# Universal and Frequently Universal Functions of Exponential Type

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# Preface

Let $\varphi(D)f = \sum_{n=0}^{\infty} \varphi_n f^{(n)}$ be a differential operator on the space of entire functions $H(\mathbb{C})$, induced by the entire function of exponential type $\varphi(z) = \sum_{n=0}^{\infty} \varphi_n z^n$. A famous result of G. Godefroy and J. H. Shapiro states that whenever $\varphi$ is non-constant, the corresponding differential operator admits universal functions (cf. [GS91, Theorem 5.1]). It is known that the lowest rate of growth that these universal functions can exhibit is closely related to the contour line

$$C_\varphi := \{z \in \mathbb{C} : |\varphi(z)| = 1\}.$$

To be specific, if all $\varphi_n \geq 0$ and $|\varphi(0)| < 1$, universal functions for $\varphi(D)$ must be at least of exponential type equal to $\tau := \operatorname{dist}(0, C_\varphi)$ (cf. [BGB02]). From a result in this thesis it turns out that, despite the different growth conditions, universal functions for different differential operators can have similarities in their algebraic structure. Precisely, for every $\varphi(D)$ induced by a non-constant $\varphi$, there are universal functions of the form $f(z)e^{\alpha z}$ with $\alpha \in C_\varphi$ and $f$ a function of exponential type zero.

The main topic of this work is the study of universal and frequently universal functions for differential operators and weighted shifts in the class of functions of exponential type. Our results provide information about the algebraic structure and growth conditions with respect to different rays. Some known results are extended in this regard. Furthermore, we provide a connection between (frequently) universal functions for different differential operators: We show that they can be derived from each other by means of a certain transform.

Our starting point is the consideration of a characteristic of functions of exponential type that has not been considered in universality so far, namely, the conjugate indicator diagram. This is a compact and convex set $K(f) \subset \mathbb{C}$, corresponding to a given entire function of exponential type $f$, which reflects certain properties of $f$.

Firstly, the set $K(f)$ reproduces the indicator function $h_f$, meaning that

$$\sup_{u\in K(f)} \mathrm{Re}(e^{i\Theta}u) = h_f(\Theta) := \limsup_{r\to\infty} \frac{\log|f(re^{i\Theta})|}{r},$$

and thus provides information about the growth of $f$ in every direction $\Theta \in [0, 2\pi)$. Secondly, the distribution of zeros of $f$ with respect to their frequency in different sectors is closely related to the size and geometry of $K(f)$. The latter becomes relevant in the research of frequently universal functions.

In Section 3.3, we prove that every compact and convex set $K$ that intersects $C_\varphi$ is the conjugate indicator diagram of a universal function for $\varphi(D)$ provided that $\varphi$ is non-constant. In particular, this shows that for every $\alpha \in C_\varphi$ there is a universal function of the form $f(z)e^{\alpha z}$ with $f$ of exponential type zero.

Things change in case of frequent universality. If $f$ is a non-constant entire function of exponential type whose zeros are distributed with positive lower density in the direction of the real axis, then $K(f)$ has some extension in the direction of the imaginary axis. This immediately implies that a function with a singleton conjugate indicator diagram cannot be frequently universal for the translation operator $f \mapsto f(\cdot + 1)$. Inspired by this easy principle, we investigate advanced conditions for the conjugate indicator diagram of these functions in Section 3.2. For instance, we show that for every non-singleton line segment $L$ of the imaginary axis there is a frequently universal function $f$ for the translation operator such that $K(f) \subset L$, whereas this is impossible in case that $L$ is some non-vertical line segment in the plane.

In Section 3.6, some of these results are extended to general differential operators. For instance, it is shown that every compact, convex set that contains a non-singleton continuum of $C_\varphi$ contains the conjugate indicator diagram of a frequently universal function for $\varphi(D)$. Conversely, this property does not hold for singleton sets. The essential tool in this context is the transform $\Phi_\varphi$, which is introduced in Section 3.5. This transform connects different differential operators and carries over their (frequently) universal functions.

The last chapter is devoted to the study of (frequently) universal functions for weighted shift operators on $H(\mathbb{C})$. These are operators mapping an entire function $\sum_{n=0}^{\infty} a_n z^n$ to $\sum_{n=0}^{\infty} w_{n+1}a_{n+1}z^n$ with a certain weight sequence $(w_n)_{n\in\mathbb{N}}$.

In a main result of this chapter, we show that (frequently) universal functions for these operators can be derived from (frequently) universal functions for the differentiation operator. This is easily proved due to a denseness result for the Hadamard product, which is provided in Section 4.2.

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# Chapter 1

# Preliminaries

## 1.1 Cauchy Cycles

This section starts with some basic notations that we use throughout this thesis. For a complex number $z$, the neighbourhood $\{w : |w - z| < \varepsilon\}$ with $\varepsilon > 0$ shall be denoted by $U_\varepsilon(z)$. By $\mathbb{D}$ we mean the unit disc $\{z : |z| < 1\}$ and by $\mathbb{T}$ the unit circle $\{z : |z| = 1\}$ in the complex plane. The extended plane $\mathbb{C} \cup \{\infty\}$ is denoted by $\mathbb{C}_\infty$ where we determine as usual $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$. For $M, L \subset \mathbb{C}$, we set $M + L := \{z + w : z \in M, w \in L\}$, $M\,L := \{zw : z \in M,\ w \in L\}$, $M^{-1} := \{w^{-1} : w \in M\}$, and $\operatorname{conv}(M)$ is the convex hull of $M$. Furthermore, the interior of $M$ is denoted by $M^\circ$, and the closure of $M$ is denoted by $\overline{M}$. For the linear hull of a family of vectors $A$, we write $\operatorname{linspan}(A)$. The intervall $[z, w]$ of two complex numbers $z, w$ shall be the closed line segment that connects these numbers. For a discrete set $\lambda \subset \mathbb{C}$, we set $n(r) := n_\lambda(r) := \#\{w \in \lambda : |w| \leq r\}$.
Then

$$\overline{\operatorname{dens}}\,(\lambda) := \limsup_{k\to\infty} \frac{n(r)}{r}$$

is the *upper density of* $\lambda$, and

$$\underline{\operatorname{dens}}\,(\lambda) := \liminf_{k\to\infty} \frac{n(r)}{r}$$

is the *lower density of* $\lambda$. If the upper density and lower density coincide, the limit

$$\operatorname{dens}(\lambda) := \lim_{k\to\infty} \frac{n(r)}{r}$$

exists and $\text{dens}(\lambda)$ is called the *density of* $\lambda$.

**Definition 1.1**

(1) For an open set $\Omega \subset \mathbb{C}_\infty$, we denote by $H(\Omega)$ the space of holomorphic functions on $\Omega$. A function $f$ is said to be holomorphic at infinity if $f(z^{-1})$ is holomorphic at the origin. The space $H(\Omega)$ is endowed with the topology of uniform convergence on compact subsets. $H(\Omega)$ is a Fréchet space.

(2) For an open set $\Omega \subset \mathbb{C}$, we denote by $H_0(\Omega)$ the space of all functions holomorphic on $\Omega$ and vanishing at $\infty$ if $\{z : |z| > R\} \subset \Omega$ for some $R > 0$. This space is endowed with the topology induced by $H(\Omega)$. As a closed subspace of $H(\Omega)$, $H_0(\Omega)$ is a Fréchet space.

(3) Let $K$ be a compact subset of $\mathbb{C}$, then $A(K)$ shall be the space $H(K^\circ) \cap C(K)$ endowed with the topology induced by the norm $||F|| := \sup_{z \in K} |F(z)|$. $A(K)$ is a Banach space.

**Remark 1.2** Let $\Omega$ be an open subset of $\mathbb{C}$. If $\{z : |z| > R\} \subset \Omega$ for some $R > 0$, Riemann's theorem on removable singularities implies that each $f \in H_0(\Omega)$ is holomorphic at $\infty$.

After this basic preliminaries, we come to the main objective of this section, namely, the introduction of cycles and, in particular, of Cauchy cycles. This concept is also used in [GE93].

**Definition 1.3**

(1) Let $\gamma : [a, b] \to \mathbb{C}$ be a piecewise continuously differentiable curve, then $\gamma$ is called a *path*. Its range $\gamma([a, b])$ is called the *trace* of $\gamma$ and shall be denoted by $|\gamma|$. The path that passes through $|\gamma|$ in the reverse direction is given by $\gamma^-(t) := \gamma(a + b - t)$, $t \in [a, b]$. If $\gamma(a) = \gamma(b)$, then $\gamma$ is said to be a *closed path*.

(2) The length of a path $\gamma$ is given by

$$\int_a^b |\gamma'(t)|\, dt$$

and shall be denoted by $\text{len}(\gamma)$.

(3) For a closed path $\gamma$ and $z \in \mathbb{C} \setminus |\gamma|$, we set

$$\text{ind}_\gamma(z) := \frac{1}{2\pi i} \int_\gamma \frac{1}{\xi - z} \, d\xi$$

and call $\text{ind}_\gamma(z)$ the *index* of $z$ with respect to $\gamma$.

The terminology in the next definition is taken from [Rud74].

**Definition 1.4**

(1) Let $\gamma_1, \ldots, \gamma_n$ be a finite family of closed paths. Then $\Gamma := (\gamma_1, \ldots, \gamma_n)$ is called a *cycle*. With $\Gamma^-$ we shall denote the cycle $(\gamma_1^-, \ldots, \gamma_n^-)$. The *trace* of $\Gamma$ is given by $|\Gamma| := \bigcup_{j=1}^{n} |\gamma_j|$.

(2) The integral of a function $f \in C(|\Gamma|)$ over a cycle $\Gamma = (\gamma_1, \ldots, \gamma_n)$ is defined by

$$\int_\Gamma f(\xi) \, d\xi := \sum_{j=1}^{n} \int_{\gamma_j} f(\xi) \, d\xi.$$

(3) The length of a cycle $\Gamma = (\gamma_1, \ldots, \gamma_n)$ is given by

$$\text{len}(\Gamma) := \sum_{j=1}^{n} \text{len}(\gamma_j).$$

(4) For a cycle $\Gamma$ and $z \in \mathbb{C} \setminus |\Gamma|$, we set

$$\text{ind}_\Gamma(z) := \frac{1}{2\pi i} \int_\Gamma \frac{1}{\xi - z} \, d\xi$$

and call $\text{ind}_\Gamma(z)$ the *index* of $z$ with respect to $\Gamma$.

(5) Let $\Gamma = (\gamma_1, \ldots, \gamma_n)$ be a cycle and $\varphi \in C^1(|\Gamma|)$. Then $\varphi \circ \Gamma$ is defined as $(\varphi \circ \gamma_1, ..., \varphi \circ \gamma_n)$. The union $\Gamma \cup \tilde{\Gamma}$ with another cycle $\tilde{\Gamma} := (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_m)$ shall be defined as $(\gamma_1, \ldots, \gamma_n, \tilde{\gamma}_1, \ldots, \tilde{\gamma}_m)$.

**Remark 1.5** Let $\Gamma$ be a cycle and $f \in C(|\Gamma|)$, then

$$\int_{\Gamma^-} f(t)\,dt = -\int_\Gamma f(t)\,dt.$$

**Theorem 1.6** *Let $\Omega \subset \mathbb{C}$ be some open set and $f \in H(\Omega)$. If $\Gamma$ is a cycle in $\Omega$ that satisfies $\operatorname{ind}_\Gamma(w) = 0$ for all $w \in \mathbb{C} \setminus \Omega$, then for all $z \in \Omega \setminus |\Gamma|$, the Cauchy integral formula*

$$f(z)\operatorname{ind}_\Gamma(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z}\,d\xi$$

*and Cauchy's integral theorem*

$$0 = \int_\Gamma f(\xi)\,d\xi$$

*are valid.*

For the proof of this theorem, we refer to [Rud74, Theorem 10.35].

**Definition 1.7** Let $\Omega \subset \mathbb{C}$ be an open set and $K$ a compact subset of $\Omega$. A cycle $\Gamma = (\gamma_1, \ldots, \gamma_n)$ such that $|\Gamma| \subset \Omega \setminus K$ and

$$\operatorname{ind}_\Gamma(z) = \begin{cases} 1 & \text{, for all } z \in K \\ 0 & \text{, for all } z \in \mathbb{C} \setminus \Omega \end{cases}$$

is called a *Cauchy cycle for $K$ in $\Omega$*.

**Remark 1.8** In this thesis, we are mainly concerned with compact sets $K \subset \mathbb{C}$ that are also convex. In this case, a Cauchy cycle for $K$ in an open neighbourhood of $K$ can be chosen as a Jordan curve having positive orientation with respect to the points in $K$.

**Theorem 1.9** *Let $\Omega$ be an open set in the complex plane and $K$ a compact subset of $\Omega$. Then there exists a Cauchy cycle for $K$ in $\Omega$.*

The above assertion is an immediate consequence of [Rud74, Theorem 13.5].

Henceforth, we say that $\varphi \in H(\Omega)$ is univalent if $\varphi$ maps one-to-one from $\Omega$ to $\varphi(\Omega)$.

**Proposition 1.10** *Let $\Omega \subset \mathbb{C}$ be an open set, $\varphi \in H(\Omega)$ univalent and $K \subset \Omega$ compact. If $\Gamma$ is a Cauchy cycle for $K$ in $\Omega$, then $\varphi \circ \Gamma$ is a Cauchy cycle for $\varphi(K)$ in $\varphi(\Omega)$.*

*Proof.* Let $f$ be a holomorphic function with a zero of order one at $w$. Then the residue of $f'/f$ at $w$ is equal to one. For $z = \varphi(w) \in \varphi(K)$, we set $f(\xi) := \varphi(\xi) - \varphi(w)$. Since $\varphi$ is univalent, $f$ has exactly one zero of order one in $\Omega$. With a version of the residue theorem for cycles (cf.[Rud74, Theorem 10.42]), we have

$$\begin{aligned} 1 &= \frac{1}{2\pi i}\int_\Gamma \frac{f'(\xi)}{f(\xi)}\,d\xi = \frac{1}{2\pi i}\int_\Gamma \frac{\varphi'(\xi)}{\varphi(\xi)-\varphi(w)}d\xi \\ &= \frac{1}{2\pi i}\int_{\varphi\circ\Gamma} \frac{1}{w-z}\,dw = \operatorname{ind}_{\varphi\circ\Gamma}(z). \end{aligned}$$

Now, let $z \in \mathbb{C} \setminus \varphi(\Omega)$. Then $f(\xi) := \varphi(\xi) - z$ has no zero in $\Omega$. Thus the above reasoning shows that $\operatorname{ind}_{\varphi\circ\Gamma}(z) = 0$. □

For holomorphic functions that vanish at infinity, we obtain a Cauchy integral representation for points located in the exterior of a cycle.

**Lemma 1.11** *Let $K \subset \mathbb{C}$ be a compact set, $f \in H_0(\mathbb{C} \setminus K)$ and $\Gamma$ a Cauchy cycle for $K$ in $\mathbb{C}$. Then for $z \in \mathbb{C} \setminus K$ with $\operatorname{ind}_\Gamma(z) = 0$, we have*

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{z-\xi}\,d\xi.$$

*Proof.* We choose $\varepsilon > 0$ such that $\overline{U_\varepsilon(z)} \cap K = \emptyset$. Then, with $\gamma(t) = z + \varepsilon e^{it}$, $t \in [0, 2\pi]$, we obtain that $\tilde{\Gamma} := \Gamma \cup (\gamma)$ is a Cauchy cycle for $K \cup \{z\}$ in $\mathbb{C}$. For every $R > \max\{|z|, \max\{|w| : w \in K\}\}$, the path $\gamma_R(t) := Re^{it}$, $t \in [0, 2\pi]$, is also a Cauchy cycle for $K \cup \{z\}$ in $\mathbb{C}$. Consequently, the index of every $w \in K \cup \{z\}$ with respect to $(\gamma_R^-) \cup \tilde{\Gamma}$ is zero. Taking into account that $\xi \mapsto f(\xi)/(\xi - z)$ is holomorphic in $\mathbb{C} \setminus (K \cup \{z\})$, Cauchy's integral theorem yields

$$\frac{1}{2\pi i}\int_{\gamma_R} \frac{f(\xi)}{\xi - z}\,d\xi = \frac{1}{2\pi i}\int_{\tilde{\Gamma}} \frac{f(\xi)}{\xi - z}\,d\xi.$$

The function $f$ is zero at infinity, and thus the integral on the left hand side vanishes for $R$ going to infinity. Since the equality holds for all $R$ that are large enough, we obtain that both integrals are equal to zero. Considering the Cauchy integral formula, this means

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z}\, d\xi = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{z - \xi}\, d\xi.$$

□

## 1.2 Functions of Exponential Type and the Conjugate Indicator Diagram

In the following section, we introduce the basic terminology and some properties of growth conditions for entire functions. Furthermore, we establish the indicator function, the conjugate indicator diagram for functions of exponential type, and its connection to growth via the support function.
Henceforth, $M_f(r)$ denotes the *maximum modulus*, $\max\{|f(z)| : |z| = r\}$, of a function $f$ for $r \geq 0$.

**Definition 1.12** Let $f$ be an entire function.

(1) Then

$$\limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r} = \rho$$

is the *order* of $f$, and $f$ is called a function of order $\rho$.

(2) If $f$ is a function of finite, positive order $\rho$ and

$$\limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho}} = \tau,$$

then $\tau$ is the *type* of $f$. The function $f$ is said to be a function of order $\rho$ and type $\tau$.

(3) Entire functions $f$ that satisfy

$$\limsup_{r\to\infty} \frac{\log M_f(r)}{r} = \tau < \infty,$$

are called *functions of exponential type.* In order to specify that the above lim sup is equal to $\tau$, $f$ is said to be a *function of exponential type* $\tau$.

We assume that the Taylor expansion of an entire function $f$ is given by

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n. \tag{1.1}$$

Then the growth of the $|a_n|$ characterizes whether $f$ is a function of exponential type (see [Boa54] Theorem 2.2.10).

**Proposition 1.13** *The entire function $f$ in (1.1) is a function of exponential type $\tau$ if and only if*

$$\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \tau < \infty.$$

In the terms introduced in Definition 1.12, the growth of an entire function is measured with respect to the maximum modulus. For the investigation of the growth on single rays or in sectors, we introduce the well-known indicator function.

**Definition 1.14** (Indicator Function) Assume that $f$ is a function of exponential type. Then the *indicator function* $h_f : [0, 2\pi] \to [-\infty, \infty)$ of $f$ is defined by

$$h_f(\Theta) := \limsup_{r\to\infty} \frac{\log |f(re^{i\Theta})|}{r}.$$

Let $f$ be an entire function of exponential type. For a fixed $\xi \in \mathbb{T}$, the growth of $f$ in the direction of $\xi$ can be estimated by means of its indicator function, and one easily verifies that the Laplace transform

$$\mathcal{B}_\xi f(z) := \xi \int_0^\infty f(t\xi)\, e^{-zt\xi}\, dt \tag{1.2}$$

defines a holomorphic function in the half-space

$$W_\xi := \{z \in \mathbb{C} : \mathrm{Re}(z\xi) > h_f(\arg(\xi))\}. \tag{1.3}$$

In [Mor94, pages 36-37], it is shown that for $\xi_1$, $\xi_2 \in \mathbb{T}$ the functions $\mathcal{B}_{\xi_1} f$ and $\mathcal{B}_{\xi_2} f$ defined in (1.2) coincide on $W_{\xi_1} \cap W_{\xi_2}$. Now, gluing these functions, we obtain a holomorphic function in $\bigcup_{\xi\in\mathbb{T}} W_\xi$ that vanishes at infinity and that we denote by $\mathcal{B}f$.

**Definition 1.15** (Borel Transform, Conjugate Indicator Diagram) Let $f$ be an entire function of exponential type. Then the corresponding function $\mathcal{B}f$ is called the *Borel transform of* $f$. The convex and closed set $\mathbb{C} \setminus \bigcup_{\xi \in \mathbb{T}} W_\xi$ is the *conjugate indicator diagram* of $f$ and shall be denoted by $K(f)$. The mapping $f \mapsto \mathcal{B}f$ is the *Borel transform.*

**Theorem 1.16** *Let $f$ in (1.1) be an entire function of exponential type. Then $\mathcal{B}f$ is the analytic continuation of $\sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}$ to $\mathbb{C} \setminus K(f)$.*

A proof of this result can be found in many books about entire functions such as [Boa54].

The conjugate indicator diagram of an entire function $f$ of exponential type is empty if and only if its Borel transform is holomorphic in $\mathbb{C}$. Since $\mathcal{B}f$ vanishes at infinity, this is equivalent to $\mathcal{B}f \equiv 0$, and thus $f \equiv 0$.

The next proposition summarizes some properties of the conjugate indicator diagram, which can be found in [Boa54, pages 75-77].

**Proposition 1.17** *Let $f, g$ be two functions of exponential type. Then the following are valid:*

*(1) The conjugate indicator diagram of $fg$ is contained in $K(f) + K(g)$. If one of these sets is a singleton, the conjugate indicator diagram of $fg$ is equal to the set $K(f) + K(g)$.*

*(2) The conjugate indicator diagram of $f + g$ is contained in* $\operatorname{conv}(K(f) \cup K(g))$.

*(3) The function $f$ is a function of exponential type $\tau = \max\{|u| : u \in K(f)\}$.*

*(4) The conjugate indicator diagram of $f'$ is contained in $K(f)$.*

**Example 1.18**

(1) The conjugate indicator diagram of $f(z) = e^{\alpha z}$ is $\{\alpha\}$, and $f$ is a function of exponential type $|\alpha|$.

(2) The conjugate indicator diagram of $f(z) = \sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$ is $[-i, i]$, and $f$ is a function of exponential type 1.

(3) The conjugate indicator diagram of every entire function of order less than one is contained in $\{0\}$.

As the half-spaces in (1.3) are defined by means of the indicator function, it is clear that the union of all these half-spaces, respectively the conjugate indicator diagram, reflects the behaviour of the indicator function. In [Pól29], G. Pólya shows a useful relation between the geometry of the conjugate indicator diagram and the indicator function. The geometry of the conjugate indicator diagram becomes relevant via the following function.

**Definition 1.19** Let $K$ be a convex and compact subset of the complex plane. Then the function

$$H_K(z) := \sup_{u \in K} \mathrm{Re}(zu)$$

is the *support function* of $K$.

**Theorem 1.20** *Let $f$ be a function of exponential type. Then*

$$h_f(\Theta) = H_{K(f)}(e^{i\Theta})$$

*for all $\Theta \in [0, 2\pi]$.*

Contrary to the above definition of the support function, G. Pólya uses the support function $k(\Theta) := \max \left\{ \mathrm{Re} \left( ue^{-i\Theta} \right) : u \in K \right\}$. The above result then states that $h_f(\Theta) = k(\Theta)$ if $K$ is the reflection of the conjugate indicator diagram with respect to the real axis (the indicator diagram). That is why the set we use is called "conjugate indicator diagram". Our definition of the support function is more common nowadays. It is also used in [BG95] and [Mor94].
We will now give some basic properties of the support function (cf. [BG95, Proposition 1.3.14]).

**Remark 1.21** For two non-empty, compact and convex subsets $K, L$ of $\mathbb{C}$, the support function has the following properties:

(1) $H_K + H_L = H_{K+L}$,

(2) $H_K(\lambda\, z) = \lambda\, H_K(z)$ for $\lambda > 0$,

(3) $H_K(\lambda z + (1-\lambda)w) \leq \lambda\, H_K(z) + (1-\lambda)\, H_K(w)$ for $0 < \lambda < 1$.

(4) If $K \subset L$, then $H_K \leq H_L$.

**Example 1.22**

(1) If $K = \{z : |z| \leq r\}$, then $H_K(z) = r|z|$.

(2) If $K = \{w\} = \{x + iy\}$, then $H_K(z) = H_K(|z|\, e^{i\Theta}) = |z|\, (x \cos(\Theta) - y \sin(\Theta))$.

(3) If $K = [-id, id]$ for $d > 0$, then $H_K(z) = H_K(|z|\, e^{i\Theta}) = d\, |z||\sin(\Theta)|$.

(4) If $K \subset \mathbb{C}$ is some compact, convex set and $\tilde{K} := K + \tau\overline{\mathbb{D}}$, then $H_{\tilde{K}}(z) = H_K(z) + \tau|z|$.

## 1.3 The Space Exp($K$)

We introduce a topological vector space, namely Exp($K$), that consists of functions of exponential type having a conjugate indicator diagram contained in $K$. The hypercyclicity and frequent hypercyclicity of operators acting on Exp($K$) is a main topic in this thesis.
We use the following notation: For any topological vector space $X$, the dual space endowed with the strong topology is denoted by $X^*$. For the action of a functional $\Lambda$ on a vector $x$, we write $\langle \Lambda, x \rangle$.
In the previous section we introduced the Borel transform of an entire function of exponential type via the Laplace transform in different directions of the complex plane. We shall now see that conversely, a function of exponential type can be reconstructed by means of an integral formula via its Borel transform. This yields the inverse of the Borel transform.

**Theorem 1.23** (Pólya Representation) *Let $f$ be a function of exponential type, then*

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi)\, e^{z\xi}\, d\xi$$

*where $\Gamma$ is a Cauchy cycle for $K(f)$ in $\mathbb{C}$.*

We easily deduce the following.

**Proposition 1.24** *Assume that $f$ is a function of exponential type and $K$ a compact, convex subset of $\mathbb{C}$. Then the conjugate indicator diagram of $f$ is contained in $K$ if and only if for each $\varepsilon > 0$ there is a constant $C_\varepsilon$ such that*

$$\sup_{z\in\mathbb{C}} |f(z)|e^{-H_K(z)-\varepsilon|z|} \leq C_\varepsilon.$$

This result leads to the definition of a space of entire functions of exponential type.

**Definition 1.25** Let $K$ be a compact and convex subset of $\mathbb{C}$. Then $\mathrm{Exp}(K)$ is the space of functions of exponential type that have a conjugate indicator diagram that is contained in $K$. For $f \in \mathrm{Exp}(K)$ and $n \in \mathbb{N}$, we define

$$||f||_{K,n} := \sup_{z\in\mathbb{C}} |f(z)|e^{-H_K(z)-\frac{1}{n}|z|}.$$

Then $||f||_{K,n}$ is well defined by Proposition 1.24, and it is a norm on $\mathrm{Exp}(K)$. The space $\mathrm{Exp}(K)$ shall be endowed with the topology induced by the sequence of norms $(||\cdot||_{K,n})_{n\in\mathbb{N}}$.

Following [Mor94], we shall now formulate the topologies on our considered spaces in terms of projective and inductive limits. This approach provides nice relations between these topologies and the topologies of their dual spaces. A short introduction to this topic is given in Appendix A.
We consider a compact and convex set $K \subset \mathbb{C}$. Then for some fixed $n \in \mathbb{N}$,

$$\mathrm{Exp}_n(K) := \{f \in H(\mathbb{C}) : ||f||_{K,n} < \infty\}$$

is a Banach space with respect to the norm $||\cdot||_{K,n}$. The intersection of all these Banach spaces generates $\mathrm{Exp}(K)$.

**Proposition 1.26** *Let $K$ be a compact and convex subset of $\mathbb{C}$. Then*

$$\mathrm{Exp}(K) = \mathrm{ProjLim}(\mathrm{Exp}_n(K) : n \in \mathbb{N}),$$

*and the embeddings $\mathrm{Exp}_n(K) \hookrightarrow \mathrm{Exp}_{n-1}(K)$ are compact. Thus $\mathrm{Exp}(K)$ is an FS space.*

**Remark 1.27** In this thesis we will not make use of the Banach spaces $\mathrm{Exp}_n(K)$ in our reasonings. Instead the following will be applied:
Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact and convex subsets of $\mathbb{C}$ such that $K_n^\circ \supset K_{n+1}$ and $\bigcap_{n\in\mathbb{N}} K_n = K$. Then $K$ is a compact, convex subset of $\mathbb{C}$ and

$$\mathrm{Exp}(K) = \bigcap_{n\in\mathbb{N}} \mathrm{Exp}(K_n).$$

Remark A.10 implies that this equality also holds in topological sense. That means, the topology on $\mathrm{Exp}(K)$ is also induced by the norms $(||\cdot||_{K_n,k})_{n,k\in\mathbb{N}}$. See also Theorem A.8 to verify this equality.

**Proposition 1.28** *Let $\Omega$ be an open subset of $\mathbb{C}$ and $(K_n)_{n\in\mathbb{N}}$ an increasing sequence of compact sets such that $\bigcup_{n\in\mathbb{N}} K_n = \Omega$ and $K_n \subset K_{n+1}^\circ$. Then*

$$H(\Omega) = \mathrm{ProjLim}(A(K_n) : n \in \mathbb{N}),$$

*and the restrictions $F \mapsto F|_{K_{n-1}}$ from $A(K_n)$ to $A(K_{n-1})$ are compact. Thus $H(\Omega)$ is an FS space.*

**Remark 1.29** Let $\Omega$ be an open subset of $\mathbb{C}$. As a closed subspace of $H(\Omega)$, $H_0(\Omega)$ is an FS space (cf. Theorem A.13).

For simplicity, if $f$ is a function of exponential type and $K$ is a compact, convex set containing $K(f)$, we also denote the mapping $f \mapsto \mathcal{B}f|_{\mathbb{C}\setminus K}$ by $\mathcal{B}$. With the above introduced spaces, we obtain an isomorphism (cf. [Mor94, Theorem 2.5.1]).

**Proposition 1.30** *Assume that $K$ is a convex and compact subset of $\mathbb{C}$. Then the Borel transform*

$$\mathcal{B} = \mathcal{B}_K : \mathrm{Exp}(K) \to H_0(\mathbb{C} \setminus K)$$

*is an isomorphism.*

**Remark 1.31** For a compact set $K \subset \mathbb{C}$ (possibly non-convex), it is clear that the restriction $F \mapsto F|_{\mathbb{C}\setminus\mathrm{conv}(K)}$ from $H_0(\mathbb{C} \setminus K)$ to $H_0(\mathbb{C} \setminus \mathrm{conv}(K))$ has dense image. Hence $\mathcal{B}^{-1}_{\mathrm{conv}(K)}(F \mapsto F|_{\mathbb{C}\setminus\mathrm{conv}(K)}) : H_0(\mathbb{C} \setminus K) \to \mathrm{Exp}(\mathrm{conv}(K))$ has dense image. The transform $\mathcal{B}^{-1}_{\mathrm{conv}(K)}(F \mapsto F|_{\mathbb{C}\setminus\mathrm{conv}(K)})$ shall also be denoted by $\mathcal{B}_K^{-1}$. The Pólya representation

$$\mathcal{B}_K^{-1}F(z) = \frac{1}{2\pi i}\int_\Gamma F(\xi)\, e^{z\xi}\, d\xi$$

for $F \in H_0(\mathbb{C} \setminus K)$ is valid for all Cauchy cycles $\Gamma$ for $K$ in $\mathbb{C}$.

A germ of holomorphic functions on a set $K \subset \mathbb{C}$ is the collection of holomorphic functions that coincide on some open neighbourhood of $K$. In terms of inductive limits, one can endow the space of germs of holomorphic functions with a natural topology.

**Definition 1.32** Let $K$ be a compact subset of $\mathbb{C}$ and $U(K)$ the collection of all open neighbourhoods of $K$. We define

$$H(K) := \mathrm{IndLim}(H(\Omega) : \Omega \in U(K)).$$

The elements $[\varphi] \in H(K)$ are called *germs of holomorphic functions* (on $K$).

Replacing $U(K)$ by an appropriate sequence of bounded neigbourhoods and $H(\Omega)$ by $A(\overline{\Omega})$, $H(K)$ becomes a DFS space (cf. [Mor94, Theorem 1.5.5]).

**Proposition 1.33** *Let $K$ be a compact subset of $\mathbb{C}$ and $(K_n)_{n\in\mathbb{N}}$ a decreasing sequence of compact sets such that $K_{n+1} \subset K_n^\circ$ and $K = \bigcap_{n\in\mathbb{N}} K_n$. Then*

$$H(K) = \mathrm{IndLim}(A(K_n) : n \in \mathbb{N})$$

*and the restrictions $f \mapsto f|_{K_{n+1}}$ from $A(K_n)$ to $A(K_{n+1})$ are compact. Thus $H(K)$ is a DFS space.*

An element $[\varphi] \in H(K)$ can be considered as a family of representatives $(\varphi_\iota)_{\iota\in I}$, each of which is holomorphic on some open neighbourhood of $K$ and each two of which coincide on an open neighbourhood of $K$. Two elements $[\varphi_1], [\varphi_2] \in H(K)$ are equal if and only if a representative $\varphi$ belongs to both $[\varphi_1]$ and $[\varphi_2]$.
In order to simplify the notation, we agree the following: An element $[\varphi] \in H(K)$ shall always be identified with some of its representatives $\varphi$. An open neighbourhood of $K$ that is a domain of holomorphy of $\varphi$ is denoted by $\Omega_\varphi$.

**Definition 1.34** Let $K$ be a compact subset of $\mathbb{C}$. The elements of $H^*(K)$ are called *locally analytic functionals on K.*

**Remark 1.35** According to Theorem A.11 and Proposition 1.33, $H^*(K)$ is an FS space. If $\Omega \subset \mathbb{C}$ is open, then $H^*(\Omega)$ is a DFS space. The elements of $H^*(\Omega)$ are called *analytic functionals (on* $\Omega$*)*.

For the sake of completeness, we introduce two further transforms that yield the isomorphism of the spaces $\operatorname{Exp}(K)$, $H_0(\mathbb{C} \setminus K)$ and $H^*(K)$.
We consider a locally analytic functional $\Lambda \in H^*(K)$, where $K$ is some compact, possibly non-convex, subset of $\mathbb{C}$. Then the function $F$ defined by

$$F(z) := \mathcal{C}\Lambda(z) := \left\langle \Lambda, \xi \mapsto \frac{1}{z - \xi} \right\rangle$$

is an element of $H_0(\mathbb{C} \setminus K)$. The mapping

$$\Lambda \mapsto \mathcal{C}\Lambda$$

is called the *Cauchy-Hilbert transform.* The function $f$ defined by

$$f(z) := \mathcal{F}\Lambda(z) := \left\langle \Lambda, \xi \mapsto e^{z\xi} \right\rangle$$

is an element of $\operatorname{Exp}(\operatorname{conv}(K))$, and the mapping

$$\Lambda \mapsto \mathcal{F}\Lambda$$

is called the *Fourier-Borel transform.*

**Proposition 1.36** *Let $K$ be a compact subset of* $\mathbb{C}$.

*(1) The Cauchy-Hilbert transform*

$$\mathcal{C} : H^*(K) \to H_0(\mathbb{C} \setminus K)$$

*is an isomorphism.*

*(2) If $K$ is convex, the Fourier-Borel transform*

$$\mathcal{F} : H^*(K) \to \operatorname{Exp}(K)$$

*is an isomorphism.*

For the proof of this result, we refer to [Mor94, Theorem 2.1.9, Theorem 2.5.2].

**Remark 1.37** For $H \in H_0(\mathbb{C} \setminus K)$, the locally analytic functional $\mathcal{C}^{-1}H \in H^*(K)$ is given by

$$\left\langle \mathcal{C}^{-1}H, \varphi \right\rangle = \frac{1}{2\pi i} \int_\Gamma H(\xi)\, \varphi(\xi)\, d\xi$$

where $\Gamma$ is a Cauchy cycle for $K$ in $\Omega_\varphi$.
To see this, let $\Lambda \in H^*(K)$ be defined by the above integral. Then for $z \in \mathbb{C} \setminus \Omega_\varphi$, we have $\operatorname{ind}_\Gamma(z) = 0$ and obtain

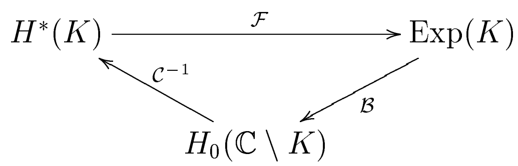
$$\mathcal{C}\Lambda(z) = \left\langle \Lambda, \xi \mapsto \frac{1}{z-\xi} \right\rangle = \frac{1}{2\pi i} \int_\Gamma \frac{H(\xi)}{z-\xi}\, d\xi = H(z)$$

by Lemma 1.11.

According to Remark 1.37, the function $H \in H_0(\mathbb{C} \setminus K)$ is mapped to a function of $\operatorname{Exp}(K)$ by

$$\mathcal{F}\,\mathcal{C}^{-1}H = \frac{1}{2\pi i} \int_\Gamma H(\xi)\, e^{z\xi}\, d\xi.$$

The right side of the above formula is equal to the Pólya representation (see Theorem 1.23) if $H$ is regarded as the Borel transform of a function that belongs to $\operatorname{Exp}(K)$. This shows $\mathcal{B}^{-1} = \mathcal{F}\,\mathcal{C}^{-1}$.
The below diagram abstracts the relation between the introduced transforms. Let $K \subset \mathbb{C}$ be compact and convex. Then the diagram




is commutative and the mappings are isomorphisms.

By Theorem A.12, DFS-spaces are reflexive and thus $H(K)$ is isomorphic to the space $(H(K)^*)^*$. Since $H_0(\mathbb{C}\setminus K)$ and $H^*(K)$ are isomorphic, we have an description of the functionals $\Lambda \in H_0^*(\mathbb{C} \setminus K)$ by means of the germs of holomorphic functions on $K$. This relation is known as the Köthe duality (see [Köt53]).

**Theorem 1.38** (Köthe Duality) *Let $K$ be a compact, convex set in $\mathbb{C}$ and $\Lambda \in H_0^*(\mathbb{C} \setminus K)$. Then there is a germ $\varphi \in H(K)$ such that*

$$\langle \Lambda, F \rangle = \frac{1}{2\pi i} \int_\Gamma \varphi(\xi)\, F(\xi)\, d\xi$$

*for all $F \in H_0(\mathbb{C} \setminus K)$. Here $\Gamma$ is a Cauchy cycle for $K$ in $\Omega_\varphi$. As $\mathrm{Exp}(K)$ and $H_0(\mathbb{C} \setminus K)$ are isomorphic via the Borel transform, any functional $\Lambda \in \mathrm{Exp}^*(K)$ is represented by*

$$\langle \Lambda, f \rangle = \frac{1}{2\pi i} \int_\Gamma \varphi(\xi)\, \mathcal{B}f(\xi)\, d\xi$$

*with $\varphi$ and $\Gamma$ as above.*

The Köthe duality actually describes the dual-space of $H(\Omega)$ for arbitrary open sets $\Omega$.

In many cases it will be convenient to consider $H_0(\mathbb{C} \setminus K^{-1})$ instead of $H_0(\mathbb{C} \setminus K)$. For that purpose, we introduce the following transform: Let $F \in H_0(\mathbb{C} \setminus K)$ where $K$ is some compact subset of $\mathbb{C}$. Then the function

$$\mathcal{U}F(z) := \frac{1}{z}\, F\left(\frac{1}{z}\right) \tag{1.4}$$

is holomorphic in $\mathbb{C} \setminus K^{-1}$ and vanishes at infinity.

**Proposition 1.39** *Let $K \subset \mathbb{C}$ be compact. Then*

$$\mathcal{U} : H_0(\mathbb{C} \setminus K) \to H_0(\mathbb{C} \setminus K^{-1})$$

*defined by (1.4) is an isomorphism.*

*Proof.* By definition, it is clear that $\mathcal{U}(F)$ is an element of $H_0(\mathbb{C} \setminus K^{-1})$ and that $\mathcal{U}$ is a bijective mapping. According to Remark 1.2, we can show the continuity in the topology of uniform convergence on compact subsets of $\mathbb{C}_\infty \setminus K$ and $\mathbb{C}_\infty \setminus K^{-1}$, respectively.

Let $L$ be a compact subset of $\mathbb{C}_\infty \setminus K$. This is equivalent to the fact that $L^{-1}$ is a compact subset of $\mathbb{C}_\infty \setminus K^{-1}$. Without loss of generality, we can assume that

$\overline{\varepsilon\,\mathbb{D}} \subset L^{-1}$ for $\varepsilon > 0$. For $F \in H_0(\mathbb{C} \setminus K)$, we obtain

$$\sup_{|z|>\varepsilon,\, z\in L^{-1}} |\mathcal{U}F(z)| = \sup_{|z|>\varepsilon,\, z\in L^{-1}} \left|\frac{1}{z}\right| \left|F\left(\frac{1}{z}\right)\right| \le \frac{1}{\varepsilon} \sup_{z\in L} |F(z)|$$

and by the maximum principle

$$\sup_{|z|\le\varepsilon} |\mathcal{U}F(z)| \le \sup_{|z|=\varepsilon} |\mathcal{U}F(z)| = \sup_{|z|=\varepsilon} \left|\frac{1}{z} F\left(\frac{1}{z}\right)\right| = \sup_{|z|=\frac{1}{\varepsilon}} \frac{1}{\varepsilon} |F(z)| \le \frac{1}{\varepsilon} \sup_{z\in L} |F(z)|\,.$$

These two inequalities imply

$$\sup_{z\in L^{-1}} |\mathcal{U}F(z)| \le \frac{1}{\varepsilon} \sup_{z\in L} |F(z)|$$

and hence the continuity of $\mathcal{U}$. The continuity of $\mathcal{U}^{-1}$ can be proved in the same way. □

**Remark 1.40** Let $f(z) = \sum\limits_{n=0}^{\infty} \frac{a_n}{n!} z^n$ be an entire function of exponential type $\tau$. Then from Theorem (1.16) we have that

$$\mathcal{B}f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^{n+1}}$$

for $|z| > \tau$. Therefrom, it is easily seen that the function $\mathcal{U}\mathcal{B}f \in H_0(\mathbb{C} \setminus K^{-1})$ has the Taylor series expansion

$$\mathcal{U}\mathcal{B}f(z) = \sum_{n=0}^{\infty} a_n\, z^{n+1}$$

for $|z| < \frac{1}{\tau}$. It will often be convenient to consider these power series representations to verify identities: For the exponential function $f(z) := e^{\alpha z} = \sum\limits_{n=0}^{\infty} \frac{\alpha^n}{n!} z^n$ with $\alpha \neq 0$, one immediatly deduces

$$\mathcal{B}f(z) = \frac{1}{z} \sum_{n=0}^{\infty} \frac{\alpha^n}{z^n} = \frac{1}{z-\alpha}$$

and

$$\mathcal{UB}f(z) = \sum_{n=0}^{\infty} \alpha^n z^n = \frac{1}{1-\alpha z}.$$

Our last result of this section is about the generic size of the set of functions in $\mathrm{Exp}(K)$ that have a conjugate indicator diagram that coincides with $K$.

A function is said to be exactly holomorphic in an open set $\Omega \subset \mathbb{C}$ if for all $z_0 \in \Omega$ the radius of convergence of the Taylor series of $f$ with center $z_0$ is equal to $\mathrm{dist}(z_0, \partial\Omega)$.

**Theorem 1.41** (cf. [Nes05, Theorem 4.5]) *Let $\Omega$ be a domain in $\mathbb{C}$. Then the set of functions of $H(\Omega)$ that are exactly holomorphic in $\Omega$ is a dense $G_\delta$-set (i.e. a countable intersection of open and dense sets) in $H(\Omega)$ .*

**Corollary 1.42** Let $K \subset \mathbb{C}$ be compact and convex. The set of functions of $\mathrm{Exp}(K)$ whose conjugate indicator diagram equals $K$ is residual in $\mathrm{Exp}(K)$.

*Proof.* The assertion is an immediate consequence of Theorem 1.41 and the fact that $\mathrm{Exp}(K)$ and $H_0(\mathbb{C} \setminus K)$ are isomorphic. □

## 1.4 Dense Subspaces of Exp(K)

In the investigation of hypercyclicity, it will be essential to have appropriate dense subspaces. By means of the Borel transform and the transform $\mathcal{U}$, we derive the denseness of the polynomials and the linear hull of exponential functions in $\mathrm{Exp}(K)$ from Runge's theorem under suitable conditions. At the end of this section, a general denseness result is given.

**Proposition 1.43** *Let $K$ be a compact and convex subset of $\mathbb{C}$ that contains the origin. Then the polynomials are dense in* $\mathrm{Exp}(K)$.

*Proof.* The space of polynomials shall be denoted by $\Pi$. Consider the transform $\mathcal{UB}$ from $\mathrm{Exp}(K)$ to $H(\mathbb{C} \setminus K^{-1})$. For a given polynomial $P(z) = \sum_{k=0}^{n} a_k z^k$, the Borel transform is given by $\mathcal{B}P(z) = \sum_{k=0}^{n} \frac{k! a_k}{z^{k+1}}$ and thus $\mathcal{UB}P(z) = \sum_{k=0}^{n} k! a_k z^k$. We can conclude that $\mathcal{UB}(\Pi) = \Pi$. As $K$ contains the origin, $\mathbb{C} \setminus K^{-1}$ is simply connected, and thus $\Pi$ is a dense subspace of $H(\mathbb{C} \setminus K^{-1})$ by Runge's theorem. According to Proposition 1.30 and Proposition 1.39, $\mathcal{UB}$ is an isomorphism and hence $(\mathcal{UB})^{-1}(\Pi)$ is also a dense subspace of $\mathrm{Exp}(K)$. □

For compact sets that do not necessarily contain the origin, the following result provides a dense subspace of $\operatorname{Exp}(K)$.
Henceforth, the exponential function $z \mapsto e^{\alpha z}$ is denoted by $e_\alpha$.

**Proposition 1.44** *Assume that $K$ is a compact and convex subset of $\mathbb{C}$. Then for every $\alpha \in K$, the subspace $\{Pe_\alpha : P \text{ polynomial }\}$ is dense in $\operatorname{Exp}(K)$.*

*Proof.* For any polynomial $P$ we have $K(Pe_\alpha) \subset \{\alpha\}$ by Proposition 1.17 (1). Hence $\{Pe_\alpha : P \text{ polynomial }\}$ is a subspace of $\operatorname{Exp}(K)$.
For arbitrary $f \in \operatorname{Exp}(K)$, $n \in \mathbb{N}$ and $\varepsilon > 0$, we have to find a polynomial $P$ such that $\|f - Pe_\alpha\|_{K,n} < \varepsilon$. Set $g = f/e_\alpha$, then $g$ is an element of $\operatorname{Exp}(K - \{\alpha\})$ by Proposition 1.17 (1). The set $K - \{\alpha\}$ contains the origin, and thus Proposition 1.43 implies the existence of a polynomial $P$ such that $\|g - P\|_{K-\{\alpha\},n} < \varepsilon$. We obtain

$$\begin{aligned}
\|f - Pe_\alpha\|_{K,n} &= \sup_{z\in\mathbb{C}} |g(z) - P(z)|\, |e^{\alpha z}|\, e^{-H_K(z) - \frac{1}{n}|z|} \\
&= \sup_{z\in\mathbb{C}} |g(z) - P(z)|\, e^{-H_K(z) - H_{\{-\alpha\}}(z) - \frac{1}{n}|z|} \\
&= |g(z) - P(z)|\, e^{-H_{K-\{\alpha\}}(z) - \frac{1}{n}|z|} \qquad (1.5)\\
&= \|g - P\|_{K-\{\alpha\},n} < \varepsilon.
\end{aligned}$$

□

**Remark 1.45**

(1) The calculation in (1.5) shows that the mapping $f \mapsto e_\alpha f$ is an isometric isomorphism from $\operatorname{Exp}(K)$ to $\operatorname{Exp}(K + \{\alpha\})$.

(2) For two compact and convex sets $K \subset \tilde{K} \subset \mathbb{C}$ and $\alpha \in K$, the above proposition states that $\{Pe_\alpha : P \text{ polynomial}\}$ is dense in $\operatorname{Exp}(K)$ and $\operatorname{Exp}(\tilde{K})$. Thus $\operatorname{Exp}(K)$ is continuously and densely embedded in $\operatorname{Exp}(\tilde{K})$.

**Proposition 1.46** *Let $\Omega$ be a simply connected region in $\mathbb{C}$. Then the functions of $\operatorname{Exp}(\{\alpha\})$ are a dense subspace of $H(\Omega)$.*

*Proof.* For a given $f \in H(\Omega)$, an arbitrary $\varepsilon > 0$ and a compact set $L \subset \Omega$, we have to show that there exists a function $h$ in $\operatorname{Exp}(\{\alpha\})$ such that $\sup_{z\in L} |f(z) - h(z)| < \varepsilon$.

We set $g = f/e_\alpha$. Since $\Omega$ is simply connected, there is a polynomial $P$ such that $\sup_{z\in L} |g(z) - P(z)| < \frac{\varepsilon}{M}$ where $M := \sup_{z\in L} |e_\alpha(z)|$. The function $h := Pe_\alpha$ belongs to $\mathrm{Exp}(\{\alpha\})$ by Proposition 1.17 (1) and satisfies

$$\sup_{z\in L} |f(z) - h(z)| = \sup_{z\in L} |g(z) - P(z)||e_\alpha(z)| \leq M \sup_{z\in L} |g(z) - P(z)| < \varepsilon.$$

□

In [GS91], G. Godefroy and J. H. Shapiro prove, by an application of the Hahn-Banach theorem and the Riesz representation theorem, the denseness of the linear hull of $\{e_\alpha : \alpha \in A\}$ in $H(\mathbb{C})$ for any $A \subset \mathbb{C}$ that has got an accumulation point. We prove a similar result for $\mathrm{Exp}(K)$ by means of Runge's theorem.

**Proposition 1.47** *Let $K$ be compact, convex subset of $\mathbb{C}$ and $A \subset K$ an infinite set. Then* $\mathrm{linspan}\{e_\alpha : \alpha \in A\}$ *is dense in* $\mathrm{Exp}(K)$.

*Proof.* Without loss of generality, we may assume $0 \notin A$. The functions $R_\alpha$ with $\alpha \in A$ shall be defined by $z \mapsto \frac{1}{z-\alpha}$. By a variant of Runge's theorem (cf. [LR84, Theorem 10.2]), $\mathrm{linspan}\{R_\alpha : \alpha \in A\}$ is a dense subspace of $H_0(\mathbb{C} \setminus K)$. Remark 1.40 yields $\mathcal{B}e_\alpha = R_\alpha$, and hence we have

$$\mathrm{linspan}\{e_\alpha : \alpha \in A\} = \mathrm{linspan}\{\mathcal{B}^{-1} R_\alpha : \alpha \in A\}.$$

Since the Borel transform $\mathcal{B} : \mathrm{Exp}(K) \to H_0(\mathbb{C} \setminus K)$ is an isomorphism, this shows the assertion. □

Replacing the exponential function $e_1$ by a more general function of exponential type with non-vanishing Taylor coefficients, we shall now generalize the above result. However, it is essential in the proof that $K$ contains the origin. At the end of Chapter 3, we will see that this condition is even necessary for that general case.

**Theorem 1.48** *Let $f$ be an entire function of exponential type. We further assume that $K$ is a compact, convex subset of $\mathbb{C}$ containing the origin and $A \subset \mathbb{C}$ is a bounded infinite set such that $f_\alpha(z) := f(\alpha z)$ belongs to* $\mathrm{Exp}(K)$ *for all $\alpha \in A$. Then* $\mathrm{linspan}\{f_\alpha : \alpha \in A\}$ *is dense in* $\mathrm{Exp}(K)$ *if and only if $f^{(k)}(0) \neq 0$ for every non-negative integer $k$.*

*Proof.* We set $Y := \operatorname{linspan}\{f_\alpha(z) : \alpha \in A\}$. Since the convergence in the topology of $\operatorname{Exp}(K)$ implies the convergence of the Taylor coefficients, the density of $Y$ ensures that the Taylor coefficients of $f$ do not vanish.
Let $\Lambda \in \operatorname{Exp}^*(K)$ vanish on $Y$. We prove that $\Lambda \equiv 0$, which implies the density of $Y$ by a standard application of the Hahn-Banach theorem.
Theorem 1.38 supplies an $\omega \in H(K)$ that represents $\Lambda$ by

$$\langle \Lambda, f \rangle = \frac{1}{2\pi i} \int_\Gamma \omega(\xi)\, \mathcal{B}f(\xi)\, d\xi \tag{1.6}$$

where $\Gamma$ is a Cauchy cycle for $K$ in $\Omega_\omega$. We show the existence of some connected open set $D$, containing $A$ and the origin, such that

$$K(f_\alpha) \subset \Omega_\omega \text{ for all } \alpha \in D. \tag{1.7}$$

A simple observation using the equality $H_{K(f)}(\xi) = h_f(\arg(\xi))$, $\xi \in \mathbb{T}$, (see Theorem 1.20) yields $K(f_\alpha) = \alpha\, K(f)$ for arbitrary $\alpha \in \mathbb{C}$. Hence the condition $f_\alpha \in \operatorname{Exp}(K)$ implies $A\, K(f) \subset K$. As $0 \in K$ and $K$ is convex, we obtain $r\, \alpha\, K(f) \subset K$ for all $r \in [0,1]$ and all $\alpha \in A$. The continuity of the multiplication and the compactness of $K(f)$ imply the existence of some open neighbourhood $U(\alpha, r)$ for each pair $(r, \alpha) \in [0,1] \times A$ such that $U(\alpha, r)\, K(f) \subset \Omega_\omega$. Now $D := \bigcup_{\alpha \in A} \bigcup_{r \in [0,1]} U(\alpha, r)$ has the claimed properties of (1.7).
Since $A$ is assumed to be bounded and infinite, it has an accumulation point. By means of the power series representation of the Borel transform and the transform $\mathcal{U}$ (see Remark 1.40), we immediately obtain

$$\mathcal{B}f_\alpha(\xi) = \frac{\mathcal{U}\mathcal{B}f(\frac{\alpha}{\xi})}{\xi}$$

for all $\xi \in \mathbb{C} \setminus K$ and $\alpha \in \mathbb{C}$. Now, with (1.6),

$$a(\alpha) := \langle \Lambda, f_\alpha \rangle = \frac{1}{2\pi i} \int_\Gamma \omega(\xi)\, \mathcal{B}f_\alpha(\xi)\, d\xi = \frac{1}{2\pi i} \int_\Gamma \frac{\omega(\xi)}{\xi} \mathcal{U}\mathcal{B}f\left(\frac{\alpha}{\xi}\right)\, d\xi$$

defines a holomorphic function on $D$. The condition $\Lambda|_Y \equiv 0$ means that $a$ vanishes on $A$, and since $A$ has an accumulation point, the identity theorem yields $a \equiv 0$.

Consequently,

$$a^{(k)}(\alpha) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\xi)}{\xi^{k+1}}\,(\mathcal{UB}f)^{(k)}\left(\frac{\alpha}{\xi}\right)\,d\xi = 0$$

for all $\alpha \in D$ and every $k \in \mathbb{N}\cup\{0\}$. With $\alpha = 0$ this means

$$(\mathcal{UB}f)^{(k)}(0)\,\frac{1}{2\pi i}\int_\Gamma \frac{\omega(\xi)}{\xi^{k+1}}\,d\xi = 0 \tag{1.8}$$

for all $k \in \mathbb{N}\cup\{0\}$. The assumption $f^{(k)}(0) \neq 0$ for all $k \in \mathbb{N}\cup\{0\}$ is equivalent to $(\mathcal{UB}f)^{(k)}(0) \neq 0$ for all $k \in \mathbb{N}\cup\{0\}$ as one can immediately see in the power series representation of $\mathcal{UB}f$. Consequently, equality (1.8) shows $\omega \equiv 0$ and that means $\Lambda \equiv 0$. This completes the proof. □

# Chapter 2

# Hypercyclic Operators and Universal Functions

## 2.1 Hypercyclicity

This section provides basic results of hypercyclic operators and universal functions. For extensive surveys we refer to [GE99] and the recently published book [BM09]. In the following we denote the $n$-times iteration of an operator $T$ by $T^n$.

**Definition 2.1** Let $X$ and $Y$ be topological spaces and $(T_n)_{n\in\mathbb{N}}$ a sequence of continuous mappings from $X$ to $Y$. Then $(T_n)_{n\in\mathbb{N}}$ is called *universal* (from $X$ to $Y$) if there is some $x \in X$ such that $\{T_n x : n \in \mathbb{N}\}$ is dense in $Y$. The elements generating such a dense sequence are called *universal elements* for $(T_n)_{n\in\mathbb{N}}$ (on $Y$).

In this thesis we are mainly concerned with a special case of universality, namely the hypercyclicity.

**Definition 2.2** Let $X$ be a topological vector space and $T : X \to X$ a continuous operator.

(1) The operator $T$ is said to be *hypercyclic* if there exists a vector $x \in X$ such that the orbit $\{T^n x : n \in \mathbb{N}\}$ is dense in $X$. The vectors generating a dense orbit are called *hypercyclic vectors* (for $T$).

(2) If there is an increasing sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ such that for all subsequences $(n_{k_j})_{j\in\mathbb{N}}$ there is a vector $x \in X$ generating a dense orbit $\{T^{n_{k_j}} x : j \in \mathbb{N}\}$, then $T$ is called *hereditarily hypercyclic* (with respect to $(n_k)_{k\in\mathbb{N}}$).

**Definition 2.3** If in Definition 2.2 or Definition 2.1 the space $X$ consists of functions, then the hypercyclic vectors for $T$ or the universal elements for $(T_n)_{n\in\mathbb{N}}$ are called *universal functions* (for $T$ or $(T_n)_{n\in\mathbb{N}}$, respectively).

The earliest and most famous examples of hypercyclic operators are the following:

**Example 2.4**

(1) For an $\alpha \in \mathbb{C} \setminus \{0\}$, the translation operator $T_{e_\alpha} f := f(\cdot + \alpha)$ is a hypercyclic operator on $H(\mathbb{C})$.

(2) For a simply connected region $\Omega \subset \mathbb{C}$, the differentiation operator is hypercyclic on $H(\Omega)$.

The first example is due to G. D. Birkhoff [Bir29] and the second one is due to G.R. MacLane [Mac52]. The hypercyclicity of these operators is shown as a consequence of a more general theorem in the following chapter. Henceforth, we shall use the notation $T_{e_\alpha}$ for the translation operator in the direction $\alpha$.
By means of Baire's theorem, a short proof yields the following result:

**Proposition 2.5**

*(1) Let $X$ be a Baire space, $Y$ a second-countable space and $(T_n)_{n\in\mathbb{N}}$ a sequence of continuous mappings from $X$ to $Y$ such that for each pair of non-empty open sets $U \subset X$ and $V \subset Y$ there exists an $n \in \mathbb{N}$ with $T_n(U) \cap V \neq \emptyset$. Then $(T_n)_{n\in\mathbb{N}}$ is universal from $X$ to $Y$, and the set of universal vectors is a dense $G_\delta$-set in $X$ (i.e. a countable intersection of open and dense sets of $X$).*

*(2) Let $X$ be a separable $F$-space and $T$ a continuous operator on $X$. Then $T$ is hypercyclic if and only if for each pair of non-empty open sets $U, V \subset X$ there is a positive integer $n$ such that $T^n(U) \cap V$ is non-empty. The set of hypercyclic vectors is either empty or a dense $G_\delta$-set in $X$.*

For invertible operators the above characterization implies the following:

**Proposition 2.6** *Let $X$ be a separable $F$-space and $T : X \to X$ a continuous and invertible operator. Then $T$ is hypercyclic if and only if $T^{-1}$ is hypercyclic.*

The characterization in Proposition 2.5 (2) builds the basis for the proof of the well-known hypercyclicity criterion. This criterion is the most frequently used tool to obtain hypercyclicity.

**Theorem 2.7** (Hypercyclicity Criterion) *Assume that $X$ is a separable $F$-space and $T : X \to X$ a continuous operator. Let further $X_0$, $Y_0$ be two dense subsets of $X$ and $(S_n)_{n\in\mathbb{N}}$ a sequence of mappings $S_n : Y_0 \to X$ such that for an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers the following hold:*

*(1)* $T^{n_k}x \to 0$ $(k \to \infty)$ *for all* $x \in X_0$*;*

*(2)* $S_{n_k}(x) \to 0$ $(k \to \infty)$ *for all* $x \in Y_0$*;*

*(3)* $T^{n_k}S_{n_k}(x) \to x$ $(k \to \infty)$ *for all* $x \in Y_0$.

*Then $T$ is hypercyclic. In the case that* (1), (2) *and* (3) *are valid, $T$ is said to satisfy the hypercyclicity criterion (with respect to the sequence $(n_k)_{k\in\mathbb{N}}$).*

In [BP99], J. Bés and A. Peris prove a useful characterization for operators that satisfy the hypercyclicity criterion.

**Theorem 2.8** (cf. [BP99, Theorem 2.3]) *Let $X$ be a separable $F$-space and $T$ a continuous operator on $X$. Then the following assertions are equivalent:*

*(i) $T$ satisfies the hypercyclicity criterion.*

*(ii) $T$ is hereditarily hypercyclic.*

*(iii) The direct sum $T \oplus T$ is hypercyclic on the product space $X \times X$ endowed with the product topology.*

The algebraic structure of the set of hypercyclic vectors is much investigated. Here we only provide a result concerning the existence of a dense subspace of hypercyclic vectors. The first versions of this result are shown in [Her91], [Bou93] and [Bés99]. The below generalization is due to J. Wengenroth [Wen03].

**Theorem 2.9** *Let $X$ be a topological vector space and $T : X \to X$ hypercyclic. Then there exists a dense subspace $M \subset X$ such that every $x \in M \setminus \{0\}$ is a hypercyclic vector for $T$.*

A remarkable result for the powers of hypercyclic operators on Banach spaces is due to I. Ansari [Ans95]. The generalization to arbitrary topological vector spaces is proved in [Wen03].

**Theorem 2.10** (Ansari's Theorem) *Let $X$ be a topological vector space and $T$ a hypercyclic operator on $X$. Then for every $n \in \mathbb{N}$, $T^n$ is hypercyclic and the set of hypercyclic vectors for $T$ and $T^n$ coincide.*

## 2.2 Frequently Hypercyclic Operators

In [BG06], F. Bayart and S. Grivaux introduce a stronger form of hypercyclicity, namely the frequent hypercyclicity. Instead of requiring the orbit $\{T^n x : n \in \mathbb{N}\}$ to intersect every non-empty open set, they imposed a condition on the frequency of recurrence to each non-empty open set.

In this section we introduce basic notations and some results for frequently hypercyclic operators. For detailed information we refer to [BG06] and [BGE07].

We start with the definition of the generalization of frequent hypercyclicity, namely the frequent universality, which is introduced in [BGE07].

**Definition 2.11** Let $X, Y$ be topological spaces and $(T_n)_{n\in\mathbb{N}}$ a sequence of continuous mappings from $X$ to $Y$. The sequence $(T_n)_{n\in\mathbb{N}}$ is called *frequently universal* (from $X$ to $Y$) if there is some $x \in X$ such that for every non-empty open subset $U$ of $Y$

$$\underline{\mathrm{dens}}(\{n \in \mathbb{N} : T_n x \in U\}) > 0.$$

In this case, $x$ is called a *frequently universal element* for $(T_n)_{n\in\mathbb{N}}$ (on $Y$).

**Definition 2.12** Let $X$ be a topological vector space and $T : X \to X$ a continuous operator. The operator $T$ is called *frequently hypercyclic* if there is some $x \in X$ such that for every non-empty open subset $U$ of $X$

$$\underline{\mathrm{dens}}(\{n \in \mathbb{N} : T^n x \in U\}) > 0.$$

In this case, $x$ is called a *frequently hypercyclic vector* for $T$.

**Definition 2.13** If in Definition 2.11 or Definition 2.12 the space $X$ consists of functions, then the frequently hypercyclic vectors or frequently universal elements are called *frequently universal functions* (for $T$ or $(T_n)_{n\in\mathbb{N}}$, respectively).

**Remark 2.14** Most of the classical hypercyclic operators, as the translation operators $T_{e_\alpha}$ $(\alpha \in \mathbb{C}\setminus\{0\})$ or the differentiation operator, are also frequently hypercyclic. But while the set of hypercyclic vectors on a separable $F$-space is a dense $G_\delta$-set (see Proposition 2.5 (2)), the set of frequently hypercyclic vectors is in most cases a set of first category.

Ansari's theorem (see Theorem 2.10) still holds for frequently hypercyclic operators (cf. [BM09, Theorem 6.30]).

**Theorem 2.15** *Let $X$ be a topological vector space and $T : X \to X$ frequently hypercyclic. Then for every $n \in \mathbb{N}$, $T^n$ is frequently hypercyclic and the frequently hypercyclic vectors for $T$ and $T^n$ coincide.*

A main result in [BG06] is a criterion for frequent hypercyclicity that has some similarities with the hypercyclicity criterion. But the proofs of these criteria are totally different since for the result below, one cannot make use of the characterization in Proposition 2.5.

**Theorem 2.16** (Frequent Hypercyclicity Criterion) *Let $X$ be a separable $F$-space and $T : X \to X$ a continuous operator. Suppose there is a dense set $Y_0 \subset X$ and a mapping $S : Y_0 \to Y_0$ such that:*

*(1)* $\sum_{k=1}^{\infty} T^k x$ *is unconditionally convergent for every* $x \in Y_0$;

*(2)* $\sum_{k=1}^{\infty} S^k x$ *is unconditionally convergent for every* $x \in Y_0$;

*(3)* $TS = \mathrm{id}$ *on* $Y_0$.

*Then $T$ is frequently hypercyclic.*

An extension of this criterion to the case of frequent universality is proved by A. Bonilla and K.-G. Grosse-Erdmann in [BGE09]. They used the following terminology for their formulation: If $X$ is an F-space, one can assume that its topology is induced by an *F-norm*. This is a mapping $||\cdot|| : X \to [0,\infty)$ so that, for all $x, y \in X$

and all scalars $\lambda$,

$$
\begin{aligned}
||x|| &> 0 \text{ for } x \neq 0, \\
||\lambda x|| &\leq ||x|| \text{ for } |\lambda| \leq 1, \\
||\lambda x|| &\to 0 \quad \text{as } \lambda \to 0, \\
||x+y|| &\leq ||x|| + ||y||.
\end{aligned}
$$

A sequence $(x_{n,k})_{k,n\in\mathbb{N}}$ is said to converge unconditionally, uniformly in $k \in \mathbb{N}$ in an F-space if for every $\varepsilon > 0$, there is a positive integer $N$ such that for every finite $F \subset \mathbb{N}$ and all $k \in \mathbb{N}$

$$\left\| \sum_{n\in F} x_{n,k} \right\| < \varepsilon$$

whenever $\{1, 2, ..., N\} \cap F = \emptyset$.

**Theorem 2.17** (Frequent Universality Criterion) *Let $X$ be an F-space, $Y$ a separable F-space and $(T_n)_{n\in\mathbb{N}}$ a sequence of continuous mappings from $X$ to $Y$. Assume there are a dense subset $Y_0$ of $Y$ and mappings $S_n : Y_0 \to X$, $n \in \mathbb{N}$, such that:*

*(1)* $\sum_{n=1}^{k} T_k S_{k-n} y$ *converges unconditionally in $Y$, uniformly in $k \in \mathbb{N}$, for all $y \in Y_0$;*

*(2)* $\sum_{n=1}^{\infty} T_k S_{k+n} y$ *converges unconditionally in $Y$, uniformly in $k \in \mathbb{N}$, for all $y \in Y_0$;*

*(3)* $\sum_{n=1}^{\infty} S_n y$ *converges unconditionally in $X$, for all $y \in Y_0$;*

*(4)* $T_n S_n y \to y$ *as $n \to \infty$ for all $y \in Y_0$.*

*Then $(T_n)_{n\in\mathbb{N}}$ is frequently universal (from $X$ to $Y$).*

As we are concerned with FS spaces, we can show the unconditional convergence by means of the absolute convergence with respect to the norms that induce the topology (cf. [BGE07]):

**Remark 2.18** Let $X$ be a Fréchet space and $(||\cdot||_n)_{n\in\mathbb{N}}$ a sequence of seminorms defining the topology on $X$. If for a sequence $(x_k)_{k\in\mathbb{N}}$ in $X$ the series $\sum_{k=1}^{\infty} ||x_k||_n$ converges for all $n \in \mathbb{N}$, then $\sum_{k=1}^{\infty} x_k$ converges unconditionally in $X$.

## 2.3 Hypercyclicity on $\mathrm{Exp}(K)$

The aim of this section is to establish two general results for hypercyclic operators on $\mathrm{Exp}(K)$.

**Proposition 2.19** *Let $K$ be a compact, convex subset of $\mathbb{C}$ and $T$ a hypercyclic operator on $\mathrm{Exp}(K)$. Then the set of universal functions (for $T$) having a conjugate indicator diagram that coincides with $K$ is residual in $\mathrm{Exp}(K)$.*

*Proof.* According to Proposition 2.5 (2) and Corollary 1.42, the set of universal functions for $T$ is a dense $G_\delta$-set in $\mathrm{Exp}(K)$, and the set of functions having a conjugate indicator diagram equal to $K$ is residual in $\mathrm{Exp}(K)$. Consequently, their intersection is still residual in $\mathrm{Exp}(K)$. □

In [BFPW05], it is shown that, under some restrictions, the hypercyclicity on projective limits is implied by the hypercyclicity on the different steps. Following the idea from [BFPW05], we will show a similar result for universality. The restrictions are adequately chosen for our needs and are much stronger than in [BFPW05].
We consider a sequence $(E_n)_{n\in\mathbb{N}}$ of Fréchet spaces with

$$E_{n+1} \subset E_n \text{ and } E_{n+1} \hookrightarrow E_n \text{ continuously for all } n \in \mathbb{N}.$$

The space $E := \bigcap\limits_{n\in\mathbb{N}} E_n$ shall be endowed with the topology induced by $(||\cdot||_{n,l})_{n,l\in\mathbb{N}}$ where $(||\cdot||_{n,l})_{l\in\mathbb{N}}$ is the system of semi-norms inducing the topology on the spaces $E_n$, i.e., $E := \mathrm{ProjLim}(E_n : n \in \mathbb{N})$.

**Proposition 2.20** *Let $X$ be a topological space and $(E_n)_{n\in\mathbb{N}}$, $E$ as above. We assume that a sequence of continuous mappings $(T_k : E_1 \to X)_{k\in\mathbb{N}}$ is such that for all $n \in \mathbb{N}$ the following holds: The mappings $T_k|_{E_n} : E_n \to X$ are continuous for all $k \in \mathbb{N}$, and for each pair of non-empty open sets $U \subset E_n$ and $V \subset X$ there is some $k \in \mathbb{N}$ with $T_k|_{E_n}(U) \cap V \neq \emptyset$. Then, if $E$ is dense in each $E_n$, for every pair of non-empty open sets $U \subset E$ and $V \subset X$ there is some $k \in \mathbb{N}$ with $T_k|_E(U) \cap V \neq \emptyset$.*

*Proof.* Let $V \subset X$ and $U \subset E$ be non-empty and open. We can assume the existence of a positive integer $n$ and an open set $W \subset E_n$ such that $U = W \cap E$. By assumption, $W \cap (T_k|_{E_n})^{-1}(V) \neq \emptyset$ for some $k \in \mathbb{N}$. Using the denseness of $E$ in $E_n$, we can conclude that $U$ intersects the open set $W \cap (T_k|_{E_n})^{-1}(V)$ and thus

$U \cap (T_k|_E)^{-1}(V) \neq \emptyset$.

□

**Corollary 2.21** Let $K, K_1, K_2, ...$ be non-empty, compact and convex subsets of $\mathbb{C}$ such that $K_n^\circ \supset K_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} K_n = K$.

(1) Let $X$ be a second-countable space and $(T_k)_{k\in\mathbb{N}}$ a sequence of continuous mappings from $\text{Exp}(K_n)$ to $X$ for all $n \in \mathbb{N}$. If for every $n \in \mathbb{N}$, the sequence $(T_k)_{k\in\mathbb{N}}$ is such that for every pair of non-empty open sets $U \subset E_n$ and $V \subset X$ there is an $k \in \mathbb{N}$ such that $T_k(U) \cap V \neq \emptyset$, then $(T_k)_{k\in\mathbb{N}}$ is universal from $\text{Exp}(K)$ to $X$, and the set of universal elements for $(T_k)_{k\in\mathbb{N}}$ is a dense $G_\delta$-set in $\text{Exp}(K)$.

(2) Let $T$ be an operator mapping continuously from $\text{Exp}(K_n)$ to $\text{Exp}(K_n)$ for all $n \in \mathbb{N}$. If $T$ is hypercyclic on each $\text{Exp}(K_n)$, then $T$ is hypercyclic on $\text{Exp}(K)$.

*Proof.* According to Remark 1.27, we have

$$\text{Exp}(K) = \bigcap_{n\in\mathbb{N}} \text{Exp}(K_n),$$

also in topological sense. For $\alpha \in K$, the subspace $\{e_\alpha P : P \text{ polynomial}\} \subset \text{Exp}(K)$ is dense in $\text{Exp}(K_n)$ for all $n \in \mathbb{N}$ by Proposition 1.44. Hence Proposition 2.20 applies, and immediately shows (1) according to Proposition 2.5 (1).
In the case of (2), Proposition 2.20 and Proposition 2.5 (1) imply that for all $n \in \mathbb{N}$ the universal elements for $(T^n)_{n\in\mathbb{N}}$ from $\text{Exp}(K)$ to $\text{Exp}(K_n)$ are a dense $G_\delta$-set, $UV_n$, in $\text{Exp}(K)$. The intersection of all sets $UV_n$ is still a dense $G_\delta$-set in $\text{Exp}(K)$, and every vector of this intersection is obviously a hypercyclic vector for $T$ on $\text{Exp}(K)$.

□

**Remark 2.22** For every non-empty, compact and convex set $K \subset \mathbb{C}$, the space $\text{Exp}(K)$ is dense in $H(\mathbb{C})$ (see Proposition 1.46). Consequently, if a continuous operator $T : \text{Exp}(K) \to \text{Exp}(K)$ can be extended to a continuous operator from $H(\mathbb{C})$ to $H(\mathbb{C})$, the hypercyclicity on $\text{Exp}(K)$ implies the hypercyclicity on $H(\mathbb{C})$.

# Chapter 3

# Hypercyclic and Frequently Hypercyclic Differential Operators

## 3.1 Introduction

As in many areas of analysis, differential operators are very extensively studied in the field of hypercyclicity and universality. In this section we present some important results that have been achieved in the study of universal and frequently universal functions for differential operators. We focus on the research of growth conditions of these functions. In the following chapters, some of these results are refined in some respects.

The (frequent) universality and (frequent) hypercyclicity are to be understood with respect to $H(\mathbb{C})$ in this section.

Let $\varphi(z) = \sum\limits_{\nu=0}^{\infty} \frac{a_\nu}{\nu!} z^\nu$ be an entire function of exponential type. Then for every $f \in H(\mathbb{C})$, the series

$$\varphi(D)f(z) := \sum_{\nu=0}^{\infty} \frac{a_\nu f^{(\nu)}(z)}{\nu!} \tag{3.1}$$

converges uniformly on arbitrary compact subsets of $\mathbb{C}$ and hence represents an entire function. Furthermore, by means of the Cauchy inequality, one easily proves (see [GS91]) that

$$\varphi(D) : H(\mathbb{C}) \to H(\mathbb{C})$$

defined by (3.1) is a continuous operator. The operator $\varphi(D)$ is called the *differential operator* induced by $\varphi$.

**Remark 3.1** Obviously, the differentiation operator is the differential operator induced by $\varphi(z) = z$. The translation operator $T_{e_\alpha} : H(\mathbb{C}) \to H(\mathbb{C})$, $f \mapsto f(\cdot + \alpha)$ is also a differential operator: For $\varphi = e_\alpha$ and $f \in H(\mathbb{C})$, we have

$$e_1(D)f(z) = \sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(z)}{\nu!}\alpha^\nu = \sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(z)}{\nu!}(z + \alpha - z)^\nu = f(z + \alpha) = T_{e_\alpha} f(z).$$

The notation $T_{e_1}$ for the operator $e_1(D)$ will become clear in Section 3.3.

A characterization of differential operators for the case $\mathbb{C}^n$ is given by G. Godefroy and J. H. Shapiro in [GS91]. We formulate this result for $n = 1$.

**Proposition 3.2** (cf. [GS91, Proposition 5.2]) *Let $T$ be a continuous operator on $H(\mathbb{C})$, then the following are equivalent:*

*(i) $T$ commutes with all translation operators $T_{e_\alpha}$.*

*(ii) $T$ commutes with the differentiation operator.*

*(iii) There is a complex Borel measure $\mu$ with compact support such that*

$$Tf(z) = \int_{\mathbb{C}} f(z + w)\, d\mu(w).$$

*(iv) There is an entire function $\varphi$ of exponential type such that $T = \varphi(D)$.*

The hypercyclicity of such operators is also proved by G. Godefroy and J. H. Shapiro.

**Theorem 3.3** (cf. [GS91, Theorem 5.1]) *Every differential operator that is not a scalar multiple of the identity is hypercyclic on $H(\mathbb{C})$.*

Remark 3.1 implies that this theorem contains the classical results of G. D. Birkhoff and G. R. MacLane (cf. Example 2.4).
The first result concerning the possible rate of growth of universal functions for differential operators is obtained by S. M. Duyos-Ruiz in [DR83] for the case of the translation operator. He shows that universal functions for the translation operator

$T_{e_\alpha}$ can have arbitrary small non-polynomial growth. In [CS91], K. C. Chan and J. H. Shapiro extend this result to Hilbert spaces containing entire functions of small growth. In both cases the growth of functions is measured via a *comparison function*. This is an entire function $a(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $a_n > 0$ and $a_{n+1}/a_n \to 0$ as $n$ tends to infinity. If further $(n+1)a_{n+1}/a_n$ is monotonically decreasing, $a$ is called an *admissible comparison function*. In [CS91], the hypercyclicity of $T_{e_\alpha}$ on the Hilbert spaces

$$E^2(a) := \left\{ f \in H(\mathbb{C}) : ||f||_a^2 := \sum_{n=0}^{\infty} \frac{|f_n|^2}{a_n^2} < \infty \right\}$$

is proved for every admissible comparison function $a$. Here $f_n$ are the Taylor coefficients of $f$ at the origin. It is also shown in [CS91] that $f \in E^2(a)$ implies

$$\sup_{z \in \mathbb{C}} \frac{|f(z)|}{a(|z|)} < \infty. \tag{3.2}$$

Consider a function $q(r) : (0,\infty) \to (0,\infty)$ with $q(r) \to \infty$ arbitrarily slow as $r \to \infty$. We set

$$m(n) := \inf_{1 \le r} \frac{r^{q(r)}}{2^n r^n},$$

which is positive since $r^{q(r)}/2^n r^n$ tends to infinity as $r \to \infty$, and choose successively

$$0 < a_n < \min \left\{ \frac{(n-1)\, a_{n-1}^2}{n\, a_{n-2}}, m(n) \right\}.$$

Then $a(z) := \sum_{n=0}^{\infty} a_n z^n$ is an admissible comparison function and (3.2) yields for $f \in E^2(a)$

$$M_f(r) \le Ca(r) = C \sum_{n=0}^{\infty} a_n r^n \le C \sum_{n=0}^{\infty} \frac{r^{q(r)}}{2^n} = 2\, C\, r^{q(r)}$$

for all $r > 1$ and some $C < \infty$. With these considerations we can formulate a consequence of [CS91, Theorem 2.1] in the following way:

**Theorem 3.4** *Let $q(r)$ be a positive function that tends to infinity as $r \to \infty$. Then for every $\alpha \in \mathbb{C} \setminus \{0\}$, there exists a universal function $f$ for $T_{e_\alpha}$ such that*

$$M_f(r) = O(r^{q(r)}) \quad \text{as } r \to \infty.$$

For the case of the differentiation operator, K.-G. Grosse-Erdmann obtained a sharp lower bound for universal functions.

**Theorem 3.5** (cf. [GE90, Theorem (A), Theorem (B)]) *There is no entire function $f$ that is universal for the differentiation operator and satisfies*

$$M_f(r) = O\left(\frac{e^r}{\sqrt{r}}\right) \quad \text{as } r \to \infty.$$

*For any $q(r) : (0, \infty) \to (0, \infty)$ with $q(r) \to \infty$ as $r \to \infty$, there is a universal function $f$ for the differentiation operator that satisfies*

$$M_f(r) = O\left(\frac{q(r)}{\sqrt{r}} e^r\right) \quad \text{as } r \to \infty.$$

Concerning the rate of growth of universal functions for general differential operators, L. Bernal-González and A. Bonilla show:

**Theorem 3.6** (cf. [BGB02, Corollary 2.4]) *Let $\varphi$ be an entire function of exponential type with $\varphi(0) = 0$ and $\tau := \operatorname{dist}(\varphi^{-1}(\mathbb{T}), 0)$. Then the following assertions hold:*

*(1) For any $d > \tau$, there is a subspace $M \subset \operatorname{Exp}(d\,\overline{\mathbb{D}})$ that is dense in $H(\mathbb{C})$ and that consists, except for zero, of universal functions for $\varphi(D)$.*

*(2) If $|\varphi(z)| = 1$ on an infinite subset of $\tau\,\mathbb{T}$, then there exists a subspace $M \subset \operatorname{Exp}(\tau\,\overline{\mathbb{D}})$ that is dense in $H(\mathbb{C})$ and that consists, except for zero, of universal functions for $\varphi(D)$.*

In [BGB02], also a negative result, similar to Theorem 3.5, is shown for general differential operators.

**Theorem 3.7** (cf. [BGB02, Corollary 2.2]) *Let $\varphi(z) = \sum_{n=0}^{\infty} c_n z^n$ be an entire function of exponential type with $|\varphi(0)| < 1$ and set $\varphi_*(z) := \sum_{n=0}^{\infty} |c_n| z^n$. Then for $\tau := \operatorname{dist}(\varphi_*^{-1}(\mathbb{T}), 0)$, there is no entire function $f$ that is universal for the differential operator $\varphi(D)$ and satisfies*

$$M_f(r) = O\left(\frac{e^{\tau r}}{\sqrt{r}}\right) \quad \text{as } r \to \infty.$$

Actually, L. Bernal-González and A. Bonilla prove more general results as they consider sequences of differential operators that are not necessarily the iteration of a single operator.
In [BGE06], A. Bonilla and K.-G. Grosse-Erdmann extend Theorem 3.6 to frequent universality in a version for $H(\mathbb{C}^n)$. We give the formulation for the case $n = 1$.

**Theorem 3.8** (cf. [BGE06, Theorem 3.4]) *Let $\varphi$ be a non-constant entire function of exponential type. Then $\varphi(D)$ is frequently hypercyclic on $H(\mathbb{C})$. Moreover, for every $d > \tau = \operatorname{dist}(\varphi^{-1}(\mathbb{T}), 0)$ there is a function $f \in \operatorname{Exp}(d\overline{\mathbb{D}})$ that is frequently universal for $\varphi(D)$.*

For the case of the differentiation operator and the translation operator, this result is refined in [BBGE10]. In this work, the growth is measured in terms of

$$M_p(f,r) := \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{\frac{1}{p}}$$

where $1 \leq p < \infty$ and $M_\infty(f,r) := M_f(r)$.

**Theorem 3.9** (cf. [BBGE10, Theorem 2.3]) *For $1 \leq p \leq \infty$, let $a := \frac{1}{2\max\{2,p\}}$. Then, for any $q : (0,\infty) \to (0,\infty)$ with $q(r) \to \infty$ as $r$ tends to infinity, there exists a frequently universal function $f$ for the differentiation operator that satisfies*

$$M_p(f,r) = O\left( q(r)\, \frac{e^r}{r^a} \right) \quad \text{as } r \to \infty.$$

**Theorem 3.10** (cf. [BBGE10, Theorem 2.4]) *For $1 \leq p \leq \infty$, let $a := \frac{1}{2\min\{2,p\}}$. Assume further that $q : (0,\infty) \to (0,\infty)$ is any function such that $q(r) \to 0$ as $r$ tends to infinity. Then there is no frequently universal function $f$ for the differentiation operator that satisfies*

$$M_p(f,r) = O\left( q(r)\, \frac{e^r}{r^a} \right) \quad \text{as } r \to \infty.$$

**Theorem 3.11** (cf. [BBGE10, Theorem 3.1 (b)]) *Let $q(r) : (0,\infty) \to (0,\infty)$ with $\liminf_{r\to\infty} q(r) = 0$. For any $\alpha \in \mathbb{C} \setminus \{0\}$, there is no frequently universal function $f$ for $T_{e_\alpha}$ such that*

$$M_1(f,r) = O\left( e^{q(r)r} \right) \quad \text{as } r \to \infty.$$

In particular, the last result implies the following.

**Corollary 3.12** There is no $f \in \operatorname{Exp}(\{0\})$ that is a frequently universal function for any translation operator $T_{e_\alpha}$.

## 3.2 The Translation Operator

The main topic of this section is the conjugate indicator diagram of frequently universal functions for the translation operator $T_{e_1}$. We provide conditions for compact, convex sets $K \subset \mathbb{C}$ that admit or exclude the existence of such functions in $\operatorname{Exp}(K)$. Furthermore, we consider additional restrictions on the rate of growth on the real axis for universal and frequently universal functions for $T_{e_1}$.
Our results below can be easily extended to arbitrary translation operators $T_{e_\alpha}$.

The result of S. M. Duyos-Ruiz and its improvement due to K. C. Chan and J. H. Shapiro provide the existence of universal functions for $T_{e_1}$ with arbitrary non-polynomial growth. Considering only the growth in the direction of the real axis, the universality of an entire function $f$ for $T_{e_1}$ implies that $|f|$ is unbounded on $\mathbb{R}$. We shall now investigate to what extent universal functions for $T_{e_1}$ can have an arbitrary slow rate of growth on the real axis.
According to Theorem B.4, the only functions of exponential type zero that have polynomial rate of growth on the real axis are the polynomials and are thus not universal for $T_{e_1}$. Hence we must allow at least arbitrarily small but positive exponential type for our desired functions. Consequently, we are in the situation where also frequent universality may be possible (cf. Theorem 3.8, Corollary 3.12).

**Theorem 3.13** *Let $q : [0, \infty) \to [1, \infty)$ such that $q(r) \to \infty$ as $r$ tends to infinity and $d > 0$. Then for every increasing sequence $(k_l)_{l \in \mathbb{N}}$ of positive integers that satisfies*

$$q(x) \geq l^c \text{ for all } x \in [(1-\delta)k_l, (1+\delta)k_l] =: I_{k_l} \tag{3.3}$$

*for some $c > 1$ and $\delta > 0$, there exists a function $f \in \operatorname{Exp}([-id, id])$ that is frequently* *universal for* $(T_{e_1}^{k_l})_{l\in\mathbb{N}}$ *on* $\mathrm{Exp}([-id, id])$ *and such that*

$$|f(x)| = O(q(|x|))$$

*on the real axis.*

*Proof.* We set $K := [-id, id]$ and $Y_0 = \mathrm{linspan}\left\{f_\alpha : \alpha \in [-i\frac{d}{2}, i\frac{d}{2}]\right\}$ where

$$f_\alpha(z) := \frac{(e^{\alpha z} - 1)^2}{z^2}.$$

Then $Y_0$ is dense in $\mathrm{Exp}(K)$ by Theorem 1.48, and $K(f_\alpha) = K$. Furthermore, let

$$E := \left\{f \in \mathrm{Exp}(K) : \sup_{x\in\mathbb{R}} \frac{|f(x)|}{q(|x|)} < \infty\right\}$$

be endowed with the topology induced by the norms $(||\cdot||_n)_{n\in\mathbb{N}}$ where

$$||f||_n := ||f||_{K,n} + \sup_{x\in\mathbb{R}} \frac{|f(x)|}{q(|x|)}.$$

Then $E$ is a Fréchet space.
We apply the frequent universality criterion (cf. Theorem 2.17) to the mappings

$$T_{e_1}^{k_l} : E \to \mathrm{Exp}(K)$$

and

$$S_{k_l} : \mathrm{Exp}(K) \to E$$

with $S_{k_l} := T_{e_{-1}}^{k_l}$.
In a first step, it is shown that in condition (1) and (2) of this criterion, we have absolute convergence for $(k_l)_{l\in\mathbb{N}} = (k)_{k\in\mathbb{N}}$. Let $n \in \mathbb{N}$ and $\alpha \in [-i\frac{d}{2}, i\frac{d}{2}]$ be fixed. Taking into account that $\alpha$ is purely imaginary, we obtain

$$\left|\left(e^{\alpha(z+k)} - 1\right)^2\right| e^{-H_K(z)-\frac{1}{2n}|z|} \le \left(\left|e^{2\alpha z}\right| + 2\left|e^{\alpha z}\right| + 1\right) e^{-H_K(z)-\frac{1}{2n}|z|} < \infty \qquad (3.4)$$

for all non-negative integers $k$. Hence

$$\sup_{|z+k|>\frac{k}{2}} |f_\alpha(z+k)|\, e^{-H_K(z)-\frac{1}{n}|z|} \leq \sup_{|z+k|>\frac{k}{2}} \frac{|e^{\alpha(z+k)}-1|^2}{\left(\frac{k}{2}\right)^2}\, e^{-H_K(z)-\frac{1}{n}|z|} = O\left(\frac{1}{k^2}\right). \tag{3.5}$$

Again from inequality (3.4), it follows that

$$\begin{aligned} &\sup_{1\leq|z+k|\leq\frac{k}{2}} |f_\alpha(z+k)|\, e^{-H_K(z)-\frac{1}{n}|z|} \\ &\leq \sup_{1\leq|z+k|\leq\frac{k}{2}} |e^{\alpha(z+k)}-1|^2\, e^{-H_K(z)-\frac{1}{2n}|z|}\, e^{-\frac{1}{2n}|z|} = O\left(e^{-\frac{k}{4n}}\right). \end{aligned} \tag{3.6}$$

Obviously,

$$\sup_{|z+k|\leq 1} |f_\alpha(z+k)|\, e^{-H_K(z)-\frac{1}{n}|z|} \leq \sup_{|z|\leq 1} |f_\alpha(z)|\, e^{-\frac{k-1}{n}} = O\left(e^{-\frac{k-1}{n}}\right). \tag{3.7}$$

The estimations (3.5), (3.6) and (3.7) now yield

$$||T_{e_1}^k f_\alpha||_{K,n} = \sup_{z\in\mathbb{C}} \left|\frac{(e^{\alpha(z+k)}-1)^2}{(z+k)^2}\right|\, e^{-H_K(z)-\frac{1}{n}|z|} = O\left(\frac{1}{k^2} + e^{-\frac{k}{4n}} + e^{-\frac{k-1}{n}}\right).$$

Thus the series $\sum_{k=1}^{\infty} ||T_{e_1}^k f_\alpha||_{K,n}$ converges for every $n \in \mathbb{N}$. By Remark 2.18, this yields the unconditional convergence of $\sum_{k=1}^{\infty} T_{e_1}^k f_\alpha$ in $\mathrm{Exp}(K)$. The unconditional convergence remains for every function $f \in Y_0$, and this implies that condition (1) of the frequent universality criterion is satisfied. With the same reasoning, one obtains the unconditional convergence of $\sum_{k=1}^{\infty} T_{e_1}^l S_{k+l} f$ in $\mathrm{Exp}(K)$ for all $f \in Y_0$. This is condition (2) of the frequent universality criterion. Obviously, the similar reasoning holds for all subsequences of $(k)_{k\in\mathbb{N}}$. As condition (4) is obvious, there is only (3) left to show.

On the real axis, the moduli of the functions $f_\alpha$ are bounded. With assumption (3.3), we obtain

$$\sup_{x\in I_{k_l}\cup -I_{k_l}} \frac{|f_\alpha(x-k_l)|}{q(|x|)} = O\left(\frac{1}{l^c}\right). \tag{3.8}$$

If $x \notin I_{k_l} \cup -I_{k_l}$, then $|x - k_l| > \delta k_l$ and hence

$$\sup_{x \in \mathbb{R} \setminus (I_{k_l} \cup -I_{k_l})} \frac{|f_\alpha(x - k_l)|}{q(|x|)} \leq \sup_{x \in \mathbb{R} \setminus (I_{k_l} \cup -I_{k_l})} \frac{|e^{\alpha(x-k_l)} - 1|^2}{(\delta k_l)^2} = O\left(\frac{1}{k_l^2}\right). \tag{3.9}$$

Combining (3.8) and (3.9), we have

$$\sum_{l=1}^{\infty} \sup_{x \in \mathbb{R}} \frac{|f_\alpha(x - k_l)|}{q(|x|)} < \infty.$$

Together with the convergence of $\sum_{l=1}^{\infty} ||S_{k_l} f_\alpha||_{K,n}$ for each $n \in \mathbb{N}$, which is shown in the first part of the proof, this implies the convergence of $\sum_{l=1}^{\infty} ||S_{k_l} f_\alpha||_n$ for every $n \in \mathbb{N}$. According to Remark 2.18 and the definition of $Y_0$, we obtain the unconditional convergence of $\sum_{l=1}^{\infty} S_{k_l} f$ in $E$ for every $f \in Y_0$. Now the frequent universality criterion yields that $(T_{e_1}^{k_l})_{l \in \mathbb{N}}$ is frequently universal from $E$ to $\mathrm{Exp}(K)$. □

**Corollary 3.14** Let $q : [0, \infty) \to [1, \infty)$ be such that $q(r) \to \infty$ as $r$ tends to infinity and $a, b \in \mathbb{R}$ with $a < b$. Then there is an $f \in \mathrm{Exp}([ia, ib])$ that is universal for $T_{e_1}$ on $\mathrm{Exp}([ia, ib])$ and that satisfies

$$|f(x)| = O(q(|x|))$$

on the real axis.

*Proof.* We choose $0 < d < b - a$. Since $q$ tends to infinity, we can find an increasing sequence $(k_l)_{l \in \mathbb{N}}$ of positive integers such that condition (3.3) holds. Now Theorem 3.13 provides a $g \in \mathrm{Exp}([-id, id])$ that is frequently universal for $(T_{e_1}^{k_l})_{l \in \mathbb{N}}$ on $\mathrm{Exp}([-id, id])$ and such that $|g(x)| = O(q(|x|))$ on $\mathbb{R}$. In particular, this implies that $g$ is universal for $(T_{e_1}^n)_{n \in \mathbb{N}}$ on $\mathrm{Exp}([-id, id])$. We choose a suitable $\beta \in \mathbb{Q}$ such that $f := e_{2\pi i \beta}\, g \in \mathrm{Exp}([ia, ib])$. Note that $f$ satisfies $|f(x)| = O(q(|x|))$ on the real axis. There exists an $m \in \mathbb{N}$ such that $\beta\, k\, m \in \mathbb{N}$, and hence $e_{2\pi i \beta}(z + k\, m) = e^{2\pi i \beta (z + k\, m)} = e_{2\pi i \beta}(z)$ for all $k \in \mathbb{N}$. We obtain that

$$(T_{e_1}^m)^k f = e_{2\pi i \beta}\, (T_{e_1}^m)^k g$$

holds for all $k \in \mathbb{N}$, and since $g$ is also universal for $T_{e_1}^m$ due to Ansari's theorem (cf. Theorem 2.10), one easily verifies that $f$ is the desired function. □

**Corollary 3.15** Let $a < b$ be real numbers. Then $T_{e_1}$ is frequently hypercyclic on $\operatorname{Exp}([ia, ib])$. Furthermore, for each $c > 1$ there is a frequently universal function $f \in \operatorname{Exp}([ia, ib])$ for $T_{e_1}$ on $\operatorname{Exp}([ia, ib])$ that satisfies

$$|f(x)| = O(|x|^c)$$

on the real axis.

*Proof.* With $q(x) := 1 + |x|^c$, condition (3.3) is satisfied for the whole sequence $(k_l)_{l\in\mathbb{N}} = (k)_{k\in\mathbb{N}}$. Consequently, for arbitrary $d > 0$, Theorem 3.13 provides a frequently universal function $g \in \operatorname{Exp}([-id, id])$ for $T_{e_1}$ on $\operatorname{Exp}([-id, id])$ satisfying the desired growth condition on $\mathbb{R}$. Now, for some $0 < d < b - a$ and a suitable choice of $\beta \in \mathbb{Q}$ we have that $f := e_{2\pi i\beta}\, g \in \operatorname{Exp}([ia, ib])$. Taking into account that Ansari's theorem still holds for frequent hypercyclicity (cf. Theorem 2.15), a similar reasoning as in the proof of Corollary 3.14 shows that $f$ is frequently universal for $T_{e_1}$ on $\operatorname{Exp}([ia, ib])$. □

The zeros of entire functions of exponential type are restricted with respect to their frequency on different rays and sectors. These restrictions are closely related to the geometry of the conjugate indicator diagram. A positive lower density of the zeros in the direction of the real axis requires some extension of the conjugate indicator diagram in the direction of the imaginary axis. We have shown that such a condition is already sufficient to allow frequent universality for the translation operator $T_{e_1}$. Our next aim is to find restrictions for the conjugate indicator diagram of a frequently universal function for $T_{e_1}$. For that purpose we make use of a classical result (cf. Theorem B.3) that connects distributions of zeros and growth conditions for holomorphic functions. This result leads to the following lemma.

**Lemma 3.16** *Let $f$ be an entire function of exponential type and*

$$c := \frac{1}{2} \max\{|\operatorname{Im}(z) - \operatorname{Im}(w)| : z, w \in K(f)\}.$$

*We further assume that $\lambda = (r_n\, e_n^{i\Theta})_{n\in\mathbb{N}} \subset \{z : |\arg(z)| < \gamma\}$ with $0 < \gamma < \frac{\pi}{2}$ is a* *sequence of zeros for* $f$ *so that* $r_{n+1} - r_n > \delta > 0$ *for all* $n \in \mathbb{N}$. *Then* $f \equiv 0$ *if*

$$d := \underline{\text{dens}}(\lambda) > \frac{c}{\pi \cos(\gamma)}.$$

*Proof.* We can assume that $K(f)$ is located such that

$$\sup\{\text{Im}(z) : z \in K(f)\} = c = \sup\{-\text{Im}(z) : z \in K(f)\}$$

and $K(f) \subset \{z : \text{Re}(z) \leq 0\}$. Otherwise, consider the function $e_\alpha f$ with a suitable $\alpha \in \mathbb{C}$. Then $|f(z)| = O\left(e^{(c+\varepsilon)|z|}\right)$ in $\{z : \text{Re}(z) \geq 0\}$ for every $\varepsilon > 0$.
We recall that $n_\lambda(r) =: n(r)$ countes the $r_n$ that are less or equal than $r$. For a given $r > 0$, we have

$$\begin{aligned}
\int_0^r \frac{n(t)}{t^2}\, dt &= \sum_{k=1}^{n(r)-1} \int_{r_k}^{r_{k+1}} \frac{k}{t^2}\, dt + \int_{r_{n(r)}}^{r} \frac{n(t)}{t^2}\, dt \\
&= \sum_{k=1}^{n(r)-1} k\left(\frac{1}{r_k} - \frac{1}{r_{k+1}}\right) + \frac{n(r)}{r_{n(r)}} - \frac{n(r)}{r} \\
&= \sum_{r_k<r} \frac{1}{r_k} - \frac{n(r)}{r}
\end{aligned}$$

and hence

$$\sum_{r_k<r} \frac{1}{r_k} \geq \int_0^r \frac{n(t)}{t^2}\, dt. \tag{3.10}$$

By assumption, we can find an $r_0 > 0$ such that $\frac{n(r)}{r} > d - \varepsilon_1 > \frac{c+\varepsilon_2}{\pi \cos(\gamma)}$ for all $r \geq r_0$ and some small $\varepsilon_1, \varepsilon_2 > 0$. With (3.10), we obtain

$$\sum_{r_k<r} \frac{1}{r_k} \geq \int_0^{r_0} \frac{n(t)}{t^2}\, dt + \int_{r_0}^{r} \frac{n(t)}{t^2}\, dt > M + (d - \varepsilon_1)\log(r)$$

with some constant $M$. Since $\lambda$ is a sequence in the sector $\{z : |\arg(z)| < \gamma\}$,

$$\sum_{r_k<r} \frac{\cos(\Theta_k)}{r_k} - \log(r)\,\frac{c+\varepsilon_2}{\pi} > \cos(\gamma)\left(M + (d-\varepsilon_1)\log(r)\right) - \log(r)\,\frac{c+\varepsilon_2}{\pi}.$$

By construction, this means

$$\sum_{r_n<r} \frac{\cos(\Theta_k)}{r_k} - \log(r)\,\frac{c+\varepsilon_2}{\pi} \to \infty \text{ as } \; r \to \infty$$

and finally implies that

$$\exp\left(\sum_{r_n<r} \frac{\cos(\Theta_n)}{r_n}\right) r^{-\frac{c+\varepsilon_2}{\pi}} \to \infty$$

as $r \to \infty$. Theorem B.3 yields $f \equiv 0$. □

Corollary 3.15 provides the existence of a frequently universal function $f$ for $T_{e_1}$ on $H(\mathbb{C})$ that is of exponential type and such that $K(f)$ is a line segment on the imaginary axis. We shall see that in this result it is essential that $K(f)$ is a vertical line segment.

**Theorem 3.17** *Let $[w,v]$ be some non-vertical line segment in $\mathbb{C}$, which means* $\mathrm{Re}(v) \neq \mathrm{Re}(w)$*. Then there is no frequently universal function $f$ for $T_{e_1}$ on $H(\mathbb{C})$ that is of exponential type and that satisfies $K(f) = [w,v]$.*

*Proof.* Suppose that $f$ is of exponential type with $K(f) = [w,v]$, and $f$ is frequently universal for $T_{e_1}$ on $H(\mathbb{C})$. Then we can find a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive integers that has positive lower density $d$ and is such that

$$\sup_{|z|\leq\frac{1}{2}} |f(z+\lambda_n) - z| < \frac{1}{2}$$

for all $n \in \mathbb{N}$. Rouché's theorem implies that $f$ has a zero $\tilde{\lambda}_n$ in $\{z : |z-\lambda_n| < \frac{1}{2}\}$ for every positive integer $n$. Obviously, also $\underline{\mathrm{dens}}((\tilde{\lambda}_n)_{n\in\mathbb{N}}) = d > 0$.
Without loss of generality, let $K(f) = \{re^{i\Theta} : r \in [0,a]\}$ with $0 \leq |\Theta| < \frac{\pi}{2}$ and some $a \in [0,\infty)$. Otherwise, consider $e_\alpha f$ with an appropriate $\alpha \in \mathbb{C}$ such that $K(e_\alpha f) = \alpha + K(f)$ has the desired position. The rotation of the arguments of $f$ by an angle of $-\Theta$ causes the rotation of the conjugate indicator diagram by the same angle. This can be easily deduced from the connection between the indicator function and the conjugate indicator diagram (cf. Theorem 1.20). That means, for $g(z) := f(e^{-i\Theta}z)$ we have $K(g) = e^{-i\Theta}K(f)$. Thus $K(g)$ is a horizontal line segment

and consequently,

$$\max\{|\mathrm{Im}(z) - \mathrm{Im}(w)| : z, w \in K(g)\} = 0. \tag{3.11}$$

Further, $(e^{i\Theta}\tilde{\lambda}_n)_{n\in\mathbb{N}}$ is a sequence of zeros of $g$. Since $|\Theta| < \frac{\pi}{2}$, we can find some $\gamma \in (|\Theta|, \frac{\pi}{2})$ such that $(e^{i\Theta}\tilde{\lambda}_n)_{n\geq n_0}$ is contained in the sector $\{z : \arg(z) < \gamma\}$ for sufficiently large $n_0$. According to the lower density of $(\tilde{\lambda}_n)_{n\in\mathbb{N}}$ and (3.11), we obtain a contradiction to Lemma 3.16. □

Theorem 3.17 in particular excludes singleton conjugate indicator diagrams for frequently universal functions for $T_{e_1}$ on $H(\mathbb{C})$ (hence on $\mathrm{Exp}(K)$).
Another restriction is given by the following result:

**Theorem 3.18** *Let $K$ be a compact, convex subset of $\{z : \mathrm{Re}(z) \leq 0\}$ such that $K \cap i\mathbb{R}$ is a singleton. Then there is no frequently universal function $f$ for $T_{e_1}$ on $H(\mathbb{C})$ that is of exponential type and that satisfies $K(f) = K$.*

*Proof.* Assuming the contrary, we suppose that $f$ is an entire function of exponential type that is frequently universal for $T_{e_1}$ on $H(\mathbb{C})$ and such that $K(f) = K$. Then there exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive integers with $\underline{\mathrm{dens}}((\lambda_n)_{n\in\mathbb{N}}) = d > 0$ and such that

$$\max_{|z|\leq\frac{1}{2}} |f(z + \lambda_n) - z| < \frac{1}{4}. \tag{3.12}$$

Let $K_1 := K(f) \cap \{z : \mathrm{Re}(z) \geq -\varepsilon\}$ with $\varepsilon > 0$ so small that

$$\max\{|\mathrm{Im}(z) - \mathrm{Im}(w)| : z, w \in K_1\} < 2\pi d. \tag{3.13}$$

This is possible since the intersection of $K(f)$ with the imaginary axis is a singleton. According to Theorem B.1 or Corollary B.2, we can decompose $\mathcal{B}f$ into the sum

$$H_1 + H_2 = \mathcal{B}f$$

where $H_1 \in H_0(\mathbb{C} \setminus K_1)$ and $H_2 \in H_0(\mathbb{C} \setminus K_2)$ with $K_2 := \overline{K(f) \setminus K_1}$. We set $h_1 = \mathcal{B}^{-1}(H_1)$ and $h_2 = \mathcal{B}^{-1}(H_2)$. The set $K_2$ is strictly contained in the left half-plane, and thus there are some $\tau > 0$ and $C > 0$ such that

$$|h_2(z)| < Ce^{-\tau|z|}$$

if $\mathrm{Re}(z) > 0$. Now (3.12) yields

$$\begin{aligned}\frac{1}{4} &> \max_{|z|\leq\frac{1}{2}} |f(z+\lambda_n) - z| = \max_{|z|\leq\frac{1}{2}} |h_1(z+\lambda_n) + h_2(z+\lambda_n) - z| \\ &> \max_{|z|\leq\frac{1}{2}} |h_1(z+\lambda_n) - z| - Ce^{-\tau(\lambda_n - \frac{1}{2})} \end{aligned} \tag{3.14}$$

for all $n \in \mathbb{N}$. Since $Ce^{-\tau(\lambda_n-\frac{1}{2})} \to 0$ as $n$ tends to infinity, (3.14) implies

$$\max_{|z|\leq\frac{1}{2}} |h_1(z+\lambda_n) - z| < \frac{1}{2}$$

for $n$ larger than some $n_0 \in \mathbb{N}$. Then $h_1$ has, by Rouché's theorem, a zero $\tilde{\lambda}_n$ in $\{z : |z-\lambda_n| \leq \frac{1}{2}\}$ for all $n > n_0$. The sequence $(\tilde{\lambda}_n)_{n>n_0}$ has the same lower density as $(\lambda_n)_{n\in\mathbb{N}}$, which is $d$. By (3.13), the conjugate indicator diagram $K(h_1)$ has an extension in the direction of the imaginary axis less than $2\pi d$, and for all $\gamma > 0$ the sequence $(\tilde{\lambda}_n)_{n>n_0}$ is contained in the sector $\{z : |\arg(z)| < \gamma\}$ if $n_0$ is sufficiently large. This contradicts Lemma 3.16. □

## 3.3 Generalized Differential Operators

For the differential operators $\varphi(D)$ introduced in Section (3.1), it is required that the inducing function $\varphi$ is of exponential type in order to ensure that $\varphi(D)$ is well-defined on $H(\mathbb{C})$. On $\mathrm{Exp}(K)$ we are concerned with functions that have a comparatively low rate of growth. This allows us to define such operators for more general functions $\varphi$.

We recall our convention that a $[\varphi] \in H(K)$ is identified with one of its representatives $\varphi$. A domain of holomorphy of $\varphi$ is denoted by $\Omega_\varphi$. In cases where $K$ is convex, we require that $\Omega_\varphi$ is simply connected.

Differentiation of parameter integrals applied to the Pólya representation (cf. Theorem 1.23) yields

$$f'(z) = \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi)\, \xi\, e^{\xi z}\, d\xi.$$

Inspired by this formula, we introduce a class of operators for entire functions of exponential type. We replace the $\xi$ caused by the differentiation in the above integral by some adequate holomorphic function. Precisely, let $f$ be an entire function of exponential type and $\varphi \in H(K(f))$. We define

$$T_\varphi f(z) := \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi)\, \varphi(\xi)\, e^{\xi z}\, d\xi \tag{3.15}$$

where $\Gamma$ is a Cauchy cycle for $K(f)$ in $\Omega_\varphi$. This definition is obviously independent from the choice of the Cauchy cycle.

**Proposition 3.19** *Let $K$ be a compact, convex set in $\mathbb{C}$ and $\varphi \in H(K)$. Then*

$$T_\varphi : \mathrm{Exp}(K) \to \mathrm{Exp}(K)$$

*defined by (3.15) is a continuous operator.*

*Proof.* For a given positive integer $n$, we choose $\Gamma$ such that $|\Gamma| \subset \frac{1}{n}\overline{\mathbb{D}} + K$. Then $H_{\mathrm{conv}(|\Gamma|)} \leq H_{K+\frac{1}{n}\overline{\mathbb{D}}}$ and that means $(\mathrm{Re}(\xi z) - H_K(z) - n^{-1}|z|) \leq 0$ for all $\xi \in |\Gamma|$ and all $z \in \mathbb{C}$ (compare Remark 1.21(4) and Example 1.22(4)). Consequently, $\left|e^{\xi z - H_K(z) - \frac{1}{n}|z|}\right| \leq 1$ for all $z \in \mathbb{C}$ and all $\xi \in |\Gamma|$. As $\mathcal{B} : \mathrm{Exp}(K) \to H_0(\mathbb{C} \setminus K)$ is an isomorphism and $|\Gamma|$ is compact in $\mathbb{C} \setminus K$, there is an $m \in \mathbb{N}$ and a constant $C > 0$ such that $\sup\{|\mathcal{B}f(\xi)| : \xi \in |\Gamma|\} \leq C\,\|f\|_{K,m}$. With $M := \frac{1}{2\pi}\int_\Gamma |\varphi(\xi)| d\xi$, we now obtain

$$\begin{aligned} \|T_\varphi f\|_{K,n} &= \sup_{z\in\mathbb{C}} \left| \frac{1}{2\pi i} \int_\Gamma \varphi(\xi)\, \mathcal{B}f(\xi)\, e^{\xi z}\, d\xi \right| e^{-H_K(z) - \frac{1}{n}|z|} \\ &\leq \sup_{z\in\mathbb{C}} \frac{1}{2\pi} \int_\Gamma |\varphi(\xi)|\, |\mathcal{B}f(\xi)| \left| e^{\xi z - H_K(z) - \frac{1}{n}|z|} \right| d\xi \leq MC\, \|f\|_{K,m}\,. \end{aligned}$$

□

**Remark 3.20**

(1) Assume that $K \subset \mathbb{C}$ is compact, not necessarily convex, and $\varphi \in H(K)$. Then the operator $T_\varphi$ is defined on $M := \{f \in \mathrm{Exp}(\mathrm{conv}(K)) : K(f) \subset K\}$, and we

can write

$$T_\varphi|_M : M \to \mathrm{Exp}(\mathrm{conv}(K)).$$

(2) In the definition of $T_\varphi$ in (3.15), let $L \subset K(f)$ be such that $\mathcal{B}f \in H_0(\mathbb{C} \setminus L)$. Then one easily observes that the Cauchy cycle for $K(f)$ in $\Omega_\varphi$ can be reduced to a Cauchy cycle for $L$ in $\Omega_\varphi$.

**Remark 3.21** The operators $T_\varphi$ are a generalization of the differential operators in Section 3.1:

(1) If $\varphi(z) = \sum_{\nu=0}^{\infty} \varphi_\nu z^\nu$ is an entire function of exponential type and $f \in \mathrm{Exp}(K)$, then

$$T_\varphi f(z) = \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\varphi(\xi)\, e^{\xi z}\, d\xi = \sum_{\nu=0}^{\infty} \frac{\varphi_\nu}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\,\xi^\nu\, e^{\xi z}\, d\xi = \sum_{\nu=0}^{\infty} \varphi_\nu f^{(\nu)}(z).$$

Hence we have $T_\varphi = \varphi(D)|_{\mathrm{Exp}(K)}$.

(2) If $\varphi$ is represented by the power series $\sum_{\nu=0}^{\infty} \varphi_\nu (z-z_0)^\nu$ in an open neighbourhood of $K$, then for $f \in \mathrm{Exp}(K)$

$$\begin{aligned} T_\varphi f(z) &= \sum_{\nu=0}^{\infty} \varphi_\nu \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\,(\xi - z_0)^\nu\, e^{\xi z}\, d\xi \\ &= \sum_{\nu=0}^{\infty} \varphi_\nu \sum_{k=0}^{\nu} \binom{\nu}{k} (-z_0)^{\nu-k}\, \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\,\xi^k\, e^{\xi z}\, d\xi \\ &= \sum_{\nu=0}^{\infty} \varphi_\nu \sum_{k=0}^{\nu} \binom{\nu}{k} (-z_0)^{\nu-k}\, f^{(k)}(z). \end{aligned}$$

Thus it is justified to call these operators *generalized differential operators.*

**Lemma 3.22** *Let $K$ be a compact, convex set in $\mathbb{C}$, $f \in \mathrm{Exp}(K)$ and $\varphi \in H(K)$. Then for all $h \in H(\Omega_\varphi)$, we have*

$$\int_\Gamma \mathcal{B}f(\xi)\,\varphi(\xi)\,h(\xi)\,d\xi = \int_\Gamma \mathcal{B}T_\varphi f(\xi)\,h(\xi)\,d\xi$$

*where $\Gamma$ is a Cauchy cycle for $K$ in $\Omega_\varphi$.*

*Proof.* As a consequence of Proposition 1.47, $E := \operatorname{linspan}\{e_\alpha : \alpha \in \mathbb{C}\}$ is dense in $H(\mathbb{C})$. Since $\Omega_\varphi$ is simply connected, Runge's theorem implies that $E$ is dense in $H(\Omega_\varphi)$. We consider the functional

$$\langle \Lambda, h\rangle := \int_\Gamma \left(\mathcal{B}f(\xi)\,\varphi(\xi) - \mathcal{B}T_\varphi f(\xi)\right)\, h(\xi)\, d\xi$$

on $H(\Omega_\varphi)$. With the Pólya representation for $T_\varphi f$, the following holds:

$$\frac{1}{2\pi i}\int_\Gamma \mathcal{B}T_\varphi f(\xi)\, e^{\xi\alpha}\, d\xi = T_\varphi f(\alpha) = \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\,\varphi(\xi)\, e^{\xi\alpha}\, d\xi.$$

Hence $\langle \Lambda, e_\alpha\rangle = 0$ for all $\alpha \in \mathbb{C}$ and consequently $\Lambda|_E = 0$. As $E$ is dense in $H(\Omega_\varphi)$, we have $\Lambda = 0$. □

Henceforth, a germ $\varphi \in H(K)$ is said to be zero-free if $\varphi$ has no zeros in $K$. For a zero-free $\varphi \in H(K)$, we always assume that $\Omega_\varphi$ is so small that $\varphi$ still has no zeros in $\Omega_\varphi$.

**Corollary 3.23** Under the assumptions of Proposition 3.19, the above lemma immediately yields

(1)
$$T_\varphi^2 f(z) = \frac{1}{2\pi i}\int_\Gamma \mathcal{B}T_\varphi f(\xi)\,\varphi(\xi)\, e^{\xi z}\, d\xi = \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\,\varphi^2(\xi)\, e^{\xi z}\, d\xi.$$

Hence, for every non-negative integer $n$, we have

$$T_\varphi^n f(z) = \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\,\varphi^n(\xi)\, e^{\xi z}\, d\xi.$$

(2) If $\varphi$ is zero-free, then $\frac{1}{\varphi} \in H(\Omega_\varphi)$ and

$$T_{\frac{1}{\varphi}} T_\varphi f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\mathcal{B}T_\varphi f(\xi)}{\varphi(\xi)}\, e^{\xi z}\, d\xi = \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\,\frac{\varphi(\xi)}{\varphi(\xi)}\, e^{\xi z}\, d\xi = f(z).$$

Thus $T_\varphi$ is invertible and $T_\varphi^{-1} = T_{\frac{1}{\varphi}}$.

**Example 3.24**

(1) In case that $\varphi = e_\alpha$ for some $\alpha \in \mathbb{C} \setminus \{0\}$, Remark 3.21 yields $T_{e_\alpha} = e_\alpha(D)|_{\mathrm{Exp}(K)}$ for each compact and convex set $K \subset \mathbb{C}$. As $e_\alpha(D)$ induces

the translation operator $f \mapsto f(\cdot + \alpha)$ (see Remark 3.1), it is now clear why we use the notation $T_{e_\alpha}$ for this operator.
The function $e_\alpha$ is zero-free in $\mathbb{C}$ so that, by Corollary 3.23 (2), $T_{e_\alpha}$ is invertible on $\operatorname{Exp}(K)$ for every compact and convex set $K$ and its inverse is given by $T_{e_{-\alpha}}$.

(2) It is clear that $T_{\mathrm{id}} = \mathrm{id}(D)|_{\operatorname{Exp}(K)} = D|_{\operatorname{Exp}(K)}$ for every $K \subset \mathbb{C}$ compact and convex. Corollary 3.23 (2) implies that the differentiation operator is invertible on $\operatorname{Exp}(K)$ if $K$ does not contain the origin. In this case, we have

$$(D^{-1})^k f(z) = T^k_{\frac{1}{\mathrm{id}}} f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\mathcal{B}f(\xi)}{\xi^k}\, e^{\xi z}\, d\xi$$

for $f \in \operatorname{Exp}(K)$.

(3) Let $\varphi(z) = e^z - 1$, then $T_\varphi f(z)$ is the well-known forward difference $\Delta(f)(z) = f(z+1) - f(z)$. The $n$-th iteration applied to an $f \in \operatorname{Exp}(K)$ is given by

$$\Delta_n(f)(z) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(z+k).$$

(4) Let $\varphi(\xi) = \frac{1}{1-\xi}$, then $T_\varphi$ is an invertible operator on $\operatorname{Exp}(K)$ for all compact and convex sets $K \subset \mathbb{C} \setminus \{1\}$. For an $f \in \operatorname{Exp}(K)$, we have

$$T^k_\varphi f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\mathcal{B}f(\xi)}{(1-\xi)^k}\, e^{\xi z}\, d\xi.$$

If $K$ is contained in the unit disc, then $\varphi(z) = \sum\limits_{n=0}^{\infty} z^n$ in an open neighbourhood of $K$ and thus

$$T_\varphi f(z) = \sum_{n=0}^{\infty} f^{(n)}(z)$$

for $f \in \operatorname{Exp}(K)$.

Similar to Proposition 3.2, we have the following result for generalized differential operators.

**Proposition 3.25** *Let $K$ be a compact, convex subset of $\mathbb{C}$ and $T$ a continuous operator on* $\operatorname{Exp}(K)$. *Then the following are equivalent:*

*(i) $T$ commutes with the differentiation operator.*

*(ii) $T$ commutes with all translation operators $T_{e_\alpha}$.*

*(iii) There is a germ $\varphi \in H(K)$ such that $T = T_\varphi$.*

*Proof.* We start with the implication $(i) \Rightarrow (ii)$. Let $f \in \operatorname{Exp}(K)$, then

$$TT_{e_\alpha} f(z) = T\left(\frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\, e^{\xi(z+\alpha)}\, d\xi\right) = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} T\left(\frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\, \xi^k\, e^{\xi z}\, d\xi\right)$$
$$= \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} T f^{(k)}(z) = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} (Tf(z))^{(k)} = T_{e_\alpha} T f(z).$$

For $(ii) \Rightarrow (iii)$, we consider $\Lambda \in \operatorname{Exp}^*(K)$ defined by $\langle \Lambda, f\rangle := Tf(0)$. Theorem 1.38 implies that $\Lambda$ is represented by a germ $\varphi \in H(K)$ via

$$\langle \Lambda, f\rangle = \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\, \varphi(\xi)\, d\xi$$

where $\Gamma$ is a Cauchy cycle for $K$ in $\Omega_\varphi$. Since $T$ commutes with all translation, we obtain

$$Tf(z) = T_{e_z} T f(0) = \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\, \varphi(\xi)\, e^{\xi z}\, d\xi$$

for all $z \in \mathbb{C}$.

In order to show the implication $(iii) \Rightarrow (i)$, we can assume that $T = T_\varphi$ with $\varphi \in H(K)$. For $f \in \operatorname{Exp}(K)$, Lemma 3.22 yields

$$DTf(z) = D\left(\frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\, \varphi(\xi)\, e^{\xi z}\, d\xi\right) = \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\, \varphi(\xi)\, \xi\, e^{\xi z}\, d\xi$$
$$= \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f'(\xi)\, \varphi(\xi)\, e^{\xi z}\, d\xi = Tf'(z).$$

□

## 3.4 Hypercyclicity of Generalized Differential Operators

In this section we give a characterization of the hypercyclicity of the operators $T_\varphi$ on $\mathrm{Exp}(K)$ depending on the location of $K$. This result provides detailed information about the growth of universal functions with respect to particular half-rays and sectors. In that sense some of the known results introduced in Section 3.1 are extended.

In addition, it will turn out that for every hypercyclic generalized differential operator there always exist some universal functions of the form $fe_\lambda$ with $f \in \mathrm{Exp}(\{0\})$ and a certain $\lambda \in \mathbb{C}$.

**Theorem 3.26** *Let $K \subset \mathbb{C}$ be compact, convex and $\varphi \in H(K)$ non-constant. Then $T_\varphi$ is hypercyclic on* $\mathrm{Exp}(K)$ *if and only if $K \cap \varphi^{-1}(\mathbb{T}) \neq \emptyset$.*

*Proof.* Firstly, assume that $K \subset \varphi^{-1}(\mathbb{D})$. Let $\Gamma$ be a Cauchy cycle for $K$ in $\varphi^{-1}(\mathbb{D})$. Then $\sup\limits_{\xi \in |\Gamma|} |\varphi(\xi)| = \delta < 1$, and hence for every $f \in \mathrm{Exp}(K)$

$$|T_\varphi^n f(0)| = \left| \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi)\, \varphi^n(\xi)\, d\xi \right| \leq \frac{\delta^n}{2\pi} \int_\Gamma |\mathcal{B}f(\xi)|\, d\xi \to 0$$

as $n$ tends to infinity. Consequently, $T_\varphi$ cannot be hypercyclic on $\mathrm{Exp}(K)$.

In the next step, we assume $K \subset \varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}})$. Then $\varphi \neq 0$ in an open neighbourhood of $K$. Thus, by Corollary 3.23 (2), the operator $T_\varphi$ is invertible on $\mathrm{Exp}(K)$ and $T_\varphi^{-1} = T_{\frac{1}{\varphi}}$. Since $1/\varphi(K) \subset \mathbb{D}$, the above reasoning implies that $T_{\frac{1}{\varphi}}$ cannot be hypercyclic on $\mathrm{Exp}(K)$. Its inverse $T_\varphi$ cannot be hypercyclic on $\mathrm{Exp}(K)$ by Proposition 2.6.

Now let $K \subset \mathbb{C}$ be a compact and convex set such that $\varphi^{-1}(\mathbb{T}) \cap K^\circ \neq \emptyset$. We define $W_1 := \varphi^{-1}(\mathbb{D}) \cap K$ and $W_2 := \varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}}) \cap K$. Then the subspaces $X_0 := \mathrm{linspan}\{e_\alpha : \alpha \in W_1\}$ and $Y_0 := \mathrm{linspan}\{e_\alpha : \alpha \in W_2\}$ are dense in $\mathrm{Exp}(K)$ by Proposition 1.47. The Borel transform of $e_\alpha$ is the Cauchy kernel $\xi \mapsto 1/(\xi - \alpha)$ (compare Remark 1.40). Thus for all $\alpha \in K$, the Cauchy integral formula yields

$$T_\varphi e_\alpha(z) = \frac{1}{2\pi i} \int_\Gamma \mathcal{B}e_\alpha(\xi)\, \varphi(\xi)\, e^{\xi z}\, d\xi = \frac{1}{2\pi i} \int_\Gamma \frac{\varphi(\xi)}{\xi - \alpha}\, e^{\xi z}\, d\xi = e_\alpha(z)\, \varphi(\alpha)$$

implying that for each $e_\alpha \in X_0$ the iteration $||T_\varphi^k e_\alpha||_{K,n} = ||e_\alpha||_{K,n}\, |\varphi(\alpha)|^k$ tend to zero. Consequently, for all $f \in X_0$,

$$T_\varphi^n f \to 0 \quad (n \to \infty). \tag{3.16}$$

We define $S : Y_0 \to Y_0$ as the linear extension of $e_\alpha \mapsto \varphi(\alpha)^{-1} e_\alpha$. (Note that $|\varphi(\alpha)| > 1$ by the definition of $Y_0$.) Then for $e_\alpha \in Y_0$, the iteration $||S^k e_\alpha||_{K,n} = ||e_\alpha||_{K,n}\, |\varphi(\alpha)|^{-k}$ converges to zero since $|\varphi(\alpha)| > 1$. Thus for all $f \in Y_0$, we have

$$S_\varphi^n f \to 0 \tag{3.17}$$

as $n$ tends to infinity. Finally, $TSf = f$, for all $f \in Y_0$, together with (3.16) and (3.17) shows that the hypercyclicity criterion (Theorem 2.7) is satisfied. This implies the hypercyclicity of $T$ on $\text{Exp}(K)$.

For an arbitrary set $K$ with $K \cap \varphi^{-1}(\mathbb{T}) \neq \emptyset$, we choose a sequence of compact, convex sets $(K_n)_{n\in\mathbb{N}}$ contained in $\Omega_\varphi$ such that $K_n^\circ \supset K_{n+1}$, the intersection of all sets $K_n$ equals $K$ and $K_n^\circ \cap \varphi^{-1}(\mathbb{T}) \neq \emptyset$. Then the above reasoning implies the hypercyclicity of $T$ on $\text{Exp}(K_n)$ for all $n$. The hypercyclicity of $T$ on $\text{Exp}(K)$ now follows from Corollary 2.21 (2).

□

**Remark 3.27** Consider a differential operator $\varphi(D)$ on $H(\mathbb{C})$. From Remark 3.21 we know that $\varphi(D)|_{\text{Exp}(K)} = T_\varphi$ for every compact, convex set $K \subset \mathbb{C}$. Let $f$ be a function of exponential type.

(1) In the first part of the above proof, we have shown that $T_\varphi^n f(0) \to 0$ as $n \to \infty$ whenever $K(f) \subset \varphi^{-1}(\mathbb{D})$. This implies that $f$ cannot be universal for $\varphi(D)$ on $H(\mathbb{C})$ in this case.

(2) Conversely, the above proof does not exclude the universality of $f$ for $\varphi(D)$ on $H(\mathbb{C})$ if $K(f) \subset \varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}})$.
One can see that the non-hypercyclicity of $T_\varphi$ on $\text{Exp}(K)$ in case that $K \subset \varphi^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}})$ is closely related to the topology of $\text{Exp}(K)$: Take $\varphi = e_1$ and $K = \{1\}$. If $f \in \text{Exp}(\{1\})$ is not identical zero, there is some $x \in \mathbb{R}$ such that

$|f(x)| > 0$. Since $H_{\{1\}}(z) = |z| \cos(\arg(z))$, we obtain

$$||T_{e_1}^k f||_{\{1\},n} = \sup_{z\in\mathbb{C}} |f(z+k)|\, e^{-H_1(z)-\frac{1}{n}|z|} \geq |f(x)|\, e^{|x-k|-\frac{1}{n}|x-k|}$$

for all $k > x$. In the case that $n > 1$, the above estimation yields that $||T_{e_1}^k f||_{\{1\},n}$ tends to infinity as $k \to \infty$ and hence excludes the universality of $f$ for $T_{e_1}$ on $\text{Exp}(\{1\})$.

Consequently, the question arises whether there is an entire function of exponential type $f$ that is universal for $T_{e_1}$ on $H(\mathbb{C})$ and such that its conjugate indicator diagram is contained in $\{z : \text{Re}(z) > 0\} = e_1^{-1}(\mathbb{C} \setminus \overline{\mathbb{D}})$.

(3) Let $\varphi$ be non-constant. Theorem 2.9 implies that for each compact and convex set $K \cap \varphi^{-1}(\mathbb{T}) \neq \emptyset$ there is a dense subspace $M \subset \text{Exp}(K)$ (hence a dense subspace of $H(\mathbb{C})$) such that each $f \in M \setminus \{0\}$ is universal for $\varphi(D)$.

**Corollary 3.28** Let $K$ be a compact, convex subset of $\mathbb{C}$ and $\varphi \in H(K)$ non-constant.

(1) Theorem 3.26 yields the hypercyclicity of $T_\varphi$ on $\text{Exp}(\{\lambda\})$ for every $\lambda$ contained in $K \cap \varphi^{-1}(\mathbb{T})$. Since all functions in $\text{Exp}(\{\lambda\})$ are of the form $fe_\lambda$ with $f \in \text{Exp}(\{0\})$, the universal functions for $T_\varphi$ on $\text{Exp}(\{\lambda\})$ are also of this form. These functions are obviously still universal for $T_\varphi$ on $\text{Exp}(K)$.

(2) Proposition 2.19 in connection with Theorem 3.26 implies the following: For every compact and convex set $L \subset K$ that intersects $\varphi^{-1}(\mathbb{T})$, there is an entire function of exponential type $f$ that is universal for $T_\varphi$ on $\text{Exp}(L)$ and such that $K(f) = L$. The set of these functions is residual in $\text{Exp}(L)$.

**Example 3.29** The following assertions are immediate consequences of Theorem 3.26:

(1) For $\alpha \in \mathbb{C} \setminus \{0\}$, the translation operator $T_{e_\alpha}$ is hypercyclic on $\text{Exp}(K)$ if and only if $K$ intersects the ray $\{i\lambda\overline{\alpha} : \lambda \in \mathbb{R}\}$.

(2) For $\alpha \in \mathbb{C} \setminus \{0\}$, the operator $\alpha D$ is hypercyclic on $\text{Exp}(K)$ if and only if $K$ intersects $|\alpha|^{-1}\mathbb{T}$.

We will go through an example showing that the above results have an interesting interpretation for the isomorphic space $H_0(\mathbb{C} \setminus K^{-1})$. Consider the operator $B : H(\Omega) \to H(\Omega)$ defined by

$$B\,F(z) := \begin{cases} \frac{F(z)-F(0)}{z}, & \text{for } z \neq 0 \\ F'(0), & \text{for } z = 0 \end{cases}$$

where $\Omega$ is a connected open set containing the origin. In terms of power series, the mapping $B$ is given by

$$\sum_{n=0}^{\infty} a_n z^n \mapsto \sum_{n=0}^{\infty} a_{n+1} z^n.$$

In [GS91], it is noted that $B$ is not hypercyclic on $H(\mathbb{C})$, and from criteria for the hypercyclicity of weighted shift operators in [GE00b], it turns out that $B$ is hypercyclic on $H(R\mathbb{D})$ if and only if $R \leq 1$. Due to the result for $\operatorname{Exp}(K)$ in Theorem 3.26, we can derive the following.

**Theorem 3.30** *Let $\alpha \neq 0$ be a complex number. Then the operator $B$ is hypercyclic on $H_0(\mathbb{C} \setminus \{\alpha^{-1}\})$ if and only if $|\alpha| = 1$.*

*Proof.* By means of Remark 1.40, it is easily seen that the diagram

$$\begin{array}{ccc} \operatorname{Exp}(\{\alpha\}) & \xrightarrow{\quad D \quad} & \operatorname{Exp}(\{\alpha\}) \\ \Big\downarrow \scriptstyle{\mathcal{UB}} & & \Big\uparrow \scriptstyle{(\mathcal{UB})^{-1}} \\ H_0(\mathbb{C} \setminus \{\alpha^{-1}\}) & \xrightarrow[\quad B \quad]{} & H_0(\mathbb{C} \setminus \{\alpha^{-1}\}) \end{array}$$

commutes. As $\mathcal{UB}$ is an isomorphism from $\operatorname{Exp}(\{\alpha\})$ to $H_0(\mathbb{C} \setminus \{\alpha^{-1}\})$ (cf. Proposition 1.30, Proposition 1.39), the hypercyclicity of $B$ on $H_0(\mathbb{C} \setminus \{\alpha^{-1}\})$ is equivalent to the hypercyclicity of $D$ on $\operatorname{Exp}(\{\alpha\})$. The assertion now follows from Theorem 3.26. □

## 3.5 The Transform $\Phi_\varphi$

We introduce a transform, namely $\Phi_\varphi$, that connects different generalized differential operators. At the end of this section, it will be easy to show that $\Phi_\varphi$ carries over

(frequent) universality for these operators.
As in the definition of $T_\varphi$ in (3.15), our starting point is the Pólya representation

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi)\, e^{\xi z}\, d\xi$$

in which we now replace the $\xi$ in the exponent: Let $f$ be an entire function of exponential type and $\varphi \in H(K(f))$. We define

$$\Phi_\varphi f(z) := \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi)\, e^{\varphi(\xi) z}\, d\xi \tag{3.18}$$

where $\Gamma$ is a Cauchy cycle for $K(f)$ in $\Omega_\varphi$. This definition is independent from the particular choice of the Cauchy cycle. If $L \subset K(f)$ is such that $\mathcal{B}f \in H_0(\mathbb{C} \setminus L)$, one easily observes that $\Gamma$ can be reduced to a Cauchy cycle for $L$ in $\Omega_\varphi$.

**Proposition 3.31** *Let $K$ be a compact, convex subset of $\mathbb{C}$ and $\varphi \in H(K)$ non-constant. Then, for each $f \in \mathrm{Exp}(K)$, the function $\Phi_\varphi f$ defined by (3.18) is an entire function of exponential type with $K(\Phi_\varphi f) \subset \mathrm{conv}(\varphi(K(f)))$. For every compact and convex set $L \subset \mathbb{C}$ that contains $\mathrm{conv}(\varphi(K))$,*

$$\Phi_\varphi : \mathrm{Exp}(K) \to \mathrm{Exp}(L)$$

*is a continuous operator that has dense image.*

*Proof.* One immediately verifies that $\Phi_\varphi f$ is an entire function.
We fix some positive integer $n$ and choose $\Gamma$ such that $\varphi(|\Gamma|)$ is contained in $\mathrm{conv}(\varphi(K(f))) + \frac{1}{n}\overline{\mathbb{D}}$. Then

$$H_{\mathrm{conv}(\varphi(|\Gamma|))}(z) \leq H_{\mathrm{conv}(\varphi(K(f)))+\frac{1}{n}\overline{\mathbb{D}}}(z) = H_{\mathrm{conv}(\varphi(K(f)))}(z) + \frac{1}{n}|z|$$

(cf. Remark 1.21 (4), Example 1.22 (4)) and thus

$$
\begin{aligned}
||\Phi_\varphi f||_{\mathrm{conv}(\varphi(K(f))),n} &= \sup_{z\in\mathbb{C}} \left| \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi)\, e^{\varphi(\xi)z} d\xi \right| \; e^{-H_{\mathrm{conv}(\varphi(K(f)))}(z) - \frac{1}{n}|z|} \\
&\leq \frac{\mathrm{len}(\Gamma)}{2\pi} \sup_{\xi\in|\Gamma|} |\mathcal{B}f(\xi)| e^{H_{\mathrm{conv}(\varphi(|\Gamma|))}(z)}\, e^{-H_{\mathrm{conv}(\varphi(K(f)))}(z) - \frac{1}{n}|z|} \qquad (3.19) \\
&\leq \frac{\mathrm{len}(\Gamma)}{2\pi} \sup_{\xi\in|\Gamma|} |\mathcal{B}f(\xi)| < \infty.
\end{aligned}
$$

As $n$ was arbitrary, we have shown that $K(\Phi_\varphi f)$ is contained in $\mathrm{conv}(\varphi(K(f)))$, which in particular implies that $\Phi_\varphi f$ is of exponential type.
For the second assertion, it is sufficient to consider $\Phi_\varphi : \mathrm{Exp}(K) \to \mathrm{Exp}(\mathrm{conv}(\varphi(K)))$ because $\mathrm{Exp}(\mathrm{conv}(\varphi(K)))$ is continuously and densely embedded in $\mathrm{Exp}(L)$ (cf. Remark 1.45 (2)) and we have shown that $\Phi_\varphi(\mathrm{Exp}(K)) \subset \mathrm{Exp}(\mathrm{conv}(\varphi(K)))$.
Taking into account that $\sup_{\xi\in|\Gamma|} |\mathcal{B}f(\xi)| \leq C\,||f||_{K,m}$ for some $C < \infty$ and $m \in \mathbb{N}$ due to the fact that $\mathcal{B} : \mathrm{Exp}(K) \to H_0(\mathbb{C} \setminus K)$ is an isomorphism, the continuity of $\Phi_\varphi$ follows from (3.19) when $K(f)$ is replaced by $K$.
It remains to show that $\Phi_\varphi(\mathrm{Exp}(K))$ is dense in $\mathrm{Exp}(\mathrm{conv}(\varphi(K)))$. Therefore, let $K_1, K_2, ...$ be a sequence of compact, convex sets in $\Omega_\varphi$ such that $K_n^\circ \supset K_{n+1}$ and the intersection of all these sets is equal to $K$. One easily deduces that the Borel transform of $e_\alpha$ is given by $\xi \mapsto \frac{1}{\xi-\alpha}$ (cf. Remark 1.40). Inserting this in (3.18), the Cauchy integral formula yields $\Phi_\varphi(e_\alpha) = e_{\varphi(\alpha)}$ for all $\alpha$ in some $K_n$. Consequently, for arbitrary $n \in \mathbb{N}$

$$\Phi_\varphi(\mathrm{linspan}\{e_\alpha : \alpha \in K_n\}) = \mathrm{linspan}\{e_{\varphi(\alpha)} : \alpha \in K_n\} \subset \mathrm{Exp}(\mathrm{conv}(\varphi(K_n)))$$

which implies that $\Phi_\varphi : \mathrm{Exp}(K_n) \to \mathrm{Exp}(\mathrm{conv}(\varphi(K_n)))$ has dense image according to Proposition 1.47 and the fact that $\varphi$ is non-constant. Since $\mathrm{Exp}(K)$ is dense in $\mathrm{Exp}(K_n)$, we obtain that $\Phi_\varphi(\mathrm{Exp}(K))$ is dense in $\mathrm{Exp}(\mathrm{conv}(\varphi(K_n)))$.
Furthermore, we have

$$\bigcap_{n\in\mathbb{N}} \mathrm{conv}(\varphi(K_n)) = \mathrm{conv}(\varphi(K))$$

and hence

$$\bigcap_{n\in\mathbb{N}} \mathrm{Exp}(\mathrm{conv}(\varphi(K_n))) = \mathrm{Exp}(\mathrm{conv}(\varphi(K))),$$

also in topological sense by Remark 1.27. It is now obvious that $\Phi_\varphi(\mathrm{Exp}(K))$ is dense in $\mathrm{Exp}(\mathrm{conv}(\varphi(K)))$. □

For the formulation of our next result, we shall introduce another convention for $\Omega_\varphi$. If $\varphi \in H(K)$ is univalent on some open neighbourhood of $K$, we assume that $\Omega_\varphi$ is so small that $\varphi$ is univalent on $\Omega_\varphi$. We recall that a holomorphic function $\varphi \in H(\Omega)$ (with $\Omega$ open) is called univalent if it maps one-to-one from $\Omega$ to $\varphi(\Omega)$. A germ $\varphi \in H(K)$ is said to be univalent if $\varphi$ is univalent on some open neighbourhood of $K$.

We consider the transform

$$\mathcal{G}_\varphi F(z) := \frac{1}{2\pi i} \int_\Gamma \frac{F(\xi)}{z - \varphi(\xi)}\, d\xi \tag{3.20}$$

for functions $F \in H_0(\mathbb{C} \setminus K)$ with $K \subset \mathbb{C}$ compact and $\varphi \in H(K)$ univalent. Here $\Gamma$ shall be a Cauchy cycle for $K$ in $\Omega_\varphi$ such that $\mathrm{ind}_{\varphi\circ\Gamma}(z) = 0$. Note that $\varphi \circ \Gamma$ is a Cauchy cycle for $\varphi(K)$ in $\varphi(\Omega_\varphi)$ by Proposition 1.10. Then $\mathcal{G}_\varphi$ maps linearly from $H_0(\mathbb{C} \setminus K)$ to $H_0(\mathbb{C} \setminus \varphi(K))$. It will turn out that $\mathcal{G}_\varphi$ is a $H_0(\mathbb{C} \setminus K)$-version of $\Phi_\varphi$. The above integral transform is inspired by the work of R.C. Buck [Buc48], in which functions of the form

$$G(z) = \frac{1}{2\pi i} \int_\Gamma \frac{F(\xi)}{1 - z\varphi(\xi)}\, d\xi$$

are considered for the investigation of integral-valued entire functions.

**Proposition 3.32** *Let $K \subset \mathbb{C}$ be compact and $\varphi \in H(K)$ univalent. Then*

$$\mathcal{G}_\varphi : H_0(\mathbb{C} \setminus K) \to H_0(\mathbb{C} \setminus \varphi(K))$$

*defined by (3.20) satisfies the following assertions:*

*(1) $\mathcal{G}_\varphi$ is an isomorphism and its inverse $\mathcal{G}_\varphi^{-1}$ is given by $\mathcal{G}_{\varphi^{-1}}$.*

*(2) If $\tilde{\varphi} \in H(\varphi(K))$ is univalent, then $\mathcal{G}_{\tilde{\varphi}}\,\mathcal{G}_\varphi = \mathcal{G}_{\tilde{\varphi}\circ\varphi}$.*

*Proof.* The transform $\mathcal{G}_\varphi$ is well-defined: Let $\Gamma$ and $\tilde{\Gamma}$ be Cauchy cycles for $K$ in $\Omega_\varphi$ with $\mathrm{ind}_{\varphi\circ\Gamma}(z) = 0$ and $\mathrm{ind}_{\varphi\circ\tilde{\Gamma}}(z) = 0$. Then the cycle $\varphi \circ \Gamma \cup \varphi \circ \tilde{\Gamma}^-$ is such that $\text{ind}_{\varphi\circ\Gamma\cup\varphi\circ\tilde{\Gamma}^-}(z) = 0$ and $\text{ind}_{\varphi\circ\Gamma\cup\varphi\circ\tilde{\Gamma}^-}(w) = 0$ for all $w \notin \varphi(\Omega_\varphi) \setminus \varphi(K)$. Since $(F \circ \varphi^{-1})\,(\varphi^{-1})'/(z - \text{id})$ is holomorphic in $\varphi(\Omega_\varphi) \setminus (\varphi(K) \cup \{z\})$, Cauchy's integral theorem yields

$$\frac{1}{2\pi i}\int_\Gamma \frac{F(\xi)}{z-\varphi(\xi)}d\xi - \frac{1}{2\pi i}\int_{\tilde{\Gamma}} \frac{F(\xi)}{z-\varphi(\xi)}d\xi$$
$$= \frac{1}{2\pi i}\int_{\varphi\circ\Gamma\cup\varphi\circ\tilde{\Gamma}^-} \frac{(F\circ\varphi^{-1})(w)\,(\varphi^{-1})'(w)}{z-w}dw = 0.$$

For an arbitrary compact set $L \subset \mathbb{C}\setminus\varphi(K)$, we choose a Cauchy cycle $\Gamma$ for $K$ in $\Omega_\varphi$ such that $\varphi(|\Gamma|) \cap L = \emptyset$ and $\text{ind}_{\varphi\circ\Gamma}(z) = 0$ for all $z \in L$. Then for $F \in H_0(\mathbb{C} \setminus K)$,

$$\sup_{z\in L}|\mathcal{G}_\varphi F(z)| \leq \frac{\text{len}(\Gamma)}{2\pi\ \text{dist}(L, \varphi(|\Gamma|))}\sup_{\xi\in|\Gamma|}|F(\xi)|.$$

This implies the continuity of $\mathcal{G}_\varphi$.

We turn now to the proof of (2). For $z \in \mathbb{C} \setminus (\tilde{\varphi} \circ \varphi(K))$, we choose a Cauchy cycle $\tilde{\Gamma}$ for $\varphi(K)$ in $\Omega_{\tilde{\varphi}} \cap \varphi(\Omega_\varphi)$ such that $\text{ind}_{\tilde{\varphi}\circ\tilde{\Gamma}}(z) = 0$. We further choose a Cauchy cycle $\Gamma$ for $K$ in $\Omega_\varphi$ with $\text{ind}_{\varphi\circ\Gamma}(w) = 0$ for all $w \in |\tilde{\Gamma}|$. By Proposition 1.10, $\varphi \circ \Gamma$ is a Cauchy cycle for $\varphi(K)$ in $\varphi(\Omega_\varphi)$ and $\varphi^{-1} \circ \tilde{\Gamma}$ is a Cauchy cycle for $K$ in $\varphi^{-1}(\Omega_{\tilde{\varphi}} \cap \varphi(\Omega_\varphi))$. Now, considering Lemma 1.11, we obtain

$$\begin{aligned}
\mathcal{G}_{\tilde{\varphi}}\,\mathcal{G}_\varphi F(z) &= \mathcal{G}_{\tilde{\varphi}}\left(w \mapsto \frac{1}{2\pi i}\int_\Gamma \frac{F(\xi)}{w-\varphi(\xi)}\,d\xi\right)(z)\\
&= \frac{1}{2\pi i}\int_{\tilde{\Gamma}} \frac{1}{z-\tilde{\varphi}(w)}\,\frac{1}{2\pi i}\int_\Gamma \frac{F(\xi)}{w-\varphi(\xi)}\,d\xi\,dw\\
&= \frac{1}{2\pi i}\int_{\tilde{\Gamma}} \frac{1}{z-\tilde{\varphi}(w)}\,\frac{1}{2\pi i}\int_{\varphi\circ\Gamma} \frac{F(\varphi^{-1}(t))}{w-t}\,(\varphi^{-1})'(t)\,dt\,dw\\
&= \frac{1}{2\pi i}\int_{\tilde{\Gamma}} \frac{F(\varphi^{-1}(w))}{z-\tilde{\varphi}(w)}\,(\varphi^{-1})'(w)\,dw\\
&= \frac{1}{2\pi i}\int_{\varphi^{-1}\circ\tilde{\Gamma}} \frac{F(\xi)}{z-\tilde{\varphi}\circ\varphi(\xi)}\,(\varphi^{-1})'(\varphi(\xi))\,\varphi'(\xi)\,d\xi\\
&= \mathcal{G}_{\tilde{\varphi}\circ\varphi}F(z)
\end{aligned}$$

for all $F \in H_0(\mathbb{C} \setminus K)$.
From (2), we obtain $\mathcal{G}_{\varphi^{-1}} \mathcal{G}_\varphi = \mathcal{G}_{\varphi^{-1} \circ \varphi} = \mathrm{id}$ and $\mathcal{G}_\varphi \mathcal{G}_{\varphi^{-1}} = \mathrm{id}$. The continuity of $\mathcal{G}_{\varphi^{-1}}$ is shown by the first part of the proof when $\varphi$ is replaced by $\varphi^{-1}$. This completes the proof. □

For the formulation of the next result we note that for $L \subset \mathbb{C}$ compact, possibly non-convex, $\mathcal{B}_L^{-1}$ is defined in Remark 1.31.

**Proposition 3.33** *Let $K$ be a compact, convex subset of $\mathbb{C}$ and $\varphi \in H(K)$ univalent. Then the following two assertions hold:*

*(1) The transform $\Phi_\varphi$ is equal to $\mathcal{B}_{\varphi(K)}^{-1} \mathcal{G}_\varphi \mathcal{B}_K$.*

*(2) If $\varphi(K)$ is convex, then $\Phi_\varphi : \mathrm{Exp}(K) \to \mathrm{Exp}(\mathrm{conv}(\varphi(K))) = \mathrm{Exp}(\varphi(K))$ is an isomorphism. Its inverse is given by $\Phi_{\varphi^{-1}}$. In particular, $\Phi_\varphi$ is isomorphic in case that $K$ is a singleton.*

The first assertion in the above result states that the diagram

$$\begin{array}{ccc} H_0(\mathbb{C} \setminus K) & \xrightarrow{\mathcal{G}_\varphi} & H_0(\mathbb{C} \setminus \varphi(K)) \\ {\scriptstyle \mathcal{B}_K} \uparrow & & \downarrow {\scriptstyle \mathcal{B}_{\varphi(K)}^{-1}} \\ \mathrm{Exp}(K) & \xrightarrow[\Phi_\varphi]{} & \mathrm{Exp}(\mathrm{conv}(\varphi(K))) \end{array}$$

commutes.

*Proof.* Let $B$ be some open neighbourhood of $K$ such that $\overline{B}$ is compact in $\Omega_\varphi$. We choose a Cauchy cycle $\Gamma_1$ for $K$ in $B$ and a Cauchy cycle $\Gamma_2$ for $\varphi(\overline{B})$ in $\varphi(\Omega_\varphi)$. Then $\varphi \circ \Gamma_1$ is a Cauchy cycle for $\varphi(K)$ in $\varphi(B)$ by Proposition 1.10 and hence $\mathrm{ind}_{\varphi \circ \Gamma_1}(w) = 0$ for all $w \in |\Gamma_2| \subset \mathbb{C} \setminus \varphi(B)$. Furthermore, $\mathrm{ind}_{\Gamma_2}(\xi) = 1$ for all $\xi$ in $\varphi(|\Gamma_1|) \subset \varphi(B)$. Now, applying the Cauchy integral formula and using the Pólya representation for $\mathcal{B}^{-1}_{\varphi(K)}$ (cf. Remark 1.31), we obtain

$$\begin{aligned}\mathcal{B}^{-1}_{\varphi(K)}\mathcal{G}_\varphi\mathcal{B}_K f(z) &= \frac{1}{2\pi i}\int_{\Gamma_2}\frac{1}{2\pi i}\int_{\Gamma_1}\frac{\mathcal{B}f(\xi)}{w-\varphi(\xi)}\,d\xi\,e^{wz}\,dw\\ &= \frac{1}{2\pi i}\int_{\Gamma_1}\mathcal{B}f(\xi)\,\frac{1}{2\pi i}\int_{\Gamma_2}\frac{e^{wz}}{w-\varphi(\xi)}\,dw\,d\xi\\ &= \frac{1}{2\pi i}\int_{\Gamma_1}\mathcal{B}f(\xi)\,e^{\varphi(\xi)z}\,d\xi\\ &= \Phi_\varphi f(z).\end{aligned}$$

This proves the assertion in (1).
In the above diagram, $\mathcal{B}_K$ and $\mathcal{G}_\varphi$ are isomorphisms by Proposition 1.30 and Proposition 3.32 (1). Since in the case that $\varphi(K)$ is convex, the mapping $\mathcal{B}^{-1}_{\varphi(K)}$ is also an isomorphism, we obtain that $\Phi_\varphi : \mathrm{Exp}(K) \to \mathrm{Exp}(\varphi(K))$ is an isomorphism due to (1). With Proposition 3.32 (1), we have $\Phi_\varphi^{-1} = (\mathcal{B}^{-1}_{\varphi(K)}\mathcal{G}_\varphi\mathcal{B}_K)^{-1} = \mathcal{B}_K^{-1}\mathcal{G}_{\varphi^{-1}}\mathcal{B}_{\varphi(K)} = \Phi_{\varphi^{-1}}$. □

**Remark 3.34** Let $K$ be a compact, convex subset of $\mathbb{C}$ and $\varphi \in H(K)$ univalent. Then Proposition 3.33 (1) implies that, for each $f \in \mathrm{Exp}(K)$, we have $\mathcal{B}\Phi_\varphi f \in H_0(\mathbb{C}\setminus\varphi(K))$.

**Remark 3.35** In the situation of Proposition 3.33, let $\tilde\varphi \in H(\varphi(K))$ be univalent and not necessarily $\tilde\varphi \in H(\mathrm{conv}(\varphi(K)))$. Then $\Phi_{\tilde\varphi\circ\varphi}$ is defined while the iteration $\Phi_{\tilde\varphi}\Phi_\varphi$ is possibly not defined. In case that $\varphi \in H(\mathrm{conv}(\varphi(K)))$ is univalent, we have

$$\Phi_{\tilde\varphi}\Phi_\varphi = \Phi_{\tilde\varphi\circ\varphi}$$

due to Proposition 3.32 (2) and Proposition 3.33 (1).

Let $f$ be an entire function of exponential type and $\varphi \in H(K(f))$. Interchanging integration and differentiation yields

$$(\Phi_\varphi f)^{(n)}(z) = \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\,\varphi^n(\xi)\,e^{\varphi(\xi)z}\,d\xi, \tag{3.21}$$

implying that the Taylor expansion of $\Phi_\varphi f$ at the origin is given by

$$\Phi_\varphi f(z) = \sum_{n=0}^{\infty} \frac{T_\varphi^n f(0)}{n!} z^n. \tag{3.22}$$

According to our conventions, if $\varphi \in H(K(f))$ is zero-free, then $\varphi$ has no zero in $\Omega_\varphi$, and since $K(f)$ is convex, $\Omega_\varphi$ is simply connected. These conditions ensure the existence of a logarithm function $\log \varphi \in H(\Omega_\varphi)$ for $\varphi$. Then for each non-negative integer $n$, we have

$$\begin{aligned} T_{e_1}^n \Phi_{\log\varphi} f(0) = \Phi_{\log\varphi} f(n) &= \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi)\, e^{n \log\varphi(\xi)} d\xi \\ &= \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi)\, \varphi^n(\xi)\, d\xi = T_\varphi^n f(0). \end{aligned} \tag{3.23}$$

We shall extend (3.22) and (3.23) to a commutative property of $\Phi_\varphi$ and generalized differential operators. The proof of this result is prepared by the following lemma.

**Lemma 3.36** *Let $f$ be an entire function of exponential type and $\varphi \in H(K(f))$ univalent. Then for each $h \in H(\varphi(\Omega_\varphi))$, we have*

$$\frac{1}{2\pi i} \int_{\tilde\Gamma} \mathcal{B}\Phi_\varphi f(\xi)\, h(\xi)\, d\xi = \frac{1}{2\pi i} \int_{\tilde\Gamma} \mathcal{B}f(\varphi^{-1}(\xi))\, (\varphi^{-1})'(\xi)\, h(\xi)\, d\xi$$

*where $\tilde\Gamma$ is a Cauchy cycle for $\varphi(K(f))$ in $\varphi(\Omega_\varphi)$.*

*Proof.* Note that, by Remark 3.34, $\mathcal{B}\Phi_\varphi f \in H_0(\mathbb{C} \setminus \varphi(K))$. Consequently, the integral on the left hand side is well-defined. Further, this integral is independent of the particular choice of the Cauchy cycle. Hence we can assume that $\tilde\Gamma = \varphi \circ \Gamma$ with $\Gamma$ being a Cauchy cycle for $K(f)$ in $\Omega_\varphi$. Considering the Pólya representation of $\Phi_\varphi f$, we obtain

$$\begin{aligned} \frac{1}{2\pi i} \int_{\varphi\circ\Gamma} \mathcal{B}\Phi_\varphi f(\xi)\, e^{\xi\alpha}\, d\xi = \Phi_\varphi f(\alpha) &= \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(w)\, e^{\varphi(w)\alpha}\, dw \\ &= \frac{1}{2\pi i} \int_{\varphi\circ\Gamma} \mathcal{B}f(\varphi^{-1}(\xi))\, (\varphi^{-1})'(\xi)\, e^{\xi\alpha}\, d\xi \end{aligned}$$

for all $\alpha \in \mathbb{C}$. Now, since $\varphi(\Omega_\varphi)$ is simply connected according to our assumptions, a similar reasoning as in the proof of Lemma 3.22 yields the assertion.

□

**Proposition 3.37** *Let $K$ be a compact, convex subset of $\mathbb{C}$ and $\varphi \in H(K)$. Then the following assertions hold:*

*(1) We have $\Phi_\varphi T_\varphi = D\,\Phi_\varphi$.*

*(2) Let $\varphi$ be zero-free. Then for every logarithm function $\log\varphi$ for $\varphi$ on $\Omega_\varphi$, we have $\Phi_{\log\varphi} T_\varphi = T_{e_1} \Phi_{\log\varphi}$.*

*(3) Let $\varphi$ be univalent. We further assume that $\tilde{K}$ is a compact, convex subset of $\mathbb{C}$ and $\tilde{\varphi} \in H(\tilde{K})$ is univalent onto some open neighbourhood of $\varphi(K)$ such that $\tilde{\varphi}^{-1} \circ \varphi(K) \subset \tilde{K}$. Then we have $\Phi_{\tilde{\varphi}^{-1}\circ\varphi} T_\varphi = T_{\tilde{\varphi}} \Phi_{\tilde{\varphi}^{-1}\circ\varphi}$.*

The above proposition states that, under the given conditions, the following diagrams commute:

(1)

$$\begin{array}{ccc} \operatorname{Exp}(K) & \xrightarrow{\Phi_\varphi} & \operatorname{Exp}(\operatorname{conv}(\varphi(K))) \\ \downarrow{\scriptstyle T_\varphi} & & \downarrow{\scriptstyle D} \\ \operatorname{Exp}(K) & \xrightarrow[\Phi_\varphi]{} & \operatorname{Exp}(\operatorname{conv}(\varphi(K))); \end{array}$$

(2)

$$\begin{array}{ccc} \operatorname{Exp}(K) & \xrightarrow{\Phi_{\log\varphi}} & \operatorname{Exp}(\operatorname{conv}(\log\varphi(K))) \\ \downarrow{\scriptstyle T_\varphi} & & \downarrow{\scriptstyle T_{e_1}} \\ \operatorname{Exp}(K) & \xrightarrow[\Phi_{\log\varphi}]{} & \operatorname{Exp}(\operatorname{conv}(\log\varphi(K))); \end{array}$$

(3)

$$\begin{array}{ccc} \operatorname{Exp}(K) & \xrightarrow{\Phi_{\tilde{\varphi}^{-1}\circ\varphi}} & \operatorname{Exp}(\tilde{K}) \\ \downarrow{\scriptstyle T_\varphi} & & \downarrow{\scriptstyle T_{\tilde{\varphi}}} \\ \operatorname{Exp}(K) & \xrightarrow[\Phi_{\tilde{\varphi}^{-1}\circ\varphi}]{} & \operatorname{Exp}(\tilde{K}). \end{array}$$

*Proof.* In order to see (1), consider the Taylor expansion for $\Phi_\varphi f$ in (3.22) and observe that

$$D\left(\sum_{n=0}^{\infty} \frac{T_\varphi^n f(0)}{n!}\, z^n\right) = \sum_{n=0}^{\infty} \frac{T_\varphi^{n+1} f(0)}{n!}\, z^n.$$

Considering Lemma 3.22, we obtain

$$\begin{aligned}
T_{e_1}^n \Phi_{\log\varphi} f(z) &= \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\ e^{(z+n)\log\varphi(\xi)} d\xi \\
&= \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\,\varphi^n(\xi)\,e^{z\log\varphi(\xi)} d\xi \\
&= \frac{1}{2\pi i}\int_\Gamma \mathcal{B}T_\varphi^n f(\xi)\,e^{z\log\varphi(\xi)} d\xi \\
&= \Phi_{\log\varphi} T_\varphi^n f(z).
\end{aligned}$$

With $n = 1$, this is the assertion in (2).
The conditions in (3) ensure that the operator $T_{\tilde\varphi}$ is defined on $\Phi_{\tilde\varphi^{-1}\circ\varphi}(\mathrm{Exp}(K))$. Let $\Gamma$ be a Cauchy cycle of $K$ in $\Omega_\varphi$ such that $\tilde\varphi^{-1}\circ\varphi(|\Gamma|) \subset \Omega_{\tilde\varphi}$. Then $\tilde\Gamma := \tilde\varphi^{-1}\circ\varphi\circ\Gamma$ is a Cauchy cycle for $\tilde\varphi^{-1}\circ\varphi(K)$ in $\tilde\varphi^{-1}\circ\varphi(\Omega_\varphi)$ due to Proposition 1.10. By Remark 3.34, $\mathcal{B}\Phi_{\tilde\varphi\circ\varphi} f \in H_0(\mathbb{C}\setminus(\tilde\varphi^{-1}\circ\varphi(K)))$ for each $f \in \mathrm{Exp}(K)$, showing that we can use $\tilde\Gamma$ to represent $T_{\tilde\varphi}\Phi_{\tilde\varphi\circ\varphi} f$ (cf. Remark 3.20 (2)). For a given $f \in \mathrm{Exp}(K)$, Lemma 3.36, substitution and Lemma 3.22, applied in this order, yield

$$\begin{aligned}
T_{\tilde\varphi}\Phi_{\tilde\varphi^{-1}\circ\varphi} f(z) &= \frac{1}{2\pi i}\int_{\tilde\Gamma} \mathcal{B}\Phi_{\tilde\varphi^{-1}\circ\varphi} f(\xi)\,\tilde\varphi(\xi)\,e^{z\xi} d\xi \\
&= \frac{1}{2\pi i}\int_{\tilde\Gamma} \mathcal{B}f(\varphi^{-1}\circ\tilde\varphi(\xi))\,(\varphi^{-1}\circ\tilde\varphi)'(\xi)\,\tilde\varphi(\xi)\,e^{z\xi} d\xi \\
&= \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(w)\,\varphi(w)\,e^{z\,\tilde\varphi^{-1}\circ\varphi(w)} dw \\
&= \frac{1}{2\pi i}\int_\Gamma \mathcal{B}T_\varphi f(w)\,e^{z\,\tilde\varphi^{-1}\circ\varphi(w)} dw \\
&= \Phi_{\tilde\varphi^{-1}\circ\varphi} T_\varphi f(z)
\end{aligned}$$

and we are done. □

**Example 3.38** We shall apply (3.22) to the cases of Example 3.24:

(1) If $\varphi = e_1$ and $f \in \operatorname{Exp}(K)$ for some compact and convex set $K \subset \mathbb{C}$, then

$$\Phi_{e_1} f(z) = \sum_{n=0}^{\infty} \frac{T_{e_1}^n f(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{f(n)}{n!} z^n.$$

(2) For $\varphi(\xi) = \xi^{-1}$ and $K \subset \mathbb{C} \setminus \{0\}$ compact and convex, we have

$$\Phi_{\varphi} f(z) = \sum_{n=0}^{\infty} \frac{T_{\varphi}^n f(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{1}{n! 2\pi i} \int_{\Gamma} \frac{\mathcal{B}f(\xi)}{\xi^n} d\xi \, z^n$$

for all $f \in \operatorname{Exp}(K)$.

(3) In the case that $\varphi(\xi) = e^{\xi} - 1$ and $K \subset \mathbb{C}$ is compact and convex, then

$$\Phi_{\varphi} f(z) = \sum_{n=0}^{\infty} \frac{\Delta_n f(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(k)}{n!} z^n$$

for all $f \in \operatorname{Exp}(K)$.

(4) If $\varphi(\xi) = \frac{1}{1-\xi}$ and $K$ is a compact and convex subset of $\mathbb{C} \setminus \{1\}$, then

$$\Phi_{\varphi} f(z) = \sum_{n=0}^{\infty} \frac{T_{\varphi}^n f(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{1}{n! 2\pi i} \int_{\Gamma} \frac{\mathcal{B} f(\xi)}{(1-\xi)^n} d\xi \, z^n$$

for all $f \in \operatorname{Exp}(K)$.

By means of Proposition 3.37 and Proposition 3.31, we are now able to show that $\Phi_{\varphi}$ carries over (frequent) universality for generalized differential operators.

**Theorem 3.39** *Let $K$ be a compact, convex set in $\mathbb{C}$ and $\varphi \in H(K)$. We further assume that $f \in \operatorname{Exp}(K)$ is (frequently) universal for $T_{\varphi}$ on $\operatorname{Exp}(K)$. Then the following assertions hold:*

*(1) The function $\Phi_{\varphi} f$ is (frequently) universal for the differentiation operator on $\operatorname{Exp}(L)$ for every compact, convex set $L \subset \mathbb{C}$ that contains $\operatorname{conv}(\varphi(K))$.*

(2) *Let* $\varphi$ *be zero-free. Then for every logarithm function* $\log\varphi$ *for* $\varphi$ *on* $\Omega_\varphi$*, the function* $\Phi_{\log\varphi} f$ *is (frequently) universal for* $T_{e_1}$ *on* $\operatorname{Exp}(L)$ *for every compact, convex set* $L \subset \mathbb{C}$ *that contains* $\operatorname{conv}(\log\varphi(K))$*.*

(3) *Let* $\varphi$ *be univalent. We further assume that* $\tilde{K}$ *is a compact, convex set in* $\mathbb{C}$ *and* $\tilde{\varphi} \in H(\tilde{K})$ *is univalent onto some open neighbourhood of* $\varphi(K)$ *and such that* $\tilde{\varphi}^{-1} \circ \varphi(K) \subset \tilde{K}$*. Then* $\Phi_{\tilde{\varphi}^{-1}\circ\varphi} f$ *is (frequently) universal for* $T_{\tilde{\varphi}}$ *on* $\operatorname{Exp}(L)$ *for every compact, convex set* $L \subset \Omega_{\tilde{\varphi}}$ *that contains* $\operatorname{conv}(\tilde{\varphi}^{-1} \circ \varphi(K))$*.*

*Proof.* Proposition 3.37 immediately yields that in the situation (1), (2) and (3) we have

$$\begin{aligned} D^n \Phi_\varphi f &= \Phi_\varphi T_\varphi^n f \text{ for all } n \in \mathbb{N}, \\ T_{e_1}^n \Phi_{\log\varphi} f &= \Phi_{\log\varphi} T_\varphi^n f \text{ for all } n \in \mathbb{N} \text{ and} \\ T_{\tilde{\varphi}}^n \Phi_{\tilde{\varphi}^{-1}\circ\varphi} f &= \Phi_{\tilde{\varphi}^{-1}\circ\varphi} T_\varphi^n f \text{ for all } n \in \mathbb{N}, \end{aligned} \tag{3.24}$$

respectively.

Assuming that $f$ is universal for $T_\varphi$ on $\operatorname{Exp}(K)$, the continuity and denseness of the image of $\Phi_\varphi$ (cf. Proposition 3.31) prove the assertions (1), (2) and (3) for the case of universality.

Now let $f$ be frequently universal for $T_\varphi$ on $\operatorname{Exp}(K)$. We consider a non-empty open set $U \subset \operatorname{Exp}(L)$. Then the preimage of $U$, with respect to the particular transform, is a non-empty open set $W \subset \operatorname{Exp}(K)$ according to Proposition 3.31. The assumption yields that the sequence $I := \{n \in \mathbb{N} : T_\varphi^n f \in W\}$ has positive lower density. With the identities in (3.24), we obtain that

$$D^n \Phi_\varphi f \in U, \quad T_{e_1}^n \Phi_{\log\varphi} f \in U \quad \text{and} \quad T_{\tilde{\varphi}}^n \Phi_{\tilde{\varphi}^{-1}\circ\varphi} f \in U$$

for each $n \in I$ in the cases (1), (2) and (3), respectively. Since $U$ was arbitrary, this shows the assertions (1), (2) and (3) for the case of frequent universality.

□

## 3.6 Applications of $\Phi_\varphi$

The main objective of this section is to extend some results from Section 3.2 to the case of generalized differential operators. The transform $\Phi_\varphi$ will be the essential tool that enables us to connect arbitrary generalized differential operators with the translation operator $T_{e_1}$. At the end of the section, we give, as another application of $\Phi_\varphi$, the counterpart to Theorem 1.48 by showing that the condition $0 \in K$ is necessary in this general situation.

In our first result, Corollary 3.15 is partly extended to generalized differential operators.

**Theorem 3.40** *Let $K \subset \mathbb{C}$ be a compact, convex set and $\varphi \in H(K)$ non-constant such that $\varphi(K)$ contains some non-singleton continuum of $\mathbb{T}$. Then $T_\varphi$ is frequently hypercyclic on* $\mathrm{Exp}(K)$*.*

*Proof.* Our assumptions ensure the existence of a compact, convex set $\tilde{K} \subset K$ such that $\varphi(\tilde{K})$ contains some non-singleton continuum of $\mathbb{T}$ and $\varphi$ is univalent as an element of $H(\tilde{K})$. We choose suitable real numbers $a < b$ so that $e^{[ia,ib]} \subset \varphi(\tilde{K})$. Now, Corollary 3.15 provides a frequently universal function $f$ for $T_{e_1}$ on $\mathrm{Exp}([ia, ib])$, and, by Theorem 3.39 (3), we have that $\Phi_{\varphi^{-1} \circ e_1} f$ is frequently universal for $T_\varphi$ on $\mathrm{Exp}(K)$. □

**Example 3.41** We consider again the operators from Example 3.24. Let $K$ be a compact and convex subset of $\mathbb{C}$. Theorem 3.26 and Theorem 3.40 imply the following assertions:

(1) If $K$ contains some non-singleton continuum of $\mathbb{T}$, then $D$ is frequently hypercyclic on $\mathrm{Exp}(K)$.

(2) If $0 \notin K$, then $T_{\mathrm{id}}^{-1} = D^{-1}$ is hypercyclic on $\mathrm{Exp}(K)$ if and only if $\mathbb{T} \cap K \neq \emptyset$. If $K$ is in addition as in (1), then $T_{\mathrm{id}}^{-1} = D^{-1}$ is frequently hypercyclic on $\mathrm{Exp}(K)$.

(3) Let $\varphi(z) = e^z - 1$, then the operator $T_\varphi f(z)$ is the forward difference $\Delta(f)(z) = f(z+1) - f(z)$ and $T_\varphi$ is hypercyclic on $\mathrm{Exp}(K)$ if and only if $K$ intersects $\{z : |e^z - 1| = 1\}$. If $K$ contains some non-singleton continuum of $\{z : |e^z - 1| = 1\}$, $T_\varphi$ is also frequently hypercyclic on $\mathrm{Exp}(K)$.

(4) For $\varphi(\xi) = \frac{1}{1-\xi}$, the corresponding operator $T_\varphi$ is defined on $\operatorname{Exp}(K)$ whenever $1 \notin K$. For hypercyclicity it is necessary and sufficient that $K$ intersects $\mathbb{T}+1$. The operator is frequently hypercyclic if $K$ contains some non-singleton continuum of $\mathbb{T}+1$.

From Theorem 3.17, it turns out that a function of exponential type having a singleton conjugate indicator diagram cannot be frequently universal for $T_{e_1}$ on $H(\mathbb{C})$. In view of Theorem 3.40, the question arises if such a condition still holds for the case of differential operators on $H(\mathbb{C})$ or generalized differential operators on $\operatorname{Exp}(K)$.

**Theorem 3.42** *Let $K$ be a compact, convex subset of $\mathbb{C}$ and $\varphi \in H(K)$.*

*(1) There is no frequently universal function for $T_\varphi$ on $\operatorname{Exp}(K)$ that has a singleton conjugate indicator diagram.*

*(2) If $\varphi$ extends to an entire function of exponential type, then there is no entire function of exponential type that is frequently universal for $\varphi(D)$ on $H(\mathbb{C})$ and that has a singleton conjugate indicator diagram.*

*Proof.* We shall start with the proof of assertion (2). Assuming the contrary, we suppose there is a function $f$ of exponential type with $K(f) = \{\lambda\}$ that is frequently universal for $\varphi(D)$ on $H(\mathbb{C})$. Then $\varphi$ is non-constant and $|\varphi(\lambda)| \geq 1$ because otherwise $\varphi(D)^n f(0) \to 0$ for $n$ tending to infinity, as it turns out from the first part of the proof of Theorem 3.26. Hence in some sufficiently small and simply connected, open neighbourhood $\Omega$ of $\lambda$, the function $\tilde{\varphi} := \varphi/\varphi(\lambda)$ is zero-free, which implies the existence of a logarithm function $\log\tilde{\varphi}$ for $\tilde{\varphi}$ on $\Omega$ with $\log\tilde{\varphi}(\lambda) = 0$. We set $h := \Phi_{\log\tilde{\varphi}} f$. Then $K(h) = \{0\}$ by Proposition 3.31 and, according to (3.23) applied to $\tilde{\varphi}$, we have

$$h(n) = \frac{1}{\varphi(\lambda)^n}\,\varphi(D)^n f(0) \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.25}$$

Let $S$ be the sector $\{z : |\arg(z)| \leq \frac{\pi}{5}\} \setminus \{0\}$. By the Casorati-Weierstrass theorem, we can choose $\alpha \in \mathbb{C}$ such that $\varphi(\alpha)$ is close enough to $\pi\varphi(\lambda)$ to ensure that

$$\frac{\varphi(\alpha)}{\varphi(\lambda)}\, S \subset \left\{z : |\arg(z) - \pi| \leq \frac{\pi}{4}\right\} \tag{3.26}$$

and $\varphi(\alpha) \neq 0$. Now, according to the continuity of $\varphi(D)$ on $H(\mathbb{C})$, for every $\varepsilon > 0$, there are some $r > 0$ and $\delta > 0$ such that for all $g \in H(\mathbb{C})$ that satisfy

$$\sup_{z \in r\,\overline{\mathbb{D}}} |g(z) - e_\alpha(z)| < \delta, \tag{3.27}$$

we have

$$|\varphi(D)g(0) - \varphi(D)e_\alpha(0)| = |\varphi(D)g(0) - \varphi(\alpha)| < \varepsilon.$$

We assume that $\delta, \varepsilon > 0$ are so small that, whenever $g$ satisfies (3.27), we have

$$g(0) \in S \text{ and } \varphi(D)g(0) \in \varphi(\alpha)\, S. \tag{3.28}$$

Our assumption implies the existence of some sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers with $\underline{\text{dens}}((n_k)_{k\in\mathbb{N}}) > 0$ and such that $\sup_{z \in r\,\overline{\mathbb{D}}} |\varphi(D)^{n_k} f(z) - e_\alpha(z)| < \delta$ for all $k \in \mathbb{N}$. The interpolating property of $h$ in (3.25) combined with (3.28) yields

$$h(n_k) \in \frac{1}{\varphi(\lambda)^{n_k}}\, S \text{ and } h(n_k + 1) \in \frac{\varphi(\alpha)}{\varphi(\lambda)^{n_k+1}}\, S \text{ for all } k \in \mathbb{N}. \tag{3.29}$$

Condition (3.26) implies that the factor $\frac{\varphi(\alpha)}{\varphi(\lambda)}$ rotates $S$ by an angle larger than $\frac{\pi}{2}$. Hence, from (3.29), it follows that for each $k \in \mathbb{N}$ either $\text{Re}(h)$ or $\text{Im}(h)$ has a sign change in $[n_k, n_k + 1]$. The intermediate value theorem yields a sequence $(w_k)_{k\in\mathbb{N}}$ of positive numbers with $w_k \in (n_k, n_k + 1)$ and

$$\text{Re}(h(w_k))\,\text{Im}(h(w_k)) = 0 \text{ for all } k \in \mathbb{N}. \tag{3.30}$$

Assuming that the Taylor series of $h$ is given by $\sum_{\nu=0}^{\infty} \frac{h_\nu}{\nu!} z^\nu$, we set $h_1(z) := \sum_{\nu=0}^{\infty} \frac{\text{Re}(h_\nu)}{\nu!} z^\nu$ and $h_2(z) := \sum_{\nu=0}^{\infty} \frac{\text{Im}(h_\nu)}{\nu!} z^\nu$. The functions $h_1, h_2$ are of exponential type zero due to Proposition 1.13 and the fact that $h$ is of exponential type zero. Thus $h_1 h_2$ is a function of exponential type zero (see Proposition 1.17 (1)), and since $\text{Re}(h(x)) = h_1(x)$ and $\text{Im}(h(x)) = h_2(x)$ for every real $x$, $h_1 h_2(w_k) = 0$ for all $k \in \mathbb{N}$ by (3.30). Taking into account that $(w_k)_{k\in\mathbb{N}}$ has obviously the same lower density as $(n_k)_{k\in\mathbb{N}}$, we have a contradiction to Lemma 3.16.

To prove assertion (1), suppose that $f \in \text{Exp}(K)$ with $K(f) := \{\lambda\}$ is frequently universal for $T_\varphi$ on $\mathrm{Exp}(K)$. Then $\Phi_\varphi f$ is frequently universal for the differentiation operator on $\mathrm{Exp}(\mathrm{conv}(\varphi(K)))$ (hence on $H(\mathbb{C})$) by Theorem 3.39 (1), and $K(\Phi_\varphi f) = \{\varphi(\lambda)\}$ by Proposition 3.31. This contradicts statement (2). □

**Remark 3.43** Theorem 3.42 shows that frequently universal functions for generalized differential operators cannot be of the form $fe_\lambda$ with $f \in \mathrm{Exp}(\{0\})$. In particular, there is no entire function of exponential type zero that is frequently universal for any generalized differential operator. On the other hand, for every non-constant generalized differential operator, there are universal functions that have the above form (compare Corollary 3.28 (1)).

We consider the special case $T_{\mathrm{id}} = D$. Theorem 3.5 states that for any $q(r)$ tending to infinity as $r \to \infty$, there exists a universal function $f$ for $D$ (on $H(\mathbb{C})$) that satisfies $M_f(r) = O\left((q(r)/\sqrt{r})e^r\right)$ as $r \to \infty$. Theorem 3.26 yields the existence of universal functions for $D$ having a "small" conjugate indicator diagram. However, concerning the growth with respect to the maximum modulus, this theorem only yields universal functions $f$ for $D$ that satisfy $M_f(r) = O(e^{(1+\varepsilon)r})$ for every $\varepsilon > 0$. By means of the transform $\Phi_\varphi$ and Corollary 3.14, we shall now refine the result of Theorem 3.26 for the case of the differentiation operator.

**Theorem 3.44** *Let $L$ be some compact and convex subset of $\mathbb{C}$ that contains a non-singleton continuum of $\mathbb{T}$. Then, for every $\varepsilon > 0$, there is an $f \in \mathrm{Exp}(L)$ that is universal for $D$ on $\mathrm{Exp}(L)$ and that satisfies*

$$M_f(r) = O(r^{\frac{1}{2}+\varepsilon}\, e^r) \quad \text{as } r \to \infty.$$

*Proof.* For a given $\varepsilon > 0$, we show that

$$m(n) := \min_{r \geq 1} \frac{e^r\, n!\, r^{\frac{1}{2}+\varepsilon}}{r^n\, n^{1+\frac{\varepsilon}{2}}} \to \infty \tag{3.31}$$

as $n$ tends to infinity.
The mapping $r \mapsto e^r/r^n$ is monotonically decreasing in $(0, n)$, and, for all $n \in \mathbb{N}$, the values $e^n\, n!/n^{n+\frac{1}{2}}$ are bounded from below by some positive constant $M$ according

to Stirling's formula. Thus, for $1 < r < \frac{n}{2}$, we have

$$\frac{e^r\, n!\, r^{\frac{1}{2}+\varepsilon}}{r^n\, n^{1+\frac{\varepsilon}{2}}} \geq \frac{e^{\frac{n}{2}}\, n!}{\left(\frac{n}{2}\right)^n\, n^{1+\frac{\varepsilon}{2}}} = \frac{e^{(\log 2-\frac{1}{2})n}}{n^{\frac{1+\varepsilon}{2}}}\, \frac{e^n\, n!}{n^{n+\frac{1}{2}}} > M\, \frac{e^{(\log 2-\frac{1}{2})n}}{n^{\frac{1+\varepsilon}{2}}}. \tag{3.32}$$

The mapping $r \mapsto e^r/r^n$ attains its minimum in $(1,\infty)$ for $r = n$. If $r \geq \frac{n}{2}$, this implies

$$\frac{e^r\, n!\, r^{\frac{1}{2}+\varepsilon}}{r^n\, n^{1+\frac{\varepsilon}{2}}} \geq \frac{e^n\, n!}{n^{n+\frac{1}{2}}}\, \frac{\left(\frac{n}{2}\right)^{\frac{1}{2}+\varepsilon}}{n^{\frac{1+\varepsilon}{2}}} \geq M\, \left(\frac{1}{2}\right)^{\frac{1}{2}+\varepsilon} n^{\frac{\varepsilon}{2}}. \tag{3.33}$$

Now, the inequalities (3.32) and (3.33) yield (3.31). Consequently, we can find a mapping $q : (0,\infty) \to [1,\infty)$ increasing to infinity and such that $q(n) \leq m(n)$ for all $n$ that are larger than some positive integer $n_0$.
Without loss of generality, we can assume that $e^{[-id,id]} \subset L$ for some $d > 0$. Corollary 3.14 provides a universal function $g$ for $T_{e_1}$ on $\mathrm{Exp}([-id,id])$ satisfying $|g(x)| = O(q(x))$ on the real axis. According to Theorem 3.39 (1) and (3.22), the function $f := \Phi_{e_1} g \in \mathrm{Exp}(\mathrm{conv}(e^{[-id,id]}))$ is universal for $D$ on $\mathrm{Exp}(L)$ and $f(z) = \sum_{n=0}^{\infty} \frac{g(n)}{n!} z^n$.
From the condition $q(n) \leq m(n)$ for all $n > n_0$, the definition of $m(n)$ in (3.31), and $|g(n)| = O(q(n))$, it follows that

$$\begin{aligned} M_f(r) &\leq O(r^{n_0}) + \sum_{n=n_0+1}^{\infty} \frac{|g(n)|}{n!}\, r^n \leq O(r^{n_0}) + C_2 \sum_{n=n_0+1}^{\infty} \frac{m(n)}{n!}\, r^n \\ &\leq O(r^{n_0}) + \sum_{n=n_0+1}^{\infty} \frac{e^r\, r^{\frac{1}{2}+\varepsilon}\, n!}{r^n\, n^{1+\frac{\varepsilon}{2}}}\, \frac{r^n}{n!} \leq O(r^{n_0}) + e^r\, r^{\frac{1}{2}+\varepsilon} \sum_{n=n_0+1}^{\infty} \frac{1}{n^{1+\frac{\varepsilon}{2}}} \\ &= O(r^{\frac{1}{2}+\varepsilon}\, e^r). \end{aligned}$$

□

As a last application of $\Phi_\varphi$, we give the counterpart to Theorem 1.48 by showing that the condition $0 \in K$ in this result is necessary in the general case.

**Theorem 3.45** *Let $f$ be an entire function of exponential type. Assume further that $K$ is a compact, convex subset of $\mathbb{C}$ such that $0 \notin K$ and $A \subset \mathbb{C}$ is an infinite set such that $f_\alpha(z) := f(\alpha z)$ belongs to $\mathrm{Exp}(K)$ for every $\alpha \in A$. Then the subspace* linspan$\{f_\alpha : \alpha \in A\}$ *is dense in* Exp$(K)$ *if and only if* $f$ *is of the form* $f(z) = c\,e^{\lambda z}$ *with* $c, \lambda \neq 0$.

*Proof.* We set $Y := \text{linspan}\{f_\alpha(z) : \alpha \in A\}$. If $f(z) = c\,e^{\lambda z}$ with $c, \lambda \neq 0$, Proposition 1.47 states that $Y$ is dense in Exp$(K)$.
A simple observation using the equality in Theorem 1.20 yields

$$K(f_\alpha) = \alpha\, K(f) \tag{3.34}$$

for arbitrary $\alpha \in \mathbb{C} \setminus \{0\}$ and $K(f_0) \subset \{0\}$. Hence the condition $f_\alpha \in \text{Exp}(K)$ implies $A\,K(f) \subset K$. Consequently, from the assumptions we can conclude that the sets $K(f)$ and $A$ do not contain the origin. We fix some $\alpha_0 \in A$ and set $A_0 := \alpha_0^{-1} A$. Our assumptions further imply that the sets $A_0$ and $K$ are both contained in certain half-spaces $H_{A_0}$ and $H_K$ that both do not contain the origin. Thus one can choose a half-ray starting from the origin that does not intersect $H_{A_0} \cup H_K$, implying that a suitable branch of the logarithm, which we denote by log in this proof, is defined on $K \cup A_0$.
We set $h_\alpha := \Phi_{\log} f_\alpha$ for $\alpha \in A$. Then, according to Remark 3.35, $\Phi_{e_1} h_\alpha = \Phi_{e_1 \circ \log} f_\alpha = f_\alpha$ and hence, with (3.22), we have

$$f_\alpha(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\,\alpha^n}{n!}\, z^n = \sum_{n=0}^{\infty} \frac{h_\alpha(n)}{n!}\, z^n \tag{3.35}$$

(see also Example 3.38 (1)). Consequently, setting $\tilde{\alpha} := \log(\alpha/\alpha_0)$, $\alpha \in A$, we obtain

$$h_\alpha(n) = h_{\alpha_0}(n) \left(\frac{\alpha}{\alpha_0}\right)^n = h_{\alpha_0}(n)\, e^{\tilde{\alpha} n}, \quad n \in \mathbb{N} \cup \{0\}. \tag{3.36}$$

For every $\alpha \in A$, (3.34) implies that

$$K(e_{\tilde{\alpha}}\, h_{\alpha_0}) = \text{conv}\left(\log\left(\frac{\alpha}{\alpha_0}\right) + \log(K(f_{\alpha_0}))\right) = \text{conv}(\log(K(f_\alpha))) \subset \text{conv}(\log(K)),$$

which shows that $e_{\tilde{\alpha}}\, h_{\alpha_0} \in \text{Exp}(\text{conv}(\log(K)))$ for all $\alpha \in A$. Considering (3.36), we immediately observe that

$$\Phi_{e_1} h_\alpha = \Phi_{e_1}(e_{\tilde{\alpha}} h_{\alpha_0}) \text{ for all } \alpha \in A. \tag{3.37}$$

Since $e_1$ is univalent as an element of $H(\mathrm{conv}(\log(K)))$, the transform $\mathcal{G}_{e_1}\mathcal{B}_{\mathrm{conv}(\log(K))}$ : $\mathrm{Exp}(\mathrm{conv}(\log(K))) \to H_0(\mathbb{C} \setminus L)$, where $L := e^{\mathrm{conv}(\log(K))}$, is well-defined and isomorphic due to Proposition 3.32 and Proposition 1.30. According to the identity $\mathcal{B}_L^{-1}\mathcal{G}_{e_1}\mathcal{B}_{\mathrm{conv}(\log(K))} = \Phi_{e_1}$ (cf. Proposition 3.33 (1)), this shows the injectivity of $\Phi_{e_1} : \mathrm{Exp}(\mathrm{conv}(\log(K))) \to \mathrm{Exp}(\mathrm{conv}(L))$ and hence, with (3.37), we obtain

$$h_\alpha = e_{\tilde{\alpha}}\, h_{\alpha_0} \tag{3.38}$$

for all $\alpha \in A$. The injectivity can also be deduced from Carlson's theorem (cf. [Boa54, Theorem 9.2.1]). Now, if $f$ is not of the form $f(z) = c\, e^{\lambda z}$ with $c, \lambda \neq 0$, then $h_{\alpha_0}$ is also not of this form and can further not be a constant. Otherwise, we would obtain a contradiction to (3.35). Thus $h_{\alpha_0}$ has at least one zero, $z_0 \in \mathbb{C}$. By the identity (3.38), $z_0$ is also a zero for all functions $h_\alpha$, implying that $\Phi_{\log}(Y) = \mathrm{linspan}\{h_\alpha : \alpha \in A\}$ is not dense in $\mathrm{Exp}(\mathrm{conv}(\log(K)))$. Since $\Phi_{\log} : \mathrm{Exp}(K) \to \mathrm{Exp}(\mathrm{conv}(\log(K)))$ has dense image, by Proposition 3.31, we have shown that $Y$ is not dense in $\mathrm{Exp}(K)$. □

# Chapter 4

# Hypercyclic and Frequently Hypercyclic Weighted Shifts

## 4.1 Introduction

In the year 1969 S. Rolewicz in [Rol69] proved the hypercyclicity of the operator $(a_0, a_1, a_2, ...) \mapsto (\lambda a_1, \lambda a_2, ...)$ for $\lambda > 1$ on $\ell^2$. Besides the differentiation operator and the translation operator on $H(\mathbb{C})$, this is one of the earliest and best known examples of hypercyclic operators. It was the starting point of the investigation of hypercyclicity of so-called weighted shift operators. This is an operator mapping a sequence $(a_0, a_1, ...)$ to $(w_1 a_1, w_2 a_2, ...)$ with a certain weight sequence $(w_n)_{n\in\mathbb{N}}$. The hypercyclicity of these operators on $\ell^2$ is characterized in a necessary and sufficient way by H. Salas in [Sal95].
In our introduction to weighted shift operators, we follow the more general approach of K.-G. Grosse-Erdmann who investigated the hypercyclicity of these operators on topological sequence spaces.
A topological sequence space $X$ is a linear subspace of $\mathbb{K}^I$ that carries a vector space topology such that the embedding $X \to \mathbb{K}^I$ is continuous if $\mathbb{K}^I$ is endowed with the product topology. Here $I$ is some countable index set. A space $X$ is called an F-sequence space, Fréchet sequence space or Banach sequence space if the topology on $X$ generates an F-space, Fréchet space or Banach space, respectively.
We use the abbreviation $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$. Our introduction is restricted to sequence spaces that are contained in $\mathbb{K}^{\mathbb{N}_0}$.

**Definition 4.1** Let $X \subset \mathbb{K}^{\mathbb{N}_0}$ be a topological sequence space. We call a sequence $w = (w_n)_{n\in\mathbb{N}}$ of non-zero scalars a *weight sequence for* $X$ if for all $(a_0, a_1, a_2, ...) \in X$ the sequence $(w_1a_1, w_2a_2, ...)$ is an element of $X$. The corresponding weighted shift operator

$$(a_0, a_1, a_2, ...) \mapsto (w_1a_1, w_2a_2, ...)$$

shall be denoted by $B_w$.

If $X$ is an F-sequence space, weighted shift operators on $X$ are always continuous, as noted in [GE00a].

We recall that a sequence $(v_n)_{n\in\mathbb{N}_0}$ in a topological vector space $X$ is called a *basis* for $X$ if for all $x \in X$ there is a unique sequence $(a_0, a_1, a_2, ...)$ of scalars such that $\lim_{n\to\infty} \sum_{j=0}^{n} a_j v_j = x$ in $X$.

**Theorem 4.2** (cf. [GE00a, Theorem 1]) *Let $X \subset \mathbb{K}^{\mathbb{N}_0}$ be an F-sequence space such that the canonical unit vectors $(u_n)_{n\in\mathbb{N}_0}$ form a basis and $w = (w_n)_{n\in\mathbb{N}}$ a weight sequence for $X$. Then the weighted shift operator $B_w$ is hypercyclic if and only if there is an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that*

$$\left(\prod_{\nu=1}^{n_k} w_\nu\right)^{-1} u_{n_k} \to 0 \text{ as } \quad k \to \infty$$

*in $X$.*

The space $H(\mathbb{C})$ can be identified with a subspace of $\mathbb{C}^{\mathbb{N}_0}$ by means of the Taylor coefficients, and a weighted shift operator on $H(\mathbb{C})$ is then given by

$$\sum_{n=0}^{\infty} a_n z^n \mapsto \sum_{n=0}^{\infty} w_{n+1} a_{n+1} z^n. \tag{4.1}$$

Obviously, a sequence $w = (w_n)_{n\in\mathbb{N}}$ of non-zero complex numbers is a weight sequence for $H(\mathbb{C})$ if and only if $w$ satisfies

$$\limsup_{n\to\infty} |w_n|^{\frac{1}{n}} < \infty. \tag{4.2}$$

Since the monomials $(z \mapsto z^n)_{n\in\mathbb{N}_0}$ are a basis of $H(\mathbb{C})$, Theorem 4.2 applies to operators that shift the Taylor coefficients.

**Corollary 4.3** (cf. [GE00a]) Let $w = (w_n)_{n\in\mathbb{N}}$ be a weight sequence for $H(\mathbb{C})$. Then the weighted shift operator $B_w$ is hypercyclic on $H(\mathbb{C})$ if and only if there is an increasing sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ such that

$$\left|\prod_{\nu=1}^{n_k} w_\nu\right|^{\frac{1}{n_k}} \to \infty$$

for $k$ tending to infinity.

**Example 4.4**

(1) In case that $w = (n)_{n\in\mathbb{N}}$, we have $B_w = D$. Corollary 4.3 immediately implies the hypercyclicity of $D$ on $H(\mathbb{C})$.

(2) In case that $w = (1)_{n\in\mathbb{N}}$, the corresponding weighted shift operator is given by

$$\sum_{n=0}^{\infty} a_n z^n \mapsto \sum_{n=0}^{\infty} a_{n+1} z^n$$

and is thus equal to the operator $B$ from Theorem 3.30. By Corollary 4.3, $B_w = B$ is not hypercyclic on $H(\mathbb{C})$.

In [GE00b], K.-G. Grosse-Erdmann investigates the rate of growth of universal functions for weighted shift operators on $H(\mathbb{C})$. In the latter article, it is assumed that a given weight sequence $w = (w_n)_{n\in\mathbb{N}}$ is such that

$$\lim_{n\to\infty}\left|\prod_{\nu=1}^{n} w_\nu\right|^{\frac{1}{n}} = \infty, \tag{4.3}$$

implying that

$$f_1(z) := \sum_{n=0}^{\infty} \frac{z^n}{w_1\, w_2 \cdots w_n} \tag{4.4}$$

is entire, where as usual $\prod_{\nu=1}^{0} w_\nu := 1$. It turns out that the critical rate of growth of universal functions for the corresponding weighted shift operator is closely related to the maximum term of $f_1$,

$$\mu(r) := \mu_{f_1}(r) = \max_{n\in\mathbb{N}} \frac{r^n}{|w_1\, w_2 \cdots w_n|} \quad (r > 0).$$

In order to obtain the sharpness of the critical rate of growth, the *weak monotonicity condition* is introduced in [GE00b]. A sequence $(w_n)_{n\in\mathbb{N}}$ of complex numbers is said to satisfy the weak monotonicity condition if there exists an increasing sequence of positive integers $(n_k)_{k\in\mathbb{N}}$ with $\sup_{k\in\mathbb{N}}(n_{k+1} - n_k) < \infty$ and such that the geometric mean of $|w_m|, |w_{m+1}|, \ldots, |w_{n_k}|$ is less or equal than the geometric mean of $|w_{n_k+1}|,\ |w_{n_k+2}|, \ldots, |w_l|$ for any $m \le n_k < l$.

**Theorem 4.5** (cf. [GE00b, Theorem 1, Theorem 2]) *Let $w = (w_n)_{n\in\mathbb{N}}$ be a weight sequence for $H(\mathbb{C})$ such that (4.3) holds. Then for every $q : (0,\infty) \to (0,\infty)$ with $q(r) \to \infty$ as $r \to \infty$, there is a universal function $f$ for $B_w$ (on $H(\mathbb{C})$) that satisfies*

$$M_f(r) = O(q(r)\, \mu_{f_1}(r)) \quad \text{as } r \to \infty.$$

*If, in addition, $w$ satisfies the weak monotonicity condition, then there is no universal function $f$ for $B_w$ (on $H(\mathbb{C})$) that satisfies*

$$M_f(r) = O(\mu_{f_1}(r)) \quad \text{as } r \to \infty.$$

The spaces $\mathrm{Exp}(K)$ and $H_0(\mathbb{C} \setminus K^{-1})$ are isomorphic via the transform $\mathcal{UB}$ (cf. Proposition 1.30, Proposition 1.39). This implies that $(z \mapsto z^n)_{n\in\mathbb{N}_0}$ is a basis in $\mathrm{Exp}(K)$ if and only if $K$ is equal to $\tau\,\overline{\mathbb{D}}$ for some $\tau \in [0,\infty)$.

Let $\tau \in [0,\infty)$ and $w = (w_n)_{n\in\mathbb{N}}$ a sequence of non-zero complex numbers. Then $w$ is a weight sequence for $\mathrm{Exp}(\tau\,\overline{\mathbb{D}})$ if and only if, for each $f \in \mathrm{Exp}(\tau\,\overline{\mathbb{D}})$, the function

$$\sum_{n=0}^{\infty} w_{n+1} \frac{f^{(n+1)}(0)}{(n+1)!} z^n$$

belongs to $\mathrm{Exp}(\tau\,\overline{\mathbb{D}})$. According to Proposition 1.13, this is equivalent to the fact that for all $f \in \mathrm{Exp}(\tau\,\overline{\mathbb{D}})$, we have

$$\limsup_{n\to\infty} \left| w_{n+1} \, \frac{f^{(n+1)}(0)}{(n+1)} \right|^{\frac{1}{n}} \leq \tau.$$

With the above condition, it is easily seen that the following assertions hold:
If $\tau \in (0,\infty)$, then $w$ is a weight sequence for $\mathrm{Exp}(\tau\,\overline{\mathbb{D}})$ if and only if

$$\limsup_{n\to\infty} |w_n|^{\frac{1}{n}} \leq 1. \tag{4.5}$$

In case that $\tau = 0$, the sequence $w$ is a weight sequence for $\mathrm{Exp}(\tau\,\overline{\mathbb{D}}) = \mathrm{Exp}(\{0\})$ if and only if

$$\limsup_{n\to\infty} |w_n|^{\frac{1}{n}} < \infty. \tag{4.6}$$

We shall now formulate the $\mathrm{Exp}(\tau\overline{\mathbb{D}})$-version of Theorem 4.2. Our formulation is chosen to be consistent to later results for $\mathrm{Exp}(K)$ with more general sets $K$.

**Corollary 4.6** For $\tau \in [0,\infty)$, let $w = (w_n)_{n\in\mathbb{N}}$ be a weight sequence for $\mathrm{Exp}(\tau\overline{\mathbb{D}})$. Then $B_w$ is hypercyclic on $\mathrm{Exp}(\tau\overline{\mathbb{D}})$ if and only if for some increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers

$$f_{1,(n_k)}(z) := \sum_{k=0}^{\infty} \frac{z^{n_k}}{w_1\, w_2 \cdots w_{n_k}}$$

is of exponential type less or equal than $\tau$.

*Proof.* We fix some positive integer $j$. The function $r \mapsto r^n\, e^{-(\tau+\frac{1}{j})r}$ attains its maximum in $[0,\infty)$ at $r = n/(\tau+\frac{1}{j})$. This yields

$$\left\| \frac{z^n}{w_1\, w_2 \cdots w_n} \right\|_{\tau\overline{\mathbb{D}},j} = \sup_{0\leq r} \frac{r^n}{|w_1\, w_2 \cdots w_n|}\, e^{-(\tau+\frac{1}{j})r} = \frac{n^n\, e^{-n}}{|w_1\, w_2 \cdots w_n|} \left(\tau + \frac{1}{j}\right)^{-n}.$$

Dividing this equation by $\frac{n!}{\sqrt{2\pi n}\,|w_1\, w_2 \cdots w_n|} \left(\tau + \frac{1}{j}\right)^{-n}$ and applying Stirling's formula, we obtain

$$\left\| \frac{z^n}{w_1\, w_2 \cdots w_n} \right\|_{\tau\overline{\mathbb{D}},j} \left( \frac{n!}{|w_1\, w_2 \cdots w_n|} \, \frac{1}{\sqrt{2\,\pi\, n}} \, \left(\tau + \frac{1}{j}\right)^{-n} \right)^{-1} \to 1$$

as $n$ tends to infinity. Thus, for an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers, the norms $\left\| \frac{z^{n_k}}{w_1\, w_2 \cdots w_{n_k}} \right\|_{\tau\overline{\mathbb{D}},j}$ converge to zero for all $j \in \mathbb{N}$ if and only if

$$\frac{n_k!}{|w_1\, w_2 \cdots w_{n_k}|} \, \frac{1}{\sqrt{2\,\pi\, n_k}} \, \left(\tau + \frac{1}{j}\right)^{-n_k} \to 0$$

as $k \to \infty$ for all $j \in \mathbb{N}$. This is the case if and only if

$$\limsup_{k\to\infty} \left( \frac{n_k!}{|w_1\, w_2 \cdots w_{n_k}|} \right)^{\frac{1}{n_k}} \leq \tau$$

which is equivalent to the condition that $f_{1,(n_k)}$ is of exponential type less or equal than $\tau$ (cf. Proposition 1.13). The result now follows from Theorem 4.2. □

**Theorem 4.7** (cf. [BGE07, Theorem 4.3]) *Let $X$ be an F-sequence space and $w = (w_n)_{n\in\mathbb{N}}$ a weight sequence for $X$. We further assume that the linear hull of the canonical unit vectors $(u_n)_{n\in\mathbb{N}_0}$ is dense in $X$. If*

$$\sum_{n=0}^{\infty} \frac{1}{w_1 w_2 \cdots w_n} u_n \tag{4.7}$$

*converges unconditionally in $X$, then $B_w$ is frequently hypercyclic on $X$.*

Applying Theorem 4.7 to $\operatorname{Exp}(\tau\,\overline{\mathbb{D}})$ leads to the following result.

**Corollary 4.8** For $\tau \in [0, \infty)$, let $w = (w_n)_{n\in\mathbb{N}}$ be a weight sequence for $\operatorname{Exp}(\tau\overline{\mathbb{D}})$. If

$$f_1(z) = \sum_{n=0}^{\infty} \frac{1}{w_1 w_2 \cdots w_n} z^n$$

is an entire function of exponential type less or equal than $\tau$, then the weighted shift operator $B_w$ is frequently hypercyclic on $\operatorname{Exp}(\tau\overline{\mathbb{D}})$.

*Proof.* Our assumption yields that

$$\limsup_{n\to\infty}\left(\frac{n!}{|w_1\,w_2\cdots w_n|}\right)^{\frac{1}{n}}\leq\tau$$

implying the unconditional convergence of

$$\sum_{n=0}^{\infty}\mathcal{UB}\left(\frac{z^n}{w_1w_2\cdots w_n}\right)=\sum_{n=0}^{\infty}\frac{n!}{w_1w_2\cdots w_n}z^n$$

in $H(\frac{1}{\tau}\mathbb{D})$. Since $\mathcal{UB}:\mathrm{Exp}(\tau\overline{\mathbb{D}})\to H(\frac{1}{\tau}\mathbb{D})$ is an isomorphism (cf. Proposition 1.30 and Proposition 1.39), this yields the unconditional convergence of

$$\sum_{n=0}^{\infty}\frac{1}{w_1\cdots w_n}z^n$$

in $\mathrm{Exp}(\tau\overline{\mathbb{D}})$. Hence we can apply Theorem 4.7 to $B_w$. □

Similarly to the above proof, one can apply $\mathcal{UB}$ for a shorter proof of Corollary 4.6.

Corollary 4.6 and Corollary 4.8 immediately yield that the differentiation operator is frequently hypercyclic on $\mathrm{Exp}(\tau\,\overline{\mathbb{D}})$ if and only if $\tau\geq 1$. Compared to the results in the previous chapter (cf. Theorem 3.26, Theorem 3.40), this does not provide detailed information about the conjugate indicator diagram of the (frequently) universal functions. This is because we were forced to assume circular sets $K$ in order to apply the sequence space criteria in Theorem 4.2 and Theorem 4.7.
It should be noted that in [GE00a], a more general criterion is proved for the universality of so-called weighted pseudo-shifts. There it is only required that the unit sequences of the considered sequence spaces are OP-bases. This means that the linear hull of the unit sequence $(u_n)_{n\in\mathbb{N}}$ is dense and the family of all coordinate projections $x\mapsto x_nu_n$ is equicontinuous. This condition does not apply on $\mathrm{Exp}(K)$ in general.
In the following, we extend the above results for weighted shift operators on $\mathrm{Exp}(K)$ by means of a different approach.

## 4.2 The Hadamard Product on $\mathrm{Exp}(K)$

Operators that map $\sum_{n=0}^{\infty} a_n z^n$ to $\sum_{n=0}^{\infty} w_{n+1} a_{n+1} z^n$ can be regarded as a composition of two mappings. The first mapping is the multiplication of the Taylor coefficients with the weight sequence, and the second one is the shift of the Taylor coefficients. The multiplication with the weight sequence is simply the Hadamard product of the power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} w_n z^n$, where we set $w_0 = 1$. For further investigations of weighted shifts, we establish some properties of the Hadamard product on $\mathrm{Exp}(K)$.
For arbitrary power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ having radius of convergence $R_1$ and $R_2$, respectively, it is obvious that $\sum_{n=0}^{\infty} a_n b_n z^n$ has a radius of convergence at least $R_1 R_2$, and it can be shown that, under certain conditions, this function is analytically continuable.
For two open sets $\Omega_1, \Omega_2 \subset \mathbb{C}$, both containing the origin, we set

$$\Omega_1 * \Omega_2 := \mathbb{C} \setminus ((\mathbb{C} \setminus \Omega_1) \cdot (\mathbb{C} \setminus \Omega_1))$$

and

$$\Omega_1^* := (\mathbb{C} \setminus \Omega_1)^{-1} \cup \{0\}.$$

Let $f \in H(\Omega_1)$ and $g \in H(\Omega_2)$, then their Hadamard product is defined by the so-called Parseval integral

$$f * g(z) := \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{\xi}\, g\left(\frac{z}{\xi}\right) d\xi \tag{4.8}$$

where $\Gamma$ is a Cauchy cycle for $z\,\Omega_2^*$ in $\Omega_1$. The Hadamard multiplication theorem states that $f * g$ is a function of $H(\Omega_1 * \Omega_2)$. A short proof of this result can be found in [Mül92] or [GE93]. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ near the origin, a simple calculation yields that $f * g$ is represented by $\sum_{n=0}^{\infty} a_n b_n z^n$ in a neighbourhood of the origin. Consequently, from the Hadamard multiplication theorem, one can conclude that $\sum_{n=0}^{\infty} a_n b_n z^n$ is analytically continuable to the component of $\Omega_1 * \Omega_2$ that contains the origin.
By means of the isomorphism of the spaces $\mathrm{Exp}(K)$ and $H_0(\mathbb{C} \setminus K)$, J. Müller,

S. Naik and S. Ponnusamy prove the following.

**Theorem 4.9** (cf. [MNP06, Theorem 2.1]) *Let $K$ be a compact, convex subset of $\mathbb{C}$ and $G \in H_0(\mathbb{C} \setminus L^{-1})$ with $L \subset \mathbb{C}$ compact. Then $f * G \in \mathrm{Exp}(\mathrm{conv}(LK))$ for all $f \in \mathrm{Exp}(K)$.*

From this result we later derive conditions ensuring that a given sequence is a weight sequence for $\mathrm{Exp}(K)$.
Let us first motivate another integral representation for the Hadamard product of entire functions of exponential type.
One easily observes that the Parseval integral (4.8) is a generalization of the Cauchy integral, in which the Cauchy kernel is replaced by a more general holomorphic function $g$. Besides the Cauchy integral formula, an entire function of exponential type is representable by means of the Pólya representation (compare Theorem 1.23),

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi)\, e^{z\xi}\, d\xi.$$

Similar to the Parseval integral, we now replace the exponential function in this integral representation by another entire function $g$ of exponential type by setting

$$H_g f(z) := \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi)\, g(z\xi)\, d\xi \tag{4.9}$$

where $\Gamma$ is a Cauchy cycle for $K(f)$ in $\mathbb{C}$. This definition is obviously independent of the particular choice of $\Gamma$. The integral formula in (4.9) is already used in [Buc47].

**Theorem 4.10** *Let $K, L$ be compact, convex subsets of $\mathbb{C}$ and $g \in \mathrm{Exp}(L)$. Then, for every compact and convex set $C \subset \mathbb{C}$ that contains $\mathrm{conv}(K\,L)$, $H_g$ in (4.9) defines a continuous operator from $\mathrm{Exp}(K)$ to $\mathrm{Exp}(C)$ and*

$$H_g f = f * \mathcal{U}\mathcal{B}g.$$

*If, in addition, $g$ is such that $g^{(n)}(0) \neq 0$ for all $n \in \mathbb{N}_0$ and $0 \in C$, then $H_g$ has dense image in $\mathrm{Exp}(C)$.*

*Proof.* From the previous chapter (cf. Remark 3.21), we know that

$$f^{(n)}(0) = T_{\mathrm{id}}^n f(0) = \frac{1}{2\pi i} \int_\Gamma \mathcal{B}f(\xi)\, \xi^n\, d\xi$$

for every non-negative integer $n$. With this identity, it follows that

$$H_g f(z) = \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\, g(z\xi)\, d\xi = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)\, z^n}{n!} \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\, \xi^n\, d\xi$$

$$= \sum_{n=0}^{\infty} \frac{g^{(n)}(0)\, f^{(n)}(0)\, z^n}{n!} = f * \mathcal{U}\mathcal{B}g(z). \tag{4.10}$$

The above equality and Theorem 4.9 yield $H_g f \in \mathrm{Exp}(\mathrm{conv}(K\, L))$.
We fix some positive integer $j$ and choose a Cauchy cycle $\Gamma$ for $K$ in $\mathbb{C}$ being so close to $K$ that $|\Gamma|\, L \subset C + \frac{1}{2j}\,\overline{\mathbb{D}}$. Then, according to Remark (1.21) (4) and Example (1.22) (4),

$$H_L(\xi z) \le H_C(z) + \frac{1}{2j}|z| \tag{4.11}$$

for all $\xi \in \Gamma$ and all $z \in \mathbb{C}$. Since $g \in \mathrm{Exp}(L)$ and $d := \max\{|\xi| : \xi \in |\Gamma|\} > 0$, we have

$$|g(z)| = O\left(e^{H_L(z) + \frac{1}{2jd}|z|}\right).$$

Together with (4.11), this implies

$$|g(\xi z)| \le M_1\, e^{H_C(z) + \frac{1}{j}|z|}, \quad \xi \in |\Gamma|,\ z \in \mathbb{C},\ M_1 < \infty. \tag{4.12}$$

The continuity of the Borel transform implies the existence of an $m \in \mathbb{N}$ and a constant $M_2$ such that $\sup\limits_{\xi \in |\Gamma|} |\mathcal{B}f(\xi)| \le M_2\, ||f||_{K,m}$. Combining this with (4.12), we obtain

$$||H_g f||_{C,j} = \sup_{z \in \mathbb{C}} \left| \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\, g(z\xi)\, d\xi \right| e^{-H_C(z) - \frac{1}{j}|z|} \le \frac{\mathrm{len}(\Gamma)}{2\pi} M_1\, M_2\, ||f||_{K,m}.$$

This proves the continuity of $H_g$. The linearity is obvious.
Now, let $g$ be such that $g^{(n)}(0) \neq 0$ for all $n \in \mathbb{N}_0$ and $0 \in C$. In a first case, we assume that $K^\circ \neq \emptyset$. One easily verifies the equation $g(\alpha z) = H_g e_\alpha$. Hence we can deduce that $H_g$ maps $\mathrm{linspan}\{e_\alpha : \alpha \in K\} \subset \mathrm{Exp}(K)$ to $\mathrm{linspan}\{g(\alpha z) : \alpha \in K\}$, which is dense in $\mathrm{Exp}(C)$ by Theorem 1.48.
In case that $K^\circ = \emptyset$, let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact, convex sets such that

$K_n^\circ \supset K_{n+1}$ and $\bigcap_{n\in\mathbb{N}} K_n = K$. Observing that the set $\operatorname{conv}((K\,L) \cup \{0\})$ equals $\bigcap_{n\in\mathbb{N}} \operatorname{conv}((K_n\,L) \cup \{0\})$ and considering Remark 1.27, we have that the equalities

$$\operatorname{Exp}(K) = \bigcap_{n\in\mathbb{N}} \operatorname{Exp}(K_n) \quad \text{and} \quad \operatorname{Exp}(\operatorname{conv}((K\,L)\cup\{0\})) = \bigcap_{n\in\mathbb{N}} \operatorname{Exp}(\operatorname{conv}((K_n\,L)\cup\{0\}))$$

do also hold in the topological sense. The denseness of $H_g(\operatorname{Exp}(K))$ in the space $\operatorname{Exp}(\operatorname{conv}(K\,L \cup \{0\}))$ now follows from the first case. According to Remark 1.45 (2), this also implies the density of $H_g(\operatorname{Exp}(K))$ in $\operatorname{Exp}(C)$ . □

## 4.3 Weighted Shifts on $\operatorname{Exp}(K)$

In this section we will extend some results from Section 4.1. Our growth condition is, as in Theorem 4.5 and Corollary 4.8, closely related to the growth of the function

$$f_1(z) := \sum_{n=0}^{\infty} \frac{1}{w_1\,w_2 \cdots w_n}\, z^n. \tag{4.13}$$

We assume that $f_1$ is an entire function of exponential type. In our main result of this section, we obtain an immediate relation between the conjugate indicator diagram of $f_1$ and the conjugate indicator diagram of (frequently) universal functions for the corresponding weighted shift operator $B_w$.

**Theorem 4.11** *Let $w := (w_n)_{n\in\mathbb{N}}$ be a weight sequence for $H(\mathbb{C})$ such that $f_1$ in (4.13) is an entire function of exponential type. We further assume that $L$ is a compact, convex subset of $\mathbb{C}$ and $h \in \operatorname{Exp}(L)$ is (frequently) universal for $D$ on $\operatorname{Exp}(L)$. Then $H_{f_1}h \in \operatorname{Exp}(\operatorname{conv}(LK(f_1)))$ is (frequently) universal for $B_w$ on $H(\mathbb{C})$.*

*Proof.* From Theorem 4.10, it turns out that $f := H_{f_1}h \in \operatorname{Exp}(\operatorname{conv}(L\,K(f_1)))$ and $H_{f_1}h = h * \mathcal{U}\mathcal{B}f_1$.

Let $\sum_{n=0}^{\infty} \frac{h_n}{n!} z^n$ be the Taylor series expansion of $h$. Then we have

$$H_{f_1} Dh(z) = H_{f_1} \left( \sum_{n=0}^{\infty} \frac{h_{n+1}}{n!} z^n \right) = \sum_{n=0}^{\infty} \frac{h_{n+1}}{w_1 w_2 \cdots w_n} z^n$$
$$= B_w \left( \sum_{n=0}^{\infty} \frac{h_n}{w_1 w_2 \cdots w_n} z^n \right) = B_w H_{f_1} h = B_w f,$$

implying that

$$B_w^k f = H_{f_1} D^k h \text{ for all } k \in \mathbb{N}_0. \tag{4.14}$$

We set $C := \operatorname{conv}(L\,K(f_1) \cup \{0\})$ and consider the operator

$$H_{f_1} : \operatorname{Exp}(L) \to \operatorname{Exp}(C).$$

Then, according to Theorem 4.10, the operator $H_{f_1}$ is continuous and has dense image. Now, if $h$ is universal for $D$ on $\operatorname{Exp}(L)$, meaning that $(D^n h)_{n\in\mathbb{N}}$ is dense in $\operatorname{Exp}(L)$, (4.14) yields that $(B_w^k f)_{k\in\mathbb{N}}$ is dense in $\operatorname{Exp}(C)$ (hence in $H(\mathbb{C})$).
For the case of frequent universality, let $U \subset \operatorname{Exp}(C)$ be non-empty and open. Then $V := H_{f_1}^{-1}(U)$ is non-empty and open in $\operatorname{Exp}(L)$. If $h$ is frequently universal for $D$ on $\operatorname{Exp}(L)$, (4.14) yields that

$$\{k \in \mathbb{N} : D^k h \in V\} = \{k \in \mathbb{N} : B_w^k f \in U\}$$

has positive lower density. This implies that $f$ is frequently universal for $B_w$ on $H(\mathbb{C})$ since $U$ was arbitrary and $\operatorname{Exp}(C)$ is dense in $H(\mathbb{C})$. □

**Corollary 4.12** Let $w$ be a weight sequence for $H(\mathbb{C})$ such that $f_1$ in (4.13) is of exponential type.

(1) For every $\alpha \in \mathbb{T}$, there is a function $f \in \operatorname{Exp}(\alpha K(f_1))$ that is universal for $B_w$ on $H(\mathbb{C})$.

(2) Let $L \subset \mathbb{C}$ be a compact and convex subset of $\mathbb{C}$ containing a non-singleton continuum of $\mathbb{T}$. Then there is a function $f \in \operatorname{Exp}(\operatorname{conv}(LK(f_1)))$ that is frequently universal for $B_w$ on $H(\mathbb{C})$.

*Proof.* Theorem 3.26 and Theorem 3.40 imply the existence of a universal function for $D$ in $\mathrm{Exp}(\{\alpha\})$ and a frequently universal function for $D$ in $\mathrm{Exp}(L)$, respectively. The assertions now follow from Theorem 4.11. □

**Remark 4.13** The above results show that one can derive a (frequently) universal function for $B_w$ from a (frequently) universal function $h$ for $D$ by setting

$$f(z) := \sum_{n=0}^{\infty} \frac{D^n h(0)}{w_1 \cdots w_n} z^n.$$

With the results from the previous chapter, we can extend this to generalized differential operators:
Consider $\alpha \in \mathbb{C}$ and $\varphi \in H(\{\alpha\})$ non-constant such that $|\varphi(\alpha)| = 1$. Then $T_\varphi$ is hypercyclic on $\mathrm{Exp}(\{\alpha\})$ by Theorem 3.26. According to Theorem 3.39 (1), the function

$$\Phi_\varphi h(z) = \sum_{n=0}^{\infty} \frac{T_\varphi^n h(0)}{n!} z^n$$

is universal for $D$ on $\mathrm{Exp}(\{\varphi(\alpha)\})$ whenever $h$ is universal for $T_\varphi$ on $\mathrm{Exp}(\{\alpha\})$. Now, as above, the function

$$f(z) := H_{f_1}\Phi_\varphi h(z) = \sum_{n=0}^{\infty} \frac{T_\varphi^n h(0)}{w_1 w_2 \cdots w_n} z^n$$

is universal for $B_w$ on $H(\mathbb{C})$. In case that $h$ is frequently universal for some $T_\varphi$ on some space $\mathrm{Exp}(L)$, we obtain a frequently universal function for $B_w$. In particular, in case that $\varphi = e_1$, the function

$$\sum_{n=0}^{\infty} \frac{h(n)}{w_1 w_2 \cdots w_n} z^n$$

is (frequently) universal for $B_w$ on $H(\mathbb{C})$, provided that $h$ is (frequently) universal for $T_{e_1}$ on some space $\mathrm{Exp}(L)$.

In the following we restrict ourselves to the case that the weighted shift operators $B_w$ are self-mappings on $\mathrm{Exp}(K)$ for a suitable choice of $K$. In other words, we consider sequences $w$ that are weight sequences for $\mathrm{Exp}(K)$. For that purpose we introduce an integral representation for $B_w$.

Let $w = (w_n)_{n\in\mathbb{N}}$ be a weight sequence for $H(\mathbb{C})$. Then, according to the condition in (4.2) and Proposition 1.13,

$$g_w(z) := \sum_{n=0}^{\infty} \frac{w_{n+1}}{(n+1)!} z^n \tag{4.15}$$

is an entire function of exponential type. For an arbitrary $f \in \mathrm{Exp}(K)$, we have, similarly to (4.10),

$$\begin{aligned} \frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\,\xi\,g_w(\xi z)\,d\xi &= \sum_{n=0}^{\infty} \frac{w_{n+1}}{(n+1)!}\,z^n\,\frac{1}{2\pi i}\int_\Gamma \mathcal{B}f(\xi)\,\xi^{n+1}\,d\xi \\ &= \sum_{n=0}^{\infty} \frac{w_{n+1}\,f^{(n+1)}(0)}{(n+1)!}\,z^n = B_w f(z) \end{aligned} \tag{4.16}$$

where $\Gamma$ is a Cauchy cycle for $K$ in $\mathbb{C}$.

**Proposition 4.14** *Let $w = (w_n)_{n\in\mathbb{N}}$ be a weight sequence for $H(\mathbb{C})$ and $g_w$ as in (4.15). Then, for any $L \subset \mathbb{C}$ compact and convex, the weighted shift operator $B_w$ maps continuously from* $\mathrm{Exp}(L)$ *to* $\mathrm{Exp}(\mathrm{conv}(L\,K(g_w)))$.

*Proof.* The proof is similar to that of Theorem 4.10, using (4.16) instead of (4.10). □

**Remark 4.15** If $K(g_w)$ is contained in $\mathbb{D}$, meaning that $g_w$ in (4.15) is of exponential type less than 1, we have

$$\limsup_{n\to\infty} |w_n|^{\frac{1}{n}} < 1$$

by Proposition 1.13. In particular, we obtain

$$\limsup_{n\to\infty} |w_1 \cdots w_n|^{\frac{1}{n}} < 1.$$

By Corollary 4.3, this excludes the hypercyclicity of $B_w$ on $H(\mathbb{C})$.

Considering Proposition 4.14, one observes that $B_w^k$ maps from $\mathrm{Exp}(K)$ to $\mathrm{Exp}(C_k)$

where $C_0 = K$ and $C_l = \mathrm{conv}(C_{l-1}\, K(g_w))$ for $l = 1, ..., k$.
Now the following assertions are obvious:

(A) If $K(g_w) = \{1\}$, the weighted shift operator $B_w$ is a self-mapping on $\mathrm{Exp}(K)$ for every compact and convex set $K$.

(B) If $K(g_w) \subset [0, 1]$, the weighted shift operator $B_w$ is a self-mapping on $\mathrm{Exp}(K)$ whenever $K$ is a compact, convex set that contains the origin.

The proof of our next result will rely on the proof of Theorem 4.11. Here it will be necessary to require that $0 \in K$ in order to ensure that $H_{f_1}$ has dense image (cf. Theorem 4.10). Thus the restriction for $g_w$ in (B) is sufficient in our case.

**Theorem 4.16** *Let $w = (w_n)_{n\in\mathbb{N}}$ be a weight sequence for $H(\mathbb{C})$ such that $f_1$ in (4.13) is an entire function of exponential type and $g_w$ in (4.15) is a function in* $\mathrm{Exp}([0, 1])$*. Furthermore, let $K$ be a compact and convex subset of $\mathbb{C}$ that contains the origin.*

*(1) If $\alpha K(f_1) \subset K$ for some $\alpha \in \mathbb{T}$, then $B_w$ is hypercyclic on* $\mathrm{Exp}(K)$*.*

*(2) If $L\,K(f_1) \subset K$ for some compact, convex set $L$ that contains some non-singleton continuum of $\mathbb{T}$, then $B_w$ is frequently hypercyclic on* $\mathrm{Exp}(K)$*.*

*Proof.* From Proposition 4.14, it turns out that our assumptions in (1) and (2) imply that $B_w$ is a continuous self-mapping on $\mathrm{Exp}(K)$.
According to Theorem 4.10, we have that

$$H_{f_1} : \mathrm{Exp}(\{\alpha\}) \to \mathrm{Exp}(\mathrm{conv}(\alpha K(f_1) \cup \{0\})) \subset \mathrm{Exp}(K)$$

has dense image in (1), and

$$H_{f_1} : \mathrm{Exp}(L) \to \mathrm{Exp}(\mathrm{conv}(LK(f_1) \cup \{0\})) \subset \mathrm{Exp}(K)$$

has dense image in (2). Consequently, from the proof of Theorem 4.11, it turns out that the universal and frequently universal functions for $B_w$ on $H(\mathbb{C})$ provided by Corollary 4.12 are even universal and frequently universal for $B_w$ on $\mathrm{Exp}(K)$, respectively. □

**Remark 4.17** In the situation (1) of the above theorem, the set of universal functions for $B_w$ that have a conjugate indicator diagram equal to $K$ is residual in $\mathrm{Exp}(K)$ due to Proposition 2.19.

The following example provides a large source for weighted shift operators that are self-mappings on $\mathrm{Exp}(K)$ provided that $K$ contains the origin.

**Example 4.18** Let $\mu$ be a finite Borel measure with support in $[0,1]$. Then its Cauchy transform $\int \frac{1}{\xi - t} d\mu(t)$ defines a function $F \in H_0(\mathbb{C} \setminus [0,1])$. We consider the entire function $g = \mathcal{B}^{-1}F \in \mathrm{Exp}([0,1])$. According to the Pólya representation (see Theorem 1.23), $g$ is represented by

$$\begin{aligned} g(z) &= \frac{1}{2\pi i} \int_\Gamma F(\xi)\, e^{z\xi}\, d\xi = \frac{1}{2\pi i} \int_\Gamma \int_{[0,1]} \frac{1}{\xi - t}\, d\mu(t)\, e^{z\xi}\, d\xi \\ &= \int_{[0,1]} \frac{1}{2\pi i} \int_\Gamma \frac{1}{\xi - t}\, e^{z\xi}\, d\xi\, d\mu(t) = \int_{[0,1]} e^{tz} d\mu(t), \end{aligned}$$

where $\Gamma$ is a Cauchy cycle for $[0,1]$ in $\mathbb{C}$. Changing integration and differentiation yields $g^{(n)}(0) = \int_{[0,1]} t^n\, d\mu(t)$ for every non-negative integer $n$. Now, if we set

$$w_n := n\, g^{(n-1)}(0) = n \int_{[0,1]} t^{n-1}\, d\mu(t) \text{ for all } n \in \mathbb{N}, \tag{4.17}$$

the function $g_w$ in (4.15) with $w := (w_n)_{n\in\mathbb{N}}$ is equal to $g$. Note that, in case that $w_n \neq 0$ for all $n \in \mathbb{N}$, the sequence $w$ is always a weight sequence for $\mathrm{Exp}(K)$ provided that $0 \in K$.
We discuss conditions of $\mu$ that imply or exclude (frequent) hypercyclicity of the weighted shift operator arising from $w = (n \int_{[0,1]} t^{n-1}\, d\mu(t))_{n\in\mathbb{N}}$.

(1) If $\mu$ is the Dirac measure $\delta_1$, we have $\int_{[0,1]} t^{n-1}\, d\mu(t) = 1$ for all $n \in \mathbb{N}$. Consequently, $w = (n)_{n\in\mathbb{N}}$ and the corresponding weighted shift $B_w$ is simply the ordinary differentiation operator. Note that in this case $f_1 = g_w$, where $f_1$ is as in (4.13). Results for the (frequent) hypercyclicity of the differentiation operator on $\mathrm{Exp}(K)$ are given in the previous chapter.

(2) In case that $\mu$ has a $\lambda$-density $h \in L_\infty([0,1])$, where $\lambda$ is the one-dimensional

Lebesgue measure, we obtain

$$\left| \int_{[0,1]} t^{n-1}\, h(t)\, d\lambda(t) \right| \leq ||h||_\infty \int_{[0,1]} t^{n-1}\, d\lambda(t) = ||h||_\infty \, \frac{1}{n}$$

for all $n \in \mathbb{N}$. Thus we have

$$\limsup_{n\to\infty} |w_1\, w_2 \cdots w_n|^{\frac{1}{n}} \leq ||h||_\infty < \infty,$$

according to (4.17). The corresponding weighted shift operator is not hypercyclic on $H(\mathbb{C})$ by Corollary 4.3.

(3) Let $\mu$ be a finite Borel measure with support in $[0, 1-\varepsilon]$ for some $\varepsilon \in (0,1)$. Then

$$\left| \int_{[0,1]} t^{n-1} d\mu(t) \right| \leq |\mu([0, 1-\varepsilon])|\, (1-\varepsilon)^{n-1},$$

implying that we have

$$\limsup_{n\to\infty} |w_1\, w_2 \cdots w_n|^{\frac{1}{n}} \leq |\mu([0, 1-\varepsilon])| \lim_{n\to\infty} \left( n!(1-\varepsilon)^{\frac{n(n-1)}{2}} \right)^{\frac{1}{n}} = 0.$$

Corollary 4.3 excludes the hypercyclicity of $B_w$ on $H(\mathbb{C})$.

(4) Let $\mu$ be a finite positive Borel measure on $[0,1]$ such that $\mu(\{1\}) > 0$. Then

$$0 < \mu(\{1\}) \leq \int_{[0,1]} t^{n-1}\, d\mu(t),$$

and we have

$$M_{f_1}(r) = \sum_{n=0}^{\infty} \frac{1}{w_1 \cdots w_n}\, r^n \leq \sum_{n=0}^{\infty} \frac{\mu(\{1\})^{-n}}{n!}\, r^n, \quad r > 0,$$

which implies that $f_1$ in (4.13) is a function of exponential type less or equal than $\mu(\{1\})^{-1}$ (see Proposition 1.13). Theorem 4.16 now yields: The weighted shift operator $B_w$ is hypercyclic on $\mathrm{Exp}(K)$ whenever $K$ contains the origin and $\alpha K(f_1) \subset K$ for some $\alpha \in \mathbb{T}$. The frequent hypercyclicity on $\mathrm{Exp}(K)$ is ensured when $0 \in K$ and $L\, K(f_1) \subset K$ for some $L$ that contains a non-singleton continuum of $\mathbb{T}$. In particular, we obtain the frequent hypercyclicity of $B_w$ on $\operatorname{Exp}(\mu(\{1\})^{-1}\,\overline{\mathbb{D}})$.

**Remark 4.19** In general, it is not easy to determine the conjugate indicator diagram of the function $f_1$ in the setting of Theorem 4.11 or Theorem 4.16. One way to do this is finding functions that interpolate the Taylor coefficients of $f_1$. For instance, in case that there is a function $h$ of exponential type such that

$$h(n) = \frac{n!}{w_1 \cdots w_n} \text{ for all } n \in \mathbb{N}_0, \tag{4.18}$$

then $K(f_1) \subset \operatorname{conv}\left(e^{K(h)}\right)$. This is a direct consequence of Proposition 3.31 and the property of $\Phi_\varphi$ in (3.22) applied to $\varphi = e_1$.

# Appendix A

# Projective and Inductive Limits

We establish the general definition of projective and inductive limits in order to classify the considered spaces in this thesis in that abstract terminology. It will turn out that some useful functional analytic properties are always guaranteed for our spaces.
This part of the appendix is an excerpt from [Mor94, pages 239-255].
In the following, we call $(I, \prec)$ a directed set if $\prec$ is an order on the set $I$ and for all $\alpha, \beta \in I$ there is a $\gamma \in I$ such that $\alpha \prec \gamma$ and $\beta \prec \gamma$.
Let $(E_\alpha : \alpha \in I)$ be a collection of linear spaces, where $I$ is a directed index set. We assume that $\rho_\alpha^\beta : E_\alpha \to E_\beta$ are linear mappings for all $\beta \prec \alpha$ from $I$ such that $\rho_\alpha^\alpha = \mathrm{id}$ and $\rho_\beta^\gamma \circ \rho_\alpha^\beta = \rho_\alpha^\gamma$ for $\gamma \prec \beta \prec \alpha$.

**Definition A.1** The above family $\left(E_\alpha, \rho_\alpha^\beta : \beta \prec \alpha; \alpha, \beta \in I\right)$ is called a *decreasing family of linear spaces.*

**Theorem A.2** *Let $\left(E_\alpha, \rho_\alpha^\beta : \beta \prec \alpha; \alpha, \beta \in I\right)$ be a decreasing family of linear spaces. Then there are a unique linear space $E$ and mappings $\rho^\alpha : E \to E_\alpha$ for all $\alpha \in I$ such that:*

*(1) $\rho_\alpha^\beta \circ \rho^\alpha = \rho^\beta$ for all $\beta \prec \alpha$ from $I$.*

*(2) For any linear space $Y$ and linear mappings $f_\alpha : Y \to E_\alpha$ such that $\rho_\alpha^\beta \circ f_\alpha = f_\beta$ for all $\beta \prec \alpha$, there is a unique linear mapping $f : Y \to E$ with $\rho^\alpha \circ f = f_\alpha$ for all $\alpha \in I$.*

**Definition A.3** The unique space $E$ in Theorem A.2 is the *projective limit* of the decreasing family, and the mappings $\rho^\alpha$ are called the *canonical mappings.*

Let $(E_\alpha : \alpha \in I)$ be a collection of linear spaces, where $I$ is a directed index set. Assume that $\rho_\alpha^\beta : E_\alpha \to E_\beta$ are linear mappings for all $\alpha \prec \beta$ from $I$ such that $\rho_\alpha^\alpha = \mathrm{id}$ and $\rho_\beta^\gamma \circ \rho_\alpha^\beta = \rho_\alpha^\gamma$ for $\alpha \prec \beta \prec \gamma$.

**Definition A.4** The above family $\left(E_\alpha, \rho_\alpha^\beta : \alpha \prec \beta; \alpha, \beta \in I\right)$ is called an *increasing family of linear spaces.*

**Theorem A.5** *Let $\left(E_\alpha, \rho_\alpha^\beta : \alpha \prec \beta; \alpha, \beta \in I\right)$ be an increasing family of linear spaces. Then there are a unique linear space $E$ and mappings $\rho_\alpha : E_\alpha \to E$ for all $\alpha \in I$ such that:*

*(1) $\rho_\beta \circ \rho_\alpha^\beta = \rho_\alpha$ for all $\alpha \prec \beta$.*

*(2) For any linear space $Y$ and linear mappings $f_\alpha : E_\alpha \to Y$ such that $f_\alpha = f_\beta \circ \rho_\alpha^\beta$ for all $\alpha \prec \beta$, there is a unique linear mapping $f : E \to Y$ with $f_\alpha = f \circ \rho_\alpha$ for all $\alpha \in I$.*

**Definition A.6** The unique space $E$ in Theorem A.5 is the *inductive limit* of the increasing family. The mappings $\rho_\alpha$ are called the *canonical mappings.*

In case that the linear spaces $E_\alpha$ are endowed with a locally convex topology, the projective and inductive limits are endowed with a topology in the following way.

**Definition A.7**

(1) Let $\left(E_\alpha, \rho_\alpha^\beta : \beta \prec \alpha;\ \alpha, \beta \in I\right)$ be a decreasing family of linear spaces where each $E_\alpha$ is endowed with a locally convex topology such that the mappings $\rho_\alpha^\beta$ are continuous. We endow the projective limit $E$ with the weakest locally convex topology such that the canonical mappings are continuous. With this topology, $E$ is called the *projective limit* of $\left(E_\alpha, \rho_\alpha^\beta : \beta \prec \alpha;\ \alpha, \beta \in I\right)$ in the category of locally convex spaces. We write $E = \mathrm{ProjLim}(E_\alpha, \rho_\alpha^\beta : \beta \prec \alpha;\ \alpha, \beta \in I)$ or shortly $\mathrm{ProjLim}(E_\alpha : \alpha \in I)$.

(2) Let $\left(E_\alpha, \rho_\alpha^\beta : \alpha \prec \beta;\ \alpha, \beta \in I\right)$ be an increasing family of linear spaces where each $E_\alpha$ is endowed with a locally convex topology such that the mappings $\rho_\alpha^\beta$ are continuous. We endow the inductive limit $E$ with the strongest locally convex topology such that the canonical mappings are continuous. With this topology, $E$ is called the *inductive limit topology* of $\left(E_\alpha, \rho_\alpha^\beta : \alpha \prec \beta;\ \alpha, \beta \in I\right)$ in the category of locally convex spaces. We write $E = \mathrm{IndLim}(E_\alpha, \rho_\alpha^\beta : \alpha \prec \beta;\ \alpha, \beta \in I)$ or shortly $\mathrm{IndLim}(E_\alpha : \alpha \in I)$.

**Theorem A.8** *Let* $E = \mathrm{ProjLim}(E_\alpha, \rho_\alpha^\beta : \beta \prec \alpha;\ \beta, \alpha \in I)$ *where the topology on* $E_\alpha$ *is induced by a system of seminorms* $\{p_{\alpha,\lambda} : \lambda \in I_\alpha\}$ *for each* $\alpha \in I$*. Then the topology on* $E$ *is induced by the system of seminorms* $\{p_{\alpha,\lambda} \circ \rho^\alpha : \alpha \in I, \lambda \in I_\alpha\}$.

In this thesis we are concerned with a special case of the above concept:

**Definition A.9**

(1) Assume that $(E_n : n \in \mathbb{N})$ is a sequence of Banach spaces and $(\rho_{n+1}^n : E_{n+1} \to E_n)_{n\in\mathbb{N}}$ a sequence of linear and compact mappings. Then $E = \mathrm{ProjLim}(E_n : n \in \mathbb{N})$ (with respect to the mappings $\rho_{n+1}^n$) is called a *Fréchet-Schwartz space* or shortly *FS space*.

(2) Assume that $(E_n : n \in \mathbb{N})$ is a sequence of Banach spaces and $(\rho_n^{n+1} : E_n \to E_{n+1})_{n\in\mathbb{N}}$ is a sequence of linear and compact mappings. Then $E = \mathrm{IndLim}(E_n : n \in \mathbb{N})$ (with respect to the mappings $\rho_n^{n+1}$) is called a *DFS space*.

**Remark A.10** Note that different decreasing or increasing families of locally convex spaces can generate the same projective or inductive limit, respectively. Consider the space $\mathrm{Exp}(K)$ from Definition 1.25 and Theorem 1.26. If $(K_n)_{n\in\mathbb{N}}$ is a decreasing family of compact and convex subsets of $\mathbb{C}$ such that $K_n^\circ \supset K_{n+1}$ and $\bigcap_{n\in\mathbb{N}} K_n = K$, then

$$\mathrm{Exp}(K) = \mathrm{ProjLim}(\mathrm{Exp}_n(K) : n \in \mathbb{N}) = \mathrm{ProjLim}(\mathrm{Exp}(K_n) : n \in \mathbb{N}).$$

This is an immediate consequence of Theorem A.8.

**Theorem A.11** *The dual space of an FS space endowed with the strong topology is a DFS space. Conversely, the strong dual of a DFS space is an FS space.*

**Theorem A.12** *Every FS space and every DFS space is a Montel space. This implies that these spaces are always reflexive.*

**Theorem A.13** *Let* $M$ *be a closed subspace of an FS space or a DFS space, then* $M$*, endowed with the relative topology, is an FS space or a DFS spaces, respectively.*

# Appendix B

# Results from Complex Analysis

We establish further results from complex analysis that are applied in this thesis: The first theorem is due to N. Aronszajn (cf. [Aro35]). For a short proof, we refer to [MW98].

**Theorem B.1** *Let $\Omega_1$, $\Omega_2$ be open sets in the extended plane and $f \in H(\Omega_1 \cap \Omega_2)$. Then there are $f_1 \in H(\Omega_1)$ and $f_2 \in H(\Omega_2)$ such that $(f_1 + f_2)|_{\Omega_1 \cap \Omega_2} = f$.*

**Corollary B.2** Let $K$ be a compact subset of $\mathbb{C}$ and $f \in H_0(\mathbb{C} \setminus K)$. Then for any two compact sets $K_1$, $K_2$ with $K_1 \cup K_2 = K$, there are functions $f_1 \in H_0(\mathbb{C} \setminus K_1)$ and $f_2 \in H_0(\mathbb{C} \setminus K_2)$ such that $(f_1 + f_2)|_{\mathbb{C} \setminus K} = f$.

The next result, due to Q. I. Rahman [Rah69], is a generalization of Fuchs' theorem (cf. [Boa54] Theorem 9.5.1).

**Theorem B.3** (cf. [Rah69, Theorem 1]) *Assume that $f$ is holomorphic in some open set that contains $\{\mathrm{Re}(z) \geq 0\}$, and $|f(z)| = O(e^{c|z|})$ for some $c > 0$. Let further $(\lambda_n = r_n\, e^{i\Theta_n})_{n \in \mathbb{N}}$ be a sequence of zeros of $f$ in $\{z : \mathrm{Re}(z) > 0\}$ ordered with respect to their modulus and such that $r_{n+1} - r_n > \delta > 0$ for all $n \in \mathbb{N}$. Then $f \equiv 0$ if and only if*

$$\limsup_{r \to \infty} \frac{\exp\left( \sum_{r_n < r} \frac{\cos(\Theta_n)}{r_n} \right)}{r^{\frac{c}{\pi}}} = \infty.$$

**Theorem B.4** (cf. [BG95, Corollary 4.1.16]) *If $f$ is a function of exponential type zero that satisfies $|f(n)| = O(|n|^p)$ for some positive integer $p$, then $f$ is a polynomial of degree at most $p$.*

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