Geometry of optimal codebooks and constructive quantization

Dissertation

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Abstract

Given a random element X in the separable Banach space $(E, \|\cdot\|)$ with law $\mathbb{P}^X = \mu$, the optimal quantization problem consists of finding a set $\alpha \subset E$ minimizing

$$e_r^r(X, E; \alpha) = \mathbb{E} \min_{a \in \alpha} \|X - a\|^r,$$

over all subsets $\alpha \subset E$ such that $\operatorname{card} \alpha \leq n$, given a natural number $n \in \mathbb{N}$ and a constant $r \in (0, \infty)$.

There are many practical problems giving rise to the analysis of such problems, beginning with the invention of Pulse Code Modulation (PCM) in the 40s of the last century, the digitalization of data (such as in the formats JPEG or in MPEG) and more recently also in financial applications.

From a theoretical point of view, the first questions being treated and answered about the quantization problem concern the asymptotic behavior of the minimal quantization error

$$e_{n,r}(X, E) = \min_{\alpha \subset E, \operatorname{card}(\alpha) \leq n} e_r(X, E; \alpha)$$

as n tends to ∞, as well as the existence of optimal quantizers, i.e. subsets $\alpha \subset E$ achieving the minimal quantization error.

While the existence of explicit formulas for optimal quantizers seems to be out of reach for most of the interesting examples, the aim of this thesis is to identify and estimate geometric properties of optimal quantizers. In particular those are for a given sequence of optimal codebooks $(\alpha_n)_{n \in \mathbb{N}}$ the asymptotic behaviour of the quantization radius

$$\rho(\alpha_n) = \max\{\|a\| : a \in \alpha_n\},$$

the convergences (in a useful sense) of the sequences

$$\left(\frac{\alpha_n}{\rho(\alpha_n)}\right)_{n \in \mathbb{N}} \quad \text{and} \quad \left(\frac{\operatorname{conv}(\alpha_n)}{\rho(\alpha_n)}\right)_{n \in \mathbb{N}},$$

as well as the asymptotics of the local characteristics

$$\begin{aligned} \mu(V_a(\alpha_n)), &\quad \mu(W_a(\alpha_n)), \\ \mu_r(V_a(\alpha_n)), &\quad \mu_r(W_a(\alpha_n)), \\ \int_{V_a(\alpha_n)} \operatorname{dist}(x, \alpha_n)^r d\mu(x), &\quad \int_{W_a(\alpha_n)} \operatorname{dist}(x, \alpha_n)^r d\mu(x), \\ \operatorname{dist}(a, \alpha_n \backslash \{a\}), & \end{aligned} \tag{1}$$

where for $a \in \alpha_n$ the Voronoi regions $V_a(\alpha_n), W_a(\alpha_n)$ are given as

$$\begin{aligned} V_a(\alpha_n) &= \{x \in E : \|x - a\| = \operatorname{dist}(x, \alpha_n)\}, \\ W_a(\alpha_n) &= \{x \in E : \|x - a\| < \operatorname{dist}(x, \alpha_n \backslash \{a\})\}. \end{aligned}$$

While chapter 1 gives an overview on some basics in quantization theory and the theory of Gaussian measures, we will establish in chapter 2 important fundamentals, including the estimation of the increments of quantization errors

$$\Delta_{n,r}(X,E) = e^r_{n,r}(X,E) - e^r_{n+1,r}(X,E),$$

which is needed to make use of so-called micro-macro inequalities. Those are inequalities, which relate asymptotics of these increments to local characteristics of optimal quantizers.

In chapter 3, we estimate in virtue of the results from chapter 2 the asymptotics for the quantization radius $\rho(\alpha_n)$ and limits of the sequences $\left(\frac{\alpha_n}{\rho(\alpha_n)}\right)_{n\in\mathbb{N}}$ and $\left(\frac{\mathrm{conv}(\alpha_n)}{\rho(\alpha_n)}\right)_{n\in\mathbb{N}}$ for sequences of optimal quantizers $(\alpha_n)_{n\in\mathbb{N}}$ for broad classes of r.e.'s X in $\mathbb{R}^d$, including e.g. the non-singular normal distribution, the exponential distribution as well as distributions with polynomial tails, such as the Students t-distributions.

In chapter 4, we extend the results given in chapter 3 for Gaussian r.e.'s to the infinite dimensional case. Here, we will particularly make use of the results established in chapter 2 concerning the estimations for $\Delta_{n,r}(X,E)$.

Chapter 5 is devoted to the analysis of the asymptotics of the local characteristic defined in equation (1) for sequences of optimal quantizers $(\alpha_n)_{n\in\mathbb{N}}$ for r.e.'s X in $\mathbb{R}^d$.

Another subject treated in this thesis (chapter 6) concerns the construction of asymptotically optimal quantizers for stochastic processes $(X_t)_{t\in I}$, being understood as the realization of a r.e. X in a (Banach) function space $E \subset \{f : I \to \mathbb{R}\}$.

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Furthermore, my thanks are due to Prof. Dr. S. Graf for being the co-advisor for the dissertation and for constructive feedback in the process of the finalization of this thesis.
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Zusammenfassung

Für eine gegebenes Zufallselement X mit zugehöriger Verteilung $\mathbb{P}^X = \mu$ und Werten in einem separablen Banachraum $(E, \|\cdot\|)$, besteht das optimale Quantisierungsproblem aus der Suche nach einer Menge $\alpha \subset E$ welche

$$e_r^r(X, E; \alpha) = \mathbb{E} \min_{a \in \alpha} \|X - a\|^r,$$

über alle Teilmengen $\alpha \subset E$ minimiert, gegeben $\operatorname{card} \alpha \leq n$ für eine gegebene natürliche Zahl $n \in \mathbb{N}$ sowie $r \in (0, \infty)$.

Beginnend mit der Erfindung der Pulse-Code-Modulation (PCM) in den 40-er Jahren des letzten Jahrhunderts, über die Digitalisierung von Daten (wie etwa in den Formaten JPEG oder MPEG) bis hin zur Anwendung in vielen Bereichen der Finanzmathematik, gibt es viele praktische Probleme welche die Untersuchung dieses Problems motivieren. Vom theoretischen Standpunkt her wurden zunächst Fragestellungen bezüglich des asymptotischen Verhaltens des minimalen Quantisierungsfehlers

$$e_{n,r}(X, E) = \min_{\alpha \subset E, \operatorname{card}(\alpha) \leq n} e_r(X, E; \alpha)$$

für n gegen ∞, sowie die Existenz optimaler Quantisierer untersucht, sprich Teilmengen $\alpha \subset E$, welche den minimalen Quantisierungsfehler erreichen.

Während die explizite Berechnung optimaler Quantisierer in den meisten interessanten Fällen nicht möglich erscheint, sollen in der vorliegenden Arbeit geometrische Eigenschaften optimaler Quantisierer herausgearbeitet werden. Im Besonderen sind dies, für eine gegebene Folge optimaler Quantisierer $(\alpha_n)_{n \in \mathbb{N}}$ das asymptotische Verhalten des Quantisierungsradius

$$\rho(\alpha_n) = \max\{\|a\| : a \in \alpha_n\},$$

die Konvergenz (in einem zu definierenden Sinne) der Folgen

$$\left(\frac{\alpha_n}{\rho(\alpha_n)}\right)_{n \in \mathbb{N}} \quad \text{und} \quad \left(\frac{\operatorname{conv}(\alpha_n)}{\rho(\alpha_n)}\right)_{n \in \mathbb{N}},$$

sowie das asymptotische Verhalten folgender "lokaler" Eigenschaften

$$\begin{aligned} \mu(V_a(\alpha_n)), &\quad \mu(W_a(\alpha_n)), \\ \mu_r(V_a(\alpha_n)), &\quad \mu_r(W_a(\alpha_n)), \\ \int_{V_a(\alpha_n)} \operatorname{dist}(x, \alpha_n)^r d\mu(x), &\quad \int_{W_a(\alpha_n)} \operatorname{dist}(x, \alpha_n)^r d\mu(x), \\ \operatorname{dist}(a, \alpha_n \backslash \{a\}), & \end{aligned} \tag{2}$$

Hierbei sind für gegebenes $a \in \alpha_n$ die Voronoizellen $V_a(\alpha_n), W_a(\alpha_n)$ gegeben durch

$$\begin{aligned} V_a(\alpha_n) &= \{x \in E : \|x - a\| = \operatorname{dist}(x, \alpha_n)\}, \\ W_a(\alpha_n) &= \{x \in E : \|x - a\| < \operatorname{dist}(x, \alpha_n \backslash \{a\})\}. \end{aligned}$$

Während im ersten Kapitel Grundkenntnisse aus dem Bereich der Quantisierungstheorie sowie der Theorie von Gaussmassen wiederholt werden, werden wir im zweiten Kapitel wichtige Grundlagen für die folgenden Untersuchungen legen, was insbesondere die Abschätzung des asymptotischen Verhaltens der Zuwächse der Quantisierungsfehler

$$\Delta_{n,r}(X,E) = e^r_{n,r}(X,E) - e^r_{n+1,r}(X,E)$$

beinhaltet. Diese werden benötigt um einen Nutzen aus den sogenannten Micro-Macro Ungleichungen zu ziehen, welche diese Zuwächse mit lokalen Eigenschaften optimaler Quantisierer in Verbindung bringen.

In Kapitel 3 werden wir vermöge der Ergebnisse aus Kapitel 2, das asymptotische Verhalten des Quantisierungsradius $\rho(\alpha_n)$ und die Grenzwerte der Folgen $\left(\frac{\alpha_n}{\rho(\alpha_n)}\right)_{n\in\mathbb{N}}$ und $\left(\frac{\mathrm{conv}(\alpha_n)}{\rho(\alpha_n)}\right)_{n\in\mathbb{N}}$ für Folgen optimaler Quantisierer $(\alpha_n)_{n\in\mathbb{N}}$ für eine grosse Klasse von Zufallselementen X in $\mathbb{R}^d$ untersuchen, welches insbesondere die nicht-singuläre Normalverteilung, die Exponentialverteilung sowie Verteilungen mit polynomiell fallenden Dichten beinhaltet, wie etwa die Students t-Verteilung.

In Kapitel 4 werden die Resultate aus Kapitel 3 auf Gaussmasse auf unendlichdimsensionalen Banachräumen erweitert. Im Besonderen werden wir dabei einen Nutzen aus der in Kapitel 2 ermittelten Asymptotik der Zuwächse $\Delta_{n,r}(X,E)$ ziehen.

Kapitel 5 ist der Untersuchung der in Gleichung (2) beschriebenen lokalen Eigenschaften von Folgen optimaler Quantisierer $(\alpha_n)_{n\in\mathbb{N}}$ für Zufallsvektoren X mit Werten in $\mathbb{R}^d$ gewidmet.

Ein weiteres Thema, welches in dieser Arbeit behandelt wird (Kapitel 6) betrifft die Konstruktion von Folgen asymptotisch optimaler Quantisierer für Stochastische Prozesse $(X_t)_{t\in I}$, betrachtet als die Realisation eines Zufallsexperimentes X in einem (Banachschen) Funktionenraum $E \subset \{f : I \to \mathbb{R}\}$.

Danksagung Zunächst möchte ich meinen Dank meinem Betreuer Prof. Dr. H. Luschgy aussprechen, für die Einführung in das interessante und breite Feld der Quantisierungstheorie, und seine Unterstützung bei der Fertigstellung meiner Dissertation auf diese unkonventionelle Weise.
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Contents

Chapter 1

Preliminaries

Prior to the analysis of the main topics of this thesis, this initial chapter shall give the reader a sufficient background enabling to follow the discussions in the subsequent chapters. Particularly, we will

- clarify the required notations,
- outline some mathematical basics to be able to analyze the quantization problems as discussed later on, and
- introduce into the range of results developed so far in quantization theory and related areas.

The Banach Space setting: Throughout this thesis, let $(E, \|\cdot\|)$ be a real separable Banach Space. We denote by $(E', \|\cdot\|_{E'})$ the topological dual space to $(E, \|\cdot\|)$, i.e.

$$E' := \{y : E \to \mathbb{R}, y \text{ continuous and linear }\}$$

equipped with the operator norm

$$\|y\|_{E'} := \sup\{\langle x, y\rangle : \; x \in \mathbb{E}, \|x\| \le 1\},$$

for $y \in E'$. Here

$$\langle \cdot, \cdot \rangle : E \times E' \to \mathbb{R}, \quad \langle x, y\rangle \mapsto y(x)$$

denotes the corresponding bilinear form. Furthermore, we write for $\epsilon \geq 0$ and $a \in E$

$$B_{\|\cdot\|}(a, \epsilon) = B(a, \epsilon) = \{x \in E : \|x - a\| \le \epsilon\}.$$

The Probability Space setting: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an abstract probability space. To ensure the existence of the random variables studied below, we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is always sufficiently rich. For a Borel random element (abbr. r.e.) $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (E, \mathcal{B}(E))$ we set

$$\mu(A) := \mathbb{P}^X(A) := \mathbb{P}(X \in A)$$

for all $A \in \mathcal{B}(E)$, where $\mathcal{B}(E)$ denotes the σ-field induced by the open sets in $(E, \|\cdot\|)$. For some $r \in (0, \infty)$, we set the r-th moment of X

$$\|X\|_{L_r(\mathbb{P},E)} := \left(\mathbb{E}\|X\|^r\right)^{\frac{1}{r}},$$

and denote by

$$\begin{aligned} L_r(\mathbb{P}, E) := L_r(\Omega, \mathcal{F}, \mathbb{P}, (E, \|\cdot\|)) := \\ \{Y : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E)), \|Y\|_{L_r(\mathbb{P},E)} < \infty\} \end{aligned}$$

set of r.e.'s in $(E, \|\cdot\|)$ with finite r-th moment.

Subsequently, the Banach space E shall always be understood as being attached with its norm $\|\cdot\|$ and the induced Borel σ-algebra $\mathcal{B}(E)$.

1.1 Gaussian random elements

Gaussian random elements play a key role in many interesting applications. The aim of this section is to present some important properties and inequalities this class of r.e.'s admits. Let $|\cdot| : \mathbb{R} \to \mathbb{R}$ denote the absolute value in $\mathbb{R}$. Given the existence, we define the mean $\mathbb{E}X$ and the variance $\operatorname{Var} X$ of a random variable X in $(\mathbb{R}, |\cdot|)$ by

$$\mathbb{E}X := \int X d\mathbb{P} = \int x d\mu(x), \qquad \text{(mean)}$$

and

$$\operatorname{Var}(X) := \int (X - \mathbb{E}X)^2 d\mathbb{P} = \int (x - \mathbb{E}X)^2 d\mu(x). \qquad \text{(variance)}$$

Definition 1.1.1. A Borel r.e. $X : (\Omega, \mathcal{F}, \mathbb{P}) \to E$ is called *Gaussian*, if for every $y \in E'$ the random variable $\langle X, y\rangle$ is either Gaussian (also called normal) or Dirac in $(\mathbb{R}, |\cdot|)$, i.e. the corresponding distribution function admits the representation

$$\Phi_{\nu,\sigma^2}(x) := \mathbb{P}(\langle X, y\rangle \in (\infty, x]) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} \exp\left(-\frac{1}{2}\left(\frac{t-\nu}{\sigma}\right)^2\right) d\lambda(t),$$

with constants $\nu \in \mathbb{R}$ and $\sigma > 0$, or

$$\mathbb{P}(\langle X, y\rangle = \nu) = \delta_\nu(\{\nu\}) = 1, \quad \mathbb{P}(\langle X, y\rangle \neq \nu) = \delta_\nu(\{\nu\}^c) = 0,$$

for a constant $\nu \in \mathbb{R}$. In both cases

$$\mathbb{E}X = \nu \quad \text{and} \quad \operatorname{Var} X = \sigma^2,$$

where $\sigma^2 = 0$ if $\langle X, y\rangle$ is Dirac.
We say X is centered if $\nu = 0$ for all $y \in E'$. Accordingly, we call the law $\mu = \mathbb{P}^X$ (centered) Gaussian if X is (centered) Gaussian.
For the Gaussian distribution $\mathbb{P}^{\langle X,y\rangle}$, we will also write $\mathcal{N}(\nu, \sigma^2) = \mathbb{P}^{\langle X,y\rangle}$ with the parameters as above.

The following famous result, which is due to Fernique, ensures the existence of the r-th moments for Gaussian random elements.

Theorem 1.1.2. *(see [Bog98, Theorem 2.8.5]) For every Gaussian r.e. X in E there exists a constant $\tau > 0$ such that*

$$\int \exp\left(\tau \|x\|^2\right) d\mu(x) < \infty. \tag{1.1}$$

As a consequence of

$$\|x\|^r \leq \frac{1}{\tau^{\lceil \frac{r}{2} \rceil}} \frac{\left\lceil \frac{r}{2} \right\rceil!}{\left\lceil \frac{r}{2} \right\rceil!} \left(\tau \|x\|^2\right)^{\lceil \frac{r}{2} \rceil} \leq \frac{1}{\tau^{\lceil \frac{r}{2} \rceil}} \left\lceil \frac{r}{2} \right\rceil! \exp\left(\tau \|x\|^2\right)$$

for $\|x\| \geq 1$, Theorem 1.1.2 implies the finiteness of $\|X\|_{L_r(\mathbb{P},E)}$ for every $r > 0$.
In the sequel, we will always consider centered Gaussian random elements. Indeed, every arbitrary Gaussian random element can be transformed into a centered one by considering $X - \mathbb{E}X$ instead of X, where

$$\mathbb{E}X := \int X d\mathbb{P} = \int x d\mu(x)$$

is called the mean of X and the integral is defined as a Bochner integral. Furthermore, we will assume throughout that $\mu = \mathbb{P}^X \neq \delta_0$. In terms of quantization theories, both limitations do not significantly change the quantization problems as we will see in the following sections.

Covariance Operator and Cameron-Martin space: We introduce some operators and subspaces that are closely related to the centered Gaussian random element X in E. First note, that $E' \subset \mathcal{L}_2(\mu)$, where

$$\mathcal{L}_2(\mu) := \{y : (E, \mathcal{B}(E)) \to (\mathbb{R}, \mathcal{B}(R)), \int (y(x))^2 d\mu(x) < \infty\},$$

which is due to the finiteness of the second moment for Gaussian r.e.'s in $\mathbb{R}$. We set

$$L_2(\mu) = \{[y], y \in \mathcal{L}_2(\mu)\}$$

where $[y] = \{y' \in \mathcal{L}_2(\mu), y = y'\mu - a.s.\}$. Then, $E'_\mu := \overline{E'}^{L_2(\mu)}$ attached with the inner product

$$(y, z)_{L_2(\mu)} := \int y(x) z(x) d\mu(x),$$

is a Hilbert space, where $\overline{E'}^{L_2(\mu)}$ denotes the closure of E' in $L_2(\mu)$.

Definition 1.1.3. (covariance operator) Let X be a (centered) Gaussian r.e. in E. Its linear and continuous covariance operator $C_X : E' \to E$ is defined by

$$C_X(y) := \int \langle X, y\rangle X d\mathbb{P},$$

where the right-hand side integral exists, again due to Fernique's Theorem (Theorem 1.1.2), as a Bochner integral. It may also be interpreted as a Pettis type integral, which is uniquely characterized by the identity

$$\left\langle \int \langle X, y\rangle X d\mathbb{P}, z \right\rangle = \int \langle X, y\rangle \langle X, z\rangle d\mathbb{P}$$

for all $z \in E'$.

Its image $C_X(E')$ is contained in the Cameron-Martin space $\mathcal{H}_\mu$ which is defined by

$$\mathcal{H}_\mu := \{x \in E : \|x\|_{\mathcal{H}_\mu} < \infty\},$$

attached with the norm

$$\|x\|_{\mathcal{H}_\mu} := \sup\{\langle x, y\rangle : y \in E', \|y\|_{L_2(\mu)} \leq 1\}.$$

We have the following characterization of $\mathcal{H}_\mu$:

Lemma 1.1.4. *Let S_μ be the formally extended operator of C_X onto ${E'}_\mu$, i.e. $S_\mu : {E'}_\mu \to E$, $f \mapsto \mathbb{E}f(X)X$. Then, S_μ is also well defined and establishes an isometric isomorphism between ${E'}_\mu$ and $\mathcal{H}_\mu$.*

Proof. On the one hand, we have by definition for each $y \in {E'}_\mu$

$$\|S_\mu(y)\|_{\mathcal{H}_\mu} = \sup\{(y, z)_{L_2(\mu)} : z \in E', \|z\|_{L_2(\mu)} \leq 1\} = \|y\|_{L_2(\mu)} < \infty \tag{1.2}$$

and thus $S_\mu({E'}_\mu) \subset \mathcal{H}_\mu$, and $\|S_\mu(y)\|_{\mathcal{H}_\mu} = \|y\|_{L_2(\mu)}$ for all $y \in {E'}_\mu$. Conversely, we may define for $x \in \mathcal{H}_\mu$ the functional $\phi_x : E' \to \mathbb{R}$ with $\phi_x(y) := \langle x, y\rangle$ which can be extended to a bounded linear functional on ${E'}_\mu$. Riesz's Theorem yields the existence of a $z \in {E'}_\mu$ with

$$\langle x, y\rangle = \phi_x(y) = (z, y)_{L_2(\mu)} = \langle S_\mu(z), y\rangle \tag{1.3}$$

for all $y \in E'$, so that $\mathcal{H}_\mu \subset S_\mu({E'}_\mu)$. In virtue of the definition of ${E'}_\mu$ this element z is unique, wherefore we are allowed to write $z = S_\mu^{-1}(x)$. □

This allows us now to define in consistence with the definition of $\|\cdot\|_{\mathcal{H}_\mu}$ (see equation 1.2) an inner product on $\mathcal{H}_\mu$ by setting for $h_i \in \mathcal{H}_\mu$

$$(h_1, h_2)_{\mathcal{H}_\mu} := \left(S_\mu^{-1}(h_1), S_\mu^{-1}(h_2)\right)_{L_2(\mu)}, \tag{1.4}$$

where $S_\mu^{-1}(h_i), i = 1, 2$ are given via equation (1.3).

Additionally, S_μ provides a factorization of C_X, i.e.

$$C_X = S_\mu S_\mu^*,$$

where $S_\mu^* : E' \to E'_\mu$ denotes the adjoint of S_μ, which is in view of

$$\langle S_\mu z, y\rangle = (y, z)_{L_2(\mu)}$$

for all $y, z \in E'$ the natural embedding from $E' \hookrightarrow E'_\mu$.

We denote by $\mathcal{K}_\mu$ the unit ball in $\mathcal{H}_\mu$ which is known to be compact in $(E, \|\cdot\|)$ (see e.g. [Bog98, Corollary 3.2.4]). The norm of the natural embedding $j_\mu : \mathcal{H}_\mu \to E$ we denote

$$\sigma(\mu) := \|j_\mu\| = \sup\{\|h\| : h \in \mathcal{K}_\mu\} = \left\|j_\mu^*\right\| = \sup\{\|y\|_{L_2(\mu)} : y \in B_{\|\cdot\|_{E'}}(0,1)\}.$$

Since $\mu \neq \delta_0$, we have $\sigma(\mu) > 0$.

Finally, note that the support of the measure μ admits a representation in form of the Cameron-Martin space $\mathcal{H}_\mu$, i.e.

$$\overline{\mathcal{H}_\mu} = \operatorname{supp}(\mu),$$

where supp denotes the support and $\overline{A}$ the closure of a set $A \subset E$ in $(E, \|\cdot\|)$.

Basic properties and some useful formulas: Properties of Gaussian random elements have been studied extensively. We consider a centered Gaussian random element X in E with law μ. For sets $A, B \subset E$ we will denote the Minkowski sum

$$A + B = \{x + y : x \in A, y \in B\}.$$

Additionally for $\lambda \in \mathbb{R}$

$$\lambda A = \{\lambda x : x \in A\}, \quad \lambda \geq 0,$$

as well as

$$\lambda A = \varnothing, \quad \lambda < 0.$$

Proposition 1.1.5. *(Erhard's inequality, see [Bog98, Theorem 4.2.2]) Let $A, B \in \mathcal{B}(E)$ be convex and $\lambda \in [0,1]$. Then*

$$\Phi^{-1}\left(\mu_*\left(\lambda A + (1-\lambda)B\right)\right) \geq \lambda\Phi^{-1}\left(\mu(A)\right) + (1-\lambda)\Phi^{-1}\left(\mu(B)\right),$$

where $\Phi(x) := \Phi_{0,1}(x)$ and μ_* denoting the inner measure of μ.

As a consequence, one obtains

Proposition 1.1.6. *(Anderson inequality, see [Bog98, Theorem 2.8.10.]) For every absolutely convex set $A \in \mathcal{B}(E), x \in E$ and every $t \in [0,1]$*

$$\mu(A + x) \leq \mu(A + tx).$$

Proposition 1.1.7. *([LP09, Theorem 1]) Let $(e_i)_{i\in\mathbb{N}}$ be a Parseval frame for $\mathcal{H}_\mu$, i.e. $\sum_{i=1}^\infty e_i(f,e_i)_{\mathcal{H}_\mu}$ converges in $\mathcal{H}_\mu$ and*

$$\sum_{i=1}^\infty e_i(f,e_i)_{\mathcal{H}_\mu} = f$$

for all $f \in \mathcal{H}_\mu$. Then, $(e_i)_{i\in\mathbb{N}}$ is admissible for X in E, i.e for every sequence $(\xi_i)_{i\in\mathbb{N}}$ of $\mathcal{N}(0,1)$-distributed r.e.'s, $\sum_{i=1}^\infty e_i\xi_i$ converges a.s. in E and

$$\mathbb{P}^{\sum_{i=1}^\infty e_i\xi_i} = \mu.$$

Proposition 1.1.8. *(Isoperimetric inequality, see [Bog98, Theorem 4.3.3.]) For every $A \in \mathcal{B}(E)$ it holds*

$$\mu(A + t\mathcal{K}_\mu) \geq \Phi(a+t)$$

with $a := \Phi^{-1}(\mu(A))$.

We set $J(x,\epsilon) := \inf_{h\in B(x,\epsilon)} J(h)$ with $J(h) := \frac{\|h\|^2_{\mathcal{H}_\mu}}{2}$, which is finite iff $h \in \mathcal{H}_\mu$. Note, that it follows by definition that

$$\{x : J(x,\epsilon) \leq \frac{t^2}{2}\} = \epsilon B(0,1) + t\mathcal{K}_\mu.$$

Proposition 1.1.9. *(Cameron Martin Formula, see [Bog98, Corrolary 2.4.3.]) For all $h \in \mathcal{H}_\mu$ and Borel sets $A \in \mathcal{B}(E)$*

$$\mu(A+h) = \exp(-\frac{\|h\|^2_{\mathcal{H}_\mu}}{2}) \int_A \exp\big((S_\mu^{-1}(h))(x)\big)\, d\mu(x).$$

Remark 1.1.10. Considering $A = B(0,\epsilon)$ in Proposition 1.1.9, one obtains in virtue of Jensen's inequality and the symmetry of μ

$$\begin{aligned}\mu(B(h,\epsilon)) &\geq \exp(-\frac{\|h\|^2_{\mathcal{H}_\mu}}{2})\mu(B(0,\epsilon))\times \\ &\qquad \exp\left(\frac{1}{\mu(B(0,\epsilon))}\left(\left(\int_{B(0,\epsilon)} (S_\mu^{-1}(h))(x)d\mu(x)\right)\right)\right) \\ &= \exp\Big(-\frac{\|h\|^2_{\mathcal{H}_\mu}}{2}\Big)\mu(B(0,\epsilon)),\end{aligned}$$

where the last equality is a consequence of the symmetry of μ and the fact that $S_\mu^{-1}(h) \in \overline{E'}^{L_2(\mu)}$.

Furthermore, one can show

Proposition 1.1.11. *(Estimation of shifted balls, see [LS01, Theorem 3.2]) For $\epsilon > 0$, $x \in E$ and $a \in [0,1]$*

$$\mu(B(x,\epsilon)) \geq \exp(-J(x,a\epsilon))\mu\left(B(0,(1-a)\epsilon)\right).$$

Furthermore,

$$\mu(B(x,\epsilon)) \leq \exp(-J(x,\epsilon))\mu\left(B(0,\epsilon)\right).$$

In addition to the existence of the r-th moments of X, which is an immediate consequence of Theorem 1.1.2, one can prove

Proposition 1.1.12. *(Equivalence of moments, see [LT91, Corollary 3.2.], [LS01, Theorem 2.5]) For each $0 < p,q < \infty$ there exist real constants $K_{p,q}$ depending on p and q solely such that for every Gaussian r.e. X in E*

$$\|X\|_{L_p(\mathbb{P},E)} \leq K_{p,q} \|X\|_{L_q(\mathbb{P},E)}.$$

As a consequence of integration by parts, one derives

Lemma 1.1.13. *For every $x > 0$,*

$$\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x}-\frac{1}{x^3}\right)\exp(-\frac{x^2}{2}) \leq 1-\Phi(x) = \Phi(-x) \leq \frac{1}{\sqrt{2\pi}}\frac{1}{x}\exp(-\frac{x^2}{2}).$$

The tail behavior of Gaussian r.e.'s may be estimated as follows:

Proposition 1.1.14. *(Large deviations, see [LT91, Theorem 3.3., Lemma 3.4.]) One has*

$$\lim_{t\to\infty}\frac{1}{t}\log\left(\mu\left(B(0,t)^c\right)\right)+\frac{t}{2(\sigma(\mu))^2} = 0.$$

Remark 1.1.15. The result slightly improves the classical large deviation result for Gaussian random elements, which reads

$$\lim_{t\to\infty}\frac{1}{t^2}\log\left(\mu\left(B(0,t)^c\right)\right) = -\frac{1}{2(\sigma(\mu))^2}.$$

This extension mainly relies on the fact that X is Radon if $(E,\|\cdot\|)$ is separable.

Proof of Proposition 1.1.14. By [LT91, Lemma 3.4.] there exists for every $\epsilon > 0$ a natural number $N(\epsilon) \in \mathbb{N}$ and a constant $K(\epsilon) \in (0,\infty)$ such that for all $t \geq \epsilon$

$$\mu\left(B(0,\epsilon+\sigma(\mu)t)^c\right) \leq K(\epsilon)(1+t)^{N(\epsilon)}\exp(-\frac{t^2}{2}),$$

which is the same as

$$\mu\left(B(0,t)^c\right) \leq K(\epsilon)\Big(1+\frac{t-\epsilon}{\sigma(\mu)}\Big)^{N(\epsilon)}\exp(-\frac{(t-\epsilon)^2}{2\sigma^2}). \tag{1.5}$$

On the other hand, we have for every $t > 0$ and every $f \in E'$ with $\|f\|_{E'} \leq 1$, by applying Lemma 1.1.13

$$\begin{aligned} \mathbb{P}(\|X\| > t) &\geq \mathbb{P}(|f(X)| > t) \geq 1 - 2\Phi(t\sigma(\mu^f)) \\ &\geq \frac{2}{\sqrt{2\pi}}\left(\frac{1}{t\sigma(\mu^f)} - \frac{1}{(t\sigma(\mu^f))^3}\right)\exp\left(-\frac{t^2}{2\sigma(\mu^f)^2}\right). \end{aligned} \tag{1.6}$$

By continuity of the embedding $j_\mu : \mathcal{H}_\mu \to E$ there exists an $f \in E', \|f\| \leq 1$ such that $\sigma(\mu^f) = \sigma(\mu)$ whereof we obtain the lower bound. Equations (1.5) and (1.6) imply for every $\epsilon > 0$

$$\mathcal{O}\left(\frac{\log(t)}{t}\right) \leq \frac{1}{t}\log\left(\mu\left(B(0,t)^c\right)\right) + \frac{t}{2(\sigma(\mu))^2} \leq \frac{\epsilon}{(\sigma(\mu))^2} + \mathcal{O}\left(\frac{\log(t)}{t}\right),$$

which yields the assertion. □

Corollary 1.1.16. *Let $r \geq 0, \epsilon > 0$. There exists a real constant $t(\epsilon) < \infty$ such that*

$$\exp\left(-\frac{t^2}{2(\sigma(\mu))^2} - \epsilon t\right) \leq \mathbb{E}\,\|X\|^r\,\mathbb{1}_{\{\|X\|>t\}} \leq \exp\left(-\frac{t^2}{2(\sigma(\mu))^2} + \epsilon t\right)$$

for all $t \geq t(\epsilon)$.

Proof. Using Fubini's Theorem, we obtain for $t \geq 1$

$$\begin{aligned} \int \|x\|^r\,\mathbb{1}_{\{\|x\|\geq t\}}d\mu(x) &= \int_0^\infty \mu(\{\|x\| \geq t\} \cap \{\|x\|^r \geq y\})d\lambda(y) \\ &= t^r\mu(\|x\| \geq t) + \int_t^\infty r y^{r-1}\mu(\|x\| \geq y)d\lambda(y) \end{aligned}$$

By applying Proposition 1.1.14 we find for every $\epsilon > 0$ a constant $t(\epsilon)$ such that

$$\mu(\|x\| \geq t) \leq \exp\left(\frac{\epsilon}{2}t - \frac{t^2}{2\sigma(\mu)^2}\right),$$

and

$$\max\left\{t^r, rt^{r-1}\right\} \leq \exp\left(\frac{\epsilon}{4}t\right)$$

for $t \geq t(\epsilon)$. Hence

$$\begin{aligned} \int \|x\|^r\mathbb{1}_{\{\|x\|\geq t\}}d\mu(x) &\leq \exp\left(\epsilon\frac{3}{4}t - \frac{t^2}{2\sigma(\mu)^2}\right) \\ &\quad + C(\epsilon)\int_t^\infty \left(y/\sigma(\mu) - \epsilon\frac{3}{4}\sigma(\mu)\right)\exp\left(-\frac{1}{2}\left(y/\sigma(\mu) - \epsilon\frac{3}{4}\sigma(\mu)\right)^2\right)d\lambda(y) \\ &\leq \exp\left(\epsilon t - \frac{t^2}{2\sigma(\mu)^2}\right) \end{aligned}$$

for all $t > t(\epsilon)$. The lower bound is a direct consequence of Proposition 1.1.14. □

References: A rigorous introduction into Gaussian measures is given by [Bog98] and [LT91], see also [VTC87], a summary of the recent research and its applications can be found in [LS01].

1.2 The quantization problem

The origin: The analysis of quantization problems goes back to the early 40s of the last century, with the invention of Pulse-Code-Modulation (PCM). Electrical engineers had to face the problem that any continuous or highly complex signal (e.g. a sound signal), which had to be either transmitted or stored, needed to be compressed, such that

1. the loss of information from the original signal would be limited, i.e. one would be able to reconstruct the original signal up to a limited error, and

2. the storage or transmission complexity would remain below a prescribed limit.

This problem is also known as the lossy source coding problem. Mathematically speaking, we consider an arbitrary Borel r.e. X in $(E, \|\cdot\|)$, also called the original, and the compressed or quantized version $\widehat{X}$, which is the transmitted or stored information and which is of lower complexity. As regards point 1), the loss of information when compressing the signal can be measured by any symmetric measurable mapping

$$\widetilde{\rho} : E \times E \to [0, \infty)$$

called distortion or distortion measure. The most studied distortion measures are difference distortion measures, only depending on the difference of the two input variables. Hereafter, we will consider norm-based difference distortion measures of the form

$$\widetilde{\rho}(x, y) = \|x - y\|^r$$

for some $r \in (0, \infty)$ and $x, y \in E$. Since the signals in question are represented by random elements, we additionally have to apply a measure for the expectation, whereby we derive error functions of the form

$$\mathbb{E} \left\| X - \widehat{X} \right\|^r .$$

As regards point 2), the complexity restriction forces the compressed signal $\widehat{X}$ to satisfy an information constraint. Let $n \in \mathbb{N}$. The most studied constraints (see e.g. Kolmogorov [Kol93]) are

1. Quantization constraint, i.e.

$$\left|\mathrm{supp}(\mathbb{P}^{\widehat{X}})\right| \le n.$$

2. Entropy constraint, i.e. for $s > 0$

$$\mathbb{H}_s(\widehat{X}) \le \log_s(n),$$

where the entropy of a discrete random variable Y admitting the representation $\mathbb{P}^Y(A) = \int_A f_Y(x) d\mathbb{P}(x) = \sum_{y_i \in A} y_i \mathbb{P}(Y = y_i)$ is defined as

$$\mathbb{H}_s(Y) := -\int \log_s(f_Y(x)) d\mathbb{P}^Y(x),$$

with $\log_s$ denoting the logarithm with basis s.

3. Mutual information constraint, i.e.

$$\mathbb{I}_s(X; \widehat{X}) \le \log_s(n),$$

where $\mathbb{I}_s(X;Y)$ for two random elements X, Y in E is defined as

$$\mathbb{I}_s(X;Y) := \int \log_s\left(\frac{\partial \mathbb{P}^{(X,Y)}}{\partial \mathbb{P}^X \otimes \mathbb{P}^Y}(x,y)\right) d\mathbb{P}^{(X,Y)}(x,y),$$

if $\mathbb{P}^{(X,Y)} \ll \mathbb{P}^X \otimes \mathbb{P}^Y$ and ∞ otherwise. Here, we denote for measures $\mathbb{P}, \mathbb{Q}$ by $\mathbb{P} \otimes \mathbb{Q}$ the product measure of $\mathbb{P}$ and $\mathbb{Q}$ as well as for a $\mathbb{P}$ absolute continuous $\mathbb{Q}$ by $\frac{\partial \mathbb{Q}}{\partial \mathbb{P}}$ the Radon-Nikodyn density of $\mathbb{Q}$.

If s equals the Euler constant e, we will omit the indexation and write $\mathbb{I}(X; \widehat{X})$ and $\mathbb{H}(\widehat{X})$ instead of $\mathbb{I}_s(X; \widehat{X})$ and $\mathbb{H}_s(\widehat{X})$.

From an intuitive point of view, the quantization constraint seems to be a natural restriction. The entropy constraint is motivated by the fact that every random element Y with a finite support can be represented by a prefix free binary code of an *average* length not longer than $\mathbb{H}_2(Y) + 1$ (e.g. by applying Huffman coding or Shannon coding). Here, one makes use of the fact that representing more likely codes with a shorter bit sequence decreases the average length of a signal represented by a binary code. In fact, the worst case for the entropy, i.e. an upper bound, is given by

$$\mathbb{H}_s(Y) \le \log_s\left(\left|\operatorname{supp}(\mathbb{P}^Y)\right|\right), \tag{1.7}$$

where equality is attained if

$$\mathbb{P}^Y = \frac{1}{|\operatorname{supp}(\mathbb{P}^Y)|} \sum_{y_i \in \operatorname{supp}(Y)} \delta_{y_i}.$$

Here $|\alpha|$ denotes the cardinality of a set $\alpha \subset E$.

The mutual information constraint has been introduced by Shannon. It is part of Shannon's source coding theorem, which will also play a role in the analysis of the asymptotics of quantization errors. In general, one has for any reconstruction $\widehat{X}$ of X the relationship

$$\mathbb{I}_s(X; \widehat{X}) \le \mathbb{H}_s(\widehat{X}) \le \log_s\left(\left|\operatorname{supp}(\mathbb{P}^{\widehat{X}})\right|\right), \tag{1.8}$$

i.e. the quantization constraint is the most restrictive one.

Besides its application in signal compression, quantization methods found their way into different other parts of applied mathematics. Apart from areas which are closely related to classical PCM-techniques, such as image compression or speech coding algorithms (see [Chu03]), it found its way into other areas of applied mathematics during the 90s of the last century, especially numerical finance, where quantizers are for instance being used through cubature formulas in option pricing models or as variance reduction techniques for Monte-Carlo simulations. The range of applications of quantization methods are treated in more detail in the paragraph Applications to numerical finance below.

Hereafter, we will consider for a given random variable X in $E = (E, \|\cdot\|, \mathcal{B}(E))$, a natural number $n \in \mathbb{N}$ and $r > 0$ the optimization problem

$$\inf_{\widehat{X}} \left\| X - \widehat{X} \right\|_{L_r(\mathbb{P},E)}$$

given the quantization constraint $\left|\operatorname{supp}(\mathbb{P}^{\widehat{X}})\right| \leq n$.

All of the results shown in this section including their proofs are well known, except for those which are explicitly mentioned as such. We present some sketches of those proofs in order to enable the reader to get a better understanding of the general theory.

Definitions and Basic facts: We consider a separable Banach space $(E, \|\cdot\|)$ and a Borel random variable $X \in L_r(\mathbb{P}, E)$ for some fixed $r \in (0, \infty)$.

Definition 1.2.1. (codebook, quantization error, α-quantization)

1. A finite subset $\alpha \subset E$ is called codebook or quantizer. If additionally $|\alpha| = n$, we call α an n-codebook or n-quantizer.

2. The $L_r(\mathbb{P}, E)$-quantization error for the r.e. X induced by the codebook α is defined as

$$e_r(X, E; \alpha) := \left\| \min_{a \in \alpha} \|X - a\| \right\|_{L_r(\mathbb{P},\mathbb{R})}. \tag{1.9}$$

The corresponding infimum

$$e_{n,r}(X, E) := \inf_{\alpha} \{ e_r(X, E; \alpha) : \alpha \subset E, |\alpha| \leq n \} \tag{1.10}$$

we denote the optimal n-th $L_r(\mathbb{P}, E)$-quantization error for X.

3. For an n-codebook α, we define the nearest neighbor projection

$$f_\alpha : E \to \alpha, \quad x \mapsto \sum_{a \in \alpha} a \mathbb{1}_{C_a(\alpha)}(x),$$

where a Voronoi partition $\{C_a(\alpha), a \in \alpha\}$ is defined as a Borel partition of $(E, \mathcal{B}(E))$ satisfying

$$C_a(\alpha) \subset V_a(\alpha) := \{ x \in E : \|x - a\| = \min_{b \in \alpha} \|x - b\| \}$$

for every $a \in \alpha$. The random variable $f_\alpha(X)$ we will call an α-quantization of X.

Remark 1.2.2. Strictly speaking, the nearest neighbor projection f_α depends on the Voronoi partition $\{C_a(\alpha), a \in \alpha\}$. Hereafter, if we do not specify the Voronoi partition, the results stated shall be understood as holding for an arbitrary one. Note that in this case, many results obtained involving a single Voronoi cell $C_a(\alpha)$ equally hold for the Voronoi region $V_a(\alpha)$ and also for

$$W_a(\alpha) := \{x \in E : \|x - a\| < \min_{b \in \alpha \setminus \{a\}} \|x - b\|\},$$

since both sets can be Voronoi cells in a specific Voronoi partition. One may see this by creating a Voronoi partition for the Voronoi diagram $\{V_a(\alpha), a \in \alpha = \{a_1, \dots, a_n\}\}$ as

$$C_{a_i}(\alpha) := V_{a_i}(\alpha) \cap \left(\bigcap_{j \le i-1} V_{a_j}(\alpha) \right)^c, \ 1 \le i \le n,$$

or

$$C_{a_i}(\alpha) := W_{a_i}(\alpha) \cup \left(V_{a_i}(\alpha) \cap \left(\bigcup_{j \ge i+1} V_{a_j}(\alpha) \right)^c \right), \ 1 \le i \le n,$$

with $\bigcup_\varnothing = \varnothing$ and $\bigcap_\varnothing = E$. By selecting a_1 as the required code a, one obtains the asserted.

The optimal n-th $L_r(\mathbb{P}, E)$-quantization error admits several equivalent representations.

Proposition 1.2.3. *(c.f. [GL00, section 3] for the case $E = \mathbb{R}^d$) Let $\mathcal{M}_r(\mathcal{B}(E))$ denote the set of all probability measures on $(E, \mathcal{B}(E))$ with finite r-th moment (i.e. $\int \|x\|^r \, d\mu(x) < \infty$), and $\rho_r^*(Q_1, Q_2)$ denote the r-th Wasserstein distance between $Q_1, Q_2 \in \mathcal{M}_r(E)$, i.e.*

$$\rho_r^*(Q_1, Q_2) := \inf \left\{ \left(\int \|x - y\|^r \, dQ(x, y) \right)^{\frac{1}{r}} : Q \in \pi(Q_1, Q_2) \right\},$$

where $\pi(Q_1, Q_2)$ denotes the set of all probability measures on $\mathcal{B}(E) \otimes \mathcal{B}(E)$ with first marginal Q_1 and second marginal Q_2. We set $\mathcal{F}_n := \{f : (E, \mathcal{B}(E)) \to (E, \mathcal{B}(E)), |f(E)| \le n\}$. Then

$$\begin{aligned}
e_{n,r}(X, E) &\overset{(1)}{=} \inf \left\{ \|X - f_\alpha(X)\|_{L_r(\mathbb{P}, E)} : \alpha \subset E, |\alpha| \le n \right\} \\
&\overset{(2)}{=} \inf \left\{ \|X - f(X)\|_{L_r(\mathbb{P}, E)} : f \in \mathcal{F}_n \right\} \\
&\overset{(3)}{=} \inf \left\{ \|X - Y\|_{L_r(\mathbb{P}, E)} : Y \text{ r.e. in } E \,, |Y(\Omega)| \le n \right\} \\
&\overset{(4)}{=} \inf \left\{ \rho_r^*(\mathbb{P}^X, Q) : Q \in \mathcal{M}_r(E), |\mathrm{supp}(Q)| \le n \right\} \\
&\overset{(5)}{=} \inf \left\{ \rho_r^*(\mathbb{P}^X, \mathbb{P}^{f(X)}) : f \in \mathcal{F}_n \right\}
\end{aligned}$$

Proof. Equality (1) follows by the definition of a Voronoi partition, $\geq$ in equations (2) and (3) is obvious. Consider a discrete r.e. Y and $\alpha := \operatorname{supp}(\mathbb{P}^Y)$. Then, for every Voronoi partition $\{C_a(\alpha), a \in \alpha\}$

$$\|X - Y\|^r = \sum_{a\in\alpha} \mathbb{1}_{C_a(\alpha)} \|X - Y\|^r \geq \sum_{a\in\alpha} \mathbb{1}_{C_a(\alpha)} \|X - a\|^r = \|X - f_\alpha(X)\|^r \quad a.s.$$

and we obtain equations (2) and (3).
Let $Q \in \mathcal{M}_r(E)$, $\alpha := \operatorname{supp}(Q)$ with $|\alpha| = n$ and $\nu \in \pi(\mathbb{P}^X, Q)$. Then

$$\begin{aligned}\left(\rho_r^*(\mathbb{P}^X, Q)\right)^r &= \int_{E\times\alpha} \|x - y\|^r \, d\nu(x,y) \geq \int_{E\times\alpha} \min_{a\in\alpha} \|x - a\|^r \, d\nu(x,y) \\ &= \int_E \min_{a\in\alpha} \|x - a\|^r \, d\mathbb{P}^X(x) = e_{n,r}^r(X, E)\end{aligned} \tag{1.11}$$

On the other hand, we have for $f \in \mathcal{F}_n$

$$\mathbb{E}\|X - f(X)\|^r = \int \|x - y\|^r \, d\mathbb{P}^X \otimes \mathbb{P}^{f(X)} \geq \left(\rho_r^*(\mathbb{P}^X, \mathbb{P}^{f(X)})\right)^r,$$

wherefore we obtain with equations (1) and (2)

$$\begin{aligned} e_{n,r}^r(X,E) &= \inf\left\{\|X - f(X)\|_{L_r(\mathbb{P})} : f \in \mathcal{F}_n\right\} \\ &\geq \inf\left\{\rho_r^*(\mathbb{P}^X, \mathbb{P}^{f(X)}) : f \in \mathcal{F}_n\right\} \\ &\geq \inf\left\{\rho_r^*(\mathbb{P}^X, Q) : Q \in \mathcal{M}_r(E), |\operatorname{supp}(Q)| \leq n\right\},\end{aligned} \tag{1.12}$$

which completes the proof. □

Remark 1.2.4. As a consequence of the latter result, the quantization error only depends on the distribution $\mu = \mathbb{P}^X$ of X, such that it is also reasonable to write

$$e_{n,r}(\mu, E) \text{ instead of } e_{n,r}(X, E),$$

as well as

$$e_r(\mu, E; \alpha) \text{ instead of } e_r(X, E; \alpha),$$

whereof we will make use hereafter.

The following well known result implies that the quantization error induced by an arbitrary Gaussian r.e. X equals the quantization error induced by the centered Gaussian r.e. $X - E(X)$.

Proposition 1.2.5. *Let E, F be separable Banach spaces, $X \in L_r(\mathbb{P}, E)$ and $T : E \to F$ linear and bounded. Then*

$$e_{n,r}(T(X), F) \leq \|T\| \, e_{n,r}(X, E).$$

If furthermore T is a bijective isometry, $c > 0$ and $f \in F$ then

$$e_{n,r}(cT(X) + f, F) = c e_{n,r}(X, E).$$

Proof. We prove the first assertion, the second follows analogously. Let α be an n-codebook in E, then

$$e^r_{n,r}(T(X),F) \leq \mathbb{E} \min_{a\in\alpha} \|T(X-a)\|^r \leq \|T\|^r \, \mathbb{E} \min_{a\in\alpha} \|X-a\|^r$$

and we obtain the assertion by taking the infimum over all n-codebooks α. □

The quantization error features the following sub-additivity property.

Proposition 1.2.6. *Let $r \geq 1$. For r.e.'s X_i in $L_r(\mathbb{P},E)$ and codebooks α_i in E with $|\alpha_i| = n_i$, $i = 1,\dots,m$ it holds*

$$e_r(X^{(m)},E;\alpha^{(m)}) \leq \sum_{i=1}^m e_r(X_i,E;\alpha_i),$$

where $X^{(m)} := \sum_{i=1}^m X_i$ and $\alpha^{(m)} := \sum_{i=1}^m \alpha_i$ as a Minkowski sum. Therefore,

$$e_{\prod_{i=1}^m n_i,r}(X^{(m)},E) \leq \sum_{i=1}^m e_{n_i,r}(X_i,E).$$

Proof. For $m = 1$, there is nothing to prove. Suppose that the assertion holds for $m \in \mathbb{N}$. Then

$$\begin{aligned}
e_r(X^{(m+1)},E;\alpha^{(m+1)}) &= \left(\mathbb{E} \min_{a^{(m)}+a_{m+1}\in\alpha^{(m)}+\alpha_{m+1}} \left\|X^{(m+1)} - a^{(m)} - a_{m+1}\right\|^r\right)^{\frac{1}{r}} \\
&\leq \left(\mathbb{E} \min_{a^{(m)}\in\alpha^{(m)}} \left\|X^{(m)} - a^{(m)}\right\|^r\right)^{\frac{1}{r}} + \left(\mathbb{E} \min_{a_{m+1}\in\alpha_{m+1}} \left\|X_{m+1} - a_{m+1}\right\|^r\right)^{\frac{1}{r}} \\
&\leq \sum_{j=1}^m e_r(X_i,E;\alpha_i) + e_r(X_{m+1},E;\alpha_{m+1}) = \sum_{i=1}^{m+1} e_r(X_i,E;\alpha_i).
\end{aligned}$$

□

Optimality and stationarity: Any optimization problem naturally gives rise to the question whether optimal solutions exist, which are in this case quantizers achieving the optimal quantization error.

Definition 1.2.7. Let E be a Banach space and $X \in L_r(\mathbb{P},E)$ for some $r > 0$.

1. An n-codebook α is called an r-optimal n-codebook for the r.e. X in $(E,\|\cdot\|)$, iff
$$e_r(X,E;\alpha) = e_{n,r}(X,E).$$

2. The set of all r-optimal n-codebooks for X in E is denoted
$$\mathcal{C}_{n,r}(X,E) := \{\alpha \subset E : |\alpha| \leq n \text{ and } e_r(X,E;\alpha) = e_{n,r}(X,E)\}.$$

In the finite dimensional setting the existence of optimal quantizers is always guaranteed (see e.g. [GL00, Theorem 4.12]). In the infinite dimensional setting, this is not always true. A sufficient condition for the existence of optimal quantizers in Banach spaces is given with the following Theorem.

Theorem 1.2.8. *[GLP07, Theorem 1, Proposition 2] Let F be a Banach subspace of E such that $\mathbb{P}^X(F) = 1$. If the set*

$$\{B(x,\epsilon),\ x \in F \text{ and } \epsilon > 0\}$$

is a compact system in E, then

$$\mathcal{C}_{n,r}(X,E) \neq \varnothing. \tag{1.13}$$

In particular, equation (1.13) holds if E is 1-complemented in its Bidual E'', i.e. if there is a linear projection T from E'' onto E with $\|T\| \leq 1$.

Remark 1.2.9.

1. As a consequence of Theorem 1.2.8, if $E = F'$ for a Banach space F, then E is 1-complemented in $E'' = F'''$ and therefore

   $$\mathcal{C}_{n,r}(X,E) \neq \varnothing$$

   (see [GLP07, Corollary 1]). This includes every Hilbert space, all finite dimensional Banach spaces and also for example the spaces

   $$L_p(I) := \{f : (I, \mathcal{B}(I)) \to (\mathbb{R}, \mathcal{B}(R)), \int_I |f(x)|^p \, d\lambda^d(x) < \infty\}$$

   for $p \in [1,\infty]$ and an interval $I = \otimes_{j=1}^d I_j \subset \mathbb{R}^d$.

2. Considering E'' instead of E, does not affect quantization errors. More precisely, for a r.e. X in the separable Banach space E, one has

   $$e_{n,r}(X,E) = e_{n,r}(X,E'')$$

   for $r > 0$ and $n \in \mathbb{N}$ (see [GLP07, Theorem 2]).

3. Optimal n-quantizers may not exist in arbitrary Banach spaces (see different examples in [GLP07]).

4. An interesting open question is, whether $\mathcal{C}_{n,r}(X,E) \neq \varnothing$ if X is Gaussian. As a consequence of the Anderson inequality (Proposition 1.1.6), this is true for $n = 1$ with $\mathcal{C}_{n,r}(X,E) = \{0\}$, since

   $$\begin{aligned} \mathbb{E}\,\|X - a\|^r &= \int_{[0,\infty]} \mu\left((B(a,t^r))^c\right) dt \\ &\geq \int_{[0,\infty]} \mu\left((B(0,t^r))^c\right) dt = \mathbb{E}\,\|X\|^r \end{aligned}$$

   for all $a \in E$.

Remark 1.2.10. We use the notations of Proposition 1.2.5. Similarly to the proof thereof, one shows for every linear, bijective isometry T

$$\mathcal{C}_{n,r}(cT(X)+f,F)=c\mathcal{C}_{n,r}(X,E)+f,$$

for all $c>0$ and $f\in F$.

A useful criterion to analyze optimality of codebooks is the necessary condition of stationarity.

Proposition 1.2.11. *Let $|\mathrm{supp}(\mu)|\geq n$ and $\alpha\in\mathcal{C}_{n,r}(X,E)$. Then $|\alpha|=n$,*

$$\mu(C_a(\alpha))>0 \text{ and } a\in\mathcal{C}_{1,r}(\mathbb{P}^X(\cdot|C_a(\alpha)),E)$$

for every $a\in\alpha$ and every Voronoi partition $\{C_a(\alpha),a\in\alpha\}$. Any quantizer α satisfying these conditions is called r-stationary n-quantizer. The set of all these quantizers will be denoted $S_{n,r}(X,E)$.

The following observation is the starting point for the analysis of the asymptotic behavior of the quantization error. For a separable Banach space E and $X\in L_r(\mathbb{P},E)$, consider a countable dense subset $\{a_i,i\in\mathbb{N}\}$ of E. Then, by the dominated convergence theorem

$$\lim_{n\to\infty}\mathbb{E}\min_{i\leq n}\|X-a_i\|^r=\mathbb{E}\lim_{n\to\infty}\min_{i\leq n}\|X-a_i\|^r=0,$$

so that

$$e_{n,r}(X,E)\to 0,\quad n\to\infty. \tag{1.14}$$

Asymptotics of the quantization error in $(E,\|\cdot\|)=(\mathbb{R}^d,\|\cdot\|)$: Starting with Panter and Dite in 1951 [PD51], the asymptotic behavior of the quantization error, also called high resolution quantization, has been studied extensively. In the finite dimensional setting $(E,\|\cdot\|)=(\mathbb{R}^d,\|\cdot\|)$ the asymptotics of the quantization error for distributions with a non-vanishing Lebesgue-continuous part is (almost) fully described by the following Theorem. It goes back to Zador ([Zad63]) (in the case $d=1$ already to Panter and Dite) and Bucklew and Wise ([BW82]). A rigorous proof has been presented by Graf and Luschgy [GL00, Theorem 6.2].

Before we present the result, we will introduce a few notations. Let $f,g:I\to\mathbb{R}$, with $I\subset\mathbb{R}^+$ such that I is unbounded (Typically $I=\mathbb{N}$ or $I=[C,\infty)$ for some constant $C\geq 0$). We write

$$\begin{aligned}
f(x)\lesssim g(x),\ x\to\infty \quad &\text{if} \quad \limsup_{x\to\infty,x\in I}\frac{f(x)}{g(x)}\leq 1,\\
f(x)\preccurlyeq g(x),\ x\to\infty \quad &\text{if} \quad \limsup_{x\to\infty,x\in I}\frac{f(x)}{g(x)}<\infty,\\
f(x)\approx g(x),\ x\to\infty \quad &\text{if} \quad 0<\liminf_{x\to\infty,x\in I}\frac{f(x)}{g(x)}<\limsup_{x\to\infty,x\in I}\frac{f(x)}{g(x)}<\infty,\\
f(x)\sim g(x),\ x\to\infty \quad &\text{if} \quad \lim_{x\to\infty,x\in I}\frac{f(x)}{g(x)}=1.
\end{aligned}$$

The symbols $\gtrsim$ and $\succcurlyeq$ are defined accordingly.

Theorem 1.2.12. *(Zador Theorem; Panter and Dite Formula) Suppose* $(E, \|\cdot\|) = (\mathbb{R}^d, \|\cdot\|)$ *for an arbitrary norm* $\|\cdot\|$ *and* $X \in L_{r+\delta}(\mathbb{P})$ *for some* $\delta > 0$. *Let* $\mathbb{P}^X = \mu = \mu_s + \mu_a$ *denote the Lebesgue decomposition of* μ, *i.e.* $\mu_s \perp \lambda^d$ *and* $\mu_a \ll \lambda^d$, *and* $f = \frac{\partial \mu_a}{\partial \lambda^d}$. *Then*

$$\lim_{n\to\infty} n^{\frac{r}{d}} e^r_{n,r}(X, \mathbb{R}^d) = Q_{r,\|\cdot\|}([0,1]^d) \, \|f\|_{\frac{d}{d+r}}, \tag{1.15}$$

where

$$Q_{r,\|\cdot\|}([0,1]^d) := \inf_{n\in\mathbb{N}} n^{\frac{r}{d}} e^r_{n,r}(U([0,1]^d), \mathbb{R}^d) \in (0,\infty). \tag{1.16}$$

Here, $U([0,1]^d)$ *denotes the uniform distribution on the hypercube* $[0,1]^d$ *and* $\|\cdot\|_s$ *denotes for* $s > 0$ *and* $f : (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$

$$\|f\|_s := \left(\int_{\mathbb{R}^d} |f|^s \, d\lambda^d \right)^{\frac{1}{s}}.$$

Remark 1.2.13.

1. For a singular distribution $\mu = \mu_s$, Theorem 1.2.12 yields

   $$e_{n,r}(X, \mathbb{R}^d) = o\left(n^{-\frac{r}{d}}\right), \quad n \to \infty.$$

   If

   $$\limsup_{n\to\infty} -\frac{\log(n)}{\log(e_{n,r}(\mu, \mathbb{R}^d))} = \liminf_{n\to\infty} -\frac{\log(n)}{\log(e_{n,r}(\mu, \mathbb{R}^d))},$$

   the limit denoted by $D_r(\mu) \le d$ is called quantization dimension of μ of order r. For a detailed study see e.g. [GL00, chapter 3]; further results in this area have also been investigated by several authors, see e.g. [GL05] and [Kre06].

2. The integrability condition $X \in L_{r+\delta}(\mathbb{P}, \mathbb{R}^d)$ cannot be dropped in general, see [GL00, Example 6.4]. Furthermore, due to Hölder's inequality, this condition ensures the finiteness of the right-hand side integral in equation (1.15), see [GL00, Remark 6.3].

3. The right-hand side of equation (1.15) is called quantization coefficient of X (or μ). It is finite due to the integrability condition on X and strictly positive if μ is not purely singular. Unfortunately, the constant $Q_{r,\|\cdot\|}([0,1]^d)$ is generally not known explicitly, unless in some special cases, where the unit balls in $(\mathbb{R}^d, \|\cdot\|)$ are space-filling or the geometry of the norm $\|\cdot\|$ is well understood. Here, space-filling means that $\mathbb{R}^d$ can be covered by translations of the unit ball that are disjoint except for a λ^d null set. Particularly, those are the l_∞-norm for all $d \ge 1$ and $r \ge 1$, where

   $$Q_{r,\|\cdot\|_{l_\infty}}([0,1]^d) = \frac{d}{(d+r)2^r}$$

and additionally, if $d = 2$ also the l_2-norm and the l_1-norm

$$Q_{r,\|\cdot\|_{l_1}}([0,1]^d) = \frac{2}{(2+r)2^{\frac{r}{2}}},$$
$$Q_{r,\|\cdot\|_{l_2}}([0,1]^d) = \frac{8\ 2^{\frac{r}{2}}}{3^{((2+r)/4)}} \int_{[0,\frac{1}{2}]} \int_{[0,\frac{1-y}{\sqrt{3}}]} (x^2+y^2)^{\frac{r}{2}} d\lambda(x) d\lambda(y),$$

see [GL00, Example 8.12, Example 8.14, Theorem 8.15].

Sketch of a proof of Theorem 1.2.12. For the readers convenience, we will give a sketch of the proof of Theorem 1.2.12 which shall give a good understanding for the results obtained. We follow the lines of argumentation in [GL00, section 6].
Step 1: One shows for $\mu = U([0,1]^d)$ that

$$\lim_{n\to\infty} e^r_{n,r}(\mu,\mathbb{R}^d) = \inf_{n\in\mathbb{N}} e^r_{n,r}(\mu,\mathbb{R}^d) = c \in (0,\infty).$$

Here, the self-similarity of λ^d is essential.
Step 2: One considers measures of the form $\mu = \sum_{i=1}^m s_i U(C_i)$, where C_i are disjoint cubes of a common length $l > 0$, hence $f := \frac{\partial\mu_a}{\partial\lambda^d} = \sum_{i=1}^m s_i \mathbb{1}_{C_i} l^{-d}$. The question is then how to allocate for a fixed $n \in \mathbb{N}$ optimally $n_i := \lfloor t_i n \rfloor$ codes to each cube A_i, where $t_i \geq 0, 1 \geq i \geq m$ with $\sum_{i=1}^m t_i = 1$, such that

$$\begin{aligned} n^{\frac{r}{d}} e^r_{n,r}(\mu,\mathbb{R}^d) &\leq n^{\frac{r}{d}} \sum_{i=1}^m s_i e^r_{n_i,r}(U(C_i),\mathbb{R}^d) = n^{\frac{r}{d}} \sum_{i=1}^m s_i l^r e^r_{n_i,r}(U([0,1]^d),\mathbb{R}^d) \\ &\lesssim Q_{r,\|\cdot\|}([0,1]^d) \sum_{i=1}^m s_i t_i^{-\frac{r}{d}} l^r = Q_{r,\|\cdot\|}([0,1]^d) \left\| f h^{-\frac{r}{d}} \right\|_1, \quad n \to \infty \end{aligned}$$

reaches its minimum over all h of the shape $h := \sum_{i=1}^m t_i l \mathbb{1}_{C_i}$. This problem has a unique minimizer of the form

$$h = \|f\|_{\frac{d}{d+r}}^{-\frac{d}{d+r}} f^{\frac{d}{d+r}},$$

since, in virtue of Hölder's inequality with exponents $p = \frac{d}{d+r}$ and $q = -\frac{d}{r}$,

$$\left\| f h^{-\frac{r}{d}} \right\|_1 \geq \|f\|_p \left\| h^{-\frac{r}{d}} \right\|_q \geq \|f\|_{\frac{d}{d+r}},$$

where equality is attained, if h has the minimizing shape.
Step 3: By using the differentiation of measures theorem, one extends the result to compactly supported Lebesgue-continuous measures.
Step 4: One shows that measures which are singular with respect to the Lebesgue measure induce an asymptotic quantization error, that is of lower order than $n^{-\frac{r}{d}}$. Furthermore, one shows that the quantization error induced by a convex combination of such a measure with a Lebesgue-continuous measure equals asymptotically the quantization error of the Lebesgue-continuous part.
Step 5: One proves a non-asymptotic upper bound for the quantization error,

which can be deduced by the Pierce lemma reading as follows:
Let $d = 1$, i.e. $\mu = \mathbb{P}^X$ is a univariate distribution, then

$$e_{n,r}^r(X,\mathbb{R}) \leq n^{-r}\left(C_1\mathbb{E}\,|X|^{r+\delta} + C_2\right)$$

for all $\delta > 0$ and $n \geq C_3$, where the constants $C_i, i = 1,2,3$ depend on r and δ solely.
In the generalized version for $d \geq 2$, which is used in the proof, the factor n^{-r} is replaced by $n^{-\frac{r}{d}}$ and the constants depend additionally on the underlying norm $\|\cdot\|$ and the dimension d.
Step 6: Step 4, 5 and 6 are combined to prove the final result.

□

Consider now a sequence of n-codebooks $(\alpha_n)_{n\in\mathbb{N}}$ such that

$$e_r(\mu,\mathbb{R}^d;\alpha_n) \to 0, \quad n \to \infty.$$

Since the convergence in $L_r(\mathbb{P},\mathbb{R}^d)$ implies the weak convergence of measures, we obtain that the distribution of the α_n-quantization of X converges weakly towards μ, i.e.

$$\mathbb{P}^{f_{\alpha_n}(X)} = \sum_{a\in\alpha_n} \mu(C_a(\alpha_n))\delta_a \overset{w}{\Rightarrow} \mu, \quad n \to \infty,$$

where $\overset{w}{\Rightarrow}$ denotes the weak convergence of measures. Replacing in the latter formula the weights of the Voronoi cells $\mu(C_a(\alpha_n))$ by $\frac{1}{n}$, the middle expression still converges and one obtains the following empirical measure Theorem. A key role, as in the proof of the Zador Theorem, will here be played by the point density measure μ_r, which is given through

$$\mu_r(A) := \left\| \frac{\partial\mu_a}{\partial\lambda^d}^{\frac{d}{d+r}} \right\|_1^{-1} \int_A \frac{\partial\mu_a}{\partial\lambda^d}^{\frac{d}{d+r}} d\lambda^d$$

for $A \in \mathcal{B}(\mathbb{R}^d)$.

Theorem 1.2.14. *(Empirical measure theorem) Suppose that $f = \frac{\partial\mu_a}{\partial\lambda^d} \not\equiv 0$. Then, for every sequence of n-codebooks $(\alpha_n)_{n\in\mathbb{N}}$ satisfying*

$$e_r(\mu,\mathbb{R}^d;\alpha_n) \sim e_{n,r}(X,\mathbb{R}^d), \quad n \to \infty, \tag{1.17}$$

one has

$$\frac{1}{n}\sum_{a\in\alpha_n} \delta_a \overset{w}{\Rightarrow} \mu_r, \quad n \to \infty.$$

Proof. The proof is a consequence of the Empirical measure Theorem for Lebesgue continuous distributions (see [GL00, Theorem 7.5]) in combination with the fact that sequences of asymptotically optimal quantizers (i.e. quantizers satisfying equation (1.17)) for distributions having a non-continuous part are also asymptotically optimal for the Lebesgue-continuous part solely (see [DGLP04, Lemma 4.1]). □

In the infinite dimensional case, there is much less known about asymptotics of quantization errors. Nevertheless, many results have been obtained so far, particularly for Gaussian r.e.'s.

Asymptotics of quantization errors in infinite dimensional Hilbert spaces: The most famous result in this area was firstly mentioned in a simplified version by Donoho in the technical report [Don00]. Its proof goes back to Luschgy and Pagès [LP04a]. Let $(E, \|\cdot\|) = (H, (\cdot, \cdot))$ be a separable Hilbert space and X be a Gaussian r.e. in H with $\dim(\mathcal{H}_\mu) = \infty$. In this case, the covariance operator C_X of X has a specific representation. Since C_X is a symmetric and compact operator, there exists in virtue of the spectral theorem a sequence $(\lambda_j)_{j\in\mathbb{N}}$ of positive non-increasing eigenvalues and a sequence $(e_j)_{j\in\mathbb{N}}$ of corresponding orthonormal eigenvectors such that

$$C_X(y) = \sum_{i=1}^{\infty} \lambda_j (e_j, y) \tag{1.18}$$

for all $y \in H$. One may expand the orthonormal system $(e_j)_{j\in\mathbb{N}}$ to a basis of H, such that (1.18) still holds and $(e_j)_{j\in\mathbb{N}}$ is an (at most countable) orthonormal basis of H. Then

$$X = \sum_{j=1}^{\infty} e_j (X, e_j) \quad a.s..$$

Since $\mathbb{E}(X, e_i)(X, e_j) = (e_j, C_X(e_i)) = \lambda_i \delta_{i,j}$ we obtain that $\left(\frac{(X,e_i)}{\sqrt{\lambda_j}}\right)_{j:\lambda_j>0}$ is a sequence of independent $\mathcal{N}(0,1)$-distributed random elements in $(\mathbb{R}, |\cdot|)$ such that

$$X \stackrel{d}{=} \sum_{j=1}^{\infty} \sqrt{\lambda_j} e_j \xi_j \tag{1.19}$$

for every sequence $(\xi_j)_{j\in\mathbb{N}}$ of independent $\mathcal{N}(0,1)$-distributed random elements. The representation (1.19) is also called the Karhunen-Loève expansion for X.

For convenience, we will write for the mutual information constraint expected distortion error

$$d_{n,r}(X, H) := \inf\left\{\left(\mathbb{E}\left\|X - \widehat{X}\right\|^r\right)^{\frac{1}{r}} : \widehat{X} \text{ r.e. in } H, \mathbb{I}(X; \widehat{X}) \le \log(n)\right\}.$$

$d^r_{n,r}(X, H)$ is also known as the (logarithmic) distortion rate function.

Theorem 1.2.15. *(c.f. [LP04a]) Suppose that the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$ of the Covariance Operator C_X of X satisfy $\lambda_j \sim \phi(j)$ for a regular varying function*

ϕ with index $-b < -1$. Then

$$d_{n,2}^2(X,H) \sim e_{n,2}^2(X,H) \sim \left(\frac{b}{2}\right)^{b-1} b \int_{\log(n)}^{\infty} \phi(x)d\lambda(x)$$
$$\sim \left(\frac{b}{2}\right)^{b-1} \frac{b}{b-1} \log(n)\phi(\log(n)), \quad n \to \infty.$$

Remark 1.2.16.

1. The result has also been extended to the case where $\phi(j)$ is regular varying with index -1 (see [LP04a]). In this case, the optimal rate for the quantization error can even be achieved by sequences of scalar product quantizers, which are much easier to be constructed.

2. In Dereich's dissertation (see [Der03, chapter 6]) the result has been extended to arbitrary moments $r > 0$ instead of 2. Surprisingly, this leads to the exact same asymptotic quantization error. Furthermore, the result has been extended to broader classes of eigenvalue sequences.

3. Additionally, it was shown by Dereich (see [Der03, chapters 6 and 7]) that the same asymptotics also holds for the coding error induced by random codebooks following an optimally chosen distribution. Furthermore, there exists an explicit relationship to the asymptotic coding error induced by random codebooks generated by the distribution μ itself. Note, that this asymptotics differs from the other asymptotic error rates.

Sketch of the proof of Theorem 1.2.15. In virtue of equation (1.8), it follows

$$d_{n,2}(X,H) \leq e_{n,2}(X,H),$$

such that we only need to prove the upper bound for $e_{n,2}(X,H)$ and the lower bound for $d_{n,2}(X,H)$.
Step 1: Sharp asymptotics of $d_{n,2}(X,H)$: (c.f. [Der09, section 4.2]) This classical result, which has its origin in information theory, goes back to Kolmogorov. The proof needs the following ingredients:

1. The distortion rate function of a Gaussian r.e. ξ in $(\mathbb{R}, \|\cdot\|)$ with variance σ^2 is well known and reads
$$d_{n,2}^2(\xi,\mathbb{R}) = \frac{\sigma^2}{n^2}.$$

2. The distortion rate function allows to estimate the distortion of a multivariate (even infinite dimensional) Gaussian r.e. through the sum of distortions of univariate Gaussian r.e.'s, i.e.
$$d_{n,2}^2\left(\sum_{i=1}^{\infty}\sqrt{\lambda_i}\xi_i e_i, H\right) = \sum_{i=1}^{\infty}\sqrt{\lambda_i} d_{n_i,2}^2(\xi_i,\mathbb{R}) = \sum_{i=1}^{\infty}\frac{\lambda_i}{n_i^2}$$
with $n_i \geq 1$ subject to $\prod_{i=1}^{\infty} n_i \leq n$.

3. The solution of the optimization problem

$$\min \quad \sum_{i=1}^{\infty} \frac{\lambda_i}{n_i^2}$$
$$c.t. \prod_{i=1}^{\infty} n_i \le n$$

can be derived by applying Lagrange multipliers and is given by

$$n_i = \left(\frac{\lambda_i}{\lambda_i \wedge \kappa^*} \right)^{\frac{1}{2}},$$

where the constant κ^* is given as the unique solution of the equation

$$\kappa^* = \inf_{\kappa > 0} \left\{ \prod_{i=1}^{\infty} \left(\frac{\lambda_i}{\kappa \wedge \lambda_i} \right)^2 \le n \right\}.$$

This result is also known as the Kolmogorov inverse water filling principle.

4. The final step is to compute the asymptotics of the distortion rate function based on the assumption about the asymptotics of the eigenvalues and the explicit representation of the distortion rate function.

Step 2: Asymptotic upper bound for $e_{n,2}(X,H)$: The approach used by Luschgy and Pagès in [LP04a] is known as a block quantization or as subband decomposition approach. The idea of the proof is as follows:

1. First note, that every quantizer $\alpha \in S_{n,2}(X,H)$ is contained in the linear span of the eigenvectors corresponding to the n largest eigenvalues of the covariance operator C_X of X (see [LP02, Theorem 3.1, Theorem 3.2]). Therefore, by orthogonality of the eigenvectors we obtain for every sequence $(\xi_j)_{j\in\mathbb{N}}$ of independent $\mathcal{N}(0,1)$-distributed r.e's

$$e_{n,2}^2(X,H) \le e_{n,2}^2\Big(\sum_{i=1}^{d} \sqrt{\lambda_i}\xi_i e_i, H\Big) + \sum_{i=d+1}^{\infty} \lambda_i$$

for every $d \in \mathbb{N}$, where equality is attained, if

$$d \ge d^*(n) := \inf\left\{ d \in \mathbb{N} : \mathcal{C}_{n,r}(X,H) \cap (\operatorname{span}\{e_1, \ldots, e_d\})^n \ne \varnothing \right\}.$$

2. Let $(l_n)_{n\in\mathbb{N}}, (m_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{N}$. By considering m_n blocks of size l_n, we obtain for natural numbers $(n_i)_{i\le m_n}$ with $\prod_{i=1}^{m_n} n_i \le n$ by using orthogonality of $(e_i)_{i\in\mathbb{N}}$ and Proposition 1.2.5

$$\begin{aligned}
e_{n,2}^2(X,H) &\le \sum_{i=1}^{m_n} e_{n_i,2}^2\left(\sum_{j=l_n(i-1)+1}^{l_n i} \sqrt{\lambda_j}\xi_j e_j, H \right) + \sum_{i=l_n m_n+1}^{\infty} \lambda_i \\
&\le \sum_{i=1}^{m_n} \lambda_{l_n(i-1)+1} e_{n_i,2}^2(N(0,I_{l_n}), \mathbb{R}^{l_n}) + \sum_{i=l_n m_n+1}^{\infty} \lambda_i \\
&\le \sum_{i=1}^{m_n} \lambda_{l_n(i-1)+1} n_i^{-\frac{2}{l_n}} C(l_n) + \sum_{i=l_n m_n+1}^{\infty} \lambda_i,
\end{aligned}$$

where $\mathbb{R}^{l_n}$ is equipped with the Euclidean norm and

$$C(l) := \sup_{k\geq 1} k^{\frac{2}{l}} e_{k,2}(N(0,I_l),\mathbb{R}^l),\ l \in \mathbb{N}.$$

3. One shows that $\frac{C(l)}{l} \to 1$ as $l \to \infty$ see ([LPW08, Proposition 1],[LP04a, Proposition 4.4]). Here, the lower bound is a classical quantization result, which can be obtained by using a random quantizer upper bound and a ball lower bound for

$$Q(l) := \lim_{k\to\infty} k^{\frac{2}{l}} e_{k,2}(N(0,I_l),\mathbb{R}^l),$$

which yields

$$\frac{Q(l)}{l} \to 1.$$

4. Finally, one has to choose for each $n \in \mathbb{N}$ the constants m_n, l_n and $(n_j)_{1\leq j\leq m_n}$ such that

$$e_{n,2}^2(X,H) \lesssim \sum_{i=1}^{m_n} \lambda_{l_n(i-1)+1} n_i^{-\frac{2}{l_n}} l_n + \sum_{i=l_n m_n+1}^{\infty} \lambda_i$$

achieves an asymptotic minimal rate. Under the assumption of regularly varying eigenvalues, one chooses for some $\theta \in (0,1)$ and $n \in \mathbb{N}$

$$\begin{aligned} l_n &= \left\lfloor \max\{1,\log(n)\}^{\theta} \right\rfloor, \\ m_n &= \max\{k \geq 1 : n^{\frac{1}{k}} \lambda_{(k-1)l_n+1}^{\frac{l_n}{2}} \left(\prod_{i=1}^{k} \lambda_{(j-1)l_n+1}\right)^{-\frac{l}{2k}} \geq 1\}, \\ n_j(n) &= \left\lfloor n^{\frac{1}{m_n}} \lambda_{(j-1)l_n+1}^{\frac{l_n}{2}} \left(\lambda_{(i-1)l_n+1}\right)^{-\frac{l_n}{2m_n}} \right\rfloor,\ j \in \{1,\ldots,m_n\}. \end{aligned}$$

The remainder of the proof is calculus, see [LPW08] or [LP04a] and the references therein.

□

Asymptotics of quantization errors in infinite dimensional Banach spaces: In case the separable Banach space $(E,\|\cdot\|)$ is non-Hilbertian, only weaker results concerning the asymptotic behavior of quantization errors have been obtained so far. Nevertheless, many relationships to other properties of random elements have been established, namely to

- the small ball function $\phi_\mu : \mathbb{R}^+ \to \mathbb{R}^+$ of the distribution $\mu = \mathbb{P}^X$, which is defined for $\epsilon > 0$ as
$$\phi_\mu(\epsilon) := -\log\left(\mu(B(0,\epsilon))\right).$$
- the Kolmogorov n-width, that is for a random element X in the Banach space E
$$k_{n,r}(X,E) := \inf\left\{\|X-Y\|_{L_r(\mathbb{P})} : Y \text{ r.e. in E },\dim(\operatorname{span}(Y(\Omega))) \leq n\right\}.$$

- the linear n-width, that is for a random element X in the Banach space E
$$l_{n,r}(X,E) := \inf\left\{\|X - f(X)\|_{L_r(\mathbb{P})} : f \in L(E,E), \dim(\operatorname{span}(f(E))) \le n\right\},$$
where $L(E,E) := \{f : E \to E, \text{ linear and continuous }\}$.

- approximation with Parseval frames for the Gaussian r.e. $X \in E$, i.e.
$$\xi_{n,r}(X,E) = \inf\left\{\left\|X - \sum_{i=1}^{n}\xi_i h_i\right\|_{L_r(\mathbb{P})} : (h_i)_{i\in\mathbb{N}} \text{ parseval frame in } \mathcal{H}_\mu\right\}.$$

Furthermore, considering the case of random elements (not necessarily Gaussian) induced as the path of a stochastic process $(X_t)_{t\in I}$ for an interval $I \subset \mathbb{R}$, relations to

- the pathwise regularity of a stochastic process, i.e.
$$\mathbb{E}\,|X_t - X_s|^r \le \phi(t-s)^r$$
for some measurable function $\phi : \mathbb{R} \to \mathbb{R}$, and

- the covariance function of a Gaussian process, that is
$$\Gamma(s,t) = \mathbb{E}X_t X_s,$$

have been established.

We will only formulate one result describing the relation to the small ball function and give an idea of the proof. In chapter 6 we will treat the cases involving stochastic processes in more detail. The Kolmogorov and linear approximation n-width are closely related to the small ball function, such that the results look similarly in many cases (see Creutzig, [Cre02, Corollary 4.7.2]).

Note, that due to a result of Tsyrelson (see [Lif95, Theorem 11] or [Bog98, Corollary 4.4.2]), the small ball function ϕ_μ of a centered Gaussian r.e. X in E is strictly decreasing, such that the inverse ϕ_μ^{-1} exists.

Theorem 1.2.17. *(cf. [GLP03, Theorem 1.2], see also [DFMS03, Theorem 3.1] and [Der03, Theorem 3.2.3]) Let X be a Gaussian r.e. in the separable Banach space $(E, \|\cdot\|)$ with law $\mu = \mathbb{P}^X$ and let $\dim \mathcal{H}_\mu = \infty$. Then, for every $\delta > 1$*
$$\phi_\mu^{-1}(\log(\delta n)) \lesssim e_{n,r}(X,E) \lesssim \phi_\mu^{-1}\left(\frac{\log(n)}{2\delta}\right), \quad n \to \infty.$$
In particular, if ϕ_μ (and thus ϕ_μ^{-1}) is regularly varying at ∞ with index $-a$, $a \in (0,\infty)$, then
$$\phi_\mu^{-1}(\log(n)) \approx e_{n,r}(X,E), \quad n \to \infty.$$

As in the previous paragraphs, we want to give a deeper insight into the proof of the Theorem. It heavily relies on the close relationship between the small ball function ϕ_μ and the metric entropy of the unit ball $\mathcal{K}_\mu$ in the Cameron-Martin-space $\mathcal{H}_\mu$, which we will introduce now.

Definition 1.2.18. (Metric Entropy) For a subset $A \in \mathcal{B}(E)$ and $\epsilon > 0$ we set

$$N_e(\epsilon, A) := \inf\{n \in \mathbb{N} : \exists \alpha \subset E, |\alpha| \leq n, A \subset \alpha + B(0,\epsilon)\}.$$

Then, $H_e(\epsilon, A) := \log(N_e(\epsilon, A))$ is called the metric ϵ-entropy of A.

The starting point of the proof of Theorem 1.2.17 is a discovery of Kuelbs and Li [KL93], who found a tight relationship between the small ball function and the metric entropy. The results have been extended by Li and Linde in [LL99]. A key lemma obtained in these papers, which is a consequence of the Isoperimetric inequality (Proposition 1.1.8) and the Estimation of shifted balls (Proposition 1.1.11), reads as follows.

Lemma 1.2.19. *For $\epsilon, \lambda > 0$ and $n \in \mathbb{N}$, one has*

$$H_e(2\epsilon, \lambda\mathcal{K}_\mu) \leq \phi_\mu(\epsilon) + \frac{\lambda^2}{2},$$

as well as

$$H_e(\epsilon, \lambda\mathcal{K}_\mu) - \phi_\mu(2\epsilon) \geq \Phi(\lambda + \Phi^{-1}(\mu(B(0,\epsilon)))).$$

Sketch of the proof of Theorem 1.2.17:

Proof. There are two inequalities to be proven, we will start with the asymptotic lower bound for $e_{n,r}(X,E)$.

- Lower bound for $e_{n,r}(X,E)$: The key argument to be used for the lower bound is the Anderson inequality. Let $n \in \mathbb{N}$, $\epsilon > 0$ and α be an n-codebook. Then, unimodality of μ, meaning $\mu(B(a,\epsilon)) \leq \mu(B(0,\epsilon))$ for all $a \in E$ and $\epsilon > 0$ (see Proposition 1.1.6), implies

$$\begin{aligned} e^r_{n,r}(X,E) &\geq \int_{(\alpha+B(0,\epsilon))^c} \min_{a\in\alpha} \|x-a\|^r \, d\mu(x) \\ &\geq \epsilon^r \left(1 - \mu\left(\alpha + B(0,\epsilon)\right)\right) \\ &\geq \epsilon^r \left(1 - \sum_{a\in\alpha} \mu\left(B(a,\epsilon)\right)\right) \\ &\geq \epsilon^r (1 - n\mu(B(0,\epsilon))) \end{aligned}$$

Choosing $\epsilon = \epsilon(n) := \phi_\mu^{-1}(\log(\delta n))$ for some $\delta > 1$, one obtains

$$(1 - n\mu(B(0,\epsilon(n)))) = (1 - \frac{1}{\delta}),$$

which yields the asserted lower bound.

- Upper bound for $e_{n,r}(X,E)$: The idea is to cover for a fixed n a multiple of the unit ball $\mathcal{K}_\mu$ in $\mathcal{H}_\mu$ with n small balls, and to control the remainder with the isoperimetric inequality, such that the quantization error is limited by the radius of the covering small balls. To be more precise, let $\epsilon, \lambda > 0$ and the codebook $\alpha = \alpha(\epsilon,\lambda) \subset E$ such that

$$|\alpha(\epsilon,\lambda)| = N_e(2\epsilon, \lambda\mathcal{K}_\mu)$$

and

$$\lambda\mathcal{K}_\mu \subset \alpha + B(0,2\epsilon).$$

Therefore, $A_{\lambda,\epsilon} := \lambda\mathcal{K}_\mu + B(0,\epsilon) \subset \alpha + B(0,3\epsilon)$ and we obtain in view of Hölder's inequality, for all $\epsilon \leq 1$

$$\begin{aligned} e^r_{|\alpha(\epsilon,\lambda)|,r}(X,E) &\leq \int_{A_{\lambda,\epsilon}} \min_{a\in\alpha} \|x-a\|^r \, d\mu(x) + \int_{A^c_{\lambda,\epsilon}} \min_{a\in\alpha} \|x-a\|^r \, d\mu(x) \\ &\leq (3\epsilon)^r + \left(\mu(A^c_{\lambda,\epsilon})\right)^{\frac{1}{2}} 2^r (\|X\|^{2r}_{L_r(\mathbb{P})} + 2^{2r})^{\frac{1}{2}}. \end{aligned}$$

Consequently, using $A^c_{\lambda,\epsilon} = (\lambda\mathcal{K}_\mu + B(0,\epsilon))^c$, the isoperimetric inequality (Proposition 1.1.8) implies

$$e^r_{|\alpha(\epsilon,\lambda)|,r}(X,E) \leq (3\epsilon)^r + \left(1 - \Phi(\Phi^{-1}(\mu(B(0,\epsilon))) + \lambda)\right)^{\frac{1}{2}} \kappa_r,$$

with $\kappa_r := 2^r(\|X\|^{2r}_{L_r(\mathbb{P})} + 2^{2r})^{\frac{1}{2}}$.
The final step is to choose $\lambda = \lambda(\epsilon)$ sufficiently large, such that

$$\log(|\alpha(\epsilon,\lambda(\epsilon))|) \lesssim 2\delta\phi_\mu(\epsilon), \quad \epsilon \to 0,$$

but still

$$\left(1 - \Phi(\Phi^{-1}(\mu(B(0,\epsilon))) + \lambda(\epsilon))\right)^{\frac{1}{2}} \kappa_r = \mathrm{o}(3\epsilon)^r$$

as $\epsilon \to 0$. In [GLP03] this sequence has been chosen for $\delta > 1$ as $\lambda(\epsilon,\delta) := (2(2\delta-1)\phi_\mu(\epsilon))^{\frac{1}{2}}$. The remainder of the upper bound of the proof consists of calculus and the application of Lemma 1.2.19.

□

Random quantizer upper bounds: As an alternative approach to the proof of the upper bound for the quantization error in general Banach spaces, one may consider quantization with random quantizers instead, which also achieves the true weak asymptotic upper bound.

Definition 1.2.20. Let E be a separable Banach space and $X \in L_r(\mathbb{P},E)$ for some $r > 0$. We define the optimal n-th $L_r(\mathbb{P},E)$ random quantization error by

$$\tau_{n,r}(X,E) := \inf\left\{\int\int \min_{i=1,\dots,n} \|x - x_i\|^r \, d\nu^n(x_1,\dots,x_n) d\mu(x)\right\}^{\frac{1}{r}},$$

where the infimum is taken over all $\nu \in \mathcal{M}_r(\mathcal{B}(E))$.

Obviously, the optimal random quantization error gives an upper bound for the optimal quantization error since, for every $n \in \mathbb{N}$, and $\nu \in \mathcal{M}_r(\mathcal{B}(E))$, one has

$$\int \min_{i=1,\dots,n} \|x - y_i\|^r \, d\mu(x) \geq \inf_{\alpha} \left\{ \int \min_{a\in\alpha} \|x - a\|^r \, d\mu(x) : \alpha \subset E, |\alpha| = n \right\} \nu^n(dy) a.s..$$

The following upper bound has been established by Dereich et al. (see [DFMS03, Theorem 2.1, Theorem 4.1]).

Theorem 1.2.21. *Let X be a Gaussian r.e. in the separable Banach space $(E, \|\cdot\|)$, $r > 0$ and let the small ball function ϕ_μ satisfy*

$$\lim_{\epsilon\to 0} \frac{\phi_\mu(\epsilon)}{(\log(\epsilon))^{\frac{1}{a}}} = \infty$$

for some $a \in (0,1)$. Then, for every $r, \delta > 0$

$$\tau_{n,r}(X,E) \lesssim 2\phi_\mu^{-1}\left(\frac{\log(n)}{2+\delta}\right), \quad n \to \infty.$$

In particular, if the small ball function admits the representation

$$\phi_\mu(x) = x^{-a} J\left(\frac{1}{x}\right), \quad x \to 0$$

for a slowly varying function $J(\cdot)$ and $-a < 0$, then

$$\tau_{n,r}(X,E) \lesssim 2^{1+\frac{1}{a}} \phi_\mu^{-1}\left(\log(n)\right), \quad n \to \infty.$$

Remark 1.2.22. In ([DFMS03]) the authors use independent copies $X_i, i = 1, \dots, n$ of X to prove Theorem 1.2.21. Generally, this approach does not yield the sharp asymptotic rate for the optimal random quantization error, see [GL00, section 9] for the finite dimensional case and [Der03, Corollary 7.5.7] for general Gaussian r.e.'s in (infinite) dimensional Hilbert spaces.

Local and global behavior of optimal codebooks: So far, we have mainly treated the asymptotic behavior of the optimal quantization error. Much less is known about the exact shape of optimal codebooks, given those exist. Unfortunately, almost no explicit formulas for optimal codebooks are known so far. Therefore, one is strongly interested to obtain at least geometric properties optimal codebooks have to fulfill. One of these properties is, for a Voronoi partition $\{C_a(\alpha_n), a \in \alpha_n\}$ for an r-optimal n-codebook α_n, the question of the μ-weights of the Voronoi cells $\mu(V_a(\alpha))$. Furthermore, questions have been raised regarding the following characteristics of (optimal) codebooks:

Definition 1.2.23. (Local and global characteristics) Let α be an n-codebook for the r.e. X in the separable Banach space $(E, \|\cdot\|)$, $n \in \mathbb{N}$ and $r > 0$. Then:

1. The maximum radius for a bounded set $A \subset E$ is defined by
$$\rho(A) := \sup\{\|a\| : a \in A\}.$$
If $A = \alpha$ a codebook, we also call $\rho(\alpha)$ the quantization radius of α.

2. Suppose that $\mathcal{C}_{n,r}(X,E) \neq \varnothing$. The n-th upper quantization radius for the r.e. X in E of order r is defined as
$$\overline{\rho}_{n,r}(X,E) := \sup\{\rho(\alpha) : \alpha \in \mathcal{C}_{n,r}(X,E)\}.$$
Accordingly, we define the n-th lower quantization radius for the r.e. X in E of order r as
$$\underline{\rho}_{n,r}(X,E) := \inf\{\rho(\alpha) : \alpha \in \mathcal{C}_{n,r}(X,E)\}.$$

3. We define the local quantization error for X in E of order r for the code $a \in \alpha$ with respect to the Voronoi partition $\{C_a(\alpha), a \in \alpha\}$ by
$$e_{r;\text{loc}}(\mu, E; \alpha, C_a(\alpha)) := \left(\int_{C_a(\alpha)} \|x-a\|^r \, d\mu(x) \right)^{\frac{1}{r}}.$$
$e_{r;\text{loc}}(X, E; \alpha, C_a(\alpha))$ is also known as the local inertia.

4. We set for $n \in \mathbb{N}$ the increments of the r-th power of the quantization error
$$\Delta_{n,r}(X,E) := e_{n,r}^r(X,E) - e_{n+1,r}^r(X,E).$$
Hereafter, we will usually speak of the increments of the quantization error and omitting the r-th power, when considering $\Delta_{n,r}(X,E)$.

5. Let $s > 0$. The random variable X is said to have the $r-s$-property, if for every sequence $(\alpha_n)_{n\in\mathbb{N}}$ of r-optimal n-codebooks for X in E
$$\limsup_{n\to\infty} \frac{e_s(X,E;\alpha_n)}{e_{n,r}(X,E)} < \infty.$$

The local quantization error, the weights of the Voronoi cells, and the quantization radius seem to be interesting characteristics describing the geometry of optimal codebooks. The increments of the quantization error does not seem to fit into this group, but, as we will see in chapter 2, there is a close relationship to the aforementioned characteristics. Similarly, we will be able to show for specific r.e.'s X a close relationship between the $r-s$-property and the quantization radius.

One key result obtained so far in the finite dimensional case $(E, \|\cdot\|) = \left(\mathbb{R}^d, \|\cdot\|\right)$ is the weak asymptotics of the increments of the quantization error, which is due to Graf, Luschgy, and Pages [GLP10]. Parts of the results have already been proven in former articles of these authors, see e.g. [PS08], [GL02], [GLP08]. We will present here a slightly extended version of the result for distributions having a non-vanishing Lebesgue-continuous part.

Theorem 1.2.24. *Let $(E, \|\cdot\|) = (\mathbb{R}^d, \|\cdot\|)$ for an arbitrary norm $\|\cdot\|$, $r \in (0,\infty)$ and $X \in L_{r+\delta}(\mathbb{P})$ for some $\delta > 0$ with $f = \frac{\partial \mu_a}{\partial \lambda^d} \not\equiv 0$. Then, there exists a constant $C_\Delta(\mu) \in [1,\infty)$ such that*

$$\frac{1}{C_\Delta(\mu)} n^{-(1+\frac{r}{d})} \le \Delta_{n,r}(X,E) \le C_\Delta(\mu) n^{-(1+\frac{r}{d})},$$

for all $n \in \mathbb{N}$.

Proof. Let $\mu = \mu_a + \mu_s$ denote the Lebesgue decomposition of μ, $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of r-optimal n-codebooks for X in $\mathbb{R}^d$ and $(\{C_a(\alpha_n), a \in \alpha_n\})_{n\in\mathbb{N}}$ be a corresponding sequence of Voronoi partitions.

Step 1: (Upper bound) We partly follow the lines of the proof in ([GLP08]). Obviously, we have

$$\left|\left\{a \in \alpha_{n+1} : \mu(C_a(\alpha_{n+1})) > \frac{4}{n+1}\right\}\right| \le \frac{n+1}{4},$$

and

$$\left|\left\{a \in \alpha_{n+1} : e^r_{r;\mathrm{loc}}(X,E;\alpha_{n+1},C_a(\alpha_{n+1})) > \frac{4e^r_{n,r}(X,E)}{n+1}\right\}\right| \le \frac{n+1}{4},$$

which implies for

$$\beta_{n+1} := \Big\{a \in \alpha_{n+1} : e^r_{r;\mathrm{loc}}(X,E;\alpha_{n+1},C_a(\alpha_{n+1})) \le \frac{4e^r_{n,r}(X,E)}{n+1},$$
$$\mu(C_a(\alpha_{n+1})) \le \frac{4}{n+1}\Big\}$$

the cardinality $|\beta_{n+1}| \ge \frac{n+1}{2}$. We choose a constant $\kappa \in \mathbb{R}$ sufficiently large, such that

$$\mu_r([-\kappa,\kappa]^d) = \left\|f^{\frac{d}{d+r}}\right\|_1^{-1} \int_{[-\kappa,\kappa]^d} f^{\frac{d}{d+r}} \, d\lambda^d \ge \frac{3}{4}.$$

$\mu_r(\partial[-\kappa,\kappa]^d) = 0$ implies in view of Theorem 1.2.14 and the Portmanteau Theorem

$$\left|\alpha_{n+1} \cap [-\kappa,\kappa]^d\right| \gtrsim \frac{3(n+1)}{4}, \quad n \to \infty,$$

and therefore

$$\left|\beta_{n+1} \cap [-\kappa,\kappa]^d\right| \gtrsim \frac{n+1}{4}, \quad n \to \infty. \tag{1.20}$$

There exists a constant $C < \infty$ such that for every $n \ge 7$ the cube $[-\kappa,\kappa]^d$ can be composed into $\left\lfloor \frac{n+1}{8} \right\rfloor$ cubes with diameter not longer than $Cn^{-\frac{1}{d}}$. In combination with equation (1.20), we find two elements $a_{n+1}, b_{n+1} \in \beta_{n+1}$ being in the same cube, i.e.

$$\|a_{n+1} - b_{n+1}\| \le Cn^{-\frac{1}{d}},$$

for all $n \geq n^* \in \mathbb{N}$. Using $a_{n+1}, b_{n+1} \in \beta_{n+1}$ we deduce for $n \geq n^*$

$$\begin{aligned}\Delta_{n,r}(X,E) &\leq e_r^r(X,E;\alpha_{n+1}\backslash\{a_{n+1}\}) - e_{n+1,r}^r(X,E) \\ &\leq \int_{C_{a_{n+1}}(\alpha_{n+1})} \|x-b_{n+1}\|^r - \|x-a_{n+1}\|^r \, d\mu(x) \\ &\leq (2^r-1)e_{r;\mathrm{loc}}^r(X,E;\alpha_{n+1}, C_{a_{n+1}}(\alpha_{n+1})) \\ &+ 2^r \|a_{n+1}-b_{n+1}\|^r \mu(C_{a_{n+1}}(\alpha_{n+1})) \\ &\leq (2^r-1)\frac{4}{n+1} e_{n+1,r}^r(X,E) + 2^r C n^{-\frac{r}{d}} \frac{4}{n+1},\end{aligned} \tag{1.21}$$

which yields in virtue of Theorem 1.2.12 the asserted upper bound.
Step 2: (Lower bound) We partly follow the lines of the proof in ([GLP10]). We set for $n \in \mathbb{N}$ and $y \in \mathbb{R}^d$ $\beta_n := \alpha_n \cup \{y\}$ and $\delta_n = \mathrm{dist}(y,\alpha_n)$, where $\mathrm{dist}(x,A) = \inf_{a\in A}\|x-a\|$ for $x \in \mathbb{R}^d, A \subset \mathbb{R}^d$. For every $b < \frac{1}{2}$ we obtain

$$\begin{aligned}\Delta_{n,r}(X,E) &\geq e_{n,r}^r(X,E) - e_r^r(X,E;\beta_n) \\ &\geq \int_{W_y(\beta_n)} \min_{a\in\alpha_n} \|x-a\|^r - \|x-y\|^r \, d\mu(x) \\ &\geq \int_{B(y,b\delta_n)} ((1-b)^r - b^r)\delta_n^r d\mu(x) \\ &\geq \mu_a\left(B(y,b\delta_n)\right)((1-b)^r - b^r)\delta_n^r.\end{aligned} \tag{1.22}$$

First note, that for every $y \in \mathrm{supp}(\mu_a)$

$$\delta_n = \mathrm{dist}(y,\alpha_n) \to 0, \quad n \to \infty,$$

since otherwise $e_r(X,E;\alpha_n) \nrightarrow 0$ in contradiction to equation (1.14). We set for $n \in \mathbb{N}$

$$f_n(y) := \frac{\mu_a\left(B(y,b\delta_n)\right)}{\lambda^d((B(y,b\delta_n)))}$$

By Lebesgue's differentiation Theorem, we obtain $f_n \overset{n\to\infty}{\longrightarrow} f$ λ^d-a.s.. Let $\kappa := \sup\{c > 0 : \mu_a\{f > c\} \geq \frac{3\mu_a(\mathbb{R}^d)}{4}\}$. Then, Egorov's Theorem (see e.g. [Kal02, Lemma 1.3.6]) implies

$$f_n(y) \to f(y), \quad n \to \infty \text{ uniformly on } A$$

for a subset $A \subset \{f > \kappa\}$ such that $\mu_a(A) \geq \frac{1}{2}\mu_a(\mathbb{R}^d)$ and therefore $\lambda^d(A) > 0$. In combination with equation (1.22) we obtain for n sufficiently large with $C(b) := ((1-b)^r - b^r)$ by integrating both sides over A with respect to $d\lambda^d$

$$\begin{aligned}\Delta_{n,r}(X,E)\lambda^d(A) &\geq \int_A \mu_a\left(B(y,b\delta_n)\right) C(b)\delta_n^r d\lambda^d(y) \\ &\geq \frac{C(b)\kappa}{2}\int_A \lambda^d\left(B(y,b\delta_n)\right)\delta_n^r d\lambda^d(y) \\ &= \frac{\kappa C(b)\lambda^d\left(B(y,b)\right)}{2}\int_A \delta_n^{r+d} d\lambda^d(y) \\ &\geq \frac{\kappa C(b) b^d \lambda^d\left(B(0,1)\right)\lambda^d(A)}{2} e_{n,r+d}^{r+d}(U(A),\mathbb{R}^d)\end{aligned}$$

which yields in view of Theorem 1.2.12 the asserted lower bound. □

Remark 1.2.25. Suppose that X is a r.e. on $\mathbb{R}^d$ with non-vanishing Lebesgue-continuous part and finite r-th moment.

1. As a direct consequence of the theory of asymptotic sequences, there is only one possible sharp asymptotic rate the increments of the quantization error could admit. As a general result (see A.10 or [BGT87, Proposition 1.5.10]), one has for every regularly varying function $f : [A, \infty) \to \mathbb{R}$ with index $\alpha < -1$ at infinity
$$\frac{xf(x)}{\int_x^\infty f(t)d\lambda(t)} \overset{x\to\infty}{\longrightarrow} -(\alpha+1). \tag{1.23}$$
Assuming that $\Delta_{n,r}(X,E)$ is regularly varying with index $\alpha < -1$, we obtain as immediate consequence by setting $f(x) = \Delta_{\lceil x\rceil,r}(X,E)$ in equation (1.23)
$$e_{n,r}^r(X,E) = \sum_{j=n}^{\infty} \Delta_{j,r}(X,E) \sim \frac{1}{-(\alpha+1)}\Delta_{n,r}(X,E)n, \quad n \to .\infty$$
Hence, one needs to have
$$\Delta_{n,r}(X,E) \sim -\frac{\alpha+1}{n} e_{n,r}^r(X,E), \quad n \to \infty.$$
In view of the Zador theorem, we obtain for distributions with non-vanishing Lebesgue-continuous part $\mu_a = f d\lambda^d$
$$\Delta_{n,r}(X,E) \sim \frac{r}{d} Q_{r,\|\cdot\|}([0,1]^d)\,\|f\|_{\frac{d}{d+r}}\, n^{-\frac{r+d}{d}}, \quad n \to \infty$$

2. The general result presented in part 1) concerning the asymptotic behavior of integrated asymptotics describes the easier direction of the relationship between sequences and its increments. The converse direction is much more difficult and does not hold in general. To obtain the converse result, one needs additional conditions (also called Tauberian conditions) to deduce from the asymptotics of a sequence the asymptotics of its increments. One famous example for such a Tauberian condition is part of the Monotone Density Theorem (see Theorem A.11).
Suppose that $e_{n,r}^r(X,E) = \sum_{i=n}^\infty \Delta_{n,r}(X,E)$ is regularly varying at infinity with index $\rho < -1$ and $\Delta_{n,r}(X,E)$ is monotone on $[N,\infty)$ for some $N \in \mathbb{N}$. Then $\Delta_{n,r}(X,E)$ is regularly varying at infinity and the asymptotics is uniquely determined via equation (1.23).
Unfortunately, it is not clear whether such a monotony holds for the increments of the quantization error of interesting classes of distributions.

Remark 1.2.26. The proof of Theorem 1.2.24 already uses two micro-macro inequalities being contained in the estimates (1.22) and (1.21) which we will use later more extensively.

Remark 1.2.27. The radius problem has already been analyzed in the thesis of Sagna (see [PS08]) for finite dimensional Hilbert spaces $(H, \|\cdot\|)$ and distributions having radial tails with respect to the Hilbert space norm $\|\cdot\|$. A main part of this thesis will be to

- solve open problems raised in [PS08],
- extend the results to arbitrary finite dimensional Banach spaces and more general distributions,
- define other characteristics of distributions giving a better understanding of the geometry of optimal codebooks,
- extend the results and related problems to infinite dimensional Banach spaces and
- make use of the solutions of the radius problems to improve the results obtained for other local quantization problems, especially the analysis of local inertia and weights of Voronoi cells (see [GLP10]).

Construction of optimal quantizers: Given the few explicit formulas known for optimal codebooks, one needs numerical algorithms to calculate optimal (or at least stationary) solutions for the quantization problem.
Historically, the first approach that was used is the so-called Lloyd-I procedure.

Let X be a r.e. in $\mathbb{R}$, $n \in \mathbb{N}$ be fixed and α_0 being an n-codebook in $\mathbb{R}$. Define inductively for $k \in \mathbb{N}$

$$\alpha_k := \left(\mathbb{E}\left(X | X \in C_{a_i}(\alpha_{k-1})\right)\right)_{1 \leq i \leq n},$$

for an arbitrary Voronoi partition $\{C_a(\alpha_{k-1}), a \in \alpha_{k-1}\}$. Under some reasonable assumptions on the distribution μ, one shows that the sequence

$$\left(e_2(X, \mathbb{R}; \alpha_k)\right)_{k \in \mathbb{N}}$$

is a non-increasing sequence and that $f_{\alpha_k}(X)$ converges to a random vector $\widehat{X}$ with $\widehat{\alpha} := \operatorname{supp}(\mathbb{P}^{\widehat{X}})$ satisfying $|\widehat{\alpha}| = n$. Furthermore, one observes easily that $\widehat{\alpha}$ is a stationary codebook. Under some additional regularity assumption (i.e. log-concavity of the density $f := \frac{\partial \mu}{\partial \lambda}$), one knows that $|S_{n,2}(X, \mathbb{R})| = 1$ which implies $C_{n,2}(X, \mathbb{R}) = \{\widehat{\alpha}\}$.

Even if the algorithm can be generalized to an arbitrary $d \in \mathbb{N}$, many problems arise. The calculation of the conditional expectation needs the explicit shape of the Voronoi partition and the calculation of integrals over complex polyhedron in $\mathbb{R}^d$, which is quite difficult. Furthermore, no condition verifying the optimality of the limiting codebook $\widehat{\alpha}$ is known.

One possible solution to tackle this problem is the so-called CLVQ-algorithm. Let X be a r.e. in the separable Banach space $(E, \|\cdot\|)$, $n \in \mathbb{N}$ be fixed and α_0 being an n-codebook in E. Suppose that $r > 1$ and the underlying norm $\|\cdot\|$ is smooth. A common shape of a gradient algorithm is

$$\alpha_{k+1} = \alpha_k - \gamma_n \nabla e_{n,r}^r(X, E; \alpha_k), \; k \in \mathbb{N} \tag{1.24}$$

with $(\gamma_n)_{n\in\mathbb{N}}$ being a step-sequence and the gradient

$$\nabla e^r_{n,r}(X,E;\alpha_k) = r\mathbb{E}\left(1_{C_{a_i}(\alpha_k)\setminus\{a_i\}}(X)\left\|X-a_i\right\|^{r-1}\nabla\left\|\cdot\right\|(X-a_i)\right) \in (E')^n$$

(see [GLP07, Proposition 4]), for an Voronoi partition $\{C_{a_i}(\alpha_k), a_i \in \alpha_k\}$. A one-point Monte-Carlo approximation of the gradient transforms equation (1.24) into

$$\alpha_{k+1} = \alpha_k - \gamma_n r\left(1_{C_{a_i}(\alpha)\setminus\{a_i\}}(\widetilde{X_k})\left\|\widetilde{X_k}-a_i\right\|^{r-1}\nabla\left\|\cdot\right\|(\widetilde{X_k}-a_i)\right) \quad k\in\mathbb{N}. \tag{1.25}$$

Under some regularity condition on the step sequence, particularly

$$\sum_{i\in\mathbb{N}}\gamma_i = \infty, \quad \sum_{i\in\mathbb{N}}\gamma_i^2 < \infty$$

the algorithm turns out to be very robust and to reach local minima of the optimization problem, even if no general convergence of the algorithm can be proven.

A key issue for the performance of the algorithms is a good initialization. As mentioned by several authors (see e.g. [PN07] , [PS08] or [Yee10]), the knowledge of the quantization radius of optimal codebooks can significantly improve the convergence of the algorithm. Particularly, the authors propose explicit initial codebooks α_0 based on the knowledge of the quantization radius for optimal codebooks.
In [PN07] the authors propose for specific exponential distributed r.e.'s X on the real line $\mathbb{R}$ an initial n-codebook $\alpha_0 = \{a_1,\dots,a_n\}$ with

$$a_1 = -\rho_n(X), \quad a_n = \rho_n(X)$$

and the remaining ordered codes $a_i, 2\le i\le n-1$ such that

$$\mu_r(V_{a_i}(\alpha_0)) = \mu_r(V_{a_j}(\alpha_0)),\ i,j\in\{2,\dots,n-1\},$$

where $\rho_n(X)$ is a reasonable approximation for the quantization radius for $\alpha \in \mathcal{C}_{n,r}(X,E)$.
Sagna proposes in his dissertation to initialize the algorithm for the case $d\ge 1$ (see [Sag08]) by setting

$$\alpha_0 = \rho_n\left\{\frac{Z_1}{\left\|Z_1\right\|_{L_r(\mathbb{P},\mathbb{R}^d)}},\dots,\frac{Z_n}{\left\|Z_n\right\|_{L_r(\mathbb{P},\mathbb{R}^d)}}\right\}$$

with again ρ_n being an asymptotic approximation for the quantization radius and $\mathbb{P}^{Z_i} = N(0,I_d)$. Hence the starting codebook is uniformly distributed over the sphere S_{d-1} in $(\mathbb{R}^d,\left\|\cdot\right\|)$.

Applications to numerical finance: Plenty of problems arising in financial applications can be expressed as the estimation of expectation of the form

$$\mathbb{E}F(X) \tag{1.26}$$

for some r.e. X in the Banach space E and a mapping $F : E \to \mathbb{R}$. Typically, the r.e. X is either a multivariate r.e. in $\mathbb{R}^d$ or a r.e. induced by the path of a stochastic process with value in $L_p(I)$ or $C(I)$ for compact interval $I \subset \mathbb{R}^d$. There is a variety of situations in which quantization methods can be applied to help solving problems of such a type. For a general overview, one may consult [PPP04] for the case of X r.e. in $\mathbb{R}^d$ or [PP05] for the functional quantization case.

(*Cubature formulae*) The first idea is to use precomputed optimal quantizers $\alpha_n \in \mathcal{C}_{n,r}(X,E)$ for some r.e. X and $r > 0$ to derive for broad classes of functionals F good approximations in the form of cubature formulae for the expectation 1.26, i.e.

$$\mathbb{E}F(X) \approxeq \sum_{i=1}^{n} \mu(C_{a_i}(\alpha_n))F(a_i).$$

To estimate the approximation error, one derives for Lipschitz continuous $F : E \to \mathbb{R}$ in view of Proposition 1.2.5

$$|\mathbb{E}F(X) - \mathbb{E}F(f_{\alpha_n}(X))| \le L_F \left\| X - f_{\alpha_n}(X) \right\|_{L_1(E,\mathbb{P})} \le L_F e_{n,r}(X,E)$$

with L_F denoting the Lipschitz constant to F. Given $r = 2$ and the Hilbertian case $(E, \|\cdot\|) = (H, (\cdot,\cdot))$, one obtains as a consequence of the stationarity of optimal codebooks the following sharper version:

Lemma 1.2.28. *(c.f. [PPP04, Section 2]) Suppose that $F : E \to \mathbb{R}$ is continuously differentiable with Lipschitz continuous differential DF. Then for $\alpha_n \in \mathcal{C}_{n,2}(X,H)$ and $n \in \mathbb{N}$*

$$|\mathbb{E}F(X) - \mathbb{E}F(f_{\alpha_n}(X))| \le L_{DF} e_{n,2}^2(X,E).$$

(*Variance reduction with optimal codebooks*) Another popular way to use optimal quantizers is to use them as a control variate in a Monte-Carlo method, i.e.

$$\mathbb{E}F(X) \approxeq \sum_{i=1}^{n} \mu(C_{a_i}(\alpha_n)F(a_i) + \frac{1}{m}\sum_{i=1}^{m}(F(X_i) - F(f_{\alpha_n}(X_i)))$$

for independent copies $X_i, i \le m$ of X. Then, (see [PP05, section 7]) or [CDMGR09, Theorem 2] for

$$R_{m,n} := \mathbb{E}F(X) - \sum_{i=1}^{n} \mu(C_{a_i}(\alpha_n)F(a_i) - \frac{1}{m}\sum_{i=1}^{m}(F(X_i) - F(f_{\alpha_n}(X_i)))$$

one estimates

$$\mathbb{E}\left|R_{m,n}\right|^2 \le \frac{\mathbb{E}\left(F(X) - F(f_{\alpha_n}(X))\right)^2}{m}$$

and in virtue of the CLT

$$\sqrt{m}R_{m,n} \overset{w}{\Rightarrow} N\big(0, \|F(X) - F(f_{\alpha_n}(X))\|_2^2\big),\ m \to \infty.$$

Hence, for Lipschitz continuous F and $\alpha_n \in \mathcal{C}_{n,2}(X,E)$

$$\mathbb{E}\left|R_{m,n}\right|^2 \leq \frac{L_F^2 e_{n,2}^2(X,E)}{m}.$$

Additional notations: For a r.e. X in the separable Banach space $(E, \|\cdot\|)$, we set the survival function

$$\overline{F^X} : \mathbb{R}_+ \to [0,1], \quad x \mapsto \mathbb{P}(\|X\| > x).$$

The generalized survival function we define for $s > 0$ by

$$\overline{F^X}_s : \mathbb{R}_+ \to [0, \mathbb{E}\|X\|^s], \quad x \mapsto \mathbb{E}1_{\|X\|>x}\|X\|^s.$$

References: A good introduction into Information theory is given by Cover and Thomas [CT91]. For historical aspects, see [Gra90], for a rigorous mathematical introduction one may consult [Iha93].
For aspects of finite dimensional quantization problems, see the summary article by Gray and Neuhoff [GN98] and the monograph of Graf and Luschgy [GL00] which contains a broad range of quantization results on absolutely continuous and fractal measures. For further results on finite dimensional fractal quantization, see e.g. [GL01], [Kre06] and [GL05].
For results on the infinite dimensional case, one may consult the dissertations by Fehringer [Feh01], Creutzig [Cre02], Dereich [Der03] and Wilberz [Wil08], as well as the research articles [LP02], [LP04a], [DFMS03], [GLP03], [DS06] for the Gaussian case, [Der08b], [Der08a], [LP06] for Gaussian diffusions, [ADSV09], [LP08], [AD09] for general Levy processes and [LGP] on fractal functional quantization.
For research articles on the radius problem, see [NN01] and [PS08], the local inertia have been treated in [GLP10], the $(r-s)$-problem in [GLP08]. Further references are also given in chapters 3,4 and 5.
A good introduction into numerical algorithms can be found in [PPP04] for the finite dimensional case and [PP05] for the functional quantization case. Both contain aspects for the variety of applications for optimal quantization.

For general references on quantization, see the homepage www.quantize.maths-fi.com which is devoted to quantization and its applications in numerical finance.

Chapter 2

Increments of quantization errors and related inequalities

This chapter is devoted to micro-macro inequalities and the estimation of the increments of the quantization error, which are two key ingredients for the study of the radius problems in chapters 3, 4 and 5.
In section 2.1 we will provide several micro-macro-inequalities, which will be useful in the subsequent sections and chapters. This includes two known micro-macro inequalities accomplished with an equivalent for the first micro-macro inequality for Gaussian r.e.'s as well as an extended version of the second micro-macro inequality. In section 2.2 we establish a general lower bound for the increments of the quantization error. This bound can equally be used in case of Lebesgue-continuous and singular distributions in $\mathbb{R}^d$ as well as r.e.'s in infinite dimensional Banach spaces. In particular, the lower bound gives an alternative proof for the lower bound in Theorem 1.2.24. Furthermore, we will show a general upper bound for the increments of the quantization error for r.e.'s satisfying a unimodality condition. Finally, we will show a few basic relations between the increments of the quantization error and the quantization radius of optimal codebooks, which is discussed in the subsequent chapters.

2.1 Micro-Macro inequalities

In this section, we will discuss optimal quantization related micro-macro inequalities. Those are inequalities which allow to deduce connections between local or geometric characteristics for optimal codebooks to global characteristics, such as the asymptotics of the quantization error, or its corresponding increments. In the proof of Theorem 1.2.24 we already saw two very important micro-macro inequalities for the finite dimensional case. We will formulate those micro-macro inequalities and add an infinite dimensional equivalent for the first micro-macro inequality for Gaussian random elements. We will need the following definitions:

Definition 2.1.1. A r.e. X in $(\mathbb{R}^d, \|\cdot\|)$ satisfies the lower peakless condition on $A \in \mathcal{B}(\mathbb{R}^d)$ if there exists a constant $c = c(A) > 0$ such that

$$\mu(B(x,s)) \geq c\frac{\partial \mu_a}{\partial \lambda^d}(x)\lambda^d(B(0,s)) = c\lambda^d(B(0,1))\frac{\partial \mu_a}{\partial \lambda^d}(x)s^d$$

for all $x \in A$ and $s > 0$ such that $B(x,s) \subset A$.

Definition 2.1.2. A function $f : [A, \infty) \to \mathbb{R}$, $A \in [0, \infty)$ is called almost decreasing on $[A, \infty)$ if for some constant $m_f \in (0, \infty)$

$$f(y) \geq m_f f(x)$$

for all $y \in [A, x]$ and $x \in [A, \infty)$.

Lemma 2.1.3. *Let X be a r.e. in $(\mathbb{R}^d, \|\cdot\|)$ such that*

$$f = \frac{\partial \mu_a}{\partial \lambda^d} \not\equiv 0$$

admits the representation $f(x) = g(\|x\|_0)$ for $x \in B_{\|\cdot\|_0}(0,\kappa)^c$, where $\|\cdot\|_0$ is an arbitrary norm on $\mathbb{R}^d$ and g is almost decreasing on $[\kappa, \infty)$ for a constant $\kappa \geq 0$. Then, X satisfies the lower peakless condition on $B_{\|\cdot\|_0}(0,\kappa)^c$.

Proof. Let C be a positive and finite constant such that $\frac{1}{C}\|\cdot\|_0 \leq \|\cdot\| \leq C\|\cdot\|_0$, $x \in B_{\|\cdot\|_0}(0,\kappa)^c$ and $s > 0$ such that $B(x,s) \subset B_{\|\cdot\|_0}(0,\kappa)^c$. First note, that

$$\{f \geq m_g f(x)\} \supset B_{\|\cdot\|_0}(0, \|x\|_0) \backslash B_{\|\cdot\|_0}(0,\kappa),$$

where m_g denotes the almost decreasing constant of g and

$$B(x,s) \supset B_{\|\cdot\|_0}(x, \frac{s}{C}),$$

such that

$$B(x,s) \cap \{f \geq m_g f(x)\} \supset \big(B_{\|\cdot\|_0}(0,\|x\|_0)\backslash B_{\|\cdot\|_0}(0,\kappa)\big) \cap B_{\|\cdot\|_0}(x, \frac{s}{C}) \tag{2.1}$$

By the triangle inequality, we have for $x' := x - \frac{x}{\|x\|_0}\frac{s}{2C}$

$$B_{\|\cdot\|_0}(x', \frac{s}{2C}) \subset B_{\|\cdot\|_0}(0,\|x\|_0)\backslash B_{\|\cdot\|_0}(0,\kappa),$$
$$B_{\|\cdot\|_0}(x', \frac{s}{2C}) \subset B_{\|\cdot\|_0}(x, \frac{s}{C}),$$

and thus with (2.1)

$$B(x,s) \cap \{f \geq m_g f(x)\} \supset B_{\|\cdot\|_0}(x', \frac{s}{2C}) \supset B(x', \frac{s}{2C^2}).$$

Consequently, we obtain

$$\begin{aligned}\mu(B(x,s)) &\geq \mu_a(B(x,s) \cap \{f \geq m_g f(x)\}) \geq m_g f(x)\lambda^d(B(x,s) \cap \{f \geq m_g f(x)\}) \\ &\geq m_g f(x)\lambda^d\left(B(x', \frac{s}{2C^2})\right) = m_g f(x)\left(\frac{1}{2C^2}\right)^d \lambda^d\left(B(x,s)\right)\end{aligned}$$

and thus the assertion with $c = m_g\left(\frac{1}{2C^2}\right)^d$. ∎

Proposition 2.1.4. *(First micro-macro inequality) Let E be a separable Banach space and $X \in L_r(\mathbb{P}, E)$ for some $r > 0$ and $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of r-optimal n-codebooks for X in E. Then, for every $0 < b < \frac{1}{2}$, $n \in \mathbb{N}$ and $y \in E$*

$$\Delta_{n,r}(X,E) \geq ((1-b)^r - b^r)\,\mu\left(B(y, b\,\mathrm{dist}(y,\alpha_n))\right)\mathrm{dist}(y,\alpha_n)^r.$$

Proof. Let $0 < b < \frac{1}{2}$ and $n \in \mathbb{N}$. For $y \in E$, consider the $(n+1)$-codebook $\beta_n := \alpha_n \cup \{y\}$ and $\delta_n := \mathrm{dist}(y,\alpha_n)$. Then, by using $B(y, b\delta_n) \subset W_y(\beta_n)$

$$\begin{aligned}\Delta_{n,r}(X,E) &\geq e^r_{n,r}(X,E) - e^r_r(X,E;\beta_n)\\ &\geq \int_{W_y(\beta_n)} \min_{a\in\alpha_n} \|x-a\|^r - \|x-y\|^r\, d\mu(x)\\ &\geq \int_{B(y,b\delta_n)} ((1-b)^r - b^r)\delta_n^r d\mu(x)\\ &= \mu\left(B(y,b\delta_n)\right)((1-b)^r - b^r)\delta_n^r.\end{aligned} \tag{2.2}$$

□

As a consequence, we obtain for r.e.'s satisfying a local peakless condition on $A \in \mathcal{B}(\mathbb{R}^d)$

Proposition 2.1.5. *(First micro-macro inequality in $\mathbb{R}^d$) Let $X \in L_r(\mathbb{P}, \mathbb{R}^d)$ and $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of r-optimal n-codebooks for X in $\mathbb{R}^d$. Suppose that X satisfies the lower peakless condition on a set $A \in \mathcal{B}(\mathbb{R}^d)$. Then, for every $0 < b < \frac{1}{2}$, $n \in \mathbb{N}$ and $y \in A$ with $B(y, b\,\mathrm{dist}(x,\alpha_n)) \subset A$*

$$\Delta_{n,r}(X,E) \geq ((1-b)^r - b^r)\, b^d f(y)\lambda^d(B(0,1))c(\mu,A)\,\mathrm{dist}(y,\alpha_n)^{r+d},$$

where $f = \frac{\partial\mu_a}{\partial\lambda^d}$ and $c(\mu, A)$ is a constant depending on μ and A only.

Proof. The proof is a consequence of the first micro-macro inequality (Proposition 2.1.4) and the definition of the local peakless condition. □

In virtue of the Cameron-Martin formula, Gaussian random elements obey a specific type of peakless condition, which reads

Proposition 2.1.6. *(First micro-macro inequality for Gaussian r.e.'s) Let X be a Gaussian r.e. in the separable Banach space $(E, \|\cdot\|)$, $r > 0$ and suppose that $\mathcal{C}_{n,r}(X,E) \neq \emptyset$ for $n \in \mathbb{N}$. Then, for every $0 < b < \frac{1}{2}$, $y \in E$ and $\alpha_n \in \mathcal{C}_{n,r}(X,E)$*

$$\Delta_{n,r}(X,E) \geq ((1-b)^r - b^r)\exp\Big(-\frac{\|y\|^2_{\mathcal{H}_\mu}}{2}\Big)\,\mathrm{dist}(y,\alpha_n)^r\mu(B(0, b\,\mathrm{dist}(y,\alpha_n))).$$

Proof. As a consequence of the first micro-macro inequality, we have for $y \in E$ and $\delta_n := \mathrm{dist}(y,\alpha_n)$

$$\Delta_{n,r}(X,E) \geq \mu\left(B(y,b\delta_n)\right)((1-b)^r - b^r)\delta_n^r. \tag{2.3}$$

In combination with the estimation of shifted balls (Proposition 1.1.11) with $a = 0$, this impliess

$$\Delta_{n,r}(X,E) \geq \mu\left(B(0, b\delta_n)\right) \exp\left(-\frac{\|y\|^2_{\mathcal{H}_\mu}}{2}\right)((1-b)^r - b^r)\delta_n^r.$$

□

The second micro-macro inequality is dimension-free and reads

Proposition 2.1.7. *(Second micro-macro inequality) Let X be a r.e. in the separable Banach space $(E, \|\cdot\|)$, $X \in L_r(\mathbb{P}, E)$ for some $r > 0$ and $\mathcal{C}_{n+1,r}(X,E) \neq \varnothing$ for a natural number $n \in \mathbb{N}$. Then, for every $\alpha_{n+1} \in \mathcal{C}_{n+1,r}(X,E)$*

$$\Delta_{n,r}(X,E) \leq \int_{W_a(\alpha_{n+1})} \operatorname{dist}(x, \alpha_{n+1}\backslash\{a\})^r - \|x-a\|^r \, d\mu(x)$$

for all $a \in \alpha_{n+1}$. Furthermore, for $r \geq 1$

$$\begin{aligned}\Delta_{n,r}(X,E) &\leq (2^r - 1) e^r_{r;\mathrm{loc}}(X, E; \alpha_{n+1}, W_a(\alpha_{n+1})) \\ &\quad + 2^r \mu(W_a(\alpha_{n+1})) \operatorname{dist}(a, \alpha_{n+1}\backslash\{a\})^r,\end{aligned}$$

and for $0 < r \leq 1$

$$\Delta_{n,r}(X,E) \leq \mu(W_a(\alpha_{n+1})) \operatorname{dist}(a, \alpha_{n+1}\backslash\{a\})^r.$$

Proof. As a consequence of equation (1.21), we obtain for an arbitrary Voronoi partition $\{C_a(\alpha_{n+1}), a \in \alpha_{n+1}\}$ and every $a \in \alpha_{n+1}$

$$\Delta_{n,r}(X,E) \leq \int_{C_a(\alpha_{n+1})} \operatorname{dist}(x, \alpha_{n+1}\backslash\{a\})^r - \|x-a\|^r \, d\mu(x).$$

Since $\operatorname{dist}(x, \alpha_{n+1}\backslash\{a\})^r = \|x-a\|^r$ on $V_a(\alpha)\backslash W_a(\alpha)$, for all $a \in \alpha_{n+1}$, we obtain the first assertion. The second and third assertions are consequences of the first in combination with the inequalities

$$\operatorname{dist}(x, \alpha_{n+1}\backslash\{a\})^r \leq (\operatorname{dist}(a, \alpha_{n+1}\backslash\{a\} + \|x-a\|)^r$$

and

$$(a+b)^r \leq 2^r(a^r + b^r)$$

for all $r > 0$ and $a, b \in \mathbb{R}^+$, which can be sharpened to

$$(a+b)^r \leq a^r + b^r$$

for all $0 \leq r \leq 1$. □

Proposition 2.1.8. *(Extended second micro-macro inequality) Let X be a r.e. in the separable Banach space $(E, \|\cdot\|)$, $X \in L_r(\mathbb{P}, E)$ for some $r > 0$ and $\mathcal{C}_{n+1,r}(X,E) \neq \varnothing$ for a natural number $n \in \mathbb{N}$. Then, for every $\alpha_{n+1} \in \mathcal{C}_{n+1,r}(X,E)$, and $a \in \alpha_{n+1}$ and every $C > 0$*

$$\begin{aligned}\Delta_{n,r}(X,E) &\leq \Big(2^r - 1 + \big(\frac{2}{C}\big)^r\Big) e^r_{r;\mathrm{loc}}(X, E; \alpha_{n+1}, W_a(\alpha_{n+1})) \\ &\quad + 2^r \mu\Big(B(a, C \operatorname{dist}(a, \alpha_{n+1}\backslash\{a\}))\Big) \operatorname{dist}(a, \alpha_{n+1}\backslash\{a\})^r.\end{aligned}$$

Proof. For convenience, we abbreviate $\delta_n(a) := \text{dist}(a, \alpha_{n+1} \backslash \{a\})$. In view of the second micro-macro inequality (Proposition 2.1.7), it is sufficient to show that

$$\mu(W_a(\alpha_{n+1}) \backslash B(a, C\delta_n(a)))(\delta_n(a))^r \le \frac{1}{C^r} e^r_{r;\text{loc}}(X, E; \alpha_{n+1}, W_a(\alpha_{n+1})).$$

In fact, we can estimate

$$\begin{aligned}
\mu\left(W_a(\alpha_{n+1}) \backslash B(a, C\delta_n(a))\right)(\delta_n(a))^r &= \frac{1}{C^r} \int_{W_a(\alpha_{n+1}) \backslash B(a, C\delta_n(a))} (C\delta_n(a))^r d\mu(x) \\
&\le \frac{1}{C^r} \int_{W_a(\alpha_{n+1}) \backslash B(a, C\delta_n(a))} \|x - a\|^r d\mu(x) \\
&\le \frac{1}{C^r} e^r_{r,\text{loc}}(X, E; \alpha_{n+1}, W_a(\alpha_{n+1})),
\end{aligned}$$

which yields the assertion. □

Notes and References: The first micro-macro inequality in $\mathbb{R}^d$ and the second micro-macro inequality have been introduced implicitly in [GLP08] as part of several proofs therein. They have been explicitly mentioned initially in [GLP10]. The first micro-macro inequality for Gaussian r.e.'s and the extended second micro-macro inequality seem to be new.

2.2 Increments of the quantization error

The main purpose of this section is to develop an infinite dimensional equivalent for Gaussian r.e.'s similar to Theorem 1.2.24. Unfortunately, we will not be able to obtain a result as sharp as Theorem 1.2.24, meaning a sharp estimate for the weak asymptotics of the increments of the r-th power of the quantization error. Still, it will be sufficient for chapter 4, where the sharp rate for the logarithmic asymptitics of the increments of quantization errors is needed. Moreover, the result may be used to generalize those obtained for the finite dimensional case.

In this section, we will assume throughout that $\mathcal{C}_{n,r}(X, E) \neq \varnothing$ for all $n \in \mathbb{N}$ and $r > 0$. In fact, as concerning estimations of the increments of the quantization error, this can be done without loss of generality, since, in view of Remark 1.2.9

$$\Delta_{n,r}(X, E) = \Delta_{n,r}(X, E''),$$

where E'' denoting the Bidual to E, in which the existence of optimal codebooks is always guaranteed , see [GLP07, Corollary 1]. Thus, estimates for $\Delta_{n,r}(X, E'')$ equally hold for $\Delta_{n,r}(X, E)$.

Upper bound for the increments of the quantization error In a general non necessary finite-dimensional setting, there are several tools which might not exist and which are used in the proof for the upper bound of the increments of the quantization error in Theorem 1.2.24, namely

- an empirical measure theorem equivalent to Theorem 1.2.14 in combination with
- a compact set (equivalent to a hypercube $[-c,c]^d$) with a positive μ-measure.

To determine similar results for general Gaussian r.e.'s, we have to make use of specific properties, namely the general version of the Anderson inequality and the Cameron-Martin formula.

Lemma 2.2.1. *Let X be a Gaussian r.e. in the separable Banach space $(E,\|\cdot\|)$. Then*

$$\mu\left(B(x,\kappa)\backslash B\left(x,\frac{\kappa}{4}\right)\right) \geq \mu\left(B\left(x,\frac{\kappa}{4}\right)\right) \tag{2.4}$$

for every $x \in \left(\overset{\circ}{B}(0,\frac{1}{2}\kappa)\right)^c$ and $\kappa > 0$. Furthermore

$$\mu\left(B(x,\kappa)\backslash B\left(x,\frac{\kappa}{4}\right)\right) \geq \mu\left(B\left(x,\frac{\kappa}{4}\right)\right)\exp\left(-\frac{\kappa^2}{2\sigma(\mu)^2}\right) \tag{2.5}$$

for all $x \in B(0,\frac{1}{2}\kappa)$ and $\kappa > 0$.

Proof. Let $\kappa > 0$ and $x \in \left(\overset{\circ}{B}(0,\frac{1}{2}\kappa)\right)^c$. We consider

$$x' := x - \frac{1}{2}\kappa\frac{x}{\|x\|} = x(1-\frac{\kappa}{2\|x\|}).$$

Then, by applying the triangle inequality

$$\mu\left(B(x',\frac{1}{4}\kappa)\cap B(x,\frac{1}{4}\kappa)\right) \leq \mu(\partial B(x',\frac{1}{4}\kappa)) = 0,$$

where the last equality follows by continuity of the function $y \mapsto \mu(\{\|x\| \leq y\})$ on $(0,\infty)$, see e.g. [Bog98, Corollary 4.4.2]. Furthermore, we have

$$B(x',\frac{1}{4}\kappa) \subset B(x,\kappa),$$

which implies

$$\mu(B(x,\kappa)) \geq \mu\left(B\left(x,\frac{\kappa}{4}\right)\right) + \mu\left(B\left(x',\frac{\kappa}{4}\right)\right).$$

By the Anderson inequality (Proposition 1.1.6) with $A = B(0,\frac{\kappa}{4})$ and $t = (1-\frac{\kappa}{2\|x\|}) \in [0,1]$ we obtain

$$\mu(B(x,\kappa)) \geq 2\left(B\left(x,\frac{\kappa}{4}\right)\right),$$

which implies the first assertion.
For $x \in B(0, \frac{1}{2}\kappa)$, by continuity of the embedding $j_\mu : \mathcal{H}_\mu \to E$ and compactness of the unit ball $\mathcal{K}_\mu$ we can select $h \in \mathcal{K}_\mu$ such that

$$\|h\|_{\mathcal{H}_\mu} = 1$$

and

$$\|h\| = \sigma(\mu).$$

Since $x \in B(0, \frac{1}{2}\kappa)$ we have

$$c^* := \sup\{c \geq 0 : hc \in B(x, \frac{1}{2}\kappa)\} \in [0, \infty)$$

which implies

$$\sigma(\mu)c^* = \|c^* h\| \leq \|x\| + \frac{1}{2}\kappa \leq \frac{1}{2}\kappa + \frac{1}{2}\kappa \leq \kappa.$$

In particular, this yields

$$\|c^* h\|_{\mathcal{H}_\mu} \leq \frac{\kappa}{\sigma(\mu)}.$$

We obtain

$$B(c^* h, \frac{\kappa}{4}) \subset B(x, \kappa), \quad \mu(B(x, \frac{\kappa}{4}) \cap B(c^* h, \frac{\kappa}{4})) = 0 \tag{2.6}$$

and may estimate in virtue of the Anderson inequality with $t = 0$ (Proposition 1.1.6) and the estimation of shifted balls with $a = 0$ (Proposition 1.1.11)

$$\begin{aligned} \mu\Big(B(x, \frac{\kappa}{4})\Big) &\leq \mu\Big(B(0, \frac{\kappa}{4})\Big) \leq \exp\Big(\frac{\|c^* h\|^2_{\mathcal{H}_\mu}}{2}\Big)\mu\Big(B(c^* h, \frac{\kappa}{4})\Big) \\ &\leq \exp\Big(\frac{\kappa^2}{2\sigma(\mu)^2}\Big)\mu\Big(B(c^* h, \frac{\kappa}{4})\Big). \end{aligned} \tag{2.7}$$

Combining equations (2.6) and (2.7) yields the second assertion. □

Theorem 2.2.2. *Let X be a Gaussian r.e. in the separable Banach space E. Then, there exists a constant $C_{\mathrm{loc}} > 0$ such that*

$$\Delta_{n-1,r}(X, E) \leq C_{\mathrm{loc}} e^r_{r,\mathrm{loc}}(X, E; \alpha_n, W_a(\alpha_n))$$

for all $n \geq 2$, $\alpha_n \in \mathcal{C}_{n,r}(X, E)$ and $a \in \alpha_n$.

Proof. Let $n \in \mathbb{N}$, $n \geq 2$, $C > 0$, $\alpha_n \in \mathcal{C}_{n,r}(X, E)$ and $a \in \alpha_n$. For convenience, we abbreviate $\delta_n(a) := \mathrm{dist}(a, \alpha_n \backslash \{a\})$. By the extended second micro-macro inequality (Proposition 2.1.8), we have

$$\begin{aligned} \Delta_{n-1,r}(X, E) \leq &\Big(\frac{2^r}{C^r} + 2^r - 1\Big) e^r_{r,\mathrm{loc}}(X, E; \alpha_n, W_a(\alpha_n)) \\ &+ 2^r \mu(B(a, C\delta_n(a)))\delta_n(a)^r. \end{aligned} \tag{2.8}$$

To estimate the second term of the right-hand side of equation (2.8), we will consider 2 cases. Let $C < \frac{1}{8}$.

Case $0 \notin B(a, 2C\delta_n(a))$**:**

By Lemma 2.2.1 equation (2.4) with $x = a$ and $\kappa = 4C\delta_n(a)$ and the fact that

$$B\big(a, 4C\delta_n(a)\big) \subset W_a(\alpha_n)$$

we obtain

$$\begin{aligned}
\mu(B(a, C\delta_n(a)))\delta_n(a)^r &= \int_{B(a,\frac{\kappa}{4})} \operatorname{dist}(a, \alpha_n \backslash \{a\})^r d\mu(x) \\
&\leq \frac{1}{C^r} \int_{B(a,\kappa)\backslash B(a,\frac{\kappa}{4})} (C \operatorname{dist}(a, \alpha_n \backslash \{a\}))^r d\mu(x) \\
&\leq \frac{1}{C^r} \int_{B(a,4C\delta_n(a))\backslash B(a,C\delta_n(a))} \|x - a\|^r d\mu(x) \qquad (2.9) \\
&\leq \frac{1}{C^r} \int_{W_a(\alpha_n)} \|x - a\|^r d\mu(x) \\
&= \frac{1}{C^r} e^r_{r,\mathrm{loc}}(X, E; \alpha_n, W_a(\alpha_n)).
\end{aligned}$$

Case $0 \in B(a, 2C\delta_n(a))$**:**

Let $\epsilon > 0$ and $h \in \mathcal{K}_\mu$ such that $\|h\|_{\mathcal{H}_\mu} = 1$ and $\|h\| = \sigma(\mu)$. Since $0, \frac{2\epsilon}{\sigma(\mu)} h \in \operatorname{supp}(\mu)$, there exists in virtue of equation (3.2) an $n_\epsilon \in \mathbb{N}$ independently of the choice of $\alpha_n \in \mathcal{C}_{n,r}(X, E)$ such that

$$\operatorname{dist}(0, \alpha_n) \leq \epsilon \text{ and } \operatorname{dist}\Big(\frac{\epsilon 2}{\sigma(\mu)} h, \alpha_n\Big) \leq \epsilon$$

for all $n \geq n_\epsilon$. Since $0 \in C_a(\alpha_n)$ we estimate

$$\operatorname{dist}(a, \alpha_n \backslash \{a\}) \leq \|a\| + \|b\| \leq 2\|b\|$$

for all $b \in \alpha_n \backslash \{a\}$. Therefore, for $b_1, b_2 \in \alpha_n \backslash \{a\}$ such that $b_1 \in B(0, \epsilon)$ and $b_2 \in B(\frac{\epsilon 2}{\sigma(\mu)} h, \epsilon)$

$$\operatorname{dist}(a, \alpha_n \backslash \{a\}) \leq 2 \max(b_1, b_2) = 2\|b_2\| \leq 6\epsilon. \qquad (2.10)$$

In view of Lemma 2.2.1 equation (2.5) with $x = a$ and $\kappa = 4C\delta_n(a)$ equation (2.10) implies

$$\mu(B(a, C\delta_n(a))) \leq \exp\left(\frac{(4C6\epsilon)^2}{2\sigma(\mu)^2}\right) \mu(B(a, 4C\delta_n(a)) \backslash B(a, C\delta_n(a))).$$

Analogously to the argumentation in equation (2.9), we obtain

$$\begin{aligned}\mu(B(a,C\delta_n(a)))\delta_n(a)^r &= \int_{B(a,\frac{\kappa}{4})} \operatorname{dist}(a,\alpha_n\backslash\{a\})^r d\mu(x)\\ &\leq \frac{\exp\left(\frac{(4C6\epsilon)^2}{2\sigma(\mu)^2}\right)}{C^r}\int_{B(a,\kappa)\backslash B(a,\frac{\kappa}{4})} (C\operatorname{dist}(a,\alpha_n\backslash\{a\}))^r d\mu(x)\\ &\leq \frac{\exp\left(\frac{8(6C\epsilon)^2}{\sigma(\mu)^2}\right)}{C^r}\int_{B(a,4C\delta_n(a))\backslash B(a,C\delta_n(a))} \|x-a\|^r d\mu(x)\\ &\leq \frac{\exp\left(\frac{8(6C\epsilon)^2}{\sigma(\mu)^2}\right)}{C^r}\int_{W_a(\alpha_n)} \|x-a\|^r d\mu(x)\\ &= \frac{\exp\left(\frac{8(6C\epsilon)^2}{\sigma(\mu)^2}\right)}{C^r} e^r_{r;\mathrm{loc}}(X,E;\alpha_n,W_a(\alpha_n))\end{aligned} \tag{2.11}$$

for all $n \geq n_\epsilon$. In summary, we obtain in virtue of equations (2.8), (2.9) and (2.11) for $n \geq n_\epsilon$ and $a \in \alpha_n$

$$\Delta_{n-1,r}(X,E) \leq \left(\frac{2^r\left(1+\exp\left(\frac{8(6C\epsilon)^2}{\sigma(\mu)^2}\right)\right)}{C^r} + 2^r - 1\right) e_{r;\mathrm{loc}}(X,E;\alpha_n,W_a(\alpha_n)).$$

Selecting $C_{\mathrm{loc}} > 0$ sufficiently large such that the assertion also holds for the finite number of $a \in \alpha_n$ and $n \leq n_\epsilon$ yields the assertion. □

As an immediate consequence, we obtain

Corollary 2.2.3. *Let X be a Gaussian r.e. in the separable Banach Space E. Then, there exists a constant $C_{\mathrm{loc}} < \infty$ such that*

$$\Delta_{n-1,r}(X,E) \leq \frac{C_{\mathrm{loc}}}{n} e^r_{n,r}(X,E)$$

for all $n \geq 2$.

Proof. For every $\alpha_n \in \mathcal{C}_{n,r}(X,E)$ there exists an $a \in \alpha_n$ such that

$$e^r_r(X,E;\alpha_n,W_a(\alpha_n)) \leq \frac{1}{n} e^r_{n,r}(X,E).$$

Thus, Theorem 2.2.2 implies

$$\Delta_{n-1,r}(X,E) \leq C_{\mathrm{loc}} e^r_{r;\mathrm{loc}}(X,E;\alpha_n,W_a(\alpha_n)) \leq \frac{C_{\mathrm{loc}}}{n} e^r_{n,r}(X,E).$$

□

Remark 2.2.4.

- Creutzig et. al. (see [CDMGR09]) establish implicitly a relationship between the n-th quantization error of order $r = 2$ and the increments of the quantization error of order $r = 1$. As a consequence of Theorem 2, Proposition 1 and Theorem 3 in the reference, the authors obtain for $N \in \mathbb{N}$

$$\sup_{m \geq 4N} \Delta_{m,1}(X,E) \leq \frac{32}{N} e_{N/2,2}(X,E).$$

The result is a combination of the second micro-macro inequality, a result of Bakhvalov for general lower bounds of approximation formulas and the Monte-Carlo approximation error for Lipschitz continuous functionals by applying variance reduction via optimal quantization. In view of Theorem 1.2.17, which implies for Gaussian r.e.'s with regularly varying small ball function

$$e_{n,2}(X,E) \approx e_{n,1}(X,E), \quad n \to \infty,$$

one sees that the result provides the same rate for the upper bound of the increments of the quantization error as Proposition 2.2.2 in the case $r = 1$.

- As a universal upper bound, the rate obtained in Corollary 2.2.3 is sharp, i.e. it cannot be improved globally for all Gaussian r.e.'s, as it is sharp in the finite dimensional case (see Theorem 1.2.24). Nonetheless, it is not a property of regularly varying sequences. Considering e.g.

$$f(n) = \frac{1}{n^2}, \text{ for } n \neq 2^k,\ k \in \mathbb{N}$$
$$f(n) = \frac{1}{n^{\frac{3}{2}}}, \text{ elsewhere}$$

for $k, n \in \mathbb{N}$. Then, as the result of a simple computation

$$g(N) = \sum_{i=N}^{\infty} f(n) \sim \frac{1}{N}, \quad n \to \infty,$$

but there is no constant $C \in (0,\infty)$ such that $f(N) \leq \frac{Cg(N)}{N}$, for all $N \in \mathbb{N}$.

- The rate obtained in Corollary 2.2.3 is generally not the true weak rate for the increments of the quantization error, particularly in the case $\dim(\mathcal{H}_\mu) = \infty$. As a general result for asymptotic sequences (see [BGT87, Proposition 1.5.9b]), one has for slowly varying functions $l : [A,\infty) \to \mathbb{R}$

$$\frac{1}{l(x)} \int_x^\infty \frac{l(t)}{t} d\lambda(t) \overset{x\to\infty}{\longrightarrow} \infty,$$

provided $\int_x^\infty \frac{l(t)}{t} d\lambda(t)$ exists for all $x \in [A,\infty)$.
Suppose now, that $e_{n,r}(X,E)$ is slowly varying at infinity, then the same holds for $l(x) = e^r_{[x],r}(X,E)\frac{x}{[x]}$. Assuming the rate obtained in Corollary

2.2.3 is the true rate, then, for some constant $C > 0$ and by denoting $\zeta(n)$ the counting measure on $\mathbb{N}$

$$
\begin{aligned}
1 &= \frac{1}{e^r_{N,r}(X,E)} \int_N^\infty \Delta_{n,r}(X,E) d\zeta(n) \geq \frac{C}{e^r_{N,r}(X,E)} \int_N^\infty \frac{e^r_{n,r}(X,E)}{n} d\zeta(n) \\
&= \frac{C}{l(N)} \int_N^\infty \frac{l(x)}{x} d\lambda(x) \to \infty, \quad N \to \infty,
\end{aligned}
$$

which is a contradiction.

- Conversely, the rate obtained for the increments of the quantization error with Corollary 2.2.3 cannot hold for arbitrary r.e.'s. Suppose that X is geometrically distributed with index $p = q = \frac{1}{2}$, i.e. for $m \in \mathbb{N}$

$$\mu(\{m\}) = \frac{1}{2^{m+1}}.$$

Considering the codebooks $\alpha_n = \{\{0\}, \ldots, \{n-1\}\}$, then

$$
\begin{aligned}
e_1(X, \mathbb{R}; \alpha_n) &= \sum_{k \geq n} (k - (n-1)) \frac{1}{2^{k+1}} \\
&= \sum_{j=0}^{\infty} (j+1) \frac{1}{2^{j+n+1}} \\
&= \frac{1}{2^{n+1}} \sum_{j=0}^{\infty} \frac{\partial}{\partial x} x^{j+1} \Big|_{x=\frac{1}{2}} \\
&= \frac{1}{2^{n+1}} \frac{\partial}{\partial x} \frac{x}{1-x} \Big|_{x=\frac{1}{2}} \\
&= \frac{1}{2^{n+1}} \frac{1}{(1-x)^2} \Big|_{x=\frac{1}{2}} = \frac{1}{2^{n-1}}.
\end{aligned}
$$

Conversely, we have for every n-codebook β_n in $\mathbb{R}$

$$\sum_{i=0}^{n} \operatorname{dist}(\{i\}, \beta_n) \geq 1$$

such that

$$
\begin{aligned}
e_1(X, \mathbb{R}, \beta_n) &\geq \sum_{i=1}^{n} \operatorname{dist}(\{i\}, \beta_n) \mu(\{i\}) \geq \sum_{i=1}^{n} \operatorname{dist}(\{i\}, \beta_n) \mu(\{n\}) \\
&\geq \mu(\{n\}) = \frac{1}{2^{n+1}}.
\end{aligned}
$$

Hence, in summary

$$\frac{1}{2^{n-1}} \geq e_{n,1}(X, \mathbb{R}) \geq \frac{1}{2^{n+1}}.$$

Suppose now that the assertion of Corollary 2.2.3 holds true. Then, for $n \geq 2$

$$\begin{aligned}
\frac{1}{2^{n+1}} \leq e_{n,1}(X,\mathbb{R}) = \sum_{j=n}^{\infty} \Delta_{j,r}(X,E) \leq C_{\text{loc}} \sum_{j=n}^{\infty} \frac{e_{j,1}(X,E)}{j} \\
&\leq 2C_{\text{loc}} \sum_{j=n}^{\infty} \frac{1}{j2^j} \\
&= 2C_{\text{loc}} \sum_{j=n}^{\infty} \int_0^{\frac{1}{2}} x^{j-1} d(x) \\
&= 2C_{\text{loc}} \int_0^{\frac{1}{2}} \sum_{k=0}^{\infty} x^{k+n-1} d(x) \\
&= 2C_{\text{loc}} \int_0^{\frac{1}{2}} \frac{x^{n-1}}{1-x} d(x) \\
&\leq 4C_{\text{loc}} \int_0^{\frac{1}{2}} x^{n-1} d(x) \\
&= \frac{4C_{\text{loc}}}{n} \frac{1}{2^n}
\end{aligned}$$

as a contradiction.

- Only under the assumption that the sharpened Anderson inequality holds, we still obtain a result similar to Corollary 2.2.3 with n replaced by $n-1$ on the right-hand side. This will be useful in chapter 5, in which we will estimate lower bounds for the local inertia $e_r^r(X,E;\alpha,W_a(\alpha_n))$.

Lower bound for the increments of the quantization errors Compared to the proof for the lower bound of the increments of the quantization error in Theorem 1.2.24, there are several difficulties one has to cope with when considering infinite dimensional Gaussian measures μ instead of finite dimensional Lebesgue-continuous ones. Particularly the fact that in general, there is no constant $C_\mu > 0$, such that

$$\mu\big(B(0,\frac{\epsilon}{2})\big) \geq C_\mu \mu\big(B(0,\epsilon)\big),$$

for all $x \in E$ and $\epsilon > 0$, complicates the derivation of sharp estimates, such as in Proposition 2.1.6. In fact, most of the interesting cases are those, in which the small ball function is regularly varying at infinity which implies estimates of the form

$$\log\Big(\mu\big(B(0,\frac{\epsilon}{2})\big)\Big) \geq C_\mu \log(\mu(B(0,\epsilon))),$$

for all $x \in E$ and $\epsilon > 0$, which are much weaker.

Thus, an alternative approach is needed. We will develop a monotonicity property for the quantization error of recursively defined codebooks of which we will derive lower bounds for the asymptotics for the increments of the r-th power of the quantization error under weak assumptions on the quantization error itself. We set for $n \in \mathbb{N}$, $r > 0$, $\alpha_n \in \mathcal{C}_{n,r}(X, E)$ and r.e.'s $X_i, i = 1, \ldots, m$ in $L_r(\mathbb{P}, E)$

$$\Delta_r^{(R)}(X, E; \alpha_n; X_1, \ldots, X_m) := \int e_{n,r}^r(X, E) - e_r^r(X, E; \alpha_n \cup \{x_1, \ldots, x_m\}) d \otimes_{i=1}^m \mathbb{P}^{X_i}(x_1, \ldots, x_m),$$

and $a_1, \ldots, a_m \in E$

$$\Delta_r(X, E; \alpha_n; a_1, \ldots, a_m) := e_{n,r}^r(X, E) - e_r^r(X, E; \alpha_n \cup \{a_1, \ldots, a_m\}).$$

Lemma 2.2.5. *For $n, m \in \mathbb{N}$, $r > 0$, i.i.d. r.e.'s $X_i, i = 1, \ldots, m$ in $L_r(\mathbb{P}, E)$ and $a_1, \ldots, a_m \in E$ one has*

$$m\Delta_r^{(R)}(X, E; \alpha_n; X_1) \geq \Delta_r^{(R)}(X, E; \alpha_n; X_1, \ldots, X_m), \tag{2.12}$$

as well as

$$m \sup_{a \in E} \Delta_r(X, E; \alpha_n; a) \geq \Delta_r(X, E; \alpha_n; a_1, \ldots, a_m), \tag{2.13}$$

for all $\alpha_n \in \mathcal{C}_{n,r}(X, E)$.

Proof. First note, that for $n \in \mathbb{N}$ and $a \in \alpha_n$

$$\min_{b \in \alpha_n \setminus \{a\}} \|x - b\|^r - \|x - a\|^r = 0$$

for all $x \in V_a(\alpha_n) \setminus W_a(\alpha)$.
We abbreviate $\alpha_n^{(m)} = \alpha_n \cup \{x_1, \ldots, x_m\}$, for $x_1, \ldots, x_m \in E$ and estimate

$$\begin{aligned}
\Delta_r^{(R)}(X, E; \alpha_n; X_1, \ldots, X_m) &= \int e_{n,r}^r(X, E) - e_r^r(X, E; \alpha_n^{(m)}) d \otimes \mathbb{P}^{X_i}(x_1, \ldots, x_m) \\
&= \int \sum_{i=1}^m \int_{V_{x_i}(\alpha_n^{(m)})} \min_{a \in \alpha_n} \|x - a\|^r - \|x - x_i\|^r \, d\mu(x) d \otimes \mathbb{P}^{X_i}(x_1, \ldots, x_m) \\
&\leq \sum_{i=1}^m \int \int_{V_{x_i}(\alpha_n \cup \{x_i\})} \min_{a \in \alpha_n} \|x - a\|^r - \|x - x_i\|^r \, d\mu(x) d \otimes \mathbb{P}^{X_i}(x_1, \ldots, x_m) \\
&= \sum_{i=1}^m \int \int_{W_{x_i}(\alpha_n \cup \{x_i\})} \min_{a \in \alpha_n} \|x - a\|^r - \|x - x_i\|^r \, d\mu(x) d \otimes \mathbb{P}^{X_i}(x_1, \ldots, x_m) \\
&= \sum_{i=1}^m \Delta_r^{(R)}(X, E; \alpha_n; X_i) = m\Delta_r^{(R)}(X, E; \alpha_n; X_1),
\end{aligned}$$

which yields equation (2.12). Similarly, we obtain for $\alpha_n^{(m)} = \alpha_n \cup \{a_1, \dots, a_m\}$

$$\begin{aligned}
\Delta_r(X, E; \alpha_n; a_1, \dots, a_m) &= e^r_{n,r}(X, E) - e^r_r(X, E; \alpha_n^{(m)}) \\
&\leq \sum_{i=1}^m \int_{V_{a_i}(\alpha_n^{(m)})} \min_{a \in \alpha_n} \|x - a\|^r - \|x - a_i\|^r \, d\mu(x) \\
&\leq \sum_{i=1}^m \int_{V_{a_i}(\alpha_n \cup \{a_i\})} \min_{a \in \alpha_n} \|x - a\|^r - \|x - a_i\|^r \, d\mu(x) \\
&= \sum_{i=1}^m \int_{W_{a_i}(\alpha_n \cup \{a_i\})} \min_{a \in \alpha_n} \|x - a\|^r - \|x - a_i\|^r \, d\mu(x) \\
&\leq \sum_{i=1}^m \sup_{a \in E} \Delta_r(X, E; \alpha_n; a) = m \sup_{a \in E} \Delta_r(X, E; \alpha_n; a),
\end{aligned}$$

which yields equation (2.13). □

In view of Lemma 2.2.5, we are now able to estimate an asymptotic lower bound for the increments of the r-th power of the quantization error based on the asymptotics of the quantization error itself. The following case particularly covers the case of finite dimensional r.e.'s with densities having a non-vanishing Lebesgue-continuous part.

Theorem 2.2.6. *(Regular variation) Let E be a separable Banach space, $r > 0$ and $X \in L_r(\mathbb{P}, E)$. Suppose that*

$$e^r_{n,r}(X, E) \approx \psi(n), \quad n \to \infty$$

for a regular varying function $\psi : \mathbb{N} \to \mathbb{R}$ with index $-b < 0$. Then, there is a constant $\kappa_\Delta > 0$ such that

$$\Delta_{n,r}(X, E) \gtrsim \frac{\kappa_\Delta}{n} e^r_{n,r}(X, E), \quad n \to \infty.$$

Proof. First note, that

$$\Delta_{n,r}(X, E) \geq \sup_{a \in E} \Delta_r(X, E; \alpha_n; \{a\})$$

for all $n \in \mathbb{N}$ and furthermore, by the assumption on the regularity of the quantization error, there is a constant $0 < C < \infty$ such that

$$\frac{1}{C} \psi(n) \lesssim e_{n,r}(X, E) \lesssim C \psi(n), \quad n \to \infty.$$

Let $k > C^{\frac{2}{b}}$. By Lemma 2.2.5 equation (2.13), we estimate with $m = \lceil kn \rceil$ and

$\{a_1, \ldots, a_m\} = \alpha_m \in C_{m,r}(X,E)$

$$\begin{aligned}\Delta_{n,r}(X,E) &\geq \sup_{a\in E} \Delta_r(X,E;\alpha_n;a) \geq \frac{1}{m}\Delta_r(X,E;\alpha_n;a_1,\ldots,a_m)\\ &\geq \frac{1}{m}\left(e^r_{n,r}(X,E) - e^r_{m,r}(X,E)\right)\\ &\gtrsim \frac{1}{m}\left(\frac{1}{C}\psi(n) - C\psi(m)\right) = \frac{1}{mC}\psi(n)\left(1 - C^2\frac{\psi(m)}{\psi(n)}\right)\\ &\sim \frac{1}{Ckn}\psi(n)\left(1 - C^2k^{-b}\right) \gtrsim \frac{1-C^2k^{-b}}{C^2kn}e^r_{n,r}(X,E), \quad n \to \infty\end{aligned}$$

which implies the assertion with $\kappa_\Delta = \frac{1-C^2k^{-b}}{C^2k}$. □

Remark 2.2.7. Suppose that $e^r_{n,r}(X,E)$ is regularly varying at infinity with index $-b$, then

$$\kappa_\Delta = \frac{1-k^{-b}}{k},$$

which attains its maximum at $\kappa^* = (1+b)^{\frac{1}{b}}$, implying

$$\kappa^*_\Delta = \frac{b}{(1+b)^{(1+\frac{1}{b})}}.$$

Remark 2.2.8. 1. In view of the Zador Theorem, Theorem 2.2.6 gives an alternative proof for the lower bound in Theorem 1.2.24.

2. For many other distributions on $\mathbb{R}^d$, the assumption of Theorem 2.2.6 on the regularity of the asymptotics of the quantization error is satisfied, such as a broad class of self-similar distributions (see e.g. [GL05] or [Kre06]).

The following Theorem particularly covers many of the interesting cases of r.e's in infinite dimensional Banach spaces, see also Example 2.2.11 below for specific examples.

Theorem 2.2.9. *(Slow variation) Suppose that there is a regular varying function $\psi : \mathbb{R}^+ \to \mathbb{R}$ with index $-a < 0$ such that*

$$\psi(\log(n)) \sim e^r_{n,r}(X,E), \quad n \to \infty \tag{2.14}$$

for some $r > 0$. Then, for every $\epsilon > 0$

$$\Delta_{n,r}(X,E) \gtrsim \frac{1}{n^{1+\epsilon}}, \quad n \to \infty.$$

In particular,

$$\log(\Delta_{n,r}(X,E)) \gtrsim -\log(n), \quad n \to \infty.$$

Proof. Let $\epsilon > 0$ and $m = \lceil n^{1+\epsilon} \rceil$. By Lemma 2.2.5 equation (2.13), we obtain for $\{a_1, \ldots, a_m\} = \alpha_m \in C_{m,r}(X, E)$ the estimate

$$\begin{aligned}
\Delta_{n,r}(X,E) &\geq \sup_{b \in E} \Delta_r(X,E;\alpha_n;b) \geq \frac{1}{m}\Delta_r(X,E;\alpha_n;a_1,\ldots,a_m) \\
&\geq \frac{1}{m}\left(e^r_{n,r}(X,E) - e^r_{m,r}(X,E)\right) \\
&\gtrsim \frac{1}{n^{1+\epsilon}}\left(\psi(\log(n)) - \psi((1+\epsilon)\log(n))\right) \\
&\sim \frac{1}{n^{1+\epsilon}}\psi(\log(n))\left(1-(1+\epsilon)^{-a}\right)), \quad n \to \infty,
\end{aligned}$$

which implies the assertion. □

Remark 2.2.10. The approach used in the previous Theorem might also be sharp enough to achieve the true weak asymptotics for the increments of the r-th power of the quantization error. Imagine that $e^r_{n,r}(X,E) = \frac{1}{\log(n)+c_n}$ for a bounded sequence $c_n, n \in \mathbb{N}$. Then for $m = \lceil kn \rceil$ for $k > 2\sup_{n\in\mathbb{N}}\{|c_n|\}$

$$\begin{aligned}
\Delta_{n,r}(X,E) &\geq \frac{1}{m}\left(e^r_{n,r}(X,E) - e^r_{m,r}(X,E)\right) \\
&\gtrsim \frac{1}{kn}\frac{\log(kn) - \log(n) - 2\sup_{n\in\mathbb{N}}\{|c_n|\}}{\log(n)\log(kn)} = \frac{1}{kn}\frac{\log(k) - 2\sup_{n\in\mathbb{N}}\{|c_n|\}}{\log(n)\log(kn)} \\
&\sim \frac{\log(k) - 2\sup_{n\in\mathbb{N}}\{|c_n|\}}{kn\log(n)^2}, \quad n \to \infty.
\end{aligned}$$

As a result of a straightforward calculation, this is the true weak rate for the increments of $e^r_{n,r}(X,E)$.

Example 2.2.11.

1. Suppose that X is a Gaussian r.e. in the separable Hilbert space $(H, (\cdot,\cdot))$ such that the non-increasing eigenvalue sequence $(\lambda_j)_{j\in\mathbb{N}}$ of the covariance operator C_X satisfies

$$\lambda_j \sim \phi(j), \quad j \to \infty,$$

for a regularly varying function ϕ with index $-b < -1$. Then, as a consequence of Theorem 1.2.15 and [Der03, chapter 6] for the case $r \neq 2$, the assumptions on the regularity of the quantization error are satisfied and thus, for every $r > 0$

$$\log(\Delta_{n,r}(X,E)) \gtrsim -\log(n), \quad n \to \infty,$$

as a consequence of Theorem 2.2.9.

2. Suppose that $(X_t)_{t\in[0,1]}$ is a fractional Brownian Motion with path in $(E, \|\cdot\|) = (C([0,1]), \|\cdot\|_{L_\infty})$, where $\|f\|_{L_\infty}$ denotes

$$\|f\|_{L_\infty} := \sup_{x\in[0,1]} |f(x)|.$$

Then, the assumptions of Theorem 2.2.9 (see [DS06]) are satisfied and thus

$$\log(\Delta_{n,r}(X,E)) \gtrsim -\log(n), \quad n \to \infty.$$

3. Similar results as under 2) also hold when replacing the fractional Brownian motion with a Brownian diffusion (see [Der08b]), the Banach space $(C([0,1]), \|\cdot\|_{L_\infty})$ by the Banach space $(L_p([0,1]), \|\cdot\|_{L_p})$ (see [DS06]) or even when exchanging both (see [LP06]).

If it is not known, whether the asymptotics of the quantization error satisfies the assumption of Theorem 2.2.9, we still obtain

Proposition 2.2.12. *(Slow variation, second version) Let $r > 0$ and $1 \le \kappa < \infty$. Suppose that there is a regular varying function $\psi : \mathbb{R}^+ \to \mathbb{R}$ with index $-a < 0$ such that*

$$\psi(\log(n)) \lesssim e^r_{n,r}(X,E) \lesssim \kappa\psi(\log(n)), \quad n \to \infty. \tag{2.15}$$

Then, for every $\epsilon > 0$

$$\Delta_{n,r}(X,E) \gtrsim \frac{1}{n^{\kappa^{\frac{1}{a}}+\epsilon}}, \quad n \to \infty.$$

Particularly,

$$\log(\Delta_{n,r}(X,E)) \gtrsim -\kappa^{\frac{1}{a}} \log(n), \quad n \to \infty.$$

Proof. Let $\epsilon > 0$ and $m = \lceil n^\theta \rceil$ for a constant $\theta > 1$ to be specified. By Lemma 2.2.5 equation (2.13), we estimate with $\{a_1, \dots, a_m\} = \alpha_m \in C_{m,r}(X,E)$

$$\begin{aligned}
\Delta_{n,r}(X,E) &\ge \sup_{a\in E} \Delta_r(X,E;\alpha_n;a) \ge \frac{1}{m}\Delta_r(X,E;\alpha_n;a_1,\dots,a_m) \\
&\ge \frac{1}{m}\left(e^r_{n,r}(X,E) - e^r_{m,r}(X,E)\right) \\
&\gtrsim \frac{1}{n^\theta}\left(\psi(\log(n)) - \kappa\psi(\theta\log(n))\right) \\
&\sim \frac{1}{n^\theta}\psi(\log(n))\left(1-\kappa\theta^{-a}\right), \quad n \to \infty.
\end{aligned}$$

$1 - \kappa\theta^{-a} > 0$ if $\theta > \kappa^{\frac{1}{a}}$ yields the assertion. ☐

A metric entropy approach: Recalling the proof for the lower bound of the increments of the quantization error in Theorem 1.2.24, one could also try to use the first micro-macro inequality (for Gaussian r.e.'s) to obtain a lower bound for the increments in the infinite dimensional case. Unfortunately, this seems to be impossible, as we will try to illustrate with the proof of the following result. We will need the following Theorems:

Theorem 2.2.13. *(See [Lif95, Section 18, Theorem 2, Example 2]) Let E be a separable Hilbert space and X be a Gaussian r.e. in E. Suppose, that the eigenvalues $\lambda_j, j \in \mathbb{N}$ of the covariance operator C_X of X admit the representation*

$$\lambda_j \sim \kappa j^{-\beta}, \quad j \to \infty$$

for constants $\kappa > 0$ and $\beta > 1$. Then

$$\phi_\mu(\epsilon) \sim C_s(\kappa, \beta)\epsilon^{-\frac{2}{\beta-1}}, \quad \epsilon \to 0,$$

for a constant $C_s(\kappa, \beta) \in (0, \infty)$.

Theorem 2.2.14. *(See [LP04b, Corollary 2.4]) Let E be a separable Hilbert space and X be a Gaussian r.e. in E. Suppose, that the eigenvalues $\lambda_j, j \in \mathbb{N}$ of the covariance operator C_X of X admit the representation*

$$\lambda_j \sim \kappa j^{-\beta}, \quad j \to \infty$$

for constants $\kappa > 0$ and $\beta > 1$. Then

$$H_e(\epsilon, \mathcal{K}_\mu) \sim C_e(\kappa, \beta)\epsilon^{-\frac{2}{\beta}}, \quad \epsilon \to 0,$$

for a constant $C_e(\kappa, \beta) \in (0, \infty)$.

Proposition 2.2.15. *Let E be a separable Hilbert space and X be a Gaussian r.e. in E. Suppose, that the eigenvalues $\lambda_j, j \in \mathbb{N}$ of the covariance operator C_X of X admit the representation*

$$\lambda_j \sim \kappa j^{-\beta}, \quad j \to \infty$$

for constants $\kappa > 0$ and $\beta > 1$. Then, there exists a constant $C \in [1, \infty)$ such that

$$-\log(\Delta e^r_{n,r}(X, E)) \lesssim C \log(n), \quad n \to \infty. \tag{2.16}$$

Remark 2.2.16. The constant C arising in equation (2.16) is, based on the proof of the following result, in general larger than 1, which is the sharp constant as a consequence of Theorem 2.2.9.

Proof of Proposition 2.2.15. Recall, that for $A \in \mathcal{B}(E)$

$$N_e(\epsilon, A) := \inf\{n \in \mathbb{N} : \exists a_1, \ldots, a_n \in E, \bigcup_{i=1}^n B(a_i, \epsilon) \supset A\}$$

and by definition

$$N_e\left(\frac{\epsilon}{\lambda}, \mathcal{K}_\mu\right) = N_e(\epsilon, \lambda\mathcal{K}_\mu), \tag{2.17}$$

for every $\lambda > 0$.

Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of r-optimal n-codebooks for X in E. Using the first micro-macro inequality for Gaussian r.e.'s (Proposition 2.1.6), we obtain for every $y_n \in E$ and $b \in (0, \frac{1}{2})$ with $\delta_n = \operatorname{dist}(y_n, \alpha_n)$

$$\Delta_{n,r}(X,E) \geq ((1-b)^r - b^r)\delta_n^r \exp(-\frac{\|y_n\|^2_{\mathcal{H}_\mu}}{2})\mu(B(0, b\delta_n)).$$

Applying the negative logarithm yields

$$-\log(\Delta_{n,r}(X,E)) \leq -\log(((1-b)^r - b^r)) - r\log(\delta_n) + \frac{\|y_n\|^2_{\mathcal{H}_\mu}}{2} + \phi_\mu(b\delta_n). \tag{2.18}$$

We want to estimate the upper bound of the right-hand side of equation (2.18). So let $\lambda \in (0,\infty)$. For $y_n \in \mathcal{K}_n := \lambda \log(n)^{\frac{1}{2}}\mathcal{K}_\mu$ one has

$$\frac{\|y_n\|^2_{\mathcal{H}_\mu}}{2} \leq \frac{\lambda^2}{2}\log(n). \tag{2.19}$$

Since $|\alpha_n| = n$, we find in virtue of Theorem 2.2.14 an $y_n \in \mathcal{K}_n$ such that

$$\log(n+1) = H_e\left(\frac{\delta_n}{\lambda \log(n)^{\frac{1}{2}}}, \mathcal{K}_\mu\right) \sim C_e(\kappa,\beta)\left(\frac{\delta_n}{\lambda\log(n)^{\frac{1}{2}}}\right)^{-\frac{2}{\beta}}, \quad n \to \infty \tag{2.20}$$

which implies

$$\delta_n \sim C_e'(\kappa,\beta)\lambda\log(n)^{\frac{-\beta+1}{2}}, \quad n \to \infty$$

for a constant $C_e'(\kappa,\beta) \in (0,\infty)$. Applying Theorem 2.2.13 we deduce

$$\phi_\mu(b\delta_n) \sim C_s(\kappa,\beta)\left(bC_e'(\kappa,\beta)\lambda\right)^{-\frac{2}{\beta-1}}\log(n), \quad n \to \infty$$

and

$$r\log(\delta_n) \sim r\frac{-(\beta-1)}{2}\log(\log(n)), \quad n \to \infty.$$

In summary, we obtain

$$-\log(\Delta_{n,r}(X,E)) \lesssim \frac{\lambda^2}{2}\log(n) + C_s(\kappa,\beta)\left(bC_e'(\kappa,\beta)\lambda\right)^{-\frac{2}{\beta-1}}\log(n), \quad n \to \infty$$

□

Remark 2.2.17.
- One may also deduce a similar result by using the tight relationship between the small ball function and the Metric entropy in the general Banach space setting (see e.g. [LS01, Theorem 3.3]).
- The constants $C_s(\kappa,\beta)$ and $C_e(\kappa,\beta)$ are also known explicitly. Considering e.g. X being a Brownian motion with path in $L_2([0,1])$, then

$$C_s(\kappa,\beta) = \frac{1}{\pi}$$

(see [LP04b, Example 3.2]) and

$$C_e(\kappa,\beta)=\frac{1}{8}$$

(see [LS01, Theorem 6.3]) which yields for $\lambda\in(0,\infty)$ and $b<\frac{1}{2}$

$$-\log(\Delta_{n,r}(X,E))\lesssim\left(\frac{\lambda^2}{2}+\frac{1}{\pi}\left(\frac{b\lambda}{8}\right)^{-2}\right)\log(n),\quad n\to\infty.$$

Minimizing the constant can be achieved by selecting $\lambda^*=\frac{2}{\pi^{\frac{1}{4}}b}$ and as a consequence

$$-\log(\Delta_{n,r}(X,E))\lesssim\frac{4}{\pi^{\frac{1}{2}}b}\log(n),\quad n\to\infty.$$

Notes and References: The estimation of the increments of quantization errors seems to be one key factor for the research to be done hereafter. Things would be much easier, if one were able to show a Tauberian condition satisfied by the increments of quantization errors, which would directly imply the sharp asymptotics of the increments (see Theorem A.11). Unfortunately, except for the fact that the increments are positive as long as the quantization error does not vanish entirely, almost nothing is known directly by definition of the quantization problem.
The results presented in this section seem to be new throughout.

2.3 A first connection to the radius and dimension-free results

In this section, we want to gather some results concerning the quantization radius for sequences of r-optimal n-codebooks for a given r.e. X, which hold equally in finite and infinite dimensional settings. Let $r>0$, E be a separable Banach space and $X\in L_r(\mathbb{P},E)$ and suppose that $\mathcal{C}_{n,r}(X,E)\neq\emptyset$ for all $n\in\mathbb{N}$.

The following Lemma is a slight generalization of a result of Pagès and Sagna [PS08], where a similar result has been stated in the Euclidean case.

Lemma 2.3.1. *For $n\in\mathbb{N}$ let $\alpha_n\in\mathcal{C}_{n,r}(X,E)$. For all $\delta>0$ and $y\in\operatorname{supp}(\mu)$ there exists $n^*(\delta,\mu,y)\in\mathbb{N}$ such that*

$$V_{a_n}(\alpha_n)\subset B\left(y,\frac{\|a_n-y\|}{2}-\delta\right)^c$$

for all $n\geq n^*(\delta,y)$ and $a_n\in B(y,2\delta)^c\cap\alpha_n$. If furthermore X has an unbounded support, and $\|a_n\|\overset{n\to\infty}{\longrightarrow}\infty$, then

$$V_{a_n}(\alpha_n)\subset B\left(y,\frac{\|a_n-y\|}{2+\delta}\right)^c$$

for n sufficiently large.

Proof. By equation (3.2), there exists for every $\delta > 0$ and $y \in \operatorname{supp}(\mu)$ a natural number $n^*(\delta, \mu, y)$ such that there exists $b_n \in \alpha_n$ satisfying

$$b_n \in B(y, \delta) \cap \alpha_n \neq \varnothing$$

for all $n \geq n^*(\delta, \mu, y)$. Then, for $a_n \in \alpha_n$ and $x \in V_{a_n}(\alpha_n)$

$$\begin{aligned}\|x-y\| &= \|x-a_n+a_n-y\| \geq \|a_n-y\| - \|x-a_n\| \geq \|a_n-y\| - \|x-b_n\| \\ &\geq \|a_n-y\| - \|x-y\| - \|y-b_n\| > \|a_n-y\| - \|x-y\| - 2\delta\end{aligned}$$

and we obtain the first assertion. The second is a consequence of equation (3.1). □

Remark 2.3.2. Pagès and Sagna used Lemma 2.3.1 to obtain the weak asymptotic upper bound for the quantization radius of sequences of r-optimal n-codebooks. Unfortunately, one needs in the most cases at least to have a sharper version of the form

$$V_{a_n}(\alpha_n) \subset B\left(y, \|a_n - y\| - \delta\right)^c$$

to obtain the sharp asymptotic upper bound.

A straightforward adoption of the second micro-macro inequality to Lemma 2.3.1 yields

Proposition 2.3.3. *Let X have an unbounded support. For every $\delta > 0$ there exists $n^*(\delta, \mu) \in \mathbb{N}$ such that*

$$\Delta_{n-1,r}(X, E) \leq \inf_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} 2^r(1+\delta)\overline{F_r^X}\left(\frac{\rho(\alpha_n)}{2+\delta}\right)$$

for all $n \geq n^*(\delta, \mu)$.

Proof. Let $n \in \mathbb{N}$, $n \geq 2$ and $a_n \in \alpha_n$ with $\|a_n\| = \rho(\alpha_n)$. By the second micro-macro inequality (Proposition 2.1.7), we have for $b_n \in \alpha_n \backslash \{a_n\}$

$$\Delta_{n-1,r}(X, E) \leq \int_{V_{a_n}(\alpha_n)} \operatorname{dist}(x, \alpha_n \backslash \{a_n\})^r d\mu(x) \leq \int_{V_{a_n}(\alpha_n)} 2^r(\|x\|^r + \|b_n\|^r) d\mu(x).$$

Let $y \in \operatorname{supp}(\mu)$ and $\delta > 0$. By equation (3.2) we can chose $b_n, n \in \mathbb{N}$ such that $\|b_n - y\| \leq \delta$ for every $n \geq n(\delta, \mu, y)$. Thus, applying Lemma 2.3.1 yields

$$\begin{aligned}\Delta_{n-1,r}(X, E) &\leq \int_{B\left(y, \frac{\|a_n - y\|}{2} - \delta\right)^c} 2^r(\|x\|^r + \|b_n\|^r) d\mu(x) \\ &\leq \int_{B\left(0, \frac{\|a_n\|}{2} - \delta - \frac{3}{2}\|y\|\right)^c} 2^r(\|x\|^r + \|b_n\|^r) d\mu(x) \\ &\leq \int_{B\left(0, \frac{\rho(\alpha_n)}{2} - \delta - \frac{3}{2}\|y\|\right)^c} 2^r(\|x\|^r + \|b_n\|^r) d\mu(x) \\ &\leq \int_{B\left(0, \frac{\rho(\alpha_n)}{2+\delta}\right)^c} 2^r(1+\delta)\|x\|^r d\mu(x)\end{aligned}$$

for all $n \geq n'(\delta, \mu, y) \in \mathbb{N}$ since $\rho(\alpha_n) \overset{n\to\infty}{\longrightarrow} \infty$. Since $y \in \operatorname{supp}(\mu)$ is arbitrary, and $n'(\delta, \mu, y)$ independent of the choice of α_n we obtain the assertion. □

Notes and References: The results presented in this section are generalizations of similar results given in [PS08] for the finite dimensional Euclidean case.

Chapter 3

Geometry of optimal codebooks in $\mathbb{R}^d$

3.1 Introduction and known results

One main motivation for the analysis of the quantization radius of optimal codebooks is, besides the geometric interpretation, the fact that the computational cost for calculating optimal codebooks by using stochastic algorithms (particularly the Lloyd-I algorithm) can be reduced significantly by selecting an appropriate initializing codebook.
A first article purely devoted to the radius problem has been written by Na and Neuhoff [NN01], which was followed up in [Na04]. Further research has been done by Peric, Nikolic and Pokrajac ([PNP07], [PN07]) concerning the initialization of the Lloyd-I procedure, a first rigorous mathematical treatment of the topic is contained in the dissertation by Sagna [Sag08], as well as in a publication by Pagès and Sagna [PS08]. More recently, some specific results in the univariate case have been treated in the dissertation by Yee [Yee10].
Obviously, the radius problem, i.e. the estimation of the asymptotics of the maximum radius for a sequence of optimal codebooks is of interest for distributions μ having an unbounded support. As a first observation, one obtains

$$\rho(\alpha_n) \to \infty, \quad n \to \infty \tag{3.1}$$

for each sequence of n-codebooks $(\alpha_n)_{n\in\mathbb{N}}$ satisfying

$$e_r(\mu, E; \alpha_n) \to 0, \quad n \to \infty.$$

In fact, one finds for each $y \in \operatorname{supp} \mathbb{P}^X$ codes $a_n \in \alpha_n, n \in \mathbb{N}$ such that

$$a_n \to y, \quad n \to \infty, \tag{3.2}$$

which particularly implies (3.1).
Na and Neuhoff presented several heuristic approaches and formulated a conjecture on the true rate for the asymptotics of the maximum radius of a sequence of

2-optimal n-quantizers for several types of hyper-exponential distributed r.e.'s in $(\mathbb{R}, |\cdot|)$. Fort and Pagès established in [FP02] semi-closed formulas for optimal codebooks for the Exponential and the Pareto distributions on the real line, which particularly imply the sharp asymptotics of the maximum radius for sequences of r-optimal n-quantizers (see [PS08]). In [Na04], this sharp asymptotics has been proven for a Laplacian source in $\mathbb{R}$. The first general results are due to Pagès and Sagna, who established the sharp asymptotics of the quantization radius for hyper-exponential distributed r.e.'s (defined as below) in $(\mathbb{R}, |\cdot|)$. Furthermore, their approach also yielded in the Euclidean $\mathbb{R}^d$ the weak asymptotics or the logarithmic asymptotics for several classes of distributions admitting polynomial or exponential decreasing tails. The main result obtained by Pagès and Sagna reads as follows:

Proposition 3.1.1. *[PS08, Theorem 1.2] Let $(E, \|\cdot\|) = (\mathbb{R}^d, \|\cdot\|)$ such that $(\mathbb{R}^d, \|\cdot\|)$ is a Hilbert space. Let X be a r.e. in $\mathbb{R}^d$ with $\mu = \mathbb{P}^X = f d\lambda^d$, $r > 0$ and $(\alpha_n)_{n\in\mathbb{N}}$ denote a sequence of r-optimal n-quantizers for X in $\mathbb{R}^d$.*

1. *Polynomial tails: Suppose for constants $\beta \in \mathbb{R}, c > r + d$ and $K, A > 0$*

$$f(x) = K \frac{\log(x)^\beta}{\|x\|^c} \quad \forall\ x \in B(0, A)^c, \tag{3.3}$$

then

$$\log(\rho(\alpha_n)) \sim \frac{r+d}{d} \frac{1}{c-r-d} \log(n), \quad n \to \infty.$$

2. *Hyper-exponential tails: Suppose for constants $\kappa, \theta > 0, c \in \mathbb{R}$ and $A, K > 0$*

$$f(x) = K \|x\|^c \exp(-\theta \|x\|^\kappa) \quad \forall\ x \in B(0, A)^c, \tag{3.4}$$

then

$$\left(\frac{r+d}{\theta d} \log(n)\right)^{\frac{1}{\kappa}} \lesssim \rho(\alpha_n) \lesssim 2 \left(\frac{r+d}{\theta d} \log(n)\right)^{\frac{1}{\kappa}}, \quad n \to \infty.$$

If furthermore $d = 1$ and $r \geq 1$, then

$$\rho(\alpha_n) \sim \left(\frac{r+1}{\theta} \log(n)\right)^{\frac{1}{\kappa}}, \quad n \to \infty.$$

Remark 3.1.2.

- One key tool used in the proofs by the authors is the $r-s$-property for the two types of distributions for $s < r + d$.
- The results obtained above also imply the same asymptotic bounds for the maximum radius for distributions having only one-sided tails.
- The method used to prove the result does not allow the norm used in the definition of the distributions tails (equations (3.4) and (3.3)) to differ from the norm equipping $\mathbb{R}^d$. Furthermore, Hilbert space arguments have been used in the proof such that no results have been deduced for the non-Hilbertian case.

- Pagès and Sagna conjectured that the true rate of the maximum radius of optimal codebooks for hyper-exponential tails in the Euclidean $\mathbb{R}^d$ coincides with the proven lower bound, for all $r > 0$ and $d \in \mathbb{N}$.

The starting points for this dissertation have been the questions raised but which have remained unsolved by the above results. Namely

1. the question, whether the conjecture about the true rate for the maximum radius for a sequence of optimal codebooks formulated by Pagès and Sagna holds true,
2. the question, whether the conjecture can be extended in a reasonable way to $(\mathbb{R}^d, \|\cdot\|)$ attached with an arbitrary norm $\|\cdot\|$,
3. the question, in which way the results could be generalized in case the quantizing norm and the norm describing the shape of the distribution tail do not coincide,
4. the question, how the results can be generalized, not depending anymore on a specific sequence of optimal codebooks $(\alpha_n)_{n\in\mathbb{N}}$,
5. the question, whether similar results could be obtained in case of r.e.'s in infinite dimensional Hilbert (Banach) spaces,
6. the question, whether more specific geometric properties could be estimated in a similar way, particularly in the case $d > 1$ and finally
7. the question, whether the results could help to obtain further results in the area of geometric (local) quantization problems.

In the present chapter and the subsequent chapter 4 we will present the results obtained for the finite dimensional and infinite dimensional case.

In the finite dimensional case, we will prove the conjecture of Pagès and Sagna and extend the results to a broader range of distributions as well as a general Banach space setting (section 3.2). Furthermore, we will introduce quantization balls (section 3.3), which admit a deeper understanding of the geometry of optimal codebooks. Finally, we will estimate the discrepancy between the quantization radius for sequences of optimal and sequences of asymptotically optimal codebooks and give some illustrations for the obtained results.

3.2 The quantization radius

Throughout this section, let $r > 0$, $X \in L_{r+\delta}(\mathbb{P}, \mathbb{R}^d)$ for some $\delta > 0$, where $\mathbb{R}^d$ is equipped with an arbitrary norm $\|\cdot\|$. Furthermore, let $\|\cdot\|_0$ be an additional arbitrary norm on $\mathbb{R}^d$. We denote by

$$C_{\|\cdot\|,\|\cdot\|_0} := \max\{\|j_{\|\cdot\|,\|\cdot\|_0}\|, \|j_{\|\cdot\|_0,\|\cdot\|}\|\},$$

where $j_{\|\cdot\|,\|\cdot\|_0}$ and $j_{\|\cdot\|_0,\|\cdot\|}$ denote the continuous embeddings from $(\mathbb{R}^d, \|\cdot\|) \hookrightarrow (\mathbb{R}^d, \|\cdot\|_0)$ and vice versa, as well as for $x \in \mathbb{R}^d$ and $\epsilon > 0$

$$B_0(x,\epsilon) := \{y \in \mathbb{R}^d : \|x-y\|_0 \le \epsilon\},$$

and for a set $a \subset \mathbb{R}^d$

$$\mathrm{dist}_0(x,A) := \inf\{\|x-y\|_0, y \in A\}.$$

The main purpose of of this section is to prove

Theorem 3.2.1. *Let $X \in L_{r+\delta}(\mathbb{P},\mathbb{R}^d)$, $\delta, A \in (0,\infty)$ such that*

$$\mu\big(\,(B_0(0,A))^c\,\big) = \mu_a\big(\,(B_0(0,A))^c\,\big) > 0$$

and

$$f(x) = \frac{\partial \mu_a}{\partial \lambda^d}(x) = g(\|x\|_0), \quad x \in (B_0(0,A))^c \tag{3.5}$$

for a function $g : [A,\infty) \to \mathbb{R}^+$ almost decreasing on $[A,\infty)$. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of r-optimal n-quantizers for X in $\mathbb{R}^d$.

1. *(Type I) Suppose that $-\log(g(x))$ is regularly varying at infinity with index $\theta > 0$, then*

$$\begin{aligned}\rho(\alpha_n) &\sim \overline{\rho}_{n,r}(X,E) \sim \underline{\rho}_{n,r}(X,E)\\ &\sim \left\|j_{\|\cdot\|_0,\|\cdot\|}\right\|(-\log(g))^-\left(\frac{r+d}{d}\log(n)\right)\\ &\sim \left\|j_{\|\cdot\|_0,\|\cdot\|}\right\|(-\log(g))^-\left(-\log(\Delta_{n,r}(X,E))\right), \quad n \to \infty\end{aligned} \tag{3.6}$$

   with $(-\log(g))^-$ denoting an arbitrary asymptotic inverse to $(-\log(g))$.

2. *(Type II) Suppose that $g(x)$ is regularly varying at infinity with index $-a < -(r+d)$, then*

$$\begin{aligned}\rho(\alpha_n) &\approx \overline{\rho}_{n,r}(X,E) \approx \underline{\rho}_{n,r}(X,E) \approx h^-\big(n^{-\frac{r+d}{d}}\big)\\ &\approx h^-\big(\Delta_{n,r}(X,E)\big), \quad n \to \infty\end{aligned} \tag{3.7}$$

   with h^- denoting an asymptotic inverse to h, where $h(x) := g(x)x^{r+d}, x \in [A,\infty)$.

Moreover, we have for specific hyper-exponential distributions the following sharpened version of Theorem 3.2.1:

Theorem 3.2.2. *(Type I') Using the notations of Theorem 3.2.1 case 1), and additionally assuming that*

- *g is strictly decreasing on $[A,\infty)$,*
- *$g \in C^2([A,\infty))$,*

- $\left((-\log(g))^{-1}\right)^{(i)}$ *are regularly varying for* $i = 1,2$, *with* $(\cdot)^{(i)}$ *denoting the* i*-fold derivative.*

Then, there exists $\gamma_1 \in \mathcal{R}$, $\gamma_1(x) \approx \left((-\log(g))^{-1}\right)'(x)$, $x \to \infty$ *with*

$$\gamma_1(x) \sim (r+d)\left\|j_{\|\cdot\|,\|\cdot\|_0}\right\|\left(\left(-\log(g)\right)^{-1}\right)'(x), \quad x \to \infty$$

if $\log\left(\left((-\log(g))^{-1}\right)'(x)\right)$ *does not converge as* $x \to \infty$, *and* $\gamma_2 \in \mathcal{R}$ *such that*

$$\begin{aligned}
&\left\|j_{\|\cdot\|_0,\|\cdot\|}\right\|\Big(\left(-\log(g)\right)^{-1}\left(\frac{r+d}{d}\log(n)\right)\\
&\quad+\log\left(\gamma_1\left(\frac{r+d}{d}\log(n)\right)\right)\left((-\log(g))^{-1}\right)'\left(\frac{r+d}{d}\log(n)\right)\Big)\\
&\quad\leq \underline{\rho}_{n,r}(X,E) \leq \overline{\rho}_{n,r}(X,E)\\
&\quad\leq \left\|j_{\|\cdot\|_0,\|\cdot\|}\right\|\Big(\left(-\log(g)\right)^{-1}\left(\frac{r+d}{d}\log(n)\right)+\\
&\quad\log\left(\gamma_2\left(\frac{r+d}{d}\log(n)\right)\right)\left((-\log(g))^{-1}\right)'\left(\frac{r+d}{d}\log(n)\right)\Big)
\end{aligned} \tag{3.8}$$

for all $n \in \mathbb{N}$. *In particular, there exists* $\gamma \in \mathcal{R}_{\frac{1}{\theta}-1}$, $\gamma(x) \approx \log(x)\left((-\log(g))^{-1}\right)'(x)$, $x \to \infty$ *such that*

$$\begin{aligned}
-\gamma\Big(\log(n)\Big) &\leq \underline{\rho}_{n,r}(X,E) - \left\|j_{\|\cdot\|_0,\|\cdot\|}\right\|\left(-\log(g)\right)^{-1}\left(\frac{r+d}{d}\log(n)\right)\\
&\leq \overline{\rho}_{n,r}(X,E) - \left\|j_{\|\cdot\|_0,\|\cdot\|}\right\|\left(-\log(g)\right)^{-1}\left(\frac{r+d}{d}\log(n)\right) \leq \gamma\Big(\log(n)\Big)
\end{aligned} \tag{3.9}$$

for all $n \in \mathbb{N}$. *All equations hold analogously with* $-\log(\Delta_{n,r}(X,E))$ *instead of* $\frac{r+d}{d}\log(n)$, $n \in \mathbb{N}$.

Remark 3.2.3.

1. The conditions assumed in Theorem 3.2.2 are satisfied by several interesting classes of distributions, including the standard Gaussian (normal) distribution on $(\mathbb{R}^d, \|\cdot\|_2)$. In this case, Theorem 3.2.2 yields

$$\rho(\alpha_n) = \left(\frac{2(r+d)}{d}\log(n)\right)^{\frac{1}{2}} + \mathcal{O}\left(\frac{\log(\log(n))}{(\log(n))^{\frac{1}{2}}}\right)$$

   for every sequence $(\alpha_n)_{n\in\mathbb{N}}$ of r-optimal n-codebooks in $\mathbb{R}^d$.

2. The third additional condition on the density in Theorem 3.2.2 seems to be the critical one. As a consequence of the monotone density Theorem (see A.11), this condition is in particular satisfied if $\left((-\log(g))^{-1}\right)^{(i)}$ are monotone on $[A,\infty)$, for $i = 1,2$.

Remark 3.2.4. Concerning the questions raised in the beginning of this chapter, we were able to

1. prove the conjecture formulated by Pagès and Sagna,

2. generalize the results to arbitrary norms on $\mathbb{R}^d$,
3. sharpen the results obtained for densities of (Type II),
4. allow the shape of the distribution to be central symmetric with respect to a different norm,
5. generalize the density type for the exponential case to regularly varying densities allowing a closed formula solution for the asymptotics of the quantization radius,
6. generalize all the results in order to be independent of choosing the specific codebook, and also to remain true for the infimum and the supremum,
7. sharpen the results for densities of the (Type I') and to give bounds for the second order asymptotics, which particularly holds for normal distributions on $(\mathbb{R}^d, \|\cdot\|)$.

Remark 3.2.5. Similarly to the approaches presented above, the results may also be extended to other (more exotic) distribution classes, such as the log-normal distribution or the Gumbel distribution, which are not covered by the two cases presented above. Unfortunately, there does not seem to be a general approach without specific assumptions on the density that yields a generally sharp result. With an equivalent argumentation to case (Type I) one estimates the following: (Log-normal distribution type) Suppose that

$$g(x) = \exp\big(-\phi(\log(x))\big), \quad x \in [A, \infty)$$

for some regularly varying $\phi \in \mathcal{R}_\theta$ and $\theta > 1$ then

$$\rho(\alpha_n) \sim \exp(\phi^-)\left(\frac{r+d}{d}\log(n)\right), \quad n \to \infty,$$

with ϕ^- denoting an asymptotic inverse to ϕ and $\alpha_n \in \mathcal{C}_{n,r}(X,E),\ n \in \mathbb{N}$. (Gumbel distribution Type) Suppose that $\log\big(-\log(g)\big)$ is regularly varying at infinity with index $\theta > 0$, then

$$\rho(\alpha_n) \sim \big(\log\big(-\log(g)\big)\big)^-\big(\log\big(\log(n)\big)\big), \quad n \to \infty,$$

for $\alpha_n \in \mathcal{C}_{n,r}(X,E),\ n \in \mathbb{N}$.

Comments on the proofs. The proofs are analogously to the proof for density (Type I). One has to use

$$(-\log(g))^-(\exp(x)) \sim -\log\left(\overline{F^X_{r,\|\cdot\|_0}}\right)^{-1}(\exp(x)), \quad x \to \infty$$

for the log-normal type, as well as

$$\log((-\log(g)))^-(x) \sim \log\left(-\log\left(\overline{F^X_{r,\|\cdot\|_0}}\right)\right)^{-1}(x), \quad x \to \infty$$

for the Gumbel type, both as consequences of Proposition A.8. □

Throughout this section, if not explicitly differently defined, we will use the notations of Theorem 3.2.1.

Now, we come to the proofs.

Lower bound for the quantization radius

Lemma 3.2.6. *1. (Type I) For every $\epsilon > 0$ there exists a constant $C(\mu,\epsilon) \in \mathbb{R}$ and a sequence $(c_n)_{n\in\mathbb{N}}$ with $c_n = \mathcal{O}(n^{-\frac{1}{d}})$ such that*

$$B_0\Big(0, (-\log(g))^{\leftarrow}\big(\frac{r+d}{d}\log(n) - C(\mu,\epsilon)\big)\Big)\backslash B_0\big(0, A + c_n\big) \subset \bigcap_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \Big(\alpha_n + B(0,\epsilon)\Big)$$

for all $n \in \mathbb{N}$.

2. (Type I') We use the notations from Theorem 3.2.2. Then, there exist sequences $(c_n)_{n\in\mathbb{N}}, (\eta_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $\eta_n \overset{n\to\infty}{\longrightarrow} 1$, $(c_n)_{n\in\mathbb{N}}$ with $c_n = \mathcal{O}(n^{-\frac{1}{d}})$ and a constant $C' \in \mathbb{R}$ such that

$$B_0\big(0,1\big)\Big((-\log(g))^{-1}\big(\frac{r+d}{d}\log(n)\big) + \eta_n\Big(\big(\log(C') + (r+d)\log(\cdot)\big)(\cdot)\Big)\Big((-\log(g))^{-1}\Big)'\big(\frac{r+d}{d}\log(n)\big)\Big) \backslash B_0\big(0, A + c_n\big) \subset \bigcap_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \Big(\alpha_n + B(0,\epsilon_n),\Big)$$

for $n \in \mathbb{N}$, with

$$\epsilon_n := \left\| j_{\|\cdot\|_0, \|\cdot\|}\right\| (r+d)\big((-\log(g))^{-1}\big)'\left(\frac{r+d}{d}\log(n)\right)$$

for all $n \in \mathbb{N}$.

3. (Type II) For every $1 > \epsilon > 0$ there exists a constant $C(\epsilon) > 0$ such that

$$h^{\leftarrow}\big(C(\epsilon)n^{-\frac{r+d}{d}}\big)B_0(0,1) \subset \bigcap_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \Big(\alpha_n + C_{\|\cdot\|,\|\cdot\|_0} B(0,(A + C(\epsilon)) + \epsilon h^{\leftarrow}\big(C(\epsilon)n^{-\frac{r+d}{d}}\big))\Big)$$

for all $n \in \mathbb{N}$.

Proof. **Step 1: (Firewall)** By Lemma 2.1.3 it follows that μ satisfies the lower peakless property on $B_0(0,A)^c$. Therefore, by Proposition 2.1.5 there exists a constant $C = C(b, r, \|\cdot\|, \|\cdot\|_0) \in (0,\infty)$ such that for $(y_n)_{n\in\mathbb{N}}$ in $B_0(0,A)^c$ with $B(y_n, \frac{1}{2}\delta_n) \subset B_0(0,A)^c$

$$\Delta_{n,r}(X,E) \geq C f(y_n)\delta_n^{r+d} = C g(\|y_n\|_0)\delta_n^{r+d} \tag{3.10}$$

where $\delta_n = \sup_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \operatorname{dist}(y_n, \alpha_n)$ for $n \in \mathbb{N}$. Then

$$c_n := 2C_{\|\cdot\|,\|\cdot\|_0} \sup_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \sup_{x \in \partial B_0(0,A)} \operatorname{dist}(x, \alpha_n) = \mathcal{O}(n^{-\frac{1}{d}}). \tag{3.11}$$

In fact, if equation (3.11) would not hold, there exists for every $\kappa > 0$ a strictly increasing sequence $(n_k)_{k \in \mathbb{N}}$ in $\mathbb{N}$ and codebooks $\alpha_{n_k} \in \mathcal{C}_{n_k,r}(X,E)$ and $x_{n_k} \in \partial B_0(0,A)$ such that

$$\operatorname{dist}(x_{n_k}, \alpha_{n_k}) \geq \kappa n_k^{-\frac{1}{d}}, \quad k \in \mathbb{N},$$

which implies with the first micro-macro inequality (Proposition 2.1.5) for any $b \in (0, \frac{1}{2})$ and some constant $C(b,r) > 0$

$$\begin{aligned}\Delta_{n_k,r}(X,E) &\geq C(b,r)\mu\big(B(x_{n_k}, b\kappa n_k^{-\frac{1}{d}})\big)\kappa^r n_k^{-\frac{r}{d}} \\ &\geq C(b,r)\frac{1}{2} m_g g(A + C_{\|\cdot\|,\|\cdot\|_0}\kappa n_k^{-\frac{1}{d}})\lambda^d\big(B(x_{n_k},1)\big) b^d \kappa^d \kappa^r n_k^{-(1+\frac{r}{d})}\end{aligned}$$

for all $k \in \mathbb{N}$ as a contradiction to Theorem 1.2.24. Therefore, we obtain that equation (3.10) holds for all $y_n \in B_0\left(0, A + c_n\right)^c$, since for every $x \in \partial B_0(0,A)$

$$\sup_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \operatorname{dist}(\alpha_n, x) \leq \frac{c_n}{2C_{\|\cdot\|,\|\cdot\|_0}} < \frac{c_n}{C_{\|\cdot\|,\|\cdot\|_0}} \leq \operatorname{dist}(y_n, x)$$

for all $n \in \mathbb{N}$, which implies

$$B\left(y_n, \frac{1}{2}\delta_n\right) \subset B_0(0,A)^c.$$

Step 2: The idea is now to derive from equation (3.10) an estimate of the form

$$\inf_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \rho(\alpha_n) \geq \|y_n\| - \sup_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \operatorname{dist}(y_n, \alpha_n) \geq \phi(\delta_n) - \delta_n \tag{3.12}$$

for some function ϕ, and to maximize it by an (asymptotically) optimal choice of δ_n. Note, that the approaches presented yielding the choice of δ_n are heuristics, which only give a hint on how to choose δ_n optimally. We have to consider different cases:

1. (Type I) Equation (3.10) implies in virtue of Lemma A.5

$$\begin{aligned}\|y_n\|_0 &\geq (-\log g)^{\leftarrow}\Big(\big(-\log g\big)(\|y_n\|_0)\Big) \geq (-\log(g))^{\leftarrow}\Big(-\log(\Delta_{n,r}(X,E)) \\ &\quad + (r+d)\log(C^{\frac{1}{r+d}}\delta_n)\Big)\end{aligned} \tag{3.13}$$

for all $n \in \mathbb{N}$. Suppose that $\delta_n \geq \epsilon$ for some $\epsilon > 0$ and every $n \in \mathbb{N}$. Then

$$\|y_n\|_0 \geq (-\log(g))^{\leftarrow}\Big(-\log(\Delta_{n,r}(X,E)) + C_1(\mu,\epsilon)\Big), \tag{3.14}$$

where $C_1(\mu,\epsilon) = \log(C) + (r+d)\log(\epsilon)$. By Theorem 1.2.24, we know that there is a constant $C_\Delta(\mu) \in [1,\infty)$ such that

$$(C_\Delta(\mu))^{-1} n^{-\frac{r+d}{d}} \le \Delta_{n,r}(X,E) \le C_\Delta(\mu) n^{-\frac{r+d}{d}} \tag{3.15}$$

for all $n \in \mathbb{N}$. Hence, we obtain

$$\|y_n\|_0 \ge (-\log(g))^{\leftarrow}\Big(-\log(n^{-\frac{r+d}{d}}) - C(\mu,\epsilon)\Big)$$

with $C(\mu,\epsilon) = \log(C_\Delta(\mu)) - C_1(\mu,\epsilon)$. By contraposition, we obtain

$$\begin{aligned}(-\log(g))^{\leftarrow}\Big(\frac{r+d}{d}\log(n) - C(\mu,\epsilon)\Big) &\, B_0(\mathring{0},1) \setminus B_0(0, A + c_n)\\ \subset \bigcap_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} &\Big(\alpha_n + B(0,\epsilon)\Big)\end{aligned}$$

for all $n \in \mathbb{N}$. The assertion follows by the closedness of the right-hand side.

2. (Type I') We use the notations from Theorem 3.2.2 and set $\epsilon = \epsilon_n, n \in \mathbb{N}$ in the proof for (Type I). In virtue of the fact that there exists $e \in B_0(0,1)$ such that

$$\left\|j_{\|\cdot\|_0,\|\cdot\|}\right\| = \|e\|$$

and the fact that $(-\log(g))^{\leftarrow} = (-\log(g))^{-1}$, the optimization problem for the optimal choice of ϵ_n under these specific assumptions reads

$$\begin{aligned}\|e\,\|y_n\|_0\| - \epsilon_n &\ge \left\|j_{\|\cdot\|_0,\|\cdot\|}\right\| (-\log(g))^{-1}\Big(\frac{r+d}{d}\log(n) - C(\mu,\epsilon_n)\Big)\\ -\epsilon_n &\ge \left\|j_{\|\cdot\|_0,\|\cdot\|}\right\| (-\log(g))^{-1}\Big(\frac{r+d}{d}\log(n)\Big)\\ &+ \left\|j_{\|\cdot\|_0,\|\cdot\|}\right\| \Big((-\log(g))^{-1}\Big)'(\xi_n)\,|C(\mu,\epsilon_n)| - \epsilon_n \to \max_{\epsilon_n}\end{aligned} \tag{3.16}$$

for some $\xi_n \in B_{|\cdot|}(\frac{r+d}{d}\log(n), |C(\mu,\epsilon_n)|)$. In view of $C(\mu,\epsilon_n) = -\log(C) + \log(C_\Delta(\mu)) - (r+d)\log(\epsilon_n)$ and disregarding $\xi_n = \xi_n(\epsilon_n)$ we obtain the almost optimal choice for ϵ_n

$$\begin{aligned}\epsilon_n &:= (r+d)\left\|j_{\|\cdot\|_0,\|\cdot\|}\right\| \Big((-\log(g))^{-1}\Big)'\big(\frac{r+d}{d}\log(n)\big)\\ &\sim (r+d)\left\|j_{\|\cdot\|_0,\|\cdot\|}\right\| \Big((-\log(g))^{-1}\Big)'(\xi_n), \quad n\to\infty,\end{aligned}$$

where the asymptotic equivalence is a consequence of

$$|C(\mu,\epsilon_n)| = \mathrm{o}(\log(n)), \quad n \to \infty.$$

Thus, arguing as in equation (3.16), we obtain

$$\begin{aligned}\|y_n\|_0 &\ge (-\log(g))^{-1}\big(\frac{r+d}{d}\log(n)\big) +\\ &\eta_n\Big(\big(\log(C') + (r+d)\log(\cdot)\big)(\cdot)\Big)\Big((-\log(g))^{-1}\Big)'\big(\frac{r+d}{d}\log(n)\big)\end{aligned}$$

for some constant $C' \in \mathbb{R}$ and a sequence $\eta_n \to 1,\ n \to \infty$, which implies the assertion by contraposition.

3. (Type II) Suppose that $\delta_n = \epsilon_n \|y_n\|_0$ for some $1 > \epsilon_n > 0, n \in \mathbb{N}$. Then, equation (3.10) implies

$$g(\|y_n\|_0)\,\|y_n\|_0^{r+d} = f(y_n) \leq \frac{1}{C}\Delta_{n,r}(X,E)\epsilon_n^{-(r+d)} \tag{3.17}$$

Let $h(x) := g(x)x^{r+d}, x \in [A,\infty)$. Then h is regularly varying at infinity with index $-a+r+d$ (see Proposition A.2), and we obtain in virtue of Lemma A.6

$$\|y_n\|_0 \geq h^{\leftarrow}\big(h(\|y_n\|_0)\big) \geq h^{\leftarrow}\Big(\frac{1}{C}\Delta_{n,r}(X,E)\,(\epsilon_n)^{-(r+d)}\Big), \tag{3.18}$$

for $n \in \mathbb{N}$, and furthermore in view of Theorem 1.2.24 for some constant $C_\Delta(\mu) \in [1,\infty)$

$$\|y_n\|_0 \geq h^{\leftarrow}\Big(\frac{C_\Delta(\mu)}{C} n^{-\frac{r+d}{d}}\,\epsilon_n^{-(r+d)}\Big), \tag{3.19}$$

for $n \in \mathbb{N}$. Similarly to the heuristics for (Type I'), we now want to maximize

$$\begin{aligned}\big\|\,\|y_n\|_0\, e\big\| - \epsilon_n \|y_n\|_0 &\geq (\|e\| - \epsilon_n)\times \\ h^{\leftarrow}\Big(\frac{C_\Delta(\mu)}{C} n^{-\frac{r+d}{d}}\,\epsilon_n^{-(r+d)}\Big) &\to \max_{\epsilon_n}.\end{aligned}$$

Using the fact that $h^{\leftarrow} \in \mathcal{R}_{-\frac{1}{a-r-d}}$ we derive (asymptotic) equivalently

$$(\|e\| - \epsilon_n) h^{\leftarrow}\Big(\frac{C_\Delta(\mu)}{C} n^{-\frac{r+d}{d}}\Big)\epsilon_n^{\frac{-(r+d)}{-(a-r-d)}} \to \max.$$

Optimization yields the unique solution $\epsilon_n = \epsilon^*$ given as

$$\epsilon^* = \frac{\|e\|}{1 + \frac{r+d}{a-r-d}},$$

such that a choice of $\epsilon_n \equiv$ const seems to yield an optimal choice, given this approach. Thus, we set for $\epsilon_n = \epsilon \in (0,1),\ n \in \mathbb{N}$

$$C(\epsilon) = \max\left\{\frac{C_\Delta(\mu)}{C}\,(\epsilon)^{-(r+d)}, c^*\right\},$$

where $c^* = \max_{i\in\mathbb{N}}\{c_i\}$ and $c_i,\ i \in \mathbb{N}$ as in Step 1. By contraposition, equation (3.18) implies for $n \in \mathbb{N}$

$$\begin{aligned}&h^{\leftarrow}\big(C(\epsilon)n^{-\frac{r+d}{d}}\big)\, B_0(\mathring{0},1) \\ &\subset \bigcap_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \Big(\alpha_n + C_{\|\cdot\|,\|\cdot\|_0} B(0,(A+c^*) + \epsilon h^{\leftarrow}\big(C(\epsilon)n^{-\frac{r+d}{d}}))\Big)\end{aligned}$$

which yields the asserted.

□

Remark 3.2.7. There are alternative approaches for the derivation of lower bounds for the quantization radius for $\alpha_n \in \mathcal{C}_{n,r}(X,E)$. Those are based on the $(r, r+\nu)$-property for distributions having the shape of Theorem 3.2.1. In general, one may replace the first micro-macro inequality (Proposition 2.1.5) and the knowledge about the asymptotics of the increments of the quantization error (Theorem 1.2.24) by an argument based on

- the asymptotics of the quantization error itself, and
- the $(r, r+\nu)$-property for $\nu < d$.

Unfortunately, it does not seem possible to derive results as sharp as given in the previous Lemma. More precisely, one obtains based on the presented approach

- similar results on the asymptotics for densities of the form (Type I) (see [Jun11]),
- but only weaker results for regularly varying densities (only the sharp rate for $\log(\rho_n)$ can be derived), and no result of the type of Theorem 3.2.2.

As for the verification of the latter statement, one replaces $Cn^{-\frac{r+d}{d}}$ by $Cn^{-\frac{r+\nu}{d}}$ in the proofs of the Theorems, for $\nu < d$ arbitrary.

Upper bound for the quantization radius: Since g is almost decreasing on $[A, \infty)$ and g and $-\log(g)$ respectively are regularly varying, we obtain that $\{g > 0\} \supset [A, \infty)$. As a consequence, we get

$$\overline{F^X_{r,\|\cdot\|_0}}(x) := \int_{B_0(0,x)^c} \|y\|_0^r \, d\mu(y)$$

is strictly decreasing on $[A, \infty)$ and therefore, its inverse $\overline{F^X_{r,\|\cdot\|_0}}^{-1}$ exists. Furthermore, by strict monotonicity of $\log(x)$, $-\log\big(C\overline{F^X_{r,\|\cdot\|_0}}\big)$ is strictly increasing for every $C \in (0, \infty)$, and thus its inverse exists.

Lemma 3.2.8. *(Type I) There exists an asymptotic inverse* $\big(-\log(g)\big)^*$ *for* $-\log(g)$ *such that*

$$\bigcup_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \alpha_n \subset \big(-\log(g)\big)^* \left(\frac{r+d}{d} \log(n)\right) B_0(0,1)$$

for all $n \in \mathbb{N}$.

Proof. Let $\epsilon > 0$ and assume that the assertion does not hold. For

$$\overline{F^X_{r,\|\cdot\|_0}}(x) = \int_{B_0(0,x)^c} \|y\|_0^r \, d\mu(y)$$

we define

$$\begin{aligned}(-\log(g))^*(x) &:= 2C^2_{\|\cdot\|,\|\cdot\|_0}G(x)\\ &\quad -\left(2C^2_{\|\cdot\|,\|\cdot\|_0}-1\right)(-\log(g))^{\leftarrow}(x-C(\mu,\epsilon))+2\epsilon C_{\|\cdot\|,\|\cdot\|_0},\end{aligned}$$

where

$$G(x) := \max\left\{\left(-\log\left(C\overline{F^X_{r,\|\cdot\|_0}}\right)\right)^{-1}(x),(-\log(g))^{\leftarrow}(x-C(\mu,\epsilon))\right\},$$

the constant $C(\mu,\epsilon)\in\mathbb{R}$ is given as in Lemma 3.2.6 and the constant $C = 2^{r+2}C_{\|\cdot\|,\|\cdot\|_0}C_{\Delta}(\mu)$, where $C_{\Delta}(\mu)$ is from Theorem 1.2.24.

Step 1: We show that $\left(-\log\left(C\overline{F^X_{r,\|\cdot\|_0}}\right)\right)^{-1}$ is an asymptotic inverse to $-\log(g)$. In fact, we have for $x\in[A,\infty)$

$$\begin{aligned}\overline{F^X_{r,\|\cdot\|_0}}(x) &= \int_{B_0(0,x)^c}\|y\|_0^r\,d\mu(y)\\ &= \int_x^{\infty}\left(\int d\partial B_0(0,t)\right)t^r g(t)d\lambda(t) \qquad (3.20)\\ &= S_{d,\|\cdot\|_0}\int_x^{\infty}\exp(\log(g(t))+\log(t^{r+d-1}))d\lambda(t).\end{aligned}$$

By [BGT87, Theorem 4.12.10 i)] with $f(t) = -\log(g(t))-\log(t^{r+d-1})$ we obtain

$$-\log\left(\overline{F^X_{r,\|\cdot\|_0}}(x)\right)\sim-\log(g(x))-\log(x^{r+d-1}),\quad x\to\infty.$$

Since $-\log(g)$ is regularly varying at infinity with index $\theta>0$ and $\log(x^{r+d-1})$ is slowly varying at infinity this implies

$$-\log\left(C\overline{F^X_{r,\|\cdot\|_0}}\right)(x)\sim-\log(g(x)),\quad x\to\infty.$$

Step 2: We show that $(-\log(g))^*$ is an asymptotic inverse to $-\log(g)$. In fact, as a general property of functions $h,h_1,\dots,h_m$ satisfying

$$h_i(x)\sim h_j(x)\sim h(x),\quad x\to\infty$$

for all $i,j\in\{1,\dots,m\}$ and $h(x)\xrightarrow{x\to\infty}\infty$, one has

$$\sum_{i=1}^m t_ih_i(x)+\kappa\sim h(x),\quad x\to\infty$$

for every constant $\kappa\in\mathbb{R}$ and $t_i\in\mathbb{R}, i\in\{1,\dots,m\}$ with $\sum_{i=1}^m t_i = 1$. Furthermore, we have

$$\frac{\max\{h_i(x),h_j(x)\}}{h(x)} = \max\left\{\frac{h_i(x)}{h(x)},\frac{h_j(x)}{h(x)}\right\}\to1,\quad x\to\infty$$

as well as

$$h(x+\kappa)\sim h(x),\quad x\to\infty$$

for h regularly varying, which implies the assertion.

Step 3: By the assumption, there exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ and codebooks $\alpha_{n_k} \in \mathcal{C}_{n_k,r}(X,E)$ such that

$$a_{n_k} \notin (-\log(g))^* \left(\frac{r+d}{d}\log(n_k)\right) B_0(0,1)$$

for codes $a_{n_k} \in \alpha_{n_k}$ and all $k \in \mathbb{N}$. We will show that

$$V_{a_{n_k}}(\alpha_{n_k}) \subset G\Big(\big(\frac{r+d}{d}\log(n_k)\big)\Big) B_0(0,1)^c \tag{3.21}$$

for all $k \in \mathbb{N}$.
We denote for $n \in \mathbb{N}$

$$\begin{aligned} A_n &= (-\log(g))^{\leftarrow}\left(\frac{r+d}{d}\log(n) - C(\mu,\epsilon)\right) B_0(0,1), \\ B_n &= G\Big(\big(\frac{r+d}{d}\log(n)\big)\Big) B_0(0,1), \\ C_n &= (-\log(g))^* \left(\frac{r+d}{d}\log(n)\right) B_0(0,1). \end{aligned}$$

First note, that $A_n \subset B_n \subset C_n,\ n \in \mathbb{N}$. By Lemma 3.2.6, we have

$$\sup_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \sup_{x \in A_n} \operatorname{dist}(\alpha_n, x) \leq \epsilon$$

for n sufficiently large. We abbreviate for a function $f,\ f^+ := \max\{f, 0\}$. Then, by definition of $A_n, B_n,\ n \in \mathbb{N}$ and for $x_n \in B_n$

$$\begin{aligned} &\operatorname{dist}(A_n, x_n) \leq C_{\|\cdot\|,\|\cdot\|_0} \operatorname{dist}_0(A_n, x_n) \\ &\leq C_{\|\cdot\|,\|\cdot\|_0} \left(\Big(-\log\big(C\overline{F^X_{r,\|\cdot\|_0}}\big)\Big)^{-1} - (-\log(g))^{\leftarrow}(\cdot - C(\mu,\epsilon))\right)^+ \big(\frac{r+d}{d}\log(n)\big), \end{aligned}$$

which implies

$$\begin{aligned} &\operatorname{dist}(\alpha_n, x_n) \leq \operatorname{dist}(x_n, A_n) + \sup_{y \in \partial A_n} \operatorname{dist}(y, \alpha_n) \leq \epsilon \\ &+ C_{\|\cdot\|,\|\cdot\|_0} \left(\Big(-\log\big(C\overline{F^X_{r,\|\cdot\|_0}}\big)\Big)^{-1} - (-\log(g))^{\leftarrow}(\cdot - C(\mu,\epsilon))\right)^+ \big(\frac{r+d}{d}\log(n)\big). \end{aligned}$$

Conversely, we have for $x_n \in C_n^c$

$$\begin{aligned} &\operatorname{dist}(B_n, x_n) \geq \frac{1}{C_{\|\cdot\|,\|\cdot\|_0}} \operatorname{dist}_0(B_n, x_n) \geq 2\epsilon + \frac{(2C^2_{\|\cdot\|,\|\cdot\|_0} - 1)}{C_{\|\cdot\|,\|\cdot\|_0}} \times \\ &\left(\Big(-\log\big(C\overline{F^X_{r,\|\cdot\|_0}}\big)\Big)^{-1} - (-\log(g))^{\leftarrow}(\cdot - C(\mu,\epsilon))\right)^+ \big(\frac{r+d}{d}\log(n)\big) \\ &\geq 2\epsilon + C_{\|\cdot\|,\|\cdot\|_0} \times \\ &\left(\Big(-\log\big(C\overline{F^X_{r,\|\cdot\|_0}}\big)\Big)^{-1} - (-\log(g))^{\leftarrow}(\cdot - C(\mu,\epsilon))\right)^+ \big(\frac{r+d}{d}\log(n)\big), \end{aligned}$$

which implies that $B_{n_k} \subset \left(V_{a_{n_k}}(\alpha_{n_k})\right)^c,\ k \in \mathbb{N}$.

Step 4: In view of equation (3.2) there exists a sequence of codes $c_n \in \alpha_n, n \in \mathbb{N}$, such that $\|c_n\|_0 \to A$. We estimate in virtue of Theorem 1.2.24, the second micro-macro inequality (Proposition 2.1.7) and the fact that $\overline{F^X_{r,\|\cdot\|_0}}$ is decreasing

$$\begin{aligned}
n_k^{-\frac{r+d}{d}} &\leq (n_k-1)^{-\frac{r+d}{d}} \leq C_\Delta(\mu)\Delta_{n_k-1,r}(X,E)\\
&\leq C_\Delta(\mu)\int_{V_{a_{n_k}}(\alpha_{n_k})} \|x-c_{n_k}\|^r \, d\mu(x)\\
&\leq C_\Delta(\mu)C_{\|\cdot\|,\|\cdot\|_0}\int_{V_{a_{n_k}}(\alpha_{n_k})} \|x-c_{n_k}\|_0^r \, d\mu(x)\\
&\leq C_\Delta(\mu)C_{\|\cdot\|,\|\cdot\|_0}\int_{V_{a_{n_k}}(\alpha_{n_k})} 2^r\left(\|x\|_0^r+\|c_{n_k}\|_0^r\right) d\mu(x)\\
&\leq C_\Delta(\mu)C_{\|\cdot\|,\|\cdot\|_0}2^{r+1}\int_{V_{a_{n_k}}(\alpha_{n_k})} \|x\|_0^r \, d\mu(x)\\
&\leq C_\Delta(\mu)C_{\|\cdot\|,\|\cdot\|_0}2^{r+1}\overline{F^X_{r,\|\cdot\|_0}}\Big(G\big(\frac{r+d}{d}\log(n_k)\big)\Big)\\
&= \frac{1}{2}C\overline{F^X_{r,\|\cdot\|_0}}\Big(G\big(\frac{r+d}{d}\log(n_k)\big)\Big)\\
&\leq \frac{1}{2}C\overline{F^X_{r,\|\cdot\|_0}}\Big(\Big(-\log\big(C\overline{F^X_{r,\|\cdot\|_0}}\big)\Big)^{-1}\big(\frac{r+d}{d}\log(n_k)\big)\Big)
\end{aligned}$$

for all $k \in \mathbb{N}$ with $\|c_{n_k}\|_0 \leq \left(-\log\left(C\overline{F^X_{r,\|\cdot\|_0}}\right)\right)^{-1}\left(\frac{r+d}{d}\log(n_k)\right)$. Applying the negative logarithm yields the contradiction

$$\begin{aligned}
\frac{r+d}{d}\log(n_k) &\geq -\log(\frac{1}{2}) - \log\Big(C\overline{F^X_{r,\|\cdot\|_0}}\Big(\Big(-\log\big(C\overline{F^X_{r,\|\cdot\|_0}}\big)\Big)^{-1}\big(\frac{r+d}{d}\log(n_k)\big)\Big)\Big)\\
&\geq -\log(\frac{1}{2}) + \frac{r+d}{d}\log(n_k)
\end{aligned}$$

for all $k \geq 2$ with $\|c_{n_k}\|_0 \leq \left(-\log\left(C\overline{F^X_{r,\|\cdot\|_0}}\right)\right)^{-1}\left(\frac{r+d}{d}\log(n_k)\right)$. Thus, the assumption is wrong and we obtain

$$\bigcup_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \alpha_n \subset C_n$$

for all n sufficiently large. Adding a function $\gamma(x) \to 0$ to $(-\log(g))^*(x) \to \infty$ does not change its asymptotics, such that the assertion also holds for all $n \in \mathbb{N}$. $\square$

Lemma 3.2.9. *(Type II) We use the notations of Theorem 3.2.1, part 2). We have*

$$\sup_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \rho(\alpha_n) \lesssim h^-\left(n^{-\frac{r+d}{d}}\right), \quad n \to \infty$$

with h^- denoting an asymptotic inverse to h, where $h(x) = x^{r+d}g(x),\ x \in [A,\infty)$.

Proof. We abbreviate $C = C_{\|\cdot\|,\|\cdot\|_0}$. By Proposition A.10 on integrated asymptotics of regularly varying functions, we have for $x \in [A, \infty)$

$$\begin{aligned}
\overline{F_r^X}(x) &\leq C \int_{B_{\|\cdot\|}(0,x)^c} \|y\|_0^r \, d\mu(y) \\
&\leq C \int_{B_0(0,\frac{1}{C}x)^c} \|y\|_0^r g(\|y\|_0) d\lambda^d(y) \\
&= C \int_{\frac{x}{C}}^{\infty} \Big(\int_{\partial B_0(0,t)} \Big) g(t) t^r d\lambda(t) \\
&= C \int_{\frac{x}{C}}^{\infty} S_{d-1,\|\cdot\|_0} g(t) t^{r+d-1} d\lambda(t) \\
&\sim C S_{d-1,\|\cdot\|_0} \frac{1}{a-(d+r)} g\Big(\frac{x}{C}\Big)\Big(\frac{x}{C}\Big)^{r+d} \\
&\sim C^{a-(r+d)+1} S_{d-1,\|\cdot\|_0} \frac{1}{a-(d+r)} h(x), \quad x \to \infty.
\end{aligned}$$

Hence $\overline{F_r^X}(x)$ is asymptotically bounded from above by the function

$$\begin{aligned}
H(x) &= C^{a-(r+d)+1} S_{d-1,\|\cdot\|_0} \frac{1}{a-(d+r)} h(x) \\
&= C^{a-(r+d)+1} S_{d-1,\|\cdot\|_0} \frac{1}{a-(d+r)} g(x) x^{r+d}
\end{aligned}$$

regularly varying at infinity with index $-a+r+d$. By Proposition 2.3.3, we have for $\delta > 0$

$$\Delta_{n-1,r}(X,E) \leq \inf_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} 2^r(1+\delta)\overline{F_r^X}\Big(\frac{\rho(\alpha_n)}{2+\delta}\Big)$$

for all $n \geq n(\delta,\mu)$. In virtue of Theorem 1.2.24 this implies for some $C' \in \mathbb{R}$ sufficiently large

$$\begin{aligned}
\Big(\frac{1}{C'}\Big)^{-\frac{1}{a-r-d}} H^{\rightarrow}\big(n^{-\frac{r+d}{d}}\big) &\sim H^{\rightarrow}\Big(\frac{1}{C'}(n-1)^{-\frac{r+d}{d}}\Big) \\
&\geq H^{\rightarrow}\big(\Delta_{n-1,r}(X,E)\big) \\
&\geq \sup_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} H^{\rightarrow}\Big(2^r(1+\delta)\overline{F_r^X}\Big(\frac{\rho(\alpha_n)}{2+\delta}\Big)\Big) \\
&\gtrsim \sup_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \big(2^r(1+\delta)\big)^{-\frac{1}{a-r-d}} H^{\rightarrow}\Big(H\Big(\frac{\rho(\alpha_n)}{2+\delta}\Big)\Big) \\
&\gtrsim \sup_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \big(2^r(1+\delta)\big)^{-\frac{1}{a-r-d}} \frac{\rho(\alpha_n)}{2+\delta} \\
&\succcurlyeq \sup_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \rho(\alpha_n), \quad n \to \infty
\end{aligned}$$

which yields the assertion, since $h^- \approx H^{\rightarrow}$ for all asymptotic inverses h^- to h. $\square$

As we have seen in the proof of Lemma 3.2.8, the critical part for the determination of a more precise rate of convergence for the quantization radius is the discrepancy between the generalized inverse of $(-\log(g))$ and the inverse of the logarithmic survival function $\left(-\log \overline{F^X_{r,\|\cdot\|_0}}\right)$.

We will need the following Lemma.

Lemma 3.2.10. *Let $f_1 \in \mathcal{R}_\alpha$ for some $\alpha > 0$, $f_1 > 0$ and strictly increasing, $f_2 \in \mathcal{R}_\beta$ for $\beta \in \mathbb{R}$ and $f_2 > 0$. Let f_1 and*

$$f_0(x) := f_1(x) + \log(f_2(x))$$

be invertible on $[A, \infty)$ for some constant $A \in \mathbb{R}$. Then,

$$f_0^{-1}(y) = f_1^{-1}\Big(y - \eta(y) - \log\big(f_2(f_1^{-1}(y))\big)\Big)$$

for some function $\eta(y) \to 0$, $y \to \infty$.

Proof. Let $y = f_0(x)$. Then

$$\begin{aligned} x &= f_1^{-1}\big(y - \log(f_2(x)\big) \\ &= f_1^{-1}\big(y - \log(f_2(f_1^{-1}\big(y - \log(f_2(x))\big)\big). \end{aligned}$$

By proposition A.2, $f_2 \circ f_1^{-1} \in \mathcal{R}$ and furthermore

$$y = f_1(x) + \log(f_2(f_0^{-1}(y))) \sim y + \log(f_2(x)), \quad y \to \infty.$$

Hence, by regular variation of $f_2(f_1^{-1})$ we obtain

$$\begin{aligned} x &= f_1^{-1}\big(y - \log(f_2(f_1^{-1}\big(y - \log(f_2(x))\big))\big) \\ &= f_1^{-1}\Big(y - \log\big(\exp(\eta(y)) f_2(f_1^{-1}(y))\big)\Big) \\ &= f_1^{-1}\Big(y - \eta(y) - \log\big(f_2(f_1^{-1}(y))\big)\Big), \end{aligned}$$

for a function $\eta(y)$ as required. □

Lemma 3.2.11. *(Type I') There exists a bounded sequence $(\eta_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that for all $n \in \mathbb{N}$*

$$\begin{aligned} (-\log(g))^*\big(\frac{r+d}{d}\log(n)\big) \le\ & (-\log(g))^{-1}\big(\frac{r+d}{d}\log(n)\big) + \\ & \eta_n\left((-\log(g))^{-1}\right)'\big(\frac{r+d}{d}\log(n)\big)\log(\log(n)), \end{aligned}$$

where $(-\log(g))^*$ is defined as in Lemma 3.2.8 and $\epsilon = \epsilon_n$ as in Lemma 3.2.6.

Proof. We have to estimate more precisely the rate of increase for $(-\log(g))^*$ as defined in Lemma 3.2.8 as a function of y and now additionally depending on $\epsilon = \epsilon_n$.

Step 1: A precise estimate for the asymptotic of $\left(-\log C\overline{F^X_{r,\|\cdot\|}}\right)^{-1}$:
In virtue of Lemma A.9 and equation (3.20) one has for $\overline{g}(x) = x^{r+d-1}g(x)$

$$\begin{aligned}\overline{F^X_{r,\|\cdot\|}}(x) &= S_{d,\|\cdot\|}\int_x^\infty t^{r+d-1}g(t)d\lambda(t)\\ &= S_{d,\|\cdot\|}\int_x^\infty \exp\big(-(-\log(g(t))) - \log(t^{r+d-1})\big)d\lambda(t)\\ &= S_{d,\|\cdot\|}g(x)x^{r+d-1}\frac{1}{(-\log(\overline{g}))'(x)}\eta_1(x)\\ &= S_{d,\|\cdot\|}g(x)x^{r+d-1}\frac{1}{(-\log(g))'(x)}\eta_2(x),\end{aligned}$$

for functions $\eta_1(x), \eta_2(x) \to 1,\ x \to \infty$. Here, the last equality is a consequence of $(-\log(g))' \in \mathcal{R}_{\theta-1}$ for $\theta > 0$ and the fact that $\frac{1}{x} = \log(x)' \in \mathcal{R}_{-1} = o(\mathcal{R}_{\theta-1})$. Multiplying with a constant $C > 0$ and applying the negative logarithm yields

$$\begin{aligned}-\log C\overline{F^X_{r,\|\cdot\|}}(x) &= -\log(g(x)) - \log(CS_{d,\|\cdot\|}x^{r+d-1}\frac{1}{(-\log(g))'(x)}\eta_2(x))\\ &= f_1(x) + \log\big(f_2(x)\big),\end{aligned}$$

where $f_1(x) = -\log(g(x))$, $f_2(x) = C'\frac{(-\log(g))(x)}{x^{r+d}}\eta_3(x)$ for a constant $C' \in \mathbb{R}$ and a function $\eta_3(x) \to 1,\ x \to \infty$. Here, we rely on the fact that

$$(-\log(g))'(x) \sim \frac{\theta(-\log(g))(x)}{x}, \quad x \to \infty,$$

see [BGT87, Proposition 1.5.8]. In view of Lemma 3.2.10, we obtain for some function $\eta_4(y) \to 0$

$$\begin{aligned}&\big(-\log C\overline{F^X_{r,\|\cdot\|}}\big)^{-1}(y) = f_1^{-1}\Big(y - \eta_4(y) - \log\big(f_2(f_1^{-1}(y))\big)\Big)\\ &\quad= (-\log(g))^{-1}\Big(y - \eta_4(y) - \log(C') - \log(y) + (r+d)\log((-\log(g))^{-1}(y))\Big).\end{aligned}$$

As a consequence, we derive in view of the mean value theorem

$$\begin{aligned}&\big(-\log C\overline{F^X_{r,\|\cdot\|}}\big)^{-1}(y) = (-\log(g))^{-1}(y)\\ &+\big((-\log(g))^{-1}\big)'(\xi)\big(-\eta_4(y) - \log(C') - \log(y) + (r+d)\log((-\log(g))^{-1}(y))\big)\end{aligned}$$

for some $\xi = \xi(y) \in B_{|\cdot|}\big(y, |-\eta_4(y) - \log(C') - \log(y) + (r+d)\log((-\log(g))^{-1}(y))|\big)$. By Proposition A.2 $-\eta_4(y) - \log(C') - \log(y) + (r+d)\log((-\log(g))^{-1}(y)) \in \mathcal{R}_0$ which implies

$$y \sim \xi(y), \quad y \to \infty.$$

By using the assumption on the regularity of $\big((-\log(g))^{-1}\big)'$ this implies

$$\begin{aligned}&\big(-\log C\overline{F^X_{r,\|\cdot\|}}\big)^{-1}(y) = (-\log(g))^{-1}(y)\\ &+\eta_5(y)\big((-\log(g))^{-1}\big)'(y)\times\\ &\big(-\eta_4(y) - \log(C') - \log(y) + (r+d)\log((-\log(g))^{-1}(y))\big)\end{aligned}$$

for some function $\eta_5(y) \to 1,\ y \to \infty$.

Step 2: A precise estimate for $(-\log(g))^{-1}(x - C(\mu,\epsilon))$

Let $\epsilon = \epsilon_n$ be as defined in the proof of Lemma 3.2.6, i.e.

$$\epsilon_n := \left\| j_{\|\cdot\|_0,\|\cdot\|} \right\| (r+d)\big((-\log g)^{-1}\big)'\big(\frac{r+d}{d}\log(n)\big).$$

We obtain with $C(\mu,\epsilon) = C'' + (r+d)\log(\epsilon)$ for some constant $C'' \in \mathbb{R}$, by using the facts that $\xi(y) \sim y$ and $\big((-\log(g))^{-1}\big)' \in \mathcal{R}$

$$\begin{aligned}(-\log(g))^{-1}(y - C(\mu,\epsilon_n)) &= (-\log(g))^{-1}(y) - C(\mu,\epsilon_n)\big((-\log(g))^{-1}\big)'(\xi(y))\\ &= -\Big(C'' + \eta_6(n) + (r+d)\log\Big(\big((-\log g)^{-1}\big)'\big(\frac{r+d}{d}\log(n)\big)\Big)\Big)\times\\ &\eta_6(y)\big((-\log(g))^{-1}\big)'(y) + (-\log(g))^{-1}(y)\end{aligned}$$

for a function $\eta_6(n) \to 1, \quad n \to \infty$.

Step 3: We set $y_n = \frac{r+d}{d}\log(n), n \in \mathbb{N}$. Let $G = G(y_n,\epsilon_n) = G(\frac{r+d}{d}\log(n),\epsilon_n)$ be as in Lemma 3.2.8 with G now depending on $\epsilon = \epsilon_n$ additionally. We estimate in virtue of Step 1 and Step 2

$$G(\frac{r+d}{d}\log(n),\epsilon_n) \le (-\log(g))^{-1}(y_n) + \big((-\log(g))^{-1}\big)'(y_n)\eta^*(y_n)\log(\log(n)),$$

$n \in \mathbb{N}$, for a bounded sequence $-\infty < \liminf_{n\to\infty}\eta^*(y_n) \le \limsup_{n\to\infty}\eta^*(y_n) < \infty$. Furthermore, we select $\eta^*(y_n)$ such that

$$\epsilon_n \le \big((-\log(g))^{-1}\big)'(y_n)\eta^*(y_n)\log(\log(n)),\ n \in \mathbb{N}.$$

Thus, we can estimate

$$\begin{aligned}(-\log(g))^*(y_n) &= 2C^2_{\|\cdot\|,\|\cdot\|_0}G\big(y_n\big)\\ &- \big(2C^2_{\|\cdot\|,\|\cdot\|_0} - 1\big)(-\log(g))^{-1}\big(y_n - C(\mu,\epsilon_n)\big) + 2\epsilon_n C_{\|\cdot\|,\|\cdot\|_0}\\ &\le \big((-\log(g))^{-1}\big)(y_n) + \eta^*(y_n)2C^2_{\|\cdot\|,\|\cdot\|_0}\big((-\log(g))^{-1}\big)'(y_n)\log(\log(n))\\ &+ \big(2C^2_{\|\cdot\|,\|\cdot\|_0} - 1\big)\eta^*(y_n)\Big((-\log(g))^{-1}\Big)'(y_n)\log(\log(n)) + 2\epsilon_n C_{\|\cdot\|,\|\cdot\|_0}\\ &\le (-\log(g))^{-1}(y_n) + 4C^2_{\|\cdot\|,\|\cdot\|_0}\eta^*(y_n)((-\log(g))^{-1})'(y_n)(\log(\log(n))).\end{aligned}$$

$\square$

Lemma 3.2.12. *(Type I') We use the notations of Theorem 3.2.2. There exists a bounded sequence $(\eta_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that*

$$\begin{aligned}\bigcup_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \alpha_n &\subset \big(-\log(g)\big)^{-1}\left(\frac{r+d}{d}\log(n)\right)B_0(0,1) + B(0,\epsilon_n)\\ &+ \eta_n\log(\log(n))\Big((-\log(g))^{-1}\Big)'\big(\frac{r+d}{d}\log(n)\big)B_0(0,1),\end{aligned}$$

with ϵ_n *as in Lemma 3.2.6, i.e.*

$$\epsilon_n := \left\| j_{\|\cdot\|_0, \|\cdot\|} \right\| (r+d)\big((-\log g)^{-1}\big)'\Big(\frac{r+d}{d}\log(n)\Big).$$

Proof. The proof is a consequence of the proof of Lemma 3.2.8 and Lemma 3.2.11. □

Proof of the Theorems Recall, that

$$\left\| j_{\|\cdot\|_0, \|\cdot\|} \right\| := \sup_{x \in B_0(0,1)} \|x\|. \tag{3.22}$$

Proof of Theorem 3.2.1.

1. (Type I) The lower bound is a consequence of equation (3.22) in combination with Lemma 3.2.6 (Type I). As for the upper bound, one combines equation (3.22) with Lemma 3.2.8.
2. (Type II) Again, in virtue of equation 3.22, the lower bound is a consequence of Lemma 3.2.6 (Type II), the upper bound follows with Lemma 3.2.9.

□

Proof of Theorem 3.2.2. Equation (3.8) is a consequence of Lemma 3.2.6 formulation (Type I') for the lower bound and Lemma 3.2.12 for the upper bound, both in combination with equation 3.22. Equation (3.9) is a consequence of the fact that for $u \in \mathcal{R}$ and $v \in \mathcal{R}_\theta$, $\log(u)v \in \mathcal{R}_\theta$. □

3.3 Quantization balls

Given that one knows (at least asymptotically) the quantization radius for a sequence of r-optimal n-codebooks $(\alpha_n)_{n\in\mathbb{N}}$ for the r.e. X in $\mathbb{R}^d$, it is natural to ask for the "remaining" geometry of the codebook, i.e. for the shape of

$$\frac{\alpha_n}{\rho(\alpha_n)}, \; n \in \mathbb{N}.$$

By definition, we have for all $n \in \mathbb{N}$

$$\frac{\alpha_n}{\rho(\alpha_n)} \subset B(0,1).$$

Conversely, it would be interesting to know in which sense a converse result of the type

$$A \subset \frac{\alpha_n}{\rho(\alpha_n)}, \tag{3.23}$$

for a set $A \subset B(0,1)$ holds as well. For a fixed $n \in \mathbb{N}$, we cannot expect to find large sets $A \subset \mathbb{R}^d$ such that equation (3.23) holds in general. Thus, we will consider specific types of limits for $\frac{\alpha_n}{\rho(\alpha_n)}$ as n tends to infinity.

We will use different types of convergences for sequences of sets in a Banach space E, as introduced in Appendix B. In addition to the common set-theoretic lim inf and lim sup we define $\liminf^{\|\cdot\|} A_n$ as the set of limit point of the sequence $(A_n)_{n\in\mathbb{N}}$, i.e.

$$\liminf_{n\to\infty}{}^{\|\cdot\|} A_n := \bigcap_{H\in\mathcal{T}} \overline{\bigcup_{m\in H} A_m},$$

where $\mathcal{T} := \{H \subset \mathbb{N} : \mathrm{card}(H) = \infty\}$, and $\limsup^{\|\cdot\|} A_n$ as the set of all cluster points of $(A_n)_{n\in\mathbb{N}}$, i.e.

$$\limsup_{n\to\infty}{}^{\|\cdot\|} A_n := \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m\geq n} A_m}.$$

If both limits coincide, we call $\lim^{\|\cdot\|} A_n$ the $\|\cdot\|$-based limit. Furthermore, we denote by

$$\delta(A,B) := \inf\{\epsilon > 0 : A \subset B + B(0,\epsilon), B \subset A + B(0,\epsilon)\}$$

the Hausdorff distance between two arbitrary sets $A, B \subset E$, with $\inf_\varnothing := \infty$.

Definition 3.3.1. Let $r > 0$ and $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of r-optimal n-quantizers for the random variable X in the Banach space $(E, \|\cdot\|)$.

1. We call a subset $\mathcal{C} = \mathcal{C}_r(X, E; (\alpha_n)_{n\in\mathbb{N}})) \subset E$ a *quantization hull* for X in $(E, \|\cdot\|)$ of order r iff

$$\mathcal{C} = \overline{\liminf_{n\to\infty} \frac{\mathrm{conv}(\alpha_n)}{\rho(\alpha_n)}} = \overline{\limsup_{n\to\infty} \frac{\mathrm{conv}(\alpha_n)}{\rho(\alpha_n)}}.$$

   If $\mathcal{C}$ is independent of $(\alpha_n)_{n\in\mathbb{N}}$ we call $\mathcal{C}$ the quantization hull for X of order r.

2. We call a subset $\mathcal{B} = \mathcal{B}_r(X, E; (\alpha_n)_{n\in\mathbb{N}}) \subset E$ a *quantization ball* for X in $(E, \|\cdot\|)$ of order r iff

$$\mathcal{B} = \liminf_{n\to\infty}{}^{\|\cdot\|} \frac{\alpha_n}{\rho(\alpha_n)} = \limsup_{n\to\infty}{}^{\|\cdot\|} \frac{\alpha_n}{\rho(\alpha_n)}.$$

   If $\mathcal{B}$ is independent of $(\alpha_n)_{n\in\mathbb{N}}$ we call $\mathcal{B}$ the quantization ball for X of order r.

Hereafter, we will use the notation of the previous section, in particular those from Theorems 3.2.1 and 3.2.2. The aim of this section is to prove

Theorem 3.3.2. *(quantization ball, quantization hull) Let $r > 0$ and $\alpha_n \in \mathcal{C}_{n,r}(X, \mathbb{R}^d)$, $n \in \mathbb{N}$.*

1. *(Type I) The quantization ball $\mathcal{B} = \mathcal{B}_r(X, E, (\alpha_n)_{n\in\mathbb{N}})$ exists, is independent of the choice of $(\alpha_n)_{n\in\mathbb{N}}$ as well as $r > 0$ and reads*

$$\mathcal{B} = \lim_{n\to\infty}{}^{\|\cdot\|} \frac{\alpha_n}{\rho(\alpha_n)} = \frac{1}{\left\| j_{\|\cdot\|_0, \|\cdot\|} \right\|} B_0(0,1),$$

which is equivalent to

$$\delta\left(\frac{\alpha_n}{\rho(\alpha_n)}, \mathcal{B}\right) \to 0,$$

with $\delta(\cdot,\cdot)$ denoting the Hausdorff distance. Furthermore, the quantization hull $\mathcal{C} = \mathcal{C}_r(X, E, (\alpha_n)_{n\in\mathbb{N}})$ exists, and

$$\mathcal{C} = \mathcal{B}.$$

The results still hold when replacing $\rho(\alpha_n)$ by its asymptotic equivalents from Theorem 3.2.1.

2. *(Type II) There exists a constant $\kappa \in (0,1]$ such that*

$$\kappa B(0,1) \subset \liminf_{n\to\infty} \frac{\operatorname{conv}(\alpha_n)}{\rho(\alpha_n)} \subset \limsup_{n\to\infty} \frac{\operatorname{conv}(\alpha_n)}{\rho(\alpha_n)} \subset B(0,1).$$

In virtue of the results developed in the previous section, it is not surprising that we are able to sharpen the results for densities of (Type I') in the following way:

Theorem 3.3.3. *(Type I') There exists $\gamma(x) \approx \log(x)\big((-\log(g))^{-1}\big)'(x), \quad x \to \infty$ such that*

$$\delta\left(\alpha_n, \left(\frac{\rho(\alpha_n)}{\left\|j_{\|\cdot\|_0,\|\cdot\|}\right\|} B_0(0,1)\right) \backslash B_0(0,A)\right) \le \gamma(\log(n))$$

for all $n \in \mathbb{N}$, with $\delta(\cdot,\cdot)$ denoting the Hausdorff distance. The result still holds when replacing $\rho(\alpha_n)$ by its asymptotic equivalents

$$\left\|j_{\|\cdot\|_0,\|\cdot\|}\right\| (-\log(g))^{-1}\big(-\log(\Delta_{n,r}(X,E))\big),$$

$$\left\|j_{\|\cdot\|_0,\|\cdot\|}\right\| (-\log(g))^{-1}\Big(\frac{r+d}{d}\log(n)\Big),$$

$\overline{\rho}_{n,r}(X,E)$ or $\underline{\rho}_{n,r}(X,E)$.

Remark 3.3.4.
1. In ([Jun11]), the quantization ball is defined differently, namely as a set $\mathcal{B} \in \mathbb{R}^d$ satisfying

$$\mathcal{B} = \overline{\liminf_{n\to\infty} \frac{\operatorname{conv}(\alpha_n)}{\rho(\alpha_n)}} = \overline{\limsup_{n\to\infty} \frac{\operatorname{conv}(\alpha_n)}{\rho(\alpha_n)}},$$

which we call here the quantization hull for X. However, the new definition seems to be more natural, since it concerns the (scaled) codebooks itself and neither the convex hull nor a closure needs to be considered.

2. Concerning the estimates for the Hausdorff distances, the results of Theorem 3.3.2 and Theorem 3.3.3 may also be formulated involving additionally $\sup_{\alpha_n \in \mathcal{C}_{n,r}(X,E)}$.

For the case of regularly varying g (i.e. Type II), there cannot be an equivalent version of Theorem 3.3.2 concerning the quantization ball $\mathcal{B}$. In fact, this is due to a result from chapter 5 which is a consequence of the first micro-macro inequality.

Proposition 3.3.5. *(Type II) For every* $\epsilon > 0$

$$\inf_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \inf_{a \in \alpha_n \cap (\epsilon\rho(\alpha_n) B(0,1))^c} \operatorname{dist}(a, \alpha_n \backslash \{a\}) \gtrsim h^{-}(n^{-\frac{r+d}{d}}),$$

where $h(x) = x^{r+d} g(x)$. *Hence, for all sequences* $(\alpha_n)_{n\in\mathbb{N}}$ *of* r*-optimal* n*-quantizers for* X *in* $\mathbb{R}^d$

$$\lambda^d \big(\liminf_{n\to\infty}{}^{\|\cdot\|} \frac{\alpha_n}{\rho(\alpha_n)} \big) = 0.$$

Proofs of the results Taking advantage of the Lemmas proven in the previous section, we only need a few simple Lemmas in order to prove the Theorems. We will formulate those in a generality such that they may equally been used in chapter 4.

Lemma 3.3.6. *Let* $(E, \|\cdot\|)$ *be a separable Banach space,* $B \subset E$ *bounded,* $A_n \subset E$, $n \in \mathbb{N}$ *such that*

$$\delta\left(\frac{A_n}{\psi_1(n)}, B\right) \to 0, \quad n \to \infty \tag{3.24}$$

for some sequence $\psi_1(n) \to \infty$, $n \to \infty$. *Then, for every sequence* $(\psi_2(n))_{n\in\mathbb{N}}$ *such that*

$$\psi_1(n) \sim \psi_2(n), \quad n \to \infty \tag{3.25}$$

one has

$$\delta\left(\frac{A_n}{\psi_2(n)}, B\right) \to 0, \quad n \to \infty$$

Proof. Let $\epsilon > 0$. By equation (3.25) there exists $n_\epsilon \in \mathbb{N}$ such that for all $n \geq n_\epsilon$

$$\psi_1(n) \leq \psi_2(n)(1+\epsilon) \leq \psi_1(n)(1+\epsilon)^2.$$

By using metric properties of $\delta(\cdot,\cdot)$ we estimate for all $n \geq n_\epsilon$

$$\begin{aligned}\delta\left(\frac{A_n}{\psi_2(n)}, B\right) = \delta\left(\frac{A_n}{\psi_1(n)}\frac{\psi_1(n)}{\psi_2(n)}, B\right) &= \frac{\psi_1(n)}{\psi_2(n)}\delta\left(\frac{A_n}{\psi_1(n)}, \frac{\psi_2(n)}{\psi_1(n)}B\right) \\ &\leq (1+\epsilon)\left(\delta\left(\frac{A_n}{\psi_1(n)}, B\right) + \delta\left(B, \frac{\psi_2(n)}{\psi_1(n)}B\right)\right).\end{aligned}$$

Since B is bounded, there exists $\kappa > 0$ such that $B \subset \kappa B(0,1)$. Hence $B \subset \frac{\psi_2(n)}{\psi_1(n)}B + B(0,\kappa\epsilon)$ and $B\frac{\psi_2(n)}{\psi_1(n)} \subset B + B(0,\kappa\epsilon)$, which yields in virtue of equation (3.24) the asserted. □

With a similar argument, we obtain

Lemma 3.3.7. *Let $(E, \|\cdot\|)$ be a separable Banach space, $B \subset E$ bounded, $A_n \subset E$, $n \in \mathbb{N}$ such that*

$$\delta\left(A_n, B\psi_1(n)\right) \le \gamma(n) \tag{3.26}$$

for sequences $\psi_1(n) \to \infty$, $n \to \infty$ and $\gamma(n)_{n\in\mathbb{N}}$ in $\mathbb{R}$. Then, for every sequence $(\psi_2(n))_{n\in\mathbb{N}}$ such that

$$|\psi_1(n) - \psi_2(n)| \le \gamma(n), \; n \in \mathbb{N} \tag{3.27}$$

one has

$$\delta\left(A_n, \psi_2(n)B\right) = \mathcal{O}\left(\gamma(n)\right), \quad n \to \infty.$$

Proof. Again, since B is bounded, there exists $\kappa > 0$ such that $B \subset \kappa B(0,1)$. We estimate

$$\begin{aligned} A_n &\subset B\psi_1(n) + B(0,\gamma(n)) \subset \psi_2(n)B + \gamma(n)B(0,\kappa) + B(0,\gamma(n)) \\ &\subset \psi_2(n)B + B(0,(1+\kappa)\gamma(n)) \end{aligned}$$

and

$$\psi_2(n)B \subset \psi_1(n)B + \gamma(n)B(0,\kappa) \subset A_n + B(0,(1+\kappa)\gamma(n))$$

for all $n \in \mathbb{N}$, which yields the assertion. □

Proof of Theorem 3.3.2. The Theorem is a consequences of the Lemmas in the previous section, in combination with the Lemmas 3.3.6 as well as some ingredients given by the general theory of convergence of sets. Let $\alpha_n \in \mathcal{C}_{n,r}(X,E)$, $n \in \mathbb{N}$.

1. Let $\delta > 0$. As a consequence of Lemma 3.2.6 (Type I), and by using the same notations, there exists for every $\epsilon > 0$ a constant $C(\mu,\epsilon)$ such that

$$\begin{aligned} &B_0\Big(0, (-\log(g))^{\leftarrow}\Big(\frac{r+d}{d}\log(n) + C(\mu,\epsilon)\Big)\Big)\backslash B_0\big(0, A + c_n\big) \\ &\quad\subset \alpha_n + B(0,\epsilon) \end{aligned}$$

for all $n \in \mathbb{N}$. Let ϵ sufficiently large such that $C(\mu,\epsilon) \ge 0$ and $n' \in \mathbb{N}$ sufficiently large such that $(-\log(g))^{\leftarrow}\big(\frac{r+d}{d}\log(n)\big) \ge A + c_n$ for all $n \ge n'$. Adding to the right hand side $B_0\big(0, A + c_n\big)$, and using the fact that $B_0(0,1) \subset C_{\|\cdot\|,\|\cdot\|_0} B(0,1)$ we obtain

$$\begin{aligned} B_0(0,1) &\subset \frac{\alpha_n + B(0, \epsilon + (A+c_n)C_{\|\cdot\|,\|\cdot\|_0})}{(-\log(g))^{\leftarrow}\big(\frac{r+d}{d}\log(n)\big)} \\ &\subset \frac{\alpha_n}{(-\log(g))^{\leftarrow}\big(\frac{r+d}{d}\log(n)\big)} + B(0,\delta) \end{aligned} \tag{3.28}$$

for all $n \ge \max\{n(\delta,\epsilon), n'\} \in \mathbb{N}$ satisfying

$$\frac{\epsilon + (A+c_n)C_{\|\cdot\|,\|\cdot\|_0}}{(-\log(g))^{\leftarrow}\big(\frac{r+d}{d}\log(n)\big)} \le \delta.$$

Conversely, by Lemma 3.2.8 and by using the notations therein, we have for $n \in \mathbb{N}$

$$\alpha_n \subset \big(-\log(g)\big)^* \left(\frac{r+d}{d}\log(n)\right) B_0(0,1),$$

which implies

$$\begin{aligned} \frac{\alpha_n}{\big(-\log(g)\big)^{\leftarrow}\big(\frac{r+d}{d}\log(n)\big)} &\subset B_0(0,1)\frac{\big(-\log(g)\big)^*\big(\frac{r+d}{d}\log(n)\big)}{\big(-\log(g)\big)^{\leftarrow}\big(\frac{r+d}{d}\log(n)\big)} \\ &\subset B_0(0,1)+B(0,\delta) \end{aligned} \tag{3.29}$$

for all $n \geq n'(\delta) \in \mathbb{N}$ satisfying

$$\left|1-\frac{\big(-\log(g)\big)^*\big(\frac{r+d}{d}\log(n)\big)}{\big(-\log(g)\big)^{\leftarrow}\big(\frac{r+d}{d}\log(n)\big)}\right| \leq \frac{\delta}{C_{\|\cdot\|,\|\cdot\|_0}}.$$

Hence, equations (3.28) and (3.29) imply by the definition of the Haussdorff distance

$$\delta\left(\frac{\alpha_n}{(-\log(g))^{\leftarrow}(\frac{r+d}{d}\log(n))}, B_0(0,1)\right) \to 0, \quad n \to \infty,$$

and also the same when scaling both sets with $\frac{1}{\|j_{\|\cdot\|_0,\|\cdot\|}\|}$. By Lemma 3.3.6, we obtain the same result with replacing $(-\log(g))^{\leftarrow}\big(\log(\frac{r+d}{d}\log(n))\big)$ with $(-\log(g))^{\leftarrow}(-\log(\Delta_{n,r}(X,E)))$, $\rho(\alpha_n)$, $\underline{\rho}_{n,r}(X,E)$ or $\underline{\rho}_{n,r}(X,E)$. The convergence in the $\|\cdot\|$-sense is a consequence of the general theory of convergences of sets, see Corollary B.7.

As for upper bound of the quantization hull, equation (3.29) implies by convexity of the unit Balls $B(0,1)$ and $B_0(0,1)$

$$\bigcap_{\delta>0}\overline{\limsup_{n\to\infty}\frac{\operatorname{conv}(\alpha_n)}{(-\log(g))^{\leftarrow}\big(\frac{r+d}{d}\log(n)\big)}} \subset \bigcap_{\delta>0} B_0(0,1)+B(0,\delta) = B_0(0,1),$$

which yields with the independence of the left hand side from δ the asserted upper bound for the assertion with $(-\log(g))^{\leftarrow}\big(\frac{r+d}{d}\log(n)\big)$. As for the lower bound, we derive form equation (3.28)

$$B_0\overset{\circ}{(0,1)} = \bigcup_{\delta>0} B_0(0,1)-B(0,\delta) \subset \overline{\liminf_{n\to\infty}\frac{\operatorname{conv}(\alpha_n)}{(-\log(g))^{\leftarrow}\big(\frac{r+d}{d}\log(n)\big)}}.$$

By the closedness of the right-hand side, we obtain the upper bound for the formulation with $(-\log(g))^{\leftarrow}\big(\frac{r+d}{d}\log(n)\big)$. In view of Lemma 3.3.6 we can replace $(-\log(g))^{\leftarrow}\big(\frac{r+d}{d}\log(n)\big)$ by an asymptotic equivalent and the assertion follows.

2. (Type II) As a consequence of Lemma 3.2.6 (Type II), by using the notations therein, there exists for every $\epsilon > 0$ a constant $C(\epsilon)$ such that

$$\begin{aligned} &h^{\leftarrow}\big(C(\epsilon)n^{-\frac{r+d}{d}}\big)B_0(0,1) \\ &\quad \subset \alpha_n + C_{\|\cdot\|,\|\cdot\|_0}B(0,(A+C(\epsilon))+\epsilon h^{\leftarrow}\big(C(\epsilon)n^{-\frac{r+d}{d}}\big)) \end{aligned}$$

for all $n \in \mathbb{N}$, where $h(x) := g(x)x^{r+d}$, $x \in [A,\infty)$. Thus, by using the fact that $h^{\leftarrow}\big(C(\epsilon)n^{-\frac{r+d}{d}}\big) \overset{n\to\infty}{\longrightarrow} \infty$ we obtain

$$B_0(0,1) \subset \frac{\operatorname{conv}(\alpha_n)}{h^{\leftarrow}\big(C(\epsilon)\Delta_{n,r}(X,E)\big)} + B(0, 2C_{\|\cdot\|,\|\cdot\|_0}\epsilon)$$

for all $n \geq n(\epsilon)$ with $\frac{h^{\leftarrow}\left(C(\epsilon)n^{-\frac{r+d}{d}}\right)}{(A+C(\epsilon))} \geq \epsilon$. The asserted lower bound follows by equivalence of the norms and the fact that $\rho(\alpha_n) \approx h^{\leftarrow}\big(C(\epsilon)n^{-\frac{r+d}{d}}\big)$, $n \to \infty$, see Theorem 3.2.1 (Type II). The upper bound follows by definition of the quantization radius.

□

Proof of Theorem 3.3.3. We use the notations from Theorem 3.2.2. As in the previous proof, we will make use of the Lemmas from the previous section, particularly the Lemmas 3.2.6 (Type I') and 3.2.12. Let $\alpha_n \in \mathcal{C}_{n,r}(X,E)$, $n \in \mathbb{N}$. By Lemma 3.2.6 there exist sequences $(\eta_n)_{n\in\mathbb{N}}, (c_n)_{n\in\mathbb{N}}$ with $\max\{c_n, |\eta_n - 1|\} \to 0$, $n \to \infty$, $c_n = \mathcal{O}(n^{-\frac{1}{d}})$ and a constant $C' \in \mathbb{R}$ such that

$$\begin{aligned} B_0\big(0,1\big)\Big(&(-\log(g))^{-1}\big(\frac{r+d}{d}\log(n)\big)+ \\ &\eta_n\,\Big(\big(\log(C') + (r+d)\log(\cdot)\big)(\cdot)\Big)\,\Big(\big((-\log(g))^{-1}\big)'\big(\frac{r+d}{d}\log(n)\big)\Big) \\ &\backslash B_0\big(0, A + c_n\big) \subset \alpha_n + B(0,\epsilon_n) \end{aligned}$$

for all $n \in \mathbb{N}$, where

$$\epsilon_n := \left\|j_{\|\cdot\|_0,\|\cdot\|}\right\|(r+d)\big((-\log(g))^{-1}\big)'\left(\frac{r+d}{d}\log(n)\right)$$

for $n \in \mathbb{N}$. With

$$\begin{aligned} \gamma(\log(n)) := \eta_n\Big(&\big(\log(C') + (r+d)\log\big(\big((-\log(g))^{-1}\big)'\big(\frac{r+d}{d}\log(n)\big)\big)\big)\times \\ &\big(\big((-\log(g))^{-1}\big)'\big(\frac{r+d}{d}\log(n)\big)\big)\Big) = \mathcal{O}\Big(\log(\log(n))\big((-\log(g))^{-1}\big)'(\log(n))\Big), \end{aligned}$$

we obtain, by adding $|\gamma(\log(n))|\,B_0(0,1)$ and the fact that $c_n = \mathcal{O}(\gamma(\log(n)))$,

$$\begin{aligned} &B_0\big(0,1\big)\big((-\log(g))^{-1}\big(\frac{r+d}{d}\log(n)\big)\big) \\ &\qquad \backslash B_0\big(0,A\big) \subset \alpha_n + B(0,\epsilon_n) + B_0(0,|\gamma(\log(n))|). \end{aligned}$$

With equivalence of the norms and the fact that $\epsilon_n = \mathcal{O}(\gamma(\log(n)))$ as well, we obtain the asserted lower bound for the formulation with $\big((-\log(g))^{-1}\big(\frac{r+d}{d}\log(n)\big)\big)$. By Lemma 3.2.12 and using the notations therein, there exists a bounded sequence $(\eta_n)_{n\in\mathbb{N}}$

$$\begin{aligned}\alpha_n \subset & \big(-\log(g)\big)^{-1}\left(\frac{r+d}{d}\log(n)\right)B_0(0,1)\\ & + \eta_n \log\log(n)\left((-\log(g))^{-1}\right)'\left(\frac{r+d}{d}\log(n)\right)B_0(0,1) + B(0,\epsilon_n),\end{aligned}$$

with ϵ_n as defined above, which yields the asserted upper bound for the formulation with $\big(-\log(g)\big)^{-1}\big(\frac{r+d}{d}\log(n)\big)$. Lemma 3.3.7 implies the alternative formulations with the asymptotic equivalents. $\square$

Proof of Proposition 3.3.5. Let $\epsilon > 0$. Applying Theorem 5.3.1 (Type II) yields in view of Theorem 3.2.1 (Type II)

$$\begin{aligned}&\inf_{\alpha_n\in\mathcal{C}_{n,r}(X,E)} \inf_{a\in\alpha_n\cap(\epsilon\rho(\alpha_n)B(0,1))^c} \operatorname{dist}(a,\alpha_n\backslash\{a\})\\ &\quad \gtrsim n^{-\frac{1}{d}} \inf_{\alpha_n\in\mathcal{C}_{n,r}(X,E)} \inf_{a\in\alpha_n\cap(\epsilon\rho(\alpha_n)B(0,1))^c} f^{-\frac{1}{r+d}}(a)\\ &\quad \gtrsim n^{-\frac{1}{d}} g^{-\frac{1}{r+d}}(\epsilon\underline{\rho}_{n,r}(X,E))\\ &\quad \gtrsim n^{-\frac{1}{d}} g^{-\frac{1}{r+d}}(h^{-}(n^{-\frac{r+d}{d}}))\\ &\quad \gtrsim n^{-\frac{1}{d}} h^{-}(n^{-\frac{r+d}{d}}) h^{-\frac{1}{r+d}}(h^{-}(n^{-\frac{r+d}{d}}))\\ &\quad \gtrsim n^{-\frac{1}{d}} h^{-}(n^{-\frac{r+d}{d}}) n^{\frac{1}{d}} = h^{-}(n^{-\frac{r+d}{d}}), \quad n\to\infty.\end{aligned} \tag{3.30}$$

Suppose now, that $y \in B(0,\epsilon)^c$ and

$$y \in \liminf_{n\to\infty}{}^{\|\cdot\|} \frac{\alpha_n}{\rho(\alpha_n)}.$$

Then, by equation (3.30) there exists an $\epsilon' > 0$ such that

$$\operatorname{dist}\left(y, \liminf_{n\to\infty}{}^{\|\cdot\|} \frac{\alpha_n}{\rho(\alpha_n)}\backslash\{y\}\right) \geq \epsilon'.$$

Hence,

$$\left|\left\{\liminf_{n\to\infty}{}^{\|\cdot\|} \frac{\alpha_n}{\rho(\alpha_n)}\right\} \cap B(0,\epsilon)^c\right| < \infty,$$

and we derive

$$\begin{aligned}\lambda^d\left(\liminf_{n\to\infty}{}^{\|\cdot\|} \frac{\alpha_n}{\rho(\alpha_n)}\right) &\leq \lambda^d(B(0,\epsilon))\\ &+ \lambda^d\left(\liminf_{n\to\infty}{}^{\|\cdot\|} \frac{\alpha_n}{\rho(\alpha_n)}\backslash B(0,\epsilon)\right) \leq \lambda^d(B(0,1))\epsilon^d, \quad n\to\infty\end{aligned}$$

for every $\epsilon > 0$. Letting $\epsilon \to 0$ yields the asserted. $\square$

3.4 Comparison to asymptotic optimal quantizers

In this section, we want to show that there is a wide range of sequences of asymptotically optimal codebooks compared to the small amount of sequences of optimal codebooks, in terms of their geometric properties. In particular, we show that for sequences of asymptotically optimal codebooks there is a lower bound for the quantization radius, which differs from the rate obtained in the previous sections derived for optimal codebooks (Theorem 3.4.3), and which is sharp in the sense that there exists a sequence of asymptotically optimal codebooks achieving that bound asymptotically. On the contrary, we will show under mild assumptions on the quantization error itself, that the quantization radius for such sequences is unbounded from above (Proposition 3.4.1). We start with the proof of the latter assertion.

Proposition 3.4.1. *Let X be a r.e. in the separable Banach space E, $r > 0$ and*

$$e_{n,r}(X,E) \sim e_{n+1,r}(X,E), \quad n \to \infty.$$

Then, for every sequence of real numbers $(\psi(n))_{n\in\mathbb{N}}$ there exists a sequence of asymptotically r-optimal n-codebooks $(\alpha_n)_{n\in\mathbb{N}}$ for X in E such that

$$\rho(\alpha_n) \geq \psi(n), \quad n \in \mathbb{N}. \tag{3.31}$$

Proof. Consider a sequence of asymptotically r-optimal n-codebooks $(\beta_n)_{n\in\mathbb{N}}$ and $x_n \in E$ such that $\|x_n\| = \psi(n)$, $n \in \mathbb{N}$. Then, the sequence of n-codebooks $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_1 := \{0\}$ and $\alpha_n := \beta_{n-1} \cup \{x_n\}$ for $n \geq 2$ is asymptotically optimal and satisfies equation (3.31), since

$$\rho(\alpha_n) \gtrsim \|x_n\| = \psi(n), \quad n \to \infty$$

and

$$e_r(X,E;\alpha_n) \leq e_r(X,E;\beta_{n-1}) \sim e_{n-1,r}(X,E) \sim e_{n,r}(X,E), \quad n \to \infty.$$

□

Remark 3.4.2. The latter proposition contains the most relevant cases, such as g or $\log(g)$ regularly varying at infinity (with g satisfying $\frac{\partial\mu}{\partial\lambda^d}(x) = f(x) = g(\|x\|_0)$), as well as the cases treated in chapter 4 for the infinite dimensional case.

Much more interesting is the question about derivation of lower bounds for the quantization radius for asymptotically optimal quantizers. Throughout this section, let $r > 0$, X be a r.e. in $\mathbb{R}^d$ whose non-vanishing Lebesgue continuous part admits the representation from the previous section, i.e. Type I, or Type II respectively; let furthermore $\alpha_n \in \mathcal{C}_{n,r}(X,\mathbb{R}^d)$, $n \in \mathbb{N}$ and $(\beta_n)_{n\in\mathbb{N}}$ be a sequence of n-codebooks satisfying

$$e_{n,r}(X,\mathbb{R}^d) \sim e_r(X,\mathbb{R}^d;\beta_n), \quad n \to \infty.$$

Hereafter, such sequences $(\beta_n)_{n\in\mathbb{N}}$ will be called sequences of asymptotically optimal codebooks.

Theorem 3.4.3. *Let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence of asymptotically optimal codebooks for X in $(\mathbb{R}^d, \|\cdot\|)$.*

1. *(Type I) There exists a sequence of n-codebooks $(\gamma_n)_{n\in\mathbb{N}}$ and a sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that*
   - $e_{n,r}(X,\mathbb{R}^d) \sim e_r(X,\mathbb{R}^d;\gamma_n), \quad n\to\infty,$
   - $c_n \sim (-\log(g))^{-}\big(-\log(e^r_{n,r}(X,\mathbb{R}^d))\big), \quad n\to\infty$ *and*
   - $\gamma_n \subset c_n B_0(0,1), \ n\in\mathbb{N},$

   where $(-\log(g))^{-}$ denotes an arbitrary asymptotic inverse to $(-\log(g))$. Additionally, there exists a constant $C\in\mathbb{R}$ independent of $(\beta_n)_{n\in\mathbb{N}}$ such that
   $$\rho(\beta_n) \geq (-\log(g))^{\leftarrow}\big(-\log(e^r_r(X,\mathbb{R}^d;\beta_n))\big) - C,$$
   and furthermore
   $$\delta_l(B_0(0,1), \frac{\beta_n}{c_n}) \to 0, \quad n\to\infty.$$

2. *(Type II) There exists a sequence of n-codebooks $(\gamma_n)_{n\in\mathbb{N}}$ and a sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that*
   - $e_{n,r}(X,\mathbb{R}^d) \approx e_r(X,\mathbb{R}^d;\gamma_n), \quad n\to\infty,$
   - $c_n \approx h^{-}\big(e^r_{n,r}(X,\mathbb{R}^d)\big), \quad n\to\infty$ *and*
   - $\gamma_n \subset c_n B_0(0,1), n\in\mathbb{N},$

   where h^{-} denotes an arbitrary asymptotic inverse to $h(x) := x^{r+d}g(x)$. Additionally, there exists a constant $C\in(0,\infty)$ such that
   $$\rho(\beta_n) \gtrsim h^{-}\big(Ce^r_{n,r}(X,\mathbb{R}^d)\big), \quad n\to\infty.$$

Remark 3.4.4.
1. (Type I) Compared to the results obtained for the quantization radius for a sequence of r-optimal n-codebooks $(\alpha_n)_{n\in\mathbb{N}}$ for X in $\mathbb{R}^d$, there is a significant discrepancy to the rates obtained in Theorem 3.4.3. We have
$$\frac{\rho(\alpha_n)}{\rho(\gamma_n)} \sim \frac{(-\log(g))^{-}(\frac{r+d}{d}\log(n))}{(-\log(g))^{-}(-\log(e^r_{n,r}(X,\mathbb{R}^d)))} \to \left(\frac{r+d}{r}\right)^{\theta}, \quad n\to\infty$$
with $(\gamma_n)_{n\in\mathbb{N}}$ as in Theorem 3.4.3 Type I.

2. (Type II) For densities of Type II we have
$$\frac{\rho(\alpha_n)}{\rho(\gamma_n)} \approx \frac{h^{-}\big(n^{-\frac{r+d}{d}}\big)}{h^{-}\big(e^r_{n,r}(X,\mathbb{R}^d)\big)}, \quad n\to\infty$$

with $(\gamma_n)_{n\in\mathbb{N}}$ as in Theorem 3.4.3 Type II. Assuming $g(x) \approx x^{-a},\ x \to \infty$, we have

$$\frac{\rho(\alpha_n)}{\rho(\gamma_n)} \approx \frac{h^{-}\big(n^{-\frac{r+d}{d}}\big)}{h^{-}\big(e^r_{n,r}(X,\mathbb{R}^d)\big)} \approx n^{\frac{1}{a-r-d}}, \quad n \to \infty.$$

Proofs of the results The proofs follow partially the arguments for the estimation for the quantization radius for optimal codebooks. The idea is to replace the first micro-macro inequality with an equivalent version for asymptotic optimal codebooks involving $e^r_{n,r}(X,\mathbb{R}^d)$ instead of $\Delta_{n,r}(X,\mathbb{R}^d)$. The following Lemma is a mirror of Lemma 3.2.6, which is the corresponding version for optimal codebooks.

Lemma 3.4.5. *1. (Type I) There exists a constant $C > 0$ such that*

$$(-\log(g))^{\leftarrow}\big(-\log(e^r_r(X,\mathbb{R}^d;\beta_n))\big)B_0(0,1) \subset \beta_n + B(0,C)$$

for all $n \in \mathbb{N}$.

2. (Type II) For every $\epsilon > 0$ there exists a constant $C(\epsilon) \in \mathbb{R}$ such that

$$h^{\leftarrow}\big(C(\epsilon)e^r_r(X,\mathbb{R}^d;\beta_n)\big)B_0(0,1) \subset \beta_n + B\left(0, \epsilon h^{\leftarrow}\big(C(\epsilon)e^r_r(X,\mathbb{R}^d;\beta_n)\big)\right)$$

for all $n \geq n'\big((e^r_r(X,\mathbb{R}^d;\beta_n))_{n\in\mathbb{N}}\big) \in \mathbb{N}$.

Proof. 1. (Type I) Let $C \in (0,\infty)$ be specified below and $(x_n)_{n\in\mathbb{N}}$ be a sequence in E such that $\operatorname{dist}(x_n,\beta_n) \geq C$ and $B(x_n, \frac{1}{2}\operatorname{dist}(x_n,\beta_n)) \subset B_0(0,A)^c$. Then, by using the lower peakless condition for X, there exists a constant C sufficiently large such that

$$\begin{aligned} e^r_r(X,E;\beta_n) &\geq \int_{B(x_n,\frac{C}{2})} \operatorname{dist}(x,\beta_n)^r d\mu(x) \\ &\geq \left(\frac{C}{2}\right)^r \mu\left(B(x_n,\frac{C}{2})\right) \\ &\geq \kappa\left(\frac{C}{2}\right)^{r+d} g(\|x_n\|_0), \end{aligned}$$

for some $\kappa > 0$ and all $n \in \mathbb{N}$. Applying the negative logarithm and selecting C sufficiently large yields for all $n \in \mathbb{N}$

$$-\log(e^r_r(X,\mathbb{R}^d;\beta_n)) \leq -\log(\kappa) - (r+d)\log(\frac{C}{2}) - \log(g(\|x_n\|_0)) \leq -\log(g(\|x_n\|_0)).$$

By using an equivalent firewall argument as in Lemma 3.2.6, the same also for sequences $(x_n)_{n\in\mathbb{N}}$ with $x_n \in B_0(0,2A)^c$ instead of $B(x_n,\frac{1}{2}\operatorname{dist}(x_n,\beta_n)) \subset B_0(0,A)^c$ for all $n \geq n' = n'\big((e^r_r(X,\mathbb{R}^d;\beta_n))_{n\in\mathbb{N}}\big)$. By applying $(-\log(g))^{\leftarrow}$ we obtain for $n \in \mathbb{N}$

$$(-\log(g))^{\leftarrow}\left(-\log(e^r_r(X,\mathbb{R}^d;\beta_n))\right) \leq \|x_n\|_0,$$

what implies by definition of $x_n,\ n\in\mathbb{N}$

$$(-\log(g))^{\leftarrow}\left(-\log(e_r^r(X,\mathbb{R}^d;\beta_n))\right)B_0(0,1)\subset\beta_n+B(0,C)+B_0(0,2A), \tag{3.32}$$

for all $n\geq n'$. The equivalence of the norms and by selecting C sufficiently large such that equation (3.32) also holds for all $n\leq n'$ yields the assertion.

2. (Type II) Let $\epsilon>0$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in E such that $\operatorname{dist}(x_n,\beta_n)\geq\epsilon\|x_n\|_0$ and $B(x_n,\frac{1}{2}\operatorname{dist}(x_n,\beta_n))\subset B_0(0,A)^c$. Then, for $n\in\mathbb{N}$ by using the lower peakless condition

$$\begin{aligned}e_r^r(X,\mathbb{R}^d;\beta_n)&\geq\int_{B(x_n,\frac{\epsilon\|x_n\|_0}{2})}\operatorname{dist}(x,\beta_n)^r\,d\mu(x)\\&\geq\left(\frac{\epsilon\|x_n\|_0}{2}\right)^r\mu\left(B(x_n,\frac{\epsilon\|x_n\|_0}{2})\right)\\&\geq\kappa\epsilon^{r+d}\|x_n\|_0^{r+d}g(\|x_n\|_0)=\kappa\epsilon^{r+d}h(\|x_n\|_0)\end{aligned}$$

for some constant $\kappa>0$. By using an equivalent firewall argument as in Lemma 3.2.6, the same also for sequences $(x_n)_{n\in\mathbb{N}}$ with $x_n\in B_0(0,2A)^c$ instead of $B(x_n,\frac{1}{2}\operatorname{dist}(x_n,\beta_n))\subset B_0(0,A)^c$ for all $n\geq n'=n'\big((e_r^r(X,\mathbb{R}^d;\beta_n))_{n\in\mathbb{N}}\big)$. Hence, by applying $h^{\leftarrow}$, there exists a constant $C(\epsilon)\in(0,\infty)$ such that

$$h^{\leftarrow}\left(C(\epsilon)e_r^r(X,E;\beta_n)\right)\leq\|x_n\|_0$$

for all $n\geq n'\big((e_r^r(X,E;\beta_n))_{n\in\mathbb{N}}\big)$. By contraposition and the definition of $x_n,\ n\in\mathbb{N}$ this implies

$$\begin{aligned}h^{\leftarrow}\left(C(\epsilon)e_r^r(X,E;\beta_n)\right)B_0(0,1)&\subset\beta_n+B(0,\epsilon\|x_n\|_0)\\&\subset\beta_n+B(0,\epsilon h^{\leftarrow}\left(C(\epsilon)e_r^r(X,E;\beta_n)\right))\end{aligned}$$

for all $n\geq n'$.

□

For the existence of specific sequences of codebooks as asserted in Theorem 3.4.3, we need an equivalent version to Lemmas 3.2.8 and 3.2.9 for sequences of asymptotically optimal codebooks.

Lemma 3.4.6. *1. (Type I) There exists an asymptotic inverse $(-\log(g))^-$ to $(-\log(g))$ and a sequence of n-codebooks $(\gamma_n)_{n\in\mathbb{N}}$ such that*

- $\gamma_n\subset(-\log(g))^-)\left(-\log(e_{n,r}^r(X,\mathbb{R}^d))\right)B_0(0,1)$ *and*
- $e_{n,r}^r(X,\mathbb{R}^d)\sim e_r^r(X,\mathbb{R}^d;\gamma_n),\quad n\to\infty.$

2. (Type II) There exists a constant $C\in(0,\infty)$ and a sequence of n-codebooks $(\gamma_n)_{n\in\mathbb{N}}$ such that

- $\gamma_n\subset Ch^{\leftarrow}\left(e_{n,r}^r(X,\mathbb{R}^d)\right)B_0(0,1)$ *and*
- $e_{n,r}^r(X,\mathbb{R}^d)\approx e_r^r(X,\mathbb{R}^d;\gamma_n),\quad n\to\infty,$

where $h(x) := x^{r+d} g(x),\ x > A.$

Proof. Let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence of asymptotically optimal codebooks for X in $(\mathbb{R}^d, \|\cdot\|)$.

1. (Type I) Let $\delta_n \to 0,\ n \to \infty$ such that

$$-\log(\delta_n) = \mathrm{o}\big(-\log(e^r_{n,r}(X, \mathbb{R}^d; \beta_n))\big), \quad n \to \infty \tag{3.33}$$

Step 1: We set

$$G(x) := \max\left\{(-\log(g))^{\leftarrow}(x), \big(-\log(\overline{F^X_{r,\|\cdot\|}})\big)^{-1}\right\}$$

and

$$\begin{aligned}(-\log(g))^*(x) := {} & 2C^2_{\|\cdot\|,\|\cdot\|_0} G(x) \\ & - \big(2C^2_{\|\cdot\|,\|\cdot\|_0} - 1\big)(-\log(g))^{\leftarrow}(x) + 2C^2_{\|\cdot\|,\|\cdot\|_0} C\end{aligned}$$

where the constant C is from Lemma 3.4.5. As in the proof of Lemma 3.2.8, one shows that $(-\log(g))^*(x)$ is an asymptotic inverse to $(-\log(g))$.
Step 2: If $\beta_n \subset (-\log(g))^*(-\log(\delta_n e^r_r(X, \mathbb{R}^d; \beta_n)))B_0(0,1)$ for all $n \geq n' \in \mathbb{N}$ there is nothing to prove, since

$$\begin{aligned}&(-\log(g))^*(-\log(\delta_n e^r_r(X, \mathbb{R}^d; \beta_n))) \\ &\qquad \sim (-\log(g))^*(-\log(e^r_r(X, \mathbb{R}^d; \beta_n))), \quad n \to \infty\end{aligned}$$

as a consequence of equation (3.33).
Step 3: Suppose now, that

$$\beta_{n_k} \cap (-\log(g))^*(-\log(\delta_n e^r_r(X, \mathbb{R}^d; \beta_{n_k})))B^c_0(0,1) \neq \varnothing$$

for an increasing subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$. We set

$$\gamma_{n_k} := \beta_{n_k} \cap (-\log(g))^* \left(-\log(\delta_n e^r_r(X, \mathbb{R}^d; \beta_{n_k}))\right) B_0(0,1) \cup \{0\}$$

and $\gamma_n = \beta_n,\ n \neq n_k, k, n \in \mathbb{N}$. Then $|\gamma_{n_k}| \leq n_k, k \in \mathbb{N}$ and

$$e^r_r(X, E; \gamma_{n_k}) \leq e^r_r(X, \mathbb{R}^d; \beta_{n_k}) + \int_{\bigcup_{a\in\beta_{n_k}\setminus\gamma_{n_k}} V_a(\beta_{n_k})} \|x\|^r \, d\mu(x).$$

Step 4: Analogously to the argumentation in the proof of Lemma 3.2.8, one shows that

$$\bigcup_{a\in\beta_{n_k}\setminus\gamma_{n_k}} V_a(\beta_{n_k}) \subset \Big(G\big(-\log(\delta_{n_k} e^r_r(X, \mathbb{R}^d; \beta_{n_k}))\big)B_0(0,1)\Big)^c$$

for all $k \in \mathbb{N}$.
Step 5: In virtue of the monotony of $\overline{F^X_{r,\|\cdot\|_0}}$, the previous steps imply

$$\begin{aligned}
\int_{\bigcup_{a\in\beta_{n_k}\setminus\gamma_{n_k}} V_a(\beta_{n_k})} \|x\|^r \, d\mu(x) &\le C_{\|\cdot\|,\|\cdot\|_0} \overline{F^X_{r,\|\cdot\|_0}} \big(G\big(-\log(\delta_{n_k} e^r_r(X,\mathbb{R}^d;\beta_{n_k}))\big)\big)\\
&\le C_{\|\cdot\|,\|\cdot\|_0} \overline{F^X_{r,\|\cdot\|_0}} \Big(\big(-\log(\overline{F^X_{r,\|\cdot\|}})\big)^{-1}\big(-\log(\delta_{n_k} e^r_r(X,\mathbb{R}^d;\beta_{n_k}))\big)\Big)\\
&= C_{\|\cdot\|,\|\cdot\|_0} \overline{F^X_{r,\|\cdot\|_0}} \Big(\Big(\overline{F^X_{r,\|\cdot\|}}\Big)^{-1} \big(\delta_{n_k} e^r_r(X,\mathbb{R}^d;\beta_{n_k})\big)\Big)\\
&= C_{\|\cdot\|,\|\cdot\|_0} \delta_{n_k} e^r_r(X,\mathbb{R}^d;\beta_{n_k}),
\end{aligned}$$

for all $k \in \mathbb{N}$. Hence,

$$\int_{\bigcup_{a\in\beta_{n_k}\setminus\gamma_{n_k}} V_a(\beta_{n_k})} \|x\|^r \, d\mu(x) = o(e^r_r(X,\mathbb{R}^d;\beta_{n_k})), \quad k \to \infty.$$

Selecting $(-\log(g))^-$ such that $e^r_{n,r}(X,E;\beta_n)$ can be replaced with $e^r_{n,r}(X,E)$ yields the assertion.

2. (Type II) The proof for the density Type II is, similar the proofs in the previous section, much easier compared to the proof for Type I. If it exists a constant $C \in (0,\infty)$ such that

$$\beta_n \subset C h^{\leftarrow}\left(e^r_r(X,\mathbb{R}^d;\beta_n))\right) B(0,1), \quad n \in \mathbb{N},$$

we set $\gamma_n = \beta_n,\ n \in \mathbb{N}$ and there is nothing to prove. Let now $C \in (0,\infty)$ arbitrary and suppose that $\beta_{n_k} \cap C h^{\leftarrow}\left(e^r_r(X,\mathbb{R}^d;\beta_{n_k}))\right) B(0,1)^c \ne \varnothing$ for an increasing subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$. We set

$$\gamma_{n_k} := \beta_{n_k} \cap C h^{\leftarrow}\left(e^r_r(X,\mathbb{R}^d;\beta_n))\right) B(0,1) \cup \{0\}.$$

Then $|\gamma_{n_k}| \le n_k, k \in \mathbb{N}$ and

$$e^r_r(X,E;\gamma_{n_k}) \le e^r_r(X,\mathbb{R}^d;\beta_{n_k}) + \int_{\bigcup_{a\in\beta_{n_k}\setminus\gamma_{n_k}} V_a(\beta_{n_k})} \|x\|^r \, d\mu(x).$$

Lemma 2.3.1 implies that

$$\bigcup_{a\in\beta_{n_k}\setminus\gamma_{n_k}} V_a(\beta_{n_k}) \subset \frac{C}{3} h^{\leftarrow}\left(e^r_r(X,\mathbb{R}^d;\beta_n))\right) B(0,1)^c$$

for all $k \in \mathbb{N}$ sufficiently large. Hence

$$\begin{aligned}
\int_{\bigcup_{a\in\beta_{n_k}\setminus\gamma_{n_k}} V_a(\beta_{n_k})} \|x\|^r \, d\mu(x) &\le \overline{F^X}_r\left(h^{\leftarrow}\left(e^r_r(X,\mathbb{R}^d;\beta_{n_k}))\right)\right)\\
&\lesssim \left(\frac{C}{3}\right)^{r+d-a} \kappa(\overline{F^X}_r) e^r_r(X,\mathbb{R}^d;\beta_{n_k}), \quad k \to \infty,
\end{aligned}$$

with $\kappa = \kappa(\overline{F^X}_r)$ independent of C.

□

Proof of Theorem 3.4.3. The proof is a consequence of the previous Lemmas. Lemma 4.3.5 yields the lower bound of the quantization radius for an asymptotically optimal sequence $(\beta_n)_{n\in\mathbb{N}}$.
Lemma 4.3.6 ensures the existence of a sequence of asymptotically optimal codebooks $(\gamma_n)_{n\in\mathbb{N}}$ having the required shape. The additional formulation for δ_l and Type I follows by Lemma 3.4.5. □

3.5 Examples and numerical illustration

Examples:

Example 3.5.1. (Hyper-exponential tails, c.f. Type I, Type I') An interesting class of distribution on $\mathbb{R}^d$ being of the shape (Type I') is the class of hyper-exponential distributions. Let X be a random variable in $(\mathbb{R}^d, \|\cdot\|)$ with $\mathbb{P}^X = f\lambda^d$ and f having the shape

$$f(x) = g(\|x\|_0) := K^{-1} \|x\|_0^c \exp(-\theta \|x\|_0^k),\ x \in \mathbb{R}^d, \tag{3.34}$$

for constants $\theta, k > 0$, $c > -d$, an arbitrary norm $\|\cdot\|_0$ and a norming constant K. In this case, the requirements of Theorem 3.2.2 are satisfied and we estimate in virtue of Lemma 3.2.10

$$\begin{aligned}\phi_{r,d,\theta,k}(n) &:= (-\log(g))^{-1}\Big(\frac{r+d}{d}\log(n)\Big)\\ &= \left(\frac{r+d}{\theta d}\log(n)\right)^{\frac{1}{k}} + \frac{c}{k^2\theta}\log(\log(n))\left(\frac{r+d}{d\theta}\log(n)\right)^{\frac{1}{k}-1}\\ &+ \mathrm{o}\left(\log(\log(n))\,(\log(n))^{\frac{1}{k}-1}\right).\end{aligned}$$

Thus, with Theorem 3.2.2 for every sequence $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n \in \mathcal{C}_{n,r}(X,E)$, $n \in \mathbb{N}$

$$\rho(\alpha_n) = \left\|j_{\|\cdot\|_0,\|\cdot\|}\right\| \left(\frac{r+d}{\theta d}\log(n)\right)^{\frac{1}{k}} + \mathcal{O}\left(\log(\log(n))\right)(\log(n))^{\frac{1}{k}-1}.$$

Furthermore, by Theorem 3.3.3

$$\delta(\alpha_n, \rho(\alpha_n)\frac{1}{\left\|j_{\|\cdot\|_0,\|\cdot\|}\right\|}B_0(0,1)) = \mathcal{O}\left(\log(\log(n))\,(\log(n))^{-\frac{1}{k}}\right), \quad n \to \infty,$$

which particularly implies

$$\mathcal{B} = \frac{1}{\left\|j_{\|\cdot\|_0,\|\cdot\|}\right\|}B_0(0,1).$$

1. (Normal distribution) Let μ be a centered d-dimensional normal distribution with regular covariance matrix Σ and corresponding non-increasing ordered eigenvalues $\lambda_1 \geq \cdots \geq \lambda_d > 0$. Its density is given by

$$f(x) = \frac{1}{\sqrt{((2\pi)^d \det \Sigma)}} \exp(-\frac{1}{2} \left\| \Sigma^{-\frac{1}{2}} x \right\|_2^2),$$

where $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^d$. Thus, it has the shape (3.34) with $c = 0, \theta = \frac{1}{2}, k = 2$ and $\|\cdot\|_0 = \left\| \Sigma^{-\frac{1}{2}} \cdot \right\|_2$. The operator norm of the natural embedding $j : (\mathbb{R}^d, \|\cdot\|_0) \to (\mathbb{R}^d, \|\cdot\|_2)$ is given as the root of the biggest eigenvalue λ_1 of the covariance matrix. Using Theorem 3.2.2 we obtain the asymptotics of the quantization radius for every sequence of r-optimal n-quantizers $(\alpha_n)_{n \in \mathbb{N}}$ for X as

$$\rho(\alpha_n) = \sqrt{\lambda_1} \left(\frac{2(r+d)}{d} \log(n) \right)^{\frac{1}{2}} + \mathcal{O}\left(\log(\log(n)) \left(\log(n)\right)^{-\frac{1}{2}} \right), \quad n \to \infty.$$

Furthermore, by Theorem 3.3.3

$$\delta(\alpha_n, \frac{\rho(\alpha_n)}{\sqrt{\lambda_1}} B_{\left\| \Sigma^{-\frac{1}{2}} \cdot \right\|_2}(0,1)) = \mathcal{O}\left(\log(\log(n)) \left(\log(n)\right)^{-\frac{1}{2}} \right), \quad n \to \infty,$$

which particularly implies

$$\mathcal{B} = \frac{1}{\sqrt{\lambda_1}} B_{\left\| \Sigma^{-\frac{1}{2}} \cdot \right\|_2}(0,1).$$

2. (Multivariate exponential distribution) Those are distributions of the type (3.34) with $c = 0, \theta = \lambda > 0, k = 1$ and an arbitrary norm $\|\cdot\|_0$. Then

$$\rho(\alpha_n) = \left\| j_{\|\cdot\|, \|\cdot\|_0} \right\| \left(\frac{r+d}{\lambda d} \log(n) \right) + \mathcal{O}\left(\log(\log(n)) \right), \quad n \to \infty.$$

The lower bound can actually be sharpened to

$$\rho(\alpha_n) \geq \left\| j_{\|\cdot\|, \|\cdot\|_0} \right\| \left(\frac{r+d}{\lambda d} \log(n) \right) + C$$

for some constant $C \in \mathbb{R}$ and $n \in \mathbb{N}$.

Example 3.5.2. (Polynomial tails, Type II)

- (Multivariate Students t-distribution) Let X be a random variable in $(\mathbb{R}^d, \|\cdot\|)$ with $\mathbb{P}^X = f\lambda^d$ and f having the shape

$$f(x) = g(\|x\|_0) := \frac{\Gamma\left(\frac{\nu+d}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\pi\nu)^{\frac{d}{2}}} \det\left(\Sigma\right)^{\frac{1}{2}} \left(1 + \frac{\left\| \Sigma^{-\frac{1}{2}} x \right\|_2^2}{\nu} \right)^{-\frac{\nu+d}{2}}, \; x \in \mathbb{R}^d, \tag{3.35}$$

for some positive definite Matrix Σ and $\nu > 0$. Then X is said to be t-distributed with ν degrees of freedom and (if $\nu > 2$) covariance matrix $\frac{\nu}{\nu-2}\Sigma$. By Theorem 3.2.1 Type II

$$\rho(\alpha_n) \approx h^{-}(n^{-\frac{r+d}{d}}) \approx n^{\frac{r+d}{d}\frac{1}{\nu-r}}, \quad n \to \infty,$$

for every sequence $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n \in \mathcal{C}_{n,r}(X,E)$, $n \in \mathbb{N}$ and $\nu > r$. Furthermore, by Theorem 3.3.2, there exists a constant $\kappa \in (0,1]$ such that

$$\kappa B(0,1) \subset \liminf_{n\to\infty} \frac{\alpha_n}{\rho(\alpha_n)} \subset \limsup_{n\to\infty} \frac{\alpha_n}{\rho(\alpha_n)} \subset B(0,1).$$

- (Cauchy distribution) Selecting $\nu = 1$ in equation (3.35) yields a multivariate Cauchy distribution. Then, for $r < 1$

$$\rho(\alpha_n) \approx h^{-}\left(n^{-\frac{r+d}{d}}\right) \approx n^{\frac{r+d}{d}\frac{1}{1-r}}, \quad n \to \infty.$$

Illustration: Finally, we want to illustrate some of our results. For the computation of the optimal codebooks presented below, we used the CLVQ-Algorithm, see [Pag98]. We consider the Euclidean $\mathbb{R}^2, r = 2$ and $X \overset{d}{=} N(0,\Sigma)$ with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{4}$. The figures show the 2-optimal n-quantizers for $n = 50, 250, 1000$. The two ellipses in the figures are the scaled quantization balls $\mathcal{B}\rho(\alpha_n)$ and $\mathcal{B}\phi_{r,d,\theta,k}(n)$ with $\phi_{r,d,\theta,k}(n)$ as in the previous example.
As already mentioned in [PS08] for the unit-covariance case, we see that also in this case the quantization radius $\rho(\alpha_n)$ seems to be for finite n smaller than its asymptotic equivalent $\phi_{r,d,\theta,k}(n)$.
Furthermore, we observe that for small n the convex hull of α_n does not completely fill the ellipse $\mathcal{B}\rho(\alpha_n)$, whereas for growing n almost the whole ellipse seems to be filled by $\mathrm{conv}(\alpha_n)$. As a consequence of the previous Lemma, we know that the inner ellipse will tend to the outer one for n growing with a rate not slower than

$$\mathcal{O}\left(\frac{\log(\log(n))}{\log(n)^{\frac{1}{2}}}\right),$$

and that the whole ellipse (and not more) will be filled with codes.

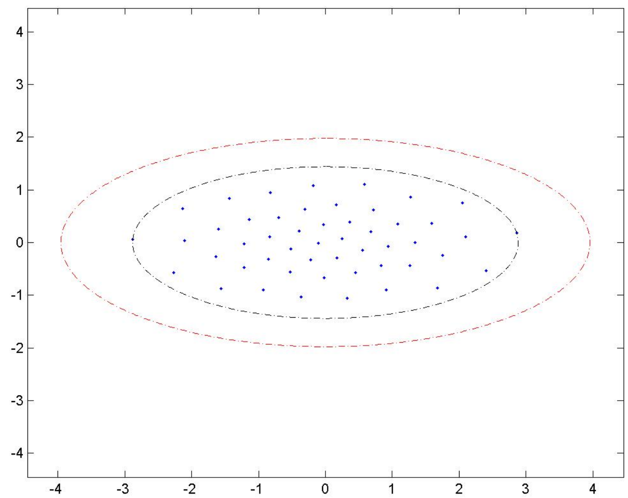

Figure 3.1: 2-optimal 50-quantizer for $X \overset{d}{=} N(0,\Sigma)$, eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{4}$, $\|\cdot\| = \|\cdot\|_2$ with the scaled ellipses $\mathcal{B}\rho(\alpha_n)$ (the inner one) and $\mathcal{B}\phi_{2,2,\frac{1}{2},2}(50)$ (the outer one)

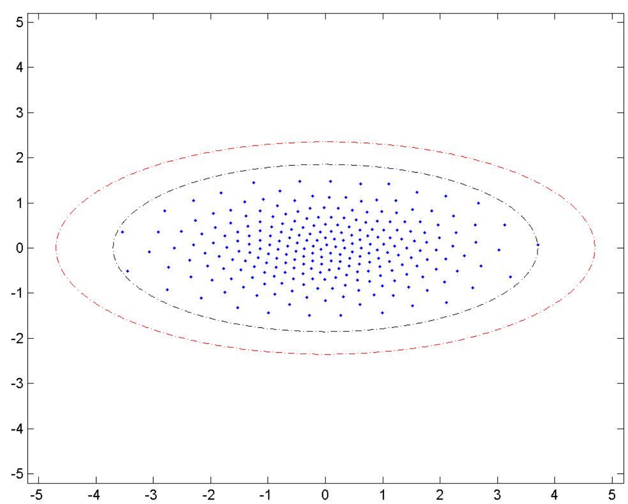

Figure 3.2: 2-optimal 250-quantizer for $X \overset{d}{=} N(0,\Sigma)$, eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{4}$, $\|\cdot\| = \|\cdot\|_2$ with the scaled ellipses $\mathcal{B}\rho(\alpha_n)$ (the inner one) and $\mathcal{B}\phi_{2,2,\frac{1}{2},2}(250)$ (the outer one)

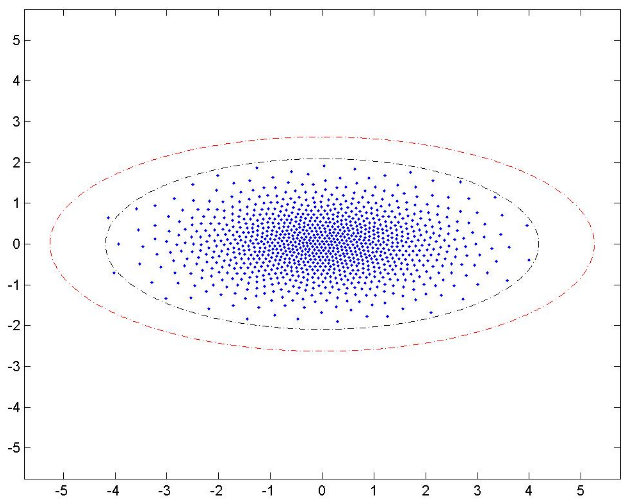

Figure 3.3: 2-optimal 1000-quantizer for $X \stackrel{d}{=} N(0,\Sigma)$, eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{4}$, $\|\cdot\| = \|\cdot\|_2$ with the scaled ellipses $\mathcal{B}\rho(\alpha_n)$ (the inner one) and $\mathcal{B}\phi_{2,2,\frac{1}{2},2}(1000)$ (the outer one)

Chapter 4

Geometry of optimal codebooks for Gaussian random elements

In this chapter, we want to analyze and estimate geometric properties for sequences of optimal codebooks $(\alpha_n)_{n\in\mathbb{N}}$ for Gaussian r.e.'s in infinite dimensional Banach spaces, namely the quantization radius and the quantization ball. For a Gaussian r.e. X in $\mathbb{R}^d$ and a sequence of optimal codebooks $(\alpha_n^{(d)})_{n\in\mathbb{N}}$, the results have been established in the previous chapter, and imply

$$\rho(\alpha_n^{(d)}) \sim \sqrt{\lambda_1}\left(\frac{2(r+d)}{d}\log(n)\right)^{\frac{1}{2}}, \quad n \to \infty,$$

with λ_1 being the largest eigenvalue of the covariance Matrix $\Sigma^{(d)}$ of $\mu = \mathbb{P}^X$ as well as

$$\mathcal{B}^{(d)} = \frac{1}{\sqrt{\lambda_1}} B_{0,d}(0,1),$$

with $B_{0,d}(0,1)$ denoting the unit ball induced by the norm $\|\cdot\|_{0,d} = \left\|\left(\Sigma^{(d)}\right)^{-\frac{1}{2}}\cdot\right\|_2$, provided the Banach space norm $\|\cdot\| = \|\cdot\|_2$. Trying to conjecture the natural expansion for an infinite dimensional case, we "derive" by exchanging thresholds

$$\rho(\alpha_n) = \lim_{d\to\infty}\rho(\alpha_n^{(d)}) = \lim_{d\to\infty}\sqrt{\lambda_1}\left(\frac{2(r+d)}{d}\log(n)\right)^{\frac{1}{2}} = \sigma(\mu)\left(2\log(n)\right)^{\frac{1}{2}}$$

as well as

$$\mathcal{B} = \lim_{d\to\infty}\mathcal{B}^{(d)} = \lim_{d\to\infty}\frac{1}{\sqrt{\lambda_1}}B_{0,d}(0,1) = \frac{1}{\sigma(\mu)}\mathcal{K}_\mu,$$

with $\sigma(\mu)$ denoting the norm of the natural embedding from $\mathcal{H}_\mu$ in E (which equals the root of the largest eigenvalue of the covariance operator C_X of X)

and $\mathcal{K}_\mu$ denoting the Strassen ball, i.e. the unit ball in the Cameron-Martin space $\mathcal{H}_\mu$.

In Theorems 4.1.1 and 4.2.1 we will prove, under some regularity requirements on the quantization error (equation (4.1)), that this conjectures hold true. As for the proofs, we will try to follow the approaches from the previous chapter. We will

- establish a tight relationship between the quantization radius and the increments of the quantization error $\Delta_{n,r}(X,E)$, in order to
- make use of the estimates for the increments of the quantization error derived for Gaussian r.e.'s in chapter 2 (see Theorem 4.1.3).

Throughout this chapter, let X be a Gaussian r.e. in the separable Banach space $(E,\|\cdot\|)$, $\dim(\mathcal{H}_\mu)=\infty$ and $r\in(0,\infty)$. We suppose that

$$\alpha_n \in \mathcal{C}_{n,r}(X,E) \neq \varnothing$$

for all $n\in\mathbb{N}$. Furthermore, we denote

$$\phi_{r,\Delta}(n) := (-2\log(\Delta_{n,r}(X,E)))^{\frac{1}{2}}, \quad n\in\mathbb{N}$$

4.1 Quantization radius

The main result of this section is

Theorem 4.1.1. *Suppose that*

$$e_{n,r}(X,E) \sim \phi(\log(n)), \quad n\to\infty, \tag{4.1}$$

for some $\phi\in\mathcal{R}_{-a}$, $a\in(0,\infty)$. Then, for every sequence of r-optimal n-codebooks $(\alpha_n)_{n\in\mathbb{N}}$ for X in E

$$\begin{aligned}\rho(\alpha_n) &\sim \underline{\rho}_{n,r}(X,E) \sim \overline{\rho}_{n,r}(X,E)\\ &\sim \sigma(\mu)\,(2\log(n))^{\frac{1}{2}} \sim \sigma(\mu)\phi_{r,\Delta}(n), \quad n\to\infty.\end{aligned}$$

One may note, that (4.1) implies $\dim(\mathcal{H}_\mu)=\infty$.

Example 4.1.2. Condition (4.1) is satisfied by essentially all cases of interest for Gaussian r.e.'s in finite dimensional Banach spaces. In particular, it holds for plenty of Gaussian r.e.'s in infinite dimensional Hilbert spaces as well as for the (fractional) Brownian motions with path in $L_p([0,T],\|\cdot\|_{L_p})$ and $C([0,T],\|\cdot\|_{L_\infty})$, $p\in[1,\infty)$ and $T\in(0,\infty)$, see Example 2.2.11.

One key ingredient for this result is the following sharp asymptotics for $\Delta_{n,r}(X,E)$ on a log-scale as derived in chapter 2.

Theorem 4.1.3. *Suppose that*

$$e^r_{n,r}(X,E) \sim \phi(\log(n)), \quad n \to \infty$$

for some $\phi \in \mathcal{R}_{-a},\ a \in (0,\infty)$*, then*

$$-\log\left(\Delta_{n,r}(X,E)\right) \sim \log(n), \quad n \to \infty.$$

Proof. The Theorem is an immediate consequence of Corollary 2.2.3 for the upper bound and Theorem 2.2.9 for the lower bound. □

Remark 4.1.4. The latter results still holds in case $\mathcal{C}_{n,r}(X,E) = \varnothing$, by an argumentation using quantization in the Bidual E'' of E.

Remark 4.1.5. For Gaussian r.e.'s in $\mathbb{R}^d$, we were able to derive estimates for the quantization radius of a sequence of optimal codebooks $(\alpha_n)_{n\in\mathbb{N}}$ including a second order asymptotics (see Theorems 3.2.2 and 3.3.3). We are not able to extend this result to this infinite dimensional case. There are several reasons for that, in particular

- The estimates for the increments of the quantization error in Theorem 4.1.3 are not as sharp as in Theorem 1.2.24.
- In contrast to the equivalence of $\|\cdot\|$ and $\|\cdot\|_0$, there is no equivalence of the norms $\|\cdot\|$ and $\|\cdot\|_{\mathcal{H}_\mu}$.

Still, we will be able to sharpen the lower bound for the quantization radius as well as to estimate a sharper upper bound for the lim inf of the quantization radius in terms of the (unknown) increments of the quantization error (Proposition 4.1.13, Corollary 4.1.15).

In the general Banach space setting, it is not known whether condition (4.1) is satisfied or not. Still, given the weak asymptotics of the quantization error is known and of such a form, we have

Theorem 4.1.6. *Suppose that* $\dim(\mathcal{H}_\mu) = \infty$*, which implies*

$$e^r_{n,r}(X,E) \gtrsim n^{-\frac{r}{d}}, \quad n \to \infty$$

for all $d > 0$*. Then*

$$-\log\left(\Delta_{n,r}(X,E)\right) \gtrsim \log(n),$$

and for every sequence of n*-codebooks* $(\alpha_n)_{n\in\mathbb{N}}$ *such that* $\alpha_n \in \mathcal{C}_{n,r}(X,E),\ n \in \mathbb{N}$

$$\rho(\alpha_n) \gtrsim \sigma(\mu)\left(2\log(n)\right)^{\frac{1}{2}}, \quad n \to \infty.$$

If furthermore

$$e_{n,r}(X,E) \approx \phi(\log(n)), \quad n \to \infty \tag{4.2}$$

for some $\phi \in \mathcal{R}_{-a},\ a \in (0,\infty)$*, then there exists a constant* $C \in [1,\infty)$ *such that*

$$C\log(n) \gtrsim -\log\left(\Delta_{n,r}(X,E)\right), \quad n \to \infty,$$

and for every sequence of optimal codebooks $(\alpha_n)_{n\in\mathbb{N}}$

$$\rho(\alpha_n) \lesssim 2\sigma(\mu)\left(C\log(n)\right)^{\frac{1}{2}}, \quad n \to \infty.$$

Remark 4.1.7. In particular, condition (4.2) holds in case the small ball function ϕ_μ is regularly varying at infinity with index $-a$, $a \in (0,\infty)$, see Theorem 1.2.17.

We come to the proofs.

Lower bounds for the quantization radius The basis for the lower bound is the following Lemma.

Lemma 4.1.8. *For every $\delta > 0$ there exists a finite positive constant $C(\delta,\mu,r)$ such that for all $n \in \mathbb{N}$ with $\log(C(\delta,\mu,r)) + \phi^2_{r,\Delta}(n) \geq 0$*

$$\left(\log(C(\delta,\mu,r)) + \phi^2_{r,\Delta}(n)\right)^{\frac{1}{2}} \mathcal{K}_\mu \subset \bigcap_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} (\alpha_n + B(0,\delta)) . \tag{4.3}$$

Hence, there exists a sequence $(\epsilon_n)_{n\in\mathbb{N}}$, $\epsilon_n \to 0$, $n \to \infty$ such that

$$\phi_{r,\Delta}(n)\mathcal{K}_\mu \subset \bigcap_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} (\alpha_n + B(0,\epsilon_n)) \tag{4.4}$$

Proof. For an arbitrary $b \in (0,\frac{1}{2})$ let

$$C(\delta,\mu,r) := (((1-b)^r - b^r)\delta^r \mu(B(0,b\delta)))^{-2} \in (0,\infty).$$

Then, the first micro-macro inequality for Gaussian r.e.'s (Proposition 2.1.6) yields for all $n \in \mathbb{N}$, $\alpha_n \in \mathcal{C}_{n,r}(X,E)$ and $y \in (\alpha_n + B(0,\delta))^c$

$$\Delta_{n,r}(X,E) > C^{-\frac{1}{2}}(\delta,\mu,r) \exp\left(-\frac{\|y\|^2_{\mathcal{H}_\mu}}{2}\right),$$

whereof we deduce by applying the negative logarithm

$$\log\left((C(\delta,\mu,r))^{\frac{1}{2}}\right) - \log(\Delta_{n,r}(X,E)) < \frac{\|y\|^2_{\mathcal{H}_\mu}}{2}.$$

By contraposition, we obtain

$$\left(\log(C(\delta,\mu,r)) + \phi^2_{r,\Delta}(n)\right)^{\frac{1}{2}} \mathcal{K}_\mu \subset \alpha_n + B(0,\delta),$$

for all n such that $\log(C(\delta,\mu,r)) + \phi^2_{r,\Delta}(n) \geq 0$, which yields in view of the independence of the left hand side from α_n the first assertion.
A first order Taylor expansion for $f(x) = (1+x)^{\frac{1}{2}}$ around $x=0$ implies for every $\epsilon \in (0,\infty)$

$$\begin{aligned}\left(\log(C(\delta,\mu,r)) + \phi^2_{r,\Delta}(n)\right)^{\frac{1}{2}} &= \phi_{r,\Delta}(n)\left(1 + \frac{\log(C(\delta,\mu,r))}{\phi^2_{r,\Delta}(n)}\right)^{\frac{1}{2}} \\ &= \phi_{r,\Delta}(n) + \mathcal{O}\left(\frac{\log(C(\delta,\mu,r))}{\phi_{r,\Delta}(n)}\right) \geq \phi_{r,\Delta}(n) - \epsilon\end{aligned}$$

for all $n \geq n'(\phi_{r,\Delta}, \delta, \epsilon)$. In view of $\mathcal{K}_\mu \subset \sigma(\mu)B(0,1)$ we obtain

$$\phi_{r,\Delta}(n)\mathcal{K}_\mu \subset \alpha_n + B(0,\delta) + B(0, \epsilon\sigma(\mu))$$

for all $n \geq n'(\phi_{r,\Delta}, \delta, \epsilon)$, which yields the second assertion as a consequence of the first. □

As an immediate consequence, we obtain with Lemma 4.1.8

Corollary 4.1.9. *For every $\delta > 0$ there exists a finite and positive constant $C(\delta, \mu, r)$ such that*

$$\sigma(\mu)\left(\log(C(\delta,\mu,r)) + \phi^2_{r,\Delta}(n)\right)^{\frac{1}{2}} \leq \underline{\rho}_{n,r}(X,E) + \delta$$

for all $n \in \mathbb{N}$ with $\log(C(\delta,\mu,r)) + \phi^2_{r,\Delta}(n) \geq 0$, which implies

$$\sigma(\mu)\phi_{r,\Delta}(n) \leq \underline{\rho}_{n,r}(X,E) + \epsilon_n$$

for a sequence $\epsilon_n \to 0$.

Proof. By definition, one has

$$\sigma(\mu) := \sup_{x \in \mathcal{K}_\mu} \{\|x\|\},$$

where the supremum is attained since $\mathcal{K}_\mu$ is compact. Therefore, Lemma 4.1.8 and the fact that

$$\|x\| \leq \inf_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \rho(\alpha_n) + \delta$$

for all $x \in \bigcap_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \alpha_n + B(0,\delta)$ imply the first assertion. The second assertion follows analogously from the second assertion of Lemma 4.1.8. □

Upper bounds for the quantization radius Combining a general dimension-free result with the tail behavior of Gaussian r.e.'s yields a first upper bound.

Remark 4.1.10. For every sequence of r-optimal n-codebooks $(\alpha_n)_{n\in\mathbb{N}}$ for X in E one has

$$\rho(\alpha_n) \lesssim 2\sigma(\mu)\left(-2\log \Delta_{n-1,r}(X,E)\right)^{\frac{1}{2}}, \quad n \to \infty.$$

Proof. By Proposition 2.3.3 one has for every $\delta > 0$

$$-\log(\Delta_{n-1,r}(X,E)) \gtrsim \sup_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} -\log\left(\overline{F^X}_r\Big(\frac{\rho(\alpha_n)}{2+\delta}\Big)\right), \quad n \to \infty.$$

Hence, by using the tail behavior for Gaussian r.e.'s (see Corollary 1.1.16) one obtains

$$-\log(\Delta_{n-1,r}(X,E)) \gtrsim \sup_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \frac{\rho(\alpha_n)^2}{2(\sigma(\mu)(2+\delta))^2}, \quad n \to \infty,$$

which yields the assertion, since $\delta > 0$ arbitrary. □

In order to improve the upper bound (i.e. to delete the additional factor 2 outside of the square root), we have to follow an approach similar to that applied in chapter 3. We will need the following lemma.

Lemma 4.1.11. *Let $A \in \mathcal{B}(E)$ and $s \geq 0$ such that $\mu(A) \leq \mu(B)$ with $B := B(0,s)^c$. Then*

$$\int_A \|x\|^r \, d\mu(x) \leq \int_B \|x\|^r \, d\mu(x),$$

for every $r \geq 0$.

Proof. For every $y \in A \cap B^c$ we have $\|y\| \leq s = \inf\{\|x\| : x \in B\}$. Furthermore, by

$$\mu(A \cap B) + \mu(A \cap B^c) = \mu(A) \leq \mu(B) = \mu(B \cap A) + \mu(B \cap A^c)$$

one obtains

$$\mu(A \cap B^c) \leq \mu(B \cap A^c).$$

Therefore

$$\begin{aligned}
\int_A \|x\|^r \, d\mu(x) &= \int_{A\cap B} \|x\|^r \, d\mu(x) + \int_{A\cap B^c} \|x\|^r \, d\mu(x) \\
&\leq \int_{A\cap B} \|x\|^r \, d\mu(x) + \int_{A\cap B^c} s^r \, d\mu(x) \\
&\leq \int_{A\cap B} \|x\|^r \, d\mu(x) + \int_{B\cap A^c} s^r \, d\mu(x) \\
&\leq \int_{A\cap B} \|x\|^r \, d\mu(x) + \int_{B\cap A^c} \|x\|^r \, d\mu(x) = \int_B \|x\|^r \, d\mu(x)
\end{aligned}$$

□

Lemma 4.1.12. *Suppose that*

$$\log(\Delta_{n,r}(X,E)) \sim \log(\Delta_{n+1,r}(X,E)), \quad n \to \infty. \tag{4.5}$$

Then, for every $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that

$$\bigcup_{\alpha_n \in \mathcal{C}_{n,r}(X,E)} \alpha_n \subset \phi_{r,\Delta}(n) \left(\mathcal{K}_\mu + B(0,\epsilon)\right)$$

for all $n \geq n(\epsilon)$.

Proof. Let $\epsilon > 0$ and suppose that the assertion does not hold.
Step 1: There exists a sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ and codebooks $\alpha_{n_k} \in \mathcal{C}_{n_k,r}(X,E)$ such that

$$a_{n_k} \in \left(\phi_{r,\Delta}(n_k) \left(\mathcal{K}_\mu + B(0,\epsilon)\right)\right)^c$$

for codes $a_{n_k} \in \alpha_{n_k}$ and $k \in \mathbb{N}$. We have for $k \in \mathbb{N}$

$$\operatorname{dist}\left(a_{n_k}, \phi_{r,\Delta}(n_k)\left(\mathcal{K}_\mu + B\left(0, \frac{\epsilon}{4}\right)\right)\right) \geq \frac{3}{4}\epsilon\phi_{r,\Delta}(n_k).$$

Conversely, for every

$$x_{n_k} + y_{n_k} \in \phi_{r,\Delta}(n_k)\left(\mathcal{K}_\mu + B\left(0, \frac{\epsilon}{4}\right)\right)$$

one has in view of Lemma 4.1.8

$$\begin{aligned}\operatorname{dist}(\alpha_{n_k}, x_{n_k} + y_{n_k}) &\le \operatorname{dist}(\alpha_{n_k}, x_{n_k}) + \frac{\epsilon}{4}\phi_{r,\Delta}(n_k) \\ &\le \frac{\epsilon}{4} + \frac{\epsilon}{4}\phi_{r,\Delta}(n_k) = \frac{\epsilon}{2}\phi_{r,\Delta}(n_k),\end{aligned}$$

for $k \ge k(\epsilon)$, which implies

$$V_{a_{n_k}}(\alpha_{n_k}) \subset \left(\phi_{r,\Delta}(n_k)\left(\mathcal{K}_\mu + B\left(0, \frac{\epsilon}{4}\right)\right)\right)^c \tag{4.6}$$

for all $k \ge k(\epsilon)$.

Step 2: We set $A_n = \phi_{r,\Delta}(n)\left(\mathcal{K}_\mu + B\left(0, \frac{\epsilon}{4}\right)\right)$ for all $n \in \mathbb{N}$. Let $(\psi(n))_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $\mu(A_{n_k}) = \mu(B_{n_k})$ for $k \in \mathbb{N}$, where $B_{n_k} = \psi(n_k)B(0,1)$. We follow now a few steps of the upper bound in the finite dimensional case. By equation (3.2), there exist $c_{n_k} \in \alpha_{n_k}, k \in \mathbb{N}$ with $\|c_{n_k}\| \to 0,\ k \to \infty$.
In virtue of Lemma 4.1.11 and equation (4.6) we obtain with $\beta_{n_k-1} := \alpha_{n_k}\backslash\{a_{n_k}\}$ and $n_k \ge 2$

$$\begin{aligned}\Delta_{n_k-1,r}(X,E) &\le \mathbb{E}\min_{a\in\beta_{n_k-1}} \|X-a\|^r - \mathbb{E}\min_{a\in\alpha_{n_k}} \|X-a\|^r \\ &\le \int_{V_{a_{n_k}}(\alpha_{n_k})} \|x - c_{n_k}\|^r \, d\mu(x) \le \int_{A^c_{n_k}} \|x - c_{n_k}\|^r \, d\mu(x) \\ &\le 2^r\left(\int_{A^c_{n_k}} \|x\|^r \, d\mu(x) + \|c_{n_k}\|^r \mu(A^c_{n_k})\right) \\ &\le 22^r\left(\int_{B^c_{n_k}} \|x\|^r \, d\mu(x)\right) = 22^r\left(\overline{F^X}_r(\psi(n_k))\right)\end{aligned}$$

for all $k \in \mathbb{N}$ with $\psi(n_k) \ge \|c_{n_k}\|$. As a consequence of Corollary 1.1.16 we obtain

$$\log\left(\left(\overline{F^X}_r(\psi(n_k))\right)\right) \sim \log(\left(\overline{F^X}(\psi(n_k))\right)), \quad k \to \infty,$$

which implies in virtue of equation (4.5)

$$-\log(\Delta_{n_k,r}(X,E)) \sim -\log(\Delta_{n_k-1,r}(X,E)) \gtrsim -\log\left(\mu(B^c_{n_k})\right), \quad k \to \infty. \tag{4.7}$$

Step 3: Since

$$A_n \supset \phi_{r,\Delta}(n)\left(\mathcal{K}_\mu\left(1 + \frac{\epsilon}{8\sigma(\mu)}\right) + B\left(0, \frac{\epsilon}{8}\right)\right)$$

we obtain in virtue of the isoperimetric inequality (Proposition 1.1.8)

$$\mu(B_{n_k}) = \mu(A_{n_k}) \geq \Phi\left(\phi_{r,\Delta}(n_k)\big(1+\frac{\epsilon}{8\sigma(\mu)}\big) + \Phi^{-1}\left(\mu\big(B\big(0,\phi_{r,\Delta}(n_k)\frac{\epsilon}{8}\big)\big)\right)\right) \tag{4.8}$$

for $k \in \mathbb{N}$, which implies in virtue of Lemma 1.1.13 for k sufficiently large

$$\begin{aligned}\mu(B_{n_k}^c) = 1-\mu(B_{n_k}) &\leq 1-\Phi\left(\phi_{r,\Delta}(n_k)(1+\frac{\epsilon}{8\sigma(\mu)}) + \Phi^{-1}\left(\mu\left(B\left(0,\phi_{r,\Delta}(n_k)\frac{\epsilon}{8}\right)\right)\right)\right)\\ &\leq \exp\left(-\frac{\left(\phi_{r,\Delta}(n_k)(1+\frac{\epsilon}{8\sigma(\mu)}) + \Phi^{-1}\left(\mu\left(B\left(0,\phi_{r,\Delta}(n_k)\frac{\epsilon}{8}\right)\right)\right)\right)^2}{2}\right).\end{aligned}$$

Since $\Phi^{-1}\left(\mu\left(B\left(0,\phi_{r,\Delta}(n_k)\frac{\epsilon}{8}\right)\right)\right) \geq 0$ for k sufficiently large, we obtain

$$-\log\left(\Delta_{n_k,r}(X,E)\right) \gtrsim \frac{\left(\phi_{r,\Delta}(n_k)(1+\frac{\epsilon}{8\sigma(\mu)})\right)^2}{2}), \quad k\to\infty$$

as a contradiction. ☐

As already mentioned above, we can sharpen the result obtained for the lim inf of the radius in the following sense.

Proposition 4.1.13. *For every constant $C_\mu \geq 1$ there exists a constant $\kappa_\mu < \infty$ such that*

$$\bigcup_{\alpha_n\in\mathcal{C}_{n,r}(X,E)} \alpha_n \subset \phi_{r,\Delta}(n)\mathcal{K}_\mu + B(0,\kappa_\mu)$$

for all $n \in \mathbb{N}_{C_\mu}$, where

$$\mathbb{N}_{C_\mu} := \{n\in\mathbb{N} : \Delta_{n,r}(X,E) \leq C_\mu \Delta_{n-1,r}(X,E)\}.$$

Proof. Let $C_\mu \geq 1$, $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ fulfilling equation (4.4) and $C>0$. We assume that the assertion does not hold.
Step 1: There exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}} \in \left(\mathbb{N}_{C_\mu}\right)^{\mathbb{N}}$ and codebooks $\alpha_{n_k} \in \mathcal{C}_{n_k,r}(X,E)$ such that

$$a_{n_k} \in \left(\phi_{r,\Delta}(n)\mathcal{K}_\mu + B(0,C)\right)^c$$

for codes $a_{n_k} \in \alpha_{n_k}$ and $k\in\mathbb{N}$. For $C \geq 4\max\{\epsilon_{n_k}, k\in\mathbb{N}\}$ and

$$a_{n_k} \in \left(\phi_{r,\Delta}(n_k)\mathcal{K}_\mu + B(0,C)\right)^c$$

for $k\in\mathbb{N}$ we have

$$\operatorname{dist}\left(a_{n_k}, \phi_{r,\Delta}(n_k)\mathcal{K}_\mu + B\big(0,\frac{C}{4}\big)\right) \geq \frac{3}{4}C.$$

Conversely, in view of Lemma 4.1.8 we obtain for $x_{n_k} + y_{n_k} \in \phi_{r,\Delta}(n_k)\mathcal{K}_\mu + B\left(0, \frac{C}{4}\right)$

$$\begin{aligned}\operatorname{dist}(\alpha_{n_k}, x_{n_k} + y_{n_k}) &\leq \operatorname{dist}(\alpha_{n_k}, x_{n_k}) + \frac{C}{4}\\ &\leq \frac{C}{4} + \frac{C}{4} = \frac{C}{2},\end{aligned}$$

which implies

$$V_{a_{n_k}}(\alpha_{n_k}) \subset \left(\phi_{r,\Delta}(n_k)\mathcal{K}_\mu + B\big(0, \frac{C}{4}\big)\right)^c \tag{4.9}$$

for all $k \in \mathbb{N}$.

Step 2: We set $A_n = \phi_{r,\Delta}(n)\mathcal{K}_\mu + B\left(0, \frac{C}{4}\right)$ for $n \in \mathbb{N}$. Let $(\psi(n))_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $\mu(A_n) = \mu(B_n)$ for $n \in \mathbb{N}$, where $B_n = \psi(n)B(0,1)$. We follow now a few steps of the upper bound in the finite dimensional case. By equation (3.2), there exist $c_{n_k} \in \alpha_{n_k}, k \in \mathbb{N}$ with $\|c_{n_k}\| \to 0,\ k \to \infty$.
In virtue of Lemma 4.1.11 and equation (4.9) we obtain for $\beta_{n_k-1} := \alpha_{n_k}\backslash\{a_{n_k}\}, k \geq 2$

$$\begin{aligned}\Delta_{n_k-1,r}(X,E) &\leq \mathbb{E}\min_{a\in\beta_{n_k-1}}\|X-a\|^r - \mathbb{E}\min_{a\in\alpha_{n_k}}\|X-a\|^r\\ &\leq \int_{V_{a_{n_k}}(\alpha_{n_k})}\|x-c_{n_k}\|^r\,d\mu(x) \leq \int_{A^c_{n_k}}\|x-c_{n_k}\|^r\,d\mu(x)\\ &\leq 2^r\left(\int_{A^c_{n_k}}\|x\|^r\,d\mu(x) + \|c_{n_k}\|^r\,\mu(A^c_{n_k})\right)\\ &\leq 22^r\left(\int_{B^c_{n_k}}\|x\|^r\,d\mu(x)\right) = 22^r\left(\overline{F^X}_r(\psi(n_k)),\right)\end{aligned} \tag{4.10}$$

for all $k \in \mathbb{N}$ such that $\psi(n_k) \geq \|c_{n_k}\|$. As a consequence of Corollary 1.1.16 with $r = 0$ and $r = r$ there exists, for every $\epsilon > 0$, a natural number $n(\epsilon)$ such that equation (4.10) implies

$$\Delta_{n_k-1,r}(X,E) \leq 22^r\left(\overline{F^X}_r(\psi(n_k))\right) \leq \left(\overline{F^X}(\psi(n_k))\right)\exp(\epsilon\psi(n_k)) \tag{4.11}$$

for every $k \in \mathbb{N}$ with $n_k \geq n(\epsilon)$.

Step 3: As a consequence of the Isoperimetric inequality (Proposition 1.1.8)

$$\mu(B_{n_k}) = \mu(A_{n_k}) \geq \Phi\left(\phi_{r,\Delta}(n_k) + \Phi^{-1}\left(\mu\left(B\left(0, \frac{C}{4}\right)\right)\right)\right) \tag{4.12}$$

for $k \in \mathbb{N}$. Equations (4.11) and (4.12) imply in virtue of Lemma 1.1.13

$$\begin{aligned}\Delta_{n_k-1,r}(X,E) &\leq \left(\overline{F^X}(\psi(n_k))\right)\exp(\epsilon\psi(n_k)) = \mu(B^c_{n_k})\exp(\epsilon\psi(n_k))\\ &\leq \left(1 - \Phi\left(\phi_{r,\Delta}(n_k) + \Phi^{-1}\left(\mu\left(B\left(0, \frac{C}{4}\right)\right)\right)\right)\right)\exp(\epsilon\psi(n_k))\\ &\leq \exp\left(-\frac{\left(\phi_{r,\Delta}(n_k) + \Phi^{-1}\left(\mu\left(B\left(0, \frac{C}{4}\right)\right)\right)\right)^2}{2}\right)\exp(\epsilon\psi(n_k))\end{aligned}$$

for all $k \in \mathbb{N}$ such that $n_k \geq n(\epsilon)$. This implies for all $C > 0$ with $\Phi^{-1}\left(\mu\left(B\left(0, \frac{C}{4}\right)\right)\right) \geq \sigma(\mu)\epsilon$ in virtue of the fact that $\psi(n_k) \lesssim \sigma(\mu)\phi_{r,\Delta}(n_k),\ k \to \infty$

$$\Delta_{n_k-1,r}(X,E) \leq \kappa \exp\left(-\frac{\phi_{r,\Delta}(n_k)^2}{2}\right) \times \exp(-2\epsilon\sigma(\mu)\phi_{r,\Delta}(n_k)) \exp(\epsilon\sigma(\mu)\phi_{r,\Delta}(n_k))$$

for some constant $\kappa \in (0,\infty)$ and all $k \in \mathbb{N}$ such that $n_k \geq n(\epsilon)$. Finally, since $n_k \in \mathbb{N}_{C_\mu}, k \in \mathbb{N}$

$$\frac{1}{C_\mu}\Delta_{n_k,r}(X,E) \leq \Delta_{n_k-1,r}(X,E) \leq \kappa \exp(-\frac{\phi_{r,\Delta}(n_k)^2}{2}) \exp(-\epsilon\sigma(\mu)\phi_{r,\Delta}(n_k)) = \kappa\Delta_{n_k,r}(X,E)\exp(-\epsilon\sigma(\mu)\phi_{r,\Delta}(n_k))$$

yields a contradiction, since $\phi_{r,\Delta}(n_k) \to \infty, k \to \infty$. □

Remark 4.1.14.
- For $r > 0$, it holds
$$\Delta_{n,r}(X,E) \leq e^r_{n,r}(X,E) \to 0, \quad n \to \infty,$$
which implies for every $C_\mu \geq 1$
$$\left|\mathbb{N}_{C_\mu}\right| = \infty.$$
- One could naturally conjecture that $\Delta_{n,r}(X,E) \leq \Delta_{n-1,r}(X,E)$ for arbitrary r.e.'s X, all $n \geq 2$ and $r > 0$. This is false. In fact, there exists a counterexample to this conjecture for specific discrete r.e.'s X in the Euclidean $\mathbb{R}^2$ for $r < 1$ and $n = 3$ (see [Kre06, Bemerkung 5.6]).

The sharpened version for the limes inferior reads as follows:

Corollary 4.1.15.

$$0 \leq \liminf_{n\to\infty}(\underline{\rho}_{n,r}(X,E) - \sigma(\mu)\phi_{r,\Delta}(n)) \leq \liminf_{n\to\infty}(\overline{\rho}_{n,r}(X,E) - \sigma(\mu)\phi_{r,\Delta}(n)) < \infty.$$

Proof. The proof is a consequence of the Remark 4.1.14 and Propositions 4.1.13 for the upper bound and Corollary 4.1.9 for the lower bound. □

Proof of the Theorems Firstly note, that Theorem 4.1.3 implies

$$-\log(\Delta_{n-1,r}(X,E)) \sim \log(n-1) \sim \log(n) \sim -\log(\Delta_{n,r}(X,E)), \quad n \to \infty. \tag{4.13}$$

Furthermore, recall that
$$\sigma(\mu) \coloneqq \sup_{x\in\mathcal{K}_\mu} \|x\|.$$

Proof of Theorem 4.1.1. The lower bound is a consequence of Corollary 4.1.9. By equation (4.13), the requirements of Lemma 4.1.12 are satisfied. Hence, the upper bounds follows in view of the definition of $\sigma(\mu)$. □

Proof of Theorem 4.1.6. The Theorem follows with Corollary 4.1.9 for the lower bound, and Remark 4.1.10 in combination with Proposition 2.2.12 for the upper bound. □

4.2 Quantization balls

Throughout this section, suppose that the regularity condition

$$e_{n,r}(X,E) \sim \phi(\log(n)), \quad n \to \infty,$$

holds for some $\phi \in \mathcal{R}_{-a}, a \in (0,\infty)$ (see (4.1)). As indicated at the beginning of this chapter, the quantization ball admits a precise description in terms of the Strassen ball $\mathcal{K}_\mu$.

Theorem 4.2.1. *(quantization ball) For every sequence of r-optimal n-quantizers $(\alpha_n)_{n\in\mathbb{N}}$ for X in E the quantization ball $\mathcal{B} = \mathcal{B}_r(X,E,(\alpha_n)_{n\in\mathbb{N}})$ exists, is independent of the choice of r and $(\alpha_n)_{n\in\mathbb{N}}$ and reads*

$$\mathcal{B} = \lim_{n\to\infty}{}^{\|\cdot\|} \frac{\alpha_n}{\rho(\alpha_n)} = \frac{1}{\sigma(\mu)}\mathcal{K}_\mu. \tag{4.14}$$

Even more,

$$\delta\left(\frac{\alpha_n}{\rho(\alpha_n)}, \frac{1}{\sigma(\mu)}\mathcal{K}_\mu\right) \to 0, \quad n \to \infty. \tag{4.15}$$

The result still holds when replacing $\rho(\alpha_n)$ by its asymptotic equivalents from Theorem 4.1.1 $\overline{\rho}_{n,r}(X,E), \overline{\rho}_{n,r}(X,E), \sigma(\mu)\phi_{r,\Delta}(n)$ and $\sigma(\mu)(2\log(n))^{\frac{1}{2}}$.

Note, that contrary to the finite dimensional case, the convergences in equations (4.14) and (4.15) are generally not equivalent, see [AB06, Example 3.83] or Example B.9.

Remark 4.2.2. Given the weaker result on the asymptotics of the increments of the quantization error, we were not able to obtain a sharp result similar to Theorem 3.3.3. Still, it is reasonable to conjecture that the result can be extended accordingly to the infinite dimensional case, i.e.

$$\delta\left(\alpha_n, \rho(\alpha_n)\frac{1}{\sigma(\mu)}\mathcal{K}_\mu\right) \to 0, \quad n \to \infty.$$

Example 4.2.3.
- Suppose that $(W_t)_{t\in[0,1]}$ is a Brownian motion with path in $C([0,1], \|\cdot\|_{L_\infty})$. As a well known fact, one has

$$\mathcal{H}_\mu = \left\{f \in W^{1,2}([0,1]) : f' \in L_2([0,1])\right\}$$

equipped with the Cameron Martin space norm

$$\|f\|_{\mathcal{H}_\mu} = \|f'\|_{L_2([0,1])}.$$

Hence, the Strassen ball $\mathcal{K}_\mu$ admits the representation

$$\mathcal{K}_\mu = \left\{f \in W^{1,2}([0,1]) : \|f'\|_{L_2([0,1])} \le 1\right\}.$$

- (See [vdVvZ08, p. 218, ff.]) Let $(W_t)_{t\in[0,1]}$ be a Brownian motion with path in $C([0,1], \|\cdot\|_{L_\infty})$ and

$$X_t^{(\alpha)} := \int_0^t (t-s)^{\alpha-\frac{1}{2}} dW_s,$$

for some $\alpha > 0$ and $t \in [0,1]$. Then $(X_t)_{t\in[0,1]}$ is a Riemann-Liouville process with Hurst parameter α whose Cameron-Martin space $\mathcal{H}_\mu$ admits the representation

$$\mathcal{H}_\mu = \{f : [0,1] \to \mathbb{R} : f = I^{\alpha+\frac{1}{2}}(g), g \in L_2([0,1])\}$$

where I^α denotes the Riemann-Liouville operator

$$(I^\alpha(f))(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

The Strassen ball $\mathcal{K}_\mu$ is given as

$$\mathcal{K}_\mu = \left\{ g \in \mathcal{H}_\mu : \left\| I^{\alpha+\frac{1}{2}}(g) \right\|_{\mathcal{H}_\mu} = \frac{\|g\|_{L_2([0,1])}}{\Gamma(\alpha+\frac{1}{2})} \le 1 \right\}.$$

For the proof of the main result of this section, we will be able to make use of the Lemmas from the previous chapter and the previous section.

Proof of Theorem 4.2.1. We show the convergence in the Hausdorff sense. Let $\epsilon > 0$. By Lemma 4.1.8 there exists an $n(\epsilon) \in \mathbb{N}$ such that for all $n \ge n(\epsilon)$ and all $\alpha_n \in \mathcal{C}_{n,r}(X, E)$

$$\frac{1}{\sigma(\mu)} \mathcal{K}_\mu \subset \frac{\alpha_n}{\phi_{r,\Delta}(n)\sigma(\mu)} + B\Big(0, \frac{\epsilon}{\phi_{r,\Delta}(n)\sigma(\mu)}\Big). \tag{4.16}$$

Conversely, by Lemma 4.1.12, there exists an $n'(\epsilon) \in \mathbb{N}$ such that

$$\frac{\alpha_n}{\sigma(\mu)\phi_{r,\Delta}(n)} \subset \frac{1}{\sigma(\mu)} \mathcal{K}_\mu + B(0, \epsilon) \tag{4.17}$$

for all $n \ge n'(\epsilon)$, which implies

$$\delta\left(\frac{1}{\sigma(\mu)} \mathcal{K}_\mu, \frac{\alpha_n}{\phi_{r,\Delta}(n)\sigma(\mu)} \right) \to 0, \quad n \to \infty.$$

By Lemma 3.3.6 the same holds when replacing $\sigma(\mu)\phi_{r,\Delta}(n)$ by its asymptotic equivalents from Theorem 4.1.1. The convergence in the $\|\cdot\|$-sense is as a general result a consequence of the convergence in the Hausdorff sense, see [AB06, Theorem 3.82] or B.8. □

4.3 Comparison to asymptotic optimal quantizers

In this section, we want to study the discrepancy between the quantization radius (and the quantization balls) for sequences of optimal and asymptotically optimal codebooks, i.e. sequences of n-codebooks $(\beta_n)_{n\in\mathbb{N}}$ satisfying

$$e_{n,r}(X,E) \sim e_r(X,E;\beta_n), \quad n \to \infty. \tag{4.18}$$

By Corollary 2.2.3, there exists a constant $C \in (0,\infty)$ such that

$$\begin{aligned} e^r_{n+1,r}(X,E) \le e^r_{n,r}(X,E) &= e^r_{n+1,r}(X,E) + \Delta_{n,r}(X,E) \\ &\le (1+\frac{C}{n})e^r_{n+1,r}(X,E) \sim e^r_{n+1,r}(X,E), \quad n \to \infty. \end{aligned}$$

such that the requirements of Proposition 3.4.1 are satisfied. Hence for Gaussian r.e.'s, there exists no asymptotic upper bound for the quantization radius for sequences of asymptotic optimal codebooks.

As for the lower bound, this question has (partly) already been treated in the dissertation of Dereich, [Der03, Lemmas 5.1.1 and 5.1.3]. More precisely, the results obtained therein imply

Proposition 4.3.1. *Let $(\beta_n)_{n\in\mathbb{N}}$ be a sequence of n-codebooks in E such that*

$$e_{n,r}(X,E) \sim e_r(X,E;\beta_n), \quad n \to \infty.$$

for some $r > 0$.

1. *There exists a sequence of n-codebooks $(\gamma_n)_{n\in\mathbb{N}}$ and a sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that*
   - $e_{n,r}(X,E) \sim e_r(X,E;\gamma_n), \quad n \to \infty,$
   - $c_n \sim \sigma(\mu)\left(-2\log(e^r_{n,r}(X,E))\right)^{\frac{1}{2}}, \quad n \to \infty$ *and*
   - $\gamma_n \subset B(0,2c_n)$, *for all $n \in \mathbb{N}$.*
2. *Furthermore, one has*

$$\rho(\beta_n) \gtrsim \sigma(\mu)\left(-2\log(e^r_{n,r}(X,E))\right)^{\frac{1}{2}}, \quad n \to \infty. \tag{4.19}$$

Remark 4.3.2. There is a discrepancy of the factor 2 between the two rates obtained in Proposition 4.3.1, i.e. the lower bound

$$\rho(\beta_n) \gtrsim \sigma(\mu)\left(-2\log(e^r_{n,r}(X,E))\right)^{\frac{1}{2}}, \quad n \to \infty.$$

and the "lowest attained" upper bound

$$\rho(\gamma_n) \sim 2\sigma(\mu)\left(-2\log(e^r_{n,r}(X,E))\right)^{\frac{1}{2}}, \quad n \to \infty.$$

Having a look at the proof of the result, one observes that this is due to the fact that the "lowest attained" upper bound has been constructed without using a sharp version of a lower bound. Comparing this with other results obtained so far, the additional factor 2 between those two rates seems to correspond to the additional factor 2 in Corollary 2.3.3, which is due to the same lack of argument in the proof. In fact, to be able to manage this problem, one needs a more precise lower bound involving the geometry of the Strassen ball $\mathcal{K}_\mu$.

Throughout this section, let $\alpha_n \in \mathcal{C}_{n,r}(X,E)$, $n \in \mathbb{N}$ and $(\beta_n)_{n\in\mathbb{N}}$ be a sequence of n-codebooks satisfying equation (4.18). We will prove that

- the lower bound given by equation (4.19) is sharp, and
- this bound can be extended to an asymptotically lower bound for $\left(\frac{\alpha_n}{\rho(\alpha_n)}\right)_{n\in\mathbb{N}}$ involving the Strassen ball $\mathcal{K}_\mu$, which asymptotically coincides with the "lowest attained" lower bound.

Furthermore, in comparison to the results obtained for sequences of asymptotically optimal quantizers for finite dimensional r.e.'s, we will prove that the discrepancy to the radius for sequences of optimal quantizers is now much bigger.

New results

Theorem 4.3.3.
There exists a sequence of n-codebooks $(\gamma_n)_{n\in\mathbb{N}}$ and a sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that

- $e_{n,r}(X,E) \sim e_r(X,E;\gamma_n), \quad n \to \infty,$
- $c_n \sim \sigma(\mu)\left(-2\log(e^r_{n,r}(X,E))\right)^{\frac{1}{2}}, \quad n \to \infty$ *and*
- $\gamma_n \subset c_n \frac{1}{\sigma(\mu))}\mathcal{K}_\mu + B(0,C)$ *for a constant* $C \in (0,\infty)$ *and* $n \in \mathbb{N}$.

In particular

$$\rho(\gamma_n) \lesssim \sigma(\mu)\left(-2\log(e^r_{n,r}(X,E))\right)^{\frac{1}{2}}, \quad n \to \infty.$$

In addition, we have for any sequence of asymptotically r-optimal n-quantizers $(\beta_n)_{n\in\mathbb{N}}$

$$\rho(\beta_n) \gtrsim \sigma(\mu)\left(-2\log(e^r_{n,r}(X,E))\right)^{\frac{1}{2}},$$

and furthermore

$$\delta_l\left(\frac{1}{\sigma(\mu)}\mathcal{K}_\mu, \frac{\beta_n}{c_n}\right) \to 0, \quad n \to \infty.$$

Remark 4.3.4. 1. The estimate presented above sharpens the rate given in 4.3.1 by the factor 2 and gives additionally a geometric limitation in terms of the Strassen ball $\mathcal{K}_\mu$.

2. Comparing the quantization radius of the sequence of asymptotically optimal codebooks $(\gamma_n)_{n\in\mathbb{N}}$ as constructed in the previous Theorem to the radius of a sequence of optimal codebooks $(\alpha_n)_{n\in\mathbb{N}}$, there is a significant discrepancy to be observed. Assuming that

$$e_{n,r}(X,E) \sim \phi(\log(n)), \quad n \to \infty$$

for some $\phi \in \mathcal{R}_{-a}$ and $a > 0$, we obtain

$$\frac{\rho(\alpha_n)}{\rho(\gamma_n)} \sim \left(\frac{2\log(n)}{-r\log(\phi(\log(n)))}\right)^{\frac{1}{2}}, \quad n \to \infty.$$

Proof of the result Even if the lower bound given by Proposition 4.3.1 is sharp, we have to develop a generalization for this bound in order to improve the upper bound. The idea is as simple as in the finite dimensional case. We replace the first micro-macro inequality with an equivalent for asymptotic optimal codebooks involving $e^r_{n,r}(X,E)$ instead of $\Delta_{n,r}(X,E)$. The following Lemma is an "equivalent" to Lemma 4.1.8 which is the corresponding version for optimal codebooks.

Lemma 4.3.5. *For any sequence of asymptotically r-optimal n-quantizers $(\beta_n)_{n\in\mathbb{N}}$ for X in E there exists a sequence $(\epsilon_n)_{n\in\mathbb{N}}, \epsilon_n \to 0$ such that*

$$(-2\log(e^r_r(X,E;\beta_n)))^{\frac{1}{2}}\,\mathcal{K}_\mu \subset \beta_n + B(0,\epsilon_n)$$

for all $n \in \mathbb{N}$.

Proof. Let $\epsilon > 0$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence in E such that $\text{dist}(x_n,\beta_n) \geq \epsilon$. Then, for $n \in \mathbb{N}$ by using the estimation of shifted balls (Proposition 1.1.11)

$$\begin{aligned}
e^r_r(X,E;\beta_n) &\geq \int_{B(x_n,\frac{\epsilon}{2})} \text{dist}(x,\beta_n)^r d\mu(x) \\
&\geq \left(\frac{\epsilon}{2}\right)^r \mu\left(B(x_n,\frac{\epsilon}{2})\right) \\
&\geq \left(\frac{\epsilon}{2}\right)^r \mu\left(B(0,\frac{\epsilon}{2})\right)\exp\left(-\frac{\|x_n\|^2_{\mathcal{H}_\mu}}{2}\right).
\end{aligned}$$

Applying the negative logarithm yields

$$-\log(e^r_r(X,E;\beta_n)) \leq C(\epsilon) + \frac{\|x_n\|^2_{\mathcal{H}_\mu}}{2},$$

for $n \in \mathbb{N}$ and some constant $C(\epsilon) \in \mathbb{R}$. Hence

$$\mathcal{K}_\mu\,(2\,(-\log(e^r_r(X,E;\beta_n)) - C(\epsilon)))^{\frac{1}{2}} \subset \beta_n + B(0,\epsilon)$$

for all $n \in \mathbb{N}$ such that $-\log(e_r^r(X,E;\beta_n)) - C(\epsilon) > 0$. By a first order Taylor expansion for $f(x) = (1+x)^{\frac{1}{2}}$ around $x = 0$ and compactness of $\mathcal{K}_\mu$, there exists a constant $C'(\epsilon) < \infty$ such that

$$(-2\log(e_r^r(X,E;\beta_n)))^{\frac{1}{2}}\,\mathcal{K}_\mu \subset \beta_n + B(0,\epsilon) + B\left(0, \frac{C'(\epsilon)}{-\log(e_r^r(X,E;\beta_n))}\right),$$

for all $n \geq n(C'(\epsilon))$. Since $\epsilon > 0$ arbitrary, the assertion follows. □

For the upper bound, we need an equivalent version to Lemma 4.1.12

Lemma 4.3.6. *For any sequence of asymptotically r-optimal n-quantizers $(\beta_n)_{n\in\mathbb{N}}$ for X in E, there exists a constant $\kappa \in (0,\infty)$ and a sequence of n-codebooks $(\gamma_n)_{n\in\mathbb{N}}$ such that*

- $\gamma_n \subset (-2\log(e_r^r(X,E;\beta_n)))^{\frac{1}{2}}\,\mathcal{K}_\mu + B(0,\kappa)$ *and*
- $e_r^r(X,E;\beta_n) \sim e_r^r(X,E;\gamma_n), \quad n \to \infty.$

Proof. Let $(\epsilon_n)_{n\in\mathbb{N}}$ be as in Lemma 4.3.5. We set

$$c_n := (-2\log(e_r^r(X,E;\beta_n)))^{\frac{1}{2}}, \; n \in \mathbb{N}$$

and

$$\delta_n = \delta_n(C,\epsilon_n) := \max\{4\epsilon_n; C\}$$

for come constant $C > 0$ specified below.
If $\beta_n \subset c_n\mathcal{K}_\mu + B(0,\delta_n)$ for all $n \geq n' \in \mathbb{N}$, we set $\gamma_n = \beta_n$ for all $n \in \mathbb{N}$ and κ sufficiently large such that the assertion also holds for all $n < n'$. Otherwise, we set for an unbounded sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$

$$\gamma_{n_k} := \beta_{n_k} \cap (c_n\mathcal{K}_\mu + B(0,\delta_{n_k})) \cup \{0\}$$

Then $|\gamma_{n_k}| \leq n_k, k \in \mathbb{N}$ and

$$e_r^r(X,E;\gamma_{n_k}) \leq e_r^r(X,E;\beta_{n_k}) + \int_{\bigcup_{a\in\beta_{n_k}\setminus\gamma_{n_k}} V_a(\beta_{n_k})} \|x\|^r \, d\mu(x).$$

Step 1: We show that

$$\bigcup_{a\in\beta_{n_k}\setminus\gamma_{n_k}} V_a(\beta_{n_k}) \subset \left(c_{n_k}\mathcal{K}_\mu + B\left(0, \frac{\delta_{n_k}}{4}\right)\right)^c.$$

For $x_{n_k} = y_{n_k} + z_{n_k} \in c_{n_k}\mathcal{K}_\mu + B(0, \frac{\delta_{n_k}}{4})$, Lemma 4.3.5 implies

$$\begin{aligned} \operatorname{dist}(x_{n_k},\beta_{n_k}) &\leq \operatorname{dist}(x_{n_k},y_{n_k}) + \operatorname{dist}(y_{n_k},\beta_{n_k}) \\ &\leq \frac{1}{4}\delta_{n_k} + \epsilon_{n_k} \leq \frac{1}{2}\delta_{n_k}. \end{aligned}$$

Conversely, for $a_{n_k} \in \beta_{n_k} \backslash \gamma_{n_k}$

$$\text{dist}(x_{n_k}, a_{n_k}) \geq \frac{3}{4}\delta_{n_k} > \frac{1}{2}\delta_{n_k}.$$

which implies

$$\bigcup_{a \in \beta_{n_k} \backslash \gamma_{n_k}} V_a(\beta_{n_k}) \subset \left(c_{n_k}\mathcal{K}_\mu + B\left(0, \frac{\delta_{n_k}}{4}\right)\right)^c \tag{4.20}$$

for all $k \in \mathbb{N}$.

Step 2: We set $A_n = c_n\mathcal{K}_\mu + B\left(0, \frac{\delta_n}{4}\right)$ for $n \in \mathbb{N}$. Let $(\psi_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $\mu(A_n) = \mu(B_n)$ for $n \in \mathbb{N}$, where $B_n = \psi_n B(0,1)$. As a consequence of Corollary 1.1.16 with $r = 0$ and $r = r$ there exists for every $\epsilon > 0$ a natural number $n(\epsilon)$ such that

$$\left(\overline{F^X}_r(\psi_n)\right) \leq \left(\overline{F^X}(\psi_n)\right) \exp\left(\frac{\epsilon}{\sigma(\mu)}\psi_n\right)$$

for every $n \geq n(\epsilon)$.

Step 3: As a consequence of the Isoperimetric inequality (Proposition 1.1.8)

$$\mu(B_n) = \mu(A_n) \geq \Phi\left(c_n + \Phi^{-1}\left(\mu\left(B\left(0, \frac{\delta_n}{4}\right)\right)\right)\right)$$

for $n \in \mathbb{N}$. Step 1 and Step 2 imply in virtue of Lemma 1.1.13 for every $\epsilon > 0$

$$\begin{aligned}
\int_{\bigcup_{a \in \beta_{n_k} \backslash \gamma_{n_k}} V_a(\beta_{n_k})} \|x\|^r \, d\mu(x) &\leq \int_{A^c_{n_k}} \|x\|^r \, d\mu(x) \\
&\leq \left(\overline{F^X_r}(\psi_{n_k})\right) \leq \left(\overline{F^X}(\psi_{n_k})\right) \exp\left(\frac{\epsilon}{\sigma(\mu)\psi_{n_k}}\right) = \mu(B^c_{n_k}) \exp\left(\frac{\epsilon}{\sigma(\mu)}\psi_{n_k}\right), \\
&\leq \left(1 - \Phi\left(c_{n_k} + \Phi^{-1}\left(\mu\left(B\left(0, \frac{\delta_{n_k}}{4}\right)\right)\right)\right)\right) \exp\left(\frac{\epsilon}{\sigma(\mu)}\psi_{n_k}\right) \\
&\leq \exp\left(-\frac{\left(c_{n_k} + \Phi^{-1}\left(\mu\left(B\left(0, \frac{\delta_{n_k}}{4}\right)\right)\right)\right)^2}{2}\right) \exp\left(\frac{\epsilon}{\sigma(\mu)}\psi_{n_k}\right)
\end{aligned}$$

for all $k \geq k(\epsilon)$. $\mathcal{K}_\mu \subset \sigma(\mu)B(0,1)$ implies $\psi_n \lesssim \sigma(\mu)c_n, \quad n \to \infty$. Hence, we obtain for all $C \in (0, \infty)$ with $\Phi^{-1}\left(\mu\left(B\left(0, \frac{\delta_{n_k}}{4}\right)\right)\right) \geq \epsilon$

$$\int_{\bigcup_{a \in \beta_{n_k} \backslash \gamma_{n_k}} V_a(\beta_{n_k})} \|x\|^r \, d\mu(x) \leq \kappa \exp\left(-\frac{c^2_{n_k}}{2}\right) \exp(-2\epsilon c_{n_k}) \exp(\epsilon c_{n_k}) \exp\left(-\frac{\epsilon^2}{2}\right)$$

for all $k \geq k(\epsilon)$ and some constant $\kappa \in (0, \infty)$. Since $c_n \to \infty$, the assertion follows. □

Proof of Theorem 4.3.3. The Theorem is a consequence of Lemmas 4.3.5 for the lower bound of the quantization radius and quantization ball for $(\beta_n)_{n \in \mathbb{N}}$ as well as Lemma 4.3.6 for the construction of a sequence of asymptotic optimal codebooks $(\gamma_n)_{n \in \mathbb{N}}$ achieving the required shape. □

Chapter 5

Local quantization problems

This chapter shall be devoted to the study of the asymptotic behavior of

- the weights of the cells $\mu(C_a(\alpha_n))$ and
- the local inertia $e_r(X, E; \alpha_n, C_a(\alpha_n))$

of Voronoi partitions $\{C_a(\alpha_n), a \in \alpha_n\}$ for sequences of r-optimal n-codebooks $(\alpha_n)_{n\in\mathbb{N}}$ for r.e.'s X in $\mathbb{R}^d$. Furthermore, as indicated by the results obtained hereafter, the behavior of

- the μ_r-weights of the Cells $C_a(\alpha_n) : \mu_r(C_a(\alpha_n))$

seem to admit a specific regularity.
Given some regularity assumptions on the underlying probability μ, the natural conjecture for the asymptotics of the local quantization error reads

$$\begin{aligned}\inf_{a\in\alpha_n} e^r_{r,\mathrm{loc}}(X, \mathbb{R}^d; \alpha_n, W_a(\alpha_n)) &\sim \sup_{a\in\alpha_n} e^r_{r,\mathrm{loc}}(X, \mathbb{R}^d; \alpha_n, V_a(\alpha_n)) \\ &\sim \frac{1}{n} e^r_{n,r}(X, \mathbb{R}^d), \quad n \to \infty,\end{aligned}$$

which goes back to Gersho (see [Ger79]); for the one-dimensional case, it is even older (see e.g. [PD51]). As for a reasonable conjecture on the asymptotics of the weights of the Voronoi cells, one derives in view of the empirical measure Theorem (Theorem 1.2.14) for measures μ with continuous and bounded Lebesgue-density f and a bounded continuous function $F : \mathbb{R}^d \to \mathbb{R}$

$$\begin{aligned}\frac{1}{n}\sum_{a\in\alpha_n} \left\| f^{\frac{d}{r+d}} \right\|_1 f^{\frac{r}{r+d}}(a) F(a) &\overset{n\to\infty}{\to} \left\| f^{\frac{d}{r+d}} \right\|_1 \int f^{\frac{r}{r+d}}(x) F(x) d\mu_r(x) \\ &= \int F(x) d\mu(x)\end{aligned}$$

(see [GLP10, chapter 1]), so that

$$\frac{1}{n}\sum_{a\in\alpha_n} \left\| f^{\frac{d}{r+d}} \right\|_1 f^{\frac{r}{r+d}}(a) \delta_a \overset{w}{\Rightarrow} \mu,$$

where $\stackrel{w}{\Rightarrow}$ denotes the weak convergence of measures. Since also

$$\sum_{a\in\alpha_n} \mu(C_a(\alpha_n))\delta_a \stackrel{w}{\Rightarrow} \mu, \quad n\to\infty$$

for every sequence of Voronoi partitions $\{C_a(\alpha_n), a\in\alpha_n\}_{n\in\mathbb{N}}$, it is reasonable to conjecture that

$$\mu(W_{a_n}(\alpha_n)) \sim \mu(V_{a_n}(\alpha_n)) \sim \frac{1}{n}\left\| f^{\frac{d}{r+d}}\right\|_1 f^{\frac{r}{r+d}}(a_n), \quad n\to\infty,$$

for every sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n\in\alpha_n,\ n\in\mathbb{N}$.

As for a conjecture on the μ_r-weights for the Voronoi cells, one has in virtue of empirical measure Theorem (Theorem 1.2.14)

$$\sum_{a\in\alpha_n}\frac{1}{n}\delta_a \stackrel{w}{\Rightarrow} \mu_r.$$

Since also

$$\sum_{a\in\alpha_n}\mu_r(C_a(\alpha_n))\delta_a \stackrel{w}{\Rightarrow} \mu_r,$$

one expects

$$\inf_{a\in\alpha_n}\mu_r(W_a(\alpha_n)) \sim \sup_{a\in\alpha_n}\mu_r(V_a(\alpha_n)) \sim \frac{1}{n}, \quad n\to\infty.$$

While the question about those asymptotics is very old, no rigorous treatment of this problem has been done until the publication of the research article of Graf, Luschgy and Pagès [GLP10]. The main results achieved in the article are presented in the following section.

Throughout this chapter, let X be a r.e. in $(E,\|\cdot\|) = (\mathbb{R}^d,\|\cdot\|)$ for an arbitrary norm $\|\cdot\|$, $\|\cdot\|_0$ be an additional arbitrary norm on $\mathbb{R}^d$, $r\in(0,\infty)$ and $X\in L_{r+\delta}(\mathbb{R}^d,\mathbb{P})$ for some $\delta>0$. We denote

$$C_{\|\cdot\|,\|\cdot\|_0} := \max\{j_{\|\cdot\|_0,\|\cdot\|}, j_{\|\cdot\|,\|\cdot\|_0}\},$$

where $j_{\|\cdot\|_0,\|\cdot\|}$ and $j_{\|\cdot\|,\|\cdot\|_0}$ denote the natural embeddings from $(\mathbb{R}^d,\|\cdot\|_0)\hookrightarrow(\mathbb{R}^d,\|\cdot\|)$ and vice versa.

5.1 Known results

For $n\in\mathbb{N}$, $\alpha_n\in\mathcal{C}_{n,r}(X,\mathbb{R}^d)$ and $a\in\alpha_n$ we denote

$$\overline{s}_{n,r}(\mu,\mathbb{R}^d;a) := \inf_{s>0}\{V_a(\alpha_n)\cap\operatorname{supp}(\mu)\subset B(x,s)\},$$
$$\underline{s}_{n,r}(\mu,\mathbb{R}^d;a) := \sup_{s>0}\{V_a(\alpha_n)\supset B(x,s)\}.$$

One easily sees that

$$\underline{s}_{n,r}(\mu,\mathbb{R}^d;a) = \frac{1}{2}\operatorname{dist}(a,\alpha_n\backslash\{a\}).$$

Furthermore, for $K \subset \mathbb{R}^d$, we set

$$\alpha_n(K) := \{a \in \alpha_n : V_a(\alpha_n) \cap K \neq \varnothing\},$$

as well as

$$\underline{\alpha}_n(K) := \{a \in \alpha_n : V_a(\alpha_n) \subset K\}.$$

Theorem 5.1.1. *(see [GLP10, Theorem 3.1, Theorem 4.1])*

1. *Suppose that $\mu = \mu_a \in \mathcal{M}_r(\mathcal{B}(\mathbb{R}^d))$ such that $f = \frac{\partial\mu}{\partial\lambda^d}$ is essentially bounded, has a connected and compact support and satisfies the following* local *peakless condition:*

$$\exists c, s_0 > 0 \text{ such that } \forall s < s_0,\ x \in \operatorname{supp}(\mu) : \mu(B(x,s)) \geq cs^d$$

Then

$$\begin{aligned} n^{-(1+\frac{r}{d})} \approx \frac{1}{n}e^r_{n,r}(X,\mathbb{R}^d) &\approx \inf_{a\in\alpha_n} e^r_{r,\mathrm{loc}}(X,\mathbb{R}^d;\alpha_n,W_a(\alpha_n)) \\ &\approx \sup_{a\in\alpha_n} e^r_{r,\mathrm{loc}}(X,\mathbb{R}^d;\alpha_n,V_a(\alpha_n)), \quad n\to\infty, \\ \frac{1}{n} &\approx \inf_{a\in\alpha_n}\mu\left(W_a(\alpha_n)\right) \approx \sup_{a\in\alpha_n}\mu\left(V_a(\alpha_n)\right), \quad n\to\infty, \end{aligned}$$

as well as

$$\overline{s}_{n,r}(\mu,\mathbb{R}^d;a) \approx \underline{s}_{n,r}(\mu,\mathbb{R}^d;a) \approx n^{-\frac{1}{d}}, \quad n\to\infty. \tag{5.1}$$

2. *Suppose that $f = \frac{\partial\mu}{\partial\lambda^d}$ satisfies the first micro-macro inequality on $\mathbb{R}^d$ in the following form: There exists a constant $c > 0$ such that for all $K \subset \mathbb{R}^d$ compact there exists $n_K \in \mathbb{N}$ such that for all $x \in K$ and $n \geq n_K$*

$$cn^{-\frac{1}{d}}f(x)^{-\frac{1}{r+d}} \geq \operatorname{dist}(x,\alpha_n).$$

Then, there exist constants $c_i \in (0,\infty)$, $i \in \{1,\dots,4\}$ such that for every *compact* $K \subset \mathbb{R}^d$

$$\max_{a\in\alpha_n(K)} \mu(V_a(\alpha_n)) \lesssim c_1 \frac{\left(\inf_{\epsilon>0} \operatorname{ess\,sup}_{K+B(0,\epsilon)} f\right)^{\frac{r}{r+d}}}{n}, \quad n\to\infty,$$

$$\begin{aligned}\max_{a\in\alpha_n(K)} & e^r_{r;\mathrm{loc}}(X,\mathbb{R}^d;\alpha_n,V_a(\alpha_n)) \\ &\lesssim c_2\left(1+\log\left(\inf_{\epsilon>0}\frac{\operatorname{ess\,sup}_{K+B(0,\epsilon)} f}{\operatorname{ess\,inf}_{K+B(0,\epsilon)} f}\right)\right) n^{-\frac{r+d}{d}}, \quad n\to\infty,\end{aligned}$$

$$\min_{a\in\alpha_n(K)} \mu(W_a(\alpha_n)) \gtrsim c_3 \frac{\sup_{\epsilon>0} \left(\operatorname{ess\,inf}_{K+B(0,\epsilon)} f\right)^{\frac{r}{r+d}}}{n}, \quad n\to\infty,$$

$$\begin{aligned}\min_{a\in\alpha_n(K)} & e^r_{r;\mathrm{loc}}(X,\mathbb{R}^d;\alpha_n,W_a(\alpha_n)) \\ &\gtrsim c_4 \sup_{\epsilon>0}\left(\frac{\operatorname{ess\,inf}_{K+B(0,\epsilon)} f}{\operatorname{ess\,sup}_{K+B(0,\epsilon)} f}\right)^{\max\{1,r\}} n^{-\frac{r+d}{d}}, \quad n\to\infty.\end{aligned}$$

Remark 5.1.2.

- Assuming for example continuity of the density f, Theorem 5.1.1 part 2) implies the weak asymptotics for the local inertia and the weights of the Voronoi cells $C_{a_n}(\alpha_n)$ for sequences of codes $(a_n)_{n\in\mathbb{N}}$ converging towards some $x\in\mathbb{R}^d$, see [GLP10, Theorem 4.1 and Corollary 4.1].
- As concerning part 2) of Theorem 5.1.1, the authors indicate that, given a regularity condition on the tail of the distribution, some of the bounds for the weights and the local inertia for sequences of Voronoi cells leaving each compact set K could hold analogously.

Remark 5.1.3. Theorem 5.1.1 Part 1) does not cover probabilities μ with unbounded Lebesgue densities. In fact, given this case, the result does not hold in general. Suppose that there is $x\in\operatorname{supp}(\mu)$ such that for every $\kappa\in(0,\infty)$ there exists an $\epsilon>0$ such that $|f|\ge\kappa$ for all $y\in B(x,\epsilon)$. For optimal codebooks α_n, let $\delta_n=\operatorname{dist}(x,\alpha_n)$, $n\in\mathbb{N}$. Then, as a consequence of the first micro-macro inequality (Proposition 2.1.4) there exists for every $b\in(0,\frac{1}{2})$ a constant $C(b,r)\in(0,\infty)$ such that

$$\Delta_{n,r}(X,\mathbb{R}^d)\ge C(b,r)\mu(B(x,b\delta_n))\delta_n^{r+d}$$

for all $n\in\mathbb{N}$. Thus, for every $n\in\mathbb{N}$ such that $b\delta_n<\epsilon$ one has

$$\Delta_{n,r}(X,\mathbb{R}^d)\ge C(b,r)\kappa\delta_n^{r+d}.$$

Hence, in contrary to (5.1)

$$\delta_n=\mathrm{o}(n^{-\frac{1}{d}}), \quad n\to\infty.$$

By assuming a non-local peakless condition, we will be able to improve the results obtained in [GLP10] for specific distributions μ having an unbounded support. In particular, we will establish a complementary result covering sequences of codes leaving every compact set $K\subset\mathbb{R}^d$.

5.2 General lower bounds

In chapter 2, we used the extended second micro-macro inequality to derive a general upper bound for the increments of the quantization error for Gaussian r.e.'s. Conversely, given that we know the asymptotics of those increments for a probability μ on $\mathcal{B}(\mathbb{R}^d)$, one may use the same inequality and an analogous argumentation to derive a lower bound for the local quantization error and the μ_r-weights of the Voronoi regions $V_a(\alpha_n)$ and $W_a(\alpha_n)$.

Throughout the remainder of this chapter, let $X \in L_{r+\delta}(\mathbb{P}, \mathbb{R}^d)$ for some constant $r \in (0, \infty)$ and $\delta > 0$. Furthermore, suppose that

$$\mu = \mu_a,$$

$\mu(B_0(0, A)^c) > 0$ and

$$f(x) = \frac{\partial \mu}{\partial \lambda^d}(x) = g(\|x\|_0), \; x \in B_0(0, A)^c,$$

for a function g almost decreasing on $[A, \infty)$ for some constant $A \geq 0$. Note, that μ satisfies under these assumptions the local peakless property on $B_0(0, A)^c$ (see Lemma 2.1.3). Concerning the generalization of Theorem 5.1.1, since

$$\begin{aligned} \alpha_n &= \alpha_n(B_0(0, A)) \cup \underline{\alpha}_n(B_0(0, A)^c) \\ &= \{a \in \alpha_n : V_a(\alpha_n) \cap B_0(0, A) \neq \varnothing\} \cup \{a \in \alpha_n : V_a(\alpha_n) \subset B_0(0, A)^c\}, \end{aligned}$$

we only need to consider sequences of codes $(a_n)_{n \in \mathbb{N}}$ with $a_n \in \underline{\alpha}_n(B_0(0, A)^c)$, $n \in \mathbb{N}$, since the remaining cells are, given an appropriate regularity assumption on μ_a on $B_0(0, A)$, covered by Theorem 5.1.1.

At first, we will establish a close relationship between the local inertia and the μ_r-weights of the Voronoi cells.

Lemma 5.2.1. *We have*

$$\begin{aligned} &\inf_{\alpha_n \in \mathcal{C}_{n,r}(X, \mathbb{R}^d)} \inf_{a \in \underline{\alpha}_n(B_0(0,A)^c)} e^r_{r;\mathrm{loc}}(X, \mathbb{R}^d; \alpha_n, W_a(\alpha_n)) \\ &\qquad \lesssim n^{-\frac{r}{d}} \inf_{\alpha_n \in \mathcal{C}_{n,r}(X, \mathbb{R}^d)} \inf_{a \in \underline{\alpha}_n(B_0(0,A)^c)} \mu_r\left(W_a(\alpha_n)\right), \quad n \to \infty. \end{aligned}$$

The same result holds when replacing $W_a(\alpha_n)$ by $V_a(\alpha_n)$, $n \in \mathbb{N}$ or all inf *by* sup.

Proof. In view of the lower peakless property of μ on $B_0(0, A)^c$ (Lemma 2.1.3), the first micro-macro inequality (Proposition 2.1.5) and the estimate for the increments of the quantization error (Theorem 1.2.24) we calculate

$$\begin{aligned}
&\inf_{\alpha_n\in\mathcal{C}_{n,r}(X,\mathbb{R}^d)}\inf_{a\in\underline{\alpha}_n(B_0(0,A)^c)} e^r_{r;\mathrm{loc}}(X,\mathbb{R}^d;\alpha_n,W_a(\alpha_n))\\
&\qquad = \inf_{\alpha_n\in\mathcal{C}_{n,r}(X,\mathbb{R}^d)}\inf_{a\in\underline{\alpha}_n(B_0(0,A)^c)} \int_{W_a(\alpha_n)} \mathrm{dist}(x,\alpha_n)^r d\mu(x)\\
&\qquad \lesssim n^{-\frac{r}{d}} \inf_{\alpha_n\in\mathcal{C}_{n,r}(X,\mathbb{R}^d)}\inf_{a\in\underline{\alpha}_n(B_0(0,A)^c)} \int_{W_a(\alpha_n)} f(x)^{-\frac{r}{r+d}} f(x) d\lambda^d(x)\\
&\qquad \approx n^{-\frac{r}{d}} \inf_{\alpha_n\in\mathcal{C}_{n,r}(X,\mathbb{R}^d)}\inf_{a\in\underline{\alpha}_n(B_0(0,A)^c)} \mu_r\left(W_a(\alpha_n)\right), \quad n\to\infty,
\end{aligned} \tag{5.2}$$

which yields the assertion. Replacing $W_a(\alpha_n)$ by $V_a(\alpha_n)$ or inf by sup in equation (5.2) yields the second assertion. □

The previous Lemma implies that any lower bound for the local inertia yields a lower bound for the μ_r-weights of the Voronoi cells. Conversely, any upper bound for the μ_r-weights yields an upper bound for the local inertia. We come now to lower bounds for the local inertia.

Theorem 5.2.2. *We have*

$$\inf_{\alpha_n\in\mathcal{C}_{n,r}(X,\mathbb{R}^d)}\inf_{a\in\underline{\alpha}_n(B_0(0,A)^c)} e^r_{r;\mathrm{loc}}(X,\mathbb{R}^d;\alpha_n,W_a(\alpha_n)) \gtrsim n^{-\frac{r+d}{d}}, \quad n\to\infty.$$

Proof. Note, that for $e,x\in\mathbb{R}^d$ and $t\in[0,1]$ such that $\|e\|_0\le\frac{1}{2}\|x\|_0$ one has

$$\begin{aligned}
\|tx+e\|_0 &\le t\|x+e\|_0+(1-t)\|e\|_0\\
&\le \|x+e\|_0-(1-t)(\|x\|_0-\|e\|_0)+(1-t)\|e\|_0\\
&\le \|x+e\|_0 .
\end{aligned} \tag{5.3}$$

Step 1: We abbreviate for $a\in\alpha_n\in\mathcal{C}_{n,r}(X,\mathbb{R}^d)$ the distance $\delta_n(a):=\mathrm{dist}(a,\alpha_n\backslash\{a\})$, $n\in\mathbb{N}$. Let $n\in\mathbb{N}$ and $a\in\underline{\alpha}_n(B_0(0,A)^c)$ for some $\alpha_n\in\mathcal{C}_{n,r}(X,\mathbb{R}^d)$. Then, for $t=(1-\frac{1}{4}\frac{\delta_n(a)}{\|a\|})$ we have

$$B\big(ta,\frac{\delta_n(a)}{8}\big)\subset B\big(a,\frac{3\delta_n(a)}{8}\big)\subset W_a(\alpha_n)$$

as well as

$$\mu\left(B\left(ta,\frac{\delta_n(a)}{8}\right)\cap B\left(a,\frac{\delta_n(a)}{8}\right)\right)=0.$$

Since $a\in\underline{\alpha}_n(B_0(0,A)^c)$ one has $0\notin B(a,\frac{1}{2}\delta_n(a))$, which also implies $\frac{1}{2}\|a\|_0\ge\frac{\delta_n(a)}{4C_{\|\cdot\|,\|\cdot\|_0}}$. Hence, we obtain in view of equation (5.3) and the fact that g is almost decreasing

$$\begin{aligned}\mu\left(B\left(a,\frac{\delta_n(a)}{8C^2_{\|\cdot\|,\|\cdot\|_0}}\right)\right) &\leq \mu\left(B_0\left(a,\frac{\delta_n(a)}{8C_{\|\cdot\|,\|\cdot\|_0}}\right)\right)\\ &= \int_{B_0\left(a,\frac{\delta_n(a)}{8C_{\|\cdot\|,\|\cdot\|_0}}\right)} g(\|x\|_0)d\lambda^d(x)\\ &\leq \frac{1}{m_g}\int_{B_0\left(ta,\frac{\delta_n(a)}{8C_{\|\cdot\|,\|\cdot\|_0}}\right)} g(\|x\|_0)d\lambda^d(x)\\ &\leq \frac{1}{m_g}\mu\left(B\left(ta,\frac{\delta_n(a)}{8}\right)\right),\end{aligned}$$

such that

$$\mu\left(B\big(a,\frac{3}{8}\delta_n(a)\big)\backslash B\big(a,\frac{\delta_n(a)}{8C^2_{\|\cdot\|,\|\cdot\|_0}}\big)\right) \geq m_g\mu\left(B\big(a,\frac{\delta_n(a)}{8C^2_{\|\cdot\|,\|\cdot\|_0}}\big)\right).$$

Step 2: In virtue of the extended second micro-macro inequality (Proposition 2.1.8), there exists for every $C \in (0,\infty)$ a constant $\kappa(C,r) \in (0,\infty)$ such that for all $n \geq 2$

$$\Delta_{n-1,r}(X,\mathbb{R}^d) \leq \kappa(C,r)\left(e^r_{r;\mathrm{loc}}(X,\mathbb{R}^d;\alpha_n,W_a(\alpha_n)) + \delta_n(a)^r\mu\left(B(a,C\delta_n(a))\right)\right) \tag{5.4}$$

independent of the choice of α_n. As a consequence of Step 1, we derive for $C = \frac{1}{8C^2_{\|\cdot\|,\|\cdot\|_0}}$

$$\begin{aligned}\delta_n(a)^r\mu\left(B\left(a,C\delta_n(a)\right)\right) &\leq \frac{1}{C^r}\int_{B(a,C\delta_n(a))}(C\delta_n(a))^r d\mu(x)\\ &\leq \frac{1}{m_gC^r}\int_{B\left(a,\frac{3}{8}\delta_n(a)\right)\backslash B\left(a,\frac{\delta_n(a)}{8C^2_{\|\cdot\|,\|\cdot\|_0}}\right)}(C\delta_n(a))^r d\mu(x)\\ &\leq \frac{1}{m_gC^r}e^r_{r;\mathrm{loc}}(X,\mathbb{R}^d;\alpha_n,W_a(\alpha_n)),\end{aligned}$$

whereof we estimate from above the second term of equation (5.4). The assertion follows with Theorem 1.2.24. □

As an immediate consequence, we derive with Lemma 5.2.1 lower bounds for the μ_r-weights of the cells $W_a(\alpha_n)$.

Theorem 5.2.3. *We have*

$$\inf_{\alpha_n\in\mathcal{C}_{n,r}(X,\mathbb{R}^d)}\ \inf_{a\in\underline{\alpha}_n(B_0(0,A)^c)}\mu_r(W_a(\alpha_n)) \succcurlyeq \frac{1}{n}, \quad n\to\infty.$$

Proof. The result follows from Theorem 5.2.2 in combination with Lemma 5.2.1. □

Another general bound we are able to derive concerns the distance between two neighbouring codes a and b in a codebook α:

Proposition 5.2.4. *There exists a constant $C(\mu, r) \in (0, \infty)$ such that*

$$\operatorname{dist}(a_n, \alpha_n \backslash \{a_n\}) \lesssim C(m_g) n^{-\frac{1}{d}} f^{-\frac{1}{r+d}}(a_n), \quad n \to \infty,$$

for all sequences $(a_n)_{n \in \mathbb{N}}$ such that $a_n \in \underline{\alpha}_n(B_0(0, A)^c)$ for codebooks $\alpha_n \in \mathcal{C}_{n,r}(X, \mathbb{R}^d)$, $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$, $a_n \in \underline{\alpha}_n(B_0(0, A)^c)$, $\alpha_n \in \mathcal{C}_{n,r}(X, \mathbb{R}^d)$ and

$$y_n \in \{x : x \in V_{a_n}(\alpha_n), \|x\|_0 \leq \inf_{z \in V_{a_n}(\alpha_n)} \|z\|_0\},$$

where the right-hand side side set is non-empty by continuity of $\|\cdot\|_0$ and compactness of $V_{a_n}(\alpha_n) \cap B_0(0, \|a_n\|_0)$. Then $\|y_n\|_0 \leq \|a_n\|_0$ and furthermore $y_n \in \partial V_{a_n}(\alpha_n)$, which implies $\operatorname{dist}(a_n, \alpha_n \backslash \{a\}) \leq 2 \operatorname{dist}(y_n, \alpha_n)$. Thus, as a consequence of the first micro-macro inequality (Proposition 2.1.5), Theorem 1.2.24 and the fact that $g(\cdot) = f(\|\cdot\|_0)$ is almost decreasing we obtain for some constant $C(r, \mu) \in (0, \infty)$ and $n \in \mathbb{N}$

$$\begin{aligned} \operatorname{dist}(a_n, \alpha_n \backslash \{a_n\}) &\leq 2 \operatorname{dist}(y_n, \alpha_n) \\ &\leq 2C(r, \mu) n^{-\frac{1}{d}} f^{-\frac{1}{r+d}}(y_n) \\ &\leq 2C(r, \mu) m_g^{-\frac{1}{r+d}} n^{-\frac{1}{d}} f^{-\frac{1}{r+d}}(a_n), \end{aligned}$$

with m_g denoting the almost decreasing constant to g. □

We will close this section with a very useful Lemma admitting a criterion to derive the (true) weak asymptotics for the local inertia and the μ and μ_r-weights for the cells $V_a(\alpha)$ and $W_a(\alpha)$.

Lemma 5.2.5. *For every sequence $(a_n)_{n \in \mathbb{N}}$ with $a_n \in \underline{\alpha}_n(B_0(0, A)^c)$ for optimal codebooks $\alpha_n \in \mathcal{C}_{n,r}(X, \mathbb{R}^d)$, $n \in \mathbb{N}$ such that*

$$f(a_n) \approx \operatorname*{ess\,sup}_{x \in V_{a_n}(\alpha_n)} f(x) \approx \operatorname*{ess\,inf}_{x \in V_{a_n}(\alpha_n)} f(x), \quad n \to \infty, \tag{5.5}$$

one has

$$\begin{aligned} \mu(V_{a_n}(\alpha_n)) \approx \mu(W_{a_n}(\alpha_n)) &\approx \frac{(f(a_n))^{\frac{r}{r+d}}}{n}, \quad n \to \infty \\ e^r_{r;\mathrm{loc}}(X, \mathbb{R}^d; \alpha_n, V_{a_n}(\alpha_n)) \approx e^r_{r;\mathrm{loc}}(X, \mathbb{R}^d; \alpha_n, W_{a_n}(\alpha_n)) &\approx n^{-\frac{r+d}{d}}, \quad n \to \infty \\ \mu_r(V_{a_n}(\alpha_n)) \approx \mu_r(W_{a_n}(\alpha_n)) &\approx \frac{1}{n}, \quad n \to \infty \\ \overline{s}_{n,r}(\mu, \mathbb{R}^d; a_n) \approx \underline{s}_{n,r}(\mu, \mathbb{R}^d; a_n) &\approx n^{-\frac{1}{d}} f^{-\frac{1}{r+d}}(a_n), \quad n \to \infty. \end{aligned}$$

Proof. We will prove the asymptotic upper bound for $\mu_r(V_{a_n}(\alpha))$ and deduce the weak asymptotics for $\mu(V_a(\alpha_n))$ from that of $\mu_r(V_{a_n}(\alpha_n))$. In virtue of the first micro-macro inequality (Proposition 2.1.5) we have

$$\begin{aligned}
\mu_r(V_{a_n}(\alpha_n)) &\leq \left\| f^{\frac{d}{d+r}} \right\|_1^{-1} \operatorname*{ess\,sup}_{x\in V_{a_n}(\alpha_n)} f^{\frac{d}{r+d}}(x)\lambda^d(V_{a_n}(\alpha_n)) \\
&\leq \left\| f^{\frac{d}{d+r}} \right\|_1^{-1} \operatorname*{ess\,sup}_{x\in V_{a_n}(\alpha_n)} f^{\frac{d}{r+d}}(x) \operatorname*{ess\,sup}_{y\in V_{a_n}(\alpha_n)} \lambda^d(B(0,\mathrm{dist}(y,\alpha_n))) \\
&\leq \left\| f^{\frac{d}{d+r}} \right\|_1^{-1} \operatorname*{ess\,sup}_{x\in V_{a_n}(\alpha_n)} f^{\frac{d}{r+d}}(x) \operatorname*{ess\,sup}_{y\in V_{a_n}(\alpha_n)} \mathrm{dist}(y,\alpha_n)^d\lambda^d(B(0,1)) \\
&\preccurlyeq \operatorname*{ess\,sup}_{x\in V_{a_n}(\alpha_n)} f^{\frac{d}{r+d}}(x)\frac{1}{n} \operatorname*{ess\,sup}_{y\in V_{a_n}(\alpha_n)} f(y)^{-\frac{d}{r+d}} \\
&\preccurlyeq \frac{1}{n}, \quad n\to\infty.
\end{aligned}$$

Hence, with Lemma 5.2.1, Theorem 5.2.2 and Theorem 5.2.3 we obtain the weak asymptotics for the local inertia and the μ_r-weights of the cells $W_a(\alpha_n)$ and $V_a(\alpha_n)$. By condition (5.5), it follows that

$$\mu_r(V_{a_n}(\alpha_n)) \approx \mu_r(W_{a_n}(\alpha_n)) \approx \frac{1}{n}, \quad n\to\infty$$

implies

$$\mu(V_{a_n}(\alpha_n)) \approx \mu(W_{a_n}(\alpha_n)) \approx \frac{(f(a_n))^{\frac{r}{r+d}}}{n}, \quad n\to\infty.$$

As for the distances $\overline{s}_{n,r}(\mu,\mathbb{R}^d;a_n)$ and $\underline{s}_{n,r}(\mu,\mathbb{R}^d;a_n)$, the upper bound is an immediate consequence of the first micro-macro inequality (Proposition 2.1.5) in combination with condition (5.5).

As for the lower bound, the second micro-macro inequality (Proposition 2.1.7) implies in virtue of the mean value Theorem

$$\begin{aligned}
n^{-(1+\frac{r}{d})} \preccurlyeq\ & r\,\mathrm{dist}(a,\alpha_n\backslash\{a_n\})\times \\
& \int_{W_{a_n}(\alpha_n)} (\mathrm{dist}(a,\alpha_n\backslash\{a_n\}) + \|x-a\|)^{r-1} d\mu(x), \quad n\to\infty.
\end{aligned}$$

By applying the first micro-macro inequality (Proposition 2.1.5) we estimate from above the right side of the integrand, which yields

$$\begin{aligned}
n^{-(1+\frac{r}{d})} \preccurlyeq\ & \mathrm{dist}(a,\alpha_n\backslash\{a_n\})\times \\
& \mu(W_{a_n}(\alpha_n))n^{-\frac{r-1}{d}} \operatorname*{ess\,sup}_{x\in W_{a_n}(\alpha_n)} f^{-\frac{r-1}{r+d}}(x), \quad n\to\infty,
\end{aligned}$$

which implies the lower bound for $\mathrm{dist}(a_n,\alpha_n\backslash\{a_n\})$, i.e.

$$n^{-\frac{1}{d}} f^{-\frac{1}{r+d}}(a_n) \preccurlyeq \mathrm{dist}(a_n,\alpha_n\backslash\{a_n\}), \quad n\to\infty. \tag{5.6}$$

□

Remark 5.2.6.
- Note, that some of the arguments being used in the previous Lemma are already contained in various results in [GLP10].
- The upper bound for the weights $\mu(V_{a_n}(\alpha_n))$ only needs the equivalence
$$f(a_n) \approx \operatorname*{ess\,sup}_{x\in V_{a_n}(\alpha_n)} f(x), \quad n\to\infty,$$
as one easily deduces from the proof of Proposition 4.3. in [GLP10].

5.3 Specific results

To estimate the weak asymptotics for the local inertia and the weights of the Voronoi cells $\mu(V_a(\alpha))$ and $\mu_r(V_a(\alpha))$ for sequences of optimal codebooks $(\alpha_n)_{n\in\mathbb{N}}$ of Lebesgue-continuous r.e's X with an unbounded support, we have to set some additional restrictions on the density f. In addition to assumptions on μ formulated at beginning of the previous section, suppose that either

1. (Type I) $-\log(g(x))$ is regularly varying at infinity with index $\theta > 0$, $-\log(g(x)) \in C_2([A,\infty))$ strictly monotone and $\big(-\log(g(x))\big)^{(i)} \in \mathcal{R}$, $i = 1,2$, or
2. (Type II) g is regularly varying at infinity with index $-a < -(r+d)$.

Note, that in both cases, given the regularity assumption on g or $\log(g)$ imply $\operatorname{supp}(\mu^{\|\cdot\|_0}) \supset [A,\infty)$. The main goal of this section is to prove

Theorem 5.3.1. *In addition to the previous assumptions, suppose that f is essentially bounded on $B_0(0,A)$ and satisfies the following local peakless condition:*
$$\exists c, s_0 > 0 \text{ such that } \forall s < s_0,\ x\in B_0(0,A) : \mu(B(x,s)) \ge cs^d.$$
Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of r-optimal n-codebooks for X in $\mathbb{R}^d$.

1. *(Type I) Let $(\eta_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$ and $(\zeta_n)_{n\in\mathbb{N}}$ defined as*
$$\begin{aligned}\zeta_n &:= (-\log(g))^{-1}\Big(\frac{r+d}{d}\log(n) + \eta_n \\ &\quad + (r+d)\log\Big(\big(((-\log(g))^{-1})'\big)\Big(\frac{r+d}{d}\log(n)\Big)\Big)\Big).\end{aligned}$$
Then, for every sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \underline{\alpha}_n(B_0(0,\zeta_n))$, $n\in\mathbb{N}$
$$\mu(V_{a_n}(\alpha_n)) \approx \mu(W_{a_n}(\alpha_n)) \approx \frac{(f(a_n))^{\frac{r}{r+d}}}{n}, \quad n\to\infty,$$
$$e^r_{r;\mathrm{loc}}(X,\mathbb{R}^d;\alpha_n,V_{a_n}(\alpha_n)) \approx e^r_{r;\mathrm{loc}}(X,\mathbb{R}^d;\alpha_n,W_{a_n}(\alpha_n)) \approx n^{-\frac{r+d}{d}}, \quad n\to\infty,$$
$$\mu_r(V_{a_n}(\alpha_n)) \approx \mu_r(W_{a_n}(\alpha_n)) \approx \frac{1}{n}, \quad n\to\infty,$$
$$\overline{s}_{n,r}(\mu,\mathbb{R}^d;a_n) \approx \underline{s}_{n,r}(\mu,\mathbb{R}^d;a_n) \approx n^{-\frac{1}{d}} f^{-\frac{1}{r+d}}(a_n), \quad n\to\infty.$$

If additionally $d = 1$, then

$$e^r_{r;\mathrm{loc}}(X, \mathbb{R}; \alpha_n, V_{a_n}(\alpha_n)) \approx e^r_{r;\mathrm{loc}}(X, \mathbb{R}; \alpha_n, W_{a_n}(\alpha_n)) \approx n^{-(r+1)}, \quad n \to \infty,$$

$$\mu_r(V_{a_n}(\alpha_n)) \approx \mu_r(W_{a_n}(\alpha_n)) \approx \frac{1}{n}, \quad n \to \infty,$$

for every sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \alpha_n$, $n \in \mathbb{N}$.

2. *(Type II) For every sequence $(a_n)_{n\in\mathbb{N}}$ with $a_n \in \alpha_n$, $n \in \mathbb{N}$*

$$\mu(V_{a_n}(\alpha_n)) \approx \mu(W_{a_n}(\alpha_n)) \approx \frac{(f(a_n))^{\frac{r}{r+d}}}{n}, \quad n \to \infty,$$

$$e^r_{r;\mathrm{loc}}(X, \mathbb{R}^d; \alpha_n, V_{a_n}(\alpha_n)) \approx e^r_{r;\mathrm{loc}}(X, \mathbb{R}^d; \alpha_n, W_{a_n}(\alpha_n)) \approx n^{-\frac{r+d}{d}}, \quad n \to \infty,$$

$$\mu_r(V_{a_n}(\alpha_n)) \approx \mu_r(W_{a_n}(\alpha_n)) \approx \frac{1}{n}, \quad n \to \infty,$$

$$\underline{s}_{n,r}(\mu, \mathbb{R}^d; a_n) \approx n^{-\frac{1}{d}} f^{-\frac{1}{r+d}}(a_n), \quad n \to \infty.$$

Remark 5.3.2. As a consequence of the general results developed in the previous section, the results concerning the asymptotic lower bounds for the local inertia and the μ_r-weights of the Voronoi cells hold without any restriction on the sequence $(a_n)_{n\in\mathbb{N}}$.

Example 5.3.3. As examples one may consider those from section 3, Examples 3.5.1 and 3.5.2.

1. (Type I) This type contains the class of hyper-exponential distributions, including the non-singular normal distribution or the multivariate exponential distributions.

2. (Type II) This type contains for example the multivariate Students t-distribution, including also the Cauchy distribution.

We come to the proof of Theorem 5.3.1.

Proof of the result We will need a few Lemmas characterizing the growth behavior of the densities f, given the different regularity assumptions (Type I) and (Type II).

Lemma 5.3.4. *Let $x_n, x_n + \Delta_n \in B_0(0, A)^c$ for all $n \in \mathbb{N}$.*

1. *(Type I) If*

$$\|\Delta_n\| = \mathcal{O}\left(\left(\left|(\log(g))'\left(\|x_n\|_0\right)\right|\right)^{-1}\right), \quad n \to \infty,$$

then

$$f(x_n) \approx f(x_n + \Delta_n), \quad n \to \infty,$$

and

$$f^{\frac{r}{r+d}}(x_n) \approx f^{\frac{r}{r+d}}(x_n + \Delta_n), \quad n \to \infty.$$

2. *(Type II) If*

$$C\,\|x_n\|_0 \gtrsim \|x_n+\Delta_n\|_0 \gtrsim \frac{1}{C}\,\|x_n\|_0\,, \quad n\to\infty$$

for some constant $C\in(0,\infty)$*, then*

$$f(x_n)\approx f(x_n+\Delta_n), \quad n\to\infty,$$

and

$$f^{\frac{r}{r+d}}(x_n)\approx f^{\frac{r}{r+d}}(x_n+\Delta_n), \quad n\to\infty.$$

Proof. 1. (Type I) By triangle inequality one has for $n\in\mathbb{N}$

$$\|x_n\|_0-\|\Delta_n\|_0\le\|x_n+\Delta_n\|_0\le\|x_n\|_0+\|\Delta_n\|_0\,.$$

Furthermore, for every $n\in\mathbb{N}$

$$\begin{aligned}\frac{f(x_n)}{f(x_n+\Delta_n)}&=\frac{\exp\big(\log(f(x_n))\big)}{\exp\big(\log(f(x_n+\Delta_n))\big)}\\&=\exp\big(\log(g(\|x_n\|_0))-\log(g(\|x_n+\Delta_n\|_0))\big)\\&=\exp\big((\log(g))'(\xi_n)\zeta_n\big),\end{aligned}$$

for some sequence $(\xi_n)_{n\in\mathbb{N}}$ in $B_{|\cdot|}(\|x_n\|_0\,,\|\Delta_n\|_0)$ and $|\zeta_n|\le\|\Delta_n\|_0\,,\ n\in\mathbb{N}$. By the assumptions on $(\Delta_n)_{n\in\mathbb{N}}$ and the fact that $(-\log(g))'$ is regularly varying, we obtain

$$\xi_n\approx\|x_n\|_0\,, \quad n\to\infty,$$

and thus

$$\big|\big(\log(g)'(\xi_n)\zeta_n\big)\big|\preccurlyeq\big|\big(\log(g)'(\|x_n\|_0)\big)\big|\,\|\Delta_n\|_0=\mathcal{O}(1), \quad n\to\infty,$$

which yields the assertion.

2. (Type II) Since $g\in\mathcal{R}_{-a}$, we estimate

$$\begin{aligned}C^{-a}\sim\frac{g(\|x_n\|_0)}{g(\frac{1}{C}\,\|x_n\|_0)}&\lesssim\frac{f(x_n)}{f(x_n+\Delta_n)}\\&\lesssim\frac{g(\|x_n\|_0)}{g(C\,\|x_n\|_0)}\sim C^a, \quad n\to\infty,\end{aligned}$$

which yields the assertion.

The second formulation is obvious. □

Proposition 5.3.5. *Let* $x_n, x_n+\Delta_n\in B_0(0,A)^c$ *for all* $n\in\mathbb{N}$.

1. *(Type I) For every bounded sequence $(\kappa_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ and $(x_n)_{n\in\mathbb{N}}$ in $B_0(0,A)^c$ satisfying*

$$\begin{aligned}\|x_n\|_0 \le (-\log(g))^{-1}\Big(\kappa_n + \frac{r+d}{d}\log(n)\\ - (r+d)\log\left(\left|\left((-\log(g))'\right)(\|x_n\|_0)\right|\right)\Big),\end{aligned}$$

and codebooks $\alpha_n \in \mathcal{C}_{n,r}(X,\mathbb{R}^d)$, $n\in\mathbb{N}$, one has

$$\operatorname{dist}(x_n,\alpha_n) = \mathcal{O}\left(\left(\left|(\log(g))'\left(\|x_n\|_0\right)\right|\right)^{-1}\right), \quad n\to\infty. \tag{5.7}$$

In particular, for every bounded sequence $(\eta_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, equation (5.7) holds for $(x_n)_{n\in\mathbb{N}}$ satisfying

$$\begin{aligned}\|x_n\|_0 \le \zeta_n &:= (-\log(g))^{-1}\Big(\frac{r+d}{d}\log(n) + \eta_n\\ &- (r+d)\log\left(\left|\left((-\log(g))'\right)\left((-\log(g))^{-1}\left(\frac{r+d}{d}\log(n)\right)\right)\right|\right)\Big)\\ &= (-\log(g))^{-1}\Big(\frac{r+d}{d}\log(n) + \eta_n\\ &+ (r+d)\log\left(\left(\left((-\log(g))^{-1}\right)'\right)\left(\frac{r+d}{d}\log(n)\right)\right)\Big).\end{aligned} \tag{5.8}$$

2. *(Type II) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $B_0(0,A)^c$ such that*

$$\|x_n\|_0 \le \kappa\rho(\alpha_n), \quad n\in\mathbb{N}$$

for some $\kappa\in(0,\infty)$ and optimal codebooks $\alpha_n\in\mathcal{C}_{n,r}(X,\mathbb{R}^d)$, $n\in\mathbb{N}$. Then, there exists a constant $\epsilon(\kappa)\in(0,1)$ such that

$$\operatorname{dist}_0(x_n,\alpha_n) \le (1-\epsilon(\kappa))\,\|x_n\|_0\,,$$

which also implies for some constant $C'\in(0,\infty)$

$$C'\,\|x_n\|_0 \ge \|a_n\|_0 \ge \frac{1}{C'}\,\|x_n\|_0$$

for all $n\in\mathbb{N}$ and every $a_n\in\alpha_n$ such that $x_n\in V_{a_n}(\alpha_n)$.

Proof. 1. In view of the first micro-macro inequality (Proposition 2.1.5) and Theorem 1.2.24, we estimate for some constant $C\in(0,\infty)$ and all $n\in\mathbb{N}$

$$\begin{aligned}\operatorname{dist}(x_n,\alpha_n) &\le C\left(\frac{n^{-(1+\frac{r}{d})}}{f(x_n)}\right)^{\frac{1}{r+d}}\\ &= Cn^{-\frac{1}{d}}\exp\left(-\frac{1}{r+d}\log(g)(\|x_n\|_0)\right)\\ &\le Cn^{-\frac{1}{d}}\exp\Big(\frac{1}{r+d}\Big(\kappa_n + \frac{r+d}{d}\log(n)\\ &- (r+d)\log\left(\left|\left((-\log(g))'\right)(\|x_n\|_0)\right|\right)\Big)\Big)\\ &= C\exp\left(\frac{\kappa_n}{r+d}\right)\left(\left|\left((-\log(g))'\right)(\|x_n\|_0)\right|\right)^{-1},\end{aligned}$$

which implies the first assertion.
Suppose now, that $\|x_n\|_0 \le \zeta_n,\ n \in \mathbb{N}$ as defined in equation (5.8) for some bounded sequence $(\eta_n)_{n\in\mathbb{N}}$. Let $\epsilon \in (0,1)$ and

$$\mathbb{N}_\epsilon := \{n \in \mathbb{N} : \|x_n\|_0 \le \epsilon\big((-\log(g))\big)^{-1}\big(\log(n)\big)\}.$$

Then

$$\begin{aligned}\|x_n\|_0 &\le (-\log(g))^{-1}\big(\frac{r+d}{d}\log(n)\big) - (1-\epsilon)(-\log(g))^{-1}\big(\frac{r+d}{d}\log(n)\big)\\ &\le (-\log(g))^{-1}\big(\frac{r+d}{d}\log(n)\big) - \big((-\log(g))^{-1}\big)'(\xi_n)\times\\ &\Big(\eta_n - (r+d)\log\Big(\big|\big((-\log(g))'\big)(\|x_n\|_0)\big|\Big)\Big)\\ &= (-\log(g))^{-1}\Big(\eta_n + \frac{r+d}{d}\log(n)\\ &- (r+d)\log\Big(\big|\big((-\log(g))'\big)(\|x_n\|_0)\big|\Big)\Big)\end{aligned}$$

for some sequence $\xi_n \sim \frac{r+d}{d}\log(n)$ and $n \in \mathbb{N}_\epsilon,\ n \ge N_1(\epsilon) \in \mathbb{N}$.
For $n \in \mathbb{N}\backslash\mathbb{N}_\epsilon$ with $n \ge N_2(\epsilon) \in \mathbb{N}$ one has

$$\epsilon(-\log(g))^{-1}\big(\log(n)\big) \le \|x_n\|_0 \le \frac{1}{\epsilon}(-\log(g))^{-1}\big(\log(n)\big).$$

Hence,

$$\begin{aligned}\|x_n\|_0 \le \zeta_n &\le (-\log(g))^{-1}\Big(\frac{r+d}{d}\log(n) + \eta_n\\ &- \eta_n' + |\theta-1|\log(\frac{1}{\epsilon}) - (r+d)\log\Big(\big|\big((-\log(g))'\big)(\|x_n\|_0)\big|\Big)\Big)\\ &,\end{aligned}$$

for some sequence $\eta_n' \to 0$. $\eta_n - \eta_n' + |\theta-1|\log(\frac{1}{\epsilon})$ is bounded and we obtain the assertion.

2. We show that the assertion holds for two types of subsequences $(x_{n_k})_{k\in\mathbb{N}}$, which will be for convenience also denoted $(x_n)_{n\in\mathbb{N}}$.
Step 1: Let $\epsilon \in (0,1)$. Let $(x_n)_{n\in\mathbb{N}}$ satisfy $\|x_n\|_0 \le h^{\leftarrow}(C(\epsilon)n^{-\frac{r+d}{d}}),\ n \in \mathbb{N}$ where $h(x) := g(x)x^{r+d}$ and a constant $C(\epsilon) \in \mathbb{R}$ specified later. Then, by the first micro-macro inequality (Proposition 2.1.5), there exists a constant

$C \in (0, \infty)$ such that

$$\begin{aligned}
\operatorname{dist}(x_n, \alpha_n) &\le C \left(\frac{n^{-(1+\frac{r}{d})}}{f(x_n)} \right)^{\frac{1}{r+d}} \\
&= C n^{-\frac{1}{d}} \frac{\|x_n\|_0}{\|x_n\|_0} \left(g(\|x_n\|_0)\right)^{-\frac{1}{r+d}} \\
&= C n^{-\frac{1}{d}} \|x_n\|_0 \left(h(\|x_n\|_0)\right)^{-\frac{1}{r+d}} \\
&\le C n^{-\frac{1}{d}} \|x_n\|_0 \left(C(\epsilon) n^{-\frac{r+d}{d}}\right)^{-\frac{1}{r+d}} \\
&= C \|x_n\|_0 \, C(\epsilon)^{-\frac{1}{r+d}}
\end{aligned}$$

for all $n \in \mathbb{N}$. Since

$$\operatorname{dist}_0(x_n, \alpha_n) \le C_{\|\cdot\|, \|\cdot\|_0} \operatorname{dist}(x_n, \alpha_n)$$

for $n \in \mathbb{N}$, we obtain the assertion by selecting $C(\epsilon)$ sufficiently large such that $C C_{\|\cdot\|, \|\cdot\|_0} C(\epsilon)^{-\frac{1}{r+d}} < 1$.

Step 2: Suppose now that the sequence $(x_n)_{n \in \mathbb{N}}$ satisfies

$$\|x_n\|_0 \ge h^{\leftarrow}(C(\epsilon) n^{-\frac{r+d}{d}}), \ n \in \mathbb{N},$$

with $C(\epsilon) \in \mathbb{R}$ such that $C C_{\|\cdot\|, \|\cdot\|_0} C(\epsilon)^{-\frac{1}{r+d}} = \frac{1}{2}$. Then, by Step 1, there exist codes $b_n \in \alpha_n$ such that

$$\|b_n - x'_n\|_0 \le \frac{1}{2} \|x'_n\|_0 ,$$

where

$$x'_n = \frac{x_n}{\|x_n\|_0} h^{\leftarrow}(C(\epsilon) n^{-\frac{r+d}{d}}).$$

Therefore

$$\begin{aligned}
\operatorname{dist}_0(x_n, \alpha_n) \le \operatorname{dist}_0(x_n, b_n) &\le \|x_n - x'_n\|_0 + \|b_n - x'_n\|_0 \\
&\le \|x_n\|_0 \left(1 - \frac{1}{\|x_n\|_0} h^{\leftarrow}(C(\epsilon) n^{-\frac{r+d}{d}}) \right) + \frac{1}{2} \|x'_n\|_0 \\
&= \|x_n\|_0 \left(1 - \frac{1}{2\|x_n\|_0} h^{\leftarrow}(C(\epsilon) n^{-\frac{r+d}{d}}) \right),
\end{aligned}$$

which yields the assertion since, by the assumption for some $\kappa < \infty$

$$\|x_n\|_0 \lesssim \kappa \rho(\alpha_n) \approx h^{\leftarrow}(C(\epsilon) n^{-\frac{r+d}{d}}), \quad n \to \infty.$$

$\square$

Unfortunately, at least for sequences of codes $(a_n)_{n\in\mathbb{N}}$ with unbounded Voronoi cells $V_{a_n}(\alpha_n)$, it seems to be impossible to satisfy condition (5.5), independently of the shape of the density f.
For densities of Type II, as a consequence of the following Proposition, we will be able to cope, for every constant $C > 0$, with sequences of cells $(V_{a_n}(\alpha_n))_{n\in\mathbb{N}}$ satisfying $V_{a_n}(\alpha_n) \subset C\rho(\alpha_n)B(0,1),\ n \in \mathbb{N}$. This will be sufficient to cover the outer cells of the codebook α_n. As for the densities of Type I, a similar result can only be achieved under some additional restrictions.

Lemma 5.3.6. *Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of r-optimal n-quantizers for X in $\mathbb{R}^d$.*

1. *(Type I) One has*
$$\mu_r(B_0(0,y_n)^c) \preccurlyeq \frac{1}{n}, \quad n \to \infty$$
for every sequence $(y_n)_{n\in\mathbb{N}}$ such that
$$y_n \geq (-\log(g))^{-1}\Big(\frac{r+d}{d}\log(n) + \kappa_n \\ + (r+d)\log(y_n) - \frac{r+d}{d}\log((-\log(g))(y_n))\Big)$$
for some arbitrary bounded sequence $(\kappa_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$. In particular, this is true for
$$y_n \geq \zeta_n' := (-\log(g))^{-1}\Big(\frac{r+d}{d}\log(n) + \kappa_n \\ + (r+d)\log((-\log(g))^{-1}(\frac{r+d}{d}\log(n)))- \\ \frac{r+d}{d}\log((\frac{r+d}{d}\log(n))))\Big).$$

2. *(Type II) For every $C > 0$ there exists a constant $C' < \infty$ such that for every sequence $(a_n)_{n\in\mathbb{N}}$ satisfying $a_n \in \alpha_n,\ n \in \mathbb{N}$*
$$\mu(B_0(0,C\rho(\alpha_n))^c) \lesssim \frac{C'}{n} f(a_n)^{\frac{r}{r+d}}, \quad n \to \infty,$$
as well as
$$\mu_r(B_0(0,C\rho(\alpha_n))^c) \lesssim C'\frac{1}{n}, \quad n \to \infty.$$
Conversely, for every $C' < \infty$ there exists a constant $C > 0$ such that
$$\mu(B_0(0,C\rho(\alpha_n))^c) \lesssim \frac{C'}{n} f(a_n)^{\frac{r}{r+d}}, \quad n \to \infty,$$
and
$$\mu_r(B_0(0,C\rho(\alpha_n))^c) \lesssim C'\frac{1}{n}, \quad n \to \infty.$$

Proof. 1. (Type I) In virtue of Proposition A.9 we estimate for $x \geq A$

$$\begin{aligned}
\mu_r(B_0(0,x)^c) &= \left\| f^{\frac{d}{r+d}} \right\|_1^{-1} \int_{B_0(0,x)^c} f^{\frac{d}{r+d}}(t) d\lambda^d(t) \\
&= \left\| f^{\frac{d}{r+d}} \right\|_1^{-1} C \int_x^\infty t^{d-1} g^{\frac{d}{r+d}}(t) d\lambda(t) \\
&\approx x^{d-1} g^{\frac{d}{r+d}}(x) \frac{1}{(-\log(g))'(x)} \\
&\approx x^d g^{\frac{d}{r+d}}(x) \frac{1}{(-\log(g))(x)}, \quad x \to \infty,
\end{aligned}$$

for some constant $C \in (0,\infty)$. Hence

$$\begin{aligned}
&\mu_r(B_0(0,y_n)^c) \preccurlyeq \frac{1}{n}, \quad n \to \infty \\
\Longleftrightarrow & -d\log(y_n) - \frac{d}{r+d}\log(g(y_n)) + \log\big((-\log(g))(y_n)\big) \\
&\geq \log(n) + \mathcal{O}(1) \\
\Longleftrightarrow & y_n \geq (-\log(g))^{-1}\Big(\frac{r+d}{d}\log(n) + \mathcal{O}(1) \\
&+ (r+d)\log(y_n) - \frac{r+d}{d}\log\big((-\log(g))(y_n)\big)\Big).
\end{aligned}$$

By an argumentation as in the proof of Proposition 5.3.5 (Type I), the result holds true for $y_n \geq \zeta_n'$, $n \in \mathbb{N}$.

2. (Type II) Let $C > 0$. For every sequence $(a_n)_{n\in\mathbb{N}}$ such that $a_n \in \alpha_n \cap B_0(0, C\rho(\alpha_n))^c$, $n \in \mathbb{N}$ one has

$$\begin{aligned}
f^{\frac{r}{r+d}}(a_n) &\geq g^{\frac{r}{r+d}}(\|a_n\|_0) \gtrsim (CC_{\|\cdot\|,\|\cdot\|_0})^{\frac{-ar}{r+d}} g^{\frac{r}{r+d}}(\rho(\alpha_n)) \\
&\approx \frac{\rho(\alpha_n)^r}{\rho(\alpha_n)^r} g^{\frac{r}{r+d}}(\rho(\alpha_n)) \\
&\approx \frac{1}{\rho(\alpha_n)^r} h^{\frac{r}{r+d}}(\rho(\alpha_n)), \quad n \to \infty,
\end{aligned}$$

where $h(x) := x^{r+d} g(x)$. Thus, by Theorem 3.2.1 (Type II)

$$f^{\frac{r}{r+d}}(a_n) \gtrsim \kappa n^{-\frac{r}{d}} \frac{1}{\rho(\alpha_n)^r}, \quad n \to \infty,$$

for some $\kappa \in (0,1)$. Moreover, we have

$$\begin{aligned}
\int_{B_0(0,C\rho(\alpha_n))^c} f(x) d\lambda^d(x) &= \int_{C\rho(\alpha_n)}^\infty \int_{\partial B_0(0,y)} g(y) d\lambda(y) \\
&\approx \int_{C\rho(\alpha_n)}^\infty \int_{\partial B_0(0,1)} y^{d-1} g(y) d\lambda(y) \qquad (5.9) \\
&\approx \frac{1}{\rho(\alpha_n)^r} n^{-\frac{r+d}{d}}, \quad n \to \infty.
\end{aligned}$$

Thus
$$\int_{B_0(0,C\rho(\alpha_n))^c} f(x)d\lambda^d(x) \preccurlyeq f^{\frac{r}{r+d}}(a_n)\frac{1}{n}, \quad n \to \infty,$$
which yields the upper bound for the weights of the outer region. As for the μ_r-weights, one estimates

$$\begin{aligned}
\mu_r(V_{a_n}(\alpha_n)) &\preccurlyeq \kappa \int_{B_0(0,C\rho(\alpha_n))^c} f^{\frac{d}{r+d}}(x)d\lambda^d(x) \\
&\preccurlyeq \int_{C\rho(\alpha_n)}^{\infty} x^{d-1} g^{\frac{d}{r+d}}(x)d\lambda(x) \\
&\approx h^{\frac{d}{r+d}}(\rho(\alpha_n)) \\
&\approx \frac{1}{n}, \quad n \to \infty.
\end{aligned}$$

The converse result for $\mu(B_0(0,C\rho(\alpha_n))^c)$ and $\mu_r(B_0(0,C\rho(\alpha_n))^c)$ follows analogously.

□

Proof of Theorem 5.3.1. The idea of the proof is as follows: The inner cells are covered by Theorem 5.1.1. As for the outer cells, each one is divided into an inner region and an outer region. The inner region will be covered by Lemma 5.2.5 in combination with Lemmas 5.3.4 and 5.3.5, the outer area will be covered, as far as possible, by Lemma 5.3.6.
Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence of r-optimal n-quantizers for X in $\mathbb{R}^d$ and $a_n \in \alpha_n$, $n \in \mathbb{N}$. By Theorem 5.1.1, the results hold true for (sub)-sequences $(a_{n_k})_{k\in\mathbb{N}}$ in $B_0(0,A)$. Hence, we have to prove the results for subsequence $(a_{n_k})_{k\in\mathbb{N}}$ such that $V_{a_{n_k}}(\alpha_{n_k}) \subset B_0(0,A)^c$. For convenience, we will denote this subsequence also $(a_n)_{n\in\mathbb{N}}$.

1. (Type I) The general results independent of the dimension $d \in \mathbb{N}$ are a consequence of Lemma 5.2.5 and Proposition 5.3.5 (Type I). For the specific result in case $d = 1$, we have to prove the upper bound for the μ_r-weights of the Voronoi cells. In fact, in this case there exist bounded sequences $(\eta_n)_{n\in\mathbb{N}}$ and $(\kappa_n)_{n\in\mathbb{N}}$ such that
$$\zeta_n' \leq \zeta_n, \quad n \in \mathbb{N},$$
where $(\zeta_n)_{n\in\mathbb{N}}$ as in Proposition 5.3.5, (Type I) and $(\zeta_n')_{n\in\mathbb{N}}$ as in Lemma 5.3.6 (Type I). To observe this, one estimates in virtue of the regularity

assumption on $(-\log(g))'$

$$\begin{aligned}\zeta_n' &= (-\log(g))^{-1}\Big(\frac{r+d}{d}\log(n)+\kappa_n\\ &\quad+(r+d)\log((-\log(g))^{-1}(\frac{r+d}{d}\log(n)))-\\ &\quad\frac{r+d}{d}\log((\frac{r+d}{d}\log(n))))\Big)\\ &\le (-\log(g))^{-1}\Big(\frac{r+d}{d}\log(n)+\eta_n\\ &\quad+(r+d)\log\Big(\big(((-\log(g))^{-1})'\big)\Big(\frac{r+d}{d}\log(n)\Big)\Big)\Big)=\zeta_n\end{aligned}$$

$\iff$

$$\begin{aligned}&\kappa_n+(r+d)\log((-\log(g))^{-1}(\frac{r+d}{d}\log(n)))-\frac{r+d}{d}\log((\frac{r+d}{d}\log(n))))\\ &\quad\le \eta_n+(r+d)\log\Big(\big(((-\log(g))^{-1})'\big)\Big(\frac{r+d}{d}\log(n)\Big)\Big)\end{aligned}$$

$\iff$

$$\begin{aligned}&\kappa_n-\frac{r+d}{d}\log\big((\frac{r+d}{d}\log(n))\big)\\ &\quad\lesssim \eta_n-(r+d)\log\Big(\frac{r+d}{d}\log(n)\Big),\quad n\to\infty,\end{aligned}$$

which is true for some bounded sequences $(\kappa_n)_{n\in\mathbb{N}}$ and $(\eta_n)_{n\in\mathbb{N}}$ in case $d=1$. We denote $V^i_{a_n}(\alpha_n):=V_{a_n}(\alpha_n)\cap B_0(0,\zeta_n)$ and $V^o_{a_n}(\alpha_n):=V_{a_n}(\alpha_n)\setminus V^i_{a_n}(\alpha_n)$. From Proposition 5.3.5 and Lemma 5.3.4 we deduce

$$\operatorname*{ess\,inf}_{x\in V^i_{a_n}(\alpha_n)} f(x)\approx f(a_n)\approx \operatorname*{ess\,sup}_{x\in V^i_{a_n}(\alpha_n)} f(x),\quad n\to\infty$$

for all sequences $(a_n)_{n\in\mathbb{N}}$ in $(\alpha_n)_{n\in\mathbb{N}}$. By the first micro-macro inequality (Proposition 2.1.5), Lemma 5.3.6 (Type I) and the fact that $\zeta_n'\le\zeta_n$, $n\in\mathbb{N}$ we derive

$$\begin{aligned}\mu_r(V_{a_n}(\alpha_n)) &\preccurlyeq \operatorname*{ess\,sup}_{x\in V^i_{a_n}(\alpha_n)} f^{\frac{d}{r+d}}(x)\lambda^d(V^i_{a_n}(\alpha_n))+\frac{1}{n}\\ &\le \operatorname*{ess\,sup}_{x\in V^i_{a_n}(\alpha_n)} f^{\frac{d}{r+d}}(x)\operatorname*{ess\,sup}_{y\in V^i_{a_n}(\alpha_n)}\lambda^d(B(0,\operatorname{dist}(y,\alpha_n)))+\frac{1}{n}\\ &\le \operatorname*{ess\,sup}_{x\in V^i_{a_n}(\alpha_n)} f^{\frac{d}{r+d}}(x)\operatorname*{ess\,sup}_{y\in V^i_{a_n}(\alpha_n)}\operatorname{dist}(y,\alpha_n)^d\lambda^d(B(0,1))+\frac{1}{n}\\ &\lesssim \operatorname*{ess\,sup}_{x\in V^i_{a_n}(\alpha_n)} f^{\frac{d}{r+d}}(x)\frac{1}{n}\operatorname*{ess\,sup}_{y\in V^i_{a_n}(\alpha_n)} f(y)^{-\frac{d}{r+d}}\lambda^d(B(0,1))+\frac{1}{n}\\ &\preccurlyeq \frac{1}{n},\quad n\to\infty.\end{aligned}$$

The upper bound for the local inertia we deduce with Lemma 5.2.1, the lower bounds we deduce from Theorems 5.2.2 and 5.2.3.

2. (Type II) Let $C \in (0,\infty)$. We denote $V^i_{a_n}(\alpha_n) := V_{a_n}(\alpha_n) \cap B_0(0, C\rho_n)$ and $V^o_{a_n}(\alpha_n) := V_{a_n}(\alpha_n) \backslash V^i_{a_n}(\alpha_n)$ for all $n \in \mathbb{N}$. From Proposition 5.3.5 and Lemma 5.3.4 we deduce

$$\operatorname*{ess\,inf}_{x\in V^i_{a_n}(\alpha_n)} f(x) \approx f(a_n) \approx \operatorname*{ess\,sup}_{x\in V^i_{a_n}(\alpha_n)} f(x), \quad n \to \infty$$

for all sequences $(a_n)_{n\in\mathbb{N}}$ such that $a_n \in \alpha_n$, $n \in \mathbb{N}$. We obtain by using the first micro-macro inequality (Proposition 2.1.5) and Lemma 5.3.6 (Type II)

$$\begin{aligned}
\mu(V_{a_n}(\alpha_n)) &\lesssim \operatorname*{ess\,sup}_{x\in V^i_{a_n}(\alpha_n)} f(x)\lambda^d(V^i_{a_n}(\alpha_n)) + f^{\frac{r}{r+d}}(a_n)\frac{1}{n} \\
&\le \operatorname*{ess\,sup}_{x\in V^i_{a_n}(\alpha_n)} f(x) \operatorname*{ess\,sup}_{y\in V^i_{a_n}(\alpha_n)} \lambda^d(B(0,\operatorname{dist}(y,\alpha_n))) + f^{\frac{r}{r+d}}(a_n)\frac{1}{n} \\
&\lesssim \operatorname*{ess\,sup}_{x\in V^i_{a_n}(\alpha_n)} f(x) \operatorname*{ess\,sup}_{y\in V^i_{a_n}(\alpha_n)} \operatorname{dist}(y,\alpha_n)^d\lambda^d(B(0,1)) + f^{\frac{r}{r+d}}(a_n)\frac{1}{n} \\
&\lesssim \operatorname*{ess\,sup}_{x\in V^i_{a_n}(\alpha_n)} f(x)\frac{1}{n} \operatorname*{ess\,sup}_{y\in V^i_{a_n}(\alpha_n)} f(y)^{-\frac{d}{r+d}}\lambda^d(B(0,1)) + f^{\frac{r}{r+d}}(a_n)\frac{1}{n} \\
&\lesssim C_2 f^{\frac{r}{r+d}}(a_n)\frac{1}{n}, \quad n \to \infty.
\end{aligned}$$

Analogously, one shows

$$\begin{aligned}
\mu_r(V_{a_n}(\alpha_n)) &\lesssim \operatorname*{ess\,sup}_{x\in V^i_{a_n}(\alpha_n)} f^{\frac{d}{r+d}}(x)\lambda^d(V^i_{a_n}(\alpha_n)) + \frac{1}{n} \\
&\le \operatorname*{ess\,sup}_{x\in V^i_{a_n}(\alpha_n)} f^{\frac{d}{r+d}}(x) \operatorname*{ess\,sup}_{y\in V^i_{a_n}(\alpha_n)} \lambda^d(B(0,\operatorname{dist}(y,\alpha_n))) + \frac{1}{n} \\
&\le \operatorname*{ess\,sup}_{x\in V^i_{a_n}(\alpha_n)} f^{\frac{d}{r+d}}(x) \operatorname*{ess\,sup}_{y\in V^i_{a_n}(\alpha_n)} \operatorname{dist}(y,\alpha_n)^d\lambda^d(B(0,1)) + \frac{1}{n} \\
&\lesssim \operatorname*{ess\,sup}_{x\in V^i_{a_n}(\alpha_n)} f^{\frac{d}{r+d}}(x)\frac{1}{n} \operatorname*{ess\,sup}_{y\in V^i_{a_n}(\alpha_n)} f(y)^{-\frac{d}{r+d}}\lambda^d(B(0,1)) + \frac{1}{n} \\
&\lesssim \frac{1}{n}, \quad n \to \infty.
\end{aligned}$$

The upper bound for the local inertia we deduce with Lemma 5.2.1.
We come to the lower bound for the μ-weights of the Voronoi cells. Applying the first and the second micro-macro-inequality (Propositions 2.1.5

and 2.1.7), we obtain for some constant $\kappa \in (0,\infty)$

$$
\begin{aligned}
\kappa n^{-(1+\frac{r}{d})} &\leq \int_{V_{a_n}(\alpha_n)} \operatorname{dist}(x,\alpha_n\backslash\{a_n\})^r d\mu(x) \\
&\leq \int_{V^0_{a_n}(\alpha_n)} \operatorname{dist}(x,\alpha_n\backslash\{a_n\})^r d\mu(x) + \int_{V^i_{a_n}(\alpha_n)} \operatorname{dist}(x,\alpha_n\backslash\{a_n\})^r d\mu(x) \\
&\leq n^{-\frac{r}{d}} \int_{V^0_{a_n}(\alpha_n)} f^{1-\frac{r}{r+d}}(x) d\lambda^d(x) + \int_{V^i_{a_n}(\alpha_n)} \operatorname{dist}(x,\alpha_n\backslash\{a_n\})^r d\mu(x) \\
&\leq \kappa_1 n^{-(1+\frac{r}{d})} + \int_{V^i_{a_n}(\alpha_n)} \operatorname{dist}(x,\alpha_n\backslash\{a_n\})^r d\mathbb{P}^X(x), \quad n\to\infty,
\end{aligned}
\tag{5.10}
$$

for some constant $\kappa_1 < \kappa$ given that $C \in (0,\infty)$ sufficiently large. Thus, there exists $\kappa_2 \in (0,\infty)$ such that

$$
\kappa_2 n^{-(1+\frac{r}{d})} \leq \int_{V^i_{a_n}(\alpha_n)} \operatorname{dist}(x,\alpha_n\backslash\{a_n\})^r d\mathbb{P}^X(x), \quad n\to\infty.
$$

By definition of a Voronoi cell, and by selecting C sufficiently large such that $\alpha_n \subset C\rho_n\mathcal{B}_0(0,1),\ n\in\mathbb{N}$, we derive for $x \in V^i_{a_n}(\alpha_n)$

$$
\begin{aligned}
\operatorname{dist}(x,\alpha_n\backslash\{a_n\}) &\leq \|x-a_n\| + \operatorname{dist}(a_n,\alpha_n\backslash\{a_n\}) \\
&\leq 3 \sup_{z\in V^i_{a_n}(\alpha_n)} \operatorname{dist}(z,\alpha_n).
\end{aligned}
$$

Combining this with the first micro-macro inequality (Proposition 2.1.5), we obtain

$$
\begin{aligned}
\kappa n^{-(1+\frac{r}{d})} &\leq 3C\mu(V^i_{a_n}(\alpha_n)) \operatorname*{ess\,sup}_{x\in V^i_{a_n}(\alpha_n)} f^{-\frac{r}{r+d}}(x) \\
&\lesssim \mu(V^i_{a_n}(\alpha_n)) f^{-\frac{r}{r+d}}(a_n), \quad n\to\infty,
\end{aligned}
$$

which implies the lower bound for the weight of the Voronoi cell. The lower bound for the quantization error and the μ_r-weight is an immediate consequence of Theorems 5.2.2 and 5.2.3. The asymptotic lower bound for $\operatorname{dist}(a,\alpha_n\backslash\{a\})$ is a consequence of Lemma 5.2.4. The upper bound follows with Lemma 5.2.5 for (sub)sequences $(a_n)_{n\in\mathbb{N}}$ in $B_0(0,\epsilon\rho(\alpha_n))$ and $\epsilon \in (0,1)$ sufficiently small, for (sub)sequences $(a_n)_{n\in\mathbb{N}}$ in $B_0(0,\epsilon\rho(\alpha_n))^c$ the result is obvious.

□

Chapter 6

Constructive quantization of stochastic processes

In this chapter, we want to discuss practical approaches for the construction of optimal quantizers for r.e.'s in infinite dimensional Banach spaces. Here, one is strongly interested in a constructive approach that allows to implement an explicit coding strategy and to compute (at least numerically) good codebooks.

In the finite dimensional setting, the algorithms and procedures presented in section 1.2, paragraph "construction of optimal quantizers" allow reasonable approaches to calculate optimal quantizers with a limited amount of effort. Without further ado, the methods become intractable when considering r.e.'s in infinite dimensional Banach spaces. One famous and fruitful method to overcome this problem is to reduce the complexity of the r.e. X Banach space E, in order to

- use the algorithms developed for the finite dimensional setting to calculate optimal quantizers for simplified r.e's $\phi_n(X) \in F_n \subset E$ a.s., where $\dim(F_n) < \infty$, and additionally
- to control the approximation error $\mathbb{E}\,\|X - \phi_n(X)\|^r$.

Considering e.g. Gaussian r.e.'s in a Hilbert space setting, the proof of Theorem 1.2.12 shows us how to construct asymptotically r-optimal n-quantizers for these processes, which means sequences of n-quantizers $\alpha_n,\ n \in \mathbb{N}$ satisfying

$$e_r(X, E, \alpha_n) \sim e_{n,r}(X, E), \quad n \to \infty. \tag{6.1}$$

These quantizers can be constructed by reducing the quantization problem to a quantization problem of a finite dimensional normal distributed random variable. Even if there are almost no explicit formulas known for optimal codebooks in finite dimensions, the existence is guaranteed and there exist a lot of deterministic and stochastic numerical algorithms to compute optimal codebooks, as the CLVQ- algorithms or the Lloyd-I procedures described in section 1.2. Unfortunately, one needs to know explicitly the eigenvalues and eigenvectors of the covariance operator C_X to pursue this approach.

If we consider other non-Hilbertian function spaces $(E, \|\cdot\|)$ or non-Gaussian random variables in an infinite dimensional Hilbert space, there is much less known on how to construct asymptotically optimal quantizers. Most approaches to calculate the asymptotics of the quantization error are either non-constructive (e.g. [Cre02], [GLP03]), are tailored to one specific process type (e.g. [Der08b], [Der08a] and [DS06]) or the constructed quantizers do not achieve the sharp rate in the sense of (6.1) (e.g. [LP08] or [Wil08]) but just the weak rate

$$e_r(X, E, \alpha_n) \approx e_{n,r}(X, E), \quad n \to \infty. \tag{6.2}$$

In this chapter, we develop a constructive approach to calculate sequences of asymptotically r-optimal n-quantizers (in the sense of (6.1)) for a broad class of random variables in infinite dimensional Banach spaces. *Constructive* means in this case that we reduce the quantization problem to the quantization problem of a $\mathbb{R}^d$-valued random variable, that can be solved numerically. This approach can either be used in Hilbert spaces in case the eigenvalues and eigenvectors of the covariance operator of a Gaussian random variable are unknown, or for quantization problems in different Banach spaces. Furthermore, we discuss Gaussian random variables in $\left(C(0,1), \|\cdot\|_{L_\infty}\right)$. This part is related to Wilbertz's PhD thesis ([Wil08]). More precisely, we sharpen his constructive results by showing that the quantizers constructed in the thesis also achieve the sharp rate for the asymptotic quantization error (in the sense of (6.1)). Moreover, we can show that the dimensions of the subspaces wherein these quantizers are contained can be lessened without loosing the sharp asymptotics property. Additionally, we use some ideas by Luschgy and Pagès ([LP08]) and develop for Gaussian random variables and for a broad class of Lévy processes asymptotically optimal quantizers in the Banach space $\left(L_p([0,1]), \|\cdot\|_{L_p}\right)$.

It is worth mentioning that all these quantizers can be constructed without knowing the true rate of the quantization error. This means precisely: If we know a (rough) lower bound for the quantization error, i.e. $e_{n,r}(X, E) \gtrsim C_1 \log(n)^{-b_1}$ and the true but unknown rate is $e_{n,r}(X, E) \sim C_2 \log(n)^{-b_2}$ for constants $C_1, C_2, b_1, b_2 \in (0, \infty)$, then, we are able to construct a sequence of n-quantizers $\alpha_n, n \in \mathbb{N}$ that satisfies

$$e_r(X, E, \alpha_n) \sim e_{n,r}(X, E) \sim C_2 \log(n)^{-b_2}, \quad n \to \infty \tag{6.3}$$

for the optimal, but still unknown constants C_2, b_2.

The crucial factors for the numerical implementation are the dimensions of the subspaces, wherein the asymptotically optimal quantizers are contained. We will calculate the dimensions of the subspaces obtained through our approach and we will see that for all analyzed Gaussian processes, and also for many Lévy processes, we are very close to the known asymptotics of the optimal dimension in the case of Gaussian processes in infinite dimensional Hilbert spaces.

Finally, we will present some important examples of Gaussian and Lévy processes and illustrate some of our results.

Notes and references: The results presented in this chapter have already been published, see [JL10].

6.1 The main Theorem

We assume throughout that $\left|\text{supp}(\mathbb{P}^X)\right| = \infty$. To formulate the main result, we need for an infinite subset $J \subset \mathbb{N}$ the following:

1. There exist linear operators $V_m : E \to F_m \subset E$ for $m \in J$ with $\|V_m\| \leq 1$, for finite dimensional subspaces F_m with $dim(F_m) = m$, where the operator norm $\|\cdot\|$ is defined as
$$\|V_m\| := \sup_{x\in E, \|x\|\leq 1} \|V_m(x)\|.$$
2. There exist linear isometric and onto operators $\phi_m : (F_m, \|\cdot\|) \to (\mathbb{R}^m, |\cdot|_m)$ with suitable norms $|\cdot|_m$ in $\mathbb{R}^m$ for all $m \in J$.
3. There exist random variables Z_m for $m \in J$ in E with $Z_m \stackrel{d}{=} X$, such that for the approximation error $\|\|X - V_m(Z_m)\|\|_{L_r(\mathbb{P},E)}$ holds
$$\|\|X - V_m(Z_m)\|\|_{L_r(\mathbb{P},E)} \longrightarrow 0,$$
as $m \to \infty$ along J.

Remark 6.1.1. The crucial point in condition (1) is the norm one restriction for the operators V_m. Condition (2) becomes important when constructing the quantizers in $\mathbb{R}^m$ equipped with some well known norm. As we will see in the proof of the subsequent Theorem, in order to show asymptotic optimality of a constructed sequence of quantizers one needs to know only a rough lower bound for the asymptotic quantization error. In fact, this lower bound allows us in combination with condition (3) to choose explicitly a sequence $m(n) \in J, n \in \mathbb{N}$ such that

$$\left\|\left\|X - V_{m(n)}(Z_{m(n)})\right\|\right\|_{L_r(\mathbb{P},E)} = o(e_{n,r}(X,E)) \quad , n \to \infty. \tag{6.4}$$

Theorem 6.1.2. *Assume that the conditions* (1) – (3) *hold for some infinite subset $J \subset \mathbb{N}$. We choose a sequence $(m(n))_{n\in\mathbb{N}} \in J^{\mathbb{N}}$ such that* (6.4) *is satisfied. For $n \in \mathbb{N}$, let α_n be an r-optimal n-quantizer for $\xi_n := \phi_{m(n)}(V_{m(n)}(Z_{m(n)}))$ in $(\mathbb{R}^{m(n)}, |\cdot|_{m(n)})$.*

Then, $\left(\phi^{-1}_{m(n)}(\alpha_n)\right)_{n\in\mathbb{N}}$ is an asymptotically r-optimal sequence of n-quantizers for X in E and

$$e_{n,r}(X,E) \sim \left(\mathbb{E}\left\|X - f_{\phi^{-1}_{m(n)}(\alpha_n)}(V_{m(n)}(Z_{m(n)}))\right\|^r\right)^{\frac{1}{r}} \sim e_r(X, E, \phi^{-1}_{m(n)}(\alpha_n)),$$

as $n \to \infty$.

Remark 6.1.3. Note, that for $n \in \mathbb{N}$ there always exist r-optimal n-quantizers for ξ_n ([GL00, Theorem 4.12]).

Proof. Using condition (3), and the fact that $e_{n,r}(X,E) > 0$ for all $n \in \mathbb{N}$ since $\left|\text{supp}(\mathbb{P}^X)\right| = \infty$, we can choose a sequence $(m(n))_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ fulfilling (6.4). Using Lemma 1.2.5 and condition(2), we see that $\phi^{-1}_{m(n)}(\alpha_n)$ is an r-optimal n-quantizer for $V_{m(n)}(Z_{m(n)})$ in $F_{m(n)}$. Then, by using condition(1), (6.4) and Lemma 1.2.5 we obtain

$$\begin{aligned}
e_{n,r}(X,E) &\le \left(\mathbb{E}\left\|X - f_{\phi^{-1}_{m(n)}(\alpha_n)}(V_{m(n)}(Z_{m(n)}))\right\|^r\right)^{\frac{1}{r}} \le \left(\mathbb{E}\left\|X - V_{m(n)}(Z_{m(n)})\right\|^r\right)^{\frac{1}{r}} \\
&+ \left(\mathbb{E}\left\|V_{m(n)}(Z_{m(n)}) - f_{\phi^{-1}_{m(n)}(\alpha_n)}(V_{m(n)}(Z_{m(n)}))\right\|^r\right)^{\frac{1}{r}} \\
&= \left(\mathbb{E}\left\|X - V_{m(n)}(Z_{m(n)})\right\|^r\right)^{\frac{1}{r}} + e_{n,r}(V_{m(n)}(Z_{m(n)}), (F_{m(n)}, \|\cdot\|)) \\
&\le \left(\mathbb{E}\left\|X - V_{m(n)}(Z_{m(n)})\right\|^r\right)^{\frac{1}{r}} + e_{n,r}(Z_{m(n)}, E) \\
&= \left(\mathbb{E}\left\|X - V_{m(n)}(Z_{m(n)})\right\|^r\right)^{\frac{1}{r}} + e_{n,r}(X, E) \\
&\sim e_{n,r}(X,E), \quad n \to \infty.
\end{aligned}$$

□

Remark 6.1.4. We will usually choose $Z_m = X$ for all $m \in \mathbb{N}$, with one exception in the following section and $J = \mathbb{N}$.

Remark 6.1.5. The crucial factor for the numerical implementation of the procedure are the dimensions $(m(n))_{n\in\mathbb{N}}$ of the subspaces $(F_{m(n)})_{n\in\mathbb{N}}$. For the well known case of the Brownian motion in the Hilbert space $H = L_2([0,1])$ it is known that this dimension sequence can be chosen as $m(n) \approx log(n)$, $n \to \infty$. In the following examples we will see that we can often obtain similar orders like $log(n)^c$ for constants c just slightly higher than one.

We point out that there is a non-asymptotic version of Theorem 6.1.2 for nearly optimal n-quantizers, that is for n-quantizers which are optimal up to $\epsilon > 0$. Its proof is analogously to the proof of Theorem 6.1.2.

Proposition 6.1.6. *Assume that conditions* (1) − (3) *hold. Let* $m(\epsilon) := \inf\{m \in \mathbb{N} : \|X - V_m(Z_m)\|_{L_r(\mathbb{P})} < \epsilon\}$ *and* $\xi_n := \phi_{m(\epsilon)}\left(V_{m(\epsilon)}(Z_{m(\epsilon)})\right)$ *for* $n \in \mathbb{N}$. *Then*

$$e_r(X, E, f_{\phi^{-1}_{m(\epsilon)}(\alpha_n)}(V_{m(\epsilon)}(Z_{m(\epsilon)}))) \le e_{n,r}(X,E) + \epsilon,$$

for every $n \in \mathbb{N}$ *and for every* r*-optimal* n*-quantizer* α_n *for* ξ_n *in* $\left(\mathbb{R}^{m(\epsilon)}, |\cdot|_{m(\epsilon)}\right)$.

6.2 Applications

The key criterion for the usability of the results of the previous section is the question of the existence of specific triples (F_m, V_m, ϕ_m) admitting the required assumptions of Theorem 6.1.2. As we will observe hereafter, there are several interesting cases, in which we will be able to pursue this approach.

Hilbertian path spaces Let X be a centered Gaussian random variable in the separable Hilbert space $(H, \langle\cdot,\cdot\rangle)$. Following the approach used in the proof of Theorem 1.2.15, we have for every sequence $(\xi_i)_{i\in\mathbb{N}}$ of independent $N(0,1)$-distributed random variables

$$X \stackrel{d}{=} \sum_{i=1}^{\infty} \sqrt{\lambda_i} f_i \xi_i, \tag{6.5}$$

where λ_i denote the eigenvalues and f_i denote the corresponding orthonormal eigenvectors of the covariance operator C_X of X. If these parameters are known, we can choose a sequence $(d_n)_{n\in\mathbb{N}}$ such that a sequence of optimal quantizer α_n for $X_n = \sum_{i=1}^{d_n} \sqrt{\lambda_i} f_i \xi_i$ is asymptotically optimal for X in E.
In order to construct asymptotically optimal quantizers for Gaussian random variables with unknown eigenvalues or eigenvectors of the covariance operator, we start with more general expansions. In fact, we just need one of the two orthogonalities, either in $L_2(\mathbb{P}, E)$ or in H.

1. Let $(h_i)_{i\in\mathbb{N}}$ be an orthonormal basis of H. Then

   $$X = \sum_{i=1}^{\infty} h_i \langle h_i, X\rangle \quad a.s. \; .$$

   Compared to (6.5) we see that $\langle h_i, X\rangle$ are still Gaussian, but generally not independent.

2. Let $(g_j)_{j\in\mathbb{N}}$ be an admissible sequence for X in H such that

   $$X \stackrel{d}{=} \sum_{i=1}^{\infty} \xi_i g_i.$$

   Compared to (6.5) the sequence $(g_i)_{i\in\mathbb{N}}$ is generally not orthogonal.

Before we will use these representations for X to find suitable triples (V_m, F_m, ϕ_m) as in Theorem 6.1.2, note that for Gaussian random variables in H fulfilling suitable assumptions we know that

$$e_{n,2}(X, H) \approx e_{n,s}(X, H), \quad n \to \infty \tag{6.6}$$

for all $s \geq 1$, see [GLP03]. Thus, we will focus on the case $s = 2$ to search for lower bounds for the quantization errors.

Orthonormal basis Let $(h_m)_{m\in\mathbb{N}}$ be an orthonormal basis of H.

1. We set $F_m = span\{h_1, \ldots, h_m\}$.

2. We set $V_m := pr_{F_m} : E \to F_m$, the orthogonal projection on F_m. It is well known that $\|V_m\| = 1$.

3. Define the linear, surjective and isometric operators ϕ_m by

$$\phi_m : (F_m, \|\cdot\|) \to (\mathbb{R}^m, \|\cdot\|_2), \quad h_i \mapsto e_i,$$

where e_i denotes the i-th unit vector in $\mathbb{R}^m$, $1 \leq i \leq m$.

Theorem 6.2.1. *Assume that the eigenvalue sequence $(\lambda_j)_{j\in\mathbb{N}}$ of the covariance operator C_X satisfies $\lambda_j \approx j^{-b}$ for $-b < -1$ and let $\epsilon > 0$ be arbitrary. Assume further that $(h_j)_{j\in\mathbb{N}}$ is a rate optimal ONS for X in H. We set $m(n) = \lceil log(n)^{1+\epsilon} \rceil$ for $n \in \mathbb{N}$. Then, for every sequence $(\alpha_n)_{n\in\mathbb{N}}$ with $\alpha_n \in \mathcal{C}_{n,r}(\phi_{m(n)}(V_{m(n)}(X)), (\mathbb{R}^{m(n)}, \|\cdot\|_2))$ one has*

$$e_{n,r}(X,H) \sim e_r(X,H,\phi^{-1}_{m(n)}(\alpha_n)) \sim \left(\mathbb{E}\left\|X - f_{\phi^{-1}_{m(n)}(\alpha_n)}(V_{m(n)}(X))\right\|^r\right)^{\frac{1}{r}},$$

as $n \to \infty$.

Proof. See [JL10, Theorem 3.1]. □

Admissible sequences Let $(g_i)_{i\in\mathbb{N}}$ be an admissible sequence for X, and assume that $\sum_{i=1}^{\infty} \xi_i g_i = X a.s.$. We use the following notations:

1. We set $F_m := span\{g_1, \dots, g_m\}$.

2. We define $V_m : H \to F_m \subset H$ by

$$V_m(f_j) := f_j^{(m)} \sqrt{\frac{\lambda_j^{(m)}}{\lambda_j}},$$

for $j \leq m$ and $V_m(f_j) := 0$ for $j > m$, where λ_j and f_j denote the eigenvalues and the corresponding eigenvectors of C_X and $\lambda_j^{(m)}, f_j^{(m)}$ the eigenvalues and the corresponding eigenvectors of C_{X_m}, with X_m defined as

$$X_m := \sum_{i=1}^{m} g_i \xi_i.$$

Note that V_m maps H onto F_m, since

$$span\{g_1, \dots, g_m\} = span\{f_1^{(m)}, \dots, f_m^{(m)}\}.$$

Furthermore, it is important to mention that one does not need to know λ_j and f_j explicitly to construct the subsequent quantizers, since we can find for any $m \in \mathbb{N}$ a random variable $Z_m \overset{d}{=} X$ such that $V_m(Z_m) = \sum_{i=1}^{m} \xi_i g_i$ (see the proof of Theorem 6.2.2), which is explicitly known and sufficient to know for the construction.

3. Define the linear, surjective and isometric operators ϕ_m by

$$\phi_m : (F_m, \|\cdot\|) \to (\mathbb{R}^m, \|\cdot\|_2), \quad f_i^{(m)} \to e_i,$$

where e_i denotes the i-th unit vector of $\mathbb{R}^m$ for $1 \leq i \leq m$.

Theorem 6.2.2. *Assume that the eigenvalue sequence* $(\lambda_j)_{j\in\mathbb{N}}$ *of the covariance operator* C_X *satisfies* $\lambda_j \approx j^{-b}$ *for* $-b < -1$ *and let* $\epsilon > 0$ *arbitrary. Assume that* $(g_j)_{j\in\mathbb{N}}$ *is a rate optimal admissible sequence for* X *in* H*. We set* $m(n) = \lceil log(n)^{1+\epsilon}\rceil$ *for* $n \in \mathbb{N}$*. Then, there exist random variables* Z_m, $m \in \mathbb{N}$ *with* $Z_m \stackrel{d}{=} X$ *such that for every sequence* $(\alpha_n)_{n\in\mathbb{N}}$ *of* r*-optimal* n*-quantizers for* $\phi_{m(n)}(V_{m(n)}(Z_{m(n)}))$ *in* $(\mathbb{R}^{m(n)}, \|\cdot\|_2)$

$$e_{n,r}(X,H) \sim e_r(X,H,\phi^{-1}_{m(n)}(\alpha_n)) \sim \left(\mathbb{E}\left\|X - f_{\phi^{-1}_{m(n)}(\alpha_n)}(V_{m(n)}(Z_{m(n)}))\right\|^r\right)^{\frac{1}{r}},$$

as $n \to \infty$*.*

Proof. See [JL10, Theorem 3.3]. □

Gaussian processes with path in $(C([0,1]), \|\cdot\|_{L_\infty})$ In the previous paragraph, where we worked with Gaussian random variables in Hilbert spaces, we saw that special Hilbertian subspaces, projections and other operators linked to the Gaussian random variable were good tools to develop asymptotically optimal quantizers based on Theorem 6.1.2. Since we now consider the non-Hilbertian separable Banach space $(C([0,1]), \|\cdot\|_{L_\infty})$, we have to find different tools that are suitable for Theorem 6.1.2.

The tool used in [Wil08] are B-splines of order $s \in \mathbb{N}$. In the case $s = 2$, that we will consider hereafter, these splines span the same subspace of $C([0,1], \|\cdot\|_{L_\infty})$ as the classical Schauder basis. We set for $x \in [0,1]$, $m \geq 2$ and $1 \leq i \leq m$ the knots $t_i^{(m)} := \frac{i-1}{m-1}$ and the hat functions

$$f_i^{(m)}(x) := 1_{[t_i^{(m)}, t_{i+1}^{(m)}]}(x)(1-(x-t_i^{(m)})(m-1)) + 1_{[t_{i-1}^{(m)}, t_i^{(m)})}(x)(x-t_{i-1}^{(m)})(m-1).$$

We use the following notations:

1. As subspaces F_m we set $F_m := span\{f_j^{(m)}, 1 \leq j \leq m\}$.

2. As linear and continuous operators $V_m : C([0,1]) \to F_m$ we set the quasi-interpolant
$$V_m(f) := \sum_{i=1}^m f_i^{(m)} \beta_i^{(m)}(f),$$
where $\beta_i^{(m)}(f) := f(t_i^{(m)})$.

3. The linear and surjective isometric mappings ϕ_m we define as
$$\begin{aligned}\phi_m : (F_m, \|\cdot\|_{L_\infty}) &\to (R^m, \|\cdot\|_\infty),\\ \sum_{i=1}^m a_i f_i^{(m)} &\to (a_1, \dots, a_m).\end{aligned}$$
It is easy to see that $\left\|\sum_{i=1}^m a_i f_i^{(m)}\right\|_{L_\infty} = \|(a_1, \dots, a_m)\|_\infty$ holds for every $a \in \mathbb{R}^m$.

For the application of Theorem 6.1.2, we need to know the error bounds for the approximation of X with the quasi-interpolant $V_m(X)$. For Gaussian random variables, we can provide the following result based on the smoothness of an admissible sequence for X in E.

By estimating the approximation power of the B-splines of order 2 for Gaussian processes (see [JL10, Proposition 4.1]), one is able to deduce

Theorem 6.2.3. *(see [JL10, Theorem 4.2]) Let X be a centered Gaussian random variable and $(g_j)_{j\in\mathbb{N}}$ be an admissible sequence for X in $C([0,1])$ fulfilling*

1. $\|g_j\|_{L_\infty} \le C_1 j^{-\theta}$ *for every* $j \ge 1$, $\theta > \frac{1}{2}$ *and* $C_1 < \infty$.
2. $g_j \in C^2([0,1])$ *with* $\left\|g_j''\right\|_{L_\infty} \le C_2 j^{-\theta+2}$ *for every* $j \ge 1$ *and* $C_2 < \infty$.

with $\theta = \frac{b}{2}$, *where the constant* $b > 1$ *satisfies* $\lambda_j \gtrsim K j^{-b}$ *with* $\lambda_j, j \in \mathbb{N}$ *denoting the monotone decreasing eigenvalues of the covariance operator* C_X *of* X *in* $H = L_2([0,1])$ *and* $K > 0$. *We set* $m(n) := \left\lceil log(n)^{\frac{5}{4}+\epsilon}\right\rceil$ *for some* $\epsilon > 0$. *Then, for every sequence* $(\alpha_n)_{n\in\mathbb{N}}$ *of* r*-optimal* n*-quantizers for* $\phi_{m(n)}(V_{m(n)}(X))$ *in* $(\mathbb{R}^{m(n)}, \|\cdot\|_\infty)$ *it holds*

$$\begin{aligned} e_{n,r}(X, (C([0,1]), \|\cdot\|_{L_\infty})) &\sim e_r(X, C([0,1]), \phi_{m(n)}^{-1}(\alpha_n)) \\ &\sim \left(\mathbb{E}\left\|X - f_{\phi^{-1}_{m(n)}(\alpha_n)}(V_{m(n)}(X))\right\|_{L_\infty}^r\right)^{\frac{1}{r}}, \end{aligned}$$

as $n \to \infty$.

Stochastic processes with path in $L_p([0,1], \|\cdot\|_{L_p})$ Another useful tool for our purposes is the Haar basis in $L_p([0,1])$ for $1 \le p < \infty$, which is defined by

$$\begin{aligned} e_0 &:= 1_{[0,1]} \quad e_1 := 1_{[0,\frac{1}{2})} - 1_{[\frac{1}{2},1]} \\ e_{2^n+k} &:= 2^{\frac{n}{2}} e_1(2^n \cdot - k) \quad , n \in \mathbb{N},\ k \in \{0, \dots, 2^n - 1\}. \end{aligned}$$

This is an orthonormal basis of $L_2([0,1])$ and a Schauder basis of $L_p([0,1])$ for $p \in [1,\infty)$, that is $\langle f, e_0\rangle + \sum_{n=0}^\infty \sum_{k=1}^{2^n-1} \langle f, e_{2^n+k}\rangle e_{2^n+k}$ converges to f in $L_p([0,1])$ for every $f \in L_p([0,1])$, see [Sin70].
The Haar basis is used in [LP08] to construct rate optimal sequences of quantizers for mean-regular processes. These processes are specified through the property that for all $0 \le s \le t \le 1$

$$\mathbb{E}|X_t - X_s|^p \le (\rho(t-s))^p, \tag{6.7}$$

where $\rho : \mathbb{R}_+ \to [0,\infty)$ is regularly varying with index $b > 0$ at 0, which means that

$$\lim_{x\to 0} \frac{\rho(cx)}{\rho(x)} = c^b,$$

for all $c > 0$. Condition (6.7) also guarantees that the paths $t \to X_t$ lie in $L_p([0,1])$.

For our approach, it will be convenient to define for $m \in \mathbb{N}$ and $1 \le i \le m+1$ the knots $t_i^{(m)} := \frac{i-1}{m}$, for $1 \le i \le m-1$ the functions

$$f_i^{(m)}(x) := 1_{[t_i^{(m)}, t_{i+1}^{(m)})}(x)\sqrt{m}, \quad f_m^{(m)}(x) := 1_{[t_m^{(m)}, 1]}(x)\sqrt{m},$$

and the operators

$$V_m(f) := \sum_{i=1}^{m} f_i^{(m)} \left\langle f_i^{(m)}, f \right\rangle.$$

Note, that for $f \in L_1([0,1])$, $m = 2^{n+1}$ and $n \in \mathbb{N}_0$

$$\langle e_0, f\rangle e_0 + \sum_{i=0}^{n} \sum_{k=0}^{2^i-1} \langle e_{2^i+k}, f\rangle e_{2^i+k} = \sum_{i=1}^{m} f_i^{(m)} \left\langle f_i^{(m)}, f \right\rangle.$$

We set the following notations:

1. We set for $m \in \mathbb{N}$ the subspaces $F_m := span\{f_1^{(m)}, \ldots, f_m^{(m)}\}$.
2. Set the linear and continuous operator V_m to
$$\begin{aligned} V_m : L_p([0,1]) &\longrightarrow F_m \\ f &\longrightarrow \sum_{i=1}^{m} \left\langle f_i^{(m)}, f \right\rangle f_i^{(m)}. \end{aligned}$$
3. For $p \in [1, \infty)$ we set the isometric isomorphisms $\phi_{m,p} : (F_m, \|\cdot\|_{L_p}) \to (\mathbb{R}^m, \|\cdot\|_p)$ as
$$\phi_{m,p}(\sum_{i=1}^{m} a_i f_i^{(m)}) := m^{(\frac{1}{2}-\frac{1}{p})}(a_1, \ldots, a_m).$$

Theorem 6.2.4. *Let X be a random variable in the Banach space $(E, \|\cdot\|) = (L_p([0,1]), \|\cdot\|_{L_p})$ for some $p \in [1,\infty)$ fulfilling the mean pathwise regularity property*

$$\|X_t - X_s\|_{L_{r \vee p}} \le C(t-s)^a,$$

for constants $C, a > 0$ and $t > s \in [0,1]$. Moreover, assume that $K\ log(n)^{-b} \lesssim e_{n,r}(X,E)$ for constants $K, b > 0$. Then, for an arbitrary $\epsilon > 0$ and $m(n) := \left\lceil (log(n))^{\frac{b}{a}+\epsilon} \right\rceil$ holds that every sequence of r-optimal n-quantizers $(\alpha_n)_{n\in\mathbb{N}}$ for $\phi_{m(n),p}(V_m(n)(X))$ in $(\mathbb{R}^{m(n)}, \|\cdot\|_p)$ satisfies

$$\begin{aligned} e_{n,r}(X, L_p([0,1])) &\sim e_r(X, L_p([0,1]), \phi^{-1}_{m(n),p}(\alpha_n)) \\ &\sim \left(\mathbb{E}\left\| X - f_{\phi^{-1}_{m(n),p}(\alpha_n)}(V_{m(n)}(X))\right\|_{L_p}^r\right)^{\frac{1}{r}}, \end{aligned}$$

as $n \to \infty$.

Proof. See [JL10, Theorem 5.1]. □

6.3 Examples

In this section, we want to present some processes that fulfill the requirements of Theorems 6.2.1, 6.2.2, 6.2.3 and 6.2.4. Firstly, we give some examples for Gaussian processes that can be applied to all of the four Theorems, and secondly we describe how our approach can be applied to Lévy processes in view of Theorem 6.2.4.

Example 6.3.1. *Gaussian processes and Brownian diffusions:*

- *Brownian motion and fractional Brownian motion:*
  Let $(X_t^{(H)})_{t\in[0,1]}$ be a fractional Brownian motion with Hurst parameter $H \in (0,1)$ (in the case $H = \frac{1}{2}$ we have an ordinary Brownian motion). Its covariance function is given by

  $$\mathbb{E}X_s^{(H)}X_t^{(H)} = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right).$$

  Note, that except from the case of an ordinary Brownian motion the eigenvalues and eigenvectors of the fractional Brownian motion are not known explicitly. Nevertheless, the sharp asymptotics of the eigenvalues has been determined (see e.g. [LP04a]).

  In [DvZ04] the authors construct an admissible sequence $(g_j)_{j\in\mathbb{N}}$ in $C([0,1])$ that satisfies the requirements of Theorem 6.2.3 with $\theta = \frac{1}{2} + H$. Furthermore, the eigenvalues λ_j of $C_{X^{(H)}}$ in $L_2([0,1])$ satisfy $\lambda_j \approx j^{-(1+2H)}$, see e.g.[LP04a] such that the requirements for Theorem 6.2.3 are satisfied. Additionally, this sequence is a rate optimal admissible sequence for $X^{(H)}$ in $L_2([0,1])$, such that the requirements for Theorem 6.2.2 are also met. Constructing recursively an orthonormal sequence $(h_j)_{j\in\mathbb{N}}$ by applying the Gram-Schmidt procedure to the sequence $(g_j)_{j\in\mathbb{N}}$ yields a rate optimal ONS for $X^{(H)}$ in $L_2([0,1])$ that can be used in the application of Theorem 6.2.1. In section 7 we will illustrate the quantizers constructed for $X^{(H)}$ with this ONS for several Hurst parameters H. Note, that there are several other admissible sequences for the fractional Brownian motion, which can be applied similarly as described above, see e.g. [DvZ05b] or [DvZ05a]. Moreover, we have for $s,t \in [0,1]$ the mean regularity property

  $$\mathbb{E}\left\|X_t^H - X_s^H\right\|^p = C_{H,p}\,|t-s|^{pH},$$

  and the asymptotics of the quantization error is given as

  $$e_{n,r}(X^H, L_p([0,1])) \approx e_{n,2}(X^H, L_2([0,1])) \approx (log(n))^{-H}, \quad n \to \infty$$

  $\forall r,p \geq 1$ (see [GLP03]), such that the requirements of Theorem 6.2.4 are met with $a = b = H$. Note, that in [DS06] the authors show the existence of constants $k(H,E)$ for $E = C([0,1])$ and $E = L_p([0,1])$ independent of r such that

  $$e_{n,r}(X^H, E) \sim k(H,E)\,(log(n))^{-H}, \quad n \to \infty.$$

Therefore, the quantization errors of the sequences of quantizers constructed via Theorems 6.2.1, 6.2.2, 6.2.3 and 6.2.4 also fulfill this sharp asymptotics.

- *Brownian bridge:*
  Let $(B_t)_{t\in[0,1]}$ be a Brownian bridge with covariance function

  $$\mathbb{E}B_sB_t = \min(s,t) - st.$$

  Since the eigenvalues and eigenvectors of the Brownian bridge are explicitly known, we do not have to search for any other admissible sequence or ONS for $(B_t)_{t\in[0,1]}$ to be applied in $H = L_2([0,1])$. This (the eigenvalue-eigenvector) admissible sequence also satisfies the requirements of Theorem 6.2.3. The mean pathwise regularity for the Brownian bridge can be deduced by

  $$\begin{aligned}\left(\mathbb{E}\,|B_t - B_s|^p\right)^{\frac{1}{p}} &\leq C_{p,2}\left(\mathbb{E}\,|B_t - B_s|^2\right)^{\frac{1}{2}} = C_{p,2}\left(|t-s| - |t-s|^2\right)^{\frac{1}{2}}\\ &\leq C\,|t-s|^{\frac{1}{2}},\end{aligned}$$

  for any $p \geq 1$. Combining [LS01, Theorem 3.7] and [GLP03, Corollary 1.3] yields

  $$e_{n,r}(B, L_p([0,1])) \approx (log(n))^{-\frac{1}{2}}, \quad n \to \infty,$$

  for all $r, p \geq 1$, such that Theorem 6.2.4 can be applied with $a = b = \frac{1}{2}$.

- *Brownian diffusions:*
  We consider a 1-dimensional Brownian diffusion $(X_t)_{t\in[0,1]}$ fulfilling the SDE

  $$X_t = \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s,$$

  where the deterministic functions $b, \sigma : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfy the growth assumption

  $$|b(t,x)| + |\sigma(t,x)| \leq C\,(1 + |x|)\,.$$

  Under some additional ellipticity assumption on σ, the asymptotics of the quantization error in $(L_p([0,1], \|\cdot\|_p)$ is then given by

  $$e_{n,r}(X, L_p([0,1])) \approx (\log(n))^{-\frac{1}{2}},$$

  as $n \to \infty$ (see [Der08a] and also [LP06]). Furthermore, one shows for $0 \leq s \leq t \leq 1$

  $$\left(\mathbb{E}\,\|X_t - X_s\|^p\right)^{\frac{1}{p}} \leq C(t-s)^{\frac{1}{2}}$$

  (see [LP08, Example 3.1]) such that Theorem 6.2.4 can be applied with $a = b = \frac{1}{2}$.

Example 6.3.2. *Lévy processes*
Let $(X_t)_{t\in[0,1]}$ be a real Lévy process, that is X is a cádlàg process with $\mathbb{P}(X_0 = 0) = 1$ and stationary and independent increments. The characteristic exponent $\psi(u)$ given through the equation

$$\int_{\mathbb{R}} \exp(iux)\mathbb{P}^{X_1}(dx) = \exp(-\psi(u)), u \in \mathbb{R}$$

is characterized by the *Lévy-Khintchine formula*

$$\psi(u) = iau + \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (1 - e^{iux} + iux\chi_{(|x|<1)})\Pi(dx),$$

where the characteristic triple (a, σ, Π) contains constants $a \in \mathbb{R}$, $\sigma \geq 0$ and a measure Π on $\mathbb{R}\backslash\{0\}$ satisfying $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty$. By definition, we know that

$$\mathbb{E}\,|X_t - X_s|^p = \mathbb{E}\,|X_{t-s}|^p\,, \tag{6.8}$$

and it is further known that the latter moment is finite if and only if

$$\int_{(|x|\geq 1)} |x|^p\,\Pi(dx) < \infty.$$

Furthermore, by the *Lévy-Ito decomposition*, X can be written as the sum of independent Lévy processes

$$X = X^{(1)} + X^{(2)} + X^{(3)},$$

where $X^{(3)}$ is a Brownian motion with drift, $X^{(2)}$ is a *Compound Poisson process* and $X^{(1)}$ is a Lévy process with bounded jumps and without Brownian component.
Firstly, we will analyze the mean pathwise regularity of these three types of Lévy processes to combine these results with lower bounds for the asymptotical quantization error.

1. *Mean pathwise regularity of the 3 components of the Lévy-Ito decomposition:*

   - According to an extended *Millar's Lemma* [LP08, Lemma 5], for all Lévy processes with bounded jumps and without Brownian component, there exists for every $p \geq 2$ a constant $C < \infty$ such that for every $t \in [0,1]$
   $$\mathbb{E}\,|X_t|^p \leq Ct = C\left(t^{\frac{1}{p}}\right)^p. \tag{6.9}$$
   Combining (6.8) and (6.9), we can choose ρ in (6.7) as $\rho_{1,p}(x) = x^{\frac{1}{p}}$. For $p \in [1,2)$ we have by using (6.9) with $p = 2$
   $$\mathbb{E}\,|X_t|^p \leq \left(\mathbb{E}\,|X_t|^2\right)^{\frac{p}{2}} \leq (Ct)^{\frac{p}{2}} = \left(C^{\frac{1}{2}} t^{\frac{1}{2}}\right)^p$$
   and thus we can choose $\rho_{1,p}(x) = Cx^{\frac{1}{2}}$. Combining these facts we get $\rho_{1,p}(x) = Cx^{\frac{1}{2\vee p}}$ for $p \geq 1$.

- We consider the Compound Poisson process

$$X_t = \sum_{k=1}^{K_t} Z_k,$$

where K denotes a standard Poisson process with intensity $\lambda = 1$ and $(U_k)_{k\in\mathbb{N}}$ is an i.i.d. sequence of random variables with $\|Z_1\|_{L_p(\mathbb{P})} < \infty$. Then, one shows that

$$\mathbb{E}\left|\sum_{k=1}^{K_t} Z_k\right|^p \leq t\,\|Z_1\|_{L_p(\mathbb{P})}^p \exp(-t)\sum_{k=1}^{\infty}\frac{t^{k-1}k^p}{k!} \leq C\left(t^{\frac{1}{p}}\right)^p,$$

so that (6.7) is satisfied with $\phi_{2,p}(x) = x^{\frac{1}{p}}$.

- We consider a Brownian Motion with drift. Using example 6.3.1 and Lemma 1.2.5 we can choose ρ in (6.7) as $\rho_{3,p}(x) = \rho_3(x) = x^{\frac{1}{2}}$ for all $p \geq 1$.

2. *Lévy processes with non-vanishing Brownian component:*
Let X be a Lévy process with a non-vanishing Brownian component, which means that σ in the characteristic triple satisfies $\sigma > 0$. By Proposition 4 in [LP08] for $r, p \geq 1$, it holds that

$$(\log(n))^{-\frac{1}{2}} \approx C e_{n,r}(W, L_p) \lesssim e_{n,r}(X, L_p), \quad n \to \infty \tag{6.10}$$

for some constant $C \in (0,\infty)$ and W denotes a Brownian Motion. We consider the Lévy-Ito decomposition $X = X^{(1)} + X^{(2)} + X^{(3)}$, and assume that for $X_t^{(2)} = \sum_{k=1}^{K_t} Z_k$ holds $\|Z_1\|_{L_{p\vee r}(\mathbb{P})} < \infty$. Therefore, we receive the mean pathwise regularity for X, all $p, r \geq 1$ and some constant $C < \infty$

$$\rho_p(x) := C x^{\frac{1}{2\vee r\vee p}}. \tag{6.11}$$

Thus, we can apply Theorem 6.2.4 with $a = \frac{1}{2\vee p\vee r}$ and $b = \frac{1}{2}$.

3. *Compound Poisson processes:*
For a Compound Poisson process X we know that the rate for the asymptotic quantization error under suitable assumptions is given by

$$e_{n,r}(X, L_p) \approx \exp(-\kappa\sqrt{log(n)log(log(n))}), \quad n \to \infty,$$

see [ADSV09, Theorems 13,14] and [LP08, Proposition 3] for a constant $\kappa \in (0,\infty)$. Thus, the sequence $(m(n))_{n\in\mathbb{N}}$ has to grow faster than in the examples above. To fulfill

$$\left\|\|X - V_m(X)\|_{L_p([0,1])}\right\|_{L_r(\mathbb{P})} = o\left(\exp(-\kappa\sqrt{\log(n)\log(\log(n))})\right),$$

as $n \to \infty$, (see the proof of Theorem 6.2.4) we need to choose $m(n) = \left\lceil (p\vee r)\exp(\kappa\sqrt{\log(n)\log(\log(n))}(1+\epsilon))\right\rceil$ for an arbitrary $\epsilon > 0$.

4. *α-stable Lévy processes with $\alpha \in (0,2)$:*
   These are Lévy processes satisfying the self similarity property
   $$X_t \overset{d}{=} t^{\frac{1}{\alpha}} X_1,$$
   and furthermore
   $$\mathbb{E}|X_1|^\alpha = \infty \text{ and } \sup_r \{\mathbb{E}|X_1|^r < \infty\} = \alpha.$$
   Thus, we can choose $\rho(x) = Cx^{\frac{1}{\alpha}}$ for any $p \geq 1$ and constants $C_p < \infty$. The asymptotics of the quantization error for X is given by
   $$e_{n,r}(X, L_p) \approx log(n)^{-\frac{1}{\alpha}}, \quad n \to \infty$$
   for $r, p \geq 1$ ([AD09]), such that we meet the requirements of Theorem 6.2.4 by setting $a = b = \alpha$.

6.4 Numerical illustrations

In this section, we want to highlight the steps needed for a numerical implementation of our approach, and also give some illustrating results.

For illustration purposes, we will concentrate on the case described in section 3.1 for $r = 2$. Examples for quantizers as constructed in section 4 can be found in [Wil08]. The quantizers shown hereafter were calculated numerically, by using the widely used CLVQ-algorithm as described in [Pag98]. To achieve a better accuracy, we finally performed a few steps of a gradient algorithm by approximating the gradient with a Monte-Carlo simulation.

Let $X^{(H)}$ be a Fractional Brownian motion with Hurst parameter H. We used the admissible sequence as described in [DvZ04]
$$X_t^{(H)} \overset{d}{=} \sum_{n=1}^{\infty} \frac{\sqrt{2}c_H}{|J_{1-H}(x_n)|} \frac{\sin(x_n t)}{x_n^{1+H}} \xi_n^1 + \sum_{n=1}^{\infty} \frac{\sqrt{2}c_H}{|J_{-H}(y_n)|} \frac{1-\cos(y_n t)}{y_n^{1+H}} \xi_n^2,$$
where c_H is given as
$$c_H^2 := \frac{\sin(\pi H)\,\Gamma(1+2H)}{\pi},$$
J_{1-H} and J_{-H} are Bessel functions with corresponding parameters and x_n and y_n are the ordered roots of the Bessel functions with parameters $-H$ and $1-H$. After ordering the elements of the two parts of the expansion in an alternating manner, and applying Gram-Schmidt's procedure for orthogonalization to construct a rate optimal ONS, we used the method as described in section 3.1. We show the results we obtained for $n = 10$, $m = 4$ and the Hurst parameters $H = 0.3, 0.5$ and 0.7. To show the effects of changing parameters, we also present the quantizers obtained after increasing the size of the containing subspace ($m = 8$) and in addition the effect of increasing the quantizer-size ($n = 30$). Since $X^{(H)}$ is for $H = 0.5$ an ordinary Brownian motion, one can compare the results with the results obtained for the Brownian motion by using the Karhunen-Loève-Expansion. (see e.g. [LPW08])

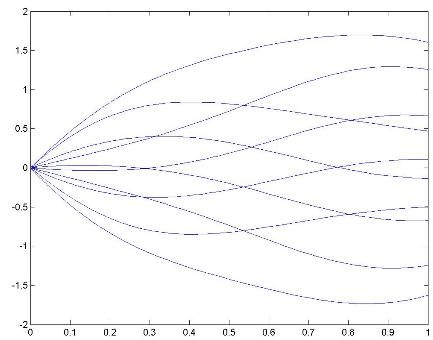

Figure 6.1: 10-quantizer for the Fractional Brownian Motion with Hurst Parameter $H = 0.3$ in a 4-dimensional subspace

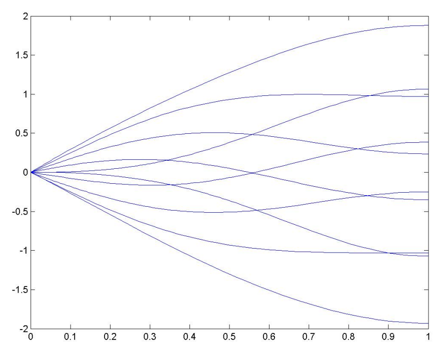

Figure 6.2: 10-quantizer for the Fractional Brownian Motion with Hurst Parameter $H = 0.5$ in a 4-dimensional subspace

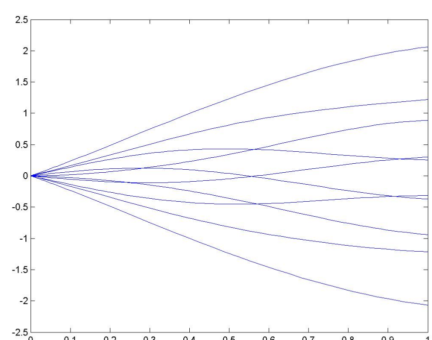

Figure 6.3: 10-quantizer for the Fractional Brownian Motion with Hurst Parameter $H = 0.7$ in a 4-dimensional subspace

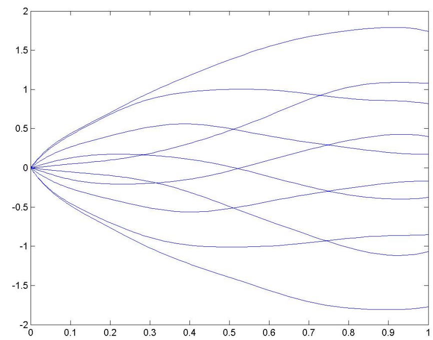

Figure 6.4: 10-quantizer for the Fractional Brownian Motion with Hurst Parameter $H = 0.3$ in an 8-dimensional subspace

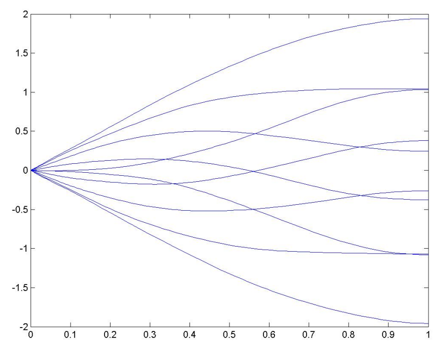

Figure 6.5: 10-quantizer for the Fractional Brownian Motion with Hurst Parameter $H = 0.5$ in an 8-dimensional subspace

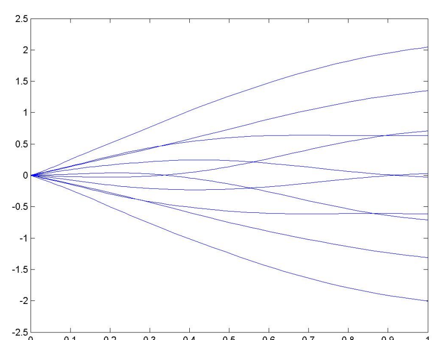

Figure 6.6: 10-quantizer for the Fractional Brownian Motion with Hurst Parameter $H = 0.7$ in an 8-dimensional subspace

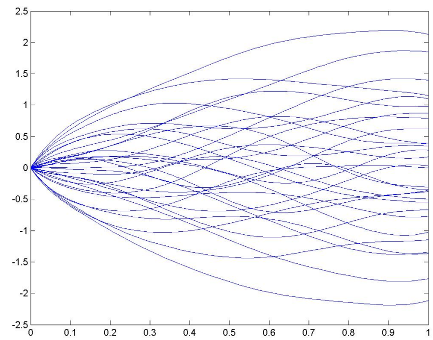

Figure 6.7: 30-quantizer for the Fractional Brownian Motion with Hurst Parameter $H = 0.3$ in an 8-dimensional subspace

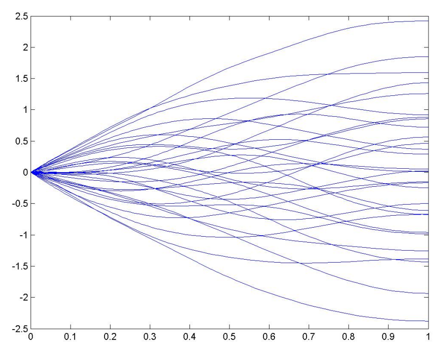

Figure 6.8: 30-quantizer for the Fractional Brownian Motion with Hurst Parameter $H = 0.5$ in an 8-dimensional subspace

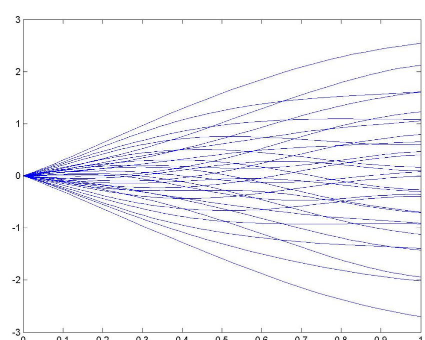

Figure 6.9: 30-quantizer for the Fractional Brownian Motion with Hurst Parameter $H = 0.7$ in an 8-dimensional subspace

Chapter 7

Open problems

Chapter 2 In chapter 2, we concentrated on the estimations of the increments of the quantization error

$$\Delta_{n,r}(X,E) = e^r_{n,r}(X,E) - e^r_{n+1,r}(X,E),$$

and its relations to local characteristics of optimal codebooks.

Problem 7.1. *Is it possible to estimate sharper versions for $\Delta_{n,r}(X,E)$ given X is Gaussian in $(E,\|\cdot\|)$, and $\dim(\mathcal{H}_\mu) = \infty$ of the form*

$$\Delta_{n,r}(X,E) \approx \phi(n), \quad n \to \infty?$$

As indicated in section 2 of chapter 2, it seems to be difficult to achieve a result as sharp as in Problem (7.1) by using arguments relating $\Delta_{n,r}(X,E)$ to the local quantization error $e_{r;\mathrm{loc}}(X,E;\alpha,C_a(\alpha_n))$. This gives rise to a different approach for the estimation of those increments, and leads to

Problem 7.2. *Does a condition exist on the probability μ such that $\Delta_{n,r}(X,E)$ is monotone in n? In this case, the monotone density theorem directly implies an even sharper version for the asymptotics of the increments.*

Chapter 3 Chapter 3 concerns the analysis of the geometry of sequences of (asymptotically) optimal codebooks for probabilities μ on $\mathbb{R}^d$, in particular, the asymptotics of the quantization radius and the quantization ball.

Problem 7.3. *In view of the results derived in chapter 3 section 2, it remains unclear whether under some weak assumptions on the shape function g satisfying $f = g(\|\cdot\|_0)$, there exists a function ϕ_g such that*

$$\rho(\alpha_n) \sim \phi_g(n), \quad n \to \infty.$$

It seems reasonable to conjecture that

$$\rho(\alpha_n) \sim \left(\overline{F_{r,\|\cdot\|_0}}\right)^{-1}(Cn^{-\frac{r+d}{d}}), \quad n \to \infty,$$

for a constant $C \in (0,\infty)$. In particular, this is true for the cases of Type I, similar cases like the Gumbel or log-normal types, and might also hold for distributions of Type II, given one assumes that $\rho(\alpha_n)$ is regularly varying at infinity.

Problem 7.4. *As concerning quantization balls and quantization hulls for densities of the Type II, it remains open whether*

$$\overline{\liminf_{n\to\infty} \frac{\operatorname{conv}(\alpha_n)}{\rho(\alpha_n)}} = \overline{\limsup_{n\to\infty} \frac{\operatorname{conv}(\alpha_n)}{\rho(\alpha_n)}} = \mathcal{B}$$

for a subset $\mathcal{B} \subset B(0,1)$. In particular, it remains uncertain whether the quantizing norm $\|\cdot\|$ is also "dominated" by the distributional norm $\|\cdot\|_0$.

Chapter 4 Chapter 4 concerns the analysis of the geometry of (asymptotically) optimal codebooks for Gaussian measures μ on a separable Banach space $(E,\|\cdot\|)$, in particular its quantization radius and quantization balls.

Problem 7.5. *Given a sharper version for the estimation of the asymptotics of the increments $\Delta_{n,r}(X,E)$, it would be interesting to know if the quantization radius of sequences of optimal codebooks also admits a sharper estimate of the form*

$$\rho(\alpha_n) = \sigma(\mu)\,(2\log(n))^{\frac{1}{2}} + o(1).$$

Problem 7.6. *It would be interesting to know, if the results can be extended to other measures, such as Lévy measures or Gaussian diffusions. To follow the approach presented above, one would need equivalents for*

- *the Anderson inequality (needed for the estimation of the increments of quantization errors),*
- *the estimation of shifted balls and*
- *the isoperimetric inequality.*

Chapter 5 In chapter 5, we use the estimations done in chapter 2 to derive estimates for local characteristics of optimal codebooks, i.e.

$$\mu(C_a(\alpha_n)),\quad \mu_r(C_a(\alpha_n)),$$
$$e_{r;\mathrm{loc}}(X,E;\alpha_n,C_a(\alpha_n)) \text{ and } \operatorname{dist}(a,\alpha_n\backslash\{a\}).$$

Problem 7.7. *Is it possible to give a useful criterion on the density μ (similar to the almost monotonicity on g) to achieve a universal upper bound for the μ_r-weights of the Voronoi cells $V_a(\alpha_n)$. As a consequence of the results developed in chapter 5, this implies*

$$\frac{1}{n}e^r_{n,r}(X,E) \approx e^r_{r;\mathrm{loc}}(X,E;\alpha_n,V_a(\alpha_n)) \approx e^r_{r;\mathrm{loc}}(X,E;\alpha_n,W_a(\alpha_n)),\quad n\to\infty,$$
$$\mu_r(V_a(\alpha_n)) \approx \mu_r(W_a(\alpha_n)) \approx \frac{1}{n},\quad n\to\infty.$$

Problem 7.8. *Under which condition on the measure μ the asymptotics $\approx$ in the previous problem may be replaced by $\sim$?*

As for the latter problem, it does not seem to be possible to achieve such a result by using the techniques used in this thesis, in particular by using the micro-macro inequalities.

Chapter 6 Chapter 6 concerns the construction of asymptotically optimal codebooks $(\alpha_n)_{n\in\mathbb{N}}$ for stochastic processes $(X_t)_{t\in I}$ achieving

$$e_{n,r}(X,E) \sim e_r(X,E;\alpha_n), \quad n \to \infty.$$

Problem 7.9. *Concerning the case of quantization in $(C([0,1], \|\cdot\|_{L_\infty}))$, is it possible to decrease the dimension for the sequence of spaces $F_{m(n)}$ to $\log(n)^{1+\epsilon}$, for an arbitrary $\epsilon \in (0,1)$?*

Appendix A

Regular variation

Definition A.1. Let $I \subset \mathbb{R}^+$ be unbounded. A function $h : I \to \mathbb{R}$ is called regularly varying at infinity with index $b \in \mathbb{R}$, if

$$\lim_{x\to\infty, x\in I} \sup \left\{ \frac{h(y)}{h(x)} : y \le cx \right\} = \lim_{x\to\infty, x\in I} \inf \left\{ \frac{h(y)}{h(x)} : y \ge cx \right\} = c^b$$

for all $c > 0$. The set of all those functions is denoted $\mathcal{R}_b(I) = \mathcal{R}_b$. If $b = 0$, we also call h slowly varying. Furthermore, we set

$$\mathcal{R} := \bigcup_{b\in\mathbb{R}} \mathcal{R}_b.$$

Regularly varying functions admit a broad range of closure properties. The proofs are straightforward.

Proposition A.2. *(closure properties for regular variation, see [BGT87, Proposition 1.5.7]) Let $a, a_i \in \mathbb{R}$ and $f_i \in \mathcal{R}_{a_i}$, $i = 1, 2$.*

1. $f_1 f_2 \in \mathcal{R}_{a_1+a_n}$.
2. *If* $f_2(x) \to \infty$, $x \to \infty$ *then* $f_1 \circ f_2 \in \mathcal{R}_{a_1 a_2}$.
3. $f_1 + f_2 \in \mathcal{R}_{\max\{a_1,a_2\}}$.
4. $f_1^a \in \mathcal{R}_{aa_1}$.
5. *Suppose that* $f_2 \to \infty$, $x \to \infty$. *As a consequence of the first points, we derive* $\log(f_2) \in \mathcal{R}_0$ *and* $\log(f_2) f_1 \in \mathcal{R}_{a_1}$.

Remark A.3. Every regularly varying function $h \in \mathcal{R}_\alpha$ can be written as a product of a function $g \in \mathcal{R}_0$ and a monomial $x \mapsto x^\alpha, \alpha \in \mathbb{R}$, by considering $g : I \to \mathbb{R}$, $x \mapsto h(x)x^{-\alpha}$.

We will assume throughout that $I = \mathbb{R}^+$. Regularly varying functions admit a specific type of inversion.

Theorem A.4. *(See [BGT87, Theorem 1.5.12]) For $\alpha \neq 0$ and $f \in \mathcal{R}_\alpha$ there exists an asymptotic inverse $g \in \mathcal{R}_{\frac{1}{\alpha}}$ such that*

$$g(f(x)) \sim f(g(x)) \sim x, \quad x \to \infty.$$

Furthermore, g is unique up to asymptotic equivalence.

Proof. The result is shown in the reference for indexes $\alpha > 0$. For $f \in \mathcal{R}_\alpha$, $\alpha < 0$, consider an asymptotic inverse g^- to $g := \frac{1}{f} \in \mathcal{R}_{-\alpha}$. Then the function h with $x \mapsto g^-(\frac{1}{x})$ is an asymptotic inverse to f, since

$$f(h(x)) = f(g^-(\frac{1}{x})) = \frac{1}{g(g^-(\frac{1}{x}))} \sim \frac{1}{\frac{1}{x}} = x, \quad x \to \infty$$

and

$$h(f(x)) = g^-(\frac{1}{f(x)}) = g^-(g(x)) \sim x, \quad x \to \infty.$$

□

There are some specific asymptotic inverse functions to $f \in \mathcal{R}_\alpha$ for $\alpha \neq 0$.

Lemma A.5. *Let $f : [A, \infty) \to \mathbb{R}^+$ be regularly varying at infinity with index $\theta > 0$. Then*

$$f^{\leftarrow}(y) := \inf\{x \in [A, \infty) : f(x) \geq y\}$$

is non-decreasing and regularly varying at infinity with index $\frac{1}{\theta}$. Furthermore, $f^{\leftarrow}$ is an asymptotic inverse to f and

$$f^{\leftarrow}\big(f(x)\big) \leq x$$

for all $x \in [A, \infty)$.

Proof. The property

$$f^{\leftarrow}\big(f(x)\big) \leq x$$

and the non-decreasing property follow by definition. To see that $f^{\leftarrow}$ is an asymptotic inverse to f, see the second part of the proof in [BGT87, Theorem 1.5.12]. Let $\lambda, \kappa > 1$, and $\delta \in (0, \infty)$. As a consequence of Potter's Theorem (see [BGT87, Theorem 1.5.6]), there exists an $u_0 > 0$ such that

$$\frac{1}{\kappa\lambda^{\theta+\delta}} f(v) \leq f(u)$$

for all $v \in [\frac{1}{\lambda(a)}u, u]$ and $u > u_0$. Since $f \to \infty$, we can choose y sufficiently large such that $f^{\leftarrow}(y) \geq u_0$. Then, by definition of $f^{\leftarrow}$, there exists $x \in [f^{\leftarrow}(y), \lambda f^{\leftarrow}(y)]$ such that $f(x) \geq y$. Selecting $u = f^{\leftarrow}(y)$ and $v = x$, we obtain

$$\frac{1}{\kappa\lambda^{\theta+\delta}} y \frac{1}{\kappa\lambda^{\theta+\delta}} f(x) \leq f(f^{\leftarrow}(y)).$$

Since $\kappa, \lambda > 1$ arbitrary, we obtain

$$y \lesssim f(f^{\leftarrow}(y)), \quad y \to \infty.$$

The $\liminf$ as well as the converse direction $f^{\leftarrow}(f(x)) \sim x, \quad x \to \infty$ can be shown analogously. □

Lemma A.6. *Let $f : [A, \infty) \to \mathbb{R}^+$ be regularly varying at infinity with index $-a < 0$. Then*

$$f^{\leftarrow}(y) := \inf\{x \in [A, \infty) : f(x) \leq y\}$$

is non-increasing and regularly varying at infinity with index $-\frac{1}{a}$. Furthermore, $f^{\leftarrow}$ is an asymptotic inverse to f and

$$f^{\leftarrow}\big(f(x)\big) \leq x$$

for all $x \in [A, \infty)$.

Proof. The property

$$f^{\leftarrow}\big(f(x)\big) \leq x$$

and the non-increasing property follow by definition. To see that $f^{\leftarrow}$ is an asymptotic inverse to f, see the second part of the proof of [BGT87, Theorem 1.5.12]. The proof is analogously to the proof of Lemma A.5. □

Lemma A.7. *Let $f : [A, \infty) \to \mathbb{R}^+$ be regularly varying at infinity with index $-a < 0$. Then*

$$f^{\rightarrow}(y) := \sup\{x \in [A, \infty) : f(x) \geq y\}$$

is non-increasing and regularly varying at infinity with index $-\frac{1}{a}$. Furthermore, $f^{\rightarrow}$ is an asymptotic inverse to f and

$$f^{\rightarrow}\big(f(x)\big) \geq x$$

for all $x \in [A, \infty)$.

Proof. The property

$$f^{\rightarrow}\big(f(x)\big) \geq x$$

and the non-increasing property follow by definition. To see that $f^{\rightarrow}$ is an asymptotic inverse to f, see the second part of the proof in [BGT87, Theorem 1.5.12]. The proof is analogously to the proof of Lemma A.5. □

Abelian Theorems A Theorem is called Abelian, if it describes the relationship between the regularity of a function to integrals involving that function. Converse results are called Tauberian, which are much more difficult to be obtained. Those are treated in the next paragraph.

A broad class of Lebesgue-densities h in $\mathcal{B}(\mathbb{R}^d)$ admit the shape

$$h(x) = \exp(-f(\|x\|_0)), \quad x \in \mathbb{R}^d,$$

for some $f \in \mathcal{R}_\alpha$ and a norm $\|\cdot\|_0$ on $\mathbb{R}^d$. A helpful tool for the analysis of those densities is

Lemma A.8. *(See [BGT87, Theorem 4.12.10]) Let $f \in \mathcal{R}_\alpha$ for some $\alpha > 0$. Then*

$$-\log\left(\int_x^\infty \exp(-f(y))d(y)\right) \sim f(x), \quad x \to \infty.$$

A careful reading of the proof of Lemma A.8 reveals the following sharpened version of the result.

Lemma A.9. *Let $f \in \mathcal{R}_\alpha$ for some $\alpha > 0$, strictly increasing and $f \in C^2([A,\infty))$ and*

$$(f^{-1})^{(i)}(x) \in \mathcal{R},\ i = 1,2.$$

Then

$$\int_x^\infty \exp(-f(y))dy = \exp(-f(x))\left(f'(x)\right)^{-1}\left(1 + \mathcal{O}\big(\frac{1}{f(x)}\big)\right), \quad x \to \infty$$

Proof. The proof is based on the argumentation in the proof in [BGT87, Theorem 4.12.10]. By substitution formula with $t = f(y)$ and differentiation of the inverse function, we obtain for $x \in f^{-1}[A,\infty)$

$$\begin{aligned}\int_{f(x)}^\infty \exp(-t)\left(f^{-1}\right)'(t)d\lambda(t) &= \int_y^\infty \exp(-f(y))\left(f^{-1}\right)'(f(y))f'(y)d\lambda(y)\\ &= \int_x^\infty \exp(-f(y))d\lambda(y).\end{aligned}$$

Integration by party yields

$$\begin{aligned}\int_{f(x)}^\infty \exp(-t)\left(f^{-1}\right)'(t)d\lambda(t) &= \exp(-f(x))\left(f^{-1}\right)'(f(x))\\ &\quad + \int_{f(x)}^\infty \exp(-t)(f^{-1})''(t)d\lambda(t).\end{aligned}$$

Since $f \in \mathcal{R}_\alpha$, we have $f^{-1} \in \mathcal{R}_{\frac{1}{\alpha}}$, which implies $\left(f^{-1}\right)^{(i)} \in \mathcal{R}_{\frac{1}{\alpha}-i}$, for $i = 1,2$. Hence,

$$\int_{f(x)}^\infty \exp(-t)(f^{-1})''(t)d\lambda(t) = \mathcal{O}\left(\frac{\int_{f(x)}^\infty \exp(-t)\left(f^{-1}\right)'(t)d\lambda(t)}{f(x)}\right)$$

and we obtain again in virtue of the differentiation formula for the inverse function

$$\int_x^\infty \exp(-f(y))dy = \exp(-f(x))\,(f'(x))^{-1}\Big(1+\mathcal{O}\big(\frac{1}{f(x)}\big)\Big), \quad x\to\infty.$$

□

The equivalent tool for regularly varying densities is

Proposition A.10. *(See [BGT87, Proposition 1.5.10]) Let $l \in \mathcal{R}_0$ and $\alpha < -1$. Then $\int_x^\infty t^\alpha l(t)d\lambda(t)$ converges and*

$$\int_x^\infty t^\alpha l(t)d\lambda(t) \sim l(x)\int_x^\infty t^\alpha d\lambda(t) = l(x)\frac{t^{\alpha+1}}{-(\alpha+1)}, \quad x\to\infty.$$

Tauberian Theorems Much more powerful results are those of the Tauberian type, where asymptotics of terms involving a function give under specific conditions asymptotic results on the function itself.

Theorem A.11. *(Monotone density Theorem) Let $F \in \mathcal{R}_\alpha$ for some $\alpha \in \mathbb{R}$ such that $F(x) = \int_x^\infty f(x)d\lambda(x)$. If f is monotone on $[A,\infty)$ for some $A \in \mathbb{R}^+$, then*

$$f(x) \sim -\alpha\frac{F(x)}{x}, \quad x\to\infty.$$

Proof. The proof is almost identical to the proof in [BGT87, Theorem 1.7.2], where $F(x) = \int_0^x f(x)d\lambda(x)$. Let f be non-increasing. Let $0 < a < b < \infty$. Then

$$F(ax) - F(bx) = \int_{ax}^{bx} f(y)d\lambda(y),$$

which implies

$$\frac{f(ax)(b-a)x}{F(x)} \lesssim \frac{F(ax)-F(bx)}{F(x)} \lesssim \frac{f(bx)(b-a)x}{F(x)}, \quad x\to\infty$$

. The middle term converges to $a^\alpha - b^\alpha$, which implies with $a = 1$

$$\limsup_{x\to\infty}\frac{f(x)x}{F(x)} \le -\frac{b^\alpha - 1}{b-1}.$$

Letting $b \to 1$ yields the asserted asymptotic upper bound for f

$$\frac{f(x)x}{F(x)} \lesssim -\alpha, \quad x\to\infty.$$

The lower bound as well as the proof for f non-decreasing is analogously. □

References: The standard monograph containing almost all key results concerning regularly varying functions is [BGT87].

Appendix B

Convergence of sets

Our motivation for considering the concepts for convergences of sets is the following observation: By definition of the maximum radius ρ, we have for any sequence of bounded sets $(A_n)_{n\in\mathbb{N}}$ in E

$$\frac{A_n}{\rho(A_n)} \subset B(0,1), \quad n \in \mathbb{N}$$

and furthermore, by convexity of the unit ball $B(0,1)$, also

$$\frac{\operatorname{conv}(A_n)}{\rho(A_n)} \subset B(0,1), \quad n \in \mathbb{N}.$$

Here conv denotes the convex hull, i.e. for a subset $A \subset E$

$$\operatorname{conv}(A) := \{x \in E : \exists \lambda \in [0,1],\ a, b \in A \text{ such that } x = \lambda a + (1 - \lambda b)\}.$$

Considering $A_n = \alpha_n, n \in \mathbb{N}$ for a sequence of r-optimal n-codebooks α_n for a r.e. X in $(E, \|\cdot\|)$, one may be interested in the behavior of these scaled codebooks when n tends to infinity. In fact, given a convergence of $\frac{\alpha_n}{\rho(\alpha_n)}$ or $\frac{\operatorname{conv}(\alpha_n)}{\rho(\alpha_n)}$ in a reasonable way towards some set $A \subset B(0,1)$, we would get a better understanding of the (asymptotic) geometry of optimal codebooks.

We will consider three types of convergence, where one is topology-free, i.e. it does not depend on the Banach space norm $\|\cdot\|$.

Definition B.1. (Set-theoretic convergence) Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets in E.

1. The classical set-theoretic limes inferior is defined as

$$\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} A_m.$$

2. The classical set-theoretic limes superior is defined as

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m. \tag{B.1}$$

3. If both limits coincide, i.e. there exists a subset $A \subset E$ such that

$$A = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$$

we call A the set-theoretic limit of the sequence $(A_n)_{n\in\mathbb{N}}$ and we write $A = \lim_{n\to\infty} A_n$.

Definition B.2. (Convergence in the $\|\cdot\|$-sense). Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets in E.

1. An element $x \in E$ is called *limit point* of $(A_n)_{n\in\mathbb{N}}$ if, for every $\epsilon > 0$ there exists an $n(\epsilon) \in \mathbb{N}$ such that

$$x + B(0,\epsilon) \cap A_n \neq \varnothing,$$

for all $n \geq n(\epsilon)$.

2. An element $x \in E$ is called *cluster point* of $(A_n)_{n\in\mathbb{N}}$ if, for every $n \in \mathbb{N}$ and every $\epsilon > 0$ there exists an $m(\epsilon,n) \in \mathbb{N}$ such that

$$x + B(0,\epsilon) \cap A_{m(\epsilon,n)} \neq \varnothing.$$

3. $\operatorname{liminf}^{\|\cdot\|}$ is defined as the set of all limit points of $(A_n)_{n\in\mathbb{N}}$.

4. $\operatorname{limsup}^{\|\cdot\|}$ is defined as the set of all cluster points of $(A_n)_{n\in\mathbb{N}}$.

5. If both limits coincide, i.e. there exists a subset $A \subset E$ such that

$$A = \operatorname*{liminf}_{n\to\infty}{}^{\|\cdot\|} A_n = \operatorname*{limsup}_{n\to\infty}{}^{\|\cdot\|} A_n$$

we call A the $\|\cdot\|$-based limit of the sequence $(A_n)_{n\in\mathbb{N}}$ and we write $A = \lim^{\|\cdot\|}{}_{n\to\infty} A_n$.

We have the following characterizations for the $\|\cdot\|$-based limes inferior and superior.

Proposition B.3. *(see [KT84, Propositions 3.2.11, 3.2.12]) Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of subsets in E.*

1. *The $\|\cdot\|$-based* $\limsup$ *admits the characterization*

$$\limsup_{n\in\mathbb{N}}{}^{\|\cdot\|} A_n := \bigcap_{n\in\mathbb{N}} \overline{\bigcup_{m\geq n} A_m}.$$

2. *The $\|\cdot\|$-based* $\liminf$ *admits the characterization*

$$\liminf_{n\in\mathbb{N}}{}^{\|\cdot\|} A_n := \bigcap_{H\in\mathcal{T}} \overline{\bigcup_{m\in H} A_m},$$

where $\mathcal{T} := \{H \subset \mathbb{N} : \operatorname{card}(H) = \infty\}$.

The $\|\cdot\|$-based convergence is closely related to the Hausdorff extended pseudometric $\delta(\cdot,\cdot)$.

Definition B.4. For non-empty subsets $A, B \subset E$ we define the Hausdorff extended pseudomertic $\delta(A,B)$ by

$$\delta(A,B) = \max\{\delta_l(A,B), \delta_u(A,B)\},$$

with $\delta_l(A,B) := \inf\{\lambda \geq 0 : A \subset B + B(0,\lambda)\}$ and $\delta_u(A,B) = \delta_l(B,A)$.

We set $\mathcal{P}^E := \{A : A \subset E\}$. Considering specific subsets of $\mathcal{M} \subset \mathcal{P}^E$, the tuple $(\mathcal{M},\delta)$ even becomes a metric space.

Lemma B.5. *The Hausdorff extended pseudometric δ is a metric on the family of sets*

$$\mathcal{F}_{b,0}(E) := \{A \subset E : A \neq \varnothing, \textit{ bounded and closed}\}.$$

In particular, for every bounded subset $F \subset E$, δ is a metric on

$$\mathcal{F}_0(F) := \{A \subset F : A \neq \varnothing \textit{ and closed}\}.$$

Proof. By definition, δ is symmetric and $\delta(A,A) = 0$. Let $A, B, C \in \mathcal{F}_{b,0}$. Since A and B are bounded and non-empty, we have $\delta(A,B) \in [0,\infty)$. Let $\lambda_1 = \delta(A,C)$ and $\lambda_2 = \delta(C,B)$. Then we have for $\epsilon > 0$ arbitrary

$$A \subset C + B(0,\lambda_1+\epsilon) \subset B + B(0,\lambda_1+\epsilon) + B(0,\lambda_2+\epsilon) = B + B(0,\lambda_1+\lambda_2+2\epsilon)$$

and thus $\delta_l(A,B) \leq \lambda_1 + \lambda_2$. The argumentation for δ_u is analogous. Finally, suppose $\delta(A,B) = 0$ and $x \in A\backslash B$. Then (since x compact and B is closed) we have $\operatorname{dist}(x,B) = \epsilon > 0$. Therefore

$$x \notin B + B(0,\frac{\epsilon}{2}),$$

which is a contradiction. The second part of the assertion follows analogously.

□

We will briefly discuss whether or not there is a topology such that the $\|\cdot\|$-based convergence corresponds to the convergence in that topology. In our cases, it will be the question whether such a topology (a metric) exists on $\mathcal{F}_0(F)$ for a bounded subset $F \subset E$. Considering the criteria for a convergence to be topological, which are well known (see e.g. [KT84, Criteria 3.3.1]), we obtain three different answers. We will formulate the results for metric spaces, even if most of the following results hold in a more general framework (see [KT84, chapters 3 and 4]).

Theorem B.6. *(See [KT84, Theorem 4.3.8, Theorem 3.3.10, Theorem 3.3.11, Corollary 4.2.4, Theorem 4.2.4, Corollary 4.5.6]) Let $F \subset E$ such that (F,d) is a complete metric space, where $d(x,y) = \|x-y\|$, $x, y \in E$.*

1. *$(\mathcal{F}_0(F), \delta)$ is complete.*
2. *If (F, d) is not a locally compact metric space, then there is no metric on $\mathcal{F}_0(F)$ which generates the $\|\cdot\|$-based convergence.*
3. *If (F, d) is a compact metric space, and $(A_n)_{n\in\mathbb{N}}$ is a sequence of subsets in F, then*
$$\lim_{n\to\infty}^{\|\cdot\|} A_n = A$$
if and only if
$$A_n \to A, \quad n \to \infty \text{ in } (\mathcal{F}_0(F), \delta(\cdot,\cdot)).$$
4. *Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of closed subset in the locally compact Banach space $(E, \|\cdot\|)$, then*
$$\lim_{n\to\infty}^{\|\cdot\|} A_n = A$$
if and only if A_n converges to A in the topology of closed convergence on $\mathcal{F}(E) := \{A \subset E : A \text{ closed}\}$. Moreover, this topology is metrizable.

Therefore, we can deduce for our problem in the finite dimensional case the following results.

Corollary B.7. *Suppose that the Banach space $(E, \|\cdot\|)$ is finite dimensional. Then, for any sequence of closed subset $(A_n)_{n\in\mathbb{N}}$ in $B(0,1)$*
$$\lim_{n\to\infty}^{\|\cdot\|} A_n = A,$$
if and only if
$$\delta(A_n, A) \to 0 \quad , n \to \infty.$$
Furthermore, A is a closed subset of $B(0,1)$.

In the general case, we still have the following.

Theorem B.8. *(see [AB06, Theorem 3.82]) Let F_n be a sequence of closed sets in $(E, \|\cdot\|)$ such that*
$$\delta(F_n, F) \to 0, \quad n \to \infty,$$
for a closed subset $F \subset E$. Then
$$\lim_{n\to\infty}^{\|\cdot\|} F_n \to F, \quad n \to \infty.$$

The converse direction is false unless E is compact.

Example B.9. (see [AB06, Example 3.83]) Let $X = \mathbb{N}$ and
$$\operatorname{dist}(A, B) = 0, \text{ if } A = B \text{ and } \operatorname{dist}(A, B) = 1, \text{ if } A \neq B.$$
Then, for $F_n = \{1, \dots, n\},\ n \in \mathbb{N}$ one has
$$\lim_{n\to\infty}^{\|\cdot\|} F_n \to \mathbb{N}, \quad n \to \infty,$$
but
$$\delta(F_n, F) = 1, \quad \forall\ n \in \mathbb{N}.$$

References: Chapters 3 and 4 of [KT84] and Chapter 3 in [AB06].

Nomenclature

Abbreviations

a.s.	almost surely	19.
abbr.	abbreviated	8.
i.e.	id est, that is	7.
i.i.d.	independent and identically distributed	32.
r.e.	random element	8.

Basics and sets

$\|A\|$	Cardinality of the set A	8.
$\mathrm{conv}(A)$	convex hull of the set A	164.
$\mathbb{N}$	Natural numbers without 0	8.
$\mathbb{R}$	Real numbers	8.
$\mathbb{R}^+$	Positive real numbers	8.

Banach space definitions

$(E', \|\cdot\|)$	the topological Dual Space Banach Space to E	7.
$(E, \|\cdot\|)$	a separable Banach Space	7.

$\mathrm{dist}(x,A)$	$\inf_{a\in A}\|x-a\|$ for $x\in E$ and $A\subset E$	36.
λA	$\{\lambda x : x\in A\},\ \lambda\geq 0;\ \varnothing$ if $\lambda<0$	11.
$\overline{A}$	closure of A in $(E,\|\cdot\|)$	11.
∂A	the boundary of the set A	35.
$\mathring{A}$	the inner of the set A	18.
$A+B$	$\{x+y : x\in A, y\in B\}$, Minkowski sum	11.
A^c	the complement of the set A	13.
$B_0(a,\epsilon)$	Norm ball around a induced by the norm $\|\cdot\|_0$	67.
$B_{\|\cdot\|}(a,\epsilon), B(a,\epsilon)$	Norm Ball with radius ϵ around a	7.
$H_e(\epsilon,A)$	Metric ϵ-entropy of A	31.
$N_e(\epsilon,A)$	$\inf\{n\in\mathbb{N} : \exists\alpha\subset E, \lvert\alpha\rvert\leq n, A\subset\alpha+B(0,\epsilon)\}$	31.

Probabilistic definitions

$(\Omega,\mathcal{F},\mathbb{P})$	an abstract Probability Space	8.
$(y,z)_{L_2(\mu)}$	Inner product in $L_2(\mu)$	9.
$\mathbb{E}X$	Mean of the random element X	9.
$\mathcal{B}(E)$	σ-field induced by the open sets in $(E,\|\cdot\|)$	8.
$\mathcal{H}_\mu$	$\{x\in E : \|x\|_{\mathcal{H}_\mu}<\infty\}$, Cameron-Martin space of μ	10.
$\mathcal{M}_r(\mathcal{B}(E))$	set of all probability measures on $\mathcal{B}(E)$ with finite r-th moment	18.
$\mathcal{N}(\nu,\sigma^2)$	normal distribution with mean ν and variance σ^2	8.
μ_a	Lebesgue-absolute continuous part of μ	23.
μ_r	Point density measure of $\mu\in\mathcal{M}_1(\mathcal{B}(\mathbb{R}^d))$	25.
μ_s	Lebesgue-singular part of μ	23.
$\|x\|_{\mathcal{H}_\mu}$	$\sup\{\langle x,y\rangle : y\in E', \|y\|_{L_2(\mu)}\leq 1\}$, norm in $\mathcal{H}_\mu$	10.
$\|X\|_{L_r(\mathbb{P},E)}$	The r-th moment of X, $\left(\mathbb{E}\|X\|^r\right)^{\frac{1}{r}}$	8.
$\overline{F^X}(x)$	$\mathbb{P}(\|X\|>x)$	41.

$\overline{F^X}_s(x)$	$\mathbb{E}1_{\Vert X\Vert>x}\Vert X\Vert^s$ . . . 41.
$\phi_\mu(\epsilon)$	$-\log(\mu(B(0,\epsilon)))$, small ball function of μ . . . 29.
Φ_{ν,σ^2}	Distribution function of the normal distribution $\mathcal{N}(\nu,\sigma^2)$. 8.
$\sigma(\mu)$	$\sup\{\Vert h\Vert : h\in\mathcal{K}_\mu\}$ norm of the embedding $j_\mu:\mathcal{H}_\mu\to E$ . . . 11.
$\mathrm{supp}(\mu)$	Support of the measure μ . . . 11.
C_X	covariance operator of the r.e. X . . . 10.
$J(h)$	$\frac{\Vert h\Vert^2_{\mathcal{H}_\mu}}{2}$, for $h\in\mathcal{H}_\mu$ . . . 12.
$J(x,\epsilon)$	$\inf_{h\in B(x,\epsilon)} J(h)$ . . . 12.
$L_2(\mu)$	Space of μ square-integrable functions . . . 9.
$L_r(\mathbb{P},E)$	$\{Y:(\Omega,\mathcal{F})\to(E,\mathcal{B}(E)), Y \text{ r.e.}, \Vert Y\Vert_{L_r(\mathbb{P},E)}<\infty\}$ . . . 8.
S_μ	formally extended operator of C_X onto E'_μ . . . 10.
$U(I)$	Uniform distribution on a bounded set $I\in\mathcal{B}(\mathbb{R}^d)$ . . . 23.

Quantization Theories

$\alpha_n(K)$	$\{a\in\alpha_n : V_a(\alpha_n)\cap K\neq\varnothing\}$ . . . 120.
$\Delta_{r,n}(X,E)$	$e^r_{n,r}(X,E)-e^r_{n+1,r}(X,E)$, increments of the quantization error 34.
$\Delta_r(X,E;\alpha_n;a_1,\dots,a_m)$	$e^r_{n,r}(X,E)-e^r_r(X,E;\alpha_n\cup\{a_1,\dots,a_m\})$ . . . 54.
$\mathcal{C}_{n,r}(X,E)$	The set of all r-optimal n-quantizers for X in E . . . 20.
$\overline{\rho}_{n,r}(X,E)$	Upper quantization radius $\sup\{\rho(\alpha):\alpha\in\mathcal{C}_{n,r}(X,E)\}$ . . . 34.
$\overline{s}_{n,r}(\mu,\mathbb{R}^d;a)$	$\inf_{s>0}\{V_a(\alpha_n)\cap\mathrm{supp}(\mu)\subset B(a,s)\}$ . . . 119.
$\rho(A)$	Maximum radius $\sup\{\Vert a\Vert : a\in A\}$ . . . 34.
$\tau_{n,r}(X,E)$	optimal n-th $L_r(\mathbb{P},E)$ random quantization error . . . 32.
$\underline{\alpha}_n(K)$	$\{a\in\alpha_n : V_a(\alpha_n)\subset K\}$ . . . 120.
$\underline{\rho}_{n,r}(X,E)$	Lower quantization radius $\inf\{\rho(\alpha):\alpha\in\mathcal{C}_{n,r}(X,E)\}$ . . . 34.
$\underline{s}_{n,r}(\mu,\mathbb{R}^d;a)$	$\sup_{s>0}\{V_a(\alpha_n)\supset B(a,s)\}$ . . . 119.
$\{C_a(\alpha), a\in\alpha\}$	Voronoi partition with respect to α . . . 17.

$\{V_a(\alpha), a \in \alpha\}$	Voronoi diagram with respect to α	17.
$e_r(\mu, E; \alpha)$	$L_r(\mathbb{P})$-quantization error for the distribution μ in E induced by the codebook α	19.
$e_r(X, E; \alpha)$	$L_r(\mathbb{P})$-quantization error for the r.e. X in E induced by the codebook α	17.
$e_{n,r}(\mu, E)$	optimal n-th $L_r(\mathbb{P})$-quantization error for the distribution μ in E	19.
$e_{n,r}(X, E)$	optimal n-th $L_r(\mathbb{P}, E)$-quantization error for the r.e. X	17.
$e_{r;\mathrm{loc}}(\mu, E; \alpha, C_a(\alpha))$	Local quantization error $\left(\int_{C_a(\alpha)} \lVert x-a \rVert^r \, d\mu(x)\right)^{\frac{1}{r}}$	34.
$f_\alpha(X)$	α-quantization of X	18.
f_α	Nearest neighbor projection on α	17.
r-s-**property**	$\limsup_{n\to\infty} \frac{e_s(X,E;\alpha_n)}{e_{n,r}(X,E)} < \infty$	34.

Real numbers

$\lvert x \rvert$	absolute value of $x \in \mathbb{R}$	8.
$\lfloor x \rfloor$	largest integer lower or equal to x	24.
$\mathcal{R}$	$\bigcup_{b\in\mathbb{R}} \mathcal{R}_b$	159.
$\mathcal{R}_\alpha$	The set of all functions f regularly varying at infinity with index $\alpha \in \mathbb{R}$	159.
$\phi_{r,\Delta}(n)$	$(-2\log(\Delta_{n,r}(X,E)))^{\frac{1}{2}}$	102.
$f(x) \gtrsim g(x)$	$\liminf_{x\to\infty, x\in I} \frac{f(x)}{g(x)} \geq 1$	22.
$f(x) \lesssim g(x)$	$\limsup_{x\to\infty, x\in I} \frac{f(x)}{g(x)} \leq 1$	22.
$f(x) \preccurlyeq g(x)$	$\limsup_{x\to\infty, x\in I} \frac{f(x)}{g(x)} < \infty$	22.
$f(x) \succcurlyeq g(x)$	$\liminf_{x\to\infty, x\in I} \frac{f(x)}{g(x)} < \infty$	22.
$f(x) \approx g(x)$	$0 < \liminf_{x\to\infty, x\in I} \frac{f(x)}{g(x)} < \limsup_{x\to\infty, x\in I} \frac{f(x)}{g(x)} < \infty$	22.
$f(x) \sim g(x)$	$\lim_{x\to\infty, x\in I} \frac{f(x)}{g(x)} = 1$	22.
$f^{\leftarrow}$	Asymptotic left inverse to f	160.

Set Theories

Norms and Spaces

Information Theories

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