V of E there is $t > 0$ such that $T(t)(U) \cap V \neq \emptyset$ and $T(t+1)(U) \cap V \neq \emptyset$.

v) There is $\alpha > 0$ such that for every pair of non-empty open subsets U, V of E there is $t > 0$ satisfying $T(t)(U) \cap V \neq \emptyset$ and $T(t+\alpha)(U) \cap V \neq \emptyset$.

vi) *If* $I \subset [0,\infty)$ *is syndetic (that is there exists* $K > 0$ *such that* $[t, t+K] \cap I \neq \emptyset$ *for all* $t \geq 0$*) then* $\{T(t);\, t \in I\}$ *is weakly mixing.*

vii) *There is* $K > 0$ *such that for every* $I \subset [0,\infty)$ *satisfying* $[t, t+K] \cap I \neq \emptyset$ *for all* $t \geq 0$ *the set* $\{T(t);\, t \in I\}$ *is weakly mixing.*

PROOF: That i) implies ii) is obvious by setting $w_k := S_{t_k} z$.

To show that ii) implies iii) choose a pair of non-empty open subsets U and V and a zeroneighbourhood W in E. Let $y \in U \cap Y$ and $z \in V \cap Z$. Since $(T(t_k)w_k)_{k\in\mathbb{N}}$ converges to z and $(T(t_k)y)_{k\in\mathbb{N}}$ as well as $(w_k)_{k\in\mathbb{N}}$ converge to 0, we have $T(t_k)y \in W$, $w_k \in W$ and $T(t_k)w_k \in V$, i.e. $T(t_k)(U) \cap W \neq \emptyset$ and $T(t_k)(W) \cap V \neq \emptyset$ for sufficiently large k.

In order to show that iii) implies iv) we note that for $U, V \subset E$ open and not empty and for a zeroneighbourhood W there is $t > 0$ with $T(t)(U) \cap W \neq \emptyset$ and $T(t)(W) \cap V \neq \emptyset$. In particular, $T(t)^{-1}(V)$ is not empty. Since $W \cap T(t)^{-1}(W)$ is a zeroneighbourhood, we can use iii) again to find $s > 0$ satisfying $T(s)(U) \cap (W \cap T(t)^{-1}(W)) \neq \emptyset$ and $T(s)(W \cap T(t)^{-1}(W)) \cap T(t)^{-1}(V) \neq \emptyset$. Using the semigroup law, this yields $T(t+s)(U) \cap W \neq \emptyset$ and $T(t+s)(W) \cap V \neq \emptyset$, so that $N(U, V, W) := \{r \geq 0; T(r)(U) \cap W \neq \emptyset \text{ and } T(r)(W) \cap V \neq \emptyset\}$ is unbounded.

Using this, we find $t > 1$ such that $\emptyset \neq T(t-1)(W) \cap T(1)^{-1}(V)$. Since V was an arbitrary non-empty open subset of E this implies that $T(1)$ has dense range.

Now, let $x \in U$ and $y \in V$. Since $T(1)$ has dense range, we find $v \in E$ and $k_0 \in \mathbb{N}$ such that $T(1)(B(v, 2/k)) \subset V$ for all $k \in \mathbb{N}, k \geq k_0$ (here and in the sequel we denote the open ball with center c and radius r by $B(c, r)$). Now iii) ensures that there is a sequence of positive numbers $(t_k)_{k\in\mathbb{N}}$ such that $T(t_k)(U) \cap [B(0, 1/k) \cap T(1)^{-1}(B(0, 1/k))] \neq \emptyset$ and $T(t_k)[B(0, 1/k) \cap T(1)^{-1}(B(0, 1/k))] \cap B(v, 1/k) \neq \emptyset$.

This last expression implies that $T(t_k)(T(1)^{-1}(B(0, 1/k))) \cap T(1)^{-1}(V) \neq \emptyset$, i.e. $T(1)^{-1}(B(0, 1/k) \cap T(t_k)^{-1}(V)) \neq \emptyset$ for $k \geq k_0$, so that we can find a sequence $(w_k)_{k\in\mathbb{N}}$ in E converging to 0 such that $T(t_k)w_k \in V$ for all $k \geq k_0$. Additionally, there is a sequence $(\tilde{w}_k)_{k\in\mathbb{N}}$ converging to 0 in E for which $\|T(t_k)\tilde{w}_k - v\| < 1/k$.

From $T(t_k)(U) \cap [B(0, 1/k) \cap T(1)^{-1}(B(0, 1/k))] \neq \emptyset$ we conclude that there is a sequence $(x_k)_{k\in\mathbb{N}}$ in U such that $\|T(t_k)x_k\| < 1/k$. If we set $u_k := x_k + w_k$ and $\tilde{u}_k := x_k + \tilde{w}_k, k \in N$, we see that $u_k, \tilde{u}_k \in U$ for large k and that $T(t_k)u_k \in V$ and $T(t_k + 1)\tilde{u}_k = T(1)T(t_k)\tilde{u}_k \in T(1)(B(v, 2/k)) \subset V$ for sufficiently large k, i.e. $T(t_k)(U) \cap V \neq \emptyset$ and $T(t_k + 1)(U) \cap V \neq \emptyset$ for large k.

That iv) implies v) is obvious.

It was shown by S. Grivaux in [27, theorem 3.2] that in the single operator case the analogue of v) implies i). This proof can be directly adapted to the C_0-semigroup case. We include it for the reader's convenience.

We will show that v) implies that T is weakly mixing, so that i) follows from theorem 2.4. Let $\emptyset \neq U_i, V_i \subset E$ be open $(i = 1, 2)$. Obviously, v) implies that T is transitive, hence hypercyclic.

Let $v_1 \in HC(T) \cap V_1$. There is $r_1 > 0$ such that $u_1 := T(r_1)v_1 \in U_1$. Since $T(r_1)$ has dense range, there is $w_2 \in E$ such that $u_2 := T(r_1)w_2 \in U_2$. Let $v_2 \in V_2$ and $\delta > 0$ be such that $B(v_2, \delta) \subset V_2$ and $B(u_2, \delta) \subset U_2$. By corollary 1.11 it follows that $(T(\alpha) - I)v_1 \in HC(T)$. Let $M > 0$ and $\omega \in \mathbb{R}$ be such that $\|T(t)\| \leq Me^{t\omega}$ for every $t \geq 0$ (see appendix A.1). We can find $q_1, p_1 > 0$ for which

$$\begin{aligned}\|T(q_1)(T(\alpha)-I)v_1-(w_2-v_2)\| &< \frac{\delta}{2Me^{r_1\omega}},\\ \|T(p_1)v_1-(v_2-T(q_1)v_1)\| &< \frac{\delta}{2Me^{r_1\omega}}.\end{aligned}$$

Now, by setting $z_2 := T(p_1)u_1+T(q_1+\alpha)u_1 = T(p_1+r_1)v_1+T(q_1+\alpha+r_1)v_1$ we obtain

$$\begin{aligned}\|z_2-u_2\| &= \|T(r_1)[T(p_1)v_1+T(q_1+\alpha)v_1-w_2]\|\\ &\leq Me^{r_1\omega}[\|T(p_1)v_1-(v_2-T(q_1)v_1)\|\\ &\qquad +\|v_2-T(q_1)v_1+T(q_1+\alpha)v_1-w_2\|]\\ &< \delta,\end{aligned}$$

that is $z_2 \in B(u_2,\delta) \subset U_2$. Analogously, one shows that $y_2 := T(p_1)v_1 + T(q_1)v_1 \in V_2$.

We define $\tilde{U}_k := B(u_1, 2^{-k})$ and $\tilde{V}_k := B(v_1, 2^{-k}), k \in \mathbb{N}$, and use v) to obtain sequences $(u_k)_{k\in\mathbb{N}}$ and $(\tilde{u}_k)_{k\in\mathbb{N}}$ converging to u_1 and a sequence $(t_k)_{k\in\mathbb{N}} \in [0,\infty)^{\mathbb{N}}$ such that $(T(t_k)u_k)_{k\in\mathbb{N}}$ and $(T(t_k+\alpha)\tilde{u}_k)_{k\in\mathbb{N}}$ converge to v_1.

It follows that $\lim_{k\to\infty} T(t_k)[T(p_1)u_k+T(q_1+\alpha)\tilde{u}_k] = T(p_1)v_1+T(q_1)v_1 = y_2 \in V_2$, i.e. $T(t_k)(U_2)\cap V_2 \neq \emptyset$ for large k. Because of $\lim_{k\to\infty} u_k = u_1 \in U_1$ and $\lim_{k\to\infty} T(t_k)u_k = v_1 \in V_1$, we also have $T(t_k)(U_1)\cap V_1 \neq \emptyset$ for sufficiently large k, so that T is weakly mixing.

In order to show that i) implies vi) we first note that the Hypercyclicity Criterion implies that the C_0-semigroup $T\oplus\cdots\oplus T$ consisting of n copies of T acting on E^n satisfies the criterion, too. Just take $Y^n, Z^n, (t_k)_{k\in\mathbb{N}}$ and $S_t\oplus\cdots\oplus S_t : Z^n \to E^n$. So, $T\oplus\ldots\oplus T$ is in particular hypercyclic.

Now, let $I \subset [0,\infty)$ be syndetic and $K > 0$ such that $[t, t+K]\cap I \neq \emptyset$ for all $t \geq 0$. Let $U_i, V_i, i = 1,2$, be non-empty open subsets of E. Since T is locally equicontinuous (see appendix A.2) we find $\delta > 0$ and non-empty open sets $\tilde{U}_i \subset U_i, i = 1,2$, such that $T(s)(\tilde{U}_i) \subset U_i$ for all $0 \leq s \leq \delta$. Choose $m \in \mathbb{N}$ satisfying $m\delta > K$. Taking $2(m+1)$ factors we have that $T\oplus\cdots\oplus T$ is hypercyclic, so there is $t > 0$ for which

$$\begin{aligned}&T\oplus\cdots\oplus T\,(t)(\tilde{U}_1\times\cdots\times\tilde{U}_1\times\tilde{U}_2\times\cdots\times\tilde{U}_2)\\ &\cap(V_1\times T(\delta)^{-1}(V_1)\times\cdots\times T(m\delta)^{-1}(V_1)\\ &\qquad \times V_2\times T(\delta)^{-1}(V_2)\times\cdots\times T(m\delta)^{-1}(V_2)) \neq \emptyset,\end{aligned}$$

i.e. there is $t > 0$ such that for every $0 \leq j \leq m$ we have

$$T(t+j\delta)(\tilde{U}_i)\cap V_i \neq \emptyset,\ i = 1,2.$$

Because I is syndetic and $m\delta > K$ there is $s \in [t, t+m\delta)\cap I$, so we can find $1 \leq j_0 \leq m$ such that $s \in [t+(j_0-1)\delta, t+j_0\delta)\cap I$. For $i = 1,2$ we then get from $0 \leq t+j_0\delta-s < \delta$ and the definition of $\tilde{U}_i$ that

$$\emptyset \neq T(t+j_0\delta)(\tilde{U}_i)\cap V_i = T(s)T(t+j_0\delta-s)(\tilde{U}_i)\cap V_i \subset T(s)(U_i)\cap V_i.$$

Since $s \in I$, this shows the transitivity of $\{T\oplus T\,(s);\ s \in I\}$, i.e. vi).

Clearly, vi) implies vii).

Next, we show that vii) implies v). Assume that v) is not true, which implies the existence of $U, V \subset E$ open and not empty such that $T(t+K)(U) \cap V = \emptyset$ whenever $T(t)(U) \cap V \neq \emptyset$ and $T(t)(U) \cap V = \emptyset$ whenever $T(t+K)(U) \cap V \neq \emptyset$. Let $I := \{t \geq 0;\, T(t)(U) \cap V = \emptyset\}$. It follows that $t + K \in I$ for all $t \notin I$ so that in particular $[t, t+K] \cap I \neq \emptyset$ for every $t \geq 0$. By vii) we have that $\{T(t);\, t \in I\}$ is weakly mixing, in particular there is $t \in I$ such that $T(t)(U) \cap V \neq \emptyset$. But this means that $t \notin I$ giving a contradiction. □

Although we are not able to answer the question whether every hypercyclic C_0-semigroup satisfies the Criterion in general, we now show that this is in fact the case under some additional "regularity condition" on T.

In order to do so we need the following theorem, which was inspired by [28, theorem 2].

Theorem 2.6 *Let T be a C_0-semigroup on a separable Banach space E. Suppose there are dense subsets Y, Z of E, a sequence of positive numbers $(t_k)_{k\in\mathbb{N}}$ and mappings $R_k : Z \to E$ such that*

a) For every $y \in Y$ the set $\{T(t_k)y;\, k \in \mathbb{N}\}$ is relatively compact.

b) For every $z \in Z$ we have that $(R_k z)_{k\in\mathbb{N}}$ converges to 0.

c) For every $z \in Z$ the sequence $(T(t_k)R_k z)_{k\in\mathbb{N}}$ converges to z.

Then, T satisfies the Hypercyclicity Criterion.

By theorem 2.5 ii), every C_0-semigroup T satisfying the Hypercyclicity Criterion satisfies the conditions of the above theorem, so that theorem 2.6 is still another equivalent formulation of the Criterion.

Proof of theorem 2.6: We first show that $\{T(t_k);\, k \in \mathbb{N}\}$ is a hypercyclic family of operators. Let $U, V \subset E$ be open and not empty. Take u from $U \cap Y$ and $(t_{k_j})_{j\in\mathbb{N}}$ such that $a := \lim_{j\to\infty} T(t_{k_j})u$ exists. Now, let v be in $V \cap (Z + a)$. Then, $(R_k(v-a))_{k\in\mathbb{N}}$ converges to 0 so that $(u + R_k(v-a))_{k\in\mathbb{N}}$ converges to u. On the other hand, $(T(t_{k_j})(u + R_{k_j}(v-a)))_{j\in\mathbb{N}}$ converges to v so that $T(t_{k_j})(U) \cap V \neq \emptyset$ for sufficiently large j. This shows that $(T(t_k))_{k\in\mathbb{N}}$ is transitive, hence hypercyclic by remark 1.2 iii).

Let $x \in HC(T(t_k); k \in \mathbb{N})$ and let $(t_{k_l})_{l\in\mathbb{N}}$ be a such that $\lim_{l\to\infty} T(t_{k_l})x = 0$. Then, $\tilde{Y} := \{T(t_k)x; k \in \mathbb{N}\}$ is dense in E and for all $y = T(t_{k_0})x \in \tilde{Y}$ we have $\lim_{l\to\infty} T(t_{k_l})y = \lim_{l\to\infty} T(t_{k_0})T(t_{k_l})x = 0$.

Clearly the family $(T(t_{k_l}))_{l\in\mathbb{N}}$ satisfies conditions a), b), and c) of the theorem, too, so that it is transitive, hence hypercyclic by the first part of the proof.

Let $u \in HC(T(t_{k_l}); l \in \mathbb{N})$ and $\tilde{Z} := \{T(t_{k_l})u;\, l \in \mathbb{N}\}$. Then, $\tilde{Z}$ is dense in E. Furthermore, there is a subsequence $(s_l)_{l\in\mathbb{N}}$ of $(t_{k_l})_{l\in\mathbb{N}}$ such that $T(s_l)(B(0, 1/l)) \cap B(u, 1/l) \neq \emptyset$, i.e. there is $(\tilde{w}_l)_{l\in\mathbb{N}}$ converging to 0 such that $(T(s_l)\tilde{w}_l)_{l\in\mathbb{N}}$ converges to u.

If we define for $z = T(t_{k_l})u \in \tilde{Z}$ the sequence $(w_n)_{n\in\mathbb{N}}$ by $w_n := T(t_{k_l})\tilde{w}_n$ we see that $\lim_{n\to\infty} w_n = 0$. Additionally, we have for $z = T(t_{k_l})u \in \tilde{Z}$ that $T(s_n)w_n = T(t_{k_l})(T(s_n)\tilde{w}_n)$ so that the sequence $(T(s_n)w_n)_{l\in\mathbb{N}}$ converges to $T(t_{k_n})u = z$. This shows that for every $z \in \tilde{Z}$ there is a sequence $(w_n)_{n\in\mathbb{N}}$ converging to 0 such that $\lim_{n\to\infty} T(s_n)w_n = z$.

Since $(s_l)_{l\in\mathbb{N}}$ is a subsequence of $(t_{k_l})_{l\in\mathbb{N}}$, it also holds that $\lim_{l\to\infty} T(s_l)y = 0$ for all $y \in \tilde{Y}$. This gives that T satisfies condition ii) of theorem 2.5. □

We are now able to prove the following theorem saying that a hypercyclic C_0-semigroup T satisfies the Criterion if some additional condition is satisfied.

Theorem 2.7 *Let T be a hypercyclic C_0-semigroup on a separable Banach space E and let $C := \{y \in E;\ orb(T, y)$ is relatively compact$\}$.*
If C is dense in E, then T satisfies the Criterion.

PROOF: Let $x \in HC(T)$, $(t_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers and $(w_k)_{k\in\mathbb{N}}$ be a sequence converging to 0 such that $\lim_{k\to\infty} T(t_k)w_k = x$.

Then, $Z := \mathrm{orb}(T, x)$ is dense in E and $R_k : Z \to E, T(t)x \mapsto T(t)w_k, k \in \mathbb{N}$, is a well-defined mapping such that $(R_k z)_{k\in\mathbb{N}}$ converges to 0 for all $z \in Z$. Furthermore, $T(t_k)R_k z = T(t)T(t_k)w_k$ for $z = T(t)x \in Z$, so that $(T(t_k)R_k z)_{k\in\mathbb{N}}$ converges to z for every $z \in Z$.

By hypothesis, $Y := C$ is dense in E and trivially for all $y \in Y$ there is a subsequence $(t_{k_j})_{j\in\mathbb{N}}$ such that $(T(t_{k_j})y)_{j\in\mathbb{N}}$ converges.

This shows that all the hypotheses of theorem 2.6 are fulfilled, so that T satisfies the Hypercyclicity Criterion. □

The analogue of the following corollary in the single operator case is due to J. Wengenroth [61] who gave a direct proof.

Corollary 2.8 *Let T be a hypercyclic C_0-semigroup on a separable Banach space E such that $E_0 := \{x \in E;\ \lim_{t\to\infty} T(t)x = 0\}$ is dense in E.*
Then T satisfies the Hypercyclicity Criterion.

PROOF: This is a direct consequence of theorem 2.7. □

The following theorem was pointed out to us by A. Peris [50]. We give two different proofs here:

Theorem 2.9 *Let T be a C_0-semigroup on a separable Banach space E. Then, the following are equivalent.*

i) T satisfies the Hypercyclicity Criterion.

ii) The operator $T(t_0)$ satisfies the Hypercyclicity Criterion for all $t_0 > 0$.

iii) There is $t_0 > 0$ such that the operator $T(t_0)$ satisfies the Hypercyclicity Criterion.

PROOF: Having in mind theorem 2.5 ii), all we have to show is that i) implies ii). So, let $t_0 > 0$. If we define $I := \{nt_0;\ n \in \mathbb{N}\}$, then I is a syndetic set so that ii) follows from theorem 2.5 and the fact that an operator on a separable Banach space is weakly mixing if and only if it satisfies the Hypercyclicity Criterion for operators (cf. [8, theorem 2.3]).

Alternative proof of i) ⇒ ii): Let Y, Z and $(t_k)_{k\in\mathbb{K}}$ be as in theorem 2.5 ii) and let $t_0 > 0$. For every $k \in \mathbb{N}$ there are exactly one $s_k \in [0, t_0)$ and $n_k \in \mathbb{N}_0$ such that $t_k = n_k t_0 + s_k$.

For every $y \in Y$ we have that $\lim_{k\to\infty} T(t_k)y = 0$. Using the equicontinuity of the family $(T(t))_{0\le t\le t_0}$ this yields

$$\begin{aligned}\lim_{k\to\infty} T(n_k t_0)T(t_0)y &= \lim_{k\to\infty} T(n_k t_0 + s_k)T(t_0 - s_k)y \\ &= \lim_{k\to\infty} T(t_0 - s_k)T(t_k)y = 0\end{aligned}$$

for every $y \in Y$. This shows that $\lim_{k\to\infty} T(n_k t_0)y = 0$ for all y from the subset $\tilde{Y} := T(t_0)(Y)$ which is dense since Y is dense and $T(t_0)$ has dense range by corollary 1.11.

Now, let $z \in Z$ and $(w_k)_{k\in\mathbb{N}}$ be a sequence converging to 0 in E such that $(T(t_k)w_k)_{k\in\mathbb{N}}$ converges to z. From the equicontinuity of the family $(T(t))_{0\le t\le t_0}$ it follows that $(T(s_k)w_k)_{k\in\mathbb{N}}$ converges to 0 in E. Since $T(n_k t_0)T(s_k)w_k = T(t_k)w_k$ it follows that $(T(n_k t_0)T(s_k)w_k)_{k\in\mathbb{N}}$ converges to z.

Using [13, theorem 1.5], this shows that $T(t_0)$ satisfies the Hypercyclicity Criterion for single operators with $\tilde{Y}, Z$ and $(n_k)_{k\in\mathbb{N}}$. □

It was shown by Oxtoby and Ulam in [44, theorem 6] that "almost every" operator from a topologically transitive flow in a separable metric space is topologically transitive.

A slight modification of their proof shows that if $x \in HC(T)$ then there is a dense G_δ-set $I_x \subset [0,\infty)$ such that $x \in HC(T(t))$ for every $t \in I_x$. This was independently shown by A. Conejero in [12] and can even be strengthened to the following.

Theorem 2.10 *Let T be a hypercyclic C_0-semigroup on E, $x \in HC(T), t_0 > 0$, and $(n_k)_{k\in\mathbb{N}}$ be a strictly increasing sequence of positive integers which satisfies $\sup_{k\in\mathbb{N}}(n_{k+1} - n_k) < \infty$. Furthermore, let $B \subset (0, t_0)$ be open.*

Then, there is a dense G_δ-subset I_x of $[0,\infty)$ such that $\{T(n_k t)x;\, k \in \mathbb{N}\}$ is dense in E for every $t \in I_x$, and $t_0 = t_1 + t_2$, where $t_1 \in B \cap I_x$ and $t_2 \in I_x$.

In particular, every operator $T(t), t > 0$, from T is the composition of two hypercyclic operators from T which share a prescribed hypercyclic vector.

PROOF: Let $(U_m)_{m\in\mathbb{N}}$ be an open base of the topology of E such that $U_m \neq \emptyset$ for all $m \in \mathbb{N}$. We define $I_m := \{t \ge 0;\, \exists\, l \in \mathbb{N} : T(n_l t)x \in U_m\}$. From the strong continuity of T it follows immediately that I_m is open in $[0,\infty)$. It is also dense, as will be shown now.

Let $(a, b) \subset [0,\infty)$ and $M := \sup_{k\in\mathbb{N}}(n_{k+1} - n_k)$. Since $(n_k)_{k\in\mathbb{N}}$ is strictly increasing there is K such that $n_k(b-a) > Ma$ for all $k \ge K$, that is

$$n_k b - n_{k+1} a = n_k(b-a) + (n_k - n_{k+1})a \ge n_k(b-a) - Ma > 0$$

for all $k \ge K$. This implies that $\bigcup_{k\ge K}(n_k a, n_k b)$ is connected, so that there is $L > 0$ such that $[L,\infty) \subset \bigcup_{k\in\mathbb{N}}(n_k a, n_k b)$.

Now, since $x \in HC(T)$ we have that $\{T(t)x;\, t \ge L\}$ is dense in E. Therefore we can find $t \ge L$ such that $U_m \ni T(t)x = T(t/n_l)^{n_l}x$, where n_l is chosen in such a way that $t/n_l \in (a, b)$, which is possible by $[L,\infty) \subset \bigcup_{k\in\mathbb{N}}(n_k a, n_k b)$. This proves $t/n_l \in I_m \cap (a, b)$.

Using Baire's Category theorem, we obtain that $I_x := \bigcap_{m\in\mathbb{N}} I_m$ is a dense G_δ-subset of $[0,\infty)$. Obviously, for every $t \in I_x$ the set $\{T(n_k t)x;\, k \in \mathbb{N}\}$ is dense in E.

Let us denote for a subset C of $[0,\infty)$ the set $\{t_0 - t;\, t \in C\}$ by $t_0 - C$. It follows from the density of I_x in $[0,\infty)$ that $I_x \cap (t_0 - B)$ as well as $t_0 - (I_x \cap B)$ are dense G_δ-subsets of $t_0 - B$. Applying Baire's theorem again, we see that $\emptyset \neq [I_x \cap (t_0 - B)] \cap [t_0 - (I_x \cap B)]$, i.e. there are $t_1 \in B \cap I_x$ and $t_2 \in I_x$ such that $t_0 = t_1 + t_2$.

From this it follows that $x \in HC(T(t_1)) \cap HC(T(t_2))$ and $T(t_0) = T(t_1) \circ T(t_2)$. □

Using theorem 2.10 and a theorem due to S. Grivaux we can now give a strengthened version of theorem 2.7.

Theorem 2.11 *Let T be a hypercyclic C_0-semigroup on a separable Banach space E and let $B := \{y \in E;\, orb(T,y)$ is bounded$\}$.*

If B is dense in E, then T satisfies the Hypercyclicity Criterion.

Proof: It follows immediately from the hypothesis that for every $t > 0$ the set $B(t) := \{y \in E;\, \text{orb}(T(t),y) \text{ is bounded}\}$ is dense in E. From theorem 2.10 it follows in particular that there is $t_0 > 0$ such that $T(t_0)$ is a hypercyclic operator on E. Using the denseness of $B(t_0)$ it follows from [27, theorem 4.4] that $T(t_0)$ satisfies the Hypercyclicity Criterion for single operators. Now the theorem follows from theorem 2.9. □

Having in mind theorem 2.10 one might ask if every operator $T(t), t > 0$, from a hypercyclic C_0-semigroup T is hypercyclic. This deep question was posed by Bermúdez et al. in [5]. It was only very recently solved by Conejero, Müller and Peris using sophisticated arguments from homotopy theory. They did not only answer the problem in [5] in the affirmative, but they also proved that $x \in HC(T)$ if and only if $x \in \bigcap_{t>0} HC(T(t))$.

Theorem 2.12 (cf. [14, theorem 2.3]) *Let T be a hypercyclic C_0-semigroup and $x \in HC(T)$. Then $x \in HC(T(t_0))$ for all $t_0 > 0$.*

Before we close this section we come back to theorem 2.4. It characterises when a C_0-semigroup on a separable Banach space E is weakly mixing. In the following theorem we give a characterisation of when a C_0-semigroup is weakly mixing on an arbitrary, not necessarily separable Banach space.

Theorem 2.13 *Let T be a C_0-semigroup on a Banach space E. Then, the following are equivalent.*

i) T is weakly mixing.

ii) For every open and non-empty subsets U, V_1, V_2 of E there is $t > 0$ such that $T(t)(U) \cap V_1 \neq \emptyset$ and $T(t)(U) \cap V_2 \neq \emptyset$.

Proof: It is clear that i) implies ii). To show the converse, let $U_i, V_i \subset E, i = 1, 2$, be open and not empty. From ii) it follows that there is $s > 0$ such that $T(s)(U_1) \cap U_2 \neq \emptyset$ and $T(s)(U_1) \cap V_2 \neq \emptyset$. From this we conclude $T(s)^{-1}(V_2) \neq \emptyset$ and using the continuity of $T(s)$ we find $\tilde{U}_1 \subset U_1$ open and non-empty such that $T(s)(\tilde{U}_1) \subset U_2$.

Now we use ii) again to find $t > 0$ such that $T(t)(\tilde{U}_1) \cap V_1 \neq \emptyset$ and $T(t)(\tilde{U}_1) \cap T(s)^{-1}(V_2) \neq \emptyset$. This yields

$$\emptyset \neq T(t)(\tilde{U}_1) \cap V_1 \subset T(t)(U_1) \cap V_1$$

and

$$\emptyset \neq T(t)(T(s)(\tilde{U}_1)) \cap V_2 \subset T(t)(U_2) \cap V_2,$$

so that T is weakly mixing. □

Remark 2.14 i) If in the above theorem E is separable, it follows from theorem 2.4 that condition ii) is still another equivalent formulation of the Hypercyclicity Criterion.

ii) Note that the above proof works exactly in the same way in a more general setting. Let I be a set and $T : I \to L(E)$ a mapping such that $T(\iota) \circ T(\kappa) = T(\kappa) \circ T(\iota)$ for every $\iota, \kappa \in I$. Then, T is weakly mixing if and only if for every open and non-empty subsets U, V_1, V_2 of E there is $\iota \in I$ such that $T(\iota)(U) \cap V_1 \neq \emptyset$ and $T(\iota)(U) \cap V_2 \neq \emptyset$.

3 Infinitely regular hypercyclic vectors

When the generator A of a hypercyclic C_0-semigroup T is an unbounded operator, it follows from the non-emptiness of its resolvent set and the Open-Mapping theorem, that $D(A)$ is of first category in E. So, we cannot use the Baire argument to show the existence of a hypercyclic vector x in $D(A)$.

Nevertheless, we will show in this short chapter that for a hypercyclic C_0-semigroup even $D(A^\infty) \cap HC(T)$ is dense in E.

We first show that $HC(T) \cap D(A^n) \neq \emptyset$ for every $n \in \mathbb{N}$ whenever T is hypercyclic.

This chapter will be published in *Proceedings of the American Mathematical Society* under the title "On chaotic C_0-semigroups and infinitely regular hypercyclic vectors".

Lemma 3.1 *Let T be a hypercyclic C_0-semigroup on E and let $(A, D(A))$ be its generator. For $n \in \mathbb{N}$ we consider the graph norm $\|x\|_n := \sum_{j=0}^n \|A^j x\|$ on $D(A^n)$, and we set $T_n := (T_n(t))_{t\geq 0} := (T(t)_{|D(A^n)})_{t\geq 0}$.*

Then, T_n is a hypercyclic C_0-semigroup on $(D(A^n), \|\cdot\|_n)$ with $HC(T_n) \subseteq HC(T)$ and $HC(T_n)$ is dense in E. In particular $HC(T) \cap D(A^n) \neq \emptyset$.

Proof: That $(D(A^n), \|\cdot\|_n)$ is a Banach space and T_n is a C_0-semigroup on $(D(A^n), \|\cdot\|_n)$ is a well known result (see e.g. [24, chapter II.5]).

Now let λ be in the resolvent set of A and $R(\lambda, A)$ the resolvent operator of A in λ. Then $R(\lambda, A)^n : (E, \|\cdot\|) \to (D(A^n), \|\cdot\|_n)$ is a continuous isomorphism and obviously $T_n(t) \circ R(\lambda, A)^n = R(\lambda, A)^n \circ T(t)$ for every $t \geq 0$, so that T_n is a hypercyclic C_0-semigroup by lemma 1.4 and $HC(T_n)$ is a dense G_δ-subset of $(D(A^n), \|\cdot\|_n)$.

Since the continuous inclusion $\iota : (D(A^n), \|\cdot\|_n) \hookrightarrow E$ has dense range and $T(t) \circ \iota = \iota \circ T_n(t), t \geq 0$, we see that $HC(T_n) \subseteq HC(T)$ and that $HC(T_n)$ is dense in E. □

We equip $D(A^\infty) = \bigcap_{n\in\mathbb{N}} D(A^n)$ with the locally convex topology induced by the increasing family of seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}_0}$, where as above $\|x\|_n = \sum_{j=0}^n \|A^j x\|$. Then $(D(A^\infty), (\|\cdot\|_n)_{n\in\mathbb{N}_0})$ is a Fréchet space and we obtain the following result.

Theorem 3.2 *Let T be a hypercyclic C_0-semigroup on E and let $(A, D(A))$ be its generator. Then $T_\infty := (T_\infty(t))_{t\geq 0} := (T(t)_{|D(A^\infty)})_{t\geq 0}$ is a hypercyclic semigroup on $(D(A^\infty), (\|\cdot\|_n)_{n\in\mathbb{N}_0})$ (where hypercyclicity of a semigroup on a Fréchet space is defined in an obvious way) with $HC(T_\infty) \subset HC(T)$ and $HC(T_\infty)$ is dense in E. In particular $D(A^\infty) \cap HC(T) \neq \emptyset$.*

Proof: Since $(D(A^\infty), (\|\cdot\|_n)_{n\in\mathbb{N}_0})$ is the projective limit of the countable family of Banach spaces $(D(A^n), \|\cdot\|_n)$ and since each of the Banach spaces $(D(A^n), \|\cdot\|_n)$ is separable (because $(D(A^n), \|\cdot\|_n)$ is isomorphic to a closed subspace of $E^{(n+1)}$), $(D(A^\infty), (\|\cdot\|_n)_{n\in\mathbb{N}_0})$ is a separable Fréchet space, hence second countable as topological space.

That T_∞ is a semigroup of continuous operators on $(D(A^\infty), (\|\cdot\|_n)_{n\in\mathbb{N}_0})$ is well known.

We will show that T_∞ is topologically transitive on $(D(A^\infty), (\|\cdot\|_n)_{n\in\mathbb{N}_0})$ and therefore hypercyclic. To do so, we choose $x, y \in D(A^\infty)$ and a neighbourhood

W of zero in $(D(A^\infty), (\|\cdot\|_n)_{n\in\mathbb{N}_0})$. Then there are $\varepsilon > 0$ and $m_0 \in \mathbb{N}_0$ such that $\{z \in D(A^\infty);\ \|z\|_{m_0} < \varepsilon\} \subseteq W$. Since $x, y \in D(A^\infty) \subset D(A^{m_0})$ and T_{m_0} is hypercyclic, hence topologically transitive, there is t_0 such that $U := (x + V) \cap T_{m_0}(t_0)^{-1}(y + V) \neq \emptyset$, where $V := \{z \in D(A^{m_0});\ \|z\|_{m_0} < \varepsilon\}$. Since U is open in $(D(A^{m_0}), \|\cdot\|_{m_0})$ and $D(A^\infty)$ is dense in $(D(A^{m_0}), \|\cdot\|_{m_0})$, there is $z \in D(A^\infty)$ with $\|x - z\|_{m_0} < \varepsilon, \|y - T_{m_0}(t_0)(z)\|_{m_0} < \varepsilon$, that is $z - x \in W$, $T_\infty(t_0)(z) - y = T_{m_0}(t_0)(z) - y \in W$ which shows $(x + W) \cap T_\infty(t_0)^{-1}(y + W) \neq \emptyset$, i.e. the topological transitivity of T_∞.

Because the inclusion $\iota : (D(A^\infty), (\|\cdot\|_n)_{n\in\mathbb{N}_0}) \hookrightarrow E$ is continuous, has dense range, and $\iota \circ T_\infty = T \circ \iota$ we have $HC(T_\infty) \subseteq HC(T)$ and $HC(T_\infty)$ is dense in E. □

Remark 3.3 In a more general setting, it can be shown that given a strongly reduced projective spectrum $\mathcal{X} = (X_\alpha)_{\alpha\in I}$ and a topological transitive semigroup $(T(t)_\alpha)_{t\geq 0}$ on each of the components, the induced semigroup on the projective limit of $\mathcal{X}$ is topologically transitive again, cf. [10, proposition 2.1].

With the same kind of arguments we used to prove theorem 3.2 we can prove the following result.

Theorem 3.4 *Let A be the generator of a C_0-semigroup T on a Banach space E. Let $\lambda \in \rho(A)$ and $R(\lambda, A)$ the resolvent operator of A in λ. Then, the following are equivalent.*

i) $R(\lambda, A)$ is hypercyclic on E.

ii) $(\lambda - A)$ is hypercyclic on $(D(A^\infty), (\|\cdot\|_n)_{n\in\mathbb{N}_0})$.

Proof: First, note that

$$R(\lambda, A) : (D(A^\infty), (\|\cdot\|_n)_{n\in\mathbb{N}_0}) \to (D(A^\infty), (\|\cdot\|_n)_{n\in\mathbb{N}_0})$$

is well defined and continuous, since it commutes with A and $R(\lambda, A) \circ (\lambda - A) = (\lambda - A) \circ R(\lambda, A) = id_{D(A^\infty)}$.

In order to show that i) implies ii) observe that

$$R(\lambda, A)^n : E \to (D(A^n), \|\cdot\|_n)$$

is a continuous isomorphism for every $n \in \mathbb{N}$ and that the following diagram commutes.

$$\begin{array}{ccc} E & \xrightarrow{R(\lambda, A)} & E \\ \downarrow{\scriptstyle R(\lambda, A)^n} & & \downarrow{\scriptstyle R(\lambda, A)^n} \\ (D(A^n), \|\cdot\|_n) & \xrightarrow{R(\lambda, A)} & (D(A^n), \|\cdot\|_n) \end{array}$$

From the single operator analogue of lemma 1.4 (see e.g. [39, lemma 2.1]) it follows that $R(\lambda, A)$ is hypercyclic on $(D(A^n), \|\cdot\|_n)$ for all $n \in \mathbb{N}$.

Let $x, y \in D(A^\infty)$ and W a neighbourhood of zero in $(D(A^\infty), (\|\cdot\|_n)_{n\in\mathbb{N}_0})$. Then there are $\varepsilon > 0$ and $m_0 \in \mathbb{N}_0$ such that $\{z \in D(A^\infty);\ \|z\|_{m_0} < \varepsilon\} \subseteq W$.

Since $x, y \in D(A^\infty) \subset D(A^{m_0})$ and $R(\lambda, A)$ is hypercyclic, hence topologically transitive, on $(D(A^{m_0}), \|\cdot\|_{m_0})$ there is $k \in \mathbb{N}$ such that $U := (x+V) \cap R(\lambda, A)^{-k}(y+V) \neq \emptyset$, where $V := \{z \in D(A^{m_0});\ \|z\|_{m_0} < \varepsilon\}$. Since U is open in $(D(A^{m_0}), \|\cdot\|_{m_0})$ and $D(A^\infty)$ is dense in $(D(A^{m_0}), \|\cdot\|_{m_0})$, there is $z \in D(A^\infty)$ with $\|x-z\|_{m_0} < \varepsilon$, $\|y - R(\lambda, A)^k(z)\|_{m_0} < \varepsilon$, that is $z - x \in W$, $R(\lambda, A)^k z - y \in W$ which shows $(x+W) \cap R(\lambda, A)^{-k} \neq \emptyset$, i.e. the topological transitivity of $R(\lambda, A)$ on $(D(A^\infty), (\|\cdot\|_n)_{n\in\mathbb{N}_0})$. From the bijectivity of $R(\lambda, A)$ on $(D(A^\infty), (\|\cdot\|_n)_{n\in\mathbb{N}_0})$ it immediately follows that its inverse $(\lambda - A)$ is transitive, too. As already observed in the proof of theorem 3.2, $(D(A^\infty), (\|\cdot\|_n)_{n\in\mathbb{N}_0})$ is separable so that $(\lambda - A)$ is hypercyclic by remark 1.2 iii).

In order to show that ii) implies i), note that by ii) $R(\lambda, A)$ is hypercyclic on $(D(A^\infty), (\|\cdot\|_n)_{n\in\mathbb{N}_0})$. Since the inclusion $\iota : (D(A^\infty), (\|\cdot\|_n)_{n\in\mathbb{N}_0}) \hookrightarrow E$ is continuous, has dense range and makes the diagram

$$\begin{array}{ccc} (D(A^\infty), (\|\cdot\|_n)_{n\in\mathbb{N}_0}) & \xrightarrow{R(\lambda, A)} & (D(A^\infty), (\|\cdot\|_n)_{n\in\mathbb{N}_0}) \\ \downarrow \iota & & \downarrow \iota \\ E & \xrightarrow{R(\lambda, A)} & E \end{array}$$

commutative, i) follows again from the single operator analogue of lemma 1.4 (cf. [39, lemma 2.1]).

□

4 Chaotic C_0-semigroups

In this chapter we turn our attention to chaotic C_0-semigroups. Parts of it will be published in *Proceedings of the American Mathematical Society* under the title "On chaotic C_0-semigroups and infinitely regular hypercyclic vectors".

There are many notions of chaos for a (discrete) dynamical system, that is for a continuous mapping $f : X \to X$ on a metric space (X, d). We recall that a continuous mapping $f : X \to X$ is called *topologically transitive* if for every pair of non-empty open subsets U, V of X there is $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$.

f is said to have *sensitive dependence on initial data* if there is $\delta > 0$ such that for every $x \in X$ and every neighbourhood U of x there is $y \in U$ and $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y)) > \delta$.

Recall that the set of periodic points of f is defined as $\operatorname{per}(f) := \{x \in X; \exists\, n \in \mathbb{N} : f^n(x) = x\}$.

Definition 4.1 Let f be a continuous mapping on the metric space (X, d). Then f is called *chaotic in sense of Devaney* if f is topologically transitive, has sensitive dependence on initial data and if the set $\operatorname{per}(f)$ is dense in X.

On the other hand, f is called *chaotic in the sense of Auslander and Yorke* if f has sensitive dependence on initial data and if there is a point $x \in X$ such that the *orbit of x under f*, i.e. $\operatorname{orb}(f, x) := \{f^n(x);\, n \in \mathbb{N}_0\}$ is dense in (X, d).

It was shown by Banks et al. in [4] that a transitive mapping f on a metric space with a dense set of periodic points already has sensitive dependence on initial data. So f is chaotic in the sense of Devaney if and only if it is transitive and has a dense set of periodic points.

Now, if E is a separable Banach space and T is an operator on it, we see that the notion of chaos in the sense of Auslander and Yorke for T is very close to hypercyclicity. In fact, Godefroy and Shapiro showed in [26, proposition 6.1] that every hypercyclic operator T has sensitive dependence on initial data in a very dramatic way. Since the proof for the analogous result for C_0-semigroups works with exactly the same arguments, we include it for the reader's convenience.

Proposition 4.2 (cf. [26, proposition 6.1.]) *Let E be a Banach space and T a hypercyclic operator, respectively a hypercyclic C_0-semigroup, on E.*

Then, for every $x \in E$ there is a dense G_δ-set $S(x) \subset E$, such that the set of orbit-differences $\{T^n x - T^n y; n \in \mathbb{N}_0\}$, resp. $\{T(t)x - T(t)y;\, t \geq 0\}$, is dense in E for every $y \in S(x)$.

Proof: Since by remark 1.2 $HC(T)$ is a dense G_δ-subset of E, the same is true for the set $S(x) := x - HC(T)$. Obviously, $x - y \in HC(T)$ for all $y \in S(x)$, so that the desired property of y follows from the linearity of T, resp. $T(t)$. □

Remark 4.3 Note that the above proposition is true in the single operator case not only for Banach spaces but more general for F-spaces.

In view of this proposition, every hypercyclic operator T on a Banach space E is chaotic in the sense of Auslander and Yorke. Because of this, one usually means "chaotic in the sense of Devaney" when speaking of a *chaotic operator* on a Banach space E.

We make the same convention for C_0-semigroups.

Definition 4.4 A C_0-semigroup T on a separable Banach space E is called *chaotic* if it is hypercyclic and if the set of periodic points $\mathrm{per}(T) := \{x \in E; \exists\, t > 0 : T(t)x = x\}$ is dense in E.

There is a number of articles devoted to chaotic C_0-semigroups. As for hypercyclic C_0-semigroups the first article dealing systematically with chaotic semigroups is the one by Desch et al. [19]. See also [5], [6], [18], [41], [42], [40], [58], or [59].

As a first result about chaotic C_0-semigroups we present the following interesting theorem.

Theorem 4.5 *Let T be a chaotic C_0-semigroup on a Banach space E. Then, T satisfies the Hypercyclicity Criterion and is therefore weakly mixing.*

PROOF: The theorem follows directly from the denseness of $\mathrm{per}(T)$ and theorem 2.7. □

Remark 4.6 The analogue of theorem 4.5 in the single operator case is due to Bès, Peris [8, proposition 2.14]. Their proof uses the deep result of Ansari [2, theorem 1] stating that T^k is a hypercyclic operator whenever T is.

Combining the above theorem and theorem 2.9 yields the following result.

Corollary 4.7 *Let T be a chaotic C_0-semigroup on E. Then, for every $t > 0$ the operator $T(t)$ is weakly mixing.*

In particular, the above corollary yields that all operators $T(t), t > 0$, of a chaotic C_0-semigroup T are hypercyclic. Despite this fact, it is not known whether a chaotic C_0-semigroup has to contain a single chaotic operator.

The most commonly used condition for proving that a given C_0-semigroup on a complex Banach space is chaotic is due to Desch, Schappacher and Webb (see e.g. [41], [40], [58], or [59]). They gave the following sufficient condition on the spectrum of the generator $(A, D(A))$ for the semigroup to be chaotic:

(DSW) For some open and connected subset U of the point spectrum $\sigma_p(A) \subseteq \mathbb{C}$ of A intersecting the imaginary axis there exist eigenvectors x_λ corresponding to $\lambda \in U$ such that for each $\phi \in E'\backslash\{0\}$ the mapping $F_\phi(\lambda) = \phi(x_\lambda)$ is holomorphic on U and does not vanish identically.

Theorem 4.8 (cf. [19, theorem 3.1]) *A C_0-semigroup T on a separable complex Banach space E is chaotic whenever its generator $(A, D(A))$ satisfies (DSW).*

We will now show that condition (DSW) actually implies the chaoticity of each $T(t), t > 0$. Its proof uses an argument of [26].

Theorem 4.9 *Let E be a separable complex Banach space and let A be the generator of the C_0-semigroup T on E. Suppose that condition (DSW) is fulfilled. Then, for every $t_0 > 0$ the operator $T(t_0)$ is chaotic.*

PROOF: Let $t_0 > 0$. One could use corollary 4.7 to obtain that $T(t_0)$ is hypercyclic, but we prefer to give a direct proof here.

We define the sets $\Omega_1 := \{z \in U;\, \mathrm{Re}\, z > 0\}$ and $\Omega_2 := \{z \in U;\, \mathrm{Re}\, z < 0\}$ which are non-empty, open subsets of U by hypothesis and therefore contain accumulation points in U. Since U is open and intersects the imaginary axis, the set $\Omega_3 := \{z \in U;\, \mathrm{Re}\, z = 0,\, t_0 \mathrm{Im}\, z \in 2\pi\mathbb{Q}\}$ contains accumulation points in U, as well.

We set $V_j := \mathrm{span}\{x_\lambda;\, \lambda \in \Omega_j\}$, $j = 1, 2, 3$ and observe that V_j is a dense subspace of E: If $\phi \in E'$ is such that $0 = \phi(x_\lambda) = F_\phi(\lambda)$ for every $\lambda \in \Omega_j$ it follows that the holomorphic function F_ϕ vanishes identically on U, since Ω_j have accumulation points in U. By hypothesis this implies $\phi = 0$ so that from the Hahn-Banach theorem we obtain the density of V_j in E.

Using the spectral mapping theorem for the point spectrum of C_0-semigroups (see appendix A.7), i.e. $\sigma_p(T(t))\backslash\{0\} = e^{t\sigma_p(A)}, t \geq 0$, we get for $\sum_{k=1}^m \alpha_k x_{\lambda_k} \in V_2$ and $n \in \mathbb{N} : T(t_0)^n(\sum_{k=1}^m \alpha_k x_{\lambda_k}) = \sum_{k=1}^m \alpha_k e^{nt_0\lambda_k} x_{\lambda_k}$ which converges to zero as n tends to infinity since $|e^{t_0\lambda_k}| < 1$.

If we set $S : V_1 \to V_1, \sum_{k=1}^m \alpha_k x_{\lambda_k} \mapsto \sum_{k=1}^m \alpha_k e^{-t_0\lambda_k} x_{\lambda_k}$ (note that S is well-defined because of the linear independence of $\{x_\lambda;\, \lambda \in U\}$) we obtain $T(t_0) \circ S = \mathrm{id}_{V_1}$ once again from the spectral mapping theorem. Because of $|e^{t_0\lambda_k}| > 1$ for $\lambda_k \in \Omega_1$ we see that $S^n x$ tends to zero as n tends to infinity for all $x \in V_1$. We have proved that $T(t_0)$ satisfies the Hypercyclicity Criterion for single operators (see theorem 2.3), so it is hypercyclic.

It remains to show that the dense subspace V_3 consists of periodic points of $T(t_0)$. To do so we take $p = \sum_{k=1}^m \alpha_k x_{\lambda_k} \in V_3$ with $\lambda_k \in \Omega_3$ and $e^{t_0\lambda_k} = e^{2\pi i \frac{j_k}{n_k}}$ (with j_k, n_k being integers). For $M := \prod_{k=1}^m n_k$ we then obtain by applying the spectral mapping theorem again $T(t_0)^M(p) = T(t_0)^M(\sum_{k=1}^m \alpha_k x_{\lambda_k}) = \sum_{k=1}^m \alpha_k x_{\lambda_k} = p$, i.e. the set of periodic vectors of $T(t_0)$ is dense in E. □

We can use the above theorem to show that under suitable assumptions on the point spectrum of its generator every operator of a C_0-semigroup is chaotic as soon as it has a non-trivial periodic point.

Corollary 4.10 *Let A be the generator of a C_0-semigroup T on a separable complex Banach space. Assume that $\sigma_p(A)$ is an open, connected, non-empty subset of $\mathbb{C}$ and that for every $\lambda \in \sigma_p(A)$ there is a corresponding eigenvector x_λ such that for each $\phi \in E'\backslash\{0\}$ the mapping $F_\phi(\lambda) = \phi(x_\lambda)$ is holomorphic on $\sigma_p(A)$ and does not vanish identically. Then, the following are equivalent.*

i) $T(t)$ is a chaotic operator for every $t > 0$.

ii) T is chaotic.

iii) T has a non-trivial periodic point.

iv) $\sigma_p(A)$ intersects the imaginary axis.

Proof: In view of theorem 4.9 it clearly suffices to show that iii) implies iv). That T has a non-trivial periodic point $x \neq 0$ means there is $s > 0$ such that $T(s)x = x$, i.e. $1 \in \sigma_p(T(s))$. By the spectral mapping theorem for the point spectrum (see appendix A.7) this means that there is $\lambda \in \sigma_p(A)$ such that $1 = e^{s\lambda}$. In particular, $\lambda \in i\mathbb{R}$ so that $\sigma_p(A)$ intersects the imaginary axis. □

Example 4.11 a) In [19, example 4.12] Desch et al. showed that the solution semigroup T on $L^2([0,\infty),\mathbb{C})$ of the partial differential equation

$$\begin{aligned} u_t(x,t) &= au_{xx}(x,t) + bu_x(x,t) + cu(x,t) \\ u(0,t) &= 0 \text{ for } t \geq 0 \\ u(x,0) &= f(x) \text{ for } x \geq 0 \text{ with some } f \in L^2([0,\infty),\mathbb{C}) \end{aligned}$$

satisfies condition (DSW) if $a, b, c > 0$ and $c < b^2/(2a) < 1$. So, by theorem 4.9, each of the operators $T(t), t > 0$, is chaotic. The unboundedness of $[0,\infty)$ herein is essential, as will be seen in 4.15.

b) Let $\rho : [0,\infty) \to (0,\infty)$ be a Lebesgue-measurable function satisfying the growth condition $\sup_{s\geq 0} \rho(s)/\rho(s+t) < Me^{\omega t}$ for some $M > 1, \omega \in \mathbb{R}$ and all $t \geq 0$, and let $p \geq 1$. We consider the weighted Lebesgue space $L^p_\rho([0,\infty),\mathbb{C})$ of measurable complex valued functions with its natural norm, i.e. $\|u\|^p := \int_0^\infty |u(t)|^p \rho(t)dt$. Then $(L^p_\rho([0,\infty),\mathbb{C}), \|\cdot\|)$ is a Banach space on which by $(T(t)u)(s) := u(t+s), t, s \geq 0$ we have defined a strongly continuous semigroup. It is a well-known fact that the domain of the generator of T is given by $D(A) = \{u \in L^p_\rho([0,\infty),\mathbb{C}); u$ is absolutely continuous and $u' \in L^p_\rho([0,\infty),\mathbb{C})\}$ and that $Au = u'$.

For the special case $\rho(t) = e^{-\alpha t}$ we see that for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda < \alpha$ the function $x_\lambda(t) := e^{\lambda t}$ belongs to $L^p_\rho([0,\infty),\mathbb{C})$, so that λ belongs to the point spectrum of $(A, D(A))$. For every $g \in L^q_\rho([0,\infty),\mathbb{C})$, where $1/p + 1/q = 1$, we have that $\{\lambda \in \mathbb{C}; \operatorname{Re}\lambda < \alpha\} \to \mathbb{C}, \lambda \mapsto \int_0^\infty e^{\lambda t} g(t)\rho(t)dt$ is holomorphic (the integrand is locally bounded in λ by an integrable function, so that one can interchange the integral and the derivative, see e.g. [20, theorem 13.8.6]). Since the considered function is a Laplace transform, the condition (DSW) is fulfilled whenever $\alpha > 0$.

It should be noted that in this particular example condition (DSW) is equivalent to the chaoticity of the translation semigroup T (cf. [18, theorem 4.6], or corollary 4.10).

c) Let

$$C_{0,\rho} := \{f : \mathbb{R} \to \mathbb{C}; f \text{ continuous}, \lim_{x\to\infty} |f(x)|\rho(x) = \lim_{x\to-\infty} |f(x)|\rho(x) = 0\}$$

be equipped with $\|f\| := \sup_{x\in\mathbb{R}} |f(x)|\rho(x)$, where $\rho : \mathbb{R} \to \mathbb{R}, x \mapsto \min\{1, 1/|x|\}$. According to [41, theorem 4], the translation semigroup $T = (T(t))_{t\geq 0}$ with $T(t)f := f(\cdot + t)$ is chaotic since $\lim_{t\to\infty} \rho(t) = \lim_{t\to-\infty} \rho(t) = 0$. It is easy to verify that the point spectrum of the generator A coincides with the imaginary axis, and that every $T(t), t > 0$, is chaotic, so that the "spectral part" of condition (DSW) is not necessary for the chaoticity of every $T(t)$ of a chaotic semigroup.

Next, we show that a C_0-semigroup T whose generator A has the property that $(D(A), \|\cdot\|_A) \hookrightarrow E$ is compact, can never be chaotic. We first give a condition ensuring non-hypercyclicity.

Theorem 4.12 *Let T be a C_0-semigroup on a Banach space E with generator A such that $\sigma(A) \neq \emptyset$. Let $\|\cdot\|_A$ denote the graphnorm on $D(A)$. If the imbedding $(D(A), \|\cdot\|_A) \hookrightarrow E$ is compact, then T is not hypercyclic.*

Proof: Since the resolvent set $\rho(A)$ is not empty, we can choose $\lambda \in \rho(A)$. By compactness of $(D(A), \|\cdot\|_A) \hookrightarrow E$ it follows that the resolvent $R(\lambda, A) : E \to E$ is compact. Because of $\sigma(R(\lambda, A))\backslash\{0\} = \frac{1}{\lambda - \sigma(A)}$ and $\sigma(A) \neq \emptyset$ there is $\mu_0 \in \sigma(R(\lambda, A))\backslash\{0\}$, which has to be an eigenvalue by the compactness of $R(\lambda, A)$. So μ_0 is an eigenvalue of the adjoint $R(\lambda, A)^* = R(\lambda, A^*)$, too. Let ϕ be a corresponding eigenvector. Then $R(\lambda, A^*)\phi = \mu_0\phi$, so that $\phi \in D(A^*)$ and $A^*\phi = (\lambda - \frac{1}{\mu_0})\phi$, i.e. A^* has an eigenvalue, or equivalently $\sigma_r(A) \neq \emptyset$. Using theorem 1.10 we see that T cannot be hypercyclic. □

Corollary 4.13 *Let T be a C_0-semigroup with generator A on a Banach space E. If the imbedding $(D(A), \|\cdot\|_A) \hookrightarrow E$ is compact, then T is not chaotic.*

Proof: We assume that T is chaotic. In particular, there are $t > 0$ and $x \in E\backslash\{0\}$ such that $T(t)x = x$. Using the spectral mapping theorem for the point spectrum, we see that $\sigma(A) \neq \emptyset$. So, by the above theorem, T cannot be hypercyclic which is a contradiction to the assumed chaoticity of T. □

We now apply our results to C_0-semigroups on $L^p(\Omega)$ generated by strongly elliptic partial differential operators, where Ω is bounded.

Let $\Omega \subset \mathbb{R}^d$ be open and bounded, $1 < p < \infty$. By $W^{m,p}(\Omega)$ we denote as usual the Sobolev space of order m and p, i.e. $W^{m,p}(\Omega)$ is the completion of the space

$$\{u \in C^m(\Omega); \int_\Omega \sum_{|\alpha|\leq m} |D^\alpha u|^p \, d\lambda^d < \infty\}$$

endowed with the norm

$$\|u\|_{m,p} := (\int_\Omega \sum_{|\alpha|\leq m} |D^\alpha u|^p \, d\lambda^d)^{1/p},$$

where $D^\alpha := D_1^{\alpha_1} \dots D_d^{\alpha_d}$ for $\alpha \in \mathbb{N}_0^d$ with $D_j := \frac{\partial}{\partial x_j}$, and λ^d denotes d-dimensional Lebesgue measure. Furthermore, let $W_0^{m,p}(\Omega)$ be the completion of $C_0^m(\Omega) := \{f \in C^m(\Omega); \operatorname{supp} f \text{ compact in } \Omega\}$ equipped with $\|\cdot\|_{m,p}$.

If $\partial\Omega$ is of class C^1, it follows from Sobolev's imbedding theorem (cf. [1, theorem 6.3]) that the inclusion $(W^{m,p}(\Omega), \|\cdot\|_{m,p}) \hookrightarrow L^p(\Omega)$ is compact for every $m \geq 1$.

Let $m \in \mathbb{N}$ and $a_\alpha \in C^\infty(\Omega) \cap C(\bar{\Omega})$ for $|\alpha| \leq 2m$. For $1 < p < \infty$ we define

$$D(A_p) := W^{2m,p}(\Omega) \cap W_0^{m,p}(\Omega)$$

and

$$A_p : D(A_p) \to L^p(\Omega), f \mapsto \sum_{|\alpha|\leq 2m} a_\alpha D^\alpha f,$$

where all the derivatives are understood in the weak sense. Note that $(D(A_p), \|\cdot\|_{2m,p})$ is a Banach space for all $1 < p < \infty$.

Now, for $1 < p < \infty$ the operator $(A_p, D(A_p))$ is called *strongly elliptic* if there is a constant $C > 0$ such that for every $x \in \bar{\Omega}$ and every $\xi \in \mathbb{R}^d$ we have

$$Re\,[(-1)^m \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha] \geq C|\xi|^{2m}.$$

Note that this definition is independent of p. For the proof of the following theorem see e.g. [46, theorem 7.3.5].

Theorem 4.14 *Let $\partial\Omega$ be C^∞ and $1 < p < \infty$. If $(A_p, D(A_p))$ is strongly elliptic, then $(-A_p, D(A_p))$ generates an (analytic) C_0-semigroup on $L^p(\Omega)$.*

If $(A_p, D(A_p))$ is strongly elliptic it follows from the above theorem that $(A_p, D(A_p))$ is closed, i.e. $D(A_p)$ equipped with the graph norm $\|\cdot\|_{A_p}$ is a Banach space. Furthermore, it follows from the above, that the inclusion $(D(A_p), \|\cdot\|_{2m,p}) \hookrightarrow L^p(\Omega)$ is compact for all $1 < p < \infty$. In order to show the compactness of $(D(A_p), \|\cdot\|_{A_p}) \hookrightarrow L^p(\Omega)$ it is therefore sufficient to show the continuity of the inclusion $(D(A_p), \|\cdot\|_{A_p}) \hookrightarrow (D(A_p), \|\cdot\|_{n,p})$.

Since Ω is bounded and the coefficients a_α of A_p belong to $C(\bar{\Omega})$ it follows that the inclusion $(D(A_p), \|\cdot\|_{2m,p}) \hookrightarrow (D(A_p), \|\cdot\|_{A_p})$ is continuous. Since the last spaces are Banach spaces, the Open-Mapping theorem yields the continuity of $(D(A_p), \|\cdot\|_{A_p}) \hookrightarrow (D(A_p), \|\cdot\|_{2m,p})$.

Corollary 4.15 *Let $\partial\Omega$ be C^∞ and $(A_p, D(A_p))$ be strongly elliptic, $1 < p < \infty$.*

The C_0-semigroup T_p on $L^p(\Omega)$ generated by $(-A_p, D(A_p))$ is not chaotic. Furthermore, if $\sigma(A_p) \neq \emptyset$ then T_p is not hypercyclic.

PROOF: This is an immediate consequence of corollary 4.13, respectively theorem 4.12, and the compactness of $(D(A_p), \|\cdot\|_{A_p}) \hookrightarrow L^p(\Omega)$. □

5 Transitive and mixing families of weighted composition operators

In this chapter we investigate when families of weighted composition operators on spaces of integrable functions and spaces of continuous functions are transitive or mixing, respectively. Parts of this chapter will be published in *Ergodic Theory and Dynamical Systems* under the title "Hypercyclic, mixing, and chaotic C_0-semigroups induced by semiflows".

Recall that a sequence of operators $(T_n)_{n\in\mathbb{N}}$ on a Banach space E is called *transitive* if for every pair of non-empty open subsets U, V of E there is some $n \in \mathbb{N}$ such that $T_n(U) \cap V \neq \emptyset$, and that $(T_n)_{n\in\mathbb{N}}$ is called *mixing* if $T_n(U) \cap V \neq \emptyset$ for all sufficiently large $n \in \mathbb{N}$. Generalising these notions we make the following definition.

Definition 5.1 Let E be a Banach space and $I \neq \emptyset$ a set. A mapping $T : I \to L(E)$ is called

i) *hypercyclic* if there is a vector x in E whose orbit $\mathrm{orb}(T, x) := \{T(\iota)x;\ \iota \in I\}$ is dense in E.

ii) *transitive* if for every pair of non-empty open subsets U, V of E there is some $\iota \in I$ such that $T(\iota)(U) \cap V \neq \emptyset$.

iii) *weakly mixing* if $T \oplus T$ is transitive.

Furthermore, if I is a topological space and T is strongly continuous, we call T *mixing* if for every pair of non-empty open subsets U, V of E there is some compact subset J of I such that $T(\iota)(U) \cap V \neq \emptyset$ for all $\iota \in J^c$.

Clearly, every weakly mixing T is transitive. The following proposition shows that every mixing T is weakly mixing.

Proposition 5.2 *Let E be a Banach space, I a topological space and $T : I \to L(E)$ be strongly continuous. If T is mixing, then T is weakly mixing.*

PROOF: Observe first that for any compact subset J of I the strong continuity of T together with the Banach Steinhaus theorem yield $\sup\{\|T(\iota)\|;\ \iota \in J\} < \infty$. This implies that I cannot be compact, because if it was compact we would have $T(\iota)(B(0,1)) \cap \{x \in E;\ \|x\| > M\} = \emptyset$ for all $\iota \in I$, where $M := \sup\{\|T(\iota)\|;\ \iota \in I\} < \infty$, contradicting the fact that T is mixing.

Now, let $U_i, V_i \subset E$ be open and not empty, $i = 1, 2$. There are compact subsets $J_i, i = 1, 2$, of I such that $T(\iota)(U_i) \cap V_i \neq \emptyset$ for all $\iota \in J_i^c$. Since I is not compact we have $I\backslash(J_1 \cup J_2) \neq \emptyset$ and obviously $T \oplus T\ (\iota)(U_1 \times U_2) \cap (V_1 \times V_2) \neq \emptyset$ for all $\iota \in I\backslash(J_1 \cup J_2)$. □

Remark 5.3 If E is separable it follows exactly as in remark 1.2 iii) that for every transitive T the set of hypercyclic vectors $HC(T)$ is a dense G_δ-subset of E.

On the other hand, if E is arbitrary and T is such that its set of hypercyclic vectors is dense, then T is obviously transitive. A sufficient condition for a hypercyclic T to have a dense set of hypercyclic vectors is for example that all the operators of T commute and have dense range. If this is the case one easily checks that $T(\iota)(HC(T)) \subset HC(T)$, which immediately yields that $HC(T)$ is dense in E.

We now state our general hypotheses for the rest of this chapter. From now on, unless stated explicitly otherwise, let X be a locally compact Hausdorff space, $I \neq \emptyset$ a set and let $\varphi : I \times X \to X$ be a mapping such that $\varphi(\iota, \cdot)$ is injective and continuous for all $\iota \in I$.

Furthermore, let $\rho : X \to (0, \infty)$ be upper semicontinuous. We will consider spaces of continuous functions $C_{0,\rho}(X, \mathbb{C})$ and $C_{0,\rho}(X, \mathbb{R})$, where $C_{0,\rho}(X, \mathbb{K}) := \{f : X \to \mathbb{K} \text{ continuous}; \forall \varepsilon > 0 : \{x \in X; |f(x)|\rho(x) \geq \varepsilon\} \text{ is compact}\}$ is equipped with the norm $\|f\| := \sup_{x \in X} |f(x)|\rho(x)$.

Moreover, we will consider spaces of p-integrable functions. In this context we will always assume that X not only is locally compact and Hausdorff but also σ-compact. Let μ be a (positive) locally finite Borel-measure on X. In particular, μ is σ-finite since now we assume X to be σ-compact. (For the case of general σ-finite measure spaces see section 5.3. We choose to start with the case of X being a topological space because it will be of importance in chapter 6.) For $1 \leq p < \infty$ let $L^p(\mu, \mathbb{R}), L^p(\mu, \mathbb{C})$ be as usual equipped with the norm $\|f\| := (\int_X |f|^p \, d\mu)^{1/p}$. Since in most occasions it will not matter whether the considered functions are real or complex valued, we will write $L^p(\mu)$ and $C_{0,\rho}(X)$ respectively for brevity. Since μ is locally finite, the set of compactly supported, continuous functions $C_c(X)$ is dense in $L^p(\mu)$ (cf. [53, theorem 3.14]), and obviously $C_c(X)$ is dense in $C_{0,\rho}(X)$, too.

We want to define a family of operators on $L^p(\mu)$ and $C_{0,\rho}(X)$ via φ by setting $T_\varphi(\iota)f := f \circ \varphi(\iota, \cdot)$ and want to characterise when it is transitive or even mixing.

First, let us recall the following well known theorem characterising when composition operators on L^p-spaces are continuous. For a proof see appendix B.1.

Theorem 5.4 *For an arbitrary σ-finite measure space (X, μ) and a measurable function $\psi : X \to X$ the composition operator*

$$T_\psi : (L^p(\mu), \|\cdot\|) \to (L^p(\mu), \|\cdot\|), f \mapsto f \circ \psi$$

is well-defined and continuous if and only if the image measure μ^ψ of μ under ψ is absolutely continuous with respect to μ and the μ-density f_ψ of μ^ψ is μ-a.e. bounded.

If T_ψ is continuous, then $\|T_\psi\| = \|f_\psi\|_\infty^{1/p}$.

Note, that in the context of the above theorem T_ψ is continuous on $L^p(\mu)$ either for all $1 \leq p < \infty$ or for none.

The corresponding theorem in the case of continuous functions reads as follows. For a proof see appendix B.2.

Theorem 5.5 *Let $\psi : X \to X$ be continuous. Then, the following are equivalent:*

i) The mapping $T_\psi : C_{0,\rho}(X) \to C_{0,\rho}(X), f \mapsto f \circ \psi$ is well-defined and continuous.

ii) *a)* *There is a constant $C > 0$ such that $\rho(x) \leq C\rho(\psi(x))$ for all $x \in X$.*

b) *For every compact subset K of X and every $\delta > 0$ the set $\psi^{-1}(K) \cap \{x \in X; \rho(x) \geq \delta\}$ is compact.*

Moreover, if T_ψ is a continuous operator on $C_{0,\rho}(X)$ then we have $\|T_\psi\| = \inf\{C > 0;$ C satisfies condition ii) a)$\}$.

Example 5.6 i) We consider $I = \mathbb{N}$ and let $X \in \{\mathbb{N}, \mathbb{Z}\}$ be equipped with the discrete topology, $\mu = \beta\, d\kappa$ where κ is the counting measure on X and $\beta(l) =: \beta_l > 0$ for all $l \in X$. In this case, we write $\ell^p_X(\beta)$ instead of $L^p(\mu)$. Let $\varphi(n, \cdot) = \psi^n = \psi \circ \ldots \circ \psi$ with $\psi(l) = l + 1$. Then it follows that $T_\varphi(n) = B^n$ with B being the backward shift.

Using theorem 5.4 we see that B is an operator on $\ell^p_X(\beta)$ if and only if $\sup_{l \in X} \beta_l/\beta_{l+1} < \infty$. Using Salas' result [55] characterising hypercyclicity of weighted backward shifts and the single operator analogue of lemma 1.4 one gets that T_φ is hypercyclic if and only if for every $l \in X$ there is a sequence $(n_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \beta_{l+n_k} = 0$ for $X = \mathbb{N}$, or $\lim_{k\to\infty} \beta_{l+n_k} = \lim_{k\to\infty} \beta_{l-n_k} = 0$ for $X = \mathbb{Z}$ respectively. A new proof of this will be obtained with corollary 5.10.

ii) Let $I = [0, \infty)$ and $X \in \{[0, \infty), \mathbb{R}\}$ and let the measure μ have a Lebesgue density ρ which is strictly positive. If $\varphi(t, x) = t + x$ theorem 5.4 gives that $T_\varphi(t)$ is continuous if and only if $\sup_{x\in X} \rho(x)/\rho(t + x) < \infty$.

In the special case that T_φ is a C_0-semigroup on $L^p(\mu)$ (the so called *left translation semigroup*) it was shown by Desch et al. in [19] that T_φ is hypercyclic if and only if for every x in X there is a sequence $(t_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \rho(x+t_k) = 0$ in case of $X = [0, \infty)$, respectively $\lim_{k\to\infty} \rho(x + t_k) = \lim_{k\to\infty} \rho(x - t_k) = 0$ in case of $X = \mathbb{R}$. Also, it was shown by Bermúdez et al. [6] that the C_0-semigroup T_φ is mixing if and only if $\lim_{x\to\infty} \rho(x) = 0$ if $X = [0, \infty)$, or $\lim_{x\to\infty} \rho(x) = \lim_{x\to\infty} \rho(-x) = 0$ if $X = \mathbb{R}$ respectively. These results will be obtained as special cases in chapter 6.

5.1 Characterising transitivity and weak mixing

For the rest of this section we assume that the Borel measure μ and the upper semicontinuous function ρ are always such that T_φ defines a family of operators on $L^p(\mu)$ and $C_\rho(X)$ respectively.

Instead of $\mu^{\varphi(\iota,\cdot)}$ we write μ_ι for the image measure of μ under $\varphi(\iota, \cdot)$. Before we characterise topological transitivity of T_φ on $L^p(\mu)$ in terms of μ, we make some observations.

Since $\varphi(\iota, \cdot)$ is one-to-one for every $\iota \in I$ it has an inverse mapping from $\varphi(\iota, X)$ to X which is denoted by $\varphi(-\iota, \cdot)$. In case of $(I, +)$ being a group we want to emphasis that in general $\varphi(-\iota, \cdot)$ is different from $\varphi(\kappa, \cdot)$ for κ being the additive inverse of ι in $(I, +)$. Nevertheless, in the most important case when $\varphi(\iota_1 + \iota_2, \cdot) = \varphi(\iota_1, \cdot) \circ \varphi(\iota_2, \cdot)$ for all $\iota_1, \iota_2 \in I$ and $\varphi(e, \cdot) = id_X$ with e being the unit in $(I, +)$ (see chapter 6), we have $\varphi(-\iota, \cdot) = \varphi(\kappa, \cdot)$.

Apart form the measures $(\mu_\iota)_{\iota\in I}$, we will need a second family of Borel measures on X. From the σ-compactness of X and the continuity of $\varphi(\iota, \cdot)$ it follows that for each closed subset C of X the image $\varphi(\iota, C)$ is an F_σ-set, and in particular, Borel measurable. Since the closed subsets of X generate the Borel σ-algebra over X, the injectivity of $\varphi(\iota, \cdot)$ now implies that $\varphi(\iota, B)$ is a Borel subset of X whenever B is. So by setting $\mu_{-\iota}(B) := \mu(\varphi(\iota, B))$ for $\iota \in I$ we get indeed a well-defined Borel measure on X. Again, note that in this context the notion $-\iota$ has nothing to do in general with a possibly existing additive inverse of ι in case of I being a group.

Remark 5.7 One should note that $\mu_{-\iota}(B) = \mu(\varphi(\iota, B)) = \mu(\varphi(-\iota, \cdot)^{-1}(B) \cap \varphi(\iota, X)) = (\mu_{|\varphi(\iota,X)})^{\varphi(-\iota,\cdot)}(B)$, where $\mu_{|\varphi(\iota,X)} := \mu(\,\cdot\, \cap \varphi(\iota, X))$. In particular, this shows that for a Borel subset B with $B \subseteq \varphi(\iota, X)$ the equation $\mu(B) = (\mu_{-\iota})^{\varphi(\iota,\cdot)}(B)$ holds.

We are now able to formulate our theorem.

Theorem 5.8 *Under the general assumptions, the following are equivalent.*

i) T_φ *is weakly mixing on* $L^p(\mu)$.

ii) T_φ *is topologically transitive on* $L^p(\mu)$.

iii) For every compact subset K *of* X *there are a sequence of measurable subsets* $(L_n)_{n\in\mathbb{N}}$ *of* K *and a sequence* $(\iota_n)_{n\in\mathbb{N}}$ *in* I *such that*

$$\lim_{n\to\infty} \mu(K \backslash L_n) = 0$$

and

$$\lim_{n\to\infty} \mu_{\iota_n}(L_n) = \lim_{n\to\infty} \mu_{-\iota_n}(L_n) = 0.$$

Proof: That i) implies ii) is obviuous. To prove that ii) implies iii) let $K \subset X$ be a fixed compact subset and let $\varepsilon \in (0, 1/2^p)$ be arbitrary. Since T_φ is transitive, there are $\iota \in I$ and v in $L^p(\mu)$ such that $\|v - \chi_K\|^p < \varepsilon^2$ as well as $\|T_\varphi(\iota)v + \chi_K\|^p < \varepsilon^2$.

By the continuity of the mapping $L^p(\mu, \mathbb{C}) \to L^p(\mu, \mathbb{R}), f \mapsto Ref$ and the fact that T_φ commutes with it, we can assume without loss of generality that v is real-valued.

Furthermore, for measurable subsets $B \subseteq X$ we have $\|T_\varphi(\kappa)(f\chi_B)\| \leq \|T_\varphi(\kappa)f\|$ for arbitrary $\kappa \in I$ and all $f \in L^p(\mu)$ and since the mapping $L^p(\mu, \mathbb{R}) \to L^p(\mu, \mathbb{R}), f \mapsto f^+$, where $f^+ := \max\{0, f\}$, satisfies $\|(f + g)^+\| \leq \|f^+ + g^+\|$ and commutes with T_φ we get

$$\begin{aligned} \|T_\varphi(\iota)(v^+\chi_B)\| &\leq \|(T_\varphi(\iota)v)^+\| = \|(T_\varphi(\iota)v - (-\chi_K) + (-\chi_K))^+\| \\ &\leq \|(T_\varphi(\iota)v - (-\chi_K))^+\| + \|(-\chi_K)^+\| \\ &= \|(T_\varphi(\iota)v - (-\chi_K))^+\| \leq \|T_\varphi(\iota)v + \chi_K\| < \varepsilon^{2/p} \end{aligned}$$

and $\|v - \chi_K\|^p < \varepsilon^2$ implies

$$\begin{aligned} \|v^-\chi_B\| &\leq \|v^-\| = \|(-v)^+\| = \|(\chi_K - v - \chi_K)^+\| \\ &\leq \|\chi_K - v\| + \|(-\chi_K)^+\| = \|\chi_K - v\| < \varepsilon^{2/p}, \end{aligned}$$

where $v^- := \max\{0, -v\}$.

Additionally, we have

$$\varepsilon^2 > \int_K |1 - v|^p \, d\mu \geq \int_{K\cap\{|1-v|^p>\varepsilon\}} |1 - v|^p \, d\mu \geq \varepsilon\, \mu(K \cap \{|1 - v|^p > \varepsilon\}),$$

that is $\mu(K \cap \{|1-v|^p > \varepsilon\}) < \varepsilon$. Using a similar argument we also obtain $\mu(K \cap \{|1+T_\varphi(\iota)v|^p > \varepsilon\}) < \varepsilon$.

If we set $L := K \cap \{|1-v|^p \leq \varepsilon\} \cap \{|1+T_\varphi(\iota)v|^p \leq \varepsilon\}$, we get $\mu(K \backslash L) < 2\varepsilon$, and $v|_L \geq 1-\varepsilon^{1/p} > 0$ a.e. as well as $(T_\varphi(\iota)v)|_L \leq \varepsilon^{1/p} - 1 < 0$ a.e..

Using $\varphi(-\iota,\cdot)(\varphi(\iota,K)) = K$ and $\mu_{|_{\varphi(-\iota,X)}} = (\mu_{-\iota})^{\varphi(\iota,\cdot)}$ (cf. remark 5.7) we conclude from all this and $\varepsilon < 1/2^p$

$$\begin{aligned}
\varepsilon^2 &> \|v^-\chi_{\varphi(\iota,K)}\|^p = \int_{\varphi(\iota,K)} |v^-|^p \, d\mu = \int_{\varphi(\iota,K)} |v^-|^p \, d(\mu_{|_{\varphi(\iota,X)}}) \\
&= \int_{\varphi(\iota,K)} |v^-|^p \, d(\mu_{-\iota})^{\varphi(\iota,\cdot)} = \int_{\varphi(-\iota,\cdot)(\varphi(\iota,K))} |v^-(\varphi(\iota,\cdot))|^p \, d\mu_{-\iota} \\
&\geq \int_L |(T_\varphi(\iota)v)^-|^p \, d\mu_{-\iota} \geq (1-\varepsilon^{1/p})^p \mu_{-\iota}(L) \geq 1/2^p \, \mu_{-\iota}(L),
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon^2 &> \|T_\varphi(\iota)(v^+\chi_L)\|^p = \int |(v^+(\varphi(\iota,\cdot)))(\chi_L(\varphi(\iota,\cdot)))|^p \, d\mu \\
&= \int_{\varphi(\iota,\cdot)^{-1}(L)} |v^+(\varphi(\iota,\cdot))|^p \, d\mu = \int_L |v^+|^p \, d\mu_\iota \geq (1-\varepsilon^{1/p})^p \mu_\iota(L) \\
&\geq 1/2^p \, \mu_\iota(L).
\end{aligned}$$

So we have found $L \subset K$ measurable and $\iota \in I$ with $\mu(K \backslash L) < 2\varepsilon, \mu_\iota(L) < 2^p\varepsilon^2$ and $\mu_{-\iota}(L) < 2^p\varepsilon^2$. Since ε was arbitrarily small this shows ii).

In order to show that iii) implies i) let $U_i, V_i, i = 1,2$, be non-empty open subsets of $L^p(\mu)$. Let f_i, g_i be in $C_c(X)$ with $f_i \in U_i$ and $g_i \in V_i, i = 1,2$. We choose a compact subset $K \subset X$ containing $\operatorname{supp} f_i$ and $\operatorname{supp} g_i$ and take $(L_n)_{n\in\mathbb{N}}$ and $(\iota_n)_{n\in\mathbb{N}}$ as in ii) for K. Define for $n \in \mathbb{N}$

$$v_n := f_1\chi_{L_n} + g_1(\varphi(-\iota_n,\cdot))\chi_{\varphi(\iota_n,L_n)}$$

and

$$\tilde{v}_n := f_2\chi_{L_n} + g_2(\varphi(-\iota_n,\cdot))\chi_{\varphi(\iota_n,L_n)}$$

which are measurable, bounded, and different from 0 at most on a subset of the compact set $K \cup \varphi(\iota_n, K)$, hence belongs to $L^p(\mu)$.

Obviously,

$$\|v_n - f_1\|^p \leq \|f_1\|_\infty^p \mu(K \backslash L_n) + \|g_1\|_\infty^p \mu_{-\iota_n}(L_n)$$

and

$$\|T_\varphi(\iota_n)v_n - g_1\|^p \leq \|f_1\|_\infty^p \mu_{\iota_n}(L_n) + \|g_1\|_\infty^p \mu(K \backslash L_n)$$

so that $(v_n)_{n\in\mathbb{N}}$ converges to f_1 and $(T_\varphi(\iota_n)v_n)_{n\in\mathbb{N}}$ converges to g_1.

The same arguments show that $(\tilde{v}_n)_{n\in\mathbb{N}}$ converges to f_2 and $(T_\varphi(\iota_n)\tilde{v}_n)_{n\in\mathbb{N}}$ converges to g_2.

Thus, for n sufficiently large $T_\varphi(\iota_n)(U_i) \cap V_i \neq \emptyset$ for $i = 1,2$, proving i). □

The question arises immediately, if in the above theorem L_n can always be chosen to be K. In general, this is not the case, as will be shown in example 6.21.

Remark 5.9 One should note that by theorem 5.8 T_φ is transitive on $L^p(\mu)$ either for all $1 \le p < \infty$ or for none.

Theorem 5.8 immediately yields a characterisation of transitivity for a single composition operator T_ψ on $L^p(\mu)$ by setting $\varphi(n,\cdot) := \psi \circ \ldots \circ \psi = \psi^n$. In particular, T_ψ is transitive if and only if it is weakly mixing. Recall that on separable Banach spaces topological transitivity is equivalent to hypercyclicity. So if $L^p(\mu)$ is separable, the above theorem characterises hypercyclicity of T_ψ. Moreover, recall that $L^p(\mu)$ is separable for example when the Borel σ-algebra is countably generated, which in turn is the case if X is a second countable topological space.

Corollary 5.10 *Let X be a locally compact, σ-compact, second countable Hausdorff space, μ a locally finite Borel measure on X and $\psi : X \to X$ continuous and injective. If the induced composition operator T_ψ on $L^p(\mu)$ is well-defined and continuous, the following are equivalent.*

i) T_ψ *is hypercyclic on* $L^p(\mu)$*.*

ii) $T_\psi \oplus T_\psi$ *is hypercyclic on* $L^p(\mu) \times L^p(\mu)$*.*

iii) For every compact subset K of X there are a sequence of measurable subsets $(L_k)_{k\in\mathbb{N}}$ of K and a sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\mu(K\backslash L_k) = 0$ *and* $\lim_{k\to\infty}\mu(\psi^{n_k}(L_k)) = \lim_{k\to\infty}\mu(\psi^{-n_k}(L_k)) = 0$.

PROOF: This follows immediately from theorem 5.8 for $I = \mathbb{N}$ and $\varphi(n,\cdot) := \psi \circ \ldots \circ \psi$ (n factors). □

Example 5.11 Let $X \in \{\mathbb{N},\mathbb{Z}\}$ be equipped with the discrete topology, let $\psi : X \to X$ be injective and $\mu(\{l\}) =: \beta_l > 0$. Instead of $L^p(\mu)$ we write $\ell^p_{\mathbb{N}}(\beta)$ or $\ell^p_{\mathbb{Z}}(\beta)$ respectively, and one checks easily that T_ψ is well-defined and continuous if and only if $\sup_{l\in X}\beta_l/\beta_{\psi(l)} < \infty$.

In this special case, the compact subsets K of X are the finite ones, and the sequence $(\mu(K\backslash L_n))_{n\in\mathbb{N}}$ tends to 0 if and only if $K = L_n$ for all but finitely many n. Furthermore, one has $\mu_n(K) = \sum_{l\in K\cap\psi^n(X)}\beta_{\psi^{-n}(l)}$ and $\mu_{-n}(K) = \sum_{l\in K}\beta_{\psi^n(l)}$.

Going back to example 5.6, that is the special case $\psi(l) := l+1$, one usually refers to T_ψ as the *unilateral shift* in case of $X = \mathbb{N}$, and in the case of $X = \mathbb{Z}$ the standard name of T_ψ is *bilateral shift*. So, in both cases, one has $\mu_{-n}(K) = \sum_{l\in K}\beta_{l+n}$ for finite subsets K, and in the unilateral case one has $\mu_n(K) = 0$ for $n > \max K$ for every finite set K, whereas $\mu_n(K) = \sum_{l\in K}\beta_{l-n}$ for finite K in the bilateral case.

If $K = \{l_1,\ldots,l_m\}$ with $l_1 < \ldots < l_m$ we get using $\sup_{l\in X}\beta_l/\beta_{l+1} =: M < \infty$ that $\mu_{-n}(K) = \sum_{l\in K}\beta_{l+n} \le mM^{l_m-l_1}\beta_{l_m+n}$ and in the bilateral case $\mu_n(K) = \sum_{l\in K}\beta_{l-n} \le mM^{l_m-l_1}\beta_{l_m-n}$

Using these inequalities and the separability of $\ell^p_{\mathbb{N}}(\beta)$ and $\ell^p_{\mathbb{Z}}(\beta)$, one immediately gets a new proof of the following, well known characterisation of hypercyclicity of weighted backward shifts due to Salas already mentioned in example 5.6 (see [55, theorem 2.1]).

Theorem 5.12 *i) The bilateral shift on $l^p_{\mathbb{Z}}(\beta)$ is hypercyclic if and only if for every $l \in \mathbb{Z}$ there is a strictly increasing sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \beta_{l+n_k} = \lim_{k\to\infty} \beta_{l-n_k} = 0$.*

The bilateral shift is hypercyclic if and only if it is weakly mixing.

ii) The unilateral shift on $l^p_{\mathbb{N}}(\beta)$ is hypercyclic if and only if $\liminf_{n\to\infty} \beta_n = 0$.

The unilateral shift is hypercyclic if and only if it is weakly mixing.

Clearly, the set of locally finite Borel measures on X for which T_φ defines a family of operators on $L^p(\mu)$ is convex, so that one may ask whether this is also true for the set of measures such that T_φ is transitive. This is in general not the case, as the following example shows.

Example 5.13 Consider $X = \mathbb{N}$ and $\psi(l) := l + 1$. For $l = 2^j + k$ with $j \in \mathbb{N}_0$ and $k \in \{0, \dots, 2^j - 1\}$ we define

$$\beta_l := \frac{2^j - k}{2^j}$$

and

$$\gamma_l := \begin{cases} \frac{2^{j-1}-k}{2^j} & , \quad 0 \le k \le 2^{j-1} - 1 \\ \frac{3\cdot 2^{j-1}-k}{2^j} & , \quad 2^{j-1} \le k \le 2^j - 1. \end{cases}$$

For $l = 2^j + k$ we then have

$$\frac{\beta_l}{\beta_{l+1}} = \begin{cases} \frac{2^j-k}{2^j-(k+1)} = 1 + \frac{1}{2^j-k-1} \le 2 & , \quad 0 \le k < 2^{j-1} - 1 \\ \frac{2^j-(2^j-1)}{2^j} = \frac{1}{2^j} \le 2 & , \quad k = 2^j - 1, \end{cases}$$

and

$$\frac{\gamma_l}{\gamma_{l+1}} = \begin{cases} \frac{2^{j-1}-k}{2^{j-1}-(k+1)} = 1 + \frac{1}{2^{j-1}-k-1} \le 2 & , \quad 0 \le k \le 2^{j-1} - 2 \\ \frac{2^{j-1}-(2^{j-1}-1)}{3\cdot 2^{j-1}-2^{j-1}} = \frac{1}{2^j} \le 2 & , \quad k = 2^{j-1} - 1 \\ \frac{3\cdot 2^{j-1}-k}{3\cdot 2^{j-1}-(k+1)} = 1 + \frac{1}{3\cdot 2^{j-1}-k-1} \le 2 & , \quad 2^{j-1} \le k \le 2^j - 2 \\ \frac{3\cdot 2^{j-1}-(2^j-1)}{2^j} = \frac{3}{2} - 1 + \frac{1}{2^j} \le 2 & , \quad k = 2^j - 1, \end{cases}$$

so that the unilateral backwardshift B is well-defined on $\ell^p_{\mathbb{N}}(\beta)$ and $\ell^p_{\mathbb{N}}(\gamma)$, hence on $\ell^p_{\mathbb{N}}(\lambda\beta+(1-\lambda)\gamma)$ for all $\lambda \in [0, 1]$, where $(\lambda\beta+(1-\lambda)\gamma)_l := \lambda\beta_l+(1-\lambda)\gamma_l,\ l \in \mathbb{N}$.

Because

$$\liminf_{l\to\infty} \beta_l = \lim_{j\to\infty} \beta_{2^{j+1}-1} = \lim_{j\to\infty} 2^{-j} = 0$$

and

$$\liminf_{l\to\infty} \gamma_l = \lim_{j\to\infty} \gamma_{2^{j-1}-1} = \lim_{j\to\infty} 2^{-j} = 0,$$

it follows from theorem 5.12 that B is hypercyclic on $\ell^p_{\mathbb{N}}(\beta)$ and $\ell^p_{\mathbb{N}}(\gamma)$.

On the other hand, for every $\lambda \in (0, 1)$ we have for $l = 2^j + k$

$$\lambda\beta_l+(1-\lambda)\gamma_l = \begin{cases} \lambda\frac{2^j-k}{2^j} + (1-\lambda)\frac{2^{j-1}-k}{2^j} \ge \lambda/2 & , \quad 0 \le k \le 2^{j-1} - 1 \\ \lambda\frac{2^j-k}{2^j} + (1-\lambda)\frac{3\cdot 2^{j-1}-k}{2^j} \ge (1-\lambda)/2 & , \quad 2^{j-1} \le k \le 2^j - 1 \end{cases}$$

so that $\liminf_{l\to\infty} \lambda\beta_l + (1-\lambda)\gamma_l \geq \min\{\lambda, 1-\lambda\}/2 > 0$.

Together with the above, theorem 5.12 now yields that B is hypercyclic on $\ell^p_{\mathbb{N}}(\lambda\beta + (1-\lambda)\gamma)$ if and only if $\lambda \in \{0,1\}$.

Now, we turn to the spaces $C_{0,\rho}(X)$. Remember that in this context we only assume X to be locally compact and Hausdorff. Before we formulate our theorem we make the following convention. We set $\sup\emptyset = 0$ in the context of sets of positive numbers.

Theorem 5.14 *Additionally to the general hypotheses assume that for all compact subsets K of X we have $\inf_{x\in K}\rho(x) > 0$. Then, among the following, i) implies ii) and ii) implies iii).*

i) T_φ is weakly mixing on $C_{0,\rho}(X)$.

ii) T_φ is topologically transitive on $C_{0,\rho}(X)$.

iii) For every compact subset K of X we can find a sequence $(\iota_n)_{n\in\mathbb{N}}$ in I such that

$$\lim_{n\to\infty} \sup_{x\in\varphi(\iota_n,K)} \rho(x) = \lim_{n\to\infty} \sup_{x\in\varphi(\iota_n,\cdot)^{-1}(K)} \rho(x) = 0.$$

Furthermore, if $\varphi(\iota,\cdot): X \to X$ is an open mapping for all $\iota \in I$, then the above are equivalent.

Proof: Obviously, i) implies ii). Now, assume that ii) holds, i.e. T_φ is transitive. Let K be a compact subset of X and $\varepsilon \in (0, \inf_{x\in K}\rho(x)/2)$. Furthermore, let $f \in C_c(X)$ be positive with $f_{|K} \equiv 1$. From the topological transitivity of T_φ it follows that there are ι and $v \in C_{0,\rho}(X)$ such that $\|v - f\| < \varepsilon$ and $\|T_\varphi(\iota)v + f\| < \varepsilon$ and by the same kind of argument as in the proof of theorem 5.8 we can assume without loss of generality that v is real valued.

From the positivity of f it follows again that

$$\|T_\varphi(\iota)v^+\| < \varepsilon$$

as well as

$$\|v^-\| < \varepsilon.$$

Moreover, from $\|T_\varphi(\iota)v + f\| < \varepsilon$ we obtain $v(\varphi(\iota,x)) < \varepsilon/\rho(x) - 1 < -1/2$ for every $x \in K$, and $\|v - f\| < \varepsilon$ implies $v(x) > 1 - \varepsilon/\rho(x) > 1/2$ whenever $x \in K$.

Using this, we obtain

$$\varepsilon > \|T_\varphi(\iota)v^+\| \geq \sup_{x\in\varphi(\iota,\cdot)^{-1}(K)} v^+(\varphi(\iota,x))\rho(x) > 1/2 \sup_{x\in\varphi(\iota,\cdot)^{-1}(K)} \rho(x)$$

and

$$\varepsilon > \|v^-\| \geq \sup_{x\in\varphi(\iota,K)} |v^-(x)|\rho(x) = \sup_{x\in\varphi(\iota,K)} v^-(x)\rho(x) \geq 1/2 \sup_{x\in\varphi(\iota,K)} \rho(x)$$

which proves iii).

Now assume that $\varphi(\iota,\cdot): X \to X$ is an open mapping for all $\iota \in I$. It follows that $\varphi(\iota, X)$ is open and $\varphi(-\iota,\cdot): \varphi(\iota,X) \to X$ is continuous for all $\iota \in I$.

Let U_i, V_i be non-empty open subsets of $C_{0,\rho}(X), i = 1,2$. Choose $f_i, g_i \in C_c(X)$ such that $f_i \in U_i, g_i \in V_i, i = 1,2$, and let K be a compact subset of X containing $\operatorname{supp} f_i$ and $\operatorname{supp} g_i, i = 1, 2$. It follows that the mappings $g_i \circ \varphi(-\iota, \cdot) : \varphi(\iota, X) \to \mathbb{K}$ are continuous and their supports are contained in the compact set $\varphi(\iota, K)$ so that $g_i \circ \varphi(-\iota, \cdot) \in C_c(\varphi(\iota, X))$. Since now by hypothesis $\varphi(\iota, X)$ is an open subset of X, we can extend $g_i \circ \varphi(-\iota, \cdot)$ to a compactly supported continuous function $\tilde{g}_{i,\iota}$ on X by setting it equal to 0 outside $\varphi(\iota, X)$. Clearly, $T_\varphi(\iota)\tilde{g}_{i,\iota} = g_i$.

Let $(\iota_n)_{n\in\mathbb{N}}$ be as in iii) for the compact set K. Then, by the above consideration, the functions $v_{i,n} := \tilde{g}_{i,\iota_n}$ belong to $C_c(X)$, hence to $C_{0,\rho}(X)$. One has

$$\|v_{i,n}\| = \sup_{x\in\varphi(\iota_n,K)} |g_i(\varphi(-\iota_n, x))|\rho(x) \leq \|g_i\|_{sup} \sup_{x\in\varphi(\iota_n,K)} \rho(x)$$

so that $\lim_{n\to\infty}(f_i + v_{i,n}) = f_i$.

On the other hand

$$\|T_\varphi(\iota_n)f_i\| = \sup_{x\in\varphi(\iota_n,\cdot)^{-1}(K)} |f_i(\varphi(\iota_n, x))|\rho(x) \leq \|f_i\|_{sup} \sup_{x\in\varphi(\iota_n,\cdot)^{-1}(K)} \rho(x)$$

so that $\lim_{n\to\infty} T_\varphi(\iota_n)f_i = 0$. Thus, $\lim_{n\to\infty} T_\varphi(\iota_n)(f_i + v_{i,n}) = g_i$ so that $T_\varphi(\iota_n)(U_i) \cap V_i \neq \emptyset, i = 1, 2$, proving i). □

In the case when $X = \Omega \subset \mathbb{R}^d$ is open, we can use the following deep result from real analysis to obtain a corollary. For a proof of an even more general statement see for example [16, theorem I.4.3].

Theorem 5.15 (Brouwer's theorem) *Let $\Omega \subset \mathbb{R}^d$ be open and $\psi : \Omega \to \mathbb{R}^d$ be continuous and injective. Then $\psi(\Omega)$ is an open subset of $\mathbb{R}^d$.*

Corollary 5.16 *Let Ω be an open subset of $\mathbb{R}^d$ and ρ a positive, upper semi-continuous function on Ω satisfying $\inf_{x\in K} \rho(x) > 0$ for every compact subset K of Ω.*

Then, the following are equivalent.

i) The family T_φ is transitive on $C_{0,\rho}(\Omega)$.

ii) For every compact subset K of Ω there exists a sequence $(\iota_n)_{n\in\mathbb{N}}$ in I such that

$$\lim_{n\to\infty} \sup_{x\in\varphi(\iota_n,K)} \rho(x) = \lim_{n\to\infty} \sup_{x\in\varphi(\iota_n,\cdot)^{-1}(K)} \rho(x) = 0.$$

iii) T_φ is weakly mixing on $C_{0,\rho}(\Omega)$.

Proof: This follows immediately from Brouwer's theorem and theorem 5.14. □

Remark 5.17 Note that if the mappings $\varphi(\iota, \cdot), \iota \in I$, have a common fixed point $x_0 \in \Omega$ then T_φ obviously cannot be transitive on $C_{0,\rho}(\Omega)$ because of $\rho(x_0) > 0$. This is completely different in the $L^p(\mu)$ setting, as example 6.21 will show.

As in the $L^p(\mu)$ case we obtain the following result for a single composition operator T_ψ on $C_{0,\rho}(X)$ which we want to state explicitly.

Corollary 5.18 *Let ρ be a positive, upper semicontinuous function on X satisfying $\inf_{x\in K}\rho(x) > 0$ for every compact subset K of X, and $\psi : X \to X$ injective and continuous such that T_ψ is a well defined operator on $C_{0,\rho}(X)$.*

Then, among the following i) implies ii) and ii) implies iii).

- *i) T_ψ is weakly mixing on $C_{0,\rho}(X)$.*
- *ii) T_ψ is transitive on $C_{0,\rho}(X)$.*
- *iii) For every compact subset K of X there exists a sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ such that*

$$\lim_{k\to\infty}\sup_{x\in\psi^{n_k}(K)}\rho(x) = \lim_{k\to\infty}\sup_{x\in\psi^{-n_k}(K)}\rho(x) = 0.$$

Furthermore, if $\psi : X \to X$ is an open mapping then the above are equivalent.

PROOF: This follows immediately from theorem 5.16 for $I = \mathbb{N}$ and $\varphi(n,\cdot) := \psi\circ\ldots\circ\psi$ (n factors). □

Example 5.19 Again, we equip $X \in \{\mathbb{N},\mathbb{Z}\}$ with the discrete topology. Let $\psi : X \to X, n \in \mathbb{N}$, be injective and $\rho := (\rho_l)_{l\in X}$ be a sequence of positive numbers. Instead of $C_{0,\rho}(X)$ we write $c_{0,\rho}(X)$. Because now the pre-image of every finite set is finite it follows from theorem 5.5 that the composition operator $c_{0,\rho}(X) \to c_{0,\rho}(X), (x_l)_{l\in X} \mapsto (x_{\psi(l)})_{l\in X}$ is continuous if and only if $\sup_{l\in X}\rho_l/\rho_{\psi(l)} < \infty$.

By using the separability of $c_{0,\rho}(X)$ and $\sup_{l\in X}\rho_l/\rho_{l+1} < \infty$ we again obtain for the special case of $\psi(l) = l+1$ the following theorem characterising hypercyclicity of the backwardshift on $c_{0,\rho}(X)$.

Theorem 5.20 *Let $X \in \{\mathbb{N},\mathbb{Z}\}$ and let $(\rho_l)_{l\in X}$ be such that $\sup_{l\in X}\rho_l/\rho_{l+1} < \infty$. For the backwardshift operator B on $c_{0,\rho}(X)$ we have the following.*

- *i) The unilateral shift on $c_{0,\rho}(\mathbb{N})$ is hypercyclic if and only if it is weakly mixing if and only if $\liminf_{l\to\infty}\rho_l = 0$.*
- *ii) The bilateral shift on $c_{0,\rho}(\mathbb{Z})$ is hypercyclic if and only if it is weakly mixing if and only if for every $l \in \mathbb{Z}$ there is a sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty}\rho_{l-n_k} = \lim_{k\to\infty}\rho_{l+n_k} = 0$.*

In the rest of this section we will study families of weighted composition operators. From now one we do not assume any longer that μ respectively ρ are such that T_φ defines a family of operators on $L^p(\mu)$, respectively $C_{0,\rho}(X)$.

Let $w : I \times X \to (0,\infty)$ be such that for every $\iota \in I$ we have $1/w(\iota,\cdot) \in L^\infty_{loc}(\mu)$. For the rest of this chapter we assume that for $\iota \in I$ the mapping $T_{w,\varphi}(\iota) : L^p(\mu) \to L^p(\mu), f \mapsto w(\iota,\cdot)f(\varphi(\iota,\cdot))$ is a well-defined operator on $L^p(\mu)$ and if all $w(\iota,\cdot)$ are continuous that the same is true for $T_{w,\varphi}(\iota) : C_{0,\rho}(X) \to C_{0,\rho}(X), f \mapsto w(\iota,\cdot)f(\varphi(\iota,\cdot))$. $T_{w,\varphi}(\iota)$ is called a *weighted composition operator*.

Before we continue, let us present two theorems characterising when weighted composition operators are well-defined and continuous. For their proofs, see appendix B. Note, that in the $L^p(\mu)$ case we now no longer have continuity on all the spaces $L^p(\mu), 1 \le p < \infty$ in general.

Theorem 5.21 *Let (X,μ) be an arbitrary σ-finite measure space and let $w : X \to (0,\infty)$, $\psi : X \to X$ be a measurable functions. The weighted composition operator $T_{w,\psi} : (L^p(\mu), \|\cdot\|) \to (L^p(\mu), \|\cdot\|), f \mapsto w(\cdot)\, f(\psi(\cdot))$ is well-defined and continuous if and only if the measure $\mu_{w^p,\psi}$ is absolutely continuous with respect to μ and the μ-density $f_{w^p,\psi}$ of $\mu_{w^p,\psi}$ is μ-a.e. bounded, where $\mu_{w^p,\psi}(A) = \int_{\psi^{-1}(A)} w^p \, d\mu$.*

If $T_{w,\psi}$ is continuous, then $\|T_{w,\psi}\| = \|f_{w^p,\psi}\|_\infty^{1/p}$.

Theorem 5.22 *Let X be a locally compact Hausdorff topological space and let $w : X \to (0,\infty)$, $\psi : X \to X$ be continuous. For a strictly positive, upper semicontinuous function ρ the following are equivalent.*

i) The mapping $T_{w,\psi} : C_{0,\rho}(X) \to C_{0,\rho}(X), f \mapsto w(\cdot)\, f(\psi(\cdot))$ is well-defined and continuous.

ii) a) There is a constant $C > 0$ such that $w(x)\rho(x) \leq C\rho(\psi(x))$ for all $x \in X$.

b) For every compact subset K of X and every $\delta > 0$ the set $\psi^{-1}(K) \cap \{x \in X; w(x)\rho(x) \geq \delta\}$ is compact.

Moreover, if $T_{w,\psi}$ defines an operator on $C_{0,\rho}(X)$ we have $\|T_{w,\psi}\| = \inf\{C > 0;$ C satisfies condition ii) a)$\}$.

In order to formulate our next results conveniently we introduce the following notation. For $p \in [1,\infty)$ and $\iota \in I$ we define the following measures

$$\nu_{p,\iota}(B) := \int_{\varphi(\iota,\cdot)^{-1}(B)} w(\iota,\cdot)^p \, d\mu$$

and

$$\nu_{p,-\iota}(B) := \int_{\varphi(\iota,B)} 1/w(\iota,\varphi(-\iota,\cdot))^p \, d\mu.$$

Note that $\varphi(\iota,B)$ is a Borel measurable subset of X whenever B is, as mentioned at the beginning of this section. So it follows that $\nu_{p,-\iota}$ is a well defined Borel measure on X. The same is obviously true for $\nu_{p,\iota}$.

By standard arguments one shows that for positive, measurable functions f on X we have

$$\int f \, d\nu_{p,\iota} = \int w(\iota,\cdot)^p f(\varphi(\iota,\cdot)) \, d\mu$$

and

$$\int f \, d\nu_{p,-\iota} = \int \chi_{\varphi(\iota,X)} f(\varphi(-\iota,\cdot))/w(\iota,\varphi(-\iota,\cdot))^p \, d\mu.$$

Observe that in the case $w(\iota,\cdot) \equiv 1$ for all $\iota \in I$ we have $\nu_{p,\iota} = \mu_\iota$ and $\nu_{p,-\iota} = \mu_{-\iota}$ for all $p \geq 1, \iota \in I$.

Now we will characterise when $T_{w,\varphi}$ is transitive on $L^p(\mu)$ or $C_{0,\rho}(X)$, respectively, where as before X is a locally compact Hausdorff space when considering $C_{0,\rho}(X)$, respectively X is a locally compact, σ-compact Hausdorff space when considering $L^p(\mu)$.

Theorem 5.23 *Under the general assumptions, the following are equivalent.*

i) $T_{w,\varphi}$ *is weakly mixing on* $L^p(\mu)$.

ii) $T_{w,\varphi}$ *is transitive on* $L^p(\mu)$.

iii) For every compact subset K *of* X *there are a sequence of measurable subsets* $(L_n)_{n\in\mathbb{N}}$ *of* K *and a sequence* $(\iota_n)_{n\in\mathbb{N}}$ *in* I *such that*

$$\lim_{n\to\infty} \mu(K\backslash L_n) = 0$$

as well as

$$\lim_{n\to\infty} \nu_{p,\iota_n}(L_n) = \lim_{n\to\infty} \nu_{p,-\iota_n}(L_n) = 0.$$

PROOF: We will write T instead of $T_{w,\varphi}$ for brevity.

Clearly, i) implies ii). To prove that ii) implies iii) let K be a compact subset of X and $\varepsilon \in (0, 1/2)$. From the transitivity of T it follows that there are ι in I and v in $L^p(\mu)$ such that $\|v - \chi_K\|^p < \varepsilon^2$ as well as $\|T(\iota)v + \chi_K\|^p < \varepsilon^2$. Since $w(\iota, \cdot)$ is a positive function, we conclude as in the proof of theorem 5.8, that without loss of generality, v can be chosen to be real-valued.

Using the positivity of $w(\iota, \cdot)$ again one shows in the same way as in the proof of theorem 5.8 that for measurable subsets $B \subset X$

$$\|T(\iota)(v^+\chi_B)\|^p < \varepsilon^2,$$

and

$$\|v^-\chi_B\|^p < \varepsilon^2.$$

Again as in the proof of theorem 5.8 this implies $\mu(K\cap\{|1-v| > \varepsilon\}) < \varepsilon$, and $\mu(K\cap\{|1+T(\iota)v| > \varepsilon\}) < \varepsilon$. Setting $L := K\cup\{|1-v|^p \leq \varepsilon\}\cup\{|1+T(\iota)v|^p \leq \varepsilon\})$ we again get $\mu(K\backslash L) < 2\varepsilon$, and $v|_L \geq 1 - \varepsilon^{1/p}$ as well as $(T(\iota)v)|_L \leq \varepsilon^{1/p} - 1$.

With this and $\int f\, d\nu_{p,-\iota} = \int \chi_{\varphi(\iota,X)} f(\varphi(-\iota, \cdot))/w(\iota, \varphi(-\iota, \cdot))^p\, d\mu$ we obtain

$$\begin{aligned}
\varepsilon^2 &> \int_{\varphi(\iota,K)} |v^-|^p\, d\mu = \int_{\varphi(\iota,K)} \frac{w(\iota, \varphi(-\iota, \cdot))^p}{w(\iota, \varphi(-\iota, \cdot))^p} |v^-|^p\, d\mu \\
&= \int \frac{w(\iota, \varphi(-\iota, \cdot))^p}{w(\iota, \varphi(-\iota, \cdot))^p} |v^-(\varphi(\iota, \varphi(-\iota, \cdot)))|^p \chi_K(\varphi(-\iota, \cdot))\, d\mu \\
&= \int |T(\iota)v^-|^p \chi_K\, d\nu_{p,-\iota} \geq \int_L |T(\iota)v^-|^p\, d\nu_{p,-\iota} \\
&\geq (1 - \varepsilon^{1/p})^p \nu_{p,-\iota}(L).
\end{aligned}$$

Using $\int f\, d\nu_{p,\iota} = \int w(\iota, \cdot)^p f(\varphi(\iota, \cdot))\, d\mu$ we get

$$\begin{aligned}
\varepsilon^2 &> \|T(\iota)(v^+\chi_L)\|^p = \int w(\iota, \cdot)^p |v^+ \circ \varphi(\iota, \cdot)|^p \chi_L \circ \varphi(\iota, \cdot)\, d\mu \\
&= \int_L (v^+)^p\, d\nu_{p,\iota} \geq (1 - \varepsilon^{1/p})^p \nu_{p,\iota}(L).
\end{aligned}$$

So, we have found a measurabel subset L of K and $\iota \in I$ for which $\mu(K\backslash L) < 2\varepsilon$ and $\nu_{p,\iota}(L) < 2^p\varepsilon^2$ as well as $\nu_{p,-\iota}(L) < 2^p\varepsilon^2$.

Note that for this implication we did not need $w(\iota, \cdot)^{-1} \in L^\infty_{loc}(\mu)$.

In order to show that iii) implies i), let $U_i, V_i, i = 1, 2$, be non-empty open subsets of $L^p(\mu)$ and $f_i, g_i \in C_c(X)$ be such that $f_i \in U, g_i \in V, i = 1, 2$. We choose $K \subset X$ containing $\operatorname{supp} f_i$ and $\operatorname{supp} g_i$ and let $(L_n)_{n\in\mathbb{N}}$ and $(\iota_n)_{n\in\mathbb{N}}$ be as in iii) for K.

For $n \in \mathbb{N}$ we set

$$v_n := \Big(\frac{g_1(\varphi(-\iota_n, \cdot))}{w(\iota_n, (\varphi(-\iota_n, \cdot)))}\Big)\chi_{\varphi(\iota_n, L_n)}$$

and

$$\tilde{v}_n := \Big(\frac{g_2(\varphi(-\iota_n, \cdot))}{w(\iota_n, (\varphi(-\iota_n, \cdot)))}\Big)\chi_{\varphi(\iota_n, L_n)}$$

which are measurable and because of $1/w(\iota, \cdot) \in L^\infty_{loc}(\mu)$ in $L^p(\mu)$. Then,

$$\|v_n\|^p \leq \|g_1\|^p_\infty \nu_{p,-\iota_n}(L_n)$$

so that $(f_1\chi_{L_n} + v_n)_{n\in\mathbb{N}}$ converges to f_1. Furthermore

$$\|T(\iota_n)(f_1\chi_{L_n})\|^p \leq \|f_1\|^p_\infty \nu_{p,\iota_n}(L_n),$$

so that $(T(\iota_n)(f_1\chi_{L_n}))_{n\in\mathbb{N}}$ converges to 0. Since $T(\iota_n)v_n - g_1 = -g_1\chi_{K\backslash L_n}$ we see that $(T(\iota_n)(f_1\chi_{L_n} + v_n))_{n\in\mathbb{N}}$ converges to g_1. The same arguments yield that $(f_2\chi_{L_n} + \tilde{v}_n)_{n\in\mathbb{N}}$ converges to f_2 and $(T(\iota_n)(f_2\chi_{L_n} + \tilde{v}_n))_{n\in\mathbb{N}}$ converges to g_2. Thus, $T(\iota_n)(U_i) \cap V_i \neq \emptyset, i = 1, 2$, for sufficiently large n, so that iii) follows. □

Corollary 5.24 *Let X be a locally compact, σ-compact, second countable Hausdorff space, μ a locally finite Borel measure on X, $\alpha \in \mathbb{K}\backslash\{0\}$ and $\psi : X \to X$ injective and continuous such that the corresponding composition operator T_ψ is a well-defined operator on $L^p(\mu)$. Then, the following are equivalent.*

- *i) αT_ψ is weakly mixing on $L^p(\mu)$.*
- *ii) αT_ψ is hypercyclic on $L^p(\mu)$.*
- *iii) For every compact subset K of X there are a sequence of measurable subsets $(L_k)_{k\in\mathbb{N}}$ of K and a sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \mu(K\backslash L_k) = 0$ and*

$$\lim_{n\to\infty} \alpha^{pn_k}\mu_{n_k}(L_k) = \lim_{k\to\infty} \alpha^{-pn_k}\mu_{-n_k}(L_k) = 0.$$

Proof: That i) implies ii) is obvious. In order to show that ii) implies iii) we need a result due to Leòn-Saavedra and Müller stating that an operator T is hypercyclic if and only if λT is hypercyclic for all $|\lambda| = 1$ (cf. [37]). Let $|\lambda| = 1$ be such that $\lambda\alpha > 0$. Then it follows from the hypercyclicity of $\lambda\alpha T$ and theorem 5.23 that for every compact subset K of X there are a sequence of measurable subsets $(L_k)_{k\in\mathbb{N}}$ of K and a sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \mu(K\backslash L_k) = 0$ and

$$\lim_{n\to\infty} (\lambda\alpha)^{pn_k}\mu_{n_k}(L_k) = \lim_{k\to\infty} (\lambda\alpha)^{-pn_k}\mu_{-n_k}(L_k) = 0.$$

From the fact that $|\lambda| = 1$ follows iii).

To show that iii) implies i) let again $|\lambda| = 1$ be such that $\lambda\alpha > 0$. Then theorem 5.23 and the single operator analogue of theorem 2.4 imply that $\lambda\alpha T$ satisfies the Hypercyclicity Criterion. Now, because of $|\lambda| = 1$ it follows easily that αT satisfies the Hypercyclicity Criterion so that αT is weakly mixing. □

Example 5.25 Let X be $\mathbb{N}$ equipped with the discrete topology and the counting measure κ. Let $\beta_l := 1$ for all $l \in \mathbb{N}$. Then the unilateral backward shift B is well-defined on $\ell^p := \ell^p_{\mathbb{N}}(\beta)$ and clearly is not hypercyclic because it is a contraction.

Let $(\gamma_n)_{n\in\mathbb{N}_0}$ be a sequence of positive numbers. We want to know when the family $\{\gamma_n B^n;\, n \in \mathbb{N}_0\}$ is hypercyclic on ℓ^p. Let K be a compact, i.e. finite, subset of $\mathbb{N}$. Then a sequence $(L_n)_{n\in\mathbb{N}}$ of subsets of K satisfies $\lim_{n\to\infty} \kappa(K\backslash L_n) = 0$ if and only if $L_n = K$ for all but finitely many $n \in \mathbb{N}$.

Clearly, $\nu_{p,-n}(K) = \int_K \gamma_n^{-p}\, d\mu_{-n} = \gamma_n^{-p} \sum_{l\in K} \beta_{l+n}$ and for sufficiently large n we have $\nu_{p,n}(K) = \int_K \chi_{\psi^n(\mathbb{N})}\gamma_n^p\, d\mu_n = 0$. Now, theorem 5.23 combined with the arguments from example 5.11 implies that $\{\gamma_n B^n;\, n \in \mathbb{N}_0\}$ is hypercyclic on ℓ^p if and only if $\{\gamma_n B^n;\, n \in \mathbb{N}_0\}$ is weakly mixing on ℓ^p if and only if $\liminf_{n\to\infty} \gamma_n^{-p}\beta_n = 0$.

By corollary 5.24, λB, where $\lambda \neq 0$, is a hypercyclic operator on ℓ^p if and only if $|\lambda| > 1$. This was already shown by Rolewicz in [52].

Remark 5.26 Leòn-Saavedra introduced in [35] the notion of Cesàro-hypercyclicity. Let T be an operator on the Banach space E and define for $n \in \mathbb{N}$ the operator $M_n := \frac{1}{n}\sum_{i=0}^{n-1} T^n$. Then T is called *Cesàro-hypercyclic* if there is $x \in E$ such that $\{M_n(T)x;\, n \in \mathbb{N}\}$ is dense in E. Leòn-Saavedra proved (cf. [35, theorem 2.4]) that T is Cesàro-hypercyclic if and only if the family $\{\frac{1}{n}T^n;\, n \in \mathbb{N}\}$ is hypercyclic. This, together with theorem 5.23 for $w(n,\cdot) \equiv 1/n$, $\varphi(n,\cdot) = \psi^n$ and the fact that $\nu_{p,n}(K) = n^{-p}\mu_n(K)$ and $\nu_{p,-n}(K) = n^p\mu_{-n}(K)$ immediately yields the following result.

Corollary 5.27 *Let X be a locally compact, σ-compact, second countable Hausdorff space, μ a locally finite Borel measure on X and $\psi : X \to X$ continuous and injective. If the induced composition operator T_ψ on $L^p(\mu)$ is well-defined and continuous, the following are equivalent.*

i) T_ψ is Cesàro-hypercyclic on $L^p(\mu)$.

ii) For every compact subset K of X there are a sequence $(L_k)_{k\in\mathbb{N}}$ of measurable subsets of K and a sequence of natural numbers $(n_k)_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} \mu(K\backslash L_k) = 0$ and

$$\lim_{k\to\infty} n_k^{-p}\mu_{n_k}(L_k) = \lim_{k\to\infty} n_k^p \mu_{-n_k}(L_k) = 0.$$

We now turn to the $C_{0,\rho}(X)$ case. Recall that in this context X is only assumed to be a locally compact Hausdorff space.

Theorem 5.28 *Under the general hypotheses and the additional assumption that for all compact subsets K of X we have $\inf_{x\in K} \rho(x) > 0$, among the following, i) implies ii) and ii) implies iii).*

i) $T_{w,\varphi}$ is weakly mixing on $C_{0,\rho}(X)$.

ii) $T_{w,\varphi}$ is topologically transitive on $C_{0,\rho}(X)$

iii) For every compact subset K of X we can find a sequence $(\iota_n)_{n\in\mathbb{N}}$ in I such that

$$\lim_{n\to\infty} \sup_{x\in\varphi(\iota_n,\cdot)^{-1}(K)} w(\iota_n, x)\rho(x) = \lim_{n\to\infty} \sup_{x\in\varphi(\iota_n,K)} \frac{\rho(x)}{w(\iota_n, \varphi(-\iota_n, x))} = 0.$$

Moreover, if $\varphi(\iota,\cdot) : X \to X$ is an open mapping for all $\iota \in I$ then the above are equivalent.

Proof: We write T instead of $T_{w,\varphi}$ for brevity. Assume that ii) holds. Let K be a compact subset of X, $\varepsilon \in (0, \inf_{x\in K} \rho(x)/2)$, and $f \in C_c(X)$ positive with $f_{|K} \equiv 1$. Since T is transitive we find $\iota \in I$ and $v \in C_{0,\rho}(X)$ such that $\|v - f\| < \varepsilon$ and $\|T(\iota)v + f\| < \varepsilon$. Using the same arguments as in the proof of theorem 5.8 we can assume without loss of generality that v is real valued.

Again, the positivity of f yields $\|T(\iota)v^+\| < \varepsilon$ and $\|v^-\| < \varepsilon$. Furthermore, $\|T(\iota)v + f\| < \varepsilon$ implies $w(\iota, x)v(\varphi(\iota, x)) < \varepsilon/\rho(x) - 1 < -1/2$ for every $x \in K$, and $\|v - f\| < \varepsilon$ gives $v(x) > 1 - \varepsilon/\rho(x) > 1/2$ for all $x \in K$.

From this we get

$$\begin{aligned} \varepsilon &> \|T(\iota)v^+\| \geq \sup_{x\in\varphi(\iota,\cdot)^{-1}(K)} w(\iota, x)v^+(\varphi(\iota, x))\rho(x) \\ &> 1/2 \sup_{x\in\varphi(\iota,\cdot)^{-1}(K)} w(\iota, x)\rho(x) \end{aligned}$$

as well as

$$\begin{aligned} \varepsilon &> \|v^-\| \geq \sup_{x\in\varphi(\iota,K)} v^-(x)\rho(x) = \sup_{x\in\varphi(\iota,K)} w(\iota, \varphi(-\iota, x))v^-(x)\frac{\rho(x)}{w(\iota, \varphi(-\iota, x))} \\ &> 1/2 \sup_{x\in\varphi(\iota,K)} \frac{\rho(x)}{w(\iota, \varphi(-\iota, x))} \end{aligned}$$

which proves iii).

The rest of the proof is done by similar arguments as the proof of theorem 5.14. Now, one only has to define $v_{i,n} := \frac{g_i(\varphi(-\iota_n,\cdot))}{w(\iota_n,\varphi(-\iota_n,\cdot))}$. □

Using Brouwer's theorem, we again obtain the following result for $X = \Omega \subset \mathbb{R}^d$ open.

Corollary 5.29 *Let Ω be an open subset of $\mathbb{R}^d$ and ρ a positive, upper semi-continuous function on Ω satisfying $\inf_{x\in K} \rho(x) > 0$ for every compact subset K of Ω.*

Then, the following are equivalent.

i) $T_{w,\varphi}$ is weakly mixing on $C_{0,\rho}(\Omega)$.

ii) $T_{w,\varphi}$ is transitive on $C_{0,\rho}(\Omega)$.

iii) For every compact subset K of Ω we can find a sequence $(\iota_n)_{n\in\mathbb{N}}$ in I such that

$$\lim_{n\to\infty} \sup_{x\in\varphi(\iota_n,\cdot)^{-1}(K)} w(\iota_n, x)\rho(x) = \lim_{n\to\infty} \sup_{x\in\varphi(\iota_n,K)} \frac{\rho(x)}{w(\iota_n, \varphi(-\iota_n, x))} = 0.$$

Remark 5.30 As mentioned in remark 5.26, using [35, theorem 2.4] and theorem 5.28, respectively corollary 5.29, one gets a characterisation of Cesàro-hypercyclicity for a single continuous composition operator on $c_{0,\beta}(X), X \in \{\mathbb{N}, \mathbb{Z}\}$, respectively $C_{0,\rho}(\Omega)$.

5.2 Characterising mixing

We now turn our attention to mixing families of weighted composition operators. From now on we will therefore assume that I is a topological space and that the mapping $I \to L(E), \iota \mapsto T_{w,\varphi}(\iota)$ is strongly continuous, where $E \in \{L^p(\mu), C_{0,\rho}(X)\}$. Remember that w is strictly positive and that in the case $E = L^p(\mu)$ we assume that $1/w(\iota,\cdot) \in L^\infty_{loc}(\mu)$ and that for $E = C_{0,\rho}(X)$ we assume $w(\iota,\cdot) \in C(X)$ for all $\iota \in I$.

Recall that

$$\nu_{p,\iota}(B) := \int_{\varphi(\iota,\cdot)^{-1}(B)} w(\iota,\cdot)^p \, d\mu$$

and

$$\nu_{p,-\iota}(B) := \int_{\varphi(\iota,B)} 1/w(\iota,\varphi(-\iota,\cdot))^p \, d\mu.$$

In the $L^p(\mu)$-setting we have the following theorem.

Theorem 5.31 *Under the general hypotheses, the following are equivalent.*

i) The family $T_{w,\varphi}$ is mixing on $L^p(\mu)$.

ii) For every compact subset K of X and every $\varepsilon > 0$ there is a compact subset $J \subset I, J \neq I$ such that for all ι in the complement of J there is a measurable subset L_ι of K satisfying $\mu(K \backslash L_\iota) < \varepsilon$, $\nu_{p,-\iota}(L_\iota) < \varepsilon$ and $\nu_{p,\iota}(L_\iota) < \varepsilon$.

PROOF: We write T instead of $T_{w,\varphi}$ for brevity. To show that i) implies ii) let K be a compact subset of X and $\varepsilon > 0$. Since T is mixing, we find a compact subset J of I (which is different from I as shown in the proof of proposition 5.2) such that $T(\iota)(B(\chi_K,\varepsilon)) \cap B(-\chi_K,\varepsilon) \neq \emptyset$ for every ι from J^c, i.e. for every $\iota \in J^c$ there is v_ι in $B(\chi_K,\varepsilon)$ such that $T(\iota)v_\iota \in B(-\chi_K,\varepsilon)$.

As in the proof of theorem 5.8 we can without loss of generality assume that v_ι is real valued so that again $\|T(\iota)(v_\iota^+\chi_B)\|^p < \varepsilon^p$, and $\|v_\iota^-\chi_B\|^p < \varepsilon^p$ for every $\iota \in J^c$ and every measurable subset B of X.

Because of

$$\varepsilon^p > \int_K |1 - v_\iota|^p \, d\mu \geq \mu(K \cap \{|1 - v_\iota| > 1/2\})/2^p$$

and

$$\varepsilon^p > \int_K |1 + T(\iota)v_\iota|^p \, d\mu \geq \mu(K \cap \{|1 + T(\iota)v_\iota| > 1/2\})/2^p$$

it follows for $L_\iota := K \cap \{|1 - v_\iota| \leq 1/2\} \cap \{|1 + T(\iota)v_\iota| \leq 1/2\}$ that $\mu(K \backslash L_\iota) < 2^{p+1}\varepsilon^p$ for $\iota \in J^c$.

From the definition of L_ι we have $T(\iota)v_\iota|_{L_\iota} \leq -1/2$ as well as $v_\iota|_{L_\iota} \geq 1/2$ μ-a.e., so that as in the proof of theorem 5.23 for $\iota \in J^c$

$$\varepsilon^p \geq \|v_\iota^- \chi_{\varphi(\iota,L_\iota)}\|^p \geq \nu_{p,-\iota}(L_\iota)/2^p$$

as well as

$$\varepsilon^p > \|T(\iota)(v_\iota^+ \chi_{L_\iota})\|^p \geq \nu_{p,\iota}(L_\iota)/2^p.$$

So for every $\iota \in J^c$ we have found a measurable subsets L_ι of K such that $\nu_{p,-\iota}(L_\iota) < 2^p\varepsilon^p, \nu_{p,\iota} < 2^p\varepsilon^p$, and $\mu(K\backslash L_\iota) < 2^{p+1}\varepsilon^p$ which shows ii). Note that we did not need $1/w(\iota,\cdot) \in L^p_{loc}(\mu)$ for this implication.

Now to prove that ii) implies i), let $f, g \in C_c(X)$, let K be a compact subset of X containing $\operatorname{supp} f$ and $\operatorname{supp} g$ and let $\varepsilon > 0$. Moreover, let J and $(L_\iota)_{\iota\in J^c}$ be as in ii) for K and ε.

For $\iota \in J^c$ we define

$$v_\iota := f\chi_{L_\iota} + (\frac{g(\varphi(-\iota,\cdot))}{w(\iota,(\varphi(-\iota,\cdot)))})\chi_{\varphi(\iota,L_\iota)},$$

which is measurable and because of $1/w(\iota,\cdot) \in L^\infty_{loc}(\mu)$ in $L^p(\mu)$. Then we have

$$v_\iota - f = (\frac{g(\varphi(-\iota,\cdot))}{w(\iota,(\varphi(-\iota,\cdot)))})\chi_{\varphi(\iota,L_\iota)} - f\chi_{K\backslash L_\iota}$$

and

$$T(\iota)v_\iota - g = w(\iota,\cdot)f(\varphi(\iota,\cdot))\chi_{\varphi(\iota,\cdot)^{-1}(L_\iota)} - g\chi_{K\backslash L_\iota}.$$

Thus,

$$\|\frac{g(\varphi(-\iota,\cdot))}{w(\iota,(\varphi(-\iota,\cdot)))})\chi_{\varphi(\iota,L_\iota)}\|^p \leq \|g\|^p_\infty \nu_{p,-\iota}(L_\iota)$$

and

$$\|w(\iota,\cdot)f(\varphi(\iota,\cdot))\chi_{\varphi(\iota,\cdot)^{-1}(L_\iota)}\|^p \leq \|f\|^p_\infty \nu_{p,\iota}(L_\iota).$$

Since $\mu(K\backslash L_\iota) < \varepsilon$ for $\iota \in J^c$, it follows that $\|v_\iota - f\|^p < \varepsilon(\|g\|^p_\infty + \|f\|^p_\infty)$ and $\|T(\iota)v_\iota - g\|^p < \varepsilon(\|f\|^p_\infty + \|g\|^p_\infty)$. The density of $C_c(X)$ now implies that T is mixing on $L^p(\mu)$. □

Example 5.32 Let T be the unilateral or bilateral shift on $\ell^p_\beta(\mathbb{N})$ or $\ell^p_\beta(\mathbb{Z})$ respectively. Combining our considerations form Example 5.11 with the above theorem for $w(n,\cdot) \equiv 1$ gives that T is mixing if and only if $\lim_{n\to\infty} \beta_n = 0$ or $\lim_{n\to\infty} \beta_n = \lim_{n\to-\infty} \beta_n = 0$, respectively.

The characterisation for $T_{w,\varphi}$ to be mixing on $C_{0,\rho}(X)$ reads as follows.

Theorem 5.33 *Under the general hypotheses and the additional assumption that for all compact subsets K of X we have $\inf_{x\in K}\rho(x) > 0$, among the following, i) implies ii).*

i) $T_{w,\varphi}$ is mixing on $C_{0,\rho}(X)$.

ii) For every compact subset K of X and every $\varepsilon > 0$ we can find a compact subset J of I different from I such that for all $\iota \in J^c$ we have

$$\sup_{x\in\varphi(\iota,\cdot)^{-1}(K)} w(\iota,x)\rho(x) < \varepsilon \text{ and } \sup_{x\in\varphi(\iota,K)} \frac{\rho(x)}{w(\iota,\varphi(-\iota,x))} < \varepsilon.$$

Moreover, if $\varphi(\iota,\cdot) : X \to X$ is an open mapping for all $\iota \in I$ then ii) also implies i).

PROOF: Again, we write T instead of $T_{w,\varphi}$ for brevity. Assume that T is mixing. Let K be a compact subset of X, $\varepsilon \in (0, \inf_{x\in K} \rho(x)/2)$, and $f \in C_c(X)$ positive with $f_{|K} \equiv 1$. Since T is mixing there is a compact subset J of I different form I such that for all $\iota \in J^c$ we have $v_\iota \in C_{0,\rho}(X)$ satisfying $\|v_\iota - f\| < \varepsilon$ and $\|T(\iota)v_\iota + f\| < \varepsilon$. Using the same arguments as in the proof of theorem 5.8 we can assume without loss of generality that v_ι is real valued. This in turn again shows $\|T(\iota)v_\iota^+\| < \varepsilon$ and $\|v_\iota^-\| < \varepsilon$.

Moreover, $\|T(\iota)v_\iota + f\| < \varepsilon$ implies $w(\iota, x)v_\iota(\varphi(\iota, x)) < \varepsilon/\rho(x) - 1 < -1/2$ for every $x \in K$, and $\|v_\iota - f\| < \varepsilon$ gives $v_\iota(x) > 1 - \varepsilon/\rho(x) > 1/2$ for all $x \in K$.

From this we get

$$\begin{aligned}\varepsilon &> \|T(\iota)v_\iota^+\| \geq \sup_{x\in\varphi(\iota,\cdot)^{-1}(K)} w(\iota, x)v_\iota^+(\varphi(\iota, x))\rho(x) \\ &> 1/2 \sup_{x\in\varphi(\iota,\cdot)^{-1}(K)} w(\iota, x)\rho(x)\end{aligned}$$

as well as

$$\begin{aligned}\varepsilon &> \|v_\iota^-\| \geq \sup_{x\in\varphi(\iota,K)} v_\iota^-(x)\rho(x) = \sup_{x\in\varphi(\iota,K)} w(\iota, \varphi(-\iota, x))v_\iota^-(x)\frac{\rho(x)}{w(\iota, \varphi(-\iota, x))} \\ &> 1/2 \sup_{x\in\varphi(\iota,K)} \frac{\rho(x)}{w(\iota, \varphi(-\iota, x))}\end{aligned}$$

for all $\iota \in J^c$ which proves ii).

Now we assume that $\varphi(\iota, \cdot) : X \to X$ is an open mapping for all $\iota \in I$. Let U, V be non-empty open subsets of $C_{0,\rho}(X)$. Choose $f, g \in C_c(X)$ and $\varepsilon > 0$ such that $B(f, \varepsilon) \subset U, B(g, \varepsilon) \subset V$ and let K be a compact subset of X containing $\operatorname{supp} f$ and $\operatorname{supp} g$. Let J be as in ii) for K and $\varepsilon/(\|f\|_{sup}+\|g\|_{sup}+1)$.

It follows from the additional hypotheses that $g(\varphi(-\iota, \cdot))/w(\iota, \varphi(-\iota, \cdot)) : \varphi(\iota, X) \to \mathbb{K}$ is continuous for all $\iota \in J^c$. Because $\varphi(\iota, X)$ is open, we can extend it as usual to a compactly supported continuous function g_ι on X for which $T(\iota)g_\iota = g_\iota$.

Additionally we have

$$\|g_\iota\| = \sup_{x\in\varphi(\iota,K)} |\frac{g(\varphi(-\iota, x))}{w(\iota, \varphi(-\iota, x))}|\rho(x) \leq \|g\|_{sup} \sup_{x\in\varphi(\iota,K)} \frac{\rho(x)}{w(\iota, \varphi(-\iota, x))} < \varepsilon$$

and

$$\begin{aligned}\|T(\iota)f\| &= \sup_{x\in\varphi(\iota,\cdot)^{-1}(K)} |w(\iota, x)f(\varphi(\iota, x))|\rho(x) \\ &\leq \|f\|_{sup} \sup_{x\in\varphi(\iota,\cdot)^{-1}(K)} w(\iota, x)\rho(x) \\ &< \varepsilon.\end{aligned}$$

This shows that $f + g_\iota \in B(f, \varepsilon) \subset U$ and $T(\iota)(f + g_\iota) \in B(g, \varepsilon) \subset V$ for all $\iota \in J^c$ so that T is mixing. □

In chapter 6 we will apply the results of this chapter to characterise hypercyclic and mixing C_0-semigroups generated by first order partial differential operators.

5.3 Weighted composition operators on arbitrary σ-finite measure spaces

Up to now, we have chosen X to be a locally compact, σ-compact Hausdorff topological space equipped with a locally finite Borel measure μ. We did this because of applications we had in mind which will be dealt with in the next chapter. Still it is possible to derive analogous characterisations for families of weighted composition operators on spaces of p-integrable functions over arbitrary σ-finite measure spaces. We will do this now for the sake of completeness.

Let $(X, \mathcal{A}, \mu)$ be an arbitrary σ-finite measure space, I a non-empty set, $w : I \times X \to (0, \infty)$ a mapping such that $w(\iota, \cdot)$ is a measurable and $1/w(\iota, \cdot) \in L^\infty_{loc}(\mu)$ (i.e. $\chi_A/w(\iota, \cdot) \in L^\infty(\mu)$ for every $A \in \mathcal{A}$ with $\mu(A) < \infty$) for every ι in I, and let $\varphi : I \times X \to X$ be a mapping such that $\varphi(\iota, \cdot)$ is bimeasurable on X for every $\iota \in I$, (that is $\varphi(\iota, \cdot)$ is bijective, measurable and $\varphi(-\iota, \cdot) := \varphi(\iota, \cdot)^{-1}$ is measurable, too) such that $T_{w,\varphi}(\iota) : L^p(\mu) \to L^p(\mu), f \mapsto w(\iota, \cdot)f(\varphi(\iota, \cdot))$ is a well-defined operator for all $\iota \in I$.

Again, let

$$\nu_{p,\iota}(B) := \int_{\varphi(\iota,\cdot)^{-1}(B)} w(\iota, \cdot)^p \, d\mu$$

and

$$\nu_{p,-\iota}(B) := \int_{\varphi(\iota,B)} 1/w(\iota, \varphi(-\iota, \cdot))^p \, d\mu.$$

Theorem 5.34 *Under the above hypotheses, the following are equivalent.*

i) $T_{w,\varphi}$ *is weakly mixing on* $L^p(\mu)$.

ii) $T_{w,\varphi}$ *is transitive on* $L^p(\mu)$.

iii) For every $A \in \mathcal{A}$ *with* $\mu(A) < \infty$ *there are a sequence of measurable subsets* $(A_n)_{n\in\mathbb{N}}$ *of* A *and a sequence* $(\iota_n)_{n\in\mathbb{N}}$ *in* I *such that* $\lim_{n\to\infty} \mu(A\backslash A_n) = 0$ *as well as*

$$\lim_{n\to\infty} \nu_{p,\iota_n}(A) = \lim_{n\to\infty} \nu_{p,-\iota_n}(A) = 0.$$

Proof: The proof that ii) implies iii) is almost verbatim the same as the proof of the corresponding implication of theorem 5.23 when replacing K by A and taking into account that now $\varphi(\iota, X) = X$ for all ι.

In order to show that ii) implies i), let U, V be non-empty open subsets of $L^p(\mu)$. Let $f \in U, g \in V$ be simple functions, that is $f = \sum_{i=1}^n \beta_i \chi_{B_i}$ and $g = \sum_{j=1}^k \gamma_j \chi_{C_j}$ with $(B_i)_{1\le i\le n}$ and $(C_j)_{1\le j\le k}$ pairwise disjoint, respectively, and $\mu(\bigcup_{1\le i\le n} B_i \cup \bigcup_{1\le j\le k} C_j) < \infty$. Let $A := \bigcup_{1\le i\le n} B_i \cup \bigcup_{1\le j\le k} C_j$ and chose $(A_n)_{n\in\mathbb{N}}$ and $(\iota_n)_{n\in\mathbb{N}}$ be as in ii) for A.

Now the proof is exactly the same as the corresponding implication of theorem 5.23 with the now obvious modifications. □

If I is a topological space we again have a characterisation of when $T_{w,\varphi}$ is mixing.

Theorem 5.35 *Under the above hypotheses, the following are equivalent for the family of operators* $(T_{w,\varphi}(\iota) : L^p(\mu) \to L^p(\mu), f \mapsto w(\iota, \cdot)f(\varphi(\iota, \cdot)))_{\iota\in I}$.

i) *The family* $T_{w,\varphi}$ *is mixing on* $L^p(\mu)$.

ii) *For every* $A \in \mathcal{A}$ *with* $\mu(A) < \infty$ *and every* $\varepsilon > 0$ *there is a compact subset* $J \subset I, J \neq I$ *such that for all* ι *in the complement of* J *there is a measurable subset* A_ι *of* A *satisfying* $\mu(A \backslash A_\iota) < \varepsilon$, $\nu_{p,-\iota}(A) < \varepsilon$ *and* $\nu_{p,\iota}(A) < \varepsilon$.

PROOF: The proof is an obvious modification of the proof of theorem 5.31 as the proof of theorem 5.34 is of theorem 5.23. □

Remark 5.36 As mentioned in remark 5.26, using [35, theorem 2.4] and theorem 5.34 one gets a characterisation of Cesàro-hypercyclicity for a single weighted composition operator $L^p(\mu)$.

6 Hypercyclic and mixing C_0-semigroups generated by first order partial differential operators

In [19] Desch et al. considered the so called left translation semigroup $T(t)f = f(\cdot + t)$ on $L^p_\rho(\mathbb{R}), L^p_\rho([0,\infty))$ and $C_{0,\rho}(\mathbb{R}), C_{0,\rho}([0,\infty))$ which is generated by the first order ordinary differential operator $f \mapsto \frac{d}{dx}f$ (here, ρ is a weight function). They showed that the translation semigroup is hypercyclic if and only if for every $x \in \mathbb{R}$, respectively $x \in [0,\infty)$, there is a strictly increasing sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers tending to infinity such that $\lim_{n\to\infty} \rho(x+t_n) = \lim_{n\to\infty} \rho(x-t_n) = 0$, respectively $\lim_{n\to\infty} \rho(x+t_n) = 0$.

Chaoticity of this semigroup on the same spaces was characterised by Matsui et al. in [41, 42], whereas Bermúdez et al. [6, theorem 4.3] characterised when the translation semigroup is mixing.

Sufficient conditions for hypercyclicity and chaoticity of C_0-semigroups on spaces of continuous functions and spaces of integrable functions generated by other first order ordinary differential operators were given by Matsui and Takeo [40], Myjak and Rudnicki [43] and Takeo [58].

In this chapter we will use the results from chapter 5 to characterise systematically when C_0-semigroups generated by first order partial differential operators are hypercyclic or mixing on spaces of integrable and spaces of continuous functions, whereas chaoticity is postponed to the next chapter. Parts of this chapter will appear in *Ergodic Theory and Dynamical Systems* under the title "Hypercyclic, mixing, and chaotic C_0-semigroups induced by semiflows".

6.1 Hypercyclic and mixing C_0-semigroups generated by gradient operators

In what follows Ω is an open subset of $\mathbb{R}^d$. We call a continuous function $\varphi : [0,\infty) \times \Omega \to \Omega$ a *semiflow* if $\varphi(0,\cdot) = id_\Omega$, $\varphi(t,\cdot) \circ \varphi(s,\cdot) = \varphi(t+s,\cdot)$ for all $t, s \geq 0$ and if $\varphi(t,\cdot)$ is injective for all $t \geq 0$. Typically, φ will be the solution semiflow of the initial value problem

$$\dot{x} = F(x),\ x(0) = x_0$$

on Ω, where F is a locally Lipschitz continuous vector field over Ω. Furthermore, let μ be a locally finite Borel measure and ρ a positive, upper semicontinuous function on Ω. We adopt the notation of chapter 5, that is we write μ_t for the image measure of μ under $\varphi(t,\cdot)$ etc. Remember that by theorem 5.4 the mapping $T_\varphi(t) : L^p(\mu) \to L^p(\mu), f \mapsto f(\varphi(t,\cdot)), 1 \leq p < \infty$ is well-defined and an operator if and only if μ_t is absolutely continuous with respect to μ and its μ-density f_t is μ-a.e. bounded.

Before we give a theorem characterising when T_φ is a C_0-semigroup on $L^p(\mu)$ we need some properties of semiflows.

Proposition 6.1 *Let φ be a semiflow on Ω. Then the following hold.*

i) For every $s \geq 0$ we have $\lim_{t\to s} \varphi(t,\cdot) = \varphi(s,\cdot)$ locally uniformly.

ii) For every compact subset K of Ω and every $\delta > 0$ satisfying $K+B(0,\delta) := \{x+y;\, x \in K, y \in B(0,\delta)\} \subset \Omega$ there is $n \in \mathbb{N}$ such that for every $t < 1/n$ we have $K \cap \varphi(t,(K+B(0,\delta))^c) = \emptyset$.

PROOF: i) Let $s \geq 0$ and K be a compact subset of Ω. From the continuity of φ it follows that $\varphi : [0, s+1] \times K \to \Omega$ is uniformly continuous. This implies in particular that $\lim_{t\to s} \varphi(t,x) = \varphi(s,x)$ uniformly for all $x \in K$.

ii) Again, let K be a compact subset of Ω and $\delta > 0$ as in ii). We assume to the contrary that for every $n \in \mathbb{N}$ there are $y_n \in (K+B(0,\delta))^c$ and $t_n \in (0,1/n)$ such that $\varphi(t_n, y_n) \in K$. Without loss of generality we can assume $(t_n)_{n\in\mathbb{N}}$ to be strictly decreasing and $(\varphi(t_n,y_n))_{n\in\mathbb{N}}$ to be convergent to some $z \in K$.

The continuity of φ implies that $(\varphi(t_1 - t_n, \cdot))_{n\in\mathbb{N}}$ is equicontinuous. Using this and $\lim_{n\to\infty} \varphi(t_n, y_n) = z$ we immediately get that $\lim_{n\to\infty} \varphi(t_1, y_n) = \lim_{n\to\infty} \varphi(t_1 - t_n, \varphi(t_n, y_n)) = \varphi(t_1, z)$. By Brouwer's theorem $\varphi(t_1,\cdot)^{-1}$ is continuous, thus $\lim_{n\to\infty} y_n = z \in K$, contradicting $y_n \in (K + B(0,\delta))^c$. □

The next theorem states when T_φ is a C_0-semigroup on $L^p(\mu)$.

Theorem 6.2 *The following are equivalent.*

i) The family of mappings $T_\varphi = (T_\varphi(t))_{t\geq 0}$ defined by $T_\varphi(t) : (L^p(\mu), \|\cdot\|) \to (L^p(\mu), \|\cdot\|), f \mapsto f \circ \varphi(t,\cdot)$ is well-defined and a C_0-semigroup.

ii) μ_t has an μ-density $f_t \in L^\infty(\mu)$ and there are constants $M \geq 1, \omega \in \mathbb{R}$ such that $\|f_t\|_\infty \leq Me^{t\omega}$ for all $t \geq 0$.

PROOF: That i) implies ii) follows immediately from theorem 5.4 and appendix A.1. To show that ii) implies i) let $f \in C_c(\Omega)$ and $K := \operatorname{supp} f$. Let $\delta > 0$ be such that $K + B_\delta := K + B(0,\delta) \subset \Omega$. Using proposition 6.1 ii) we obtain for sufficiently small $t \geq 0$

$$\int |f(\varphi(t,x)) - f(x)|^p \, d\mu(x) = \int_{K+B_\delta} |f(\varphi(t,x)) - f(x)|^p \, d\mu(x).$$

Thus, from the local finiteness of μ and Lebesgue's dominated convergence theorem we get $\lim_{t\to\infty} \|T(t)f - f\| = 0$. Since $C_c(\Omega)$ is dense in $L^p(\mu)$ it follows from ii) that $(T_\varphi(t))_{t\geq 0}$ is a C_0-semigroup on $L^p(\mu)$. □

Definition 6.3 Let φ be a semiflow on Ω. A Borel measure μ on Ω is called *(L^p-)admissible (for φ)* if it is locally finite and satisfies condition ii) of the above theorem. Then, the C_0-semigroup T_φ from the above theorem will be referred to as the one *induced by* φ and we will sometimes write T instead of T_φ when it is clear from the context, which semiflow φ we consider.

The following theorem describes when T_φ is a C_0-semigroup on $C_{0,\rho}(\Omega)$.

Theorem 6.4 *Under the general assumptions the following are equivalent.*

i) The family of operators $T_\varphi(t) : C_{0,\rho}(\Omega) \to C_{0,\rho}(\Omega), f \mapsto f(\varphi(t,\cdot)), t \geq 0$, is well-defined and forms a C_0-semigroup T_φ.

ii) a) There are constants $M \geq 1, \omega \in \mathbb{R}$ such that $\rho(x) \leq Me^{t\omega}\rho(\varphi(t,x))$ for all $x \in \Omega$ and $t \geq 0$.

b) For every compact subset K of Ω and every $\delta > 0$ the set $\varphi(t,\cdot)^{-1}(K) \cap \{x \in X; \rho(x) \geq \delta\}$ is compact for every $t \geq 0$.

Proof: Having in mind that the norms of the operators from a C_0-semigroup grow exponentially (see appendix A.1), theorem 5.5 immediately shows that i) implies ii).

Now, if ii) holds it follows from theorem 5.5 that T_φ is a mapping form $[0,\infty)$ into the space of operators on $C_{0,\rho}(\Omega)$ satisfying $\|T_\varphi(t)\| \leq Me^{t\omega}$.

Let $f \in C_c(\Omega)$ and $K := \operatorname{supp} f$. Let $\delta > 0$ be such that $K + B_\delta := K + B(0,\delta) \subset \Omega$. Proposition 6.1 ii) gives the existence of $n \in \mathbb{N}$ such that $\varphi(t,y) \notin K$ for every $y \in (K + B_\delta)^c$ and $0 \leq t < 1/n$. We set $L := \sup_{z \in K + \overline{B_\delta}} \rho(z) + 1$ which is positive and finite since ρ is positive and upper semicontinuous.

Let $\varepsilon > 0$. By the uniform continuity of f there is $\eta > 0$ such that $|f(x) - f(y)| < \varepsilon/L$ for all $|x - y| < \eta$. Proposition 6.1 i) implies that there is $t_0 < 1/n$ such that $|\varphi(t,x) - x| < \eta$ for all $x \in K + \overline{B_\delta}$ and $0 \leq t \leq t_0$ so that $|f(\varphi(t,x)) - f(x)|\rho(x) < \varepsilon$ for all $x \in K + \overline{B_\delta}$ and $0 \leq t \leq t_0$. Since $f(\varphi(t,x)) = f(x) = 0$ for every $x \in (K + B_\delta)^c$ and $0 \leq t \leq t_0$ this implies $\|T(t)f - f\| \leq \varepsilon$ whenever $0 \leq t \leq t_0$.

Because $C_c(\Omega)$ is dense in $C_{0,\rho}(\Omega)$ it follows from $\|T(t)\| \leq Me^{t\omega}$ that T_φ is a C_0-semigroup on $C_{0,\rho}(\Omega)$. □

As in the case of $L^p(\mu)$-spaces we make the following definition.

Definition 6.5 A positive-valued, upper semicontinuous function ρ on Ω is called a *(C_0-)admissible weight function (for the semiflow φ)* if it satisfies conditions *a*) and *b*) of theorem 6.4 ii). Then the C_0-semigroup T_φ from the above theorem will be referred to as the one *induced by* φ and we will sometimes write T instead of T_φ.

To proceed, let $F : \Omega \to \mathbb{R}^d$ be a locally Lipschitz continuous vector field such that the unique solution $\varphi(\cdot, x_0)$ of the initial value problem $\dot{x} = F(x)$, $x(0) = x_0$ exists for all $t \geq 0$ and all $x_0 \in \Omega$. Clearly, φ then is a semiflow and induces a C_0-semigroup T_φ on $L^p(\mu)$ or $C_{0,\rho}(\Omega)$ if μ or ρ are admissible.

The following proposition contains sufficient conditions when $\varphi(\cdot, x_0)$ exists for all $t \geq 0$ and all $x_0 \in \Omega$. Its proof will be given in appendix C.

Proposition 6.6 *Let Ω be an open, star-like subset of $\mathbb{R}^d$ and $F : \Omega \to \mathbb{R}^d$ be a differentiable vector field such that $\sup_{x\in\Omega} |DF(x)| < \infty$ and $F \in C(\bar{\Omega})$. Furthermore assume that $\partial\Omega = \partial\bar{\Omega}$ is C^1 and that $\langle F(y), n(y)\rangle < 0$ for all $y \in \partial\Omega$ where $n(y)$ denotes the outer normal in y.*

Then, for each $x_0 \in \Omega$ the solution of the initial value problem $\dot{x} = F(x), x(0) = x_0$ exists for all $t \geq 0$.

We are interested in the dynamical behaviour of solutions of the following Cauchy problem

$$\forall\,(t,x) \in [0,\infty) \times \Omega : \qquad \frac{\partial}{\partial t} u(t,x) = \sum_{i=1}^{d} F_i(x) \frac{\partial}{\partial x_i} u(t,x)$$
$$u(0,\cdot) = f_0$$

on $E \in \{L^p(\mu), C_{0,\rho}(\Omega)\}$, where $f_0 \in E$ is given.

The connection between this Cauchy problem and the C_0-semigroup T_φ induced by φ is given in the following theorem.

Theorem 6.7 *Let F be a locally Lipschitz continuous vector field on Ω such that the unique solution $\varphi(\cdot, x_0)$ of the initial value problem $\dot{x} = F(x)$, $x(0) = x_0$ exists for all $t \geq 0$ and all $x_0 \in \Omega$. Let $E \in \{L^p(\mu), C_{0,\rho}(\Omega)\}$ where μ and ρ are admissible for φ.*

Then the generator $(A, D(A))$ of the C_0-semigroup T_φ is an extension of the operator $B : \mathcal{D} \to E, f \mapsto \langle F, \nabla f\rangle$, where $\mathcal{D} := \{f \in C^1(\Omega) \cap E; \langle F, \nabla f\rangle \in E\}$.

We therefore call T_φ the solution semigroup of the Cauchy problem

$$\forall\, (t, x) \in [0, \infty) \times \Omega : \ \frac{\partial}{\partial t} u(t, x) = \sum_{i=1}^{d} F_i(x) \frac{\partial}{\partial x_i} u(t, x), \ u(0, \cdot) = f_0.$$

Proof: Let $(A, D(A))$ be the generator of T_φ. For sufficiently large $\lambda \in \mathbb{R}$ and $f \in \mathcal{D}$ it follows from the resolvent representation (see appendix A.6) and integration by parts:

$$\begin{aligned}
R(\lambda, A)(\lambda f - Bf) &= \int_0^\infty e^{-\lambda t} T_\varphi(t)(\lambda f - Bf)\, dt \\
&= \int_0^\infty e^{-\lambda t} (\lambda f \circ \varphi(t, \cdot) - \langle F(\varphi(t, \cdot)), (\nabla f) \circ \varphi(t, \cdot)\rangle)\, dt \\
&= \lambda \int_0^\infty e^{-\lambda t} T_\varphi(t) f\, dt \\
&\quad - \int_0^\infty e^{-\lambda t} \langle \partial_t \varphi(t, \cdot), (\nabla f) \circ \varphi(t, \cdot)\rangle\, dt \\
&= -e^{-\lambda t} T_\varphi(t) f|_0^\infty + \int_0^\infty e^{-\lambda t} \frac{d}{dt}[f \circ \varphi(t, \cdot)]\, dt \\
&\quad - \int_0^\infty e^{-\lambda t} \langle \partial_t \varphi(t, \cdot), (\nabla f) \circ \varphi(t, \cdot)\rangle\, dt = f,
\end{aligned}$$

which shows $B \subset A$. □

The next proposition gives sufficient conditions under which the generator of T_φ is the closure of an explicitly known operator. Its proof will be given in appendix C.

Proposition 6.8 *Let F be a locally Lipschitz continuous vector field on Ω such that the solution $\varphi(\cdot, x_0)$ of the initial value problem $\dot{x} = F(x)$, $x(0) = x_0$ exists for all $t \geq 0$ and all $x_0 \in \Omega$. Let $E \in \{L^p(\mu), C_{0,\rho}(\Omega)\}$ where μ and ρ are admissible for φ. Then the following holds.*

i) If F is bounded, continuously differentiable and satisfies $\sup_{x\in\Omega} |DF(x)| < \infty$ then the generator $(A, D(A))$ of T_φ is given by the closure of the operator $B : \mathcal{D} \to E, f \mapsto \langle F, \nabla f\rangle$, where $\mathcal{D} := \{f \in C^1(\Omega) \cap E; |\nabla f| \in E\}$.

ii) If F is continuously differentiable and such that the unique solution $\varphi(\cdot, x_0)$ of the initial value problem $\dot{x} = F(x)$, $x(0) = x_0$ exists not only for all $t \geq 0$ but for all $t \in \mathbb{R}$ then the generator $(A, D(A))$ of T_φ is given by the closure of the operator $B : C_c^1(\Omega) \to E, f \mapsto \langle F, \nabla f\rangle$.

Before we continue we fix some notation. If the semiflow φ is such that for all $t \geq 0$ the mapping $\varphi(t,\cdot)$ is continuously differentiable we call φ *continuously differentiable* and denote by $D\varphi(t,\cdot)$ the Jacobian of $\varphi(t,\cdot)$. We define $C_t := \{x \in \Omega;\ \det D\varphi(t,\cdot) = 0\}$. Obviously, $\Omega_t := \Omega \backslash C_t$ is an open subset of $\mathbb{R}^d$.

We now want to characterise when μ is admissible for φ if it has a Lebesgue density ρ. To do so, we need the following result. Note that there will be no danger of confusing a Lebesgue density with a C_0-admissible weight function.

Proposition 6.9 *Let μ be an admissible Borel measure with Lebesgue density ρ and let φ be a continuously differentiable semiflow. Then a μ-density of μ_t is given by $\chi_{\varphi(t,\Omega_t)}\rho(\varphi(-t,\cdot))|\det D\varphi(-t,\cdot)|/\rho$, and a μ-density of μ_{-t} is given by $\rho(\varphi(t,\cdot))|\det D\varphi(t,\cdot)|/\rho$.*

Proof: By the fact that $\varphi(t,\cdot)$ is one-to-one, $\psi : \Omega_t \to \varphi(t,\Omega_t), x \mapsto \varphi(t,x)$ is a C^1-diffeomorphism and Sard's theorem (cf. [22, theorem V.4.8]) states that $\lambda^d(\varphi(t,C_t)) = 0$, where λ^d denotes d-dimensional Lebesgue measure.

Since μ is admissible, μ_t has a μ-density f_t, so that μ_t is absolutely continuous with respect to λ^d, in particular $\mu_t(B \cap \varphi(t,C_t)) = 0$ for every Borel-subset B of Ω. Using that $\varphi(t,\Omega_t) = \psi(\Omega_t)$ is open we get from this for every open subset B of Ω and $t > 0$

$$\begin{aligned}
\mu_t(B) &= \mu_t(B \cap \varphi(t,\Omega_t)) = \mu_t(B \cap \psi(\Omega_t)) = \int_{\psi^{-1}(B\cap\psi(\Omega_t))} \rho\, d\lambda^d \\
&= \int_{B\cap\psi(\Omega_t)} \rho \circ \psi^{-1} |\det D(\psi^{-1})|\, d\lambda^d \\
&= \int_B \chi_{\varphi(t,\Omega_t)} \frac{\rho(\varphi(-t,\cdot))|\det D\varphi(-t,\cdot)|}{\rho}\, d\mu,
\end{aligned}$$

where we used substitution in $\mathbb{R}^d$ (see e.g. [53, theorem 7.26]) for the mapping $\psi^{-1} : \psi(\Omega_t) \to \Omega$ in the fourth equality. Because the open subsets of Ω generate the Borel σ-algebra, we conclude that $\chi_{\varphi(t,\Omega_t)}\rho(\varphi(-t,\cdot))|\det D\varphi(-t,\cdot)|/\rho$ is a μ-density of μ_t.

Furthermore, for open subsets B of Ω we have

$$\mu_{-t}(B) = \mu(\varphi(t,B)) = \int_{\varphi(t,B)} \rho\, d\lambda^d = \int_B \frac{\rho(\varphi(t,\cdot))|\det D\varphi(t,\cdot)|}{\rho}\, d\mu,$$

where we used substitution in $\mathbb{R}^d$ once more. Again, this implies that μ_{-t} has a μ-density which is given by $\rho(\varphi(t,\cdot))|\det D\varphi(t,\cdot)|/\rho$. □

Proposition 6.10 *Let $\rho : \Omega \to (0,\infty)$ be Borel measurable such that $\mu := \rho\, d\lambda^d$ is locally finite and let the semiflow φ be continuously differentiable. Then, the following are equivalent.*

i) μ is admissible for φ.

ii) There are $M \geq 1, \omega \in \mathbb{R}$ such that for every $t > 0$ the inequality

$$\rho \leq M e^{t\omega} \rho(\varphi(t,\cdot))|\det D\varphi(t,\cdot)|$$

holds λ^d-a.e..

PROOF: Because ρ is strictly positive μ is equivalent to λ^d.

i) $\Rightarrow$ *ii)*: From theorem 6.2 and proposition 6.9 it follows that the admissibility of μ is equivalent to $\chi_{\varphi(t,\Omega_t)}\rho(\varphi(-t,\cdot))|\det D\varphi(-t,\cdot)|/\rho \leq Me^{t\omega}$ λ^d-a.e. for some $M \geq 1$ and $\omega \in \mathbb{R}$, which is equivalent to

$$\chi_{\varphi(t,\Omega_t)}\rho(\varphi(-t,\cdot)) \leq \chi_{\varphi(t,\Omega_t)}Me^{t\omega}\rho|\det[(D\varphi(t,\cdot))\circ\varphi(-t,\cdot)]| \;\; \lambda^d\text{-a.e..}$$

So, there is a Borel subset N of Ω such that $\lambda^d(N) = 0$ and

$$\chi_{\varphi(t,\Omega_t)}(x)\rho(\varphi(-t,x)) \leq \chi_{\varphi(t,\Omega_t)}(x)Me^{t\omega}\rho(x)|\det D\varphi(t,\varphi(-t,x))|$$

for every $x \in N^c$, i.e.

$$\chi_{\varphi(t,\Omega_t)}(\varphi(t,y))\rho(y) \leq \chi_{\varphi(t,\Omega_t)}(\varphi(t,y))Me^{t\omega}\rho(\varphi(t,y))|\det D\varphi(t,y)|$$

for every $y \in \varphi(-t,N^c)$.

Because $\lambda^d(\varphi(t,C_t)) = 0$, we have $\lambda^d(N \cup \varphi(t,C_t)) = 0$, so that $0 = \mu_t(N \cup \varphi(t,C_t)) = \mu(\varphi(t,\cdot)^{-1}(N \cup \varphi(t,C_t))$ since $\mu_t \ll \lambda^d$. Now, because of the equivalence of μ and λ^d we get $\lambda^d(\varphi(t,\cdot)^{-1}(N \cup \varphi(t,C_t))) = 0$ and for every $y \in \varphi(t,\cdot)^{-1}((N \cup \varphi(t,C_t))^c) = \varphi(-t,N^c) \cap \Omega_t$ we have

$$\rho(y) = \chi_{\varphi(t,\Omega_t)}(\varphi(t,y))\rho(y) \leq Me^{t\omega}\rho(\varphi(t,y))|\det D\varphi(t,y)|,$$

that is $\rho \leq Me^{t\omega}\rho(\varphi(t,\cdot))|\det D\varphi(t,\cdot)|$ λ^d-a.e.. This shows that i) implies ii).

Using substitution in $\mathbb{R}^d$ it follows immediately that ii) implies i). □

Definition 6.11 Let φ be a differentiable semiflow. A Borel measurable function $\rho : \Omega \to (0,\infty)$ is called an *(L^p-)admissible weight function (for the semiflow φ)* if it satisfies condition ii) of the preceeding theorem and $\mu := \rho\, d\lambda^d$ is locally finite. We write $L^p_\rho(\Omega)$ instead of $L^p(\mu)$.

Using our results from chapter 5 we will now characterise when T_φ is hypercyclic or mixing.

Theorem 6.12 *Let φ be a continuously differentiable semiflow and ρ an L^p-admissible weight function. Then, the following are equivalent.*

i) The C_0-semigroup T_φ induced by φ on $L^p_\rho(\Omega)$ is hypercyclic.

ii) The C_0-semigroup T_φ induced by φ on $L^p_\rho(\Omega)$ is weakly mixing.

iii) For every compact subset K of Ω there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K and a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\int_{K\setminus L_n}\rho\, d\lambda^d = 0$,

$$\lim_{n\to\infty}\int_{L_n}\chi_{\varphi(t_n,\Omega_{t_n})}\rho(\varphi(-t_n,\cdot))|\det D\varphi(-t_n,\cdot)|\,d\lambda^d = 0$$

and

$$\lim_{n\to\infty}\int_{L_n}\rho(\varphi(t_n,\cdot))|\det D\varphi(t_n,\cdot)|\,d\lambda^d = 0.$$

PROOF: This is a direct consequence of the separability of $L^p_\rho(\Omega)$, proposition 6.9 and theorem 5.8. □

Theorem 6.13 *Let φ be a continuously differentiable semiflow and ρ an L^p-admissible weight function. Then, the following are equivalent.*

i) The C_0-semigroup T_φ induced by φ on $L^p_\rho(\Omega)$ is mixing.

ii) For every compact subset K of Ω we can find a family of measurable subsets $(L_t)_{t\geq 0}$ of K such that $\lim_{t\to\infty}\int_{K\setminus L_t}\rho\, d\lambda^d = 0$,

$$\lim_{t\to\infty}\int_{L_t}\chi_{\varphi(t,\Omega_t)}\rho(\varphi(-t,\cdot))|\det D\varphi(-t,\cdot)|\, d\lambda^d = 0$$

and

$$\lim_{t\to\infty}\int_{L_t}\rho(\varphi(t,\cdot))|\det D\varphi(t,\cdot)|\, d\lambda^d = 0.$$

Proof: The theorem follows immediately from proposition 6.9 and theorem 5.31. □

For the case of continuous functions we have the following theorems.

Theorem 6.14 *Let φ be a semiflow and ρ a C_0-admissible weight function on Ω such that $\inf_{x\in K}\rho(x) > 0$ for all compact subsets K of Ω. Then, the following are equivalent.*

i) The C_0-semigroup T_φ induced by φ on $C_{0,\rho}(\Omega)$ is hypercyclic.

ii) The C_0-semigroup T_φ induced by φ on $C_{0,\rho}(\Omega)$ is weakly mixing.

iii) For every compact subset K of Ω we can find a sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers such that

$$\lim_{n\to\infty}\sup_{x\in\varphi(t_n,\cdot)^{-1}(K)}\rho(x) = \lim_{n\to\infty}\sup_{x\in\varphi(t_n,K)}\rho(x) = 0.$$

Proof: The theorem follows directly from the separability of $C_{0,\rho}(\Omega)$ and corollary 5.16. □.

Theorem 6.15 *Let φ be a semiflow and ρ a C_0-admissible weight function on Ω such that $\inf_{x\in K}\rho(x) > 0$ for all compact subsets K of Ω. Then, the following are equivalent.*

i) The C_0-semigroup T_φ induced by φ on $C_{0,\rho}(\Omega)$ is mixing.

ii) For every compact subset K of Ω we have

$$\lim_{t\to\infty}\sup_{x\in\varphi(t,\cdot)^{-1}(K)}\rho(x) = \lim_{t\to\infty}\sup_{x\in\varphi(t,K)}\rho(x) = 0.$$

Proof: This is a direct consequence of theorem 5.33. □

Example 6.16 Let Ω be one of the sets $\mathbb{R}$ or $(0,\infty)$ and $\varphi : [0,\infty)\times\Omega \to \Omega, (t,x)\mapsto t+x$. Then we have $D\varphi(t,\cdot)\equiv 1$, so that $\Omega_t = \Omega$ for every $t>0$. By proposition 6.10 a measurable weight function ρ is L^p-admissible if and only if there are $M\geq 1$ and $\omega\in\mathbb{R}$ such that $\rho(x)\leq Me^{t\omega}\rho(t+x)$ a.e.. One should note that in this special context $T(t)f = f(\cdot + t)$ defines a C_0-semigroup on $C_{0,\rho}(\mathbb{R})$ if and only if $\rho(x)\leq Me^{\omega t}\rho(x+t)$ for some $M\geq 1$ and $\omega\in\mathbb{R}$ even if ρ is not upper semicontinuous.

In both cases we call the induced C_0-semigroup T *left translation semigroup*. By theorem 6.7 its generator is given by the ordinary differential operator $f\mapsto \frac{d}{dx}f$ defined on a suitable domain.

Desch et al. [19] investigated the left translation semigroup on $L^p_\rho([0,\infty))$, $C_{0,\rho}([0,\infty))$, $L^p_\rho(\mathbb{R})$ and $C_{0,\rho}(\mathbb{R})$. Obviously, there is no difference in considering $L^p_\rho((0,\infty))$ or $L^p_\rho([0,\infty))$. But when dealing with weighted spaces of continuous functions on the positive real axis, it is important to distinguish between $C_{0,\rho}([0,\infty))$ and $C_{0,\rho}((0,\infty))$. Again, $T(t)f = f(\cdot+t)$ defines a C_0-semigroup on $C_{0,\rho}([0,\infty))$ if and only if $\rho(x)\leq Me^{\omega t}\rho(x+t)$ for some $M\geq 1$ and $\omega\in\mathbb{R}$ even if ρ is not upper semicontinuous.

We will now show that (when considering hypercyclicity, respectively mixing, of the left translation semigroup) $C_{0,\rho}([0,\infty))$ is a special case of $C_{0,\rho}((0,\infty))$. To do so, let $\rho : [0,\infty)\to(0,\infty)$ be a fixed measurable function such that $\rho(x)\leq Me^{\omega t}\rho(x+t)$ for some $M\geq 1$ and $\omega\in\mathbb{R}$. We define

$$\tilde\rho : (0,\infty)\to(0,\infty), x\mapsto \begin{cases} \rho(x-1) & , \quad x\geq 1 \\ x\,\rho(0) & , \quad 0<x<1. \end{cases}$$

It is not hard to see that $\tilde\rho(x)\leq \tilde M e^{\omega t}\tilde\rho(x+t)$ for some $\tilde M\geq 1$. From this it follows that for all $t>0$ and $f\in C_{0,\tilde\rho}((0,\infty))$ we have

$$\lim_{x\to\infty}|f(x+t)|\tilde\rho(x)\leq \tilde M e^{\omega t}\lim_{x\to\infty}|f(x+t)|\tilde\rho(x+t) = 0$$

as well as

$$\lim_{x\to 0}|f(x+t)|\tilde\rho(x) = |f(t)|\lim_{x\to 0}\tilde\rho(x) = 0,$$

so that $T(t)f\in C_{0,\tilde\rho}((0,\infty))$, showing that $T(t)$ is a well-defined operator on $C_{0,\tilde\rho}((0,\infty))$ satisfying $\|T(t)\|\leq \tilde M e^{\omega t}$.

Moreover, using $\tilde\rho(x)\leq\tilde M e^{\omega t}\tilde\rho(x+t)$ it is not hard to see that $\tilde\rho$ is bounded above on compact intervals of $(0,\infty)$. From this it follows immediately that for $f\in C_c((0,\infty))$ we have $\lim_{t\to 0}\|T(t)f - f\| = 0$. Using $\|T(t)\|\leq\tilde M e^{\omega t}$ we finally get that the left translation semigroup is a well-defined C_0-semigroup on $C_{0,\tilde\rho}((0,\infty))$.

Now let $\Phi_1 : C_{0,\rho}([0,\infty))\to C_{0,\tilde\rho}((0,\infty)), f\mapsto f$ and $\Phi_2 : C_{0,\tilde\rho}((0,\infty))\to C_{0,\rho}([0,\infty)), f\mapsto f(\cdot+1)$. For $i=1,2$, Φ_i is continuous, linear, has a dense image and satisfies $T(t)\circ\Phi_i = \Phi_i\circ T(t)$ for all $t\geq 0$. So, by lemma 1.4 the left translation semigroup is hypercyclic, respectively weakly mixing, respectively mixing, on $C_{0,\rho}([0,\infty))$ if and only if it is on $C_{0,\tilde\rho}((0,\infty))$.

Note that contrary to the case $\Omega=\mathbb{R}$, for $\Omega=(0,\infty)$ it is not sufficient to only have $\rho(x)\leq Me^{\omega t}\rho(x+t)$ for some $M\geq 1$ and $\omega\in\mathbb{R}$ for the left translation semigroup to be well-defined on $C_{0,\rho}((0,\infty))$. Take for example $\rho\equiv 1$. Then $T(t)f\in C_{0,\rho}((0,\infty))$ for all $t\geq 0$ if and only if $f\equiv 0$. However, it is not necessary that ρ is upper semicontinuous for the left translation semigroup to be well defined on $C_{0,\rho}((0,\infty))$ as long as ρ satisfies a) and b) of theorem 6.4.

Desch et al. showed in [19, Theorems 4.7 and 4.8] that T is hypercyclic on $L^p_\rho([0,\infty))$, respectively $C_{0,\rho}([0,\infty))$, if and only if $\liminf_{x\to\infty}\rho(x) = 0$, and that T is hypercyclic on $L^p_\rho(\mathbb{R})$, respectively $C_{0,\rho}(\mathbb{R})$, if and only if for every $x \in \mathbb{R}$ there is a sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers such that $\lim_{n\to\infty}\rho(x+t_n) = \lim_{n\to\infty}\rho(x-t_n) = 0$.

Furthermore, Bermúdez et al. showed in [6, theorem 4.3] that T is mixing on $L^p_\rho(\mathbb{R})$ or $C_{0,\rho}(\mathbb{R})$, respectively $L^p_\rho([0,\infty))$ or $C_{0,\rho}([0,\infty))$, if and only if $\lim_{t\to\pm\infty}\rho(t) = 0$, respectively $\lim_{t\to\infty}\rho(t) = 0$.

We will now derive these results from theorems 6.12, 6.13, 6.14 and 6.15. Let $I \in \{\mathbb{R}, [0,\infty)\}$ and let $\rho : I \to (0,\infty)$ be measurable such that there are $M \geq 1, \omega \in \mathbb{R}$ for which $\rho(x) \leq Me^{\omega t}\rho(x+t)$ a.e. for all $t \geq 0$ and $x \in I$. These functions are called *admissible* by Desch, Schappacher and Webb in [19]. In [19] they showed the following lemma.

Lemma 6.17 (cf. [19, lemma 4.2]) *Let $\rho : I \to (0,\infty)$ be measurable such that there are $M \geq 1, \omega \in \mathbb{R}$ satisfying $\rho(x) \leq Me^{\omega t}\rho(x+t)$ a.e. for all $t \geq 0$.*

Then for each $l > 0$ there are constants $0 < c_l < C_l$ such that for each $x \in I$ and each $z \in [x, x+l]$ we have $c_l\rho(x) \leq \rho(z) \leq C_l\rho(x+l)$.

In particular, the above lemma shows that the measure $\mu := \rho\, d\lambda^d$ is locally finite. Hence, admissible weight functions in the sense of Desch, Schappacher and Webb are L^p-admissible for the semiflow $\varphi(t,x) = t+x$ by proposition 6.10. Furthermore, we already noted above that the left translation semigroup is a well-defined C_0-semigroup on $C_{0,\rho}(\mathbb{R})$, respectively $C_{0,\rho}([0,\infty))$ if ρ is admissible in the sense of Desch, Schappacher and Webb.

Corollary 6.18 (cf. [19, Theorems 4.7 and 4.8], [6, theorem 4.3]) *Let ρ be admissible in the sense of Desch, Schappacher and Webb on $I \in \{\mathbb{R}, [0,\infty)\}$.*

a) The following are equivalent.

i) The left translation semigroup is weakly mixing on $L^p_\rho(I)$, respectively $C_{0,\rho}(I)$.

ii) The left translation semigroup is hypercyclic on $L^p_\rho(I)$, respectively $C_{0,\rho}(I)$.

iii) For every $x \in I$ there is a sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers such that $\lim_{n\to\infty}\rho(x+t_n) = \lim_{n\to\infty}\rho(x-t_n) = 0$ if $I = \mathbb{R}$, or $\lim_{n\to\infty}\rho(x+t_n) = 0$ if $I = [0,\infty)$ respectively.

b) The following are equivalent.

i) The left translation semigroup is mixing on $L^p_\rho(I)$, respectively $C_{0,\rho}(I)$.

ii) $\lim_{x\to\infty}\rho(x) = \lim_{x\to\infty}\rho(-x) = 0$ if $I = \mathbb{R}$, or $\lim_{x\to\infty}\rho(x) = 0$ if $I = [0,\infty)$ respectively.

Proof: We first consider the spaces $L^p_\rho(I)$. Recall that in this situation it does not matter if we consider $L^p_\rho([0,\infty))$ or $L^p_\rho((0,\infty))$ since the spaces are the same. So we can use theorems 6.12 and 6.13 although $[0,\infty)$ is not open.

Let x be in I. The above lemma gives the existence of constants c_1, C_1 such that $(*)$ $c_1\rho(z+t) \leq \rho(y+t) \leq C_1\rho(z+t+1)$ for all $z \in I, y \in [z, z+1]$ and $t \in \mathbb{R}$ for which $z+t \in I$. For all $t \geq 0$ and every measurable subset L of $[x, x+1]$ it follows that

$$\int_L \frac{\rho(y+t)}{\rho(y)}\, d\mu(y) \geq \mu(L)\frac{c_1\rho(x+t)}{C_1\rho(x+1)}.$$

If $I = \mathbb{R}$ we can apply the above inequality (*) to $z = x - t$ with $t \geq 0$ to obtain

$$\int_L \frac{\rho(y-t)}{\rho(y)}\, d\mu(y) \geq \mu(L)\frac{c_1\rho(x-t)}{C_1\rho(x+1)}$$

again for every measurable subset L of $[x, x+1]$.

Now, if T is hypercyclic, respectively mixing, on $L^p_\rho(I)$ let $(t_n)_{n\in\mathbb{N}}$ and $(L_n)_{n\in\mathbb{N}}$ be as in theorem 6.12, respectively $(L_t)_{t\geq 0}$ be as in theorem 6.13, for $K = [x, x+1]$.

In case of $I = [0, \infty)$ it follows from the first of the above inequalities applied to t_n and L_n, respectively L_t, that $\lim_{n\to\infty} \rho(x+t_n) = 0$, respectively $\lim_{t\to\infty} \rho(x+t) = 0$.

If $I = \mathbb{R}$ we have that $\varphi(t, \cdot)$ is a mapping onto I, so that $\chi_{\varphi(t,I)} = 1$ for all $t \geq 0$. It follows from the second of the above inequalities applied to t_n and L_n, respectively L_t, that $\lim_{n\to\infty} \rho(x-t_n) = 0$, respectively $\lim_{t\to\infty} \rho(x-t) = 0$. This shows that a) ii) implies a) iii) and b) i) implies b) ii) in the $L^p_\rho(I)$-case.

To show that a) iii) implies a) i) and b) ii) implies b) i) in the $L^p_\rho(I)$-case let K be a non-empty compact subset of I. Let $a, b \in I$ such that $K \subset [a, b]$. Using the above lemma we find constants c_b, C_b such that $c_b\rho(a+t) \leq \rho(y+t) \leq C_b\rho(b+t)$ for all $y \in [a, b]$ and $t \in \mathbb{R}$ for which $a + t \in I$.

By the hypothesis there is a sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers such that $\lim_{n\to\infty} \rho(b+t_n) = 0$ and if $I = \mathbb{R}$ such that additionally $\lim_{n\to\infty} \rho(b-t_n) = 0$, respectively $\lim_{t\to\infty} \rho(b+t) = 0$ and if $I = \mathbb{R}$ additionally $\lim_{t\to\infty} \rho(b-t) = 0$.

Now if $I = [0, \infty)$ clearly $K \cap (I + t) = \emptyset$ for every compact subset K of I and $t > \max K$, so that $\mu_t(K) = 0$ for all $t > \max K$. In particular $\lim_{n\to\infty} \mu_{t_n}(K) = 0$ and $\lim_{t\to\infty} \mu_t(K) = 0$.

If $I = \mathbb{R}$ the mapping $\varphi(t, \cdot)$ is surjective for all $t \geq 0$, so that $\chi_{\varphi(t,I)} = 1$. It follows that

$$\mu_t(K) = \int_K \frac{\rho(y-t)}{\rho(y)}\, d\mu(y) \leq \mu(K)\frac{C_b\rho(b-t)}{c_b\rho(a)},$$

for all $t \geq 0$ so that $\lim_{n\to\infty} \mu_{t_n}(K) = 0$, respectively $\lim_{t\to\infty} \mu_t(K) = 0$.

Moreover, from

$$\mu_{-t}(K) = \int_K \frac{\rho(y+t)}{\rho(y)}\, d\mu(y) \leq \mu(K)\frac{C_b\rho(b+t)}{c_b\rho(a)}$$

for all $t \geq 0$ we also get $\lim_{n\to\infty} \mu_{-t_n}(K) = 0$, respectively $\lim_{t\to\infty} \mu_{-t}(K) = 0$.

So, for both cases $I \in \{[0, \infty), \mathbb{R}\}$ we have

$$\lim_{n\to\infty} \mu_{t_n}(K) = \lim_{n\to\infty} \mu_{-t_n}(K) = 0,$$

respectively

$$\lim_{t\to\infty} \mu_t(K) = \lim_{n\to\infty} \mu_{-t}(K) = 0,$$

so that a) i), respectively b) i), follows from theorem 6.12, respectively 6.13.

To treat the $C_{0,\rho}(I)$ case we distinguish between $I = \mathbb{R}$ and $I = [0, \infty)$. We first consider $I = \mathbb{R}$. Let $x \in \mathbb{R}$. If T is hypercyclic, respectively mixing, it follows from theorem 6.14, respectively theorem 6.15, that for the compact set $\{x\}$ there is a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \rho(x+t_n) = \lim_{n\to\infty} \rho(x - t_n) = 0$, respectively $\lim_{t\to\infty} \rho(x + t) = \lim_{t\to\infty} \rho(x - t) = 0$, so that a) ii) implies a) iii) and b) i) implies b) ii) for $C_{0,\rho}(\mathbb{R})$.

Now, assume that a) iii), respectively b) ii) holds for $C_{0,\rho}(\mathbb{R})$. Let K be a non-empty compact subset of $\mathbb{R}$. Let $a, b \in \mathbb{R}$ be such that $K \subset [a, b]$. There is a sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers such that $\lim_{n\to\infty} \rho(b + t_n) = 0$ and $\lim_{n\to\infty} \rho(b-t_n) = 0$, respectively $\lim_{t\to\infty} \rho(b+t) = 0$ and $\lim_{t\to\infty} \rho(b-t) = 0$. Using the above lemma we find constants c_b, C_b such that $c_b\rho(a+t) \le \rho(y+t) \le C_b\rho(b + t)$ for all $y \in [a, b]$ and $t \in \mathbb{R}$.

Therefore,

$$\lim_{n\to\infty} \sup_{y\in K} \rho(y + t_n) \le C_b \lim_{n\to\infty} \rho(b + t_n) = 0$$

and

$$\lim_{n\to\infty} \sup_{y\in K} \rho(y - t_n) \le C_b \lim_{n\to\infty} \rho(b - t_n) = 0,$$

respectively

$$\lim_{t\to\infty} \sup_{y\in K} \rho(y + t) \le C_b \lim_{t\to\infty} \rho(b + t) = 0$$

and

$$\lim_{t\to\infty} \sup_{y\in K} \rho(y - t) \le C_b \lim_{t\to\infty} \rho(b - t) = 0,$$

so that by theorem 6.14, respectively theorem 6.15, a) iii) implies a) i) and b) ii) implies b) i).

Finally, let $I = [0, \infty)$. Define $\tilde{\rho}$ as at the beginning of this example. We have already seen that the left translation semigroup T is a well-defined C_0-semigroup on $C_{0,\tilde{\rho}}((0, \infty))$ and that it is hypercyclic, weakly mixing, respectively mixing, on $C_{0,\rho}([0, \infty))$ if and only if it is on $C_{0,\tilde{\rho}}((0, \infty))$. Taking into account that $K \cap ((0, \infty) + t) = \emptyset$ for compact subsets K of $(0, \infty)$ and sufficiently large $t \ge 0$, one shows by similar arguments as for the case $I = \mathbb{R}$ that T is hypercyclic, respectively mixing, on $C_{0,\tilde{\rho}}((0, \infty))$ if and only if for all $x \in (0, \infty)$ there is a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \tilde{\rho}(x+t_n) = 0$, respectively $\lim_{t\to\infty} \tilde{\rho}(x + t) = 0$ and that hypercyclicity is equivalent to weak mixing. Recalling that $\tilde{\rho}(y) = \rho(y - 1)$ for $y \ge 1$ the above are equivalent to $\lim_{n\to\infty} \rho(x + t_n) = 0$, respectively $\lim_{t\to\infty} \rho(x + t) = 0$, which finally proves the corollary. □

Let us note that the dynamical behaviour of the left translation semigroup is the same on spaces of continuous and p-integrable functions for all $p \ge 1$.

We now give a more "calculable" criterion for hypercyclicity or mixing of T, provided that φ is "nice" and the admissible weight function ρ behaves tamely in a certain sense.

Theorem 6.19 *Let φ be a continuously differentiable semiflow on the open subset Ω of $\mathbb{R}^d$ that satisfies one of the following conditions.*

A) For every $t > 0$ we have $\varphi(t, \Omega) = \Omega$ and $\Omega_t = \Omega$.

B) For every compact subset K of Ω there is $t_K > 0$ such that $K \cap \varphi(t, \Omega_t) = \emptyset$ whenever $t > t_K$.

Furthermore, let ρ be an L^p-admissible weight function such that for every compact subset K of Ω there is $\varepsilon > 0$ with

$$(*) \qquad \varepsilon < \frac{\rho(\varphi(t,x))|\det D\varphi(t,x)|}{\rho(\varphi(t,y))|\det D\varphi(t,y)|} < 1/\varepsilon,$$

for all $x, y \in K$ and $t \in I$, where $I = \mathbb{R}$ in case of A), or $I = [0, \infty)$ in case of B), respectively.

a) The following are equivalent.

i) T_φ is weakly mixing on $L^p_\rho(\Omega)$.

ii) T_φ is hypercyclic on $L^p_\rho(\Omega)$.

iii) For every $x \in \Omega$ there is a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \rho(\varphi(t_n, x)) \det D\varphi(t_n, x) = 0$$

and if A) holds

$$\lim_{n\to\infty} \rho(\varphi(-t_n, x)) \det D\varphi(-t_n, x) = 0,$$

too.

b) The following are equivalent.

i) T_φ is mixing on $L^p_\rho(\Omega)$.

ii) For every $x \in \Omega$ we have

$$\lim_{t\to\infty} \rho(\varphi(t, x)) \det D\varphi(t, x) = 0$$

and if A) holds

$$\lim_{t\to\infty} \rho(\varphi(-t, x)) \det D\varphi(-t, x) = 0,$$

too.

Proof: We only prove part a). The proof of part b) is done similarly. Obviously, i) implies ii). To prove that ii) implies iii) let $x \in \Omega$ and set $K := \prod_{j=1}^{d} [x_j, x_j + \delta]$, where $\delta > 0$ is so that $K \subset \Omega$. Let $\varepsilon > 0$ be as in $(*)$. We find $(L_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ for K according to iii) of theorem 6.12 and observe that

$$\begin{aligned}
\mu_{-t_n}(L_n) &= \int_{L_n} \frac{\rho(\varphi(t_n,z))|\det D\varphi(t_n,z)|}{\rho(z)} \, d\mu(z) \\
&= \int_{L_n} \frac{\rho(\varphi(t_n,z))|\det D\varphi(t_n,z)|}{\rho(\varphi(t_n,x))|\det D\varphi(t_n,x)|} \frac{\rho(x)}{\rho(z)} \, d\mu(z) \cdot \\
&\quad \cdot \frac{\rho(\varphi(t_n,x))|\det D\varphi(t_n,x)|}{\rho(x)} \\
&\geq \varepsilon^2 \mu(L_n) \frac{\rho(\varphi(t_n,x))|\det D\varphi(t_n,x)|}{\rho(x)}
\end{aligned}$$

where we used $\varphi(0,y)=y$. Because $\lim_{n\to\infty}\mu(L_n)=\mu(K)$ and $\rho>0$, this shows $\lim_{n\to\infty}\rho(\varphi(t_n,x))\det D\varphi(t_n,x)=0$.

Now, assume that condition A) is satisfied. Then $\varphi(t_n,C_{t_n})=\emptyset$, hence $1\equiv\chi_{\varphi(t_n,\Omega)}=\chi_{\varphi(t_n,\Omega_{t_n})}$. Thus,

$$\begin{aligned}\mu_{t_n}(L_n) &= \int_{L_n}\chi_{\varphi(t_n,\Omega_{t_n})}\frac{\rho(\varphi(-t_n,z))|\det D\varphi(-t_n,z)|}{\rho(z)}\,d\mu(z)\\ &= \int_{L_n}\frac{\rho(\varphi(-t_n,z))|\det D\varphi(-t_n,z)|}{\rho(\varphi(-t_n,x))|\det D\varphi(-t_n,x)|}\frac{\rho(x)}{\rho(z)}\,d\mu(z)\cdot\\ &\quad\cdot\frac{\rho(\varphi(-t_n,x))|\det D\varphi(-t_n,x)|}{\rho(x)}\\ &\geq \varepsilon^2\mu(L_n)\frac{\rho(\varphi(-t_n,x))|\det D\varphi(-t_n,x)|}{\rho(x)},\end{aligned}$$

so that $\lim_{n\to\infty}\rho(\varphi(-t_n,x))\det D\varphi(-t_n,x)=0$, too, proving iii).

To show that iii) implies i) let $K\subset\Omega$ be compact and non-empty and $\varepsilon>0$ as in $(*)$. Let $x\in K$ and if $(t_n)_{n\in\mathbb{N}}$ is taken as under iii) for x, we have

$$\begin{aligned}\mu_{-t_n}(K) &= \int_K\frac{\rho(\varphi(t_n,z))|\det D\varphi(t_n,z)|}{\rho(z)}\,d\mu(z)\\ &= \int_K\frac{\rho(\varphi(t_n,z))|\det D\varphi(t_n,z)|}{\rho(\varphi(t_n,x))|\det D\varphi(t_n,x)|}\frac{\rho(x)}{\rho(z)}\,d\mu(z)\cdot\\ &\quad\cdot\frac{\rho(\varphi(t_n,x))|\det D\varphi(t_n,x)|}{\rho(x)}\\ &\leq \varepsilon^2\mu(K)\frac{\rho(\varphi(t_n,x))|\det D\varphi(t_n,x)|}{\rho(x)},\end{aligned}$$

which shows $\lim_{n\to\infty}\mu_{-t_n}(K)=0$.

Under condition A), we have

$$\begin{aligned}\mu_{t_n}(K) &= \int_K\chi_{\varphi(t_n,\Omega_{t_n})}\frac{\rho(\varphi(-t_n,z))|\det D\varphi(-t_n,z)|}{\rho(z)}\,d\mu(z)\\ &= \int_K\frac{\rho(\varphi(-t_n,z))|\det D\varphi(-t_n,z)|}{\rho(\varphi(-t_n,x))|\det D\varphi(-t_n,x)|}\frac{\rho(x)}{\rho(z)}\,d\mu(z)\cdot\\ &\quad\cdot\frac{\rho(\varphi(-t_n,x))|\det D\varphi(-t_n,x)|}{\rho(x)}\\ &\leq \varepsilon^2\mu(K)\frac{\rho(\varphi(-t_n,x))|\det D\varphi(-t_n,x)|}{\rho(x)},\end{aligned}$$

so that $\lim_{n\to\infty}\mu_{t_n}(K)=0$, too.

If condition B) holds, then

$$\mu_t(K)=\int\chi_{K\cap\varphi(t,\Omega_t)}\frac{\rho(\varphi(-t,z))|\det D\varphi(-t,z)|}{\rho(z)}\,d\mu(z)=0$$

for every $t>t_K$, so that $\lim_{n\to\infty}\mu_{t_n}(K)=0$. Now, theorem 6.12 shows that T is weakly mixing on $L^p_\rho(\Omega)$. □

For the $C_{0,\rho}(\Omega)$ case we have an analogous result. Its proof is so similar to that of theorem 6.19 that we omit it.

Theorem 6.20 *Let φ be a continuously differentiable semiflow on the open subset Ω of $\mathbb{R}^d$ that satisfies one of the following conditions.*

A) For every $t > 0$ we have $\varphi(t, \Omega) = \Omega$.

B) For every compact subset K of Ω there is $t_K > 0$ such that $K \cap \varphi(t, \Omega) = \emptyset$ whenever $t > t_K$.

Furthermore, let ρ be a C_0-admissible weight function such that for every compact subset K of Ω we have $\inf_{x \in K} \rho(x) > 0$ and there is $C < \infty$ with

$$\frac{\rho(\varphi(t, x))}{\rho(\varphi(t, y))} < C,$$

for all $x, y \in K$ and $t \in I$, where $I = \mathbb{R}$ in case of A), or $I = [0, \infty)$ in case of B), respectively.

a) The following are equivalent.

i) T_φ is weakly mixing on $C_{0,\rho}(\Omega)$.

ii) T_φ is hypercyclic on $C_{0,\rho}(\Omega)$.

iii) For every $x \in \Omega$ there is a sequence of positive numbers $(t_n)_{n \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} \rho(\varphi(t_n, x)) = 0$$

and if A) holds

$$\lim_{n \to \infty} \rho(\varphi(-t_n, x)) = 0,$$

too.

b) The following are equivalent.

i) T_φ is mixing on $C_{0,\rho}(\Omega)$.

ii) For every $x \in \Omega$ we have

$$\lim_{t \to \infty} \rho(\varphi(t, x)) = 0$$

and if A) holds

$$\lim_{t \to \infty} \rho(\varphi(-t, x)) = 0,$$

too.

Clearly, the set of L^p-admissible weight functions for φ is convex, so one might ask if this is also true for the admissible weight functions with respect to which φ induces a hypercyclic C_0-semigroup. As in the discrete case (see example 5.13), this is not true in general. Take for example the left translation semigroup on $L^p_{\rho_i}(0, \infty), i = 1, 2$, with the piecewise defined weight functions

$$\rho_1(x) := \frac{1 - 2^{j+1}}{2^{2j+1}} x + \frac{2^{j+2} - 1}{2^{j+1}}, \ x \in [2^j, 2^{j+1}), j \in \mathbb{N}_0$$

and

$$\rho_2(x) := \begin{cases} \frac{1-2^{j-1}}{2^{2j-1}} x + \frac{3 \cdot 2^{j-2} - 1}{2^{j-1}} & , \quad x \in [2^j, 3 \cdot 2^{j-1}) \ (j \in \mathbb{N}_0) \\ \frac{5}{2} - \frac{1}{2^j} x & , \quad x \in [3 \cdot 2^{j-1}, 2^{j+1}) \end{cases}$$

As in Example 5.13 one shows that ρ_i are L^p-admissible weight functions for $\varphi(t,x) = x + t$.

We have $\lim_{x\to 2^{j+1}-} \rho_1(x) = \frac{1}{2^{j+1}}$ and $\lim_{x\to 3\cdot 2^{j-1}-} \rho_2(x) = \frac{1}{2^j}$, so that $\liminf_{x\to\infty} \rho_i(x) = 0$ for $i = 1,2$, which means by corollary 6.18 that the left translation semigroup is hypercyclic on both $L^p_{\rho_1}(0,\infty)$ and $L^p_{\rho_2}(0,\infty)$.

On the other hand, for $x \in [2^j, 3\cdot 2^{j-1}), j \in \mathbb{N}_0$, we have

$$\rho_1(x) \geq \frac{1-2^{j+1}}{2^{2j+1}} 3\cdot 2^{j-1} + \frac{2^{j+2}-1}{2^{j+1}} \geq \frac{1}{2}$$

and for $x \in [3\cdot 2^{j-1}, 2^{j+1}), j \in \mathbb{N}_0$, we have

$$\rho_2(x) \geq \frac{5}{2} - \frac{1}{2^j} 2^{j+1} = \frac{1}{2},$$

so that for all $\lambda \in (0,1)$ and $x > 0$ we have

$$\liminf_{x\to\infty} [\lambda\rho_1(x) + (1-\lambda)\rho_2(x)] \geq \min\{\lambda, 1-\lambda\}/2 > 0.$$

From corollary 6.18 it therefore follows that for $\lambda \in [0,1]$ the left translation semigroup is hypercyclic on $L^p_{\lambda\rho_1+(1-\lambda)\rho_2}(0,\infty)$ if and only if $\lambda \in \{0,1\}$.

Example 6.21 We consider the semiflow $\varphi : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d, (t,x) \mapsto e^t x$. Then $\det D\varphi(t,\cdot) \equiv e^t$ so that $\Omega_t = \Omega$ for all $t > 0$. It follows that a weight function ρ is L^p-admissible if and only if there is $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\rho(x) \leq M e^{t\omega} \rho(e^t x)$ a.e..

For example, $\rho : \mathbb{R}^d \to (0,\infty), x \mapsto \frac{1}{1+|x|^2}$ is admissible, because

$$\rho(e^t x) = \frac{1}{1+e^{2t}|x|^2} \geq \frac{e^{-2t}}{1+|x|^2} = e^{-2t}\rho(x).$$

So, one may take $M = 1$ and $\omega = 2$.

If $K \subset \mathbb{R}^d$ is compact, then for $t > 0$ we have

$$\mu_t(K) = e^{-t} \int_K \rho(e^{-t}x)\, d\lambda^d(x) \leq e^{-t}\lambda^d(K),$$

which tends to 0 as t tends to infinity. On the other hand, if K does not contain 0, we have $\min_{x\in K} |x|^2 =: \delta > 0$ so that

$$\mu_{-t}(K) = e^t \int_K \rho(e^t x)\, d\lambda^d(x) \leq \frac{e^t}{1+e^{2t}\delta} \lambda^d(K).$$

This last expression tends to 0 as t tends to infinity.

Since for every compact subset K of $\mathbb{R}^d$ and every $\varepsilon > 0$ we can find a compact subset $0 \notin L \subset K$ such that $\int_{K\setminus L} \rho\, d\lambda^d < \varepsilon$, we get:

Theorem 6.22 *The solution C_0-semigroup of the Cauchy problem*

$$\frac{\partial}{\partial t} u(t,x) = \sum_{i=1}^d x_i \frac{\partial}{\partial x_i} u(t,x), \;\; u(0,x) = f_0(x)$$

in $L^p_\rho(\mathbb{R}^d)$ is mixing, where $\rho : \mathbb{R}^d \to (0,\infty), x \mapsto \frac{1}{1+|x|^2}$.

One should note that for compact subsets K with $0 \in \overset{\circ}{K}$ it follows from the monotone convergence theorem, that $\lim_{t\to\infty} \mu_{-t}(K) = \lim_{t\to\infty} \int_{e^t K} \rho(x)\, d\lambda^d(x) = \int_{\mathbb{R}^d} \rho\, d\lambda^d \geq \frac{\pi}{4}\sigma_d$, where σ_d is the surface of the d-dimensional unit sphere. This shows that in general the measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K in theorem 5.8 are different from K.

Clearly, $\rho(x) = (1+|x|^2)^{-1}$ is C_0-admissible, too. But because of $\varphi(t, 0) = 0$ for all $t \geq 0$ we see that T_φ is not hypercyclic on $C_{0,\rho}(\mathbb{R}^d)$. This stands in complete contrast to the left translation semigroup as observed above.

Note, that one could have used theorem 6.19 in order to derive the results of this example as well, since $L^p_\rho(\mathbb{R}^d) = L^p_\rho(\mathbb{R}^d\backslash\{0\})$.

In the preceding examples $\rho \equiv 1$ is an admissible weight function for the considered semiflows φ, but the induced C_0-semigroups are not hypercyclic. In the latter example, this follows from $\mu_t(B) = e^t\lambda^d(B)$. We now present an example, where $\rho \equiv 1$ yields an hypercyclic semigroup.

Example 6.23 Let $\Omega = (0, 1)$ and

$$\varphi : [0,\infty) \times (0,1) \to (0,1), (t,x) \mapsto \frac{x}{(x + (1-x)e^{-t})}.$$

One checks that φ is well-defined and the solution (semi)flow of the ordinary differential equation $\dot{x} = x(1-x)$.

It follows that $D\varphi(t,x) = e^{-t}(x + (1-x)e^{-t})^{-2}$, so that $\Omega_t = \Omega$ for each $t > 0$ and one easily shows that $\varphi(t,\Omega) = \Omega$ for all $t > 0$.

Obviously, a weight function ρ is admissible for φ only if

$$\rho(x)e^t(x + (1-x)e^{-t})^2 \leq Me^{t\omega}\rho(\frac{x}{(x+(1-x)e^{-t})})\ a.e.$$

for some $M \geq 1, \omega \in \mathbb{R}$ and every $t > 0$. Since for all $t > 0, x \in (0, 1)$ always $0 < (x + (1-x)e^{-t})^2 < 1$ holds, we see that $\rho \equiv 1$ is admissible for φ.

Moreover, for every $0 < a \leq x \leq y \leq b < 1$ and $t \in \mathbb{R}$ we have

$$\frac{D\varphi(t,x)}{D\varphi(t,y)} = (\frac{y + (1-y)e^{-t}}{x + (1-x)e^{-t}})^2 \leq (\frac{b + (1-a)e^{-t}}{a + (1-b)e^{-t}})^2.$$

This last expression is bounded in $t \in \mathbb{R}$, since it converges for t to infinity to $(b/a)^2$ and for t to minus infinity to $(1-a)^2/(1-b)^2$.

Furthermore,

$$\frac{D\varphi(t,x)}{D\varphi(t,y)} \geq (\frac{a + (1-b)e^{-t}}{b + (1-a)e^{-t}})^2$$

and the last expression is a monoton function in t, so that $D\varphi(t,x)/D\varphi(t,y) \geq \min\{a^2/b^2, (1-b)^2/(1-a)^2\}$.

So for all $t \in \mathbb{R}$ we have

$$\min\{a^2/b^2,(1-b)^2/(1-a)^2\} \leq D\varphi(t,x)/D\varphi(t,y) \leq \max\{b^2/a^2,(1-a)^2/(1-b)^2\}$$

which shows that the hypotheses of theorem 6.19 are satisfied, so that T_φ is hypercyclic on $L^p_\rho(0, 1)$ if and only if for each $x \in (0, 1)$ there is a strictly increasing sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ tending to infinity such that $\lim_{n\to\infty} D\varphi(\pm t_n, x) = 0$. Since

$$\lim_{t\to\pm\infty} D\varphi(t,x) = \lim_{t\to\pm\infty} \frac{e^{-t}}{(x+(1-x)e^{-t})^2} = 0,$$

theorem 6.19 even yields

Theorem 6.24 *The solution C_0-semigroup of the Cauchy problem*

$$\frac{\partial}{\partial t}u(t,x) = x(1-x)\,\frac{\partial}{\partial x}u(t,x),\ \ u(0,x) = f_0(x)$$

is mixing on $L^p(0,1)$.

One should note that the semiflow considered in this example does not induce a hypercyclic C_0-semigroup on $C_0(0,1)$, because all operators $T(t)$ are contractions.

We conclude this section with two more examples.

Example 6.25 On $\Omega := \{x \in \mathbb{R}^d;\ |x|^2 < 1\}$ we consider the solution (semi)flow of the ordinary differential equation $\dot{x} = (1-|x|^2)x$ which is given by

$$\varphi : \mathbb{R}\times\Omega \to \Omega, (t,x) \mapsto \frac{1}{\sqrt{(|x|^2 + (1-|x|^2)e^{-2t})}}x.$$

One easily checks that φ is well-defined and obviously injective and one calculates

$$\det D\varphi(t,x) = \frac{e^{-2t}}{(|x|^2 + (1-|x|^2)e^{-2t})^{d/2+1}} \geq 0.$$

We consider the weight function $\rho : \Omega \to (0,\infty), x \mapsto 1-|x|^2$. Then, $\mu := \rho\, d\lambda^d$ is locally finite and we will show that ρ is admissible.

Indeed, we have for $|x| < 1$ and $t \in \mathbb{R}$ that

$$(|x|^2 + (1-|x|^2)e^{-2t})^{d/2+1} \leq (1+e^{-2t})^{d/2+1},$$

so that

$$|\det D\varphi(t,x)| \geq e^{-2t}(1+e^{-2t})^{-d/2-1} \geq e^{-2t}2^{-d/2-1}$$

for $t \geq 0$. Since

$$\rho(\varphi(t,x)) = (1-|x|^2)e^{-2t}(|x|^2+(1-|x|^2)e^{-2t})^{-1} = \rho(x)e^{-2t}(|x|^2+(1-|x|^2)e^{-2t})^{-1},$$

this implies for $t > 0$ and $x \in \Omega$

$$\begin{aligned}\rho(x) &\leq 2^{d/2+1}e^{2t}|\det D\varphi(t,x)|\rho(x)\\ &= 2^{d/2+1}e^{2t}|\det D\varphi(t,x)|\rho(\varphi(t,x))(|x|^2 + (1-|x|^2)e^{-2t})e^{2t}\\ &\leq 2^{d/2+1}e^{4t}\rho(\varphi(t,x))|\det D\varphi(t,x)|,\end{aligned}$$

showing the admissibility of ρ for φ.

Now, let $K \neq \emptyset$ be a compact subset of Ω with $0 \notin K$. We set $r := \min_{x\in K}|x| > 0$ and $R := \max_{x\in K}|x| < 1$ and observe that for all $t \in \mathbb{R}$ we have

$$\begin{aligned}\frac{\rho(\varphi(t,x))}{\rho(x)}|\det D\varphi(t,x)| &= \frac{e^{-4t}}{(|x|^2+(1-|x|^2)e^{-2t})^{d/2+2}}\\ &\leq \frac{e^{-4t}}{(r^2+(1-R^2)e^{-2t})^{d/2+2}},\end{aligned}$$

so that

$$\lim_{t\to\pm\infty}\sup_{x\in K}\frac{\rho(\varphi(t,x))}{\rho(x)}|\det D\varphi(t,x)| \leq \lim_{t\to\pm\infty}\frac{e^{-4t}}{(r^2+(1-R^2)e^{-2t})^{d/2+2}} = 0.$$

In particular, we have $\lim_{t\to\pm\infty}\mu_t(K) = 0$. Since for every compact subset K of Ω and for every $\varepsilon > 0$ we can find a compact subset L of K not containing 0 such that $\mu(K\backslash L) < \varepsilon$, we obtain:

Theorem 6.26 *If $\rho(x) = 1-|x|^2$ on $\Omega = \{x \in \mathbb{R}^d;\ |x|^2 < 1\}$ then the solution semigroup to the Cauchy problem*

$$\frac{\partial}{\partial t}u(t,x) = (1-|x|^2)\sum_{i=1}^d x_i\frac{\partial}{\partial x_i}u(t,x),\ \ u(0,x) = f_0(x)$$

is mixing on $L^p_\rho(\Omega)$.

Example 6.27 Let $\Omega = (0,\infty)\times(-1,1)$ and $\varphi : [0,\infty)\times\Omega \to \Omega, (t,x,y) \mapsto (x+t,y)$. Then, $|\det D\varphi(t,x,y)| = 1$ and if we choose $\rho(x,y) = \exp(-|y|x)$ we conclude from $\rho(\varphi(t,x,y)) = \rho(x,y)\exp(-|y|t)$ that ρ is L^p-admissible for φ ($M = 1, \omega = 1$).

If K is a compact subset of Ω and $C := \min_{(x,y)\in K}|y|$ we calculate

$$\mu_{-t}(K) = \int_K \exp(-|y|t)d(x,y) \leq e^{-tC}\lambda^2(K).$$

For t large enough we have $\varphi(t,\cdot)^{-1}(K) = \emptyset$ so that $\mu_t(K) = 0$. It follows from theorem 6.12 that T_φ is hypercyclic on $L^p_\rho(\Omega)$.

Clearly, for every $x \in (0,\infty)$ we have $\rho(\varphi(t,x,0)) = 1$, so that a1) iii) of theorem 6.19 does not hold, hence the hypotheses of theorem 6.19 a) cannot be satisfied.

6.2 Generators with non-vanishing zero order terms

In the previous section we essentially considered C_0-semigroups generated by gradient operators. In order to consider C_0-semigroups generated by first order partial differential operators with a non-vanishing zero order term, we have to perturb the generator of our C_0-semigroups by a multiplication operator.

Let φ again be a continuous semiflow on an open subset Ω of $\mathbb{R}^d$ and $h : \Omega \to \mathbb{R}$ be continuous. Let $E \in \{L^p(\mu), C_{0,\rho}(\Omega)\}$, where μ is a locally finite Borel measure on Ω and ρ a positive upper semicontinuous function. Then, $B : D(B) := \{f \in E;\ hf \in E\} \to E, f \mapsto hf$ is a linear operator whose domain contains $C_c(\Omega)$, hence is densely defined.

If we set $h_t : \Omega \to \mathbb{R}, x \mapsto \exp(\int_0^t h(\varphi(r,x))\,dr)$ for $t \geq 0$, then $h_{s+t}(x) = h_s(x)h_t(\varphi(s,x))$ and h_t as well as $1/h_t$ are continuous, hence in $L^\infty_{loc}(\mu)$, and positive for every $t \geq 0$. As in section 5.1 we need the following Borel measures on Ω

$$\nu_{p,t}(B) := \int_{\varphi(t,\cdot)^{-1}(B)} h_t^p\,d\mu$$

and

$$\nu_{p,-t}(B) := \int_{\varphi(t,B)} 1/h_t(\varphi(-t,\cdot))^p\,d\mu.$$

Note that $\nu_{p,t}$ and μ_t are equivalent measures because of $h_t > 0$. Recall that for measurable $f : \Omega \to [0,\infty)$ we have

$$\int f\,d\nu_{p,t} = \int h_t(\cdot)^p f(\varphi(t,\cdot))\,d\mu$$

and

$$\int f\,d\nu_{p,-t} = \int \chi_{\varphi(t,\Omega)} f(\varphi(-t,\cdot))/h_t(\varphi(-t,\cdot))^p\,d\mu.$$

We assume that $S(t) : E \to E, f \mapsto h_t(\cdot)\,f(\varphi(t,\cdot))$ is a well-defined operator for $t \geq 0$ and that $S = (S(t))_{t\geq 0}$ is a C_0-semigroup on E. The proof of the following theorem is done by the same arguments as the one of theorem 6.2 using theorem 5.21 instead of theorem 5.4, so that we omit it.

Theorem 6.28 *Under the general assumptions the following are equivalent.*

i) The family of mappings $S(t) : L^p(\mu) \to L^p(\mu), f \mapsto h_t(\cdot)\,f(\varphi(t,\cdot)), t \geq 0$ is well-defined and a C_0-semigroup.

ii) $\nu_{p,t}$ has a μ-density $f_{p,t} \in L^\infty(\mu)$ and there are constants $M \geq 1, \omega \in \mathbb{R}$ such that $\|f_{p,t}\|_\infty \leq Me^{t\omega}$ for all $t \geq 0$.

One should note that the property of S being a C_0-semigroup on $L^p(\mu)$ now depends on p in general. In the $C_{0,\rho}(\Omega)$ case the corresponding theorem is proved in the same way as theorem 6.4.

Theorem 6.29 *Under the general assumptions the following are equivalent.*

i) The family of mappings $S(t) : C_{0,\rho}(X) \to C_{0,\rho}(X), f \mapsto h_t(\cdot)\,f(\varphi(t,\cdot))$ is well-defined and a C_0-semigroup.

ii)
   a) There are constants $M \geq 1, \omega \in \mathbb{R}$ such that $h_t(x)\rho(x) \leq Me^{t\omega}\rho(\varphi(t,x))$ for all $x \in \Omega$ and $t \geq 0$.

   b) For every compact subset K of X and every $\delta > 0$ the set $\varphi(t,\cdot)^{-1}(K) \cap \{x \in X; h_t(x)\rho(x) \geq \delta\}$ is compact for every $t \geq 0$.

Note that as a direct consequence of the above two theorems S always is a C_0-semigroup if μ, respectively ρ, are admissible for the semiflow φ and if h is bounded above.

Definition 6.30 i) The locally finite Borel measure μ on Ω is called *L^p-admissible for φ and h* if the measures $(\nu_{p,t})_{t\geq 0}$ satisfy condition ii) of theorem 6.28.

ii) ρ is called C_0-*admissible for* φ *and* h if it satisfies condition ii) of theorem 6.29.

Again, if φ is the solution semiflow of the ordinary differential equation $\dot{x} = F(x)$ one shows as in theorem 6.7 that the generator $(A, D(A))$ of the C_0-semigroup S is an extension of the densely defined operator $B : C_c^1(\Omega) \to E, f \mapsto \langle F, \nabla f\rangle + hf$, where $E \in \{L^p(\mu), C_{0,\rho}(\Omega)\}$. As in the case $h \equiv 0$ the following proposition gives sufficient conditions under which the generator of S is the closure of an explicitly known operator. Its proof will be given in appendix C.

Proposition 6.31 *Let F be a locally Lipschitz continuous vector field on Ω such that the solution $\varphi(\cdot, x_0)$ of the initial value problem $\dot{x} = F(x)$, $x(0) = x_0$ exists for all $t \geq 0$ and all $x_0 \in \Omega$. Let $h : \Omega \to \mathbb{R}$ be continuous and $E \in \{L^p(\mu), C_{0,\rho}(\Omega)\}$ where μ and ρ are admissible for φ and h. Then the following holds.*

i) Assume F and h are both bounded, continuously differentiable and satisfy $\sup_{x\in\Omega} |DF(x)| < \infty$ and $\sup_{x\in\Omega} |\nabla h(x)| < \infty$. Then the generator $(A, D(A))$ of S is given by the closure of the operator $B : \mathcal{D} \to E, f \mapsto \langle F, \nabla f\rangle + hf$, where $\mathcal{D} := \{f \in C^1(\Omega) \cap E; |\nabla f| \in E\}$.

ii) If F is continuously differentiable and such that the unique solution $\varphi(\cdot, x_0)$ of the initial value problem $\dot{x} = F(x)$, $x(0) = x_0$ exists not only for all $t \geq 0$ but for all $t \in \mathbb{R}$ and if h is continuously differentiable then the generator $(A, D(A))$ of S is given by the closure of the operator $B : C_c^1(\Omega) \to E, f \mapsto \langle F, \nabla f\rangle + hf$.

If one of the conditions of the above proposition is satisfied we call S the *solution semigroup of the Cauchy problem*

$$\begin{aligned} \forall\, t \geq 0, x \in \Omega : \frac{\partial}{\partial t} u(t, x) &= \langle F(x), \nabla_x u(t, x)\rangle + h(x)u(t, x), \\ u(0, x) &= f_0(x), f_0 \in E. \end{aligned}$$

As in the case of $h \equiv 0$ treated in section 6.1, one shows the following proposition by the same kind of arguments as propositions 6.9 and 6.10, where one has to use that $\nu_{p,t}$ and μ_t are equivalent for all $t \geq 0$ to show that i) implies ii) in part b).

Proposition 6.32 *a) Let μ be L^p-admissible for φ and h. Assume that φ is continuously differentiable and that μ has a Lebesgue density ρ. Then, a μ-density of $\nu_{p,t}$, respectively $\nu_{p,-t}$, is given by*

$$\chi_{\varphi(t,\Omega_t)} \frac{h_t^p(\varphi(-t, \cdot))\rho(\varphi(-t, \cdot))|\det D\varphi(-t, \cdot)|}{\rho},$$

respectively

$$\frac{\rho(\varphi(t, \cdot))|\det D\varphi(t, \cdot)|}{h_t^p(\cdot)\rho}.$$

b) Let $\rho : \Omega \to (0, \infty)$ be Borel measurable such that $\mu := \rho\, d\lambda^d$ is locally finite. If φ is continuously differentiable, the following are equivalent.

i) μ is admissible for φ and h.

ii) There are $M \geq 1$ and $\omega \in \mathbb{R}$ such that for every $t \geq 0$ the inequality

$$h_t^p \rho \leq M e^{t\omega} \rho(\varphi(t,\cdot)) |\det D\varphi(t,\cdot)|$$

holds λ^d-a.e..

Using the results of section 5.1 and the separability of $L^p(\mu)$ we obtain a characterisation of when S is hypercyclic.

Theorem 6.33 *Under the general assumptions the following are equivalent.*

i) The C_0-semigroup S is weakly mixing on $L^p(\mu)$.

ii) The C_0-semigroup S is hypercyclic on $L^p(\mu)$.

iii) For every compact subset K of Ω there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K and a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \mu(K \backslash L_n) = 0$ as well as

$$\lim_{n\to\infty} \nu_{p,t_n}(L_n) = \lim_{n\to\infty} \nu_{p,-t_n}(L_n) = 0.$$

Proof: The theorem follows directly from theorem 5.23 for $I = [0,\infty)$ and $w(t,\cdot) = h_t$. □

If μ has a λ^d-density ρ and φ is such that $\varphi(t,\cdot)$ is continuously differentiable on Ω the above theorem together with proposition 6.32 immediately yield:

Corollary 6.34 *The C_0-semigroup S is hypercyclic on $L^p_\rho(\Omega)$ if and only if for every compact subset K of Ω there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K and a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ satisfying $\lim_{n\to\infty} \mu(K \backslash L_n) = 0$,*

$$\lim_{n\to\infty} \int_{L_n} \chi_{\varphi(t_n,\Omega_{t_n})}(x) h_{t_n}^p(\varphi(-t_n,x)) \rho(\varphi(-t_n,x)) |\det D\varphi(-t_n,x)|\, dx = 0$$

and

$$\lim_{n\to\infty} \int_{L_n} h_{t_n}^{-p}(x) \rho(\varphi(t_n,x)) |\det D\varphi(t_n,x)|\, dx = 0.$$

Up to now we have seen that C_0-semigroups generated by gradient operators are hypercyclic either on all the spaces $L^p(\mu)$, $(1 \leq p < \infty)$ or on none. This breaks down if we allow perturbations by multiplication operators, as the following example shows.

Example 6.35 Consider on $\Omega = \mathbb{R}$ the semiflow $\varphi(t,x) := t + x$ and the weight function $\rho(x) := e^x \chi_{(-\infty,0)}(x) + \chi_{[0,\infty)}(x)$. Clearly, ρ is L^p-admissible and the C_0-semigroup T induced by φ is a semigroup of contractions on $L^p_\rho(\mathbb{R})$, in particular, it is not hypercyclic.

Now, we perturb its generator by $h \equiv c$, where $c \in \mathbb{R}$. The resulting operators form a C_0-semigroup on $L^p_\rho(\mathbb{R})$ for every $1 \leq p < \infty$ according to the remark following theorem 6.29.

We have $h_t \equiv e^{tc}$ and for every compact subset K of $\mathbb{R}$ and every measurable subset L of K we have $\int_L e^{t_n cp}\rho(x-t_n)dx = e^{t_n(cp-1)}\int_L e^x dx$ for sufficiently large t_n. Now, provided that $\lambda(L) > 0$, this expression tends to 0 as t_n tends to infinity if and only if $c < 1/p$

On the other hand, for every compact subset K of $\mathbb{R}$ and every measurable subset L of K we have $\int_L e^{-t_n cp}\rho(x+t_n)dx = e^{-t_n cp}\lambda(L)$ for sufficiently large t_n. In the case of $\lambda(L) > 0$, this converges to 0 as t_n tends to infinity if and only if $c > 0$. Corollary 6.34 yields:

Theorem 6.36 *The solution C_0-semigroup of the Cauchy problem*

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial}{\partial x}u(t,x) + cu(t,x), u(0,x) = f_0(x)$$

is hypercyclic on $L^p_\rho(\mathbb{R})$ for $\rho(x) := e^x\chi_{(-\infty,0)}(x) + \chi_{[0,\infty)}(x)$ if and only if $c \in (0, 1/p)$.

Note that this example is interesting for another reason apart from the above theorem. As already noted, the unperturbed semigroup T generated by $f \mapsto f'$ is not hypercyclic on $L^p_\rho(\mathbb{R})$. The same is true for the C_0-semigroup generated by the bounded operator $f \mapsto cf$. But the semigroup generated by $f \mapsto f' + cf$ is hypercyclic. It is even mixing and chaotic, as will be seen in example 6.39 and at the end of example 7.5, respectively.

Theorem 6.37 *Let ρ be such that $\inf_{x\in K}\rho(x) > 0$ for every compact subset K of Ω. Then, under the general assumptions, the following are equivalent.*

i) The C_0-semigroup S is weakly mixing on $C_{0,\rho}(\Omega)$.

ii) The C_0-semigroup S is hypercyclic on $C_{0,\rho}(\Omega)$.

iii) For every compact subset K of Ω there exists a sequence $(t_n)_{n\in\mathbb{N}}$ of positive numbers such that

$$\lim_{n\to\infty}\sup_{x\in\varphi(t_n,\cdot)^{-1}(K)} h_{t_n}(x)\rho(x) = \lim_{n\to\infty}\sup_{x\in\varphi(t_n,K)}\frac{\rho(x)}{h_{t_n}(\varphi(-t_n,x))} = 0.$$

Proof: The theorem follows immediately from corollary 5.29. □

Theorem 6.38 *a) Under the general assumptions the following are equivalent.*

i) The C_0-semigroup S on $L^p(\mu)$ is mixing.

ii) For every compact subset K of Ω there exists a family of measurable subsets $(L_t)_{t\geq 0}$ of K such that $\lim_{t\to\infty}\nu_{p,t}(L_t) = \lim_{t\to\infty}\nu_{p,-t}(L_t) = 0$ and $\lim\limits_{t\to\infty}\mu(K\backslash L_t) = 0$.

b) Let ρ be such that $\inf_{x\in K}\rho(x) > 0$. Then, under the general hypotheses the following are equivalent.

i) The C_0-semigroup S is mixing on $C_{0,\rho}(\Omega)$.

ii) For every compact subset K of Ω we have

$$\lim_{t\to\infty} \sup_{x\in\varphi(t,\cdot)^{-1}(K)} h_t(x)\rho(x) = \lim_{t\to\infty} \sup_{x\in\varphi(t,K)} \frac{\rho(x)}{h_t(\varphi(-t,x))} = 0.$$

PROOF: The theorem follows immediately from theorem 5.31 and theorem 5.33. □

Example 6.39 Consider on $\Omega = \mathbb{R}$ again the semiflow $\varphi(t,x) := t+x$ and the weight function $\rho(x) := e^x\chi_{(-\infty,0)}(x)+\chi_{[0,\infty)}(x)$. In example 6.35 we have seen that for $h \equiv c$, where $c \in (0,1/p)$, the resulting C_0-semigroup is hypercyclic on $L^p_\rho(\mathbb{R})$.

Moreover, for every compact subset K of $\mathbb{R}$ and every measurable subset L of K we have $\int_L e^{tcp}\rho(x-t)dx = e^{t(cp-1)}\int_L e^x dx$ for sufficiently large t. Now, provided that $\lambda(L) > 0$, this expression tends to 0 as t tends to infinity since $c < 1/p$.

On the other hand, for every compact subset K of $\mathbb{R}$ and every measurable subset L of K we have $\int_L e^{-tcp}\rho(x+t)dx = e^{-tcp}\lambda(L)$ for sufficiently large t. In the case of $\lambda(L) > 0$, this converges to 0 as t tends to infinity since $c > 0$.

From theorem 6.38 it follows that the C_0-semigroup is mixing on $L^p_\rho(\mathbb{R})$.

Remark 6.40 Again, the above conditions characterising hypercyclicity and mixing of S become more convenient if φ and h are nice and ρ behaves tamely.

For $L^p_\rho(\Omega)$, if in theorem 6.19 one replaces

$$\varepsilon < \frac{\rho(\varphi(t,x))|\det D\varphi(t,x)|}{\rho(\varphi(t,y))|\det D\varphi(t,y)|} < 1/\varepsilon$$

by

$$\varepsilon < \frac{h_t(x)^{-p}\rho(\varphi(t,x))|\det D\varphi(t,x)|}{h_t(y)^{-p}\rho(\varphi(t,y))|\det D\varphi(t,y)|} < 1/\varepsilon,$$

respectively

$$\varepsilon < \frac{h_t(x)^{-p}\rho(\varphi(t,x))|\det D\varphi(t,x)|}{h_t(y)^{-p}\rho(\varphi(t,y))|\det D\varphi(t,y)|} < 1/\varepsilon$$

and

$$\varepsilon < \frac{h_t^p(\varphi(-t,x))\rho(\varphi(-t,x))|\det D\varphi(-t,x)|}{h_t^p(\varphi(-t,y))\rho(\varphi(-t,y))|\det D\varphi(-t,y)|} < 1/\varepsilon,$$

then one can show by the same kind of arguments used in the proof of theorem 6.19 that the following are equivalent.

i) S is weakly mixing on $L^p_\rho(\Omega)$.

ii) S is hypercyclic on $L^p_\rho(\Omega)$.

iii) For every $x \in \Omega$ there is a strictly increasing sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} h_{t_n}(x)^{-p}\rho(\varphi(t_n,x))\det D\varphi(t_n,x) = 0,$$

respectively

$$\lim_{n\to\infty} h_{t_n}(x)^{-p}\rho(\varphi(t_n,x))\det D\varphi(t_n,x) = 0$$

and

$$\lim_{n\to\infty} h_{t_n}(\varphi(-t_n,x))^p \rho(\varphi(-t_n,x)) \det D\varphi(-t_n,x) = 0.$$

For $C_{0,\rho}(\Omega)$ one has to replace

$$\frac{\rho(\varphi(t,x))}{\rho(\varphi(t,y))} < C$$

in theorem 6.20 by

$$\frac{\rho(\varphi(t,x))h_t(y)}{h_t(x)\rho(\varphi(t,y))} < C,$$

respectively

$$\frac{\rho(\varphi(t,x))h_t(y)}{h_t(x)\rho(\varphi(t,y))} < C \text{ and } \frac{\rho(\varphi(-t,x))h_t(\varphi(-t,y))}{h_t(\varphi(-t,x))\rho(\varphi(-t,y))} < C,$$

to obtain the equivalence of:

i) S is weakly mixing on $C_{0,\rho}(\Omega)$.

ii) S is hypercyclic on $C_{0,\rho}(\Omega)$.

iii) For every $x \in \Omega$ there is a strictly increasing sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \frac{\rho(\varphi(t_n,x))}{h_{t_n}(x)} = 0,$$

respectively

$$\lim_{n\to\infty} \frac{\rho(\varphi(t_n,x))}{h_{t_n}(x)} = \lim_{n\to\infty} h_{t_n}(\varphi(-t_n,x))\rho(\varphi(-t_n,x)) = 0.$$

Moreover, analogous characterisations to the ones of theorems 6.19 and 6.20 of when S is mixing are true.

It follows from theorem 2.9 and theorems 6.33 or 6.37, respectively, that the C_0-semigroup S is hypercyclic on $E \in \{L^p(\mu), C_{0,\rho}(\Omega)\}$ if and only if for each $t_0 > 0$ the operator $S(t_0)$ is weakly mixing. This can also be done directly, as we want to show now. In order to do so, we need the following result.

Lemma 6.41 *Let μ be such that S is a C_0-semigroup on $L^p(\mu)$. Then there are constants $M \geq 1, \omega \in \mathbb{R}$ such that for every Borel measurable subset B of a compact subset K of Ω one has*

$$\nu_{p,t}(B) \leq M e^{(t-r)\omega} \nu_{p,r}(B)$$

for all $0 \leq r \leq t$ and

$$\nu_{p,-t}(B) \leq M e^{s\omega} \nu_{p,-(t+s)}(B)$$

for all $0 \leq s, t$.

PROOF: Since S is a C_0-semigroup on $L^p(\mu)$ there are constants $M \geq 1, \omega' \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{t\omega'}$ for all $t \geq 0$ (see appendix A.1). Let K be a compact subset of Ω, B a Borel measurable subset of K and $0 \leq r \leq t$. Then we have $\chi_B \in L^p(\mu)$ and

$$\begin{aligned}\nu_{p,t}(B) &= \|S(t)\chi_B\|^p = \|S(t-r)S(r)\chi_B\|^p \leq Me^{(t-r)p\omega'}\|S(r)\chi_B\|^p \\ &= Me^{(t-r)p\omega'}\nu_{p,r}(B).\end{aligned}$$

Furthermore, for $0 \leq s,t$ we have $\chi_{\varphi(t+s,B)}/h_{t+s}(\varphi(-(t+s),\cdot)) \in L^p(\mu)$ because B is a subset of a compact set and $\varphi(t+s,\cdot)$ is continuous. Using $h_{t+s}(x) = h_t(x)h_s(\varphi(t,x))$ we get

$$\begin{aligned}\nu_{p,-t}(B) &= \int_{\varphi(t,B)} \frac{1}{h_t^p(\varphi(-t,\cdot))}\,d\mu = \int_{\varphi(s,\cdot)^{-1}(\varphi(t+s,B))} \Big(\frac{h_s}{h_t(\varphi(-t,\cdot))h_s}\Big)^p\,d\mu \\ &= \int_{\varphi(s,\cdot)^{-1}(\varphi(t+s,B))} \Big(\frac{h_s}{h_{t+s}(\varphi(-t,\cdot))}\Big)^p\,d\mu \\ &= \int h_s^p(x)\Big(\chi_{\varphi(t+s,B)}(\varphi(s,x))\frac{1}{h_{t+s}(\varphi(-(t+s),\varphi(s,x)))}\Big)^p\,d\mu \\ &= \|S(s)\Big(\chi_{\varphi(t+s,B)}\frac{1}{h_{t+s}(\varphi(-(t+s),\cdot))}\Big)\|^p \\ &\leq Me^{sp\omega'}\|\chi_{\varphi(t+s,B)}\frac{1}{h_{t+s}(\varphi(-(t+s),\cdot))}\|^p \\ &= Me^{sp\omega'}\int_{\varphi(t+s,B)} \frac{1}{h_{t+s}^p(\varphi(-(t+s),\cdot))}\,d\mu = Me^{sp\omega'}\nu_{p,-(t+s)}(B).\end{aligned}$$

Setting $\omega := p\omega'$ gives the desired result. □

Theorem 6.42 *Under the general assumptions the following are equivalent.*

i) *S is a hypercyclic C_0-semigroup on $L^p(\mu)$*

ii) *$S(t)$ is a weakly mixing operator on $L^p(\mu)$ for every $t > 0$*

iii) *There is $t_0 > 0$ such that $S(t_0)$ is a weakly mixing operator on $L^p(\mu)$.*

PROOF: *i)* ⇒ *ii)* : Let $t > 0$ and K be a compact subset of X. Since $\varphi(t/2,\cdot)$ is continuous $\varphi(t/2,K)$ is compact so that from the hypercyclicity of S we get from theorem 6.33 a sequence of measurable subsets $(\tilde{L}_n)_{n\in\mathbb{N}}$ of $\varphi(t/2,K)$ and a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty}\mu(\varphi(t/2,K)\backslash\tilde{L}_n) = 0$ and $\lim_{n\to\infty}\nu_{p,t_n}(\tilde{L}_n) = \nu_{p,-t_n}(\tilde{L}_n) = 0$.

From the injectivity of $\varphi(t/2,\cdot)$ we get that $L_n := \varphi(t/2,\cdot)^{-1}(\tilde{L}_n)$ is a measurable subset of K satisfying

$$\begin{aligned}\lim_{n\to\infty}\mu(K\backslash L_n) &= \lim_{n\to\infty}\int_{K\backslash L_n} h_{t/2}^p/h_{t/2}^p\,d\mu \leq \lim_{n\to\infty} C\int_{K\backslash L_n} h_{t/2}^p\,d\mu \\ &= \lim_{n\to\infty} C\int_{\varphi(t/2,\cdot)^{-1}(\varphi(t/2,K)\backslash\tilde{L_n})} h_{t/2}^p\,d\mu \\ &= \lim_{n\to\infty} C\nu_{p,t/2}(\varphi(t/2,K)\backslash\tilde{L_n}) \\ &= \lim_{n\to\infty} C\int_{\varphi(t/2,\cdot)^{-1}(\varphi(t/2,K)\backslash\tilde{L_n})} f_{p,t/2}\,d\mu \\ &\leq \lim_{n\to\infty} C\|f_{p,t/2}\|_\infty\mu(\varphi(t/2,K)\backslash\tilde{L_n}) = 0,\end{aligned}$$

where we used that $1/h_{t/2} \in L^\infty_{loc}(\mu)$.

Let M and ω be as in lemma 6.41. Now, for every $n \in \mathbb{N}$ there is a (unique) $j_n \in \{k - 1/2;\, k \in \mathbb{N}_0\}$ such that $j_n t < t_n \le (j_n + 1)t$. In particular, $l_n := j_n + 1/2$ belongs to $\mathbb{N}_0$. Note that from the injectivity of $\varphi(l_n t, \cdot)$ and the fact that $h_{t/2}^{-p}$ is bounded by some constant D_t on K we have $h_{t/2}(\varphi(-l_n t, \cdot))^{-p}\chi_{\varphi(l_n t, K)} \le D_t$. From this and $h_{j_n t + t/2}(x) = h_{t/2}(x) h_{j_n t}(\varphi(t/2, x))$ we get using $\tilde{L}_n \subseteq \varphi(t/2, K)$, $l_n = j_n + 1/2$ and $L_n = \varphi(t/2, \cdot)^{-1}(\tilde{L}_n)$

$$\begin{aligned}
\nu_{p,-l_n t}(L_n) &= \int_{\varphi(l_n t, L_n)} h_{l_n t}(\varphi(-l_n t, \cdot))^{-p}\, d\mu \\
&= \int_{\varphi(j_n t, \tilde{L}_n)} h_{t/2}(\varphi(-l_n t, \cdot))^{-p} h_{j_n t}(\varphi(-l_n t, \varphi(t/2, \cdot)))^{-p}\, d\mu \\
&= \int_{\varphi(j_n t, \tilde{L}_n)} \chi_{\varphi(j_n t, \varphi(t/2, K))} h_{t/2}(\varphi(-l_n t, \cdot))^{-p} h_{j_n t}(\varphi(-j_n t, \cdot))^{-p}\, d\mu \\
&= \int_{\varphi(j_n t, \tilde{L}_n)} \chi_{\varphi(l_n t, K)} h_{t/2}(\varphi(-l_n t, \cdot))^{-p} h_{j_n t}(\varphi(-j_n t, \cdot))^{-p}\, d\mu \\
&\le D_t \int_{\varphi(j_n t, \tilde{L}_n)} h_{j_n t}(\varphi(-j_n t, \cdot))^{-p}\, d\mu \\
&= D_t \nu_{p,-j_n t}(\tilde{L}_n) \le D_t M e^{t\omega} \nu_{p,-t_n}(\tilde{L}_n),
\end{aligned}$$

where we used the second inequality of lemma 6.41 and $t_n - j_n t < t$ in the last step. This shows that $\lim_{n\to\infty} \nu_{p,-l_n t}(L_n) = 0$.

Again, from the injectivity of $\varphi(l_n t, \cdot)$ and the fact that $h_{t/2}^{-p}$ is bounded by D_t on K it follows that $\chi_{\varphi(l_n t, \cdot)^{-1}(K)}(x) h_{t/2}(\varphi(l_n t, x))^{-p} \le D_t$. Using this, $h_{(j_n+1)t}(x) = h_{l_n t}(x) h_{t/2}(\varphi(l_n t, x))$, $L_n = \varphi(t/2, \cdot)^{-1}(\tilde{L}_n)$ and $L_n \subset K$ we get

$$\begin{aligned}
\nu_{p,l_n t}(L_n) &= \int_{\varphi(l_n t, \cdot)^{-1}(L_n)} h_{l_n t}^p\, d\mu = \int_{\varphi(l_n t, \cdot)^{-1}(L_n)} \chi_{\varphi(l_n t, \cdot)^{-1}(K)} h_{l_n t}^p\, d\mu \\
&= \int_{\varphi((j_n+1)t, \cdot)^{-1}(\tilde{L}_n)} \frac{\chi_{\varphi(l_n t, \cdot)^{-1}(K)}(x)}{h_{t/2}(\varphi(l_n t, x))^p} h_{(j_n+1)t}^p(x)\, d\mu(x) \\
&\le D_t \int_{\varphi((j_n+1)t, \cdot)^{-1}(\tilde{L}_n)} h_{(j_n+1)t}^p\, d\mu = D_t \nu_{p,(j_n+1)t}(\tilde{L}_n) \\
&\le D_t M e^{\omega t} \nu_{p,t_n}(L_n),
\end{aligned}$$

where we used the first inequality of lemma 6.41 and $(j_n + 1)t - t_n < t$ in the last step. This shows $\lim_{n\to\infty} \nu_{p,t_n}(L_n) = 0$, too. So, by theorem 5.23 it follows that $S(t)$ is a weakly mixing operator on $L^p(\mu)$.

Clearly, ii) implies iii) and iii) implies i), so that the theorem holds true. □

Theorem 6.43 *Let ρ be a C_0-admissible weight function such that $\inf_{x\in K} \rho(x) > 0$ for every compact subset K of Ω. Under the general assumptions, the following are equivalent.*

i) S is a hypercyclic C_0-semigroup on $C_{0,\rho}(\Omega)$

ii) $S(t)$ is a weakly mixing operator on $C_{0,\rho}(\Omega)$ for every $t > 0$

iii) There is $t_0 > 0$ such that $S(t_0)$ is a weakly mixing operator on $C_{0,\rho}(\Omega)$.

PROOF: Clearly, all we have to show is that i) implies ii). So let $t > 0$ and K be a compact subset of Ω. Then, $K \cup \varphi(t, K)$ is compact so that by the hypercyclicity of S it follows from theorem 6.37 that there is a sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \sup_{x\in\varphi(t_n,K\cup\varphi(t,K))} \frac{\rho(x)}{h_{t_n}(\varphi(-t_n,x))} = \lim_{n\to\infty} \sup_{x\in\varphi(t_n,\cdot)^{-1}(K\cup\varphi(t,K))} h_{t_n}(x)\rho(x) = 0.$$

For all $n \in \mathbb{N}$ let $j_n \in \mathbb{N}$ be such that $j_n t \le t_n < (j_n+1)t$. Since S is a C_0-semigroup on $C_{0,\rho}(\Omega)$ it follows from theorem 6.32 that there are $M \ge 1, \omega \in \mathbb{R}$ such that $h_r(x)\rho(x) \le Me^{r\omega}\rho(\varphi(r,x))$ for all $r \ge 0, x \in \Omega$. Since S is hypercyclic it follows that $\omega > 0$. Applying this for $r = t_n - j_n t \le t$ we obtain

$$h_{t_n-j_nt}(\varphi((j_n+1)t,x))\rho(\varphi((j_n+1)t,x)) \le Me^{t\omega}\rho(\varphi(t_n,\varphi(t,x)))$$

for all $x \in \Omega$ and $n \in \mathbb{N}$.

This inequality together with $h_{r+s}(x) = h_r(x)h_s(\varphi(r,x))$ implies

$$\begin{aligned}\frac{\rho(\varphi((j_n+1)t,x))}{h_{(j_n+1)t}(x)} &\le \frac{Me^{t\omega}\rho(\varphi(t_n,\varphi(t,x)))}{h_{(j_n+1)t}(x)h_{t_n-j_nt}(\varphi((j_n+1)t,x))}\\ &= \frac{Me^{t\omega}\rho(\varphi(t_n,\varphi(t,x)))}{h_{t_n+t}(x)}\\ &= \frac{Me^{t\omega}\rho(\varphi(t_n,\varphi(t,x)))}{h_t(x)h_{t_n}(\varphi(t,x))}\end{aligned}$$

for every $x \in \Omega$ and $n \in \mathbb{N}$. Since h_t is continuous and positive it follows that there is $C_t < \infty$ such that $1/h_t$ in bounded above by C_t on K. From all this follows

$$\begin{aligned}\sup_{x\in K}\frac{\rho(\varphi((j_n+1)t,x))}{h_{(j_n+1)t}(x)} &\le Me^{t\omega}\sup_{x\in K}\frac{\rho(\varphi(t_n,\varphi(t,x)))}{h_t(x)h_{t_n}(\varphi(t,x))}\\ &\le C_tMe^{t\omega}\sup_{x\in K}\frac{\rho(\varphi(t_n,\varphi(t,x)))}{h_{t_n}(\varphi(t,x))}\\ &= C_tMe^{t\omega}\sup_{x\in\varphi(t,K)}\frac{\rho(\varphi(t_n,x))}{h_{t_n}(x)}\\ &\le C_tMe^{t\omega}\sup_{x\in K\cup\varphi(t,K)}\frac{\rho(\varphi(t_n,x))}{h_{t_n}(x)}\\ &= C_tMe^{t\omega}\sup_{x\in\varphi(t_n,K\cup\varphi(t,K))}\frac{\rho(x)}{h_{t_n}(\varphi(-t_n,x))}.\end{aligned}$$

The last term in the above inequality tends to 0 as n tends to infinity, so that

$$\lim_{n\to\infty}\sup_{x\in\varphi((j_n+1)t,K)}\frac{\rho(x)}{h_{(j_n+1)t}(\varphi(-(j_n+1)t,x))} = 0.$$

On the other hand, for $y = \varphi(-(j_n+1)t,x) \in \varphi((j_n+1)t,\cdot)^{-1}(K)$ we have using $h_{r+s}(z) = h_r(z)h_s(\varphi(r,z))$ for $r = t_n, s = (j_n+1)t - t_n$ and $z =$

$\varphi(-(j_n+1)t,x)$

$$\begin{aligned}
&h_{(j_n+1)t}(\varphi(-(j_n+1)t,x)\rho(\varphi(-(j_n+1)t,x)))\\
=\ & h_{t_n}(\varphi(-t_n,x))h_{(j_n+1)t-t_n}(\varphi(-(j_n+1)t,x))\rho(\varphi(-(j_n+1)t,x))\\
\leq\ & h_{t_n}(\varphi(-t_n,x))Me^{((j_n+1)t-t_n)\omega}\rho(\varphi(-(j_n+1)t,\varphi((j_n+1)t-t_n,x)))\\
\leq\ & Me^{t\omega}h_{t_n}(\varphi(-t_n,x))\rho(\varphi(-t_n,x)).
\end{aligned}$$

This inequality finally yields

$$\begin{aligned}
&\sup_{y\in\varphi((j_n+1)t,\cdot)^{-1}(K)} h_{(j_n+1)t}(y)\rho(y) \leq \sup_{x\in K\cap\varphi(t_n,\Omega)} Me^{t\omega}h_{t_n}(\varphi(-t_n,x))\rho(\varphi(-t_n,x))\\
\leq\ & \sup_{x\in(K\cup\varphi(t,K))\cap\varphi(t_n,\Omega)} Me^{t\omega}h_{t_n}(\varphi(-t_n,x))\rho(\varphi(-t_n,x))\\
=\ & \sup_{y\in\varphi(t_n,\cdot)^{-1}(K\cup\varphi(t,K))} Me^{t\omega}h_{t_n}(y)\rho(y).
\end{aligned}$$

The right hand side of the above inequality tends to 0 as n tends to infinity so that

$$\lim_{n\to\infty}\sup_{y\in\varphi((j_n+1)t,\cdot)^{-1}(K)} h_{(j_n+1)t}(y)\rho(y)=0.$$

Since K was an arbitrary compact subset of Ω and $((j_n+1))_{n\in\mathbb{N}}$ is a sequence of positive integers it follows from corollary 5.29 applied to the single weighted composition operator $S(t)$ that $S(t)$ is weakly mixing. □

6.3 Non-autonomous Cauchy problems

The results of chapter 5 can also be used to characterise when the evolution family $U=(U(t,s))_{s\in\mathbb{R},t\geq s}$ of non-autonomous Cauchy problems of the form

$$\begin{aligned}
\frac{\partial}{\partial t}u(t,s,x) &= \langle F(t,x),\nabla_x u(t,s,x)\rangle + h(t,x)u(t,s,x), t\geq s, x\in\Omega\\
u(s,s,x) &= u_s(x), s\in\mathbb{R},
\end{aligned}$$

is hypercyclic, where again $\Omega\subset\mathbb{R}^d$ is open, $F:\mathbb{R}\times\Omega\to\Omega$ is locally Lipschitz continuous with respect to x and $h:\mathbb{R}\times\Omega\to\mathbb{R}$ is continuous.

Recall, that a mapping $U:\{(t,s)\in\mathbb{R}^2;t\geq s\}\to E$, where E is again a Banach space, is called an *evolution family*, if it satisfies

i) $U(s,s)=id_E$ for all $s\in\mathbb{R}$

ii) $U(t,r)\circ U(r,s)=U(t,s)$ for all $s\leq r\leq t$

iii) U is continuous when we equip $L(E)$ with the strong operator topology.

Furthermore, let $D(A(t))\subset E$ be dense subspaces and $A(t):D(A(t))\to E$ be linear mappings, $t\in\mathbb{R}$. The evolution family U is said to *solve the non-autonomous Cauchy problem*

$$\text{(nCP)}\qquad \begin{aligned}
\frac{d}{dt}u(t) &= A(t)u(t)\\
u(s) &= x,\ x\in E, t\geq s
\end{aligned}$$

(on the spaces E_t) if there are dense subspaces $(E_t)_{t\in\mathbb{R}}$ of E such that $U(t,s)E_s \subset E_t \subset D(A(t)), t \geq s$, and the function $t \mapsto U(t,s)x$ solves (nCP) for fixed $s \geq 0, x \in E_s$ (cf. [24, Chapter VI.9]).

Now, let Ω again be an open subset of $\mathbb{R}^d$ and let $F : \mathbb{R} \times \Omega \to \Omega$ be locally Lipschitz continuous with respect to x and such that for every $s \in \mathbb{R}$ and every $x_0 \in \Omega$ the unique solution $\varphi(\cdot, s, x_0)$ of the initial value problem

$$\dot{x}(t) = F(t, x(t)),\ x(s) = x_0$$

exists on all $\mathbb{R}$.

The uniqueness of the solution implies that $\varphi(t,s,\cdot)$ is a bijective mapping on Ω and that its inverse mapping in given by $\varphi(s,t,\cdot)$.

Again, we call φ *continuously differentiable* if $\varphi(t,s,\cdot)$ is continuously differentiable for all $t \geq s, s \in \mathbb{R}$.

Furthermore, let $h : \mathbb{R} \times \Omega \to \mathbb{R}$ be a continuous function. We define $h_{t,s}(x) := \exp(\int_s^t h(r, \varphi(r,s,x))\,dr), t \geq s$, and observe that $h_{t,s}$ is measurable, real-valued and satisfies $1/h_{t,s} \in L^\infty_{loc}(\mu)$.

Let μ again be a locally finite Borel measure on Ω. We assume that the mapping $U(t,s)f := h_{t,s}(\cdot)f(\varphi(t,s,\cdot)), t \geq s$, is a well-defined linear operator on $L^p(\mu)$ for all $s \in \mathbb{R}$ and $t \geq s$. It is obvious, that $U(s,s)f = f$ for all $s \in \mathbb{R}$ and that $U(t,r)(U(r,s)f) = U(t,s)f$ for all $s \leq r \leq t$, so that U is an evolution family if and only if the mapping $(t,s) \mapsto U(t,s)f$ is continuous for every $f \in L^p(\mu)$.

For $s \in \mathbb{R}$ and $t \geq s$ we define the Borel measures

$$\nu_{p,(t,s)}(A) := \int_{\varphi(s,t,A)} h^p_{t,s}\,d\mu$$

and

$$\nu_{p,-(t,s)}(A) := \int_{\varphi(t,s,A)} 1/h^p_{t,s}(\varphi(s,t,\cdot))\,d\mu.$$

Theorem 6.44 *a) Under the general assumptions, the following are equivalent.*

i) U is hypercyclic on $L^p(\mu)$.

ii) For every compact subset K of Ω there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K and a sequence $((t_n,s_n))_{n\in\mathbb{N}}$ in $\{(u,v) \in \mathbb{R}^2; u \geq v\}$ such that $\lim_{n\to\infty} \mu(K\backslash L_n) = 0$ as well as

$$\lim_{n\to\infty} \nu_{p,(t_n,s_n)}(L_n) = \lim_{n\to\infty} \nu_{p,-(t_n,s_n)}(L_n) = 0.$$

b) Under the general hypotheses, the following are equivalent for fixed $s \in \mathbb{R}$.

i) $\{U(t,s); t \geq s\}$ is hypercyclic on $L^p(\mu)$.

ii) For every compact subset K of Ω there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K and a sequence $(t_n)_{n\in\mathbb{N}}$ in $[s,\infty)$ such that $\lim_{n\to\infty} \mu(K\backslash L_n) = 0$ as well as

$$\lim_{n\to\infty} \nu_{p,(t_n,s)}(L_n) = \lim_{n\to\infty} \nu_{p,-(t_n,s)}(L_n) = 0.$$

PROOF: Having in mind that $\varphi(t,s,\Omega)=\Omega$ and $\varphi(t,s,\cdot)^{-1}=\varphi(s,t,\cdot)$ for all $t,s\in R$ the theorem is a direct consequence of theorem 5.23. □

If ρ is a strictly positive, upper semicontinuous function such that $U(t,s)f := h_{t,s}(\cdot)f(\varphi(t,s,\cdot))$ defines an evolution family on $C_{0,\rho}(\Omega)$, the characterisation of when U is hypercyclic on $C_{0,\rho}(\Omega)$ can be handled analogously.

Theorem 6.45 *a) Under the general assumptions, the following are equivalent:*

i) U is hypercyclic on $C_{0,\rho}(\Omega)$.

ii) For every compact subset K of Ω there are a sequence $((t_n,s_n))_{n\in\mathbb{N}}$ in $\{(u,v)\in\mathbb{R}^2; u\geq v\}$ such that

$$\lim_{n\to\infty}\sup_{x\in\varphi(t_n,s_n,K)}\frac{\rho(x)}{h_{t_n,s_n}(\varphi(s_n,t_n,x))}=\lim_{n\to\infty}\sup_{x\in\varphi(s_n,t_n,K)}h_{t_n,s_n}(x)\rho(x)=0.$$

b) Under the general hypotheses, the following are equivalent for $s\in\mathbb{R}$:

i) $\{U(t,s); t\geq s\}$ is hypercyclic on $C_{0,\rho}(\Omega)$.

ii) For every compact subset K of Ω there is a sequence $(t_n)_{n\in\mathbb{N}}$ in $[s,\infty)$ such that

$$\lim_{n\to\infty}\sup_{x\in\varphi(s,t_n,K)}h_{t_n,s}(x)\rho(x)=\lim_{n\to\infty}\sup_{x\in\varphi(t_n,s,K)}\frac{\rho(x)}{h_{t_n,s}(\varphi(s,t_n,x))}=0.$$

Having in mind theorem 6.7 the following theorem is no surprise.

Theorem 6.46 *Let $E\in\{L^p(\mu),C_{0,\rho}(\Omega)\}$. Additionally to the general hypotheses, assume that U is an evolution family on E and that φ is continuously differentiable.*

For $t\in\mathbb{R}$ let $D(A(t)) := \{f\in E; \langle F(t,\cdot),\nabla_x f\rangle + h(t,\cdot)f\in E\}$ and $A(t): D(A(t))\to E, f\mapsto \langle F(t,\cdot),\nabla_x f\rangle + h(t,\cdot)f$, where $\nabla_x f$ is to be understood in the weak sense. Then U solves the non-autonomous Cauchy problem

$$\begin{aligned}\frac{d}{dt}u(t) &= A(t)u(t),\ t\geq s,\\ u(s) &= f_0\end{aligned}$$

on the spaces $E_t := C_c^1(\Omega), t\in\mathbb{R}$, in $L^p(\mu)$.

PROOF: We only prove the $L^p(\mu)$-case. The case $E=C_{0,\rho}(\Omega)$ is dealt with in an analogous way. Since μ is locally finite $C_c^1(\Omega)$ is dense in $L^p(\mu)$. From the fact that $\varphi(t,s,\cdot)$ is continuously differentiable it follows that it is a C^1-diffeomorphism of Ω onto itself so that $U(t,s)(C_c^1(\Omega))\subset C_c^1(\Omega)\subset D(A(t))$ for all $t\in\mathbb{R}$.

So, all that remains to be shown is that for every $f\in C_c^1(\Omega)$ and fixed $s\in\mathbb{R}$ the mapping $u(t) := U(t,s)f, t\geq s$, is differentiable and satisfies $\frac{d}{dt}u(t) = A(t)u(t)$. Let $f\in C_c^1(\Omega)$.

Then

$$U(t,s)(A(t)f) = h_{t,s}\langle F(t,\varphi(t,s,\cdot)),(\nabla f)(\varphi(t,s,\cdot))\rangle + h_{t,s}h(t,\varphi(t,s,\cdot))f(\varphi(t,s,\cdot)).$$

Since f has compact support we can use Lebesgue's theorem to conclude that

$$\begin{aligned}
&\lim_{r\to 0}\int_\Omega |1/r(U(t+r,s)f - U(t,s)f) - U(t,s)(A(t)f)|^p \, d\mu \\
= \;& \int_\Omega \lim_{r\to 0} |1/r(h_{t+r,s}(x)f(\varphi(t+r,s,x)) - h_{t,s}(x)f(\varphi(t,s,x))) \\
&-h_{t,s}(x)\langle F(t,\varphi(t,s,x)), (\nabla f)(\varphi(t,s,x))\rangle - h_{t,s}(x)h(t,\varphi(t,s,x))f(\varphi(t,s,x))|^p \, d\mu(x) \\
= \;& \int_\Omega h_{t,s}(x)f(\varphi(t,s,x))h(t,\varphi(t,s,x)) + h_{t,s}(x)\langle(\nabla f)(\varphi(t,s,x)), F(t,\varphi(t,s,x))\rangle \\
&-h_{t,s}(x)\langle F(t,\varphi(t,s,x), (\nabla f)(\varphi(t,s,x)\rangle - h_{t,s}(x)h(t,\varphi(t,s,x))f(\varphi(t,s,x))|^p \, d\mu(x) \\
= \;& 0,
\end{aligned}$$

so that $u : [s,\infty) \to L^p(\mu)$ is indeed differentiable with $u'(t) = U(t,s)(A(t)f)$. Now, since $U(t,s)f \in C_c^1(\Omega)$ it follows that for fixed $t \geq s$ the function

$$v : [t,\infty) \to L^p(\mu), r \mapsto U(r,t)(U(t,s)f)$$

is differentiable, too, and that its derivative at t is given by

$$v'(t) = U(t,t)(A(t)U(t,s)f) = A(t)(U(t,s)f).$$

Since $v(r) = u(r)$ for all $r \geq t$ this finally yields $u'(t) = v'(t) = A(t)(U(t,s)f) = A(t)u(t)$. □

Example 6.47 Let $\Omega = \mathbb{R}$, $F \equiv 1$, $h(t,x) := 2ct$ where $c \in \mathbb{R}$ and $\rho(x) := e^{-|x|}$. Then $\varphi(t,s,x) = x + (t-s)$, $h_{t,s}(x) = \exp(c(t^2 - s^2))$ and obviously $U(t,s)f := h_{t,s}f(\varphi(t,s,\cdot)$ defines an evolution family. $\nu_{p,(t,s)}$ has Lebesgue density $\exp(pc(t^2-s^2)-|\cdot-(t-s)|)$ and $\nu_{p,-(t,s)}$ has Lebesgue density $\exp(-pc(t^2-s^2) - |\cdot+(t-s)|)$ for all $s \in \mathbb{R}, t \geq s$.

For a bounded, measurable subset B of $\mathbb{R}$ with $\lambda(B) > 0$ and a sequence $(t_n)_{n\in\mathbb{N}}$ in $[s,\infty)$ we have $\lim_{n\to\infty} \nu_{p,(t_n,s)}(B) = 0$ if and only if $(t_n)_{n\in\mathbb{N}}$ converges to infinity and $pc \leq 0$ because for sufficiently large t_n

$$\nu_{p,(t_n,s)}(B) = \exp(pc(t_n^2 - s^2) - (t_n - s)) \int_B \exp(x)\, dx.$$

Furthermore, we have $\lim_{n\to\infty} \nu_{p,-(t_n,s)}(B) = 0$ if and only if $(t_n)_{n\in\mathbb{N}}$ converges to infinity and $pc \geq 0$ because for sufficiently large t_n

$$\nu_{p,-(t_n,s)}(B) = \exp(-pc(t_n^2 - s^2) - (t_n - s)) \int_B \exp(-x)\, dx.$$

From this and theorem 6.44 b) it follows that for fixed $s \in \mathbb{R}$ the family of operators $(U(t,s))_{t\geq s}$ is hypercyclic on $L^p_\rho(\mathbb{R})$ if and only if $c = 0$.

On the other hand, if $t > 0$ is sufficiently large and $s := -t$ we clearly get

$$\nu_{p,(t,-t)}(B) = \exp(-2t) \int_B \exp(x)\, dx$$

and

$$\nu_{p,-(t,-t)}(B) = \exp(-2t) \int_B \exp(-x)\, dx,$$

which both converge to 0 when t tends to infinity, so that by theorem 6.44 a) the family $(U(t,s))_{s\in\mathbb{R},t\geq s}$ is hypercyclic on $L^p(\mu)$ for all values of c.

Now, let $(t,s) \in \{(u,v) \in \mathbb{R}^2;\, u > v\}$ be fixed. It is easily verified that $U(t,s)^n f = \exp(cn(t^2-s^2))f(\cdot + n(t-s))$. Setting $h_n := \exp(cn(t^2-s^2))$ and $\varphi(n,x) = x + n(t-s)$ and adapting the notation from section 5 we get for sufficiently large n

$$\nu_{p,n}(B) = \exp(n(cp(t^2-s^2)-(t-s)))\int_B \exp(x)\,dx,$$

and

$$\nu_{p,-n}(B) = \exp(-n(cp(t^2-s^2)+(t-s)))\int_B \exp(-x)\,dx$$

so that for a sequence $(n_k)_{k\in\mathbb{N}}$ of natural numbers both $\nu_{p,n_k}(B)$ and $\nu_{p,-n_k}(B)$ converge to 0 as k tends to infinity if and only if $(n_k)_{k\in\mathbb{N}}$ converges to infinity and $cp(t^2-s^2)-(t-s) < 0$ and $cp(t^2-s^2)+(t-s) > 0$. From theorem 5.23 if follows that $U(t,s)$ is therefore a hypercyclic operator on $L^p_\rho(\mathbb{R})$ if and only if $|c(t+s)| < 1/p$.

So by fixing $c = 1$ this gives an example of an evolution family $(U(t,s))_{s\in\mathbb{R},t\geq s}$ on $L^p_\rho(\mathbb{R})$ which is hypercyclic but for which none of the families $(U(t,s))_{t\geq s}, s \in \mathbb{R}$ is hypercyclic and for which a single operator $U(t,s), t > s$, is hypercyclic on $L^p_\rho(\mathbb{R})$ if and only if $|t+s| < 1/p$.

7 Chaotic C_0-semigroups generated by first order partial differential operators

In this chapter we investigate chaoticity of C_0-semigroups generated by first order partial differential operators. Recall that a C_0-semigroup T, respectively an operator T, on a Banach space E is *chaotic* if it is transitive and if the set of periodic points, i.e. the set $\text{per}(T) := \{x \in E;\, \exists\, t > 0 : T(t)x = x\}$, respectively $\text{per}(T) := \{x \in E;\, \exists\, n \in \mathbb{N} : T^n x = x\}$, is dense in E.

Parts of this chapter will be published in *Ergodic Theory and Dynamical Systems* under the title "Hypercyclic, mixing, and chaotic C_0-semigroups induced by semiflows".

We adapt the notions and notation from chapter 5. In particular X is a locally compact Hausdorff topological space and ρ an upper semicontinuous, positive valued function on X. Again, if we are considering $L^p(\mu)$ spaces, where μ is a locally finite Borel measure on X, we additionally assume X to be σ-compact. The only difference to chapter 5 is that instead of I being an arbitrary set we now always assume $I \in \{[0,\infty), \mathbb{N}\}$. Moreover, we assume that for all $t, s \in I$ we have $\varphi(t+s,\cdot) = \varphi(t, \varphi(s,\cdot))$. Apart from these differences, we again assume $\varphi(t,\cdot)$ to be injective and continuous for all $t \in I$.

Furthermore, let $w : I \times X \to (0,\infty)$ be as in chapter 5, that is w is continuous when considering $C_{0,\rho}(X)$, or such that $1/w(t,\cdot) \in L^\infty_{loc}(\mu)$ for all $t \in I$ when dealing with $L^p(\mu)$, respectively. We assume that $w(t+s,\cdot) = w(t,\cdot)w(s,\varphi(t,\cdot))$ for all $t, s \in I$ and that the family of weighted composition operators $T_{w,\varphi}$ defined as in chapter 5 is a well-defined family of operators on $L^p(\mu)$ or $C_{0,\rho}(X)$, respectively. In particular, we have $T_{w,\varphi}(t+s) = T_{w,\varphi}(t) \circ T_{w,\varphi}(s)$ for all $t, s \in I$.

Remark 7.1 Note that if $T_{v,\psi}$ is a single weighted composition operator and if one defines for $n \in \mathbb{N}$ $\varphi(n,\cdot) := \psi \circ \cdots \circ \psi$ with n factors and $w(n,\cdot) := \prod_{j=0}^{n-1} v(\psi^j(\cdot))$ that then trivially $\varphi(n+m,\cdot) = \varphi(n, (\varphi(m,\cdot))$ and $w(n+m,\cdot) = w(n,\cdot)w(m,\varphi(n,\cdot))$ for all $n, m \in \mathbb{N}$. So, the above conditions are satisfied by $T_{w,\varphi}$ and obviously $T_{w,\varphi}(n) = T^n_{v,\psi}$ for all $n \in \mathbb{N}$.

Taking into account the characterisations of chaoticity for the left translation semigroup on $L^p_\rho(\mathbb{R})$ given in [41, 42], and the characterisation of chaoticity for weighted shifts given in [29] one might guess that a summability condition will enter. The next theorem shows that this is indeed the case, but we have to impose an extra condition on the mapping φ.

Recall that for $t \in I$

$$\nu_{p,t}(B) = \int_{\varphi(t,\cdot)^{-1}(B)} w(t,\cdot)^p \, d\mu$$

and

$$\nu_{p,-t}(B) = \int_{\varphi(t,B)} 1/w(\iota, \varphi(-t,\cdot))^p \, d\mu.$$

Furthermore, we have

$$\int f \, d\nu_{p,t} = \int w(t,\cdot)^p f(\varphi(t,\cdot)) \, d\mu$$

and

$$\int f\, d\nu_{p,-t} = \int \chi_{\varphi(t,X)} f(\varphi(-t,\cdot))/w(t,\varphi(\iota,\cdot))^p\, d\mu$$

for positive, measurable functions f on X.

Theorem 7.2 *Additional to the general assumptions, let φ be such that for every compact subset K of X there is $t_K > 0$ such that $\varphi(t,K) \cap K = \emptyset$ for every $t > t_K$. Then, the following are equivalent.*

i) $T_{w,\varphi}$ is chaotic on $L^p(\mu)$.

ii) per$(T_{w,\varphi})$ is dense in $L^p(\mu)$

iii) For every compact subset K of X there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K and a strictly increasing sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ tending to infinity such that $\lim_{n\to\infty} \mu(K\backslash L_n) = 0$, and $\lim_{n\to\infty} s_n = 0$, where

$$s_n := \sum_{l=1}^{\infty} \nu_{p,lt_n}(L_n) + \sum_{l=1}^{\infty} \nu_{p,-lt_n}(L_n).$$

Proof: Instead of $T_{w,\varphi}$ we simply write T. It is clear from the definition that i) implies ii). To show that ii) implies iii) let K be a compact subset of X. Since per(T) is dense, we can find a sequence $(v_n)_{n\in\mathbb{N}}$ of periodic points of T satisfying $\|\chi_K - v_n\| < 1/4^n$. Let $(t_n)_{n\in\mathbb{N}} \in (0,\infty)^{\mathbb{N}}$ be such that $T(t_n)v_n = v_n$, where we can assume without loss of generality that $t_{n+1} > t_n$ and $\varphi(rt_n,K)\cap\varphi(st_n,K) = \emptyset$ for all $n \in \mathbb{N}$ and $r,s \in \mathbb{Z}, r \neq s$.

From $\|\chi_K - v_n\| < 1/4^n$ it follows that for $L_n := K \cap \{|1 - v_n| \leq 1/2^n\}$ we have $\mu(K\backslash L_n) \leq 1/2^{np}$, so that $\lim_{n\to\infty} \mu(K\backslash L_n) = 0$. Since $T(t_n)v_n = v_n$ we obtain

$$\begin{aligned}
1/4^{np} \quad > \quad & \|\chi_K - v_n\|^p = \int |\chi_K - v_n|^p\, d\mu \geq \int_{X\backslash K} |v_n|^p\, d\mu \\
\geq \quad & \sum_{l=1}^{\infty} \int_{\varphi(-lt_n,K)} |v_n|^p\, d\mu + \sum_{l=1}^{\infty} \int_{\varphi(lt_n,K)} |v_n|^p\, d\mu \\
= \quad & \sum_{l=1}^{\infty} \int_{\varphi(-lt_n,K)} |T(lt_n)v_n|^p\, d\mu \\
& + \sum_{l=1}^{\infty} \int_{\varphi(lt_n,K)} |\frac{w(lt_n,\varphi(-lt_n,\cdot))}{w(lt_n,\varphi(-lt_n,\cdot))} v_n(\varphi(lt_n,\varphi(-lt_n,\cdot))|^p\, d\mu \\
= \quad & \sum_{l=1}^{\infty} \int w(lt_n,\cdot)^p |v_n(\varphi(lt_n,\cdot)|^p \chi_K(\varphi(lt_n,\cdot))\, d\mu \\
& + \sum_{l=1}^{\infty} \int |\frac{w(lt_n,\varphi(-lt_n,\cdot)) v_n(\varphi(lt_n,\varphi(-lt_n,\cdot)))}{w(lt_n,\varphi(-lt_n,\cdot))}|^p \chi_K(\varphi(-lt_n,\cdot))\, d\mu \\
= \quad & \sum_{l=1}^{\infty} \int |v_n|^p \chi_K\, d\nu_{p,lt_n} + \sum_{l=1}^{\infty} \int |w(lt_n,\cdot) v_n(\varphi(lt_n,\cdot))|^p \chi_K\, d\nu_{p,-lt_n} \\
\geq \quad & \sum_{l=1}^{\infty} \int_{L_n} |v_n|^p \chi_K\, d\nu_{p,lt_n} + \sum_{l=1}^{\infty} \int_{L_n} |T(t_n)v_n|^p\, d\nu_{p,-lt_n}
\end{aligned}$$

$$\geq \quad (1-1/2^n)^p \sum_{l=1}^{\infty} \nu_{p,lt_n}(L_n) + (1-1/2^n)^p \sum_{l=1}^{\infty} \nu_{p,-lt_n}(L_n),$$

where we used in the last inequality that $1/2^n \geq |1-v_n|$ on L_n, that is $|v_n| \geq v_n \geq 1-1/2^n$, so that iii) follows.

To show that iii) implies i) note that it follows from theorem 5.23 that T is topologically transitive, so it remains to prove that per(T) is dense in $L^p(\mu)$. To this end, we choose $f \in C_c(X)$ and a compact subset K of X containing supp f. Let $(L_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ be as in iii) for K, where we can assume without loss of generality, that $t_1 > t_K$.

Thus, $(f(\varphi(lt_n,\cdot))\chi_{\varphi(-lt_n,L_n)})_{l\in\mathbb{Z}}$ is a locally finite sequence of functions, so that

$$\begin{aligned} f_n \quad := \quad & f\chi_{L_n} + \sum_{l=1}^{\infty} w(lt_n,\cdot) f(\varphi(lt_n,\cdot))\chi_{\varphi(-lt_n,L_n)} \\ & + \sum_{l=1}^{\infty} \frac{1}{w(lt_n,\varphi(-lt_n,\cdot))} f(\varphi(-lt_n,\cdot))\chi_{\varphi(lt_n,L_n)} \end{aligned}$$

is well-defined and measurable.

From

$$\begin{aligned} \|f\|_\infty^p \nu_{p,lt_n}(L_n) \quad & \geq \quad \int |f|^p \chi_{L_n}\, d\nu_{p,lt_n} \\ & = \quad \int w(lt_n,\cdot)|f(\varphi(lt_n,\cdot))|^p \chi_{L_n}(\varphi(lt_n,\cdot))\, d\mu \end{aligned}$$

and

$$\begin{aligned} \|f\|_\infty^p \nu_{p,-lt_n}(L_n) \quad & \geq \quad \int_{L_n} |f|^p\, d\nu_{p,-lt_n} \\ & = \quad \int_{\varphi(lt_n,L_n)} |f(\varphi(-lt_n,\cdot))|^p (\varphi(-lt_n,\cdot))/w(lt_n,\varphi(-lt_n,\cdot))^p\, d\mu \end{aligned}$$

it follows using the properties of $(L_n)_{n\in\mathbb{N}}$ and $(t_n)_{n\in\mathbb{N}}$ that $f_n \in L^p(\mu)$ and $\lim_{n\to\infty} f_n = f$.

It remains to show that $f_n \in$ per(T). Because of $w(t+s,\cdot) = w(t,\cdot)w(s,\varphi(t,\cdot))$ for every $s,t \in I$, it is a straight forward calculation to show that $T(t_n)f_n = f_n$.□

As a direct consequence we obtain for the "unweighted" family T_φ induced by φ the following result.

Corollary 7.3 *Additionally to the general assumptions, let φ be such that for every compact subset K of X there is $t_K > 0$ such that $\varphi(t,K)\cap K = \emptyset$ for every $t > t_K$. Then, the following are equivalent.*

i) T_φ is chaotic on $L^p(\mu)$.

ii) per(T_φ) is dense in $L^p(\mu)$.

iii) For every compact subset K of X there are a sequence of measurable subsets $(L_n)_{n\in\mathbb{N}}$ of K and a strictly increasing sequence of positive numbers

$(t_n)_{n\in\mathbb{N}}$ tending to infinity such that $\lim_{n\to\infty}\mu(K\backslash L_n) = 0$ as well as $\lim_{n\to\infty} s_n = 0$, where $s_n := \sum_{l\in\mathbb{Z}\backslash\{0\}}^{\infty}\mu_{lt_n}(L_n)$ and μ_{lt_n} is defined as in section 5.1.

Remark 7.4 i) If N is a closed subset of X with $\varphi(t, N) = N$ for every $t \in I$, and $\mu(N) = 0$ then clearly $L^p(X, \mu) = L^p(X\backslash N, \mu)$. So, we can weaken the extra condition $\varphi(t, K)\cap K = \emptyset$ for every compact subset K of X and sufficiently large t in theorem 7.2 to: there is a closed μ-zero set N with $\varphi(t, N) = N$ for every $t > 0$ such that for every compact subset K of $X\backslash N$ and sufficiently large t $\varphi(t, K) \cap K = \emptyset$.

ii) In case of the C_0-semigroup S on $L^p(\mu)$ from section 6.2, that is

$$S(t)f := \exp\Big(\int_0^t h(\varphi(r, \cdot))\, dr\Big) f(\varphi(t, \cdot)),\ t \geq 0,$$

the above theorem characterises chaoticity of S provided that for every compact subset K of Ω there is $t_K > 0$ such that $\varphi(t, K) \cap K = \emptyset$ for every $t > t_K$.

Example 7.5 In example 6.25 we have seen that the C_0-semigroup T induced by the semiflow

$$\varphi : [0, \infty) \times \Omega \to \Omega, (t, x) \mapsto \frac{1}{\sqrt{|x|^2 + (1 - |x|^2)e^{-2t}}}x$$

on $\Omega := \{x \in \mathbb{R}^d; |x|^2 < 1\}$ is hypercyclic in $L^p_\rho(\Omega)$ where $\rho(x) := 1 - |x|^2$. We also showed in 6.25 that

$$\frac{\rho(\varphi(t, x))}{\rho(x)}|\det D\varphi(t, x)| \leq \frac{e^{-4t}}{(r^2 + (1 - R^2)e^{-2t})^{\frac{d}{2}+2}}$$

for all $0 < r < |x| < R$ and $t \in \mathbb{R}$.

Thus, it follows that for every compact subset K of $\Omega\backslash\{0\}$ there is a constant C_K such that for $a := \min\{4, d\}$ both

$$\sup_{x\in K}\frac{\rho(\varphi(t, x))}{\rho(x)}|\det D\varphi(t, x)|$$

and

$$\sup_{x\in K}\frac{\rho(\varphi(-t, x))}{\rho(x)}|\det D\varphi(-t, x)|$$

are bounded by $C_K e^{-at}$ for all $t > 0$.

From this and proposition 6.9 we conclude that $\mu_t(K) \leq C_K e^{-at}$ and $\mu_{-t}(K) \leq C_K e^{-at}$ for all $t > 0$. Choosing $L_n := K$ and $t_n := n$, we find that

$$s_n := \sum_{l\in\mathbb{Z}\backslash\{0\}} \mu_{ln}(K) \leq 2C_K \frac{e^{-2an}}{1 - e^{-2an}}$$

which converges to zero as n tends to infinity. Now, since for every $x \in \Omega\backslash\{0\}$ we have $\lim_{t\to\infty} |\varphi(t, x)| = 1$, it follows that $\varphi(t, K) \cap K = \emptyset$ for sufficiently large t. From corollary 7.3 and remark 7.4 i) we conclude that T is a chaotic C_0-semigroup on $L^p_\rho(\Omega)$.

A similar argument shows the chaoticity of the examples 6.21, 6.23 and 6.35.

In order to give a characterisation of chaoticity of $T_{w,\varphi}$ on $C_{0,\rho}(X)$, we have to provide the following lemma.

Lemma 7.6 *Let φ be such that for every compact subset K of X there is a positive number s_K satisfying $\varphi(t,K)\cap K=\emptyset$ for every $t>t_K$.*

Then, for every compact subset K of X and every $f\in C_{0,\rho}(X)$ we have

$$\lim_{t\to\infty}\sup_{x\in\varphi(t,K)}|f(x)|\rho(x)=\lim_{t\to\infty}\sup_{x\in\varphi(t,\cdot)^{-1}(K\cap\varphi(t,X))}|f(x)|\rho(x)=0.$$

PROOF: Let K be a compact subset of X, $f\in C_{0,\rho}(X)$, and let $\varepsilon>0$. Then, the set $K_\varepsilon:=\{x\in X;\ |f(x)|\rho(x)\geq\varepsilon\}$ is compact, as is $K\cup K_\varepsilon$. By the hypotheses, there is a positive number t_ε such that $\varphi(t,K\cup K_\varepsilon)\cap(K\cup K_\varepsilon)=\emptyset$, which is equivalent to $\varphi(t,\cdot)^{-1}(K\cup K_\varepsilon)\cap(K\cup K_\varepsilon)=\emptyset$, for every $t>t_\varepsilon$. In particular, for every $x\in\varphi(t,K)\cup\varphi(t,\cdot)^{-1}(K)$ we have $x\notin K_\varepsilon$ whenever $t>t_\varepsilon$, that is $|f(x)|\rho(x)<\varepsilon$. □

Theorem 7.7 *Additionally to the general assumptions let ρ satisfy $\inf_{x\in K}\rho(x)>0$ for every compact subset K of X. Furthermore, assume that for every compact subset K of X there is a positive number t_K satisfying $\varphi(t,K)\cap K=\emptyset$ for every $t>t_K$.*

Then, among the following, i) implies ii) and ii) implies iii).

i) $T_{w,\varphi}$ is chaotic on $C_{0,\rho}(X)$.

ii) per($T_{w,\varphi}$) is dense in $C_{0,\rho}(X)$.

iii) For every compact subset K of X there is $p>0$ such that

$$\lim_{n\to\infty}\sup_{x\in\varphi(np,K)}\frac{\rho(x)}{w(np,\varphi(-np,x))}=\lim_{n\to\infty}\sup_{x\in\varphi(np,\cdot)^{-1}(K)}w(np,x)\rho(x)=0.$$

Moreover, if $\varphi(t,\cdot)$ is an open mapping for every $t\in I$, the above are equivalent.

PROOF: For brevity we write T instead of $T_{w,\varphi}$. Obviously i) implies ii). To show that ii) implies iii) let K be a compact subset of X and $\varepsilon\in(0,\inf_{x\in K}\rho(x)/2)$. Let U_K be a relatively compact, open neighbourhood of K and t_K be such that $\varphi(t,\overline{U_K})\cap\overline{U_K}=\emptyset$ for $t\geq t_K$. Let $f\in C_c(X)$ be positive with $f_{|K}\equiv 1$ and $f_{|\Omega\setminus U_K}\equiv 0$.

Since per($T_{w,\varphi}$) is dense in $C_{0,\rho}(X)$ it follows that there are $v\in C_{0,\rho}(X)$ and $p>0$ such that $T(p)v=v$ and $\varepsilon>\|f-v\|$, where we can assume without loss of generality that $p>t_K$.

Because of $\varepsilon>\|f-v\|$ and $\varepsilon\in(0,\inf_{x\in K}\rho(x)/2)$ it follows that $v(x)\geq 1/2$ for all $x\in K$. Using $T(np)v=w(np,\cdot)v(\varphi(np,\cdot))=v$ this implies

$$\begin{aligned}&\forall n\in\mathbb{N},x\in\varphi(np,K):\ 1/2\leq w(np,\varphi(-np,x))v(x)\\&\forall n\in\mathbb{N},x\in\varphi(np,\cdot)^{-1}(K):\ 1/2\leq v(\varphi(np,x))=\frac{v(x)}{w(np,x)}.\end{aligned}$$

Thus, we have for every $n\in\mathbb{N}$

$$\sup_{x\in\varphi(np,\cdot)^{-1}(\overline{U_K})}|v(x)|\rho(x)\quad\geq\quad\sup_{x\in\varphi(np,\cdot)^{-1}(K)}|v(x)|\rho(x)$$

$$
\begin{aligned}
&= \sup_{x\in\varphi(np,\cdot)^{-1}(K)} h_{np}(x)|\frac{v(x)}{h_{np}(x)}|\rho(x)\\
&\geq 1/2 \sup_{x\in\varphi(np,\cdot)^{-1}(K)} h_{np}(x)\rho(x)
\end{aligned}
$$

and

$$
\begin{aligned}
\sup_{x\in\varphi(np,\overline{U_K})} |v(x)|\rho(x) &\geq \sup_{x\in\varphi(np,K)} |v(x)|\rho(x)\\
&= \sup_{x\in\varphi(np,K)} h_{np}(\varphi(-np,x))|v(x)|\frac{\rho(x)}{h_{np}(\varphi(-np,x))}\\
&\geq 1/2 \sup_{x\in\varphi(np,K)} \frac{\rho(x)}{h_{np}(\varphi(-np,x))}.
\end{aligned}
$$

Lemma 7.6 applied to $\overline{U_K}$ now yields

$$\lim_{n\to\infty} \sup_{x\in\varphi(np,\cdot)^{-1}(K)} w(np,x)\rho(x) = 0$$

as well as

$$\lim_{n\to\infty} \sup_{x\in\varphi(np,K)} \frac{\rho(x)}{w(np,\varphi(-np,x))} = 0$$

so that iii) follows.

To show that under the additional assumption iii) implies i), note that by theorem 6.37 T is transitive. To show that $\mathrm{per}(T)$ is dense in $C_{0,\rho}(X)$ let $f \in C_c(X)$ and $K := \operatorname{supp} f$. Let p be as in iii) for K where without loss of generality we can assume $p > t_K$. As in the proof of theorem 5.16 we can understand $f(\varphi(-np,\cdot))$ as an element of $C_c(X)$ for every $n \in \mathbb{N}_0$ so that by $p > t_K$ the family $(f(\varphi(np,\cdot)))_{n\in\mathbb{Z}}$ is in particular locally finite and in $C(X)$. Therefore,

$$v_m := f + \sum_{l=1}^{\infty} w(lmp,\cdot)f(\varphi(lmp,\cdot)) + \sum_{l=1}^{\infty} \frac{f(\varphi(-lmp,\cdot))}{w(lmp,\varphi(-lmp,\cdot))}$$

is a continuous function on X and using $w(t+s,\cdot) = w(t,\cdot)w(s,\varphi(t,\cdot)$ one calculates $T(mp)v_m = v_m$.

Because of $\varphi(lmp,K)\cap\varphi(lnp,K) = \emptyset$ for every $n,m \in \mathbb{Z}, n \neq m$ we have

$$
\begin{aligned}
&\sup_{x\in X} |f(x) - v_m(x)|\rho(x)\\
&= \sup_{x\in X} |\sum_{l=1}^{\infty} w(lmp,x)f(\varphi(lmp,x)) + \sum_{l=1}^{\infty} \frac{f(\varphi(-lmp,x))}{w(lmp,x)(\varphi(-lmp,x))}|\rho(x)\\
&= \max\{\sup_{l\in\mathbb{N}} \sup_{x\in\varphi(lmp,\cdot)^{-1}(K)} |w(lmp,x)f(\varphi(lmp,x))|\rho(x),\\
&\qquad \sup_{l\in\mathbb{N}} \sup_{x\in\varphi(lmp,K)} |\frac{f(\varphi(-lmp,x))}{w(lmp,\varphi(-lmp,x))}|\rho(x)\}\\
&\leq \|f\|_\infty\{\sup_{l\in\mathbb{N}} \sup_{x\in\varphi(lmp,\cdot)^{-1}(K)} w(lmp,\cdot)\rho(x)\\
&\qquad +\sup_{l\in\mathbb{N}} \sup_{x\in\varphi(lmp,K)} \frac{\rho(x)}{w(lmp,\varphi(-lmp,x))}\}
\end{aligned}
$$

which converges to 0 as m tends to infinity by our hypothesis iii).

So, all that remains to be shown is that v_m belongs to $C_{0,\rho}(X)$. Note that for all $l \in \mathbb{N}$ we have

$$f(\varphi(-lmp, \cdot))/(w(lmp, \varphi(-lmp, \cdot))) \in C_c(X)$$

and

$$w(lmp, \cdot)f(\varphi(lmp, \cdot)) = T(lmp)f,$$

i.e. all summands in the series defining v_m belong to $C_{0,\rho}(X)$. We will show that the series converges in $C_{0,\rho}(X)$ which then implies $v_m \in C_{0,\rho}(X)$.

As above, we have

$$\begin{aligned}
&\sup_{x \in X} \Big| \sum_{l=N}^{\infty} w(lmp, x) f(\varphi(lmp, x)) + \sum_{l=N}^{\infty} \frac{f(\varphi(-lmp, x))}{w(lmp, \varphi(-lmp, x))} \Big| \rho(x) \\
\leq\ & \|f\|_\infty \Big(\sup_{l \geq N} \sup_{x \in \varphi(lmp, \cdot)^{-1}(K)} w(lmp, x)\rho(x) \\
&\qquad + \sup_{l \geq N} \sup_{x \in \varphi(lmp, K)} \frac{\rho(x)}{w(lmp, \varphi(-lmp, x))} \Big),
\end{aligned}$$

which tends to 0 as N tends to infinity by iii) showing that

$$\begin{aligned}
v_m = \lim_{N \to \infty} f \quad &+ \quad \sum_{l=1}^{N} w(lmp, \cdot) f(\varphi(lmp, \cdot)) \\
&+ \quad \sum_{l=1}^{N} f(\varphi(-lmp, \cdot))/(w(lmp, \varphi(-lmp, \cdot)))
\end{aligned}$$

in $C_{0,\rho}(X)$. □

Remark 7.8 Using Brouwer's theorem in case of the C_0-semigroup S on $C_{0,\rho}(\Omega)$ from section 6.2, that is

$$S(t)f := \exp\Big(\int_0^t h(\varphi(r, \cdot))\, dr\Big) f(\varphi(t, \cdot)),\ t \geq 0,$$

the above theorem characterises chaoticity of S provided that for every compact subset K of Ω there is an open subset U_K of Ω containing K and a positive number t_K satisfying $\varphi(t, U_K) \cap U_K = \emptyset$ for every $t > t_K$.

Remark 7.9 In the situation of section 6.2, that is $X = \Omega \subset \mathbb{R}^d$ open, the Borel measure μ has a Lebesgue density ρ and one considers the C_0-semigroup S defined via

$$S(t)f = \exp\Big(\int_0^t h(\varphi(r, \cdot))\Big) f(\varphi(t, \cdot)), t \geq 0,$$

on $L^p_\rho(\Omega)$ or $C_{0,\rho}(\Omega)$ the above theorems characterising chaoticity of S under suitable additional conditions again can be formulated more conveniently, if the semiflow φ is continuously differentiable and ρ behaves tamely.

For $L^p_\rho(\Omega)$, if in theorem 6.19 one replaces

$$\varepsilon < \frac{\rho(\varphi(t, x))|\det D\varphi(t, x)|}{\rho(\varphi(t, y))|\det D\varphi(t, y)|} < 1/\varepsilon$$

by

$$\varepsilon < \frac{h_t(x)^{-p}\rho(\varphi(t,x))|\det D\varphi(t,x)|}{h_t(y)^{-p}\rho(\varphi(t,y))|\det D\varphi(t,y)|} < 1/\varepsilon,$$

respectively

$$\varepsilon < \frac{h_t(x)^{-p}\rho(\varphi(t,x))|\det D\varphi(t,x)|}{h_t(y)^{-p}\rho(\varphi(t,y))|\det D\varphi(t,y)|} < 1/\varepsilon$$

and

$$\varepsilon < \frac{h_t^p(\varphi(-t,x))\rho(\varphi(-t,x))|\det D\varphi(-t,x)|}{h_t^p(\varphi(-t,y))\rho(\varphi(-t,y))|\det D\varphi(-t,y)|} < 1/\varepsilon,$$

for every x, y in a compact set K then one can show by the same kind of arguments used in the proof of theorem 6.19 that the following are equivalent.

i) S is chaotic on $L^p_\rho(\Omega)$.

ii) $\mathrm{per}(S)$ is dense in $L^p_\rho(\Omega)$.

iii) For every $x \in \Omega$ there is a strictly increasing sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ such that for all $n \in \mathbb{N}$

$$s_n := \sum_{l\in\mathbb{N}} h_{lt}(x)^{-p}\rho(\varphi(lt,x))\det D\varphi(lt,x) < \infty \text{ and } \lim_{n\to\infty} s_n = 0$$

respectively

$$\begin{aligned} s_n &:= \sum_{l\in\mathbb{N}} h_{lt_n}(x)^{-p}\rho(\varphi(lt_n,x))\det D\varphi(lt_n,x) \\ &+ \sum_{l\in\mathbb{N}} h_{lt_n}(\varphi(lt_n,x))^p\rho(\varphi(-lt_n,x))\det D\varphi(-lt_n,x) < \infty \\ &\text{and } \lim_{n\to\infty} s_n = 0. \end{aligned}$$

For $C_{0,\rho}(\Omega)$ one has to replace

$$\frac{\rho(\varphi(t,x))}{\rho(\varphi(t,y))} < C$$

in theorem 6.20 by

$$\frac{\rho(\varphi(t,x))h_t(y)}{h_t(x)\rho(\varphi(t,y))} < C,$$

respectively

$$\frac{\rho(\varphi(t,x))h_t(y)}{h_t(x)\rho(\varphi(t,y))} < C \text{ and } \frac{\rho(\varphi(-t,x))h_t(\varphi(-t,y))}{h_t(\varphi(-t,x))\rho(\varphi(-t,y))} < C,$$

to obtain the equivalence of:

i) S is chaotic on $C_{0,\rho}(\Omega)$.

ii) $\mathrm{per}(S)$ is dense in $C_{0,\rho}(\Omega)$.

iii) For every $x \in \Omega$ there is $t > 0$ such that

$$\lim_{n\to\infty} \frac{\rho(\varphi(nt,x))}{h_{nt}(x)} = 0,$$

respectively

$$\lim_{n\to\infty} \frac{\rho(\varphi(nt,x))}{h_{nt}(x)} = \lim_{n\to\infty} h_{nt}(\varphi(-nt,x))\rho(\varphi(-nt,x)) = 0.$$

Example 7.10 Let $\Omega := (0,\infty)$, $\rho(x) := 1/x$ and $\varphi(t,x) := x + t$. Then, it follows from proposition 6.10 and theorem 6.4 that ρ is both L^p-admissible and C_0-admissible for φ. Obviously, $\Omega_t = \Omega$ and $K \cap \varphi(t,\Omega) = \emptyset$ for every compact subset K of Ω and sufficiently large t. Moreover, $\rho(\varphi(t,x))/\rho(\varphi(t,y)) = (y+t)/(x+t)$ is bounded above and bounded away from 0 for all x, y from a compact subset of Ω and $t \geq 0$, which shows that all requirements of the above remark are satisfied.

Since for all $x > 0$ we have $\lim_{n\to\infty} \rho(x+n) = 0$ it follows from remark 7.9 that the C_0-semigroup induced by φ, i.e. the left translation semigroup, is chaotic on $C_{0,\rho}(0,\infty)$.

On the other hand, if $t > 0$ we have $\sum_{l=1}^{\infty} \rho(1+lt) > \sum_{l=1}^{\infty} 1/lt = \infty$, so that the left translation semigroup is not chaotic on $L^p_\rho(0,\infty)$. But it is mixing on $L^p_\rho(0,\infty)$ by theorem 6.19 b) because of $\lim_{t\to\infty} \rho(x+t) = 0$ for every $x > 0$.

Again, as in section 5.3 it is possible to prove an analogue of theorem 7.2 in the context of an arbitrary σ-finite measure space $(X, \mathcal{A}, \mu)$.

Let $\varphi : I \times X \to X$ be such that $\varphi(t,\cdot)$ is bimeasurable for all $t \in I$ and $w : I \times X \to \mathbb{R}$ is such that $1/w(t,\cdot) \in L^p_{loc}(\mu)$ for all $t \in I$. Assume that $\varphi(t+s,\cdot) = \varphi(t,\varphi(s,\cdot))$ and $w(t+s,\cdot) = w(t,\cdot)w(s,\varphi(t,\cdot))$ for all $t, s \in I$. Moreover, assume that the mappings $T_{w,\varphi}(t), t \in I$, defined in the usual way, are well-defined operators on $L^p(\mu)$. Then we have the following theorem, whose proof is done by the same arguments as the ones of theorem 7.2 and theorem 5.34 so that we omit it.

Theorem 7.11 *Additionally to the above hypotheses, assume that for every measurable subset A of X with $\mu(A) < \infty$ there is $t_A > 0$ such that $\mu(\varphi(t,A) \cap A) = 0$ for every $t > t_A$. Then, the following are equivalent.*

i) $T_{w,\varphi}$ is chaotic on $L^p(\mu)$.

ii) $per(T_{w,\varphi})$ is dense in $L^p(\mu)$

iii) For measurable subset A of X with $\mu(A) < \infty$ there is a sequence of measurable subsets $(A_n)_{n\in\mathbb{N}}$ of A and a strictly increasing sequence of positive numbers $(t_n)_{n\in\mathbb{N}}$ tending to infinity such that $\lim_{n\to\infty} \mu(A \backslash A_n) = 0$, and $\lim_{n\to\infty} s_n = 0$, where

$$s_n := \sum_{l=1}^{\infty} \nu_{p,lt_n}(L_n) + \sum_{l=1}^{\infty} \nu_{p,-lt_n}(L_n).$$

A C_0-semigroups

In this appendix we present some of the results about C_0-semigroups used throughout the text. Let E be a Banach space. A family $T = (T(t))_{t\geq 0}$ of continuous linear operators on E is called a *C_0-semigroup* on E if the following three properties hold.

i) $T(0) = id$

ii) $T(t+s) = T(t) \circ T(s)$ for all $t, s \geq 0$

iii) For all $x \in E$ the mapping $[0, \infty) \to E, t \mapsto T(t)x$ is continuous.

For example, if A is a continuous linear operator on E, then $T(t) : E \to E, x \mapsto e^{tA}x$ forms a C_0-semigroup on E, where $e^{tA} := \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$.

One important property of C_0-semigroups is their exponential growth.

Theorem A.1 (cf. [24, proposition I.5.5]) *Let T be a C_0-semigroup on E. Then, there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{t\omega}$.*

A direct consequence of the above theorem is the following.

Corollary A.2 *Every C_0-semigroup T is locally equicontinuous, i.e. for every $\varepsilon > 0$ and every $t > 0$ there is $\delta > 0$ such that $T(s)(B(0,\delta)) \subset B(0,\varepsilon)$ for all $s \in [0,t]$, where $B(x,r)$ denotes the open ball with center x and radius r.*

Related to a C_0-semigroup T is its generator.

Definition A.3 The *generator* $(A, D(A))$ of a C_0-semigroup T is defined as

$$A : D(A) \to E, x \mapsto \lim_{h\downarrow 0} \frac{1}{h}(T(h)x - x)$$

on its *domain* $D(A) := \{x \in E; \lim_{h\downarrow 0} \frac{1}{h}(T(h)x - x) \text{ exists in } E\}$.

Obviously, $D(A)$ is a subspace of E and A is linear. Furthermore we have the following important theorem.

Theorem A.4 (cf. [24, theorem II.1.4]) *The generator of a C_0-semigroup is a closed and densely defined linear operator that determines the semigroup uniquely, i.e. if T and S are two C_0-semigroups on E with the same generator $(A, D(A))$ then $T(t) = S(t)$ for all $t \geq 0$.*

The famous Hille Yosida theorem characterises which operators are generators of C_0-semigroups.

Hille Yosida generation theorem A.5 cf. [24, theorem II.3.8] *Let $A : D(A) \subset E \to E$ be a linear mapping defined on a subspace $D(A)$ of E and let $M \geq 1, \omega \in \mathbb{R}$. Then the following are equivalent.*

i) $(A, D(A))$ generates a C_0-semigroup T satisfying

$$\|T(t)\| \leq Me^{t\omega} \text{ for } t \geq 0.$$

ii) $(A, D(A))$ *is closed, densely defined and for every* $\lambda \in \mathbb{C}$ *with* $Re\,\lambda > \omega$ *one has* $\lambda \in \rho(A)$, *i.e.* λ *belongs to the resolvent set of* $(A, D(A))$, *and*

$$\|R(\lambda, A)^n\| < \frac{M}{(Re\,\lambda - \omega)^n} \text{ for all } n \in \mathbb{N},$$

where $R(\lambda, A)$ *denotes the resolvent operator of* A *in* λ, *i.e.* $R(\lambda, A) = (\lambda - A)^{-1}$.

Sometimes it is useful to have an explicit representation of the resolvent operators $R(\lambda, A)$ of the generator $(A, D(A))$ of a C_0-semigroup T. The next theorem says that they are given by the Laplace transform of T, provided the real part of λ is large enough.

Theorem A.6 (cf. [24, theorem I.1.10]) *Let* $(A, D(A))$ *be the generator of the* C_0*-semigroup* T, *which satisfies* $\|T(t)\| \leq Me^{t\omega}$ *for all* $t \geq 0$. *For* $\lambda \in \mathbb{C}$ *with* $Re\,\lambda > \omega$, *the resolvent* $R(\lambda, A)$ *of* A *in* λ *is given by*

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x\,dt \text{ for all } x \in E.$$

Another important theorem concerning the relation between the spectra of the generator A and the semigroup operators $T(t)$ is stated in the next theorem. Herein, $\sigma_p(A)$ and $\sigma_r(A)$ denote the point spectrum and the residual spectrum respectively, i.e. $\sigma_p(A) := \{\lambda \in \mathbb{K};\ (\lambda - A) : D(A) \to E \text{ is not injective}\}$ and $\sigma_r(A) := \{\lambda \in \mathbb{K};\ (\lambda - A) : D(A) \to E \text{ has no dense range}\}$.

Spectral mapping theorem for point and residual spectrum A.7 (cf. [24, theorems IV.3.7, 3.8]) *For the generator* $(A, D(A))$ *of a strongly continuous* C_0*-semigroup* T, *we have the identities*

$$\sigma_p(T(t))\backslash\{0\} = e^{t\sigma_p(A)} \text{ for all } t \geq 0$$

and

$$\sigma_r(T(t))\backslash\{0\} = e^{t\sigma_r(A)} \text{ for all } t \geq 0.$$

Moreover, for $\mu \in \mathbb{C}$ *we have*

$$\ker(\mu - A) = \bigcap_{t \geq 0} \ker(e^{\mu t} - T(t)).$$

In order to underline the importance of C_0-semigroups with respect to evolution problems, we make the following definition.

Definition A.8 i) Let $A : D(A) \subset E \to E$ be a linear mapping defined on a subspace $D(A)$ of E. The initial value problem

$$\text{(CP)} \qquad \begin{aligned} \frac{d}{dt}u(t) &= Au(t) \text{ for } t \geq 0 \\ u(0) &= u_0,\ u_0 \in E \end{aligned}$$

is called the *abstract Cauchy problem* associated to $(A, D(A))$ and the initial value u_0.

ii) A function $u : [0, \infty) \to E$ is called a *(classical) solution* of (CP) if u is continuously differentiable, $u(t) \in D(A)$ for all $t \geq 0$ and (CP) holds.

Theorem A.9 (cf. [24, proposition II.6.2, theorem II.6.7]) *Let $D(A)$ be a subspace of E and $A : D(A) \subset E \to E$ a closed linear mapping. For the abstract Cauchy problem*

$$(CP) \qquad \begin{cases} \frac{d}{dt}u(t) = Au(t) \quad , \quad t \geq 0 \\ u(0) = u_0 \end{cases}$$

we consider the following existence and uniqueness condition:

(EU) For every $u_0 \in D(A)$, there exists a unique solution $u(\cdot, u_0)$ of (CP).

Then the following are equivalent.

i) $(A, D(A))$ is the generator of a C_0-semigroup T.

ii) $(A, D(A))$ satisfies (EU) and $\rho(A) \neq \emptyset$.

iii) $(A, D(A))$ satisfies (EU), has dense domain, and for every sequence $(x_n)_{n \in \mathbb{N}}$ in $D(A)$ satisfying $\lim_{n\to\infty} x_n = 0$, one has $\lim_{n\to\infty} u(t, x_n) = 0$ uniformly on compact intervals $[0, t_0]$.

If one of the conditions is satisfied the unique solution of (CP) for $u_0 \in D(A)$ is given by $u(t, u_0) = T(t)u_0$.

Property iii) of the previous theorem expresses what one may call a "well-posed" problem: there exists a unique solution which depends continuously on the initial data.

B Composition Operators

In chapter 5 we investigate the dynamical behavior of families of composition operators on spaces of integrable functions and spaces of continuous functions, respectively. For the reader's convenience, we present in this appendix the proofs of theorems 5.4 and 5.5.

Theorem B.1 (cf. [57, theorem 2.1.1]) *Let (X,μ) be an arbitrary σ-finite measure space and let $\psi : X \to X$ be a measurable function. The composition operator $T_\psi : (L^p(\mu), \|\cdot\|) \to (L^p(\mu), \|\cdot\|), f \mapsto f \circ \psi$ is well-defined and continuous if and only if the image measure μ^ψ of μ under ψ is absolutely continuous with respect to μ and the μ-density f_ψ of μ^ψ is μ-a.e. bounded.*

If T_ψ is continuous, then $\|T_\psi\| = \|f_\psi\|_\infty^{1/p}$.

Proof: The condition is sufficient, because if it holds, we have that $\|T_\psi f\|^p = \int |f\circ\psi|^p \, d\mu = \int |f|^p \, d\mu^\psi \le \|f_\psi\|_\infty \|f\|^p$ for every $f \in L^p(\mu)$.

The condition is necessary as well, because from the continuity of T_ψ it follows that there is $C > 0$ such that $\mu^\psi(A) = \|T_\psi \chi_A\|^p \le C\|\chi_A\|^p = C\,\mu(A)$ for every measurable subset A of X with $\mu(A) < \infty$. Since this inequality is trivially satisfied for measurable subsets A of X with $\mu(A) = \infty$ we obtain that μ^ψ is absolutely continuous with respect to μ. Hence, from the Radon Nikodym theorem we get an μ-density f_ψ of μ^ψ which is in $L^\infty(\mu)$ and is less than C μ-a.e..

It is not hard to show that in the case of continuity the estimate for $\|T_\psi\|$ holds. □

Theorem B.2 (cf. [57, theorem 4.2.4]) *Let X be a locally compact Hausdorff topological space and let $\psi : X \to X$ be continuous. For a strictly positive, upper semicontinuous function ρ let $C_{0,\rho}(X)$ be as in chapter 5. Then, the following are equivalent.*

i) The mapping $T_\psi : C_{0,\rho}(X) \to C_{0,\rho}(X), f \mapsto f\circ\psi$ is well-defined and continuous.

ii) a) There is a constant $C > 0$ such that $\rho(x) \le C\rho(\psi(x))$ for all $x \in X$.

b) For every compact subset K of X and every $\delta > 0$ the set $\psi^{-1}(K) \cap \{x \in X; \rho(x) \ge \delta\}$ is compact.

Moreover, if T_ψ is an operator on $C_{0,\rho}(X)$ then we have $\|T_\psi\| = \inf\{C > 0;\ C$ satisfies condition ii) a)$\}$.

Proof: *i)* $\Rightarrow$ *ii)* : Fix x in X and let $U := \{y \in X;\ \rho(y) < 2\rho(\psi(x))\}$. Because ρ is upper semicontinuous U is an open neighbourhood of $\psi(x)$ and since X is locally compact, we can find $g \in C_c(X)$ such that $0 \le g \le 1, g(\psi(x)) = 1$ and $g_{|X\setminus U} \equiv 0$. Let $f := \frac{1}{2\rho(\psi(x))} g$. Then f obviously belongs to $C_c(X), 0 \le f$ and $\|f\| \le 1$. Because $g(\psi(x)) = 1$ we have $\rho(x)/2\rho(\psi(x)) \le \|T_\psi f\| \le \|T_\psi\|$, which shows ii) a).

Now, to show ii) b) let K be a compact subset of X and $\delta > 0$. Because X is locally compact, there is $f_K \in C_c(X)$ with $0 \le f_K \le 1$ and $f_{K|K} \equiv 1$. Because of *i)* we have $T_\psi f_K \in C_{0,\rho}(X)$, so that by the positivity of f_K the set $F := \{x \in X;\ f_K(\psi(x))\rho(x) \ge \delta\}$ is compact. Obviously, $\psi^{-1}(K) \cap \{x \in X;\ \rho(x) \ge \delta\}$ is a subset of F and because of the continuity of ψ and the upper semicontinuity of ρ it is closed, hence compact, showing ii) b).

ii) $\Rightarrow$ *i)* : We first show that T_ψ is well-defined. Let f be in $C_{0,\rho}(X)$ and $\varepsilon > 0$. Because of ii) a) we have

$$\begin{aligned}\{x \in X;\, |f(\psi(x))|\rho(x) \geq \varepsilon\} &\subset \{x \in X;\, |f(\psi(x))|\rho(\psi(x)) \geq \varepsilon/C\} \\ &= \psi^{-1}(\{x \in X;\, |f(x)|\rho(x) \geq \varepsilon/C\}).\end{aligned}$$

Since $f \in C_{0,\rho}(X)$ the set $K := \{x \in X;\, |f(x)|\rho(x) \geq \varepsilon/C\}$ is compact so that $M := \sup_{x\in K} |f| < \infty$. Therefore, if $x_0 \in \{x \in X;\, |f(\psi(x))|\rho(x) \geq \varepsilon\}$ we have $x_0 \in \psi^{-1}(K)$ as well as $x_0 \in \{x \in X;\, \rho(x) \geq \varepsilon/(M+1)\}$, that is

$$\{x \in X;\, |f(\psi(x))|\rho(x) \geq \varepsilon\} \subset \psi^{-1}(K) \cap \{x \in X;\, \rho(x) \geq \frac{\varepsilon}{M+1}\}.$$

The right hand side of this is a compact subset of X because of ii) b).

Now, by the upper semicontinuity of $|f(\psi(\cdot))|\rho(\cdot)$ we see that the set $\{x \in X;\, |f(\psi(x))|\rho(x) \geq \varepsilon\}$ is a closed subset of a compact set, hence compact. This shows $T_\psi f \in C_{0,\rho}(X)$.

Clearly, the continuity of T_ψ now follows from ii) a), showing i).

An inspection of the proof so far yields the equality concerning $\|T_\psi\|$. □

The proofs of the corresponding theorems for weighted composition operators are done by the same kind of arguments. For the $L^p(\mu)$ case, note that by a standard argument we have $\int |f|\, d\mu_{w^p,\psi} = \int w^p |f \circ \psi|\, d\mu$ for Borel measurable $f : X \to \mathbb{K}$, where $\mu_{w^p,\psi}(A) = \int_{\psi^{-1}(A)} w^p\, d\mu$. For the $C_{0,\rho}(X)$ case one has to replace the set F in the proof of theorem B.2 by $\{x \in X;\, w(x) f_K(\psi(x))\rho(x) \geq \delta\}$ and $\{x \in X;\, \rho(x) \geq \varepsilon/(M+1)\}$ has to be replaced by $\{x \in X;\, w(x)\rho(x) \geq \varepsilon/(M+1)\}$.

Theorem B.3 *Let (X,μ) be an arbitrary σ-finite measure space and let $w : X \to (0,\infty)$, $\psi : X \to X$ be measurable functions. The weighted composition operator $T_{w,\psi} : (L^p(\mu), \|\cdot\|) \to (L^p(\mu), \|\cdot\|), f \mapsto w(\cdot)\, f(\psi(\cdot))$ is well-defined and continuous if and only if the measure $\mu_{w^p,\psi}$ is absolutely continuous with respect to μ and the μ-density $f_{w^p,\psi}$ of $\mu_{w^p,\psi}$ is μ-a.e. bounded.*

If T_ψ is continuous, then $\|T_\psi\| = \|f_{w^p,\psi}\|_\infty^{1/p}$.

Theorem B.4 *Let X be a locally compact Hausdorff topological space and let $w : X \to (0,\infty)$, $\psi : X \to X$ be continuous. For a strictly positive, upper semicontinuous function ρ the following are equivalent.*

i) The mapping $T_{w,\psi} : C_{0,\rho}(X) \to C_{0,\rho}(X), f \mapsto w(\cdot)\, f(\psi(\cdot))$ is well-defined and continuous.

ii) a) There is a constant $C > 0$ such that $w(x)\rho(x) \leq C\rho(\psi(x))$ for all $x \in X$.

b) For every compact subset K of X and every $\delta > 0$ the set $\psi^{-1}(K) \cap \{x \in X; w(x)\rho(x) \geq \delta\}$ is compact.

Moreover, if $T_{w,\psi}$ is an operator on $C_{0,\rho}(X)$ then we have $\|T_{w,\psi}\| = \inf\{C > 0;\ C$ satisfies condition ii) a)$\}$.

C Miscellanea

In this short appendix we present the proofs of proposition 6.6 and 6.8.

Proposition C.1 *Let Ω be an open, star-like subset of $\mathbb{R}^d$ and $F : \Omega \to \mathbb{R}^d$ be a differentiable vector field such that $\sup_{x\in\Omega} |DF(x)| < \infty$ and $F \in C(\bar{\Omega})$. Furthermore assume that $\partial\Omega = \partial\bar{\Omega}$ is C^1 and that $\langle F(y), n(y)\rangle < 0$ for all $y \in \partial\Omega$ where $n(y)$ denotes the outer normal in y.*

Then, for each $x_0 \in \Omega$ the solution of the initial value problem

$$\dot{x} = F(x), x(0) = x_0$$

exists for all $t \geq 0$.

Proof: Let $x_0 \in \Omega$ and $\varphi : (a, b) \to \Omega$ be the maximal solution of the initial value problem $\dot{x} = F(x), x(0) = x_0$. Assume that $b < \infty$. Then, either $\{\varphi(t);\, t \in [0, b)\}$ is unbounded or $\lim_{t\to b} \operatorname{dist}(\varphi(t), \partial\Omega) = 0$ (cf. [3, Satz 2.5.1]). We will show that in both cases we get a contradiction.

Assume that $\{\varphi(t);\, t \in [0, b)\}$ is unbounded. If $z \in \Omega$ is such that Ω is star-like with respect to z we obtain with $C := \sup_{x\in\Omega} |DF(x)|$

$$\begin{aligned} |\varphi(t)| &\leq |x_0| + \int_0^t |F(\varphi(s))|\, ds \\ &\leq |x_0| + \int_0^t |F(\varphi(s)) - F(z)|\, ds + t|F(z)| \\ &\leq |x_0| + b|F(z)| + bC|z| + C\int_0^t |\varphi(s)|\, ds. \end{aligned}$$

Setting $A := |x_0| + b|F(z)| + bC|z|$ Gronwall's lemma now ensures $|\varphi(t)| \leq Ae^{tC}$ for all $t \in [0, b)$ contradicting the unboundedness of $\{\varphi(t);\, t \in [0, b)\}$.

On the other hand, if we assume that the set $\{\varphi(t);\, t \in [0, b)\}$ is bounded then we have $\lim_{t\to b} \operatorname{dist}(\varphi(t), \partial\Omega) = 0$. From the boundedness of $\{\varphi(t);\, t \in [0, b)\}$ and $F \in C(\bar{\Omega})$ it follows that there is $M < \infty$ such that $|F(\varphi(t))| < M$ for all $t \in [0, b)$. For $0 \leq s < t < b$ we therefore have

$$|\varphi(t) - \varphi(s)| \leq \int_s^t |F(\varphi(r))|\, dr \leq M(t - s).$$

Combining this with $\lim_{t\to b} \operatorname{dist}(\varphi(t), \partial\Omega) = 0$ we see that there is $y \in \partial\Omega$ with $\lim_{t\to b} \varphi(t) = y$.

Since $\partial\Omega$ is C^1 there is an open subset U of $\mathbb{R}^d$ containing y and a continuously differentiable $g : U \to \mathbb{R}$ such that

$$U \cap \Omega = \{g < 0\}, U \cap \partial\Omega = \{g = 0\}, U \cap \bar{\Omega}^c = \{g > 0\}, \text{ and } \nabla g(y) \neq 0.$$

It follows that $|\nabla g(y)|n(y) = \nabla g(y)$, so that by hypothesis $\langle F(y), \nabla g(y)\rangle < 0$. Without loss of generality we can assume that $\langle F(x), \nabla g(x)\rangle < 0$ for all $x \in U \cap \bar{\Omega}$.

Because of $\lim_{t\to b} \varphi(t) = y$ there is t_0 such that $\varphi(t) \in U \cap \Omega$ for all $t \in [t_0, b)$, so that $\Phi : [t_0, b) \to \mathbb{R}, t \mapsto g(\varphi(t))$ is well-defined. Obviously $\Phi(t) < 0$ and

$$\Phi'(t) = \langle \nabla g(\varphi(t)), \varphi'(t)\rangle = \langle F(\varphi(t)), \nabla g(\varphi(t))\rangle < 0$$

for all $t \in [t_0, b)$, so that Φ is strictly decreasing. Finally, $g(y) = \lim_{t\to b} \Phi(t) < \Phi(t_0) < 0$ contradicting $g(y) = 0$. □

Proposition C.2 *Let F be a locally Lipschitz continuous vector field on Ω such that the solution $\varphi(\cdot, x_0)$ of the initial value problem $\dot{x} = F(x)$, $x(0) = x_0$ exists for all $t \geq 0$ and all $x_0 \in \Omega$. Let $h : \Omega \to \mathbb{R}$ be continuous and $E \in \{L^p(\mu), C_{0,\rho}(\Omega)\}$ where μ and ρ are admissible for φ and h. Then the following holds.*

i) Assume F and h are both bounded, continuously differentiable and satisfy $\sup_{x\in\Omega} |DF(x)| < \infty$ and $\sup_{x\in\Omega} |\nabla h(x)| < \infty$. Then the generator $(A, D(A))$ of S is given by the closure of the operator $B : \mathcal{D} \to E, f \mapsto \langle F, \nabla f\rangle + hf$, where $\mathcal{D} := \{f \in C^1(\Omega) \cap E; |\nabla f| \in E\}$.

ii) If F is continuously differentiable and such that the unique solution $\varphi(\cdot, x_0)$ of the initial value problem $\dot{x} = F(x)$, $x(0) = x_0$ exists not only for all $t \geq 0$ but for all $t \in \mathbb{R}$ and if h is continuously differentiable then the generator $(A, D(A))$ of S is given by the closure of the operator $B : C^1_c(\Omega) \to E, f \mapsto \langle F, \nabla f\rangle + hf$.

Proof: i) Since F is bounded it follows that B is well-defined. Now, because F is continuously differentiable it follows that $\varphi(t, \cdot)$ is continuously differentiable and that $\frac{d}{dt} D\varphi(t, x) = DF(\varphi(t, x)) \cdot D\varphi(t, x)$ with $D\varphi(0, x) = id$ (cf. [32, p. 300]). We therefore have

$$|D\varphi(t, x)| \leq 1 + \int_0^t |DF(\varphi(s, x))|\, |D\varphi(s, x)| ds \leq 1 + C \int_0^t |D\varphi(s, x)| ds$$

for suitable $C > 0$. Thus, Gronwall's inequality gives $|D\varphi(t, x)| \leq e^{tC}$.

Now, if $f \in \mathcal{D}$ it follows from the differentiability of $\varphi(t, \cdot)$ and h that $S(t)f$ is differentiable, so that $S(t)f \in C^1(\Omega) \cap E$. Furthermore, we have

$$\nabla(S(t)f) = \nabla(h_t(\cdot)f(\varphi(t, \cdot))) = (\nabla h)(\cdot)S(t)f + h_t(\cdot)(\nabla f)(\varphi(t, \cdot))D\varphi(t, \cdot),$$

so that

$$|\nabla(S(t)f)| \leq |\nabla h||S(t)f| + e^{tC}|h_t\, (\nabla f)(\varphi(t, \cdot))| = |\nabla h||S(t)f| + e^{tC}S(t)(|\nabla f|).$$

Since $\sup_{x\in\Omega} |\nabla h(x)| < \infty$ and $f \in E$ the right hand side of this inequality belongs to E, implying $|\nabla(S(t)f)| \in E$ for all $t \geq 0$. Because h is bounded this yields that $\mathcal{D}$ is S-invariant. Since $C^1_c(\Omega) \subset \mathcal{D}$ it follows that $\mathcal{D}$ is a dense subspace of E. Being dense and S-invariant [24, proposition II.1.7] gives that $\mathcal{D}$ is a core for the generator of S. As in the proof of theorem 6.7 one shows that $B \subset A$ so that i) follows.

ii) Obviously, B is well-defined. From the continuous differentiability of F it again follows that $\varphi(t, \cdot)$ is continuously differentiable. Because h is continuously differentable it follows that $S(t)f \in C^1(\Omega)$ for every $f \in C^1_c(\Omega)$. Since now we assume that the solution $\varphi(\cdot, x_0)$ of the initial value problem $\dot{x} = F(x)$, $x(0) = x_0$ exists for all $t \in \mathbb{R}$ it follows from the flow equation $\varphi(s + t, \cdot) = \varphi(s, \varphi(t, \cdot))$ for all $s, t \in \mathbb{R}$ and $\varphi(0, \cdot) = id$ that the pre-image of compact subsets of Ω under $\varphi(t, \cdot), t \geq 0$ is again compact, so that $S(t)f \in C_c(\Omega)$ for all $t \geq 0$ and $f \in C_c(\Omega)$. Therefore, $C^1_c(\Omega)$ is S-invariant and obviously dense in E so that it is a core for the generator of T_φ by [24, proposition II.1.7]. Again as in the proof of theorem 6.7 one shows that $B \subset A$ so that ii) follows. □

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Note added in proof: During the revision of this thesis we were informed that M. De La Rosa and C. Read answered in the negative the "great open problem in hypercyclicity" for the single operator case, i.e. they showed that there are hypercyclic operators T on a Banach space E such that $T \oplus T$ is not hypercyclic (cf. [17]).