A Homological Approach to the Splitting Theory of PLS-spaces

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Dedicated to the memory of Susanne Dierolf.

Abstract

The subject of this thesis is a homological approach to the splitting theory of PLS-spaces, i.e. to the question for which topologically exact short sequences

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

of PLS-spaces X, Y, Z the map g admits a right inverse. We show that the category (PLS) of PLS-spaces and continuous linear maps is an additive category in which every morphism admits a kernel and a cokernel, i.e. it is pre-abelian. However, we also show that it is neither quasi-abelian nor semi-abelian. As a foundation for our homological constructions we show the more general result that every pre-abelian category admits a largest exact structure in the sense of Quillen [25]. In the pre-abelian category (PLS) this exact structure consists precisely of the topologically exact short sequences of PLS-spaces. Using a construction of Ext^k-functors due to Yoneda [41], we show that one can define for each PLS-space A and every $k \geq 1$ a covariant additive functor

$$\mathrm{Ext}^k_{\mathrm{PLS}}(A, -)\colon (\mathrm{PLS}) \to (\mathrm{AB})$$

and a contravariant additive functor

$$\mathrm{Ext}^k_{\mathrm{PLS}}(-, A)\colon (\mathrm{PLS}) \to (\mathrm{AB})$$

inducing for every topologically exact short sequence of PLS-spaces a long exact sequence of abelian groups and group morphisms. These functors are studied in detail and we establish a connection between the functors $\mathrm{Ext}^k_{\mathrm{PLS}}$ and the functors $\mathrm{Ext}^k_{\mathrm{LS}}$ for LS-spaces. Through this connection we arrive at an analogue of a result for Fréchet spaces which connects the functors Proj^1 and Ext^1 and also gives sufficient conditions for the vanishing of the higher Ext^k. Finally, we show that $\mathrm{Ext}^k_{\mathrm{PLS}}(E, F) = 0$ for $k \geq 1$ whenever E is a closed subspace and F is a Hausdorff quotient of the space of distributions, which generalizes a result of Wengenroth [40] that is itself a generalization of results due to Domański and Vogt [9].

Zusammenfassung

In der vorliegenden Arbeit wird ein homologischer Ansatz für die Splittingtheorie von PLS-Räumen vorgestellt, die sich mit der Frage beschäftigt, für welche topologisch exakten kurzen Sequenzen

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

von PLS-Räumen X, Y, Z die Abbildung g eine stetige lineare Rechtsinverse besitzt. Wir zeigen, dass die Kategorie (PLS) der PLS-Räume und stetigen linearen Abbildungen prä-abelsch ist, also jeder Morphismus in (PLS) sowohl einen Kern als auch einen Kokern besitzt. Gleichzeitig zeigen wir, dass diese Kategorie weder quasi-abelsch noch semi-abelsch ist. Um eine Grundlage für homologische Konstruktionen in (PLS) zu schaffen, zeigen wir das allgemeinere Ergebnis, dass jede prä-abelsche Kategorie eine maximale Exaktheitsstruktur im Sinne von Quillen [25] besitzt. In der prä-abelschen Kategorie (PLS) besteht diese maximale Exaktheitsstruktur gerade aus den topologisch exakten kurzen Sequenzen von PLS-Räumen. Indem wir eine Konstruktion der Ext^k-Funktoren benutzen, die auf Yoneda [41] zurückgeht, zeigen wir, wie für jeden PLS-Raum A und jedes $k \geq 1$ ein additiver kovarianter Funktor

$$\mathrm{Ext}^k_{\mathrm{PLS}}(A, -)\colon (\mathrm{PLS}) \to (\mathrm{AB})$$

und ein additiver kontravarianter Funktor

$$\mathrm{Ext}^k_{\mathrm{PLS}}(-, A)\colon (\mathrm{PLS}) \to (\mathrm{AB})$$

derart definiert werden können, dass sie für jede topologisch exakte kurze Sequenz von PLS-Räumen eine lange exakte Sequenz von abelschen Gruppen und Gruppenhomomorphismen induzieren. Wir untersuchen diese Funktoren im Detail und stellen eine Verbindung zwischen den Funktoren $\mathrm{Ext}^k_{\mathrm{PLS}}$ und den Funktoren $\mathrm{Ext}^k_{\mathrm{LS}}$ für LS-Räume her. Diese Verbindung erlaubt uns für PLS-Räume die Entsprechung eines Ergebnisses für Fréchet Räume zu zeigen, welches die Funktoren Ext^1 und Proj^1 verbindet und zusätzlich hinreichende Bedingungen für das Verschwinden der höheren Ext^k-Gruppen angibt. Zum Abschluss zeigen wir $\mathrm{Ext}^k_{\mathrm{PLS}}(E, F) = 0$ für $k \geq 1$, falls E ein abgeschlossener Unterraum und F ein Hausdorff Quotient des Raumes der Distributionen sind. Dies verallgemeinert ein Ergebnis von Wengenroth [40], welches selbst auf Resultaten von Domański und Vogt [9] beruht.

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Contents

Chapter 1

Introduction

In the context of linear functional analysis, the so-called splitting theory is concerned with the following problem: Characterize the pairs (Z, X) of locally convex spaces X and Z such that for every topologically exact sequence of locally convex spaces and continuous linear maps

$$0 \to X \stackrel{f}{\to} Y \stackrel{g}{\to} Z \to 0 \qquad (1)$$

the map g admits a right inverse, in which case one says that the sequence (1) splits. Recall that the above sequence is topologically exact if f is a topological embedding and g an open surjection with $f(X) = g^{-1}(\{0\})$. The map g admits a right inverse if and only if the map

$$\mathcal{L}(Z, g)\colon \mathcal{L}(Z, Y) \to \mathcal{L}(Z, Z),\ h \mapsto g \circ h$$

is surjective, which can also be formulated in terms of the exactness of the sequence

$$0 \to \mathcal{L}(Z, X) \xrightarrow{\mathcal{L}(Z,f)} \mathcal{L}(Z, Y) \xrightarrow{\mathcal{L}(Z,g)} \mathcal{L}(Z, Z) \to 0.$$

Hence, the splitting problem for the sequence (1) is the question wether the functor $\mathcal{L}(Z, -)$ preserves the exactness of this sequence. This makes the splitting problem a natural candidate for the methods of homological algebra, especially derived functors, which measure to what degree an additive functor preserves exactness.

The first to use homological methods for the splitting problem in the context of functional analysis, and the one who introduced homological methods to functional analysis, in general, was Palamodov in 1968 [22, 23]. Using injective resolutions, he computed for a locally convex space A the right derived functors $\mathrm{Ext}^k(A, -)$ of the functor $\mathcal{L}(A, -)$ which acts from the (non-abelian) category of locally convex spaces to the category of vector spaces and associates with a locally convex space X the vector space of continuous linear maps $\mathcal{L}(A, X)$. The vector space $\mathrm{Ext}^1(Z, X)$ then characterizes the splitting of topologically exact sequences of the form (1), i.e. one has $\text{Ext}^1(Z, X) = 0$ if and only if every exact sequence (1) splits. Starting in the eighties, the splitting theory for Fréchet spaces, i.e. for sequences of the form (1) were X, Y, Z are Fréchet spaces, was reinvented and developed further by Vogt [36, 37] with a strong emphasis on the functional analytic aspects. The classical splitting result for Fréchet spaces is due to Vogt and Wagner [35] and states that $\text{Ext}^1(E, F) = 0$ whenever E is a Fréchet space with (DN), F is a Fréchet space with (Ω), and one of them is nuclear. This led to important results about the structure of nuclear Fréchet spaces like, e.g., for subspaces and quotients of the space s of rapidly decreasing sequences. The splitting theory for Fréchet spaces has been further refined by Frerick and Wengenroth [11, 40] and is now rather complete.

A PLS-space is a locally convex space that arises as the projective limit of a sequence of strong duals of Fréchet-Schwartz spaces. This class contains many spaces arising in classic analytic problems like Fréchet-Schwartz spaces, spaces of real analytic functions, spaces of holomorphic functions and smooth functions, spaces of distributions, and various spaces of ultradifferentiable functions and ultradistributions which are important for the theory of partial differential equations. For topologically exact sequences

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

of PLS-spaces X, Y, Z the splitting problem has been investigated by different authors, but the theory is far from being complete. In [9] Vogt and Domanski, and Wengenroth in [40] obtained splitting results for X being a Hausdorff quotient of the space of distributions $\mathcal{D}'(\Omega)$ and Z being a subspace of $\mathcal{D}'(\Omega)$ as well as for the case where Z is isomorphic to the space of all sequences. The dissertation of Kunkle [17] deals with the case where X and Z are so-called power series Köthe PLS-sequence spaces and gives a nearly complete solution in this case. In [4] Bonet and Domański give, under some mild assumptions, a complete solution for the splitting problem when Z is a Fréchet-Schwartz space. In all the mentioned papers the authors obtain their results using functional analytic arguments only und make use of an ad-hoc definition $\text{Ext}^1_{\text{PLS}}(Z, X) = 0$ when every sequence of the form (1) splits. This lack of using homological methods is due to the fact that one cannot define the right derived functors in the sense of Palamodov of the functors $\mathcal{L}(A, -)$ and $\mathcal{L}(-, A)$ for the category of PLS-spaces since it neither has enough injective nor enough projective objects (in fact there are also other obstacles to this construction; see chapter 2).

The aim of this thesis is a homological approach to the splitting theory of PLS-spaces. For this purpose we will show that one can define for each PLS-space A and every $k \geq 1$ a covariant additive functor

$$\text{Ext}^k_{\text{PLS}}(A, -) \colon (\text{PLS}) \to (\text{AB})$$

and a contravariant additive functor

$$\mathrm{Ext}^k_{\mathrm{PLS}}(-,A)\colon (\mathrm{PLS}) \to (\mathrm{AB})$$

inducing for every topologically exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ of PLS-spaces a long algebraically exact sequence

$$0 \to \mathcal{L}(A,X) \to \mathcal{L}(A,Y) \to \mathcal{L}(A,Z) \to \mathrm{Ext}^1_{\mathrm{PLS}}(A,X) \to \mathrm{Ext}^1_{\mathrm{PLS}}(A,Y) \to \mathrm{Ext}^1_{\mathrm{PLS}}(A,Z) \to \mathrm{Ext}^2_{\mathrm{PLS}}(A,X) \to \dots,$$

respectively a long algebraically exact sequence

$$0 \to \mathcal{L}(Z,A) \to \mathcal{L}(Y,A) \to \mathcal{L}(X,A) \to \mathrm{Ext}^1_{\mathrm{PLS}}(Z,A) \to \mathrm{Ext}^1_{\mathrm{PLS}}(Y,A) \to \mathrm{Ext}^1_{\mathrm{PLS}}(X,A) \to \mathrm{Ext}^2_{\mathrm{PLS}}(Z,A) \to \dots$$

of abelian groups and group morphisms. Unlike in the categories of locally convex spaces, Fréchet spaces or Banach spaces, these Ext^k-functors will not be constructed as derived functors by using injective or projective resolutions since these are not available in the category of PLS-spaces. Instead, we use a construction of the groups $\mathrm{Ext}^k_{\mathrm{PLS}}(Z,X)$ that is due to Yoneda [41] and define these groups in terms of equivalence classes of topologically exact sequences of the form

$$0 \to X \to Y_{k-1} \to Y_{k-2} \to \dots \to Y_1 \to Y_0 \to Z \to 0.$$

This approach, which does neither depend on injective nor projective resolutions, is not so well suited for a direct calculation of the Ext^k-groups as the one using derived functors. Nevertheless, it allows, under mild assumptions, to connect the vanishing of $\mathrm{Ext}^k_{\mathrm{PLS}}(Z,X)$ with the vanishing of $\mathrm{Ext}^k_{\mathrm{LS}}(Z_n,X_m)$ in the category of LS-spaces, where Z_n and X_m are LS-spaces giving rise to the PLS-spaces Z and X, respectively. These $\mathrm{Ext}^k_{\mathrm{LS}}$-groups in turn can be investigated through the well-established splitting theory for Fréchet spaces simply by a duality argument.

In the first chapter we introduce the fundamental notions of category theory with which we will be concerned. Our basic setting will be that of a pre-abelian category, i.e. an additive category in which every morphism admits a kernel and a cokernel. We recall some important properties of a pre-abelian category, like, e.g., the existence of pushouts and pullbacks. Then, we introduce the notion of an exact category in the sense of Quillen [25], of which the well-treated class of quasi-abelian categories is a special case. A kernel-cokernel pair in an additive category is a pair of composable morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

such that f is a kernel of g and g is a cokernel of f. An exact category $(\mathcal{A}, \mathcal{E})$ is an additive category $\mathcal{A}$ together with a distinguished class $\mathcal{E}$ of kernel-cokernel pairs which are subject to certain stability properties. The class $\mathcal{E}$ is then called an exact structure and its elements are called short exact sequences. The notion of an exact category is very useful, since it allows the construction of almost the complete toolset of homological algebra, like the derived category and (total) derived functors. The class of all split exact kernel-cokernel pairs provides the smallest exact structure on every additive category. Of course, this exact structure is not very profitable. Especially with respect to the splitting problem one is rather interested in a largest exact structure. It is quite often the case that the class of all kernel-cokernel pairs is an exact structure. This is the case for every quasi-abelian category, like the categories of locally convex spaces, Fréchet spaces, and Banach spaces. However, this unfortunately fails in general. Especially, we will show in the second chapter that it fails for the category of PLS-spaces. For this category we will have to work with an exact structure that is a proper subclass of all kernel-cokernel pairs. To arrive at this exact structure for PLS-spaces we will prove in the second section of the first chapter the far more general result that every pre-abelian category $\mathcal{C}$ admits a largest exact structure $\mathcal{E}$. This is a joint result with Sven-Ake Wegner and is the main subject of the article [31].

In the second chapter we look into the structure of the category (PLS) of PLS-spaces and continuous linear maps. We will show that every morphism $f\colon X \to Y$ in (PLS) admits a kernel and a cokernel and that therefore (PLS) is a pre-abelian category. The cokernel of f in (PLS) though will turn out to be the canonical morphism $Y \to C(Y/f(X))$ to the Hausdorff-completion of the quotient $Y/f(X)$. As a consequence of this, the notion of a topologically exact sequence $0 \to X \to Y \to Z \to 0$ of PLS-spaces differs from the categorical notion of a kernel-cokernel pair. In the classical categories appearing in the context of functional analysis, these two notions always coincide. In fact, Palamodov uses the properties of a kernel-cokernel pair for his homological constructions in quasi-abelian and semi-abelian categories. The kernel-cokernel pairs in (PLS) though, do not admit Palamodov's approach, in fact we will show that the category (PLS) is neither quasi-abelian nor semi-abelian. As a pre-abelian category however, it has a largest exact structure, as shown in the first chapter, and it turns out that this exact structure $\mathcal{E}_{\mathrm{PLS}}$ is none other than the class of topologically exact sequences of PLS-spaces. Therefore, the topologically exact sequences of PLS-spaces, in which one is ultimately interested from a functional analytic point of view, is well-behaved enough to allow a meaningful homological approach to the splitting problem, which is the subject of the last two chapters.

In the third chapter we will give a short introduction to the Yoneda-Ext-functors for exact categories. We will introduce the terminology necessary for the Ext^k-functors and the long exact sequences described above. In the second section of this chapter we will provide characterizations of $\mathrm{Ext}^k(Z,X)=0$ that do not use the, rather unwieldy, description of the elements of $\mathrm{Ext}^k(Z,X)$ as equivalence classes, but use properties of exact sequences. For example, one has the familiar characterization that $\mathrm{Ext}^1(Z,X)=0$ if and only if every exact sequence $0\to X\to Y\to Z\to 0$ splits. In the case $k>1$ the characterizations are somewhat more complex.

In the last chapter we will apply the homological constructions of the third chapter to the exact category $(\mathrm{PLS},\mathcal{E}_{\mathrm{PLS}})$ and then investigate more closely the functors $\mathrm{Ext}^k_{\mathrm{PLS}}$. In the first section we show that the vanishing of the Ext^k-groups is a three space property and then establish a connection between the functors $\mathrm{Ext}^k_{\mathrm{PLS}}$ for PLS-spaces and the functors $\mathrm{Ext}^k_{\mathrm{LS}}$ for LS-spaces. We show that $\mathrm{Ext}^k_{\mathrm{LS}}(Z,X)$ is isomorphic to $\mathrm{Ext}^k_{\mathrm{PLS}}(Z,X)$ if Z and X are LS-spaces and that for the vanishing of $\mathrm{Ext}^k_{\mathrm{PLS}}(Z,\prod_{n\in\mathbb{N}}X_n)$, where the X_n are LS-spaces, it is sufficient that $\mathrm{Ext}^k_{\mathrm{LS}}(Z_n,X_m)=0$ for all $n,m\in\mathbb{N}$, where the Z_n are the LS-spaces giving rise to the PLS-space Z. In the second section we will apply the results of the first section to the canonical resolution

$$0\to X\to \prod_{n\in\mathbb{N}} X_n\to \prod_{n\in\mathbb{N}} X_n$$

of a PLS-space X and arrive at an analogue of a result for Fréchet spaces that connects the functors Proj^1 and Ext^1 and also yields sufficient conditions for the vanishing of the higher Ext^k (see [40, Proposition 5.1.5]). More precisely, we will show that if $(X_n)_{n\in\mathbb{N}}$ is a sequence of LS-spaces giving rise to a PLS-space X and $(Z_m)_{m\in\mathbb{N}}$ is a sequence of LS-spaces giving rise to a PLS-space Z such that $\mathrm{Proj}^1(X)=0$ and $\mathrm{Ext}^k_{\mathrm{PLS}}(Z_m,X_n)=0$ for all $n,m\in\mathbb{N}$ and $1\le k\le k_0$, where $1\le k_0\le\infty$, then there is an isomorphism $\mathrm{Ext}^1_{\mathrm{PLS}}(Z,X)\cong\mathrm{Proj}^1(\mathcal{Y})$, where $\mathcal{Y}$ is a spectrum of spaces of continuous linear maps, and in addition one has $\mathrm{Ext}^k_{\mathrm{PLS}}(Z,X)=0$ for $2\le k\le k_0$. This reduces, under the (mild) assumption $\mathrm{Proj}^1(X)=0$, the problem of the vanishing of the higher Ext^k-groups in the category of PLS-spaces to the problem of the vanishing of the higher Ext^k-groups in the category of LS-spaces. The latter problem is accessible by a duality argument. Finally, we will show in the last section that for $k\ge 2$ the group $\mathrm{Ext}^k_{\mathrm{PLS}}(E,F)$ always vanishes if E is a closed subspace of the space of distributions $\mathcal{D}'(\Omega)$ and F is a Hausdorff quotient of $\mathcal{D}'(\Omega)$. By using the long exact sequences, this also yields a new proof of a splitting result for PLS-spaces due to Wengenroth [40], which is based on the work of Domański and Vogt [9] and states $\mathrm{Ext}^1_{\mathrm{PLS}}(E,F)=0$ in the above setting. Alltogether, we have shown that for $k\ge 1$ the groups $\mathrm{Ext}^k_{\mathrm{PLS}}(E,F)$ always vanish for subspaces E and quotients F of $\mathcal{D}'(\Omega)$.

Chapter 2

Pre-abelian and Exact Categories

The use of homological methods in functional analysis was started by Palamodov [22, 23], re-invented by Vogt [38] and expanded by many others (see the book of Wengenroth [40] for detailed references and concrete applications). What distinguishes the homological algebra in functional analysis from the classical one used in the purely algebraic context (as presented in, e.g., [7, 18, 20, 39]), is the fact that its categories are not abelian. Recall that an additive category $\mathcal{A}$ with kernels and cokernels is abelian, if for every morphism $f\colon X \to Y$ the induced morphism $\bar{f}\colon \operatorname{coim} f \to \operatorname{im} f$ is an isomorphism. The additive categories that are important for functional analysis, like, e.g., locally convex spaces, Banach spaces or Fréchet spaces, almost never fulfill this property. For example in the category of locally convex spaces and continuous linear maps, the induced morphism $\bar{f}$ of a continuous linear map $f\colon X \to Y$ is given by

$$\bar{f}\colon X/f^{-1}(\{0\}) \to f(X),\ x + f^{-1}(\{0\}) \mapsto f(x),$$

and it is easy to see that $\bar{f}$ is an isomorphism if and only if f is open onto its range.

In this chapter we will briefly introduce the basic types of additive categories appearing in the context of functional analysis, especially in the theory of locally convex spaces. Our starting point will be the notion of a pre-abelian category, i.e. of an additive category with kernels and cokernels. Then we introduce the concept of an exact category in the sense of Quillen [25]. This is an additive category in which a distinguished class of short exact sequences (a so-called exact structure) is specified and which allows for the construction of almost the complete homological toolset, as long as one remains within this exact structure. The most prominent special case of an exact category is that of a quasi-abelian category, which we also introduce together with the weaker notion of a semi-abelian category. In the second section of this chapter we show that every pre-abelian category admits a largest exact structure. As a consequence of this, meaningful homological algebra is not only possible in the well-treated context of quasi-abelian and semi-abelian categories, but in the much broader context of pre-abelian categories. We will use this maximal exact structure in the subsequent chapters, when dealing with PLS-spaces.

2.1 Pre-abelian Categories

We assume that the reader is familiar with fundamental categorical concepts, like that of additive categories, kernels, cokernels, pullbacks and pushouts. Familiarity with basic category theory as presented, e.g., in Weibel [39, Appendix A] should suffice for a complete understanding of the categorical arguments used in the text.

Definition 2.1.1. An additive category $\mathcal{C}$ is called *pre-abelian*, if every morphism in $\mathcal{C}$ has a kernel and a cokernel.

In the whole section, $\mathcal{C}$ denotes a pre-abelian category. For a morphism $f\colon X \to Y$ in $\mathcal{C}$, we will always denote its kernel by $k_f\colon \ker f \to X$ and its cokernel by $c_f\colon Y \to \operatorname{cok} f$. Furthermore, we adopt the notation of Richman, Walker [27] and call a morphism f a *kernel*, if it is the kernel of some morphism $g\colon Y \to Z$. Dually, we call f a *cokernel*, if it is the cokernel of some morphism $h\colon W \to X$. It is easy to see that f is a kernel if and only if it is a kernel of its cokernel and dually f is a cokernel if and only if it is a cokernel of its kernel. An *image* of f is a kernel of a cokernel of f and will be denoted by $i_f\colon \operatorname{im} f \to Y$. Dually, a *coimage* of f, which will be denoted by $ci_f\colon X \to \operatorname{coim} f$, is a cokernel of a kernel of f. In Schneider's notation [30, Definition 1.1.1] a morphism $f\colon X \to Y$ in $\mathcal{C}$ is called *strict* if the induced morphism $\bar{f}\colon \operatorname{coim} f \to \operatorname{im} f$ is an isomorphism. From his remarks in [30, Remark 1.1.2] it follows that a morphisms f is a strict epimorphism if and only if it is a cokernel and that f is a strict monomorphism if and only if it is a kernel. Note that strict morphisms are called homomorphisms by Wengenroth [40] in analogy to the notation of Köthe [16, p. 91] for the category of locally convex spaces (see also Meise, Vogt [19, p. 307]). As noted by Kelly [15, p. 126] a morphism in a pre-abelian category is a cokernel if and only if it is a regular epimorphism in his terminology. Clearly, the notations of Schneiders and Kelly are more general than those of Richman, Walker. However, the latter notation will turn out to be the most convenient one for our purpose in this treatise.

An important property of pre-abelian categories is the existence of pullbacks and pushouts. Given two morphisms $f\colon X \to Z$, $g\colon Y \to Z$ having the same codomain Z, the pullback (P, p_X, p_Y) of f and g can be constructed as the kernel-object of the morphism $(\,f,\, -g\,)\colon X \times Y \to Z$, together with the composition of the kernel-morphism followed by the canonical morphisms from the product to X, resp. to Y. Dually, the pushout (S, s_X, s_Y) of two morphisms $f\colon W \to X$, $g\colon W \to Y$ having the same domain W, can be obtained as the cokernel-object of the morphism $\binom{f}{-g}\colon W \to X \times Y$, together with the composition of the morphisms from X, resp. from Y, to the product and the cokernel-morphism. We will call a commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{p_Y} & Y \\ {\scriptstyle p_X}\big\downarrow & & \big\downarrow{\scriptstyle s_Y} \\ X & \xrightarrow[s_X]{} & S \end{array}$$

a pullback square, if (P, p_X, p_Y) is a pullback of s_X and s_Y. It will be called a pushout square if (S, s_X, s_Y) is a pushout of p_X and p_Y.

For later use we note the following lemma about kernels and cokernels of parallel arrows in pullback and pushout squares, which is a well-known result (see, e.g., [27, Theorem 5]):

Lemma 2.1.2.

i) If $g\colon Y \to Z$, $t\colon T \to Z$ are morphisms in $\mathcal{C}$ and (P, p_T, p_Y) is their pullback, then there is a morphism $j\colon \ker g \to P$ making the diagram

$$\begin{array}{ccccc} \ker g & \xrightarrow{j} & P & \xrightarrow{p_T} & T \\ \Big\| & & {\scriptstyle p_Y}\big\downarrow & \text{PB} & \big\downarrow{\scriptstyle t} \\ \ker g & \xrightarrow[k_g]{} & Y & \xrightarrow[g]{} & Z \end{array}$$

commutative and being a kernel of p_T.

ii) If $f\colon X \to Y$, $t\colon X \to T$ are morphisms in $\mathcal{C}$ and (S, s_T, s_Y) is their pushout, then there is a morphism $c\colon S \to \operatorname{cok} f$ making the diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{c_f} & \operatorname{cok} f \\ {\scriptstyle t}\big\downarrow & \text{PO} & \big\downarrow{\scriptstyle s_Y} & & \Big\| \\ T & \xrightarrow[s_T]{} & S & \xrightarrow[c]{} & \operatorname{cok} f \end{array}$$

commutative and being a cokernel of s_T.

A pair (f, g) of composable morphisms $X \overset{f}{\to} Y \overset{g}{\to} Z$ is called a *kernel-cokernel pair* if f is a kernel of g and g is a cokernel of f. In many important categories in functional analysis, like that of locally convex spaces, Banach spaces or Fréchet spaces, respectively, the kernel-cokernel pairs coincide with the topologically exact sequences, i.e. the sequences of continuous linear maps

$$0 \to X \overset{f}{\to} Y \overset{g}{\to} Z \to 0$$

such that f is injective and open onto its range, g is an open surjection and $f(X) = g^{-1}(\{0\})$. However, this need not be the case in every category. For example, we will see in the second chapter that in the category of PLS-spaces and continuous linear maps the class of topologically exact short sequences is a proper subclass of the kernel-cokernel pairs.

The construction of the diverse tools of homological algebra in additive categories, like, e.g., derived functors, spectral sequences and the derived category, requires some basic stability properties of kernel-cokernel pairs. The essence of what a "good class" of kernel-cokernel pairs should have, was captured by Quillen [25] and is contained within the following definition:

Definition 2.1.3. Let $\mathcal{A}$ be an additive category. If a class $\mathcal{E}$ of kernel-cokernel pairs on $\mathcal{A}$ is fixed, an *admissible kernel* is a morphism f such that there exists a morphism g with $(f, g) \in \mathcal{E}$. *Admissible cokernels* are defined dually. An *exact structure* on $\mathcal{A}$ is a class $\mathcal{E}$ of kernel-cokernel pairs which is closed under isomorphisms and satisfies the following axioms.

(E0) For each object X, $\mathrm{id}_X \colon X \to X$ is an admissible kernel.

$(\mathrm{E}0)^{\mathrm{op}}$ For each object X, $\mathrm{id}_X \colon X \to X$ is an admissible cokernel.

(E1) If $f \colon X \to Y$ and $f' \colon Y \to Y'$ are admissible kernels, then $f' \circ f$ is an admissible kernel.

$(\mathrm{E}1)^{\mathrm{op}}$ If $g \colon Y \to Z$ and $g' \colon Z \to Z'$ are admissible cokernels, then $g' \circ g$ is an admissible cokernel.

(E2) If $f \colon X \to Y$ is an admissible kernel and $t \colon X \to T$ is a morphism, then the pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle t}\downarrow & \text{PO} & \downarrow{\scriptstyle s_Y} \\ T & \xrightarrow[s_T]{} & S \end{array}$$

of f and t exists and s_T is an admissible kernel.

$(\mathrm{E}2)^{\mathrm{op}}$ If $g \colon Y \to Z$ is an admissible cokernel and $t \colon T \to Z$ is a morphism, then the pullback

$$\begin{array}{ccc} P & \xrightarrow{p_T} & T \\ {\scriptstyle p_Y}\downarrow & \text{PB} & \downarrow{\scriptstyle t} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

of g and t exists and p_T is an admissible cokernel.

An *exact category* is an additive category $\mathcal{A}$ together with an exact structure $\mathcal{E}$; the kernel-cokernel pairs in $\mathcal{E}$ are called *short exact sequences*.

Note, that the above definition of an exact category is self-dual, i.e. $(\mathcal{C}, \mathcal{E})$ is an exact category if and only if $(\mathcal{C}^{\mathrm{op}}, \mathcal{E}^{\mathrm{op}})$ is an exact category, which allows for reasoning by dualization.

The most prominent cases of exact categories, especially in the context of functional analysis, are those pre-abelian categories whose exact structure consists of all kernel-cokernel pairs. These categories are called *quasi-abelian* and have been studied independently and under different names by several authors, see the historical remarks in Rump [29, section 2]. Quasi-abelian categories are the natural starting point for the use of homological methods in functional analysis since the categories of topological vector spaces and locally convex spaces as well as several important subcategories, like Fréchet or Banach spaces, are all quasi-abelian (but not abelian). The main subject of this treatise though, the category of PLS-spaces, is not a quasi-abelian category, as will be shown in the second chapter.

The usefulness of the notion of an exact category is based on the fact that its definition yields the prerequisites necessary to prove the diverse diagram lemmas of homological algebra, like, e.g., the five-, nine-, horseshoe- and snake-lemma. This in turn yields almost the complete homological toolset, as it allows the construction of the derived category (see [21]) and the notion of (total) derived functors in the sense of Grothendieck-Verdier [33]. A comprehensive and self-contained exposition of the theory of exact categories can be found in the article of Bühler [5]. We will refer to this article for most of the results about exact categories.

With regard to applications, it is of special importance that classical derived functors can be defined in the familiar way for exact categories:

Remark 2.1.4 (Derived Functors). The theory of classical derived functors, see, e.g., Cartan-Eilenberg [7], Mac Lane [18] or Weibel [39] for the abelian case and Palamodov [22, 23] and Wengenroth [40] for the non-abelian cases used in the theory of locally convex spaces, can also be adapted to the more general setting of exact categories.

First note that the notion of injective and projective objects in an exact category is made with regard to the exact structure, i.e. an object I of an exact category $(\mathcal{C}, \mathcal{E})$ is called *injective*, if for every admissible kernel $f\colon X \to Y$ every morphism $X \to I$ factors as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \swarrow & \\ I & & \end{array}$$

and one says that the category $\mathcal{C}$ *has enough injectives* if for every object X there exists an admissible kernel $X \to I$ into an injective object I. The notions of projective objects and of enough projectives are defined dually. Let then $(\mathcal{C}, \mathcal{E})$ be an exact category with enough injectives and let $F\colon \mathcal{C} \to \mathcal{A}$ be an additive functor to an abelian category $\mathcal{A}$. In analogy to the abelian case, one can show that every object X of $\mathcal{C}$ has an injective resolution $X \hookrightarrow I^\bullet$ and that this resolution is well-defined up to homotopy equivalence (see [5, Proposition 12.3 and Theorem 12.9]). As a consequence, the complexes $F(I^\bullet)$ and $F(J^\bullet)$ are chain homotopy equivalent for any two injective resolutions $X \hookrightarrow I^\bullet$ and $X \hookrightarrow J^\bullet$. One can therefore define the right derived functors of F as

$$R^i F(X) = H^i(F(I^\bullet))$$

on the objects. Given a morphism $f: X \to Y$ in $\mathcal{C}$ and injective resolutions $X \hookrightarrow I^\bullet$ and $Y \hookrightarrow J^\bullet$, the morphism f extends uniquely up to chain homotopy equivalence to a chain map $f^\bullet: I^\bullet \to J^\bullet$, which then in turn yields a morphism $R^i F(f): R^i F(X) \to R^i F(Y)$. The uniqueness property of the chain map $f^\bullet$ also gives $R^i F(g \circ f) = R^i F(g) \circ R^i F(f)$ and $R^i F(\mathrm{id}_X) = \mathrm{id}_{R^i F(X)}$, which shows that $R^i F: \mathcal{C} \to \mathcal{A}$ is a functor.
Given a short exact sequence $0 \to X \to Y \to Z \to 0$ one can show, with the help of the horseshoe lemma for exact categories (see [5, Theorem 12.9]), the existence of a long exact sequence

$$\ldots \to R^{i-1}F(Z) \to R^i F(X) \to R^i F(Y) \to R^i F(Z) \to R^{i+1} F(X) \to \ldots$$

of objects and morphisms in $\mathcal{A}$. In addition, the functor $R^0 F$ is left-exact, i.e. it sends elements of the exact structure $\mathcal{E}$ to left-exact sequences in $\mathcal{A}$, such that the connected sequence $(R^i F)_{i \geq 0}$ forms a universal δ-functor in the sense of Grothendieck [12]. In the terminology of Palamodov [23], a left-exact functor is called injective. Moreover, if F is left-exact, there is an isomorphism of functors $R^0 F \cong F$ and the $R^i F$ measure the failure of F to be exact. If F sends admissible kernels to monomorphisms, i.e. F is semi-injective in the terminology of Palamodov [23], one can also define the additional right derived functor $F^+: \mathcal{C} \to \mathcal{A}$ in the above setting. This functor measures the failure of F to be left-exact, i.e. F is left-exact if and only if F^+ vanishes. Dually to the above, one can define the left derived functors of an additive functor, if the category $\mathcal{C}$ has enough projectives.

Note that for the above construction of the derived functors one does not necessarily need an exact structure. Palamodov [22, 23] introduced them in the weaker context of semi-abelian categories. A pre-abelian category is called *semi-abelian*, if for every morphism $f: X \to Y$ the induced morphism $\bar{f}: \operatorname{coim} f \to \operatorname{im} f$ is both a monomorphism and an epimorphism. Every quasi-abelian category is semi-abelian (see [30, Corollary 1.1.5.]) and for a long time all the known examples of semi-abelian categories in the literature also enjoyed the property of being quasi-abelian. This lead Raikov [26] to conjecture that every semi-abelian category is quasi-abelian, i.e. that the kernel-cokernel pairs of a semi-abelian category always form an exact structure. That this is not the case is contained in the article [3] of Bonet and Dierolf, who show that the category of bornological locally convex spaces and continuous linear maps is a counterexample to Raikov's conjecture (see also [31, Example 4.1] for a discussion of this counterexample). Later, Rump [29] constructed a purely algebraic counterexample.

2.2 The Maximal Exact Structure of a Pre-abelian Category

Our basic tool for the investigation of the splitting theory of PLS-spaces will be Ext^k-functors and the associated long exact sequence. In the context of pre-abelian categories, one way to construct these is as the derived functors of the functor Hom (see, e.g., [23, 37, 40]) in the way of 2.1.4. For this, the minimal prerequisites are that the category in question is semi-abelian and has enough injectives. When one does not have these, as is the case for PLS-spaces (see chapter 2), one can nonetheless construct Ext^k-functors and the long exact sequence, but the properties of an exact structure will then be needed.

In every additive category $\mathcal{A}$ the class of split exact sequences, i.e. the kernel-cokernel pairs (f, g) such that g has a right inverse, form an exact structure and this is contained within every other exact structure on $\mathcal{A}$ (see [5, Lemma 2.7 and Remark 2.8]), hence it is the minimal exact structure. Of course, this exact structure does not yield much useful information. Especially for splitting theory, i.e. the question which elements of an exact structure are the split exact ones, it is rather pointless. Of much more interest is the existence of maximal exact structures, as for example the class of all kernel-cokernel pairs in the case of quasi-abelian categories. In this section we will show that every pre-abelian category $\mathcal{C}$ admits a largest exact structure $\mathcal{E}$ in such a way, that $\mathcal{C}$ is quasi-abelian if and only if $\mathcal{E}$ consists of all kernel-cokernel pairs. This result is a joint work with Sven-Ake Wegner and is the main subject of the article [31].

In the whole section, $\mathcal{C}$ denotes a pre-abelian category. The following definition will be essential for this section and was introduced by Richman, Walker [27, p. 522].

Definition 2.2.1.

i) A cokernel $g\colon Y \to X$ in $\mathcal{C}$ is said to be *semi-stable*, if for every pullback square

$$\begin{array}{ccc} P & \xrightarrow{p_T} & T \\ {\scriptstyle p_Y}\downarrow & \text{PB} & \downarrow{\scriptstyle t} \\ Y & \xrightarrow[g]{} & X \end{array}$$

the morphism p_T is also a cokernel.

ii) A kernel $f\colon X \to Y$ in $\mathcal{C}$ is said to be *semi-stable*, if for every pushout square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle t}\downarrow & \text{PO} & \downarrow{\scriptstyle s_Y} \\ T & \xrightarrow[s_T]{} & S \end{array}$$

the morphism s_T is also a kernel.

The notion of a semi-stable cokernel coincides with that of a totally regular epimorphism, as defined by Kelly [15, p. 124] in the case of a pre-abelian category. Kelly remarks that this notion was defined by Grothendieck (under a different name), see [15, p. 124].

Recall that a morphism $g\colon Y \to Z$ is called a *retraction*, if there exists a morphism $r\colon Z \to Y$ with $g \circ r = \mathrm{id}_Z$, i.e. if it admits a right inverse. Dually, a morphism $f\colon X \to Y$ is called a *coretraction*, if there exists a morphism $l\colon Y \to X$ with $l \circ f = \mathrm{id}_X$, i.e. if it admits a left inverse. Because retractions are stable under pullbacks, coretractions are stable under pushouts, isomorphisms are stable under both and because pullbacks and pushouts are transitive, we obtain the following:

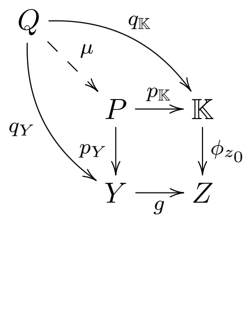
Remark 2.2.2.

i) In the situation of 2.2.1, p_T and s_T are again semi-stable.
ii) Retractions are semi-stable cokernels.
iii) Coretractions are semi-stable kernels.
iv) Isomorphisms are semi-stable cokernels and semi-stable kernels.

If $\mathcal{C}$ is a full additive subcategory of the category (TVS) of topological vector spaces and continuous linear maps that contains the ground field $\mathbb{K}$, it is easily seen that every kernel $f\colon X \to Y$ of a morphism $g\colon Y \to Z$ in $\mathcal{C}$ is an injective map and that the kernel object X is algebraically isomorphic to $g^{-1}(\{0\})$. In contrast, a cokernel $g\colon Y \to Z$ of f need not be algebraically isomorphic to the quotient $Z/f(Y)$, it does not even need to be surjective (see 3.1.5.(ii) for an example). However, if g is semi-stable we can at least establish surjectivity:

Proposition 2.2.3. Let $\mathcal{C}$ be a pre-abelian category that is a full additive subcategory of the category (TVS) containing the ground field $\mathbb{K}$. Then every semi-stable cokernel $g\colon Y \to Z$ in $\mathcal{C}$ is surjective.

Proof. Suppose g is not surjective. Choose $z_0 \in Z$ with $z_0 \notin g(Y)$. Then the mapping $\phi_{z_0}\colon \mathbb{K} \to Z$, $\lambda \mapsto \lambda z_0$ is linear and continuous, hence it is a morphism in $\mathcal{C}$. The pullback $(P, p_Y, p_{\mathbb{K}})$ of ϕ_{z_0} and g in (TVS) is the space $P = \{(y, \lambda) \in Y \times \mathbb{K} \,|\, g(y) = \phi_{z_0}(\lambda)\,\}$ together with the restrictions p_Y and $p_{\mathbb{K}}$ of the projections to P. Since $z_0 \notin g(Y)$ we have $P = g^{-1}(\{0\}) \times \{0\}$ and therefore $p_{\mathbb{K}}(P) = \{0\}$. The pullback $(Q, q_Y, q_{\mathbb{K}})$ of ϕ_{z_0} and g in $\mathcal{C}$ factors as follows



for a unique morphism $\mu\colon Q \to P$. Then $q_{\mathbb{K}}(Q) = \{0\}$, hence it is not an epimorphism in $\mathcal{C}$, in contradiction to g being a semi-stable cokernel. $\square$

For later use we recall the following stability properties of semi-stable kernels and cokernels which will be essential for the proof of 2.2.5 and are due to Kelly.

Proposition 2.2.4. (Kelly [15, Proposition 5.11 and 5.12]) Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms in $\mathcal{C}$. Put $h := g \circ f\colon X \to Z$.

i) If f and g are semi-stable cokernels, then h is a semi-stable cokernel.
ii) If f and g are semi-stable kernels, then h is a semi-stable kernel.
iii) If h is a semi-stable cokernel, then g is a semi-stable cokernel.
iv) If h is a semi-stable kernel, then f is a semi-stable kernel.

Let us remark that 2.2.4 can be proved in an elementary way by very slight modifications of the proofs of Schneiders [30, Propositions 1.1.7 and 1.1.8] and by using the fact that in a diagram of the form

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & \text{PB} & \downarrow{\scriptstyle c} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' \end{array}$$

the left-hand square is a pullback if and only if this is true for the exterior rectangle (cf. Kelly [15, Lemma 5.1]).

We will now show that $\mathcal{C}$ admits a largest exact structure $\mathcal{E}$. The proof of [E1$^{\text{op}}$] in the following was inspired by that of Keller [14, Proposition after A.1].

Theorem 2.2.5. If $\mathcal{C}$ is a pre-abelian category then the class

$$\mathcal{E} = \big\{ (f,g) \mid (f,g) \text{ is a kernel-cokernel pair, } f \text{ is a semi-stable kernel and } g \text{ is a semi-stable cokernel} \big\}$$

is an exact structure on $\mathcal{C}$. Moreover, $\mathcal{E}$ is maximal in the sense that all exact structures on $\mathcal{C}$ are contained in it. In the notation of Richman, Walker [27, p. 524] the pairs in $\mathcal{E}$ are called *stable*.

Proof. We show that $\mathcal{E}$ is closed under isomorphisms. Let $(f,g) \in \mathcal{E}$ and let

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} & & \downarrow{\scriptstyle i_Z} \\ X' & \xrightarrow[f']{} & Y' & \xrightarrow[g']{} & Z' \end{array}$$

be a commutative square in $\mathcal{C}$ with isomorphisms i_X, i_Y and i_Z. Then (f', g') belongs to $\mathcal{C}$. In fact, every commutative square

$$\begin{array}{ccc} E & \xrightarrow{h} & F \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ E' & \xrightarrow[h']{} & F' \end{array}$$

in $\mathcal{C}$ with isomorphisms ϕ and ψ is a pullback square as well as a pushout square, hence f' is a semi-stable kernel and g' a semi-stable cokernel by 2.2.2.(i) and it is easy to see that f' is the kernel of g'. This shows $(f', g') \in \mathcal{E}$.

By Bühler [5, Remark 2.4] (cf. Keller [14, App. A]) the axioms in the definition of exact category are somewhat redundant and it is in fact enough to check the axioms [E0], $[\mathrm{E0}]^{\mathrm{op}}$, $[\mathrm{E1}]^{\mathrm{op}}$, [E2] and $[\mathrm{E2}]^{\mathrm{op}}$ in order to show that $\mathcal{E}$ is an exact structure.

[E0] and $[\mathrm{E0}]^{\mathrm{op}}$ are satisfied by 2.2.2.(iv).

$[\mathrm{E2}]^{\mathrm{op}}$: Since $\mathcal{C}$ is pre-abelian, the pullback of any two morphisms is defined. Let $(f, g) \in \mathcal{E}$, assume that

$$\begin{array}{ccc} P & \xrightarrow{p_T} & T \\ {\scriptstyle p_Y}\downarrow & \text{PB} & \downarrow{\scriptstyle t} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

is a pullback square. According to 2.1.2.(i) we get a kernel $k\colon X \to P$ of p_T such that

$$\begin{array}{ccccc} X & \xrightarrow{k} & P & \xrightarrow{p_T} & T \\ \| & & {\scriptstyle p_Y}\downarrow & \text{PB} & \downarrow{\scriptstyle t} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z \end{array}$$

is commutative. Since g is a semi-stable cokernel, the same is true for p_T by 2.2.2.(i). Hence, the first row in the above diagram is a kernel-cokernel pair. Moreover, $p_Y \circ k = f$ and thus by 2.2.4.(iv) the morphism k is a semi-stable kernel, which shows $(k, p_T) \in \mathcal{E}$.

[E2]: Since $\mathcal{C}$ is pre-abelian, the pushout of any two morphisms does exist. The pair (f, g) is in $\mathcal{E}$ if and only if $(g^{\mathrm{op}}, f^{\mathrm{op}})$ is in $\mathcal{E}^{\mathrm{op}}$. Then [E2] follows from $[\mathrm{E2}]^{\mathrm{op}}$ by duality.

$[\mathrm{E1}]^{\mathrm{op}}$: Let $(f, g), (f', g') \in \mathcal{E}$ be pairs such that $g' \circ g$ is defined and let $k\colon K \to Y$ be a kernel of $g' \circ g$. Then $g' \circ g$ is a semi-stable cokernel by 2.2.4.(i) and $(k, g' \circ g)$ is a kernel-cokernel pair. Thus it remains to be shown that k is a semi-stable kernel.

Since $g' \circ g \circ k = 0$, there exists a unique $\alpha\colon K \to X'$ with $f' \circ \alpha = g \circ k$.

Claim A. The diagram

$$\begin{array}{ccc} K & \xrightarrow{\alpha} & X' \\ {\scriptstyle k}\downarrow & (1) & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

is a pullback square.

Let $l_Y : L \to Y$ and $l_{X'} : L \to X'$ be morphisms with $f' \circ l_{X'} = g \circ l_Y$. Then $g' \circ g \circ l_Y = g' \circ f' \circ l_{X'} = 0$, hence there exists a unique $\eta\colon L \to K$ with $l_Y = k \circ \eta$. This yields $f' \circ l_{X'} = g \circ l_Y = g \circ k \circ \eta = f' \circ \alpha \circ \eta$ and from this it follows $l_{X'} = \alpha \circ \eta$ since f' is a monomorphism. Since k is a monomorphism, the morphism η is unique with this property, hence (1) is a pullback square. Thus, Claim A is established.

Claim B. Let $(f, g) \in \mathcal{E}$ and

$$\begin{array}{ccc} P & \xrightarrow{p_R} & R \\ {\scriptstyle p_Y}\downarrow & \text{PB} & \downarrow{\scriptstyle r} \\ Y & \xrightarrow[g]{} & Z \end{array}$$

be a pullback square. Then $(\binom{-p_R}{p_Y}, (\, r,\ g\,)) \in \mathcal{E}$.

By 2.1.2.(i) we have a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{k} & P & \xrightarrow{p_R} & R \\ \| & & {\scriptstyle p_Y}\downarrow & \text{PB} & \downarrow{\scriptstyle r} \\ X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z \end{array}$$

such that k is a kernel of p_R and by $[\text{E2}]^{\text{op}}$ the pair (k, p_R) is in $\mathcal{E}$. We show that

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle k}\downarrow & (2) & \downarrow{\scriptstyle \omega_Y} \\ P & \xrightarrow[\binom{-p_R}{p_Y}]{} & R \oplus Y \end{array}$$

is a pushout, where ω_Y denotes the canonical morphism. Let $l_Y : Y \to L$ and $l_P\colon P \to L$ be morphisms such that $l_Y \circ f = l_P \circ k$. Then $(l_Y \circ p_Y - l_P) \circ k = 0$ holds and there is a unique morphism $\gamma\colon R \to L$ with $\gamma \circ p_R = l_Y \circ p_Y - l_P$ since p_R is the cokernel of k. This in turn gives rise to a unique morphism $\mu\colon R \oplus Y \to S$ with $\gamma = \mu \circ \omega_R$ and $l_Y = \mu \circ \omega_Y$, where ω_R denotes the canonical morphism. We compute $l_P = l_Y \circ p_Y - \gamma \circ p_R = l_Y \circ p_Y - \mu \circ \omega_R \circ p_R = \mu \circ (\omega_Y \circ p_Y - \omega_R \circ p_R) = \mu \circ \binom{-p_R}{p_Y}$. The uniqueness of μ follows from the fact that γ is unique and that p_R is an epimorphism. Hence, (2) is a pushout and therefore $\binom{-p_R}{p_Y}$ is a semi-stable kernel. It remains to be shown that

$(\,r,\,g\,)$ is a cokernel of $\left(\begin{smallmatrix}-p_R\\ p_Y\end{smallmatrix}\right)$ since then the claim follows by [E2]. We show that the pullback diagram in Claim B is also a pushout. Let $l_R\colon R\to L$ and $l_Y\colon Y\to L$ be morphisms with $l_Y\circ p_R = l_Y\circ p_Y$. Then we have $l_Y\circ f = l_Y\circ p_Y\circ k = l_R\circ p_R\circ k = 0$, hence the universal property of the cokernel g gives rise to a unique morphism $\lambda\colon Z\to L$ with $l_Y = \lambda\circ g$. In addition, we have $l_R\circ p_R = l_Y\circ p_Y = \lambda\circ g\circ p_Y = \lambda\circ r\circ p_R$ and therefore $l_R = \lambda\circ r$ since p_R is an epimorphism. This establishes Claim B.

As a consequence of Claim B we know that the pair (p,q) of morphisms $p := \left(\begin{smallmatrix}-\alpha\\ k\end{smallmatrix}\right)\colon K\to X'\oplus Y$ and $q := (\,f',\,g\,)\colon X'\oplus Y\to Z$ belongs to $\mathcal{E}$. We put $r := \left(\begin{smallmatrix}f' & 0\\ 0 & \mathrm{id}_Y\end{smallmatrix}\right)$ and obtain the commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\ f'\ } & Z \\ {\scriptstyle\omega_{X'}}\downarrow & (3) & \downarrow{\scriptstyle\omega_Z} \\ X'\oplus Y & \xrightarrow[\ r\ ]{} & Z\oplus Y \end{array}$$

where $\omega_{X'}$ and ω_Z are the canonical morphisms.

Claim C. (3) is a pushout square.

Let $l_{X'\oplus Y}\colon X'\oplus Y\to L$ and $l_Z\colon Z\to L$ be morphisms in $\mathcal{C}$ such that $l_Z\circ f' = l_{X'\oplus Y}\circ\omega_{X'}$. Denote $l_Y := l_{X'\oplus Y}\circ\omega_Y$ where $\omega_Y\colon Y\to Z\oplus Y$ is the canonical morphism. We have $\delta := (\,l_Z,\,l_Y\,)\colon Z\oplus Y\to L$ and thus

$$\begin{aligned} l_{X'\oplus Y}\circ\omega_{X'} &= l_Z\circ f' = \delta\circ\omega_Z\circ f' = \delta\circ r\circ\omega_{X'}, \\ l_{X'\oplus Y}\circ\omega_Y &= \delta\circ\omega_Y = \delta\circ\omega_Y\circ\pi_Y\circ r\circ\omega_Y \\ &= \delta\circ(id_{Z\oplus Y} - \omega_Z\circ\pi_Z)\circ r\circ\omega_Y = \delta\circ r\circ\omega_Y \end{aligned}$$

where $\pi_Y\colon Z\oplus Y\to Y$ and $\pi_Z\colon Z\oplus Y\to Z$ are the canonical morphisms. Hence the universal property of the coproduct yields $l_{X'\oplus Y} = \delta\circ r$. The uniqueness of δ follows from the universal property of the coproduct, which yields Claim C.

Now, r is a semi-stable kernel and by 2.2.4.(iii) the composition $r\circ p$ is also a semi-stable kernel. We put $\sigma := \left(\begin{smallmatrix}-g\\ \mathrm{id}_Y\end{smallmatrix}\right)$ and obtain

$$r\circ p = \left(\begin{smallmatrix}f' & 0\\ 0 & \mathrm{id}_Y\end{smallmatrix}\right)\left(\begin{smallmatrix}-\alpha\\ k\end{smallmatrix}\right) = \left(\begin{smallmatrix}-f'\circ\alpha\\ k\end{smallmatrix}\right) = \left(\begin{smallmatrix}-g\circ k\\ k\end{smallmatrix}\right) = \sigma\circ k.$$

Since $r\circ p$ is a semi-stable kernel, it follows from 2.2.4.(iv) that k is a semi-stable kernel, which yields that $(k, g'\circ g)\in\mathcal{E}$ and thus that $\mathcal{E}$ is an exact structure on $\mathcal{C}$.

It remains to check the maximality of $\mathcal{E}$. Let $\mathcal{E}'$ be a second exact structure on $\mathcal{C}$ and let $(f,g)\in\mathcal{E}'$. If

$$\begin{array}{ccc} P & \xrightarrow{\ p_T\ } & T \\ {\scriptstyle p_Y}\downarrow & \mathrm{PB} & \downarrow{\scriptstyle t} \\ Y & \xrightarrow[\ g\ ]{} & Z \end{array}$$

is a pullback square, the morphism p_T is an admissible cokernel by [E2$^{\text{op}}$] and as such a cokernel, which shows that g is semi-stable. Analogously, one can show with [E2] that the morphism f is a semi-stable kernel, hence $(f, g) \in \mathcal{E}$. This shows $\mathcal{E}' \subseteq \mathcal{E}$. □

Chapter 3

PLS-spaces

PLS-spaces are a special class of locally convex spaces that are defined as the projective limit of a sequence of strong duals of Fréchet-Schwartz spaces. It is the smallest class of locally convex spaces that contains all the duals of Fréchet-Schwartz spaces and is closed with respect to taking closed subspaces and countable products. Moreover, it contains the Fréchet-Schwartz spaces itself. This class also contains many natural examples arising in the field of analysis, like the spaces of real analytic functions, the spaces of holomorphic functions and the smooth functions, the spaces of distributions and various spaces of ultradifferentiable functions and ultradistributions, which are important for the theory of partial differential equations. An overview of the theory of PLS-spaces and its field of applications can be found, e.g., in the survey article of Domański [8].

In this chapter, we will investigate the structure of the category (PLS) of PLS-spaces and continuous linear maps. We will limit ourselves only to those properties necessary for the construction of basic homological tools, like, e.g., the existence of kernels and cokernels. A major difficulty in this context arises from the fact that a Hausdorff quotient of a PLS-space is not necessarily again a PLS-space. We will show that as a consequence of this, the category (PLS) is neither quasi-abelian nor semi-abelian, but only pre-abelian. As a pre-abelian category however, it is endowed with the maximal exact structure of 2.2.5, and we will provide a characterization of this exact structure in functional analytic terms. This exact structure, which will turn out to be quite natural, will permit a homological treatment of the splitting theory of PLS-spaces, which will be the subject of the last two chapters.

3.1 The Category of PLS-spaces

In what follows, we will use the notations of Wengenroth [40]. First, let us recall the concept of a locally convex projective spectrum and the projective limit of such a spectrum:

Remark and Definition 3.1.1. A *locally convex projective spectrum* $\mathcal{X}$ consists of a sequence $(X_n)_{n\in\mathbb{N}}$ of locally convex spaces and continuous linear maps $X_m^n\colon X_m \to X_n$, that are defined for $n \le m$, such that

i) $X_n^n = \mathrm{id}_{X_n}$ for all $n \in \mathbb{N}$,

ii) $X_n^k \circ X_m^n = X_m^k$ for $k \le n \le m$.

For two projective spectra $\mathcal{X} = (X_n, X_m^n)$ and $\mathcal{Y} = (Y_n, Y_m^n)$ a *morphism* $f\colon \mathcal{X} \to \mathcal{Y}$ *of locally convex projective spectra* is a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous linear maps $f_n\colon X_n \to Y_n$ such that $f_n \circ X_m^n = Y_m^n \circ f_m$ for $n \le m$, i.e. the diagram

$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ {\scriptstyle X_m^n}\downarrow & & \downarrow{\scriptstyle Y_m^n} \\ X_n & \xrightarrow[f_n]{} & Y_n \end{array}$$

is commutative. A sequence

$$0 \to \mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z} \to 0$$

of projective spectra $\mathcal{X} = (X_n, X_m^n), \mathcal{Y} = (Y_n, Y_m^n), \mathcal{Z} = (Z_n, Z_m^n)$ and morphisms of spectra is called *exact*, if for all $n \in \mathbb{N}$ the sequences

$$0 \to X_n \xrightarrow{f_n} Y_n \xrightarrow{g_n} Z_n \to 0$$

are topologically exact. The *projective limit* of a projective spectrum $\mathcal{X}$ is the set

$$\mathrm{Proj}(\mathcal{X}) := \{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n \mid X_m^n(x_m) = x_n \text{ for all } m \ge n\},$$

equipped with the topology induced by the product topology, which makes it a locally convex space. For a morphism of projective spectra $f\colon \mathcal{X} \to \mathcal{Y}$, the map

$$\mathrm{Proj}(f)\colon\ \mathrm{Proj}(\mathcal{X}) \to \mathrm{Proj}(\mathcal{Y}),\ (x_n)_{n\in\mathbb{N}} \mapsto (f_n(x_n))_{n\in\mathbb{N}}$$

is linear and continuous and its easy to see that $\mathrm{Proj}(g\circ f) = \mathrm{Proj}(g)\circ\mathrm{Proj}(f)$ and $\mathrm{Proj}(\mathrm{id}_{\mathcal{X}}) = \mathrm{id}_{\mathrm{Proj}(\mathcal{X})}$. Therefore, the rule which maps a projective spectrum $\mathcal{X}$ to the projective limit $\mathrm{Proj}(\mathcal{X})$ and a morphism of spectra $f\colon \mathcal{X} \to \mathcal{Y}$ to the map $\mathrm{Proj}(f)$ is a functor acting on the category of locally convex projective spectra with values in the category of locally convex spaces.

For the projective limit $\mathrm{Proj}(\mathcal{X})$ of a projective spectrum $\mathcal{X} = (X_n, X_m^n)$ we will denote by X_∞^k the canonical morphism

$$X_\infty^k\colon\ \mathrm{Proj}(\mathcal{X}) \to X_k,\ (x_n)_{n\in\mathbb{N}} \mapsto x_k$$

to the k-th component. Furthermore, a locally convex projective spectrum $\mathcal{X} = (X_n, X_m^n)$ is called *strongly reduced* if for every $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that

$$X^n_m(X_m) \subseteq \overline{X^n_\infty(\mathrm{Proj}(\mathcal{X}))},$$

where the closure is taken in X_n.

A locally convex space X is called an LS-*space*, if there exists a Fréchet-Schwartz space Y such that the strong dual Y' is isomorphic to X. In the terminology of Meise and Vogt [19], LS-spaces are called (DFS)-spaces. We will denote by (LS) the category of LS-spaces and continuous linear maps. The class of LS-spaces is stable with respect to closed subspaces, Hausdorff quotients, and finite products (see [10, §14]). For a morphism $f\colon X \to Y$ in (LS) the inclusion $f^{-1}(\{0\}) \hookrightarrow X$ is therefore a kernel of f in (LS) and the quotient map $Y \to Y/\overline{f(X)}$ is a cokernel of f in (LS). As a consequence, the category (LS) has the same kernels, cokernels and products as the quasi-abelian category (LCS) of Hausdorff locally convex spaces and continuous linear maps. Since the pullback is a kernel of the product and the pushout is a cokernel of the product (see 1.1.), the category (LS) also has the same pullbacks and pushouts as (LCS). This in turn implies that the category (LS) is itself quasi-abelian.

PLS-spaces are defined as the projective limits of LS-spaces:

Definition 3.1.2. A locally convex space X is called a PLS-space, if there exists a strongly reduced projective spectrum $\mathcal{X} = (X_n, X^n_m)$, consisting of LS-spaces X_n, such that $\mathrm{Proj}(\mathcal{X}) \cong X$.

We will denote by (PLS) the category of PLS-spaces and continuous linear maps. Note that it follows from the definition of the projective limit that PLS-spaces are automatically Hausdorff and complete. In the rest of this section we will look into the properties of this category. The basic result is:

Proposition 3.1.3. (PLS) is a pre-abelian category.

For the proof of the above we will need the following:

Lemma 3.1.4. (Wengenroth [40, Theorem 3.3.1]) For a projective spectrum of locally convex spaces $\mathcal{X} = (X_n, X^n_m)$ the following are equivalent:

i) The morphism
$$\sigma_{\mathcal{X}}\colon \prod_{n\in\mathbb{N}} X_n \to \prod_{n\in\mathbb{N}} X_n,\ (x_n)_{n\in\mathbb{N}} \mapsto (X^n_{n+1}(x_{n+1}) - x_n)_{n\in\mathbb{N}}$$
is open onto its range.

ii) For all $n \in \mathbb{N}$ and all $U \in \mathcal{U}_0(X_n)$ there exists $m \geq n$ such that $X^n_m(X_m) \subseteq X^n_\infty(\mathrm{Proj}(\mathcal{X})) + U$.

iii) For every exact sequence
$$0 \to \mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z} \to 0$$
of projective spectra of locally convex spaces the induced morphism $\mathrm{Proj}(g)\colon \mathrm{Proj}(\mathcal{Y}) \to \mathrm{Proj}(\mathcal{Z})$ is open onto its range.

Proof of 3.1.3. The product of two PLS-spaces is easily seen to be a PLS-space, hence (PLS) is a full additive subcategory of the category (LCS) of locally convex spaces and continuous linear maps. In addition, closed subspaces of PLS-spaces are again PLS-spaces (see [9, Proposition 1.2]). For a morphism of PLS-spaces $f\colon X \to Y$ the closed subspace $f^{-1}(\{0\})$ is therefore again a PLS-space, hence the inclusion $f^{-1}(\{0\}) \hookrightarrow X$ is a kernel of f in (PLS). This shows that the the category (PLS) has kernels and that these kernels coincide with those of (LCS). It remains to be shown that every morphism of (PLS)-spaces has a cokernel. In order to show this we will first establish the following:

Claim: Let X be a PLS-space and $A \subseteq X$ be a subspace of X. The Hausdorff-completion of X/A is a PLS-space.

Since $X/\overline{A}$ is the associated Hausdorff space of X/A, we can assume A to be a closed subspace. Let $\mathcal{X} = (X_n, X_m^n)$ be a strongly reduced spectrum of LS-spaces with $\mathrm{Proj}(\mathcal{X}) = X$ and define $A_n := \overline{X_\infty^n(A)}$ and $A_m^n := X_m^n|_{A_m}$. Then $\mathcal{A} = (A_n, A_m^n)$ is a strongly reduced spectrum of LS-spaces with $\mathrm{Proj}(\mathcal{A}) = A$. Define then the LS-space $Y_n := X_n/A_n$ and

$$Y_m^n\colon X_m/A_m \to X_n/A_n,\ x + A_m \mapsto X_m^n(x) + A_n$$

for $m \geq n$, which is well-defined since $X_m^n \circ X_\infty^m = X_\infty^n$. Then $\mathcal{Y} = (Y_n, Y_m^n)$ is a strongly reduced spectrum of LS-spaces. For $n \in \mathbb{N}$ the sequence

$$0 \to A_n \xrightarrow{i_n} X_n \xrightarrow{q_n} Y_n \to 0,$$

where i_n is the inclusion and q_n the quotient map, is exact, hence it induces an exact sequence

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{X} \xrightarrow{q} \mathcal{Y} \to 0$$

of projective spectra. Since the functor Proj is semi-injective (see [40, 3.3]), the induced sequence of projective limits

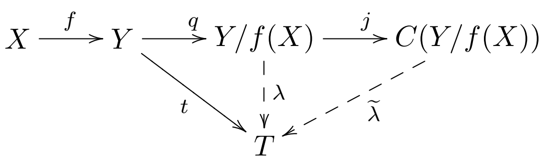
$$0 \to A \xrightarrow{\mathrm{Proj}(i)} X \xrightarrow{\mathrm{Proj}(q)} \mathrm{Proj}(\mathcal{Y}) \tag{1}$$

is algebraically exact and $\mathrm{Proj}(i)$ is open onto its range. The spectrum $\mathcal{A}$ is strongly reduced, hence it has the property (ii) of 3.1.4, which yields that $\mathrm{Proj}(q)$ is open onto its range. Then the morphism

$$j\colon X/A \to \mathrm{Proj}(\mathcal{Y})$$

with $j(x+A) = (X_\infty^n(x)+A_n)_{n\in\mathbb{N}}$, induced by the sequence (1), is also open onto its range and therefore an isomorphism onto its range. Since $j(X/A)$ is obviously dense in $\mathrm{Proj}(\mathcal{Y})$ and because $\mathrm{Proj}(\mathcal{Y})$ is a complete space, it follows that it is isomorphic to the Hausdorff-completion of X/A and it is a PLS-space by construction. This establishes the claim.

Let then $f\colon X \to Y$ be a morphism of PLS-spaces. If $t\colon Y \to T$ is a morphism of PLS-spaces with $t \circ f = 0$, there is a unique continuous linear map $\lambda\colon Y/f(X) \to T$ with $t = \lambda \circ q$, where q denotes the quotient map. Since T is a complete Hausdorff space, the morphism λ extends uniquely to the Hausdorff-completion $C(Y/f(X))$, i.e. there is a unique morphism $\widetilde{\lambda}\colon C(Y/f(X)) \to T$ making the diagram



commutative. This shows that the morphism $j \circ q\colon Y \to C(Y/f(X))$ is a cokernel of f in the category (PLS). □

Remark 3.1.5.

i) The proof of 3.1.3 immediately shows that complete quotients of PLS-spaces by closed subspaces are again PLS-spaces, which is a well-known result (see, e.g., [9, Prop. 1.2]).

ii) There exist PLS-spaces X having a closed subspace $A \subseteq X$, such that the quotient X/A is not complete and therefore not a PLS-space. In fact, any non-surjective linear partial differential operator with constant coefficients $P(D)\colon \mathcal{D}'(\Omega) \to \mathcal{D}'(\Omega)$, for an open subset $\Omega \subseteq \mathbb{R}^d$, which is surjective as an operator $P(D)\colon \mathcal{C}^\infty(\Omega) \to \mathcal{C}^\infty(\Omega)$ induces such an example since the kernel-object $\ker P(D)$ is then a closed subspace of $\mathcal{D}'(\Omega)$ with $\mathrm{Proj}^1 \neq 0$ (cf. [40, 3.4.5]). This in turn implies that the quotient $\mathcal{D}'(\Omega)/\ker P(D)$ is not a complete space by [9, 1.4]. Note that it is only possible to construct an operator with the above properties for dimension $d > 2$, as was shown by Kalmes in [13] (thus solving an old conjecture of Trèves). See example 12 in the same paper for a concrete example of such an operator in the case $d \geq 3$.

iii) Since (PLS) does not reflect the cokernels of the category (LCS), it also has different coimages and images. Let $f\colon X \to Y$ be a morphism in (PLS). The coimage of f, i.e. the cokernel of its kernel, is the canonical morphism $X \to C(X/f^{-1}(\{0\}))$ to the completion of the quotient, which arises as the composition of the quotient map followed by the morphism into the completion. The image of the morphism f, i.e. the kernel of its cokernel, is the inclusion $\overline{f(X)} \hookrightarrow Y$.

iv) Since a morphism in a pre-abelian category is an epimorphism if and only if its cokernel-object is zero and it is a monomorphism if and only if its kernel-object is zero, it follows that the epimorphisms of (PLS) are those with dense image and the monomorphisms are the injective ones.

v) The pullback of two morphisms $f\colon X \to Z$, $g\colon Y \to Z$ in (PLS) is the kernel of the morphism $p\colon X \times Y \to Z$ with $p(x,y) = f(x) - g(y)$, together with the restrictions of the projections onto the components. Since (PLS) reflects the kernels of (LCS), the pullback in (PLS) coincides with the usual pullback of locally convex spaces. On the other hand, the pushout of two morphisms $f\colon X \to Y$, $g\colon X \to Z$ in (PLS) is the cokernel of the morphism $s\colon X \to Y \times Z$ with $s(x) = (f(x), g(x))$ together with the compositions of the inclusions of Y and Z into the product and the cokernel morphism. Therefore, the pushout of f and g in (PLS) is the Hausdorff-completion $S := C((Y \times Z)/s(X))$ of the quotient $(Y \times Z)/s(X)$ together with the canonical morphisms $Y \to S$ and $Z \to S$. This shows, in particular, that the pushout in (PLS) coincides with the pushout in (LCS) if and only if $(Y \times Z)/s(X)$ is a complete Hausdorff space.

The above remark (ii) shows that there are cokernels in the category (PLS) that are not surjective and by 2.2.3 these cokernels cannot be semi-stable. This implies that (PLS) is not a quasi-abelian category, as is erroneously stated in [40, Proposition 5.3.1], where the cokernels of (PLS) that are not semi-stable are neglected and only topologically exact sequences of PLS-spaces are used. We will explain in the next section why these are just the right sequences to consider and why the other results about PLS-spaces in [40] remain untouched. But not enough that (PLS) fails to be quasi-abelian, a slight modification of the proof of 2.2.3 also reveals that it has even weaker properties:

Proposition 3.1.6. The category (PLS) is not semi-abelian.

Proof. Let X be a PLS-space that has a closed subspace $A \subseteq X$, such that the quotient X/A is not complete (see 3.1.5.(ii) for an example) and let $Y := C(X/A)$ be its completion. Then the canonical morphism $c\colon X \to Y$ is open onto its range with dense range, but is not surjective. Let $y_0 \in Y$ with $y_0 \notin c(X)$ and define

$$f\colon X \times \mathbb{C} \to Y,\ (x,\lambda) \mapsto c(x) - \lambda y_0.$$

Then f is a morphism of PLS-spaces and its kernel is

$$f^{-1}(\{0\}) = \{(x,\lambda) \,|\, c(x) = \lambda y_0\} = A \times \{0\}.$$

By remark 3.1.5.(iii), the coimage-object of the morphism f is the completion $C((X \times \mathbb{C})/(A \times \{0\})) = Y \times \mathbb{C}$ and the coimage of f is the morphism $ci_f(x,\lambda) = (c(x),\lambda)$. Furthermore, $f(X \times \mathbb{C})$ is dense in Y, hence the image of f is the identity on Y. The canonical factorization of the morphism f in

(PLS) then is

$$\begin{array}{ccc} X \times \mathbb{C} & \xrightarrow{f} & Y \\ {\scriptstyle ci_f}\downarrow & & \uparrow{\scriptstyle \mathrm{id}_Y} \\ Y \times \mathbb{C} & \xrightarrow[\bar{f}]{} & Y \end{array}$$

with $\bar{f}(y, \lambda) = y - \lambda y_0$. Since $\bar{f}(y_0, 1) = 0$, this map is not injective and is therefore not a bimorphism in (PLS), showing that this category is not semi-abelian. □

In quasi-abelian and semi-abelian categories one can make homological constructions, as those of Palamodov [22, 23] on the class of all kernel-cokernel pairs, which are then called the "short exact sequences" of these categories. However, since the category (PLS) is neither quasi-abelian nor semi-abelian, its class of kernel-cokernel pairs does not allow these constructions. In addition to these problems from the categorical point of view, the kernel-cokernel pairs are also not the "short exact sequences" that one would want from the functional analytic point of view since they do not coincide with the topologically exact short sequences. Of course, every topologically exact sequence

$$0 \to X \to Y \to Z \to 0$$

is a kernel-cokernel pair, but there are more kernel-cokernel pairs in (PLS) than just these: Consider again a PLS-space X that has a closed subspace $A \subseteq X$ such that the quotient X/A is not complete and let $C(X/A)$ be its completion. If $c\colon X \to C(X/A)$ is the canonical morphism and $i\colon A \to X$ is the inclusion, then

$$A \xrightarrow{i} X \xrightarrow{c} C(X/A)$$

is a kernel-cokernel pair by 3.1.3. However, the above pair of morphisms does not form a topologically exact sequence since c is not a surjection. In the next section we will show that one does not need to bother oneself with the particular kernel-cokernel pairs described above and that the class of topologically exact short sequences of PLS-spaces are also the natural "short exact sequences" of PLS-spaces from a categorical point of view.

3.2 The Maximal Exact Structure of (PLS)

The category (PLS) is pre-abelian, as we have shown in the previous section, and therefore the class of all stable kernel-cokernel pairs is its maximal exact structure by 2.2.5. Since (PLS) is not quasi-abelian, it is a proper subclass of its kernel-cokernel pairs. As it turns out, this exact structure is just what one would want for short exact sequences from the functional analytic point of view:

Proposition 3.2.1. The class $\mathcal{E}_{\mathrm{PLS}}$ of topologically exact sequences

$$0 \to X \to Y \to Z \to 0$$

of PLS-spaces X, Y, Z and continuous linear maps is the maximal exact structure of the pre-abelian category (PLS).

Proof. The maximal exact structure $\mathcal{E}$ of (PLS) is the class of stable kernel-cokernel pairs by 2.2.5. Since a semi-stable cokernel in (PLS) has to be surjective by 2.2.3, it follows from the particular form of the kernels and cokernels in (PLS) that every stable kernel-cokernel pair in (PLS) is a topologically exact sequence, hence $\mathcal{E} \subseteq \mathcal{E}_{\mathrm{PLS}}$.
On the other hand, let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a topologically exact sequence of (PLS)-spaces and continuous linear maps. For a morphism $t\colon T \to Z$, the pullback of t and g in (PLS) coincides with the one in (LCS) by 3.1.5.(v), hence g is semi-stable as this is true for every cokernel in the quasi-abelian category (LCS). For a morphism $t\colon X \to T$ we can form the pushout (S, s_Y, s_T) of f and t in the category (LCS) and obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s_Y} & & \| & & \\ 0 & \longrightarrow & T & \xrightarrow[s_T]{} & S & \xrightarrow[c]{} & Z & \longrightarrow & 0 \end{array}$$

whose rows are topologically exact sequences. Since being a complete Hausdorff space is a three space property (see [28, Prop.1.3]), it follows from 3.1.5.(v) that (S, s_T, s_Y) is also a pushout in the category (PLS), which shows that f is a semi-stable kernel. □

The above shows that the topologically exact sequences of PLS-spaces are the "right" exact sequences to consider in the category (PLS) when using homological constructions, in contrary to the numerous quasi-abelian categories appearing in functional analysis like locally convex spaces, Banach spaces and Fréchet spaces, where the notions of "topologically exact sequence" and "kernel-cokernel pair" always coincide. The fact that the topologically exact short sequences of (PLS) form an exact structure, is also the reason why the results about PLS-spaces in [40] remain valid. Almost all constructions that are possible in quasi-abelian categories, can also be made in exact categories without leaving the exact structure and the particular constructions in [40] remain within the exact structure $\mathcal{E}_{\mathrm{PLS}}$. Hence, being quasi-abelian is not necessary. Only the properties of an exact structure are used.

Chapter 4

Yoneda Ext-functors in Exact Categories

An extension of an object X by another object Z in an abelian category is a short exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$. Two such extensions are called equivalent if they fit into a commutative diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & X & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & 0. \end{array}$$

In 1934, Baer [2] defined an addition on the class $\mathrm{Ext}^1(Z, X)$ of equivalence classes of extensions of abelian groups, named after him, in such a way that $\mathrm{Ext}^1(Z, X)$ becomes an abelian group. Cartan and Eilenberg [7] introduced the higher Ext^k-groups as a connected sequence of group valued bifunctors $(\mathrm{Ext}^k(-,-))_{k\geq 0}$, which could be defined in any abelian category having either enough injective or enough projective objects. Yoneda [41] then showed that the groups $\mathrm{Ext}^k(Z, X)$ could also be defined without injectives or projectives in terms of equivalence classes of exact sequences of the form

$$0 \to X \to Y_{k-1} \to Y_{k-2} \to \ldots \to Y_1 \to Y_0 \to Z \to 0.$$

With further work done by Schanuel and (independently) Buchsbaum [6], this led to the connected sequence of group-valued bifunctors $\mathrm{Ext}^k(-,-)$ for arbitrary abelian categories.

These constructions of the Ext^k-groups, named after Yoneda, can also be done for the more general setting of exact categories. The proofs can essentially be carried over from the abelian case (see, e.g., [20]) by substituting the basic diagram lemmas of exact categories (see [5]) for their abelian counterparts. The exposition of Mitchell [20] will also serve as a basis of the short introduction to the Yoneda Ext-functors for exact categories, which we will give in this chapter.

In pre-abelian categories the construction of Yoneda Ext-groups was done first by Richman, Walker [27]. Note that their construction is a special case of the one for exact categories when considering the maximal exact structure 2.2.5.

4.1 The Long Exact Sequence

In this section we give a brief exposition of the Yoneda Ext-functors for exact categories as these will provide the basic tool for our investigation of the splitting theory of PLS-spaces in the next chapter. For the proofs we will mostly refer to the abelian counterparts found in [20]. The basic aim will be to introduce the notions that are necessary for the following result:

Theorem 4.1.1. Let $(\mathcal{C}, \mathcal{E})$ be an exact category and let A be an object of $\mathcal{C}$. For $k \geq 1$ there are covariant additive functors

$$\mathrm{Ext}^k(A, -)\colon \mathcal{C} \to (\mathrm{AB}) , \begin{cases} X \mapsto \mathrm{Ext}^k(A, X) \\ \alpha \mapsto \mathrm{Ext}^k(A, \alpha) \end{cases} ,$$

inducing for every short exact sequence $0 \to X \to Y \to Z \to 0$ a long exact sequence

$$\begin{gathered} 0 \to \mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y) \to \mathrm{Hom}(A, Z) \xrightarrow{\delta_0} \mathrm{Ext}^1(A, X) \to \\ \mathrm{Ext}^1(A, Y) \to \mathrm{Ext}^1(A, Z) \xrightarrow{\delta_1} \mathrm{Ext}^2(A, X) \to \ldots \to \mathrm{Ext}^{k-1}(A, Z) \xrightarrow{\delta_{k-1}} \\ \mathrm{Ext}^k(A, X) \to \mathrm{Ext}^k(A, Y) \to \mathrm{Ext}^k(A, Z) \xrightarrow{\delta_k} \mathrm{Ext}^{k+1}(A, X) \to \ldots \end{gathered}$$

of abelian groups and group morphisms. Dually, there are contravariant functors

$$\mathrm{Ext}^k(-, A)\colon \mathcal{C} \to (\mathrm{AB}) , \begin{cases} Z \mapsto \mathrm{Ext}^k(Z, A) \\ \gamma \mapsto \mathrm{Ext}^k(\gamma, A) \end{cases} ,$$

inducing for every short exact sequence $0 \to X \to Y \to Z \to 0$ a long exact sequence

$$\begin{gathered} 0 \to \mathrm{Hom}(Z, A) \to \mathrm{Hom}(Y, A) \to \mathrm{Hom}(X, A) \xrightarrow{\delta_0^*} \mathrm{Ext}^1(Z, A) \to \\ \mathrm{Ext}^1(Y, A) \to \mathrm{Ext}^1(X, A) \xrightarrow{\delta_1^*} \mathrm{Ext}^2(Z, A) \to \ldots \to \mathrm{Ext}^{k-1}(X, A) \xrightarrow{\delta_{k-1}^*} \\ \mathrm{Ext}^k(Z, X) \to \mathrm{Ext}^k(Y, A) \to \mathrm{Ext}^k(X, A) \xrightarrow{\delta_k^*} \mathrm{Ext}^{k+1}(Z, A) \to \ldots \end{gathered}$$

of abelian groups and group morphisms.

The long exact sequences of the above theorem entail many useful corollaries. For example, it follows immediately that the vanishing of Ext^k is a three space property. Another important consequence is that, given a short exact sequence $0 \to X \to Y \to Z \to 0$ with $\mathrm{Ext}^1(A,Y) = 0$, the vanishing of $\mathrm{Ext}^2(A,X)$ implies $\mathrm{Ext}^1(A,Z) = 0$. Dually, if $\mathrm{Ext}^1(Y,A) = 0$ then the vanishing of $\mathrm{Ext}^2(Z,A)$ implies $\mathrm{Ext}^1(X,A) = 0$. Hence, when asking whether the vanishing of Ext^1 is passed on to quotients or subspaces, it is natural to consider the functor Ext^2, as will be done in chapter 4 for the space of distributions.

For objects X, Z the groups $\mathrm{Ext}^k(Z,X)$ in the above theorem are constructed as collections of equivalence classes of exact sequences of the form

$$0 \to X \to Y_{k-1} \to Y_{k-2} \to \ldots \to Y_1 \to Y_0 \to Z \to 0.$$

In an exact category a sequence as above is called exact if it can be written as a composition of elements of the exact structure, i.e. of short exact sequences:

Definition 4.1.2. Let $(\mathcal{C},\mathcal{E})$ be an exact category. A sequence

$$E\colon\ 0 \to X \xrightarrow{f_k} Y_{k-1} \xrightarrow{f_{k-1}} Y_{k-2} \to \ldots \to Y_1 \xrightarrow{f_1} Y_0 \xrightarrow{f_0} Z \to 0$$

is called *exact*, if every morphism f_l factors as $f_l = m_l \circ e_l$ for an admissible cokernel $e_l\colon Y_l \to I_l$ and an admissible kernel $m_l\colon I_l \to Y_{l-1}$ such that $(m_l, e_{l-1}) \in \mathcal{E}$ for $0 \le l \le k$ (treat X as Y_k and Z as Y_{-1}). In addition, k is called the *length* of the sequence E.

Note that if $(\mathcal{C},\mathcal{E})$ is the exact category $(\mathrm{PLS}, \mathcal{E}_{\mathrm{PLS}})$ described in the last chapter, the above notion of an exact sequence of length k coincides with the usual notion of a topologically exact sequence of length k since the exact structure $\mathcal{E}_{\mathrm{PLS}}$ consists by 3.2.1 of the sequences $0 \to F \to G \to H \to 0$ of PLS-spaces F, G, H that are topologically exact.

On the class of all exact sequences of length k who start with X on the left and end with Z on the right, one can then define an equivalence relation in the following way:

Definition 4.1.3. Let $(\mathcal{C},\mathcal{E})$ be an exact category and let

$$E\colon\ 0 \to X \xrightarrow{f_k} Y_{k-1} \xrightarrow{f_{k-1}} Y_{k-2} \to \ldots \to Y_1 \xrightarrow{f_1} Y_0 \xrightarrow{f_0} Z \to 0$$

be an exact sequence in $\mathcal{C}$. We then call X the left-end of E and Z the right-end of E. A morphism with fixed ends $\phi\colon E \to E'$ between exact sequences E and E' of length k is a commutative diagram

$$\begin{array}{ccccccccccccccc} E\colon & 0 & \to & X & \to & Y_{k-1} & \to & Y_{k-2} & \to \cdots \to & Y_1 & \to & Y_0 & \to & Z & \to 0 \\ & & & \| & & \downarrow{\scriptstyle \phi_{k-1}} & & \downarrow{\scriptstyle \phi_{k-2}} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \| & \\ E'\colon & 0 & \to & X' & \to & Y'_{k-1} & \to & Y'_{k-2} & \to \cdots \to & Y'_1 & \to & Y'_0 & \to & Z' & \to 0. \end{array}$$

For $X, Z \in \mathrm{Ob}(\mathcal{C})$ we define $\mathrm{E}^k(Z,X)$ to be the exact sequences of length k with right end Z and left end X. On $\mathrm{E}^k(Z,X)$ we define the following equivalence relation:

$E \sim E' \quad \Leftrightarrow$ There is a sequence $E = E_0, E_1, \ldots, E_{l-1}, E_l = E'$ of elements of $\mathrm{E}^k(Z, X)$, so that for every $0 \leq i \leq l-1$ there is either a morphism $E_i \to E_{i+1}$ with fixed ends or a morphism $E_{i+1} \to E_i$ with fixed ends.

We define

$$\mathrm{Ext}^k(Z, X) = \mathrm{E}^k(Z, X)/\sim$$

and denote by $[E]$ the equivalence class of E in $\mathrm{Ext}^k(Z, X)$.

Note that for an exact sequence $E \in \mathrm{E}^1(Z, X)$ a morphism with fixed ends is a diagram of the form

$$\begin{array}{lccccccccc} E\colon & 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & & \| & & \downarrow{\scriptstyle\gamma} & & \| & & \\ E'\colon & 0 & \longrightarrow & X & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & 0. \end{array} \tag{1}$$

In this case the short five lemma for exact categories (see [5, Corollary 3.2]) states that γ is an isomorphism. Therefore, for two elements E and E' one has $E \sim E'$ if and only if they fit into a diagram of the form (1). This shows that $\mathrm{Ext}^1(Z, X)$ is indeed the well-known class of extensions of X by Z.

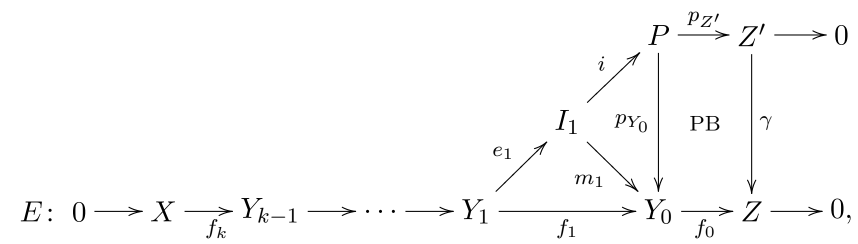
A logical difficulty, even in the classical case of abelian categories, arises from the fact that $\mathrm{Ext}^k(Z, X)$ may not be a set. Of course, if the category $\mathcal{C}$ is small, i.e. the class of its objects is a set, then $\mathrm{Ext}^k(Z, X)$ will be a set. Likewise, if the category $(\mathcal{C}, \mathcal{E})$ has enough injectives or enough projectives, then $\mathrm{Ext}^k(Z, X)$ will always be a set since then the Ext^k-functors can also be constructed as the derived functors of Hom, which take values in the category of sets (see the remark after 4.1.7). In addition, it can be shown that $\mathrm{Ext}^1(Z, X)$ is always a set, provided that the category $\mathcal{C}$ possesses either a generator or a cogenerator. Since in every full subcategory of the topological vector spaces (TVS) the ground field $\mathbb{K}$ is a generator, this will usually be the case in a functional analytic setting. However, in order to not restrict ourselves to any particular class of exact categories we introduce the notion of a big abelian group. This is defined in the same way as an ordinary abelian group, except that the underlying class need not be a set. We are prevented from talking about "the category of big abelian groups" because the class of morphisms between a given pair of big abelian groups need not be a set. They form only a so-called quasi-category (cf. [1]). We will not burden the reader with the theory of quasi-categories, but nevertheless we will talk about kernels, cokernels, images, etc., for big abelian groups. These can be defined in the same set-theoretic terms in which the corresponding notions for ordinary abelian groups are defined. Nor will we be kept from speaking of a big abelian group valued functor from a category, and a natural transformation of two such functors, since all of these can be defined for quasi-categories. Hence, from now on the term group will be understood to mean big group and we will denote the quasi-category of big abelian groups by (AB).

In order to define an addition on $\mathrm{Ext}^k(Z, X)$, we need first to introduce pullback- and pushout-constructions for exact sequences of arbitrary length:

Remark and Definition 4.1.4. Let $(\mathcal{C}, \mathcal{E})$ be an exact category, let X and Z be two objects of $\mathcal{C}$ and let

$$E\colon\ 0 \to X \xrightarrow{f_k} Y_{k-1} \xrightarrow{f_{k-1}} Y_{k-2} \to \ldots \to Y_1 \xrightarrow{f_1} Y_0 \xrightarrow{f_0} Z \to 0$$

be an exact sequence of length k. For a morphism $\gamma\colon Z' \to Z$ we find a commutative diagram



where $f_1 = m_1 \circ e_1$ is the canonical factorization of the morphism f_1, $(P, p_{Y_0}, p_{Z'})$ is the pullback of f_0 and γ and where i is the kernel of $p_{Z'}$ induced by 2.1.2. The axioms of an exact category show that $(i, p_{Z'}) \in \mathcal{E}$. Then the sequence

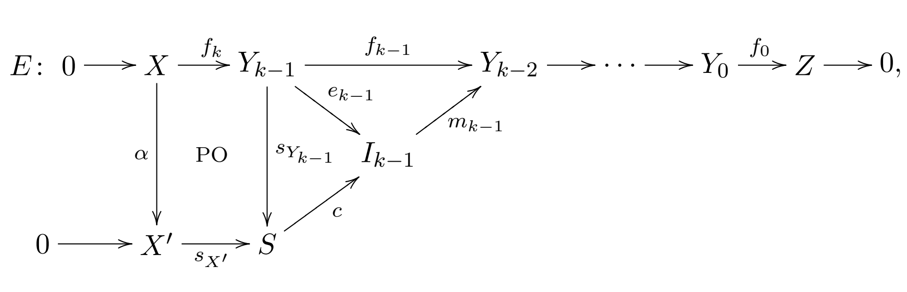
$$E_{\gamma,P}\colon\ 0 \to X \xrightarrow{f_k} Y_{k-1} \xrightarrow{f_{k-1}} Y_{k-2} \to \ldots \to Y_1 \xrightarrow{i \circ e_1} P \xrightarrow{p_{Z'}} Z' \to 0$$

is also exact. In addition, one can show that this pullback-construction does preserve the equivalence relation $\sim$, therefore the mapping

$$\mathrm{Ext}^k(Z, X) \to \mathrm{Ext}^k(Z', X),\ [E] \mapsto [E_{\gamma,P}] =: [E]\gamma$$

is well-defined.

By duality we find for every morphism $\alpha\colon X \to X'$ a commutative diagram



where $f_{k-1} = m_{k-1} \circ e_{k-1}$ is the canonical factorization of the morphism f_{k-1}, $(S, s_{Y_{k-1}}, s_{X'})$ is the pushout of f_k and α and with $(s_{X'}, c) \in \mathcal{E}$. Then the sequence

$$E_{S,\alpha}\colon\ 0 \to X' \xrightarrow{s_{X'}} S \xrightarrow{m_{k-1}\circ c} Y_{k-2} \to \ldots \to Y_1 \xrightarrow{f_1} Y_0 \xrightarrow{f_0} Z \to 0$$

is also exact and the mapping

$$\mathrm{Ext}^k_{\mathcal{E}}(Z,X) \to \mathrm{Ext}^k_{\mathcal{E}}(Z,X'),\ [E] \mapsto [E_{S,\alpha}] =: \alpha[E]$$

is well-defined.

Using these operations one can now define an addition on $\mathrm{Ext}^k(Z,X)$:

Remark and Definition 4.1.5 (Baer-sum). Given two exact sequences E, $F \in \mathrm{E}^k(Z,X)$ the component-wise direct sum of objects and morphisms

$$E \oplus F\colon\ 0 \to X \oplus X \xrightarrow{f_k \oplus f'_k} Y_{k-1} \oplus Y'_{k-1} \to \ldots \to Y_0 \oplus Y_0 \xrightarrow{f_0 \oplus f'_0} Z \oplus Z \to 0$$

is again an exact sequence (see [5, Proposition 2.9]). In addition, it is easy to show that this construction is well-behaved with respect to the equivalence relation $\sim$ and that therefore the mapping

$$\begin{array}{rcl}\oplus\colon\ \mathrm{Ext}^k(Z,X) \times \mathrm{Ext}^k(Z,X) & \to & \mathrm{Ext}^k(Z \oplus Z, X \oplus X)\\ ([E],[F]) & \mapsto & [E \oplus F]\end{array}$$

is well-defined. For an object X let

$$\Delta_X = \binom{\mathrm{id}_X}{\mathrm{id}_X}\colon X \to X \oplus X \quad \text{and} \quad \nabla_X = (\,\mathrm{id}_X,\, \mathrm{id}_X\,)\colon X \oplus X \to X$$

denote the canonical morphisms. Then it follows from the above and 4.1.4 that the mapping

$$\begin{array}{rcl}+\colon\ \mathrm{Ext}^k(Z,X) \times \mathrm{Ext}^k(Z,X) & \to & \mathrm{Ext}^k(Z,X)\\ ([E],[F]) & \mapsto & \Delta_X[E \oplus F]\nabla_Z\end{array}$$

is well-defined. $[E] + [F]$ is called the *Baer-sum* of $[E]$ and $[F]$. It makes $\mathrm{Ext}^k(Z,X)$ an abelian group.

The neutral element of $\mathrm{Ext}^k(Z,X)$ is given by the equivalence class $[0]$ of the exact sequence

$$0 \to X \xrightarrow{\binom{1}{0}} X \oplus Z \xrightarrow{(0,1)} Z \to 0$$

for $n = 1$ and by the equivalence class of the exact sequence

$$0 \to X \xrightarrow{\mathrm{id}_X} X \to 0 \to \ldots \to 0 \to Z \xrightarrow{\mathrm{id}_Z} Z \to 0$$

for $k > 1$. The additive inverse of a given equivalence class $[E]$ is the class $(-\,\mathrm{id}_X)[E]$.

It is easy to see that for $[E] \in \mathrm{Ext}^k(Z,X)$ one has

$$\mathrm{id}_X[E] = [E] = [E]\,\mathrm{id}_Z, \quad (\alpha' \circ \alpha) = \alpha'(\alpha[E]), \quad [E](\gamma \circ \gamma') = ([E]\gamma)\gamma',$$

whenever these are defined. In addition, it can be shown that the operations $[E] \mapsto [E]\gamma$ and $[E] \mapsto \alpha[E]$ are group morphisms with respect to the Baer-sum, so one actually has a covariant abelian-group-valued additive functor

$$\mathrm{Ext}^k(A,-)\colon \mathcal{C} \to (\mathrm{AB})\ , \ \begin{cases} X \mapsto \mathrm{Ext}^k(A,X) \\ \alpha \mapsto \mathrm{Ext}^k(A,\alpha) \end{cases},$$

where $\mathrm{Ext}^k(A,\alpha)$ is for a morphism $\alpha\colon X \to X'$ defined as

$$\mathrm{Ext}^k(A,\alpha)\colon\ \mathrm{Ext}^k(A,X) \to \mathrm{Ext}^k(A,X'),\ [E] \mapsto \alpha[E]$$

and a contravariant abelian-group-valued additive functor

$$\mathrm{Ext}^k(-,A)\colon \mathcal{C} \to (\mathrm{AB})\ , \ \begin{cases} Z \mapsto \mathrm{Ext}^k(Z,A) \\ \gamma \mapsto \mathrm{Ext}^k(\gamma,A) \end{cases},$$

where $\mathrm{Ext}^k(\gamma,A)$ is for a morphism $\gamma\colon Z' \to Z$ defined as

$$\mathrm{Ext}^k(\gamma,A)\colon\ \mathrm{Ext}^k(Z,A) \to \mathrm{Ext}^k(Z',A),\ [E] \mapsto [E]\gamma.$$

We have now introduced almost all the terms appearing in the long exact sequences 4.1.1, the only thing remaining is the definition of the connecting morphisms $\delta_k\colon \mathrm{Ext}^k(A,Z) \to \mathrm{Ext}^k(A,X)$ and $\delta_k^*\colon \mathrm{Ext}^k(A,Z) \to \mathrm{Ext}^k(A,X)$. We will remedy this shortly, but before this we have to say something about the composition of exact sequences:

Remark 4.1.6. Given two exact sequences

$$E\colon\ 0 \longrightarrow X \xrightarrow{f_n} Y_{n-1} \longrightarrow \cdots \longrightarrow Y_0 \xrightarrow{f_0} Z \longrightarrow 0$$
$$F\colon\ 0 \longrightarrow Z \xrightarrow{f'_m} Y'_{m-1} \longrightarrow \cdots \longrightarrow Y'_0 \xrightarrow{f'_0} Z' \longrightarrow 0$$

such that E is of length n with right-end Z and F is of length m with left-end Z, then the composed sequence

$$EF\colon\ 0 \longrightarrow X \xrightarrow{f_n} \cdots \xrightarrow{f_1} Y_0 \xrightarrow{f'_m \circ f_0} Y'_{m-1} \xrightarrow{f'_{m-1}} \cdots \xrightarrow{f'_0} Z' \longrightarrow 0$$
$$Y_0 \xrightarrow{f_0} Z \xrightarrow{f'_m} Y'_{m-1}$$

is an an exact sequence of length $m+n$. It can be shown that this composition is well-behaved with respect to the equivalence relation $\sim$ and that the pairing

$$\begin{aligned} \sigma\colon\ \mathrm{Ext}^n(W,X) \times \mathrm{Ext}^m(Z,W) &\to \mathrm{Ext}^{m+n}(Z,X) \\ ([E],[F]) &\mapsto [EF] =: [E][F] \end{aligned}$$

is bilinear with respect to the Baer-sum.

With the help of the above we can finally define the connecting morphisms:

Remark and Definition 4.1.7. Given a short exact sequence

$$E\colon\ 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

we obtain for $A \in \mathrm{Ob}(\mathcal{C})$ the connecting morphisms

$$\delta_0\colon\ \mathrm{Hom}(A,Z) \to \mathrm{Ext}^1(A,X),\ \ \gamma \mapsto [E]\gamma$$
$$\delta_0^*\colon\ \mathrm{Hom}(X,A) \to \mathrm{Ext}^1(Z,A),\ \ \alpha \mapsto \alpha[E]$$

of degree zero and the connecting morphisms

$$\delta_k\colon\ \mathrm{Ext}^k(A,Z) \to \mathrm{Ext}^{k+1}(A,X),\ [F] \mapsto [E][F]$$
$$\delta_k^*\colon\ \mathrm{Ext}^k(X,A) \to \mathrm{Ext}^{k+1}(Z,A),\ [H] \mapsto [H][E]$$

of degree $k \geq 1$. That the latter ones are group morphisms is a consequence of 4.1.6.

We now have gathered all that is necessary for the long exact sequence 4.1.1. Let us remark that the so-called Schanuel lemma (see [20, Lemma 4.1]) is crucial for the proof of the exactness of this sequence. This lemma states that a composition $[E][F]$ of equivalence classes is zero if and only if $[E]$ can be written as $[E] = [G]\gamma$ with $\gamma[F] = [0]$, or equivalently, if $[F]$ can be written as $[F] = \alpha[H]$ with $[E]\alpha = [0]$. The rest of the proof of 4.1.1 is a straightforward calculation.

In addition, let us remark that one can show that the collections of connected functors $(\mathrm{Ext}^k(A,-))_{k\geq 0}$ and $(\mathrm{Ext}^k(-,A))_{k\geq 0}$ form universal δ-functors in the sense of Grothendieck [12] when one defines $\mathrm{Ext}^0(A,-)$ as $\mathrm{Hom}(A,-)$ and $\mathrm{Ext}^0(-,A)$ as $\mathrm{Hom}(-,A)$. Therefore, if the exact category $(\mathcal{C},\mathcal{E})$ has enough injectives, it follows from the fact that universal δ-functors are unique up to an isomorphism of δ-functors that the functors $\mathrm{Ext}^k(A,-)$ are isomorphic to the derived functors of $\mathrm{Hom}(A,-)$ and, if $(\mathcal{C},\mathcal{E})$ has enough projectives, the functors $\mathrm{Ext}^k(-,A)$ are isomorphic to the derived functors of $\mathrm{Hom}(-,A)$. Especially one has

$$\mathrm{Ext}^k(Z,X) \cong R^k\,\mathrm{Hom}(Z,X)$$

in every exact category having either enough injectives or enough projectives. A direct construction of the above isomorphism of abelian groups can also be found in [20, VII.7].

4.2 The Characterization of Ext^k(Z,X)=0

In this section we will provide characterizations of $\mathrm{Ext}^k(Z,X) = 0$ using only the representants of the equivalence classes forming this group. These characterizations will be the basis for our investigation of the splitting theory of the exact category $((\mathrm{PLS}), \mathcal{E}_{\mathrm{PLS}})$ in the next chapter.

As one would expect, the vanishing of the Ext^1-group is strongly connected with the splitting of short exact sequences. To see this, let us recall the definition of the latter:

Remark and Definition 4.2.1. In any exact category $(\mathcal{C}, \mathcal{E})$ it is easy to show that for a short exact sequence

$$E\colon\ 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

the following are equivalent:

i) f has a left inverse.

ii) g has a right inverse.

iii) There is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\binom{1}{0}} & X \oplus Z & \xrightarrow{(0,1)} & Z & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\beta} & & \| & & \\ 0 & \longrightarrow & X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \longrightarrow & 0 \end{array}$$

such that β is an isomorphism.

A short exact sequence is called *split exact* if it satisfies the above properties.

The property iii) shows that a short exact sequence E in an exact category is split exact if and only if $[E] = [0]$ in the corresponding Ext^1-group. Therefore, the vanishing of this group characterizes the splitting of short exact sequences:

Proposition 4.2.2. Let $(\mathcal{C}, \mathcal{E})$ be an exact category. For $X, Z \in \mathrm{Ob}(\mathcal{C})$ the following are equivalent:

i) $\mathrm{Ext}^1(Z, X) = 0$.

ii) Every exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is split exact.

For $k > 1$ the equivalence relation $\sim$ needed for the definition of $\mathrm{Ext}^k(Z, X)$ is rather unwieldy and does, at first sight, not seem very useful for calculations. The following characterization of $[E] = [0]$, which is also a consequence of the Schanuel lemma (see [20, VII, Lemma 4.1]), improves this situation:

Lemma 4.2.3 ([20], VII, Theorem 4.2). Let $(\mathcal{C}, \mathcal{E})$ be an exact category and $[E] \in \mathrm{Ext}^k(Z, X)$. The following are equivalent:

i) $[E] = [0]$.

ii) There is an $F \in \mathrm{E}^k(Z, X)$ and a sequence $E \to F \leftarrow 0$ of morphisms with fixed ends, i.e. there is a commutative diagram with exact rows

$$\begin{array}{rccccccccccccccccc}
E\colon & 0 & \to & X & \to & Y_{n-1} & \to & Y_{n-2} & \to \cdots \to & Y_1 & \to & Y_0 & \xrightarrow{f_0} & Z & \to & 0 \\
& & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \| & & \\
F\colon & 0 & \to & X & \to & W_{n-1} & \to & W_{n-2} & \to \cdots \to & W_1 & \to & W_0 & \to & Z & \to & 0 \\
& & & \| & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \| & & \\
0\colon & 0 & \to & X & = & X & \to & 0 & \to \cdots \to & 0 & \to & Z & = & Z & \to & 0.
\end{array}$$

iii) There is a $G \in \mathrm{E}^k(Z, X)$ and a sequence $E \leftarrow G \to 0$ of morphisms with fixed ends, i.e. there is a commutative diagram with exact rows

$$\begin{array}{rccccccccccccccccc}
E\colon & 0 & \to & X & \to & Y_{n-1} & \to & Y_{n-2} & \to \cdots \to & Y_1 & \to & Y_0 & \xrightarrow{f_0} & Z & \to & 0 \\
& & & \| & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \| & & \\
G\colon & 0 & \to & X & \to & V_{n-1} & \to & V_{n-2} & \to \cdots \to & V_1 & \to & V_0 & \to & Z & \to & 0 \\
& & & \| & & \downarrow & & \updownarrow & & \downarrow & & \downarrow & & \| & & \\
0\colon & 0 & \to & X & = & X & \to & 0 & \to \cdots \to & 0 & \to & Z & = & Z & \to & 0.
\end{array}$$

The above characterization of $[E] = [0]$ using morphisms with fixed ends also gives a very useful characterization of $[E] = [0]$ using only a representant E of this equivalence class. This characterization already appeared in the work of Yoneda [41] for abelian categories. We nonetheless provide a proof of it in the setting of exact categories since Yoneda's terminology widely differs from the one commonly used today.

Lemma 4.2.4. Let $(\mathcal{C}, \mathcal{E})$ be an exact category. For every exact sequence

$$E\colon\ 0 \to X \xrightarrow{f_n} Y_{n-1} \xrightarrow{f_{n-1}} Y_{n-2} \to \cdots \to Y_1 \xrightarrow{f_1} Y_0 \xrightarrow{f_0} Z \to 0$$

in $\mathcal{C}$ with $n > 1$ the following are equivalent:

i) $[E] = 0$.

ii) There is a commutative diagram with exact rows

$$\begin{array}{rcccccccccccccc}
E\colon & 0 & \to & X & \xrightarrow{f_n} & Y_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-2} & \to \cdots \to & Y_1 & \xrightarrow{f_1} & Y_0 & \xrightarrow{f_0} & Z & \to 0 \\
& & & \| & & \downarrow{\scriptstyle \phi_{n-1}} & & \downarrow{\scriptstyle \phi_{n-2}} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & & \\
F\colon & 0 & \to & X & \xrightarrow[g_n]{} & V_{n-1} & \xrightarrow[g_{n-1}]{} & V_{n-2} & \to \cdots \to & V_1 & \xrightarrow[g_1]{} & V_0 & \to 0. & &
\end{array}$$

iii) There is a commutative diagram with exact rows

$$\begin{array}{rcccccccccccccc}
E\colon & 0 & \to & X & \xrightarrow{f_n} & Y_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-2} & \to \cdots \to & Y_1 & \xrightarrow{f_1} & Y_0 & \xrightarrow{f_0} & Z & \to 0 \\
& & & \| & & \downarrow{\scriptstyle \phi_{n-1}} & & \downarrow{\scriptstyle \phi_{n-2}} & & \downarrow{\scriptstyle \phi_1} & & \| & & & \\
F\colon & 0 & \to & X & \xrightarrow[g_n]{} & V_{n-1} & \xrightarrow[g_{n-1}]{} & V_{n-2} & \to \cdots \to & V_1 & \xrightarrow[g_1]{} & Y_0 & \to 0. & &
\end{array}$$

iv) There is a commutative diagram with exact rows

$$
\begin{array}{lcccccccccccccc}
F\colon & & 0 \to & V_{n-1} & \xrightarrow{g_{n-1}} & V_{n-2} & \to \cdots \to & V_1 & \xrightarrow{g_1} & V_0 & \xrightarrow{g_0} & Z & \to 0 \\
 & & & \downarrow \phi_{n-1} & & \downarrow \phi_{n-2} & & \downarrow \phi_1 & & \downarrow \phi_0 & & \| & \\
E\colon & 0 \to & X \xrightarrow{f_n} & Y_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-2} & \to \cdots \to & Y_1 & \xrightarrow{f_1} & Y_0 & \xrightarrow{f_0} & Z & \to 0.
\end{array}
$$

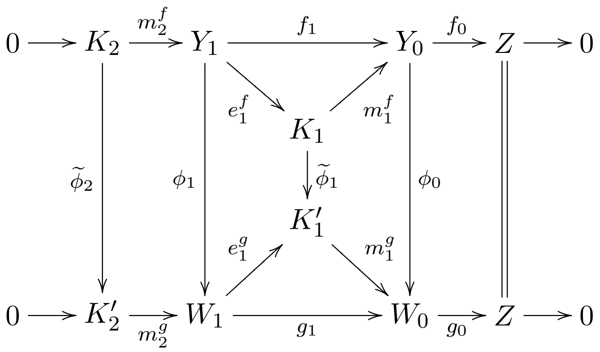
v) There is a commutative diagram with exact rows

$$
\begin{array}{lcccccccccccccc}
F\colon & & 0 \to & Y_{n-1} & \xrightarrow{g_{n-1}} & V_{n-2} & \to \cdots \to & V_1 & \xrightarrow{g_1} & V_0 & \xrightarrow{g_0} & Z & \to 0 \\
 & & & \| & & \downarrow \phi_{n-2} & & \downarrow \phi_1 & & \downarrow \phi_0 & & \| & \\
E\colon & 0 \to & X \xrightarrow{f_n} & Y_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-2} & \to \cdots \to & Y_1 & \xrightarrow{f_1} & Y_0 & \xrightarrow{f_0} & Z & \to 0.
\end{array}
$$

Proof. (i)⇒(ii) If E is an element of $\mathrm{E}^n(Z,X)$ with $[E]=0$, there is by 4.2.3 a sequence $E \to F \leftarrow 0$ of elements of $\mathrm{E}^n(Z,X)$ and morphisms with fixed ends, i.e. there is a commutative diagram with exact rows

$$
\begin{array}{lccccccccccccc}
E\colon & 0 \to & X & \xrightarrow{f_n} & Y_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-2} & \to \cdots \to & Y_1 & \xrightarrow{f_1} & Y_0 & \xrightarrow{f_0} & Z & \to 0 \\
 & & \| & & \downarrow \phi_{n-1} & & \downarrow \phi_{n-2} & & \downarrow \phi_1 & & \downarrow \phi_0 & & \| & \\
F\colon & 0 \to & X & \xrightarrow{g_n} & W_{n-1} & \xrightarrow{g_{n-1}} & W_{n-2} & \to \cdots \to & W_1 & \xrightarrow{g_1} & W_0 & \xrightarrow{g_0} & Z & \to 0 \\
 & & \| & & \uparrow g_n & & \uparrow & & \uparrow & & \uparrow \psi_0 & & \| & \\
0\colon & 0 \to & X & = & X & \longrightarrow & 0 & \to \cdots \to & 0 & \longrightarrow & Z & = & Z & \to 0.
\end{array}
$$

Therefore, ψ_0 is a right inverse of the morphism g_0. Let then $f_k = m_k^f \circ e_k^f$ and $g_k = m_k^g \circ e_k^g$ be the canonical factorizations of the morphisms f_k and g_k and let $\widetilde{\phi}_k$ be the unique morphism with $\phi_{k-1} \circ m_k^f = m_k^g \circ \widetilde{\phi}_k$ and $\widetilde{\phi}_k \circ e_k^f = e_k^g \circ \phi_{k+1}$ for $k = 1, \ldots, n-1$, which are inductively obtained as the induced morphisms between the cokernels or the kernels in the short exact sequences forming the longer exact sequences, respectively. Then we have the commutative diagram with exact rows



induced by the upper right corner of the previous diagram. Since g_0 has a right inverse, we know by 4.2.1 that m_1^g has a left inverse. Choose a left inverse $h\colon W_0 \to K_1'$ and define $\mu := h \circ \phi_0$. This yields

$$m_1^g \circ \mu \circ m_1^f = m_1^g \circ h \circ \phi_0 \circ m_1^f = m_1^g \circ h \circ m_1^g \circ \widetilde{\phi}_1 = m_1^g \circ \widetilde{\phi}_1$$

and therefore $\mu \circ m_1^f = \widetilde{\phi}_1$ since m_1^g is a monomorphism. Then the diagram

$$\begin{array}{rcccccccccccccc}
E\colon & 0 \longrightarrow & X & \xrightarrow{f_n} & Y_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-2} & \longrightarrow \cdots \longrightarrow & Y_1 & \xrightarrow{f_1} & Y_0 & \xrightarrow{f_0} & Z & \longrightarrow 0 \\
& & \| & & \downarrow{\scriptstyle \phi_{n-1}} & & \downarrow{\scriptstyle \phi_{n-2}} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \mu} & & & \\
F'\colon & 0 \longrightarrow & X & \xrightarrow[g_n]{} & W_{n-1} & \xrightarrow[g_{n-1}]{} & W_{n-2} & \longrightarrow \cdots \longrightarrow & W_1 & \xrightarrow[e_1^g]{} & K_1' & \longrightarrow 0 & &
\end{array}$$

is commutative, which shows (ii).
(ii)⇒(iii) Let

$$\begin{array}{ccccccccc}
& & & & K_1 & & & & \\
& & & \nearrow{\scriptstyle e_1^f} & & \searrow{\scriptstyle m_1^f} & & & \\
0 \longrightarrow & K_2 & \xrightarrow{m_2^f} & Y_1 & \xrightarrow{\quad f_1 \quad} & Y_0 & \xrightarrow{f_0} & Z & \longrightarrow 0 \\
& \downarrow{\scriptstyle \widetilde{\psi}_2} & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_0} & & & \\
0 \longrightarrow & K_2' & \xrightarrow[m_2^g]{} & V_1 & \xrightarrow[\quad g_1 \quad]{} & V_0 & \longrightarrow 0 & &
\end{array}$$

be the commutative diagram with exact rows induced by the right hand side of the diagram of the assumption and the canoncial factorizations of the morphisms of this diagram. Since g_1 is an admissible cokernel, the pullback (P, p_{Y_0}, p_{V_1}) of ψ_0 and g_1 exists and we have a commutative diagram with exact rows

$$\begin{array}{ccccc}
K_2' & \xrightarrow{k} & P & \xrightarrow{p_{Y_0}} & Y_0 \\
\| & & \downarrow{\scriptstyle p_{V_1}} & & \downarrow{\scriptstyle \psi_0} \\
K_2' & \xrightarrow[m_2^g]{} & V_1 & \xrightarrow[g_1]{} & V_0.
\end{array}$$

Since $\psi_0 \circ f_1 = g_1 \circ \psi_1$ the universal property of the pullback gives rise to a unique morphism $\lambda\colon Y_1 \to P$ with $f_1 = p_{Y_0} \circ \lambda$ and $\psi_1 = p_{V_1} \circ \lambda$. Then we also have

$$p_{V_1} \circ \lambda \circ m_2^f = \psi_1 \circ m_2^f = m_2^g \circ \widetilde{\psi}_2$$
$$p_{Y_0} \circ \lambda \circ m_2^f = f_1 \circ m_2^f = 0 = p_{Y_0} \circ k \circ \widetilde{\psi}_2$$

and therefore $\lambda \circ m_2^f = k \circ \widetilde{\psi}_2$ by the universal property of the pullback. This shows that the diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & K_2 & \xrightarrow{m_2^f} & Y_1 & \xrightarrow{f_1} & Y_0 & \longrightarrow 0 \\
& \| & & \downarrow{\scriptstyle \lambda} & & \| & \\
0 \longrightarrow & K_2' & \xrightarrow[k]{} & V_1 & \xrightarrow[p_{Y_0}]{} & Y_0 & \longrightarrow 0
\end{array}$$

is commutative and it fits together with the rest of the diagram of the assumption to give the desired commutative diagram.

(iii)⇒(i) Let $[E]$ be an element of $\mathrm{Ext}^n(Z,X)$. If the diagram of the assumption is commutative, then the diagram

$$\begin{array}{rccccccccccccccccc}
E\colon & 0 & \to & X & \xrightarrow{f_n} & Y_{n-1} & \xrightarrow{f_{n-1}} & Y_{n-2} & \to & \cdots & \to & Y_1 & \xrightarrow{f_1} & Y_0 & \xrightarrow{f_0} & Z & \to & 0 \\
 & & & \| & & \downarrow{\scriptstyle \phi_{n-1}} & & \downarrow{\scriptstyle \phi_{n-2}} & & & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \binom{\phi_0}{0}} & & \| & & \\
F\colon & 0 & \to & X & \xrightarrow[g_n]{} & V_{n-1} & \xrightarrow[g_{n-1}]{} & V_{n-2} & \to & \cdots & \to & V_1 & \xrightarrow[\binom{g_1}{0}]{} & V_0 \oplus Z & \xrightarrow[(g_0,1)]{} & Z & \to & 0 \\
 & & & \| & & \uparrow{\scriptstyle g_n} & & \uparrow & & & & \uparrow & & \uparrow{\scriptstyle \omega_Z} & & \| & & \\
0\colon & 0 & \to & X & = & X & \to & 0 & \to & \cdots & \to & 0 & \to & Z & = & Z & \to & 0
\end{array}$$

is also commutative, which shows $[E]=0$.

By following the dual arguments of the proofs (i)⇒(ii)⇒(iii)⇒(i) one can analogously show (i)⇒(iv)⇒(v)⇒(i). □

The above directly yields a characterization for the vanishing of the higher Ext-groups which does not involve the equivalence relation $\sim$:

Corollary 4.2.5. For an exact category $(\mathcal{C},\mathcal{E})$, $k>1$ and X, $Z \in \mathrm{Ob}(\mathcal{C})$ the following are equivalent:

i) $\mathrm{Ext}^k(Z,X)=0$.

ii) Every exact sequence

$$0 \to X \to Y_{k-1} \to Y_{k-2} \to \cdots \to Y_1 \to Y_0 \to Z \to 0$$

in $\mathcal{C}$ has the equivalent properties of lemma 4.2.4.

We note the following for later use:

Proposition 4.2.6. Let $(\mathcal{C},\mathcal{E})$ be an exact category and let X be an object of $\mathcal{C}$ such that $\mathrm{Ext}^2(Z,X)=0$ for all objects Z of $\mathcal{C}$. Then $\mathrm{Ext}^k(Z,X)=0$ for $k \geq 2$ and all objects Z of $\mathcal{C}$.

Proof. We prove the assertion by induction on k. For $k>2$ let

$$E\colon\ 0 \to X \to Y_{k-1} \to Y_{k-2} \to \ldots \to Y_1 \xrightarrow{f_1} Y_0 \to Z \to 0$$

be an exact sequence and let $f_1 = m_1 \circ e_1$ be the canonical factorization of f_1 into an admissible epimorphism $e_1\colon Y_1 \to I_1$ and an admissible monomorphism $m_1\colon I_1 \to Y_0$. Then we also have the exact sequence of length $k-1$

$$\widetilde{E}\colon\ 0 \to X \to Y_{k-1} \to Y_{k-2} \to \ldots \to Y_1 \xrightarrow{e_1} I_1 \to 0.$$

By induction we have $\operatorname{Ext}^{k-1}(I_1, X) = 0$, hence there exists a commutative diagram

$$\begin{array}{cccccccccccccccc}
\widetilde{E}\colon & 0 & \longrightarrow & X & \longrightarrow & Y_{k-1} & \longrightarrow & Y_{k-2} & \longrightarrow & \cdots & \longrightarrow & Y_1 & \longrightarrow & I_1 & \longrightarrow & 0 \\
& & & \| & & \downarrow & & \downarrow & & & & \downarrow & & & & \\
\widetilde{F}\colon & 0 & \longrightarrow & X & \longrightarrow & V_{k-1} & \longrightarrow & V_{k-2} & \longrightarrow & \cdots & \longrightarrow & V_1 & \longrightarrow & 0 & &
\end{array}$$

by 4.2.4. Then one also has the commutative diagram with exact rows

$$\begin{array}{cccccccccccccccccc}
E\colon & 0 & \longrightarrow & X & \longrightarrow & Y_{k-1} & \longrightarrow & Y_{k-2} & \longrightarrow & \cdots & \longrightarrow & Y_1 & \longrightarrow & Y_0 & \longrightarrow & Z & \longrightarrow & 0 \\
& & & \| & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \\
F\colon & 0 & \longrightarrow & X & \longrightarrow & V_{k-1} & \longrightarrow & V_{k-2} & \longrightarrow & \cdots & \longrightarrow & V_1 & \longrightarrow & 0 & \longrightarrow & 0, & &
\end{array}$$

which shows $[E] = [0]$ by 4.2.4 and thus $\operatorname{Ext}^k(Z, X) = 0$. □

Chapter 5

Ext^k-functors for PLS-spaces

The splitting theory of PLS-spaces is concerned with the following problem: Characterize the pairs (Z, X) of PLS-spaces X and Z such that every topologically exact sequence

$$0 \to X \to Y \to Z \to 0 \tag{1}$$

is split exact. The problem is of special interest if one of the spaces X or Z is a classical PLS-space, like the space of distributions $\mathcal{D}'(\Omega)$ or the space of analytic functions $\mathcal{A}(\Omega)$. The splitting theory of PLS-spaces has been investigated by different authors (see, e.g., [9, 40, 4, 17]), but the theory is far from being complete. Up to now, the splitting problem was investigated by using an ad-hoc definition of $\mathrm{Ext}^1_{\mathrm{PLS}}(Z, X) = 0$, i.e. one uses the splitting characterization of 4.2.2 as a definition of $\mathrm{Ext}^1_{\mathrm{PLS}}(Z, X) = 0$. The abstract tools of the third chapter applied to the pre-abelian category (PLS) and the maximal exact structure $\mathcal{E}_{PLS}$ of 3.2.1 show that one gets, in analogy to locally convex spaces, Banach spaces or Fréchet spaces, abelian group valued functors $\mathrm{Ext}^k_{\mathrm{PLS}}$. In this way, every exact sequence (1) splits, if and only if the group $\mathrm{Ext}^1_{\mathrm{PLS}}(Z, X)$ is trivial.

In this last chapter we will investigate the functors $\mathrm{Ext}^k_{\mathrm{PLS}}$ more closely. In the first section we will establish a connection between the functors Ext^k_{PLS} for PLS-spaces and the functors Ext^k_{LS} for LS-spaces. Amongst other things, we will show that for the vanishing of $\mathrm{Ext}^k_{PLS}(Z, \prod_{n\in\mathbb{N}} X_n)$, where the X_n are LS-spaces, it is sufficient that $\mathrm{Ext}^k_{LS}(Z_n, X_m) = 0$ for all $n, m \in \mathbb{N}$, where the Z_n are LS-spaces giving rise to the PLS-space Z. In the second section we will apply these results to the canonical resolution

$$0 \to X \to \prod_{n\in\mathbb{N}} X_n \xrightarrow{\sigma_X} \prod_{n\in\mathbb{N}} X_n$$

of a PLS-space X to arrive at an analogue of a result for Fréchet spaces, which connects the functors Ext^1 and Proj^1 and also gives sufficient conditions for the vanishing of the higher Ext^k (see [40, Proposition 5.1.5]). In the last section we make use of this result and compute for $k \geq 2$ the groups $\mathrm{Ext}^2_{\mathrm{PLS}}(E, F)$ for a closed subspace E and a Hausdorff quotient F of the space of distributions $\mathcal{D}'(\Omega)$. This in turn yields a new proof of a result of Wengenroth [40, Theorem 5.3.8], which states that $\mathrm{Ext}^1_{\mathrm{PLS}}(E, F) = 0$ in the above setting.

5.1 $\mathrm{Ext}_{\mathrm{PLS}}$ and $\mathrm{Ext}_{\mathrm{LS}}$

We have seen in the second chapter that the class of all topologically exact sequences $0 \to X \to Y \to Z \to 0$ of PLS-spaces X, Y, Z form the maximal exact structure $\mathcal{E}_{\mathrm{PLS}}$ of the pre-abelian category (PLS) of PLS-spaces and continuous linear maps. Applying 4.1.1 to the exact category $(\mathrm{PLS}, \mathcal{E}_{\mathrm{PLS}})$ gives us for each PLS-space A the covariant additive functors

$$\mathrm{Ext}^k_{\mathrm{PLS}}(A, -)\colon (\mathrm{PLS}) \to (\mathrm{AB}) \,, \left\{ \begin{array}{c} X \mapsto \mathrm{Ext}^k_{\mathrm{PLS}}(A, X) \\ \alpha \mapsto \mathrm{Ext}^k_{\mathrm{PLS}}(A, \alpha) \end{array} \right. ,$$

and the contravariant additive functors

$$\mathrm{Ext}^k_{\mathrm{PLS}}(-, A)\colon (\mathrm{PLS}) \to (\mathrm{AB}) \,, \left\{ \begin{array}{c} X \mapsto \mathrm{Ext}^k_{\mathrm{PLS}}(A, X) \\ \alpha \mapsto \mathrm{Ext}^k_{\mathrm{PLS}}(A, \alpha) \end{array} \right. ,$$

which induce for every short exact sequence of PLS-spaces the covariant and contravariant long exact sequences of 4.1.1.

An immediate consequence of these long exact sequences is that the vanishing of $\mathrm{Ext}^k_{\mathrm{PLS}}$ is a three space property:

Proposition 5.1.1. Let $0 \to X \to Y \to Z \to 0$ be a topologically exact sequence of PLS-spaces and continuous linear maps and let A be a PLS-space. Then for all $k \geq 1$ we have

i) $\mathrm{Ext}^k_{\mathrm{PLS}}(A, X) = 0$ and $\mathrm{Ext}^k_{\mathrm{PLS}}(A, Z) = 0$ imply $\mathrm{Ext}^k_{\mathrm{PLS}}(A, Y) = 0$,

ii) $\mathrm{Ext}^k_{\mathrm{PLS}}(X, A) = 0$ and $\mathrm{Ext}^k_{\mathrm{PLS}}(Z, A) = 0$ imply $\mathrm{Ext}^k_{\mathrm{PLS}}(Y, A) = 0$.

Proof. The covariant long exact sequence 4.1.1 reads

$$\ldots \to \mathrm{Ext}^k_{\mathrm{PLS}}(A, X) \to \mathrm{Ext}^k_{\mathrm{PLS}}(A, Y) \to \mathrm{Ext}^k_{\mathrm{PLS}}(A, Z) \to \ldots,$$

which immediately shows (i). Analogously, (ii) follows from the contravariant long exact sequence. □

Note that the assertion (i) in the above proposition was already shown in the case $k = 1$ by Domański and Vogt [9, Proposition 1.12] under the additional assumption that the PLS-space A is ultrabornological.

Since the Ext^k-functors and long exact sequences of 4.1.1 are defined for every exact category, they are defined for every quasi-abelian category. Especially, they are defined for the category (LS) of LS-spaces and continuous linear maps. In the rest of this section we will investigate the connection between the functors $\mathrm{Ext}^k_{\mathrm{LS}}$ and $\mathrm{Ext}^k_{\mathrm{PLS}}$. Recall that a short exact sequence of projective spectra and morphisms

$$0 \to \mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z} \to 0$$

consists of connected topologically exact sequences of locally convex spaces of the form $0 \to X_n \xrightarrow{f_n} Y_n \xrightarrow{g_n} Z_n \to 0$ on all steps of the spectra. It has been shown by Domański and Vogt that every short exact sequence of PLS-spaces arises as the projective limit of a short exact sequence of projective spectra (see [9, section 1]):

Lemma 5.1.2. Given an exact sequence of PLS-spaces

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \tag{1}$$

there are strongly reduced spectra $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ of LS-spaces and a sequence

$$0 \to \mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \to 0 \tag{2}$$

which is exact in the category of locally convex spectra such that (1) is the projective limit of the sequence (2). Moreover, if $\widetilde{\mathcal{X}}$, $\widetilde{\mathcal{Y}}$, $\widetilde{\mathcal{Z}}$ are strongly reduced spectra of LS-spaces having X, Y, Z as projective limits we can take either $\mathcal{Y} = \widetilde{\mathcal{Y}}$ or $\mathcal{X}$ and $\mathcal{Z}$ as subsequences of $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{Z}}$.

Remark 5.1.3. The above lemma also implies that being an LS-space is a three space property in the category (PLS), i.e. given a short exact sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \tag{1}$$

of PLS-spaces with X, Z being LS-spaces it follows that Y is also an LS-space. In fact, 5.1.2 gives strongly reduced projective spectra $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ such that (1) arises as the projective limit of a short exact sequence

$$0 \to \mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \to 0$$

of locally convex spectra, and since X and Z are LS-spaces one can take $\mathcal{X}$, $\mathcal{Z}$ to be the constant spectra $\mathcal{X} = (X, \mathrm{id}_X)$, $\mathcal{Z} = (Z, \mathrm{id}_Z)$. This yields a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle Y^1_\infty} & & \| & & \\ 0 & \longrightarrow & X & \xrightarrow[f_1]{} & Y_1 & \xrightarrow[g_1]{} & Z & \longrightarrow & 0 \end{array}$$

where the lower row is the first step of the exact sequence of projective spectra and consists therefore only of LS-spaces. The five lemma for exact categories (see [5, Corollary 3.2]) then implies that Y^1_∞ is an isomorphism, hence Y is an LS-space.

The above connection between the short exact sequences of PLS- and LS-spaces yields the following result, which will be crucial for our further investigation:

Proposition 5.1.4. Let X be an LS-space, Z a PLS-space and

$$E\colon\ 0 \longrightarrow X \xrightarrow{f_n} Y_{n-1} \xrightarrow{f_{n-1}} Y_{n-2} \longrightarrow \cdots \longrightarrow Y_1 \xrightarrow{f_1} Y_0 \xrightarrow{f_0} Z \longrightarrow 0$$

an exact sequence of PLS-spaces. If $\mathscr{Z} = (Z_n, Z_m^n)$ is a strongly reduced spectrum with $\mathrm{Proj}(\mathscr{Z}) = Z$, there exists $n_0 \in \mathbb{N}$ and an exact sequence

$$H\colon\ 0 \longrightarrow X \xrightarrow{g_n} W_{n-1} \xrightarrow{g_{n-1}} W_{n-2} \longrightarrow \cdots \longrightarrow W_1 \xrightarrow{g_1} W_0 \xrightarrow{g_0} Z_{n_0} \longrightarrow 0$$

of LS-spaces with $[E] = [H]\, Z_\infty^{n_0}$.

The proof of the above will also make use of the following factorization lemma for morphisms of short exact sequences:

Lemma 5.1.5 ([5], Proposition 3.1). Let $(\mathcal{C}, \mathcal{E})$ be an exact category and let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \longrightarrow & 0 \end{array}$$

be a commutative diagram with exact rows. Then there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle a} & (1) & \downarrow{\scriptstyle b'} & & \| & & \\ 0 & \longrightarrow & X & \xrightarrow[m]{} & D & \xrightarrow{e} & Z' & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle b''} & (2) & \downarrow{\scriptstyle c} & & \\ 0 & \longrightarrow & X & \xrightarrow[f]{} & Y & \xrightarrow[g]{} & Z & \longrightarrow & 0, \end{array}$$

such that the diagrams (1) and (2) are pullback as well as pushout squares and $b'' \circ b' = b$.

Proof of 5.1.4. First, we decompose the exact sequence E into short exact sequences of PLS-spaces

(1) $0 \longrightarrow X \xrightarrow{f_n} Y_{n-1} \xrightarrow{e_{n-1}} I_{n-1} \longrightarrow 0$,

(2) $0 \longrightarrow I_k \xrightarrow{m_k} Y_{k-1} \xrightarrow{e_{k-1}} I_{k-1} \longrightarrow 0$ for $2 \leq k \leq n-1$,

(3) $0 \longrightarrow I_1 \xrightarrow{m_1} Y_0 \xrightarrow{f_0} Z \longrightarrow 0$.

By 5.1.2 there exist strongly reduced spectra $\mathscr{X}$, $\mathscr{Y}_{n-1}$, $\mathscr{I}_{n-1}$ of LS-spaces and a sequence

$$0 \to \mathscr{X} \to \mathscr{Y}_{n-1} \to \mathscr{I}_{n-1} \to 0 \tag{I}$$

which is exact in the category of locally convex spectra such that (1) is the projective limit of (I). Furthermore, we can take $\mathscr{X}$ to be the constant spectrum $\mathscr{X} = (X, \mathrm{id}_X)_{n\in\mathbb{N}}$. The fact that (1) arises as the projective limit of (I) implies that we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f_n} & Y_{n-1} & \xrightarrow{e_{n-1}} & I_{n-1} & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle Y^1_{n-1,\infty}} & & \downarrow{\scriptstyle I^1_{n-1,\infty}} & & \\
0 & \longrightarrow & X & \xrightarrow[f_{n,1}]{} & Y_{n-1,1} & \xrightarrow[e_{n-1,1}]{} & I_{n-1,1} & \longrightarrow & 0,
\end{array}$$

where the lower row is the first step of the sequence (I) of projective spectra of LS-spaces and where the vertical arrows are the canonical morphisms from the projective limits to the respective steps.
Again, use 5.1.2 to find strongly reduced spectra $\widetilde{\mathscr{I}}_{n-1}$, $\mathscr{Y}_{n-2}$, $\mathscr{I}_{n-2}$ of LS-spaces and an exact sequence of projective spectra

$$0 \to \widetilde{\mathscr{I}}_{n-1} \to \mathscr{Y}_{n-2} \to \mathscr{I}_{n-2} \to 0 \tag{II}$$

such that (2), in the case $k = n-1$, is the projective limit of (II) and choose $\widetilde{\mathscr{I}}_{n-1}$ as a subsequence of $\mathscr{I}_{n-1}$. As before, we find a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_{n-1} & \xrightarrow{m_{n-1}} & Y_{n-2} & \xrightarrow{e_{n-2}} & I_{n-2} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle I^{m_0}_{n-1,\infty}} & & \downarrow{\scriptstyle Y^{m_0}_{n-2,\infty}} & & \downarrow{\scriptstyle I^{m_0}_{n-2,\infty}} & & \\
0 & \longrightarrow & I_{n-1,m_0} & \xrightarrow[m_{n-1,m_0}]{} & Y_{n-2,m_0} & \xrightarrow[e_{n-2,m_0}]{} & I_{n-2,m_0} & \longrightarrow & 0,
\end{array}$$

where the lower row is the m_0-th step of the sequence (II) for an $m_0 \geq 1$ such that the connecting morphism from the space I_{n-1,m_0} (belonging to the subsequence $\widetilde{\mathscr{I}}_{n-1}$ of $\mathscr{I}_{n-1}$) to the space $I_{n-1,1}$ is defined. Let $(S_{n-2}, s_{Y_{n-2}}, s_{I_{n-1}})$ be the pushout of m_{n-1,m_0} and the connecting morphism I^1_{n-1,m_0} of the spectrum $\mathscr{I}_{n-1}$, then we get a commutative diagram of LS-spaces with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_{n-1,m_0} & \xrightarrow{m_{n-1,m_0}} & Y_{n-2,m_0} & \xrightarrow{e_{n-2,m_0}} & I_{n-2,m_0} & \longrightarrow & 0\\
 & & {\scriptstyle I^1_{n-1,m_0}}\downarrow & \mathrm{PO} & \downarrow{\scriptstyle s_{Y_{n-2}}} & & \| & & \\
0 & \longrightarrow & I_{n-1,1} & \xrightarrow[s_{I_{n-1}}]{} & S_{n-2} & \xrightarrow[c_{n-2}]{} & I_{n-2,m_0} & \longrightarrow & 0.
\end{array}$$

Define $W_{n-1} := Y_{n-1,1}$, $W_{n-2} := S_{n-2}$, $g_n := f_{n,1}$ and $g_{n-1} := s_{I_{n-1}} \circ e_{n-1,1}$, then the extended diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "f_n"] \arrow[dddd, equal] & Y_{n-1} \arrow[rr, "f_{n-1}"] \arrow[dr, "e_{n-1}"'] \arrow[dddd, "Y^1_{n-1,\infty}"'] & & Y_{n-2} \arrow[r, "e_{n-2}"] \arrow[dddd, "s_{Y_{n-2}} \circ Y^{m_0}_{n-2,\infty}"'] & I_{n-2} \arrow[r] \arrow[dddd, "I^{m_0}_{n-2,\infty}"] & 0 \\
& & & I_{n-1} \arrow[ur, "m_{n-1}"'] \arrow[dd, "I^1_{n-1,\infty}"] & & & \\
& & & & & & \\
& & & I_{n-1,1} \arrow[dr, "s_{I_{n-1}}"] & & & \\
0 \arrow[r] & X \arrow[r, "g_n"'] & W_{n-1} \arrow[rr, "g_{n-1}"'] \arrow[ur, "e_{n-1,1}"] & & W_{n-2} \arrow[r, "c_{n-2}"'] & I_{n-2,m_0} \arrow[r] & 0
\end{tikzcd}$$

is commutative with exact rows and its lower row is an exact sequence of LS-spaces. Proceding inductively in this way, we get a commutative diagram with exact rows

$$\begin{tikzcd}
E\colon\ 0 \arrow[r] & X \arrow[r, "f_n"] \arrow[d, equal] & Y_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & Y_{n-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & Y_1 \arrow[r, "f_1"] \arrow[d] & Y_0 \arrow[r, "f_0"] \arrow[d] & Z \arrow[r] \arrow[d, "Z^{n_0}_\infty"] & 0 \\
H\colon\ 0 \arrow[r] & X \arrow[r, "g_n"'] & W_{n-1} \arrow[r, "g_{n-1}"'] & W_{n-2} \arrow[r] & \cdots \arrow[r] & W_1 \arrow[r, "g_1"'] & W_0 \arrow[r, "g_0"'] & Z_{n_0} \arrow[r] & 0
\end{tikzcd}$$

for an $n_0 \in \mathbb{N}$, whose lower row consists of LS-spaces. Let then

$$\begin{tikzcd}
0 \arrow[r] & I_0 \arrow[r] \arrow[d] & Y_0 \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, "Z^{n_0}_\infty"] & 0 \\
0 \arrow[r] & I_{0,k_0} \arrow[r] & W_0 \arrow[r] & Z_{n_0} \arrow[r] & 0
\end{tikzcd}$$

be the right end of this commutative diagram, then lemma 5.1.5 yields a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & I_0 \arrow[r] \arrow[d] & Y_0 \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I_{0,k_0} \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{PB}"] & Z \arrow[r] \arrow[d, "Z^{n_0}_\infty"] & 0 \\
0 \arrow[r] & I_{0,k_0} \arrow[r] & W_0 \arrow[r] & Z_{n_0} \arrow[r] & 0,
\end{tikzcd}$$

such that the lower-right square is a pullback. This in turn shows that the diagram with exact rows

$$\begin{tikzcd}
E\colon\ 0 \arrow[r] & X \arrow[r, "f_n"] \arrow[d, equal] & Y_{n-1} \arrow[r, "f_{n-1}"] \arrow[d] & Y_{n-2} \arrow[r] \arrow[d] & \cdots \arrow[r] & Y_1 \arrow[r, "f_1"] \arrow[d] & Y_0 \arrow[r, "f_0"] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\
E'\colon\ 0 \arrow[r] & X \arrow[r, "g_n"'] & W_{n-1} \arrow[r, "g_{n-1}"'] & W_{n-2} \arrow[r] & \cdots \arrow[r] & W_1 \arrow[r] \arrow[dr, "g_1"'] & D \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{PB}"] & Z \arrow[r] \arrow[d, "Z^{n_0}_\infty"] & 0 \\
H\colon\ 0 \arrow[r] & X \arrow[r, "g_n"'] & W_{n-1} \arrow[r, "g_{n-1}"'] & W_{n-2} \arrow[r] & \cdots \arrow[r] & W_1 \arrow[r, "g_1"'] & W_0 \arrow[r, "g_0"'] & Z_{n_0} \arrow[r] & 0
\end{tikzcd}$$

is commutative, which implies $[E] = [E'] = [H]\, Z_\infty^{n_0}$. □

A first consequence of the above proposition 5.1.4 is that for two LS-spaces the Ext^k-group considered in the category (LS) coincides with that in (PLS):

Corollary 5.1.6. If X, Z are LS-spaces, then there is an isomorphism of abelian groups

$$\mathrm{Ext}^k_{\mathrm{LS}}(Z,X) \cong \mathrm{Ext}^k_{\mathrm{PLS}}(Z,X)$$

for all $k \in \mathbb{N}$.

Proof. For two LS-spaces X, Z and $k \in \mathbb{N}$ the map

$$\phi_k\colon\ \mathrm{Ext}^k_{\mathrm{LS}}(Z,X) \to \mathrm{Ext}^k_{\mathrm{PLS}}(Z,X),\ [E] \mapsto [E]$$

is well-defined, and, since the Baer-sum is constructed on the representants of the equivalence classes, it is a group morphism. The surjectivity of ϕ_k follows from the proof of 5.1.4 which shows how we can write $[E] \in \mathrm{Ext}^k_{\mathrm{PLS}}(Z,X)$ as $[E] = [H]\, Z_\infty^{n_0}$ for an exact sequence H of LS-spaces whose left-end is X and it is clear from the construction that we can take $Z_\infty^{n_0}$ to be the identity on Z, in the case of Z being an LS-space. It remains to check the injectivity of ϕ_k. Let $[E] \in \mathrm{Ext}^k_{\mathrm{LS}}(Z,X)$ with $\phi_k([E]) = [0]$. In the case $k = 1$ it follows from the fact that being an LS-space is a three space property in (PLS) (see 5.1.3) that $[E] = [0]$. For $k > 1$ we know by 4.2.4 that $\phi_k([E]) = [0]$ is equivalent to the existence of a commutative diagram with exact rows

$$\begin{array}{ccccccccccccccccc}
E\colon & 0 & \to & X & \xrightarrow{f_k} & Y_{k-1} & \xrightarrow{f_{k-1}} & Y_{k-2} & \to & \cdots & \to & Y_1 & \xrightarrow{f_1} & Y_0 & \xrightarrow{f_0} & Z & \to 0\\
 & & & \| & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & \\
F\colon & 0 & \to & X & \xrightarrow[g_k]{} & V_{k-1} & \xrightarrow[g_{k-1}]{} & V_{k-2} & \to & \cdots & \to & V_1 & \xrightarrow[g_1]{} & V_0 & \to & 0 &
\end{array}$$

such that $V_{k-1}, \ldots, V_0$ are PLS-spaces. The proof of 5.1.4 shows that we then can construct a commutative diagram with exact rows

$$\begin{array}{cccccccccccccc}
F\colon & 0 & \to & X & \xrightarrow{g_k} & V_{k-1} & \xrightarrow{g_{k-1}} & V_{k-2} & \to & \cdots & \to & V_1 & \xrightarrow{g_1} & V_0 \to 0\\
 & & & \| & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow\\
\widetilde{F}\colon & 0 & \to & X & \xrightarrow[\widetilde{g}_k]{} & \widetilde{V}_{k-1} & \xrightarrow[\widetilde{g}_{k-1}]{} & \widetilde{V}_{k-2} & \to & \cdots & \to & \widetilde{V}_1 & \xrightarrow[\widetilde{g}_1]{} & \widetilde{V}_0 \to 0
\end{array}$$

such that $\widetilde{V}_{k-1}, \ldots, \widetilde{V}_0$ are LS-spaces. The composition of these two diagrams shows that $[E] = [0]$ in $\mathrm{Ext}^k_{\mathrm{LS}}(Z,X)$, hence ϕ_k is injective and therefore an isomorphism of abelian groups. □

A second consequence of 5.1.4 is that for the vanishing of $\mathrm{Ext}^k_{\mathrm{PLS}}(Z,X)$, where X is an LS-space, it suffices that all the $\mathrm{Ext}^k_{\mathrm{LS}}(Z_n,X)$ vanish. Note that this was already observed in the case $k = 1$ by Kunkle in his dissertation [17, Proposition 2.11].

Corollary 5.1.7. Let X be an LS-space, Z a PLS-space and $\mathscr{Z} = (Z_n, Z_m^n)$ a strongly reduced spectrum with $\operatorname{Proj}(\mathscr{Z}) = Z$. Then $\operatorname{Ext}^k_{\mathrm{LS}}(Z_n, X) = 0$ for all $n \in \mathbb{N}$ implies $\operatorname{Ext}^k_{\mathrm{PLS}}(Z, X) = 0$.

Proof. For $[E] \in \operatorname{Ext}^k_{\mathrm{PLS}}(Z, X)$ we find by 5.1.4 an $n_0 \in \mathbb{N}$ and an equivalence class $[H] \in \operatorname{Ext}^k_{\mathrm{LS}}(Z_{n_0}, X)$ with $[E] = [H]\, Z_\infty^{n_0}$. As we have seen in the last chapter, the map

$$\operatorname{Ext}^k_{\mathrm{PLS}}(Z_{n_0}, X) \to \operatorname{Ext}^k_{\mathrm{PLS}}(Z, X),\ [F] \mapsto [F]\, Z_\infty^{n_0}$$

is a group morphism. Since $\operatorname{Ext}^k_{\mathrm{PLS}}(Z_{n_0}, X)$ and $\operatorname{Ext}^k_{\mathrm{LS}}(Z_{n_0}, X)$ are isomorphic by 5.1.6 and we have $\operatorname{Ext}^k_{\mathrm{LS}}(Z_{n_0}, X) = 0$ by assumption, it follows that $[E] = [H]\, Z_\infty^{n_0} = [0]\, Z_\infty^{n_0} = [0]$. □

The above result also implies the following:

Proposition 5.1.8. If $(X_n)_{n\in\mathbb{N}}$ is a sequence of LS-spaces, Z a PLS-space and $\mathscr{Z} = (Z_n, Z_m^n)$ is a strongly reduced spectrum with $Z = \operatorname{Proj}(\mathscr{Z})$ and $\operatorname{Ext}^k_{\mathrm{LS}}(Z_n, X_m) = 0$ for all $n, m \in \mathbb{N}$, then

$$\operatorname{Ext}^k_{\mathrm{PLS}}(Z, \prod_{n\in\mathbb{N}} X_n) = 0.$$

Proof. By 5.1.7 we have $\prod_{n\in\mathbb{N}} \operatorname{Ext}^k_{\mathrm{PLS}}(Z, X_n) = 0$. Let $[E]$ be an element of $\operatorname{Ext}^k_{\mathrm{PLS}}(Z, \prod_{n\in\mathbb{N}} X_n)$, then for all $l \in \mathbb{N}$ the pushout construction of 4.1.4 yields a commutative diagram

$$\begin{array}{llllllllllllll}
E\colon & 0 & \longrightarrow & \prod\limits_{n\in\mathbb{N}} X_n & \xrightarrow{f_k} & Y_{k-1} & \xrightarrow{f_{k-1}} & Y_{k-2} & \longrightarrow \cdots \longrightarrow & Y_0 & \xrightarrow{f_0} & Z & \longrightarrow & 0 \\
 & & & \downarrow{\scriptstyle \pi_l} & & \downarrow & & \| & & \| & & \| & & \\
E_l\colon & 0 & \longrightarrow & X_l & \longrightarrow & S_l & \longrightarrow & Y_{k-2} & \longrightarrow \cdots \longrightarrow & Y_0 & \xrightarrow[f_0]{} & Z & \longrightarrow & 0.
\end{array}$$

Since $\prod_{n\in\mathbb{N}} \operatorname{Ext}^k_{\mathrm{PLS}}(Z, X_n) = 0$, we have for all $l \in \mathbb{N}$ a commutative diagram with exact rows

$$\begin{array}{llllllllllllllll}
E_l\colon & 0 & \longrightarrow & X_l & \longrightarrow & S_k & \longrightarrow & Y_{k-2} & \longrightarrow \cdots \longrightarrow & Y_1 & \xrightarrow{f_1} & Y_0 & \xrightarrow{f_0} & Z & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \\
F\colon & 0 & \longrightarrow & X_l & \longrightarrow & V_l^{k-1} & \longrightarrow & V_l^{k-2} & \longrightarrow \cdots \longrightarrow & V_l^1 & \longrightarrow & V_l^0 & \longrightarrow & 0 & &
\end{array}$$

by 4.2.4. The countable product of topologically exact sequences is again a topologically exact sequence (algebraic exactness is clear and the topological properties only depend on finitely many factors), hence the rows of the commutative diagram

$$\begin{array}{llllllllllllll}
E\colon & 0 & \longrightarrow & \prod\limits_{n\in\mathbb{N}} X_n & \xrightarrow{f_k} & Y_{k-1} & \xrightarrow{f_{k-1}} & Y_{k-2} & \longrightarrow \cdots \longrightarrow & Y_0 & \xrightarrow{f_0} & Z & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow & & \downarrow & & & & \\
 & 0 & \longrightarrow & \prod\limits_{n\in\mathbb{N}} X_n & \longrightarrow & \prod\limits_{n\in\mathbb{N}} V_n^{k-1} & \longrightarrow & \prod\limits_{n\in\mathbb{N}} V_n^{k-2} & \longrightarrow \cdots \longrightarrow & \prod\limits_{n\in\mathbb{N}} V_n^0 & \longrightarrow & 0 & &
\end{array}$$

are exact. By 4.2.4 this implies $[E] = 0$. □

5.2 The connection between $\mathbf{Ext}^k_{\mathrm{PLS}}$ and $\mathbf{Proj}^1$

For Fréchet spaces, there is a close connection between Ext^k and the derived functors of Proj. Under conditions which are often fulfilled in relevant cases, one can show for two Fréchet spaces X, Z that there is an isomorphism $\mathrm{Ext}^1(Z,X) \cong \mathrm{Proj}^1(\mathcal{Y})$, where $\mathcal{Y} = (\mathcal{L}(Z,X_n), \mathcal{L}(Z,X^n_m))$ is a projective spectrum of spaces of continuous linear maps (see [40, Proposition 5.1.5]). In this section we will show that the formalism of the long exact sequence 4.1.1 and the results about $\mathrm{Ext}^k_{\mathrm{PLS}}$ obtained so far yield an analogous result for PLS-spaces. Note, that the connection between the vanishing of $\mathrm{Ext}^1(Z,X)$ and the vanishing of $\mathrm{Proj}^1(\mathcal{Y})$ for PLS-spaces is a known result of Kunkle (see [17, Corollary 2.8 and Corollary 2.9]), who established it by using the splitting characterization of $\mathrm{Ext}^1(Z,X) = 0$ and who then made use of this connection to get splitting results for power series spaces of PLS-type. This connection between Ext^1 and Proj^1 is also used by Bonet and Domański in [4]. We will improve Kunkle's result by showing the isomorphy $\mathrm{Ext}^1(Z,X) \cong \mathrm{Proj}^1(\mathcal{Y})$ for PLS-spaces, even if both do not vanish, and we will bring in the higher Ext^k-groups to arrive at a complete analogue of [40, Proposition 5.1.5] for PLS-spaces.

Let X be a PLS-space and let $\mathcal{X} = (X_n, X^n_m)$ be a strongly reduced spectrum with $\mathrm{Proj}(\mathcal{X}) = X$. Since $\mathcal{X}$ is strongly reduced, it satisfies the condition

$$\forall_{n\in\mathbb{N}}\ \forall_{U\in\mathcal{U}_0(F_n)}\ \exists_{m\geq n}\ X^n_m(X_m) \subseteq X^n_\infty(\mathrm{Proj}(\mathcal{X})) + U.$$

Therefore, it follows from 3.1.4 that the morphism

$$\sigma_X\colon \prod_{n\in\mathbb{N}} X_n \to \prod_{n\in\mathbb{N}} X_n,\ (x_n)_{n\in\mathbb{N}} \mapsto (X^n_{n+1}(x_{n+1}) - x_n)_{n\in\mathbb{N}}$$

is open onto its range. Since the PLS-space $X = \mathrm{Proj}(\mathcal{X})$ is the kernel object of the morphism σ_X, the so-called *canonical resolution*

$$0 \to X \to \prod_{n\in\mathbb{N}} X_n \xrightarrow{\sigma_X} \prod_{n\in\mathbb{N}} X_n$$

of X is a topologically exact sequence. The first derived functor of Proj evaluated at $\mathcal{X}$, which can be identified with the space

$$\mathrm{Proj}^1(\mathcal{X}) = \textstyle\prod_{n\in\mathbb{N}} X_n / \operatorname{im}\sigma_X,$$

measures the failure of surjectivity of the map σ_X, i.e. one has $\mathrm{Proj}^1(\mathcal{X}) = 0$ if and only if σ_X is surjective, which is equivalent to

$$0 \to X \to \prod_{n\in\mathbb{N}} X_n \xrightarrow{\sigma_X} \prod_{n\in\mathbb{N}} X_n \to 0$$

being an exact sequence. Then it follows directly from the long exact sequence 4.1.1 that for $\mathrm{Proj}^1(\mathcal{X}) = 0$ the vanishing of the space $\mathrm{Proj}^1(\mathcal{Y})$, with $\mathcal{Y} = (\mathcal{L}(Z,X_n), \mathcal{L}(Z,X^n_m))$, is a necessary condition for the vanishing of $\mathrm{Ext}^1_{\mathrm{PLS}}(Z,X)$:

Proposition 5.2.1. Let X, Z be PLS spaces and let $\mathcal{X} = (X_n, X_m^n)$ and $\mathcal{Z} = (Z_n, Z_m^n)$ be strongly reduced projective spectra with $\mathrm{Proj}(\mathcal{X}) = X$ and $\mathrm{Proj}(\mathcal{Z}) = Z$. If $\mathrm{Proj}^1(\mathcal{X}) = 0$ and $\mathrm{Ext}^1_{\mathrm{PLS}}(Z, X) = 0$, then $\mathrm{Proj}^1(\mathcal{Y}) = 0$, where $\mathcal{Y} = (\mathcal{L}(Z, X_n), \mathcal{L}(Z, X_m^n))$.

Proof. Since $\mathrm{Proj}^1(\mathcal{X}) = 0$, the canonical resolution

$$0 \to X \to \prod_{n\in\mathbb{N}} X_n \xrightarrow{\sigma_X} \prod_{n\in\mathbb{N}} X_n \to 0$$

is a short exact sequence. Applying the functor $\mathcal{L}(Z, -)$ to this sequence yields the long exact sequence

$$0 \to \mathcal{L}(Z, X) \to \prod_{n\in\mathbb{N}} \mathcal{L}(Z, X_n) \xrightarrow{\sigma_Y} \prod_{n\in\mathbb{N}} \mathcal{L}(Z, X_n) \to \mathrm{Ext}^1_{\mathrm{PLS}}(Z, X) \to \dots$$

by 4.1.1. Then $\mathrm{Ext}^1_{\mathrm{PLS}}(Z, X) = 0$ implies that the map σ_Y is surjective, hence $\mathrm{Proj}^1(\mathcal{Y}) = 0$. □

Remark 5.2.2. Note that in the analogue of the above proposition 5.2.1 for Fréchet spaces [40, Proposition 5.1.5.] the assumption $\mathrm{Proj}^1(\mathcal{X}) = 0$ is not explicitly stated, as it is always fulfilled for Fréchet spaces (see [37, Lemma 1.1]).

Keeping in mind the above remark, the long exact sequence 4.1.1 and the results of the first section of this chapter now give the complete analogue of [40, Proposition 5.1.5] for PLS-spaces:

Theorem 5.2.3. Let X, Z be PLS spaces and $\mathcal{X} = (X_n, X_m^n)$, $\mathcal{Z} = (Z_n, Z_m^n)$ strongly reduced projective spectra with $\mathrm{Proj}(\mathcal{X}) = X$ and $\mathrm{Proj}(\mathcal{Z}) = Z$. If we have

a) $\mathrm{Proj}^1(\mathcal{X}) = 0$,
b) $\mathrm{Ext}^k_{\mathrm{LS}}(Z_n, X_m) = 0$ for all $n, m \in \mathbb{N}$ and all $1 \le k \le k_0$,

for a $1 \le k_0 \le \infty$, then

i) $\mathrm{Ext}^1_{\mathrm{PLS}}(Z, X) \cong \mathrm{Proj}^1(\mathcal{Y})$ with $\mathcal{Y} = (\mathcal{L}(Z, X_n), \mathcal{L}(Z, X_m^n))$,
ii) $\mathrm{Ext}^k_{\mathrm{PLS}}(Z, X) = 0$ for all $2 \le k \le k_0$.

Proof. The assumption $\mathrm{Proj}^1(\mathcal{X}) = 0$ yields that the canonical resolution

$$0 \to X \to \prod_{n\in\mathbb{N}} X_n \xrightarrow{\sigma_X} \prod_{n\in\mathbb{N}} X_n \to 0$$

of X is a short exact sequence. Applying the functor $\mathcal{L}(Z, -)$ to this sequence gives the long exact sequence

$$0 \to \mathcal{L}(Z, X) \to \prod_{n\in\mathbb{N}} \mathcal{L}(Z, X_n) \to \prod_{n\in\mathbb{N}} \mathcal{L}(Z, X_n) \to \mathrm{Ext}^1_{\mathrm{PLS}}(Z, X) \to \dots$$

$$\to \mathrm{Ext}^{k-1}_{\mathrm{PLS}}(Z, \prod_{n\in\mathbb{N}} X_n) \to \mathrm{Ext}^k_{\mathrm{PLS}}(Z, X) \to \mathrm{Ext}^k_{\mathrm{PLS}}(Z, \prod_{n\in\mathbb{N}} X_n) \to \dots$$

of abelian groups. By proposition 5.1.8 we have $\mathrm{Ext}^k_{\mathrm{PLS}}(Z, \prod_{n\in\mathbb{N}} X_n) = 0$ for $1 \le k \le k_0$, which implies $\mathrm{Ext}^k_{\mathrm{PLS}}(Z, X) = 0$ for $2 \le k \le k_0$. Furthermore, it follows from the exactness of the sequence

$$0 \to \mathcal{L}(Z, X) \to \prod_{n\in\mathbb{N}} \mathcal{L}(Z, X_n) \xrightarrow{\sigma_Y} \prod_{n\in\mathbb{N}} \mathcal{L}(Z, X_n) \to \mathrm{Ext}^1_{\mathrm{PLS}}(Z, X) \to 0$$

that $\mathrm{Ext}^1_{\mathrm{PLS}}(Z, X)$ is isomorphic to the cokernel of σ_Y, which is just the space $\mathrm{Proj}^1(\mathcal{Y})$. □

5.3 $\mathrm{Ext}_{\mathrm{PLS}}$ for the Space of Distributions

We have briefly mentioned in the last chapter, that the long exact sequence 4.1.1 gives a natural answer to the question when the splitting property $\mathrm{Ext}^1_{\mathrm{PLS}}(H, G) = 0$ is passed on to quotients of G and to subspaces of H: Given a short exact sequence

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

of PLS-spaces and continuous linear maps, one gets for each PLS-space E the covariant long exact sequence

$$\ldots \mathrm{Ext}^1_{\mathrm{PLS}}(E, Y) \to \mathrm{Ext}^1_{\mathrm{PLS}}(E, Z) \to \mathrm{Ext}^2_{\mathrm{PLS}}(E, X) \to \mathrm{Ext}^2_{\mathrm{PLS}}(E, Y) \to \ldots$$

and the contravariant long exact sequence

$$\ldots \mathrm{Ext}^1_{\mathrm{PLS}}(Y, E) \to \mathrm{Ext}^1_{\mathrm{PLS}}(X, E) \to \mathrm{Ext}^2_{\mathrm{PLS}}(Z, E) \to \mathrm{Ext}^2_{\mathrm{PLS}}(Y, E) \to \ldots$$

of 4.1.1. This gives the following implications:

$$\mathrm{Ext}^1_{\mathrm{PLS}}(E, Y) = 0,\ \mathrm{Ext}^2_{\mathrm{PLS}}(E, X) = 0 \implies \mathrm{Ext}^1_{\mathrm{PLS}}(E, Z) = 0,$$

$$\mathrm{Ext}^1_{\mathrm{PLS}}(E, Y) = 0,\ \mathrm{Ext}^2_{\mathrm{PLS}}(E, Y) = 0 \implies \mathrm{Ext}^1_{\mathrm{PLS}}(E, Z) \cong \mathrm{Ext}^2_{\mathrm{PLS}}(E, X).$$

Dually, the second of the above long sequences gives analogous results for $\mathrm{Ext}^1_{\mathrm{PLS}}(X, E)$. Therefore, it is natural to consider the functor Ext^2 in this context.

In this section we will compute for $k \ge 2$ the groups $\mathrm{Ext}^k_{\mathrm{PLS}}(E, F)$ for a closed subspace E and a Hausdorff quotient F of the space $\mathcal{D}'(\Omega)$ of distributions on an open subset $\Omega \subseteq \mathbb{R}^n$ (in the sense of L. Schwartz). This in turn gives, by application of the above reasoning, a new proof of a splitting theorem for the space of distributions due to Wengenroth [40, Theorem 5.3.8], that states $\mathrm{Ext}^1_{\mathrm{PLS}}(E, F) = 0$ in the above setting. This result is itself an improvement of a result by Domański and Vogt [9, Theorem 2.1 and Theorem 3.1], who showed that $\mathrm{Ext}^1_{\mathrm{PLS}}(E, \mathcal{D}'(\Omega)) = 0$ and $\mathrm{Ext}^1_{\mathrm{PLS}}(\mathcal{D}'(\Omega), F) = 0$ for a closed subspace E of $\mathcal{D}'(\Omega)$ and a Hausdorff quotient F of $\mathcal{D}'(\Omega)$.

We will make use of the isomorphy $\mathcal{D}'(\Omega) \cong (s')^{\mathbb{N}}$ to get this results , where

s' is the strong dual of the space s of rapidly decreasing sequences. This isomorphism was discovered by Valdivia [32] and (independently) Vogt [36]. We will only work on the right-hand side of this isomorphism and will therefore use the abbreviation $\mathcal{D}' := (s')^{\mathbb{N}}$.

Another tool for our investigation will be the following connection between exact sequences of LS-spaces and Fréchet-Schwartz spaces, which contains only well-known facts (see [19, Theorem 25.19 and Theorem 26.4]):

Proposition 5.3.1. For a sequence of LS-spaces and continuous linear maps

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \tag{1}$$

the following are equivalent:

i) (1) is topologically exact,

ii) The dual sequence $0 \longrightarrow Z' \xrightarrow{g^*} Y' \xrightarrow{f^*} Y' \longrightarrow 0$ of Fréchet Schwartz spaces is topologically exact.

Furthermore the sequence (1) is split exact if and only if the dual sequence is split exact.

The Yoneda-Ext-functors introduced in the third chapter can be constructed for every exact category and they coincide with the derived functors of $\mathcal{L}(A, -)$, $\mathcal{L}(-, A)$ if these can be defined. For the quasi-abelian categories (F) of Fréchet spaces and (FS) of Fréchet-Schwartz spaces we will denote the respective Yoneda-Ext-functors by $\mathrm{Ext}^k_{\mathrm{F}}$ and $\mathrm{Ext}^k_{\mathrm{FS}}$. The above result 5.3.1 allows us to access the vanishing of $\mathrm{Ext}^k_{\mathrm{LS}}(Z_n, X_m)$ by dualizing to the catagory of Fréchet-Schwartz spaces:

Proposition 5.3.2. For two LS-spaces X, Z and $k \geq 1$ the following are equivalent:

i) $\mathrm{Ext}^k_{\mathrm{FS}}(X', Z') = 0$,

ii) $\mathrm{Ext}^k_{\mathrm{LS}}(Z, X) = 0$.

Proof. For $k = 1$, this is already stated in 5.3.1. For $k > 1$, assume that $\mathrm{Ext}^k_{\mathrm{FS}}(X', Z') = 0$ and let

$$E\colon\ 0 \to X \to Y_{k-1} \to Y_{k-2} \to \ldots \to Y_1 \to Y_0 \to Z \to 0$$

be an exact sequence of LS-spaces. By 5.3.1 the dual sequence

$$E'\colon\ 0 \to Z' \to Y_0' \to Y_1' \to \ldots \to Y_{k-2}' \to Y_{k-1}' \to X' \to 0$$

of Fréchet-Schwartz spaces is also exact. Since $\mathrm{Ext}^k_{\mathrm{FS}}(X', Z') = 0$, there is by 4.2.4 a commutative diagram with exact rows

$$\begin{array}{ccccccccccccccc}
E'\colon & 0 & \longrightarrow & Z' & \longrightarrow & Y_0' & \longrightarrow & Y_1' & \longrightarrow \cdots \longrightarrow & Y_{k-2}' & \longrightarrow & Y_{k-1}' & \longrightarrow & X' & \longrightarrow 0 \\
 & & & \| & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
F'\colon & 0 & \longrightarrow & Z' & \longrightarrow & V_0 & \longrightarrow & V_1 & \longrightarrow \cdots \longrightarrow & V_{k-2} & \longrightarrow & V_{k-1} & \longrightarrow & 0 &
\end{array}$$

consisting of Fréchet-Schwartz spaces. Dualizing this diagram again to the category (LS), we get a commutative diagram with exact rows

$$\begin{array}{lccccccccccccccc}
F\colon & & & 0 & \to & V'_{k-1} & \to & V'_{k-2} & \to & \cdots & \to & V'_1 & \to & V'_0 & \to & Z \to 0 \\
 & & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \| \\
E\colon & 0 & \to & X & \to & Y_{k-1} & \to & Y_{k-2} & \to & \cdots & \to & Y_1 & \to & Y_0 & \to & Z \to 0
\end{array}$$

of LS-spaces. Then 4.2.4 yields $[E] = [0]$. This shows i)⇒ii). Analogously one shows ii)⇒i). □

We have seen in 5.2.3 that the vanishing of the groups $\mathrm{Ext}^k_{\mathrm{LS}}(Z_n, X_m)$ is, under mild assumptions, sufficient for the vanishing of $\mathrm{Ext}^k_{\mathrm{PLS}}(Z, X)$ for $k \geq 2$. The above proposition will allow us to access the vanishing of $\mathrm{Ext}^k_{\mathrm{LS}}(Z_n, X_m)$ by first dualizing to the catagory of Fréchet-Schwartz spaces and then applying the well-established splitting theory for the category of Fréchet spaces. In the case $X \cong \mathcal{D}'$ this is also possible for $k = 1$:

Remark 5.3.3. Due to Domański and Vogt one has $\mathrm{Ext}^1_{\mathrm{PLS}}(E, \mathcal{D}') = 0$ for every closed subspace E of $\mathcal{D}'$ (see [9, Theorem 3.1]). This result can also be obtained by making use of 5.3.1 and 5.1.8 : Since E is a closed subspace of $\mathcal{D}'$, there is a strongly reduced projective spectrum $\mathcal{E} = (E_n, E^n_m)$ consisting of closed subspaces of s' such that $\mathrm{Proj}(\mathcal{E}) = E$. Because of 5.1.8, it is enough to show $\mathrm{Ext}^1_{\mathrm{LS}}(E_n, s') = 0$ for all $n \in \mathbb{N}$. By 5.3.1 a short exact sequence of LS-spaces splits if and only if the dual sequence of Fréchet-Schwartz spaces splits and it is a well-known result of Vogt and Wagner [35] that one has $\mathrm{Ext}^1_{\mathrm{F}}(s, E'_n) = 0$ in the category (F) of Fréchet spaces. Being a Schwartz space is a three space property in (F) (see [28, Proposition 3.7]), hence one also has $\mathrm{Ext}^1_{\mathrm{FS}}(s, E'_n) = 0$ in the category (FS) of Fréchet-Schwartz spaces and thus $\mathrm{Ext}^1_{\mathrm{LS}}(E_n, s') = 0$.

To compute the higher $\mathrm{Ext}^k_{\mathrm{PLS}}$-groups for subspaces and quotients of $\mathcal{D}'$ we make use of the following lemma:

Lemma 5.3.4. Let E, F be two Fréchet-Schwartz spaces such that F is nuclear. Then $\mathrm{Ext}^k_{\mathrm{FS}}(E, F) = 0$ for $k \geq 2$.

Proof. In the category of Fréchet spaces one knows $\mathrm{Ext}^k_{\mathrm{F}}(E, F) = 0$ for $k \geq 2$, which is a result of Vogt [37, Theorem 1.2 and Corollary 1.3]. For $k = 2$, let

$$H\colon\ 0 \to F \to Y_1 \to Y_0 \to E \to 0$$

be an exact sequence of Fréchet-Schwartz spaces. Since $\mathrm{Ext}^2_{\mathrm{F}}(E, F) = 0$, there is by 4.2.4 a commutative diagram with exact rows

$$\begin{array}{lccccccccc}
H\colon & 0 & \to & F & \to & Y_1 & \to & Y_0 & \to & E \to 0 \\
 & & & \| & & \downarrow & & \| & & \\
G\colon & 0 & \to & F & \to & V_1 & \to & Y_0 & \to & 0
\end{array}$$

for a Fréchet space V_1. Being a Schwartz space is a three space property in the category of Fréchet spaces, hence V_1 is a Fréchet-Schwartz space. By 4.2.4 the above diagram implies $[H] = [0]$, hence $\mathrm{Ext}^2_{\mathrm{FS}}(E, F) = 0$. Then 4.2.6 shows $\mathrm{Ext}^k_{\mathrm{FS}}(E, F) = 0$ for $k \geq 2$. □

Proposition 5.3.5. Let E and F be PLS-spaces such that E is isomorphic to a subspace of $\mathcal{D}'$ and F is isomorphic to a quotient of $\mathcal{D}'$. Then $\mathrm{Ext}^k_{\mathrm{PLS}}(E, F) = 0$ for $k \geq 2$.

Proof. There is a strongly reduced spectrum $\mathcal{E} = (E_n, E^n_m)$ consisting of closed subspaces of s' with $\mathrm{Proj}(\mathcal{E}) = E$ and a strongly reduced spectrum $\mathcal{F} = (F_n, F^n_m)$ consisting of Hausdorff quotients of s' with $\mathrm{Proj}(\mathcal{F}) = F$. Since the space $\mathcal{D}'$ is ultrabornological and since this property is stable with respect to taking quotients, the space F is also ultrabornological, which implies $\mathrm{Proj}^1(\mathcal{F}) = 0$ by [9, Theorem 1.1]. By 5.3.1 we have $\mathrm{Ext}^1_{\mathrm{LS}}(E_n, F_m) = 0$ if and only if $\mathrm{Ext}^1_{\mathrm{FS}}(F'_m, E'_n) = 0$ in the category (FS) of Fréchet Schwartz spaces. Since being a Schwartz space is a three space property in the category (F) of Fréchet spaces, it is enough to show $\mathrm{Ext}^1_{\mathrm{F}}(F'_m, E'_n) = 0$. Here, the classical splitting result for subspaces and quotients of the space s due to Vogt and Wagner [34, 35] yields $\mathrm{Ext}^1_{\mathrm{F}}(F'_m, E'_n) = 0$ since F'_m is a closed subspace of s and E'_n is a Hausdorff quotient of s. Hence, we have $\mathrm{Ext}^1_{\mathrm{LS}}(E_n, F_m) = 0$ for all $n, m \in \mathbb{N}$. For $k \geq 2$ we know by 5.3.4 that $\mathrm{Ext}^k_{\mathrm{FS}}(F'_m, E'_n) = 0$ for all $n, m \in \mathbb{N}$ since the E'_n are nuclear Fréchet spaces. By 5.3.2 one also has $\mathrm{Ext}^k_{\mathrm{LS}}(E_n, F_m) = 0$ for $k \geq 2$ and all $n, m \in \mathbb{N}$. Applying theorem 5.2.3 yields $\mathrm{Ext}^k_{\mathrm{PLS}}(E, F) = 0$ for $k \geq 2$. □

Together with the long exact sequences, the above also gives the already mentioned new proof of the splitting theorem for subspaces and quotients of the space of distributions [40, Theorem 5.3.8], i.e. one has $\mathrm{Ext}^1_{\mathrm{PLS}}(E, F) = 0$ in the above setting. Hence, we see that all $\mathrm{Ext}^k_{\mathrm{PLS}}$-groups do vanish:

Theorem 5.3.6. Let E and F be PLS-spaces such that E is isomorphic to a subspace of $\mathcal{D}'$ and F is isomorphic to a quotient of $\mathcal{D}'$. Then one has $\mathrm{Ext}^k_{\mathrm{PLS}}(E, F) = 0$ for $k \geq 1$.

Proof. We have shown the assertion of the theorem for $k \geq 2$ in 5.3.5, it remains to show the case $k = 1$. By 5.3.5 we have $\mathrm{Ext}^1_{\mathrm{PLS}}(E, \mathcal{D}') = 0$ and by [9, Theorem 2.1] there is a topologically exact sequence

$$0 \to \mathcal{D}' \to \mathcal{D}' \to F \to 0$$

of PLS-spaces. Applying the functor $\mathcal{L}(E, -)$ yields the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^1_{\mathrm{PLS}}(E, \mathcal{D}') \longrightarrow \mathrm{Ext}^1_{\mathrm{PLS}}(E, F) \longrightarrow \mathrm{Ext}^2_{\mathrm{PLS}}(E, \mathcal{D}') \longrightarrow \cdots$$

of abelian groups and group morphisms. We have $\mathrm{Ext}^2_{\mathrm{PLS}}(E, \mathcal{D}') = 0$ by 5.3.5, hence the exactness implies $\mathrm{Ext}^1_{\mathrm{PLS}}(E, F) = 0$. □

Remark 5.3.7.

i) If in the above theorem 5.3.6 the space E is ultrabornological one can also prove 5.3.6 using the contravariant long exact sequence of 4.1.1. In fact, by [9, Theorem 3.1] there is a topologically exact sequence

$$0 \to E \to \mathcal{D}' \to \mathcal{D}' \to 0$$

in the case of E being an ultrabornological subspace and one also has $\text{Ext}^1_{\text{PLS}}(\mathcal{D}', F) = 0$ by [9, Theorem 2.1]. Applying the functor $\mathcal{L}(-, F)$ yields the long exact sequence

$$\cdots \longrightarrow \text{Ext}^1_{\text{PLS}}(\mathcal{D}', F) \longrightarrow \text{Ext}^1_{\text{PLS}}(E, F) \longrightarrow \text{Ext}^2_{\text{PLS}}(\mathcal{D}', F) \longrightarrow \cdots$$

of abelian groups. By 5.3.5 we have $\text{Ext}^2_{\text{PLS}}(\mathcal{D}', F) = 0$, hence the exactness of this sequence also implies $\text{Ext}^1_{\text{PLS}}(E, F) = 0$.

ii) It follows from 5.3.5, 5.3.6, and the long exact sequences that one also has $\text{Ext}^k_{\text{PLS}}(Z, X) = 0$ for $k \geq 2$ if either X, Z are both closed subspaces of $\mathcal{D}'$, with X ultrabornological, or if X, Z are both Hausdorff quotients of $\mathcal{D}'$.

iii) Given two PLS-spaces X, Z one can also consider the abelian groups $\text{Ext}^k_{\text{LCS}}(Z, X)$, where (LCS) is the quasi-abelian category of locally convex spaces. Since (LCS) has enough injective objects, these groups can also be computed as the derived functors of $\mathcal{L}(Z, -)$ (see [23] or [40]). Assuming the continuum hypothesis, the above remark ii) gives an example where the groups $\text{Ext}^k_{\text{PLS}}(Z, X)$ and $\text{Ext}^k_{\text{LCS}}(Z, X)$ do not coincide: The space of all sequences ω is a closed subspace of $\mathcal{D}'$ and the space of finite sequences φ is a closed subspace of $\mathcal{D}'$ that is ultrabornological, hence ii) yields $\text{Ext}^2_{\text{PLS}}(\omega, \varphi) = 0$. However, it is a result of Wengenroth [40, Proposition 5.1.13] that $\text{Ext}^2_{\text{LCS}}(\omega, \varphi) \neq 0$ under the continuum hypothesis.

Bibliography

[1] J. Adamek, H. Herrlich and G.E. Strecker, *Abstract and Concrete Categories*, Pure and applied mathematics (John Wiley&Sons) (1989).

[2] R. Baer, *Erweiterung von Gruppen und ihren Isomorphismen*, Math. Z. 38, 375-416 (1934).

[3] J. Bonet and S. Dierolf, *The pullback for bornological and ultra-bornological spaces*, Note Mat. **25**(1), 63–67 (2005/06).

[4] J. Bonet and P. Domański, *The splitting of exact sequences of* PLS-*spaces and smooth dependence of solutions of linear partial differential equations*, Adv. Math. 217, No. 2, 561-585 (2008).

[5] T. Bühler, *Exact categories*, Expositiones Mathematicae (2009), In Press, Accepted Manuscript, DOI: 10.1016/j.ex math.2009.04.004, URL: http://dx.doi.org/10.1016/j.exmath.2009.04.004

[6] D. A. Buchsbaum, *A note on homology in categories*, Ann. of Math. 71, 199-209 (1960).

[7] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton Univ. Press, Princeton, New Jersey, (1956).

[8] P. Domański, *Classical* PLS-*spaces: Spaces of distributions, real analytic functions and their relatives*, Z. Ciesielski, A. Pełczyński, L. Skrzypczak (Eds.), Orlicz Centenary Volume, Proceedings of the Conferences: Wladyslaw Orlitz Centenary Conference and Function Spaces VII held in Poznań, July 21-25, 2003, Banach Center Publ., vol 64, Institute of Mathematics, Polish Academy of Sciences, Warszawa, pp. 51-70 (2004).

[9] P. Domański and D. Vogt, *A splitting theorem for the space of distributions*, Studia Math., 140(1), 57-77, 2000.

[10] K. Floret, *Lokalkonvexe Sequenzen mit kompakten Abbildungen*, J. reine Angew. Math. 247, 155-195, (1971).

[11] L. Frerick and J. Wengenroth, *A sufficient condition for the vanishing of the derived projective limit functor*, Arch. Math. 67, 296-301, (1996).

[12] A. Grothendieck, *Sur quelques points d'algébre homologique*, Tohoku Math. J. (2)9, 119-221 (1957).

[13] T. Kalmes, *Every P-convex subset of $\mathbb{R}^2$ is already strongly P-convex*, Preprint (2009); arXiv:0907:3037v1.

[14] B. Keller, *Chain complexes and stable categories*, Manuscripta Math. **67**(4), 379–417 (1990).

[15] G. M. Kelly, *Monomorphisms, epimorphisms, and pull-backs*, J. Austral. Math. Soc. **9**, 124–142 (1969).

[16] G. Köthe, *Topological vector spaces. I*, (Springer-Verlag New York Inc., New York, 1969), Translated from the German by D. J. H. Garling. Die Grundlehren der mathematischen Wissenschaften, Band 159.

[17] D. Kunkle, *Splitting of power series spaces of* (PLS)*-type*, Dissertation, Bergische Universität Wuppertal (2001).

[18] S. Mac Lane, *Homology*, Die Grundlehren der mathematischen Wissenschaften, Bd. 114, Academic Press Inc., Publishers, New York (1963).

[19] R. Meise and D. Vogt, *Introduction to functional analysis*, Oxford Graduate Texts in Mathematics, Vol. 2 (The Clarendon Press Oxford University Press, New York, 1997), Translated from the German by M. S. Ramanujan and revised by the authors.

[20] B. Mitchell, *Theory of categories*, New York: Acad. Pr. (Pure and applied mathematics;17) (1965).

[21] A. Neeman, *The derived category of an exact category*, J. Algebra 135, no. 2, 388-394 (1990).

[22] V. P. Palamodov, *The projective limit functor in the category of topological linear spaces*, Mat. Sb. **75** (1968) 567–603 (in Russian), English transl., Math. USSR Sbornik **17**, 189–315 (1972).

[23] V. P. Palamodov, *Homological methods in the theory of locally convex spaces*, Uspekhi Mat. Nauk **26** (1) (1971) 3–66 (in Russian), English transl., Russian Math. Surveys **26** (1) pp. 1–64 (1971).

[24] F. Prosmans, *Derived categories for functional analysis*, Publ. Res. Inst. Math. Sci. **36**(5-6), 19–83 (2000).

[25] D. Quillen, *Higher algebraic K-theory. I*, in: Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972), pp. 85–147. Lecture Notes in Math., Vol. 341 (Springer, Berlin, 1973).

[26] D. A. Raikov, *Semiabelian Categories*, Soviet Math. Doklady 10, 1242-1245 (1969).

[27] F. Richman and E. A. Walker, Ext in pre-Abelian categories, Pacific J. Math. **71**(2), 521–535 (1977).

[28] W. Roelcke and S. Dierolf, *On the three-space-problem for topological vector spaces*, Collect. Math. **32**(1), 13–35 (1981).

[29] W. Rump, *A counterexample to Raikov's conjecture*, Bull. Lond. Math. Soc. **40**(6), 985–994 (2008).

[30] J. P. Schneiders, *Quasi-abelian categories and sheaves*, Mém. Soc. Math. Fr. (N.S.)(76) (1999).

[31] D. Sieg and S. A. Wegner, *Maximal exact structures on additive categories*, to appear in Math. Nachr.

[32] M. Valdivia, *Representations of the spaces $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$*, Rev. Real Acad. Cienc. Exact. Fis. Natur. Madrid, 72(3):385-414 (1978).

[33] J. L. Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque (1996), no. 239, xii+253 pp. (1997).

[34] D. Vogt, *Charakterisierung der Unterräume von s*, Math. Z., 155(2), 109-117 (1977).

[35] D. Vogt and M. J. Wagner, *Charakterisierung der Quotientenräume von s und eine Vermutung von Martineau*, Studia Math., 67(3):225-240 (1980).

[36] D. Vogt, *Sequence space representations of spaces of test functions and distributions*, In Functional analysis, holomorphy, and approximation theory (Rio de Janeiro, 1979), pages 405-443. Dekker, New York (1983).

[37] D. Vogt, *On the functors* $\operatorname{Ext}^1(E, F)$ *for Fréchet spaces*, Studia Math.,85(2):163-197 (1987).

[38] D. Vogt, *Lectures on projective spectra of (DF)-spaces*, Seminar lectures, Wuppertal (1987).

[39] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge (1994).

[40] J. Wengenroth, *Derived Functors in Functional Analysis*, Lecture Notes in Mathematics **1810**, Springer, Berlin (2003).

[41] N. Yoneda, *On Ext and exact sequences*, J. Fac. Sci. Univ. Tokyo Sect. I 8, 507-576 (1960).