A logarithmic barrier approach and its regularization applied to convex semi-infinite programming problems

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Contents

Chapter 1

Introduction

In this thesis we consider semi-infinite programming problems of the following general form:

$$
\begin{aligned}
\text{minimize} \quad & f(x) \\
\text{s.t.} \quad & x \in \mathbb{R}^n, \quad Ax = b, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \\
& g_i(x, t) \leq 0 \quad \text{for all } t \in T^i \, (i = 1, \dots, l),
\end{aligned}
\tag{1.1}
$$

where f is convex, each g_i is convex in x as well as continuous in t and each T^i is a nonempty compact set. Thus we deal with finitely many variables and infinitely many constraints. Such problems occur in various fields, for instance we point at the following applications:

- *Least-cost strategies for air pollution abatement* studied, e.g. by Gorr et al. [13] and Kortanek, Gorr [29];

- *Robot trajectory planning* studied, e.g. by Hettich, Still [18] and Haaren-Retagne [15];

- Engineering design problems like *Seismic resistant design of structures*, *electronic circuit design* and the *design of SISO/MIMO control systems* studied by Polak [37];

- *Digital filter design* studied, e.g. by Potchinkov [40, 41] and Kortanek, Moulin [30];

- Applications in finance studied, e.g. by Tichatschke et al. [56].

Besides many problems (including some of that given above) arise in the field of Chebyshev-approximations or optimal control problems. While Chebyshev-approximation problems are often linear semi-infinite programming problems (cf., e.g., Hettich, Zencke [19]), the discretized optimal control problems are mostly of a more difficult structure due to the involved differential equations (cf., e.g., Sachs [49]). For details and more applications we refer also to the collection papers [8] and [44] as well as to the extensive survey by Hettich, Kortanek [17].

1.1   Review of literature

As consequence of the variety of applications particular methods for solving semi-infinite problems were developed. A survey is given by Hettich, Kortanek [17] again. Thereby it turns out that the

numerical methods typically generate sequences of finite optimization problems and, following Hettich, Kortanek [17], we can classify them into three types: *exchange methods*, *discretization methods* and *methods based on local reduction*. Nevertheless, particularly caused by the infinite number of constraints each of these methods has critical points for a practical realization. So, the exchange methods require the solution of a global optimization problem in each step, the discretization methods typically lead to finite problems with a very large amount of constraints and the local reduction methods are based on the necessary optimality conditions and use a further knowledge of the local behaviour of the constraints. Each of these methods may cause a (very) high computational effort so that no standard method for solving semi-infinite problems is currently available.

Regardless the work in the field of semi-infinite programming the *interior-point approach* for solving finite (convex) problems was developed during the last decades. This research was first initiated by proposing the *logarithmic barrier method* by Frisch [11] in 1955. The fundamental results of the intensive study during the following years were summarized by the monograph of Fiacco, McCormick [9], published in 1968. A qualitatively new stage in the development of interior point methods has been started with the paper of Gill et al. [12], where the relationship between Karmarkar's method and the logarithmic barrier methods for linear programs was shown. This fact brought to the light the polynomial complexity of logarithmic barrier methods for some classes of problems so that competitive interior-point methods for solving finite convex, especially linear and quadratic, problems could be developed. A survey of such methods for (mostly) linear problems is given by Andersen et al. [1].

Motivated by these powerful methods for finite problems including large-scale problems it was natural to try to transfer ideas from interior point methods to the field of semi-infinite programming problems. So the first algorithm in this context was an extension of an affine-scaling algorithm to linear semi-infinite problems suggested by Ferris, Philpott [6, 7]. But it is not easily possible to extend each interior-point approach to semi-infinite problems. For instance Powell [42] showed that the application of Karmarkar's algorithm to linear semi-infinite problems does not have to work. Additionally, a survey of interior-point approaches which can naturally be extended to semi-infinite problems is given by Todd [57] and Tunçel, Todd [58]. A further approach originates from the *method of analytic centers* which was introduced by Sonnevend [53] and extensively studied by Jarre [22] for finite convex problems. In order to tackle the semi-infinite problem of the form (1.1) directly, Sonnevend [54, 55] and Schättler [50, 51] extended this approach to convex semi-infinite problems by introducing an integral form of the logarithmic barrier. But, unfortunately the barrier property may be lost due to the smoothing effect of the integral (cf., e.g, Tunçel, Todd [58] and Jarre [23]).

Usually boundedness (or in fact compactness) of the feasible set or at least of the solution set of the given problems is assumed in all interior point approaches for semi-infinite problems mentioned above. Dropping this restrictive assumption Kaplan, Tichatschke [26] suggested a combination of the logarithmic barrier method with a discretization procedure for the constraint set and the *proximal point method* which was introduced by Martinet [32, 33]. Furthermore, due to the regularization, the approach of Kaplan, Tichatschke allows to treat ill-posed semi-infinite problems with ill-posedness in the sense of Hadamard. Especially the case where the finite auxiliary problems are not solvable is

of interest in that field. A further advantage of the method proposed by Kaplan, Tichatschke is given by the fact that convergence of the iterates can be established. This is not clear in each case if one applies the pure interior-point methods for convex problems excepting linear and quadratic ones.

All methods stated above are based on the smooth problem formulation (1.1) and make use of differentiability properties of the involved functions. In contrast to this, Polak [37] suggested a nondifferentiable reformulation of semi-infinite problems by means of using the *max*-function in the description of the constraints. This leads in fact to optimization problems with finitely many but nondifferentiable constraints which cause some difficulties. Nevertheless this reformulation will be the basis of the thesis which is outlined in the sequel.

1.2   Outline of the thesis

In Chapter 2 we start with a review of the classical logarithmic barrier method for convex problems since we intend to apply this method to convex semi-infinite programming problems. In particular the method is briefly stated and two basic convergence results are given.

Then several approaches transferring the logarithmic barrier method to semi-infinite programming problems, given in the smooth formulation (1.1), are discussed in detail. Thereby it turns out that certain difficulties from the theoretical and/or numerical point of view occur in each of these approaches.

In order to avoid these difficulties we apply the logarithmic barrier method directly to the nondifferentiable reformulation of the semi-infinite problems. Consequently, we consider barrier problems with nondifferentiable objective functions so that a method for minimizing nondifferentiable convex functions under convex constraints is required. Such methods often use subgradient information, more exact they often assume the existence of bounded subgradients or even subdifferentials on the feasible set. Due to the logarithmic part in the objective function of the barrier problems such a property does not hold in our case so that we enforce it by doing the following: The logarithmic barrier function is minimized on successively determined nonempty compact sets which are located in the relative interior of the feasible set. Introducing this procedure a conceptual algorithm for solving convex semi-infinite problems is finally presented.

In Chapter 3 the minimization of a convex nondifferentiable function on a nonempty convex compact set is in the focus of interest. Based on the assumption that the input data like objective function and subgradient information are exactly available, several known published methods can be used, one of which is the proximal level bundle method of Kiwiel [28]. Problematic in our case is that the objective function at hand contains a term whose evaluation requires the exact solution of a global maximization problem. In order to avoid this we extend Kiwiel's bundle method to the situation of inexact given input data. In doing so an inexact determination of the global maximum is permitted.

In Chapter 4 our conceptual algorithm is first described in detail for one semi-infinite constraint. This also includes the required specification of the assumptions. One essential assumption is the compactness of the solution set of the given problem. As stated above such an assumption (or the stronger condition that the feasible set is compact) is quite usual in the field of interior point methods.

Thus, after showing that the extended bundle method can be in fact applied, a convergence analysis based on the results of Fiacco, McCormick [9] is presented. Since these results only ensure the convergence of the iterates to the solution set in general, it is not surprising that we cannot prove convergence to one certain point of this set if the given problem is not uniquely solvable. But in each case convergence to the solution set can be established.

Finally, the straight-forward extension of the implementable algorithm to convex problems with finitely many semi-infinite constraints is presented. Thus, without specifying detailed assumptions at this point, we are able to solve convex problems of the form (1.1) if they possess a nonempty compact set of optimal solutions.

In Chapter 5 we drop this restrictive condition on the solution set. Then, following the ideas of Kaplan, Tichatschke [24–27], our method developed in Chapter 4 is coupled with the proximal point regularization technique. This procedure leads to auxiliary problems with strongly convex objective functions so that these problems are uniquely solvable and the method suggested in Chapter 4 is applicable to them. Based on this fact a combined algorithm is stated in detail. Therein we additionally make use of the multi-step technique introduced by Kaplan, Tichatschke [24] which allows to do more steps of the algorithm with large barrier parameters. Since the conditioning of the barrier problems is getting worse when the barrier parameter tends to zero, the multi-step approach stabilizes the combined method.

A convergence analysis based on that of Chapter 4 and that of Kaplan, Tichatschke [27] is established. Thereby it turns out that, in contrast to the method presented in Chapter 4, the regularized algorithm generates a sequence which converges to an optimal solution of the given problem under certain conditions.

Furthermore, a result with respect to the rate of convergence of the values of the objective function holds under more restrictive conditions than before. But, considering only the class of problems with quadratic growth we can even show linear convergence of the values of the objective function as well as the iterates. This reflects a well-known result in the theory of the proximal point method (cf., e.g., Rockafellar [47]).

In Chapter 6 we perform the numerical analysis of the discussed algorithms. In particular, we first determine the nonempty compact sets on which the minimization of the (regularized) logarithmic barrier function has to be done. Furthermore, based on the previously determined compact sets, we investigate how to compute the required constants. Then we have a closer look at the inexact maximization procedure which is required for each inexact evaluation of the logarithmic barrier function. The inexact maximization of a function on a nonempty compact set is usually carried out by maximizing this function on a finite grid which discretizes the given compact set. Since these grids can be very large, a deletion rule for excluding certain grid points from the maximization process is developed. This deletion rule should accelerate the evaluation of the logarithmic barrier function and consequently the whole iteration process.

The previously developed logarithmic barrier algorithms require strictly feasible starting points. Finding such points is in general a difficult task and we discuss their determination in detail.

In Chapter 7 we apply our algorithms to model examples in order to show the typical behaviour of the considered methods. Most of the examples are previously investigated by Voetmann [61] in

the context of the proximal interior point method of Kaplan, Tichatschke [26].

In Chapter 8 an application arising in the field of finance (cf. Tichatschke et al. [56]) is presented. In particular we approximate the run of the curve of the German stock index DAX over a given time interval. The approximation is based on a differential equation under uncertainty. So, by means of some simplifications we obtain a linear Chebyshev approximation problem.

In Chapter 9 we discuss the design of digital filters. We first give an introduction into the mathematical model of the design of perfect reconstruction filter banks. This leads to a semi-infinite program with a single constraint which was previously investigated by Kortanek, Moulin [30].

Chapter 2

The logarithmic barrier approach for convex optimization problems

In this chapter we first summarize basic results of the classical logarithmic barrier method for finite convex optimization problems. Further several trials for an extension of this method to semi-infinite problems are reported and a conceptual algorithm for solving convex semi-infinite problems is developed.

2.1 Finite problems

In this section the classical logarithmic barrier approach for solving finite convex programming problems is reviewed. In order to do this we consider the problem

$$\text{minimize} \quad f(x) \quad \text{s.t.} \quad x \in \mathbb{R}^n, \quad Ax = b, \quad g_j(x) \leq 0 \ (j = 1, \ldots, l) \tag{2.1}$$

with $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ as well as convex functions $f : \mathbb{R}^n \to \mathbb{R}$ and $g_j : \mathbb{R}^n \to \mathbb{R}$ for $j = 1, \ldots, l$. Then the classical logarithmic barrier approach can be described as follows (see, e.g., Wright [62], Section 3.2):

Algorithm 2.1

- Given $\mu_1 > 0$.

- For $i = 1, 2, \ldots$:

    - Compute a minimizer x^i of the barrier problem

    $$\text{minimize} \quad f_i(x) := f(x) - \mu_i \sum_{j=1}^{l} \ln(-g_j(x)) \tag{2.2}$$
    $$\text{s.t.} \quad x \in \mathbb{R}^n, \quad Ax = b, \quad g_j(x) < 0 \ (j = 1, \ldots, l).$$

    - Choose $\mu_{i+1} \in (0, \mu_i)$.

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The algorithm is practicable if problem (2.2) is solvable in each step. This ensures the following lemma which corresponds to Lemma 12 in Fiacco, McCormick [9] and Theorem 4 in Wright [62], although we are able to use a weaker assumption. Fiacco, McCormick [9], Wright [62] as well as other authors assume that the feasible region of (2.1) is bounded. Instead of this restrictive condition we only assume that the set of optimal solutions of (2.1) is bounded (which is in fact equivalent to the compactness since we deal with continuous functions in a finite dimensional space).

Lemma 2.2 *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *and* $g_j : \mathbb{R}^n \to \mathbb{R}$ *for* $j = 1, \ldots, l$ *be convex functions and* $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ *be given. Assume that the solution set of* (2.1) *is nonempty and compact. Moreover, assume that the Slater Constraint Qualification is fulfilled, i.e., there exists* $\hat{x} \in \mathbb{R}^n$ *with* $A\hat{x} = b$ *and* $g_j(\hat{x}) < 0$ *for all* $j = 1, \ldots, l$. *Then the level set*

$$\mathcal{L}_i(\tau) := \{x \in \mathbb{R}^n : f_i(x) \leq \tau, \, Ax = b, \, g_j(x) < 0 \, (j = 1, \ldots, l)\} \tag{2.3}$$

is compact for all $\tau \in \mathbb{R}$ *and fixed* $i \in \mathbb{N}$. *Especially problem* (2.2) *is solvable with a compact set of optimal solutions.*

Proof: Let $\tau \in \mathbb{R}$ be arbitrarily given. To show the compactness of $\mathcal{L}_i(\tau)$ we prove that it is bounded and closed.

We first show that it is bounded. Suppose that $\mathcal{L}_i(\tau)$ were unbounded, then there exists a sequence $\{z^k\}$ with $z^k \in \mathcal{L}_i(\tau)$ and $\|z^k\| > k$. $\|\cdot\|$ is an arbitrary but fixed norm on $\mathbb{R}^n$. Setting $y^k := z^k/\|z^k\|$ we have $\|y^k\| = 1$ for all $k \in \mathbb{N}$ and the sequence $\{y^k\}$ has at least one accumulation point y with $\|y\| = 1$. Without loss of generality we assume that $\{y^k\}$ converges to y. Let x^* be an optimal solution of (2.1). Then we want to show that each point $x^* + sy$ with $s > 0$ is also an optimal solution of (2.1) which contradicts our assumption of the compactness of the solution set since $y \neq 0$.

In order to show the feasibility of such points $x^* + sy$ for (2.1) let $s > 0$ be fixed. Then, taking the convexity of g_j into account, we have for all $k > s$ and $j = 1, \ldots, l$:

$$g_j\left(\left(1 - \frac{s}{\|z^k\|}\right)x^* + s\frac{z^k}{\|z^k\|}\right) \leq \left(1 - \frac{s}{\|z^k\|}\right)g_j(x^*) + \frac{s}{\|z^k\|}g_j(z^k) < 0.$$

Thus $k \to \infty$ leads to $g_j(x^* + sy) \leq 0$ for all $j = 1, \ldots, l$. Additionally, it holds

$$Ay = \lim_{k \to \infty} Ay^k = \lim_{k \to \infty} \frac{Az^k}{\|z^k\|} = \lim_{k \to \infty} \frac{b}{\|z^k\|} = 0$$

such that $x^* + sy$ is a feasible solution of (2.1). Further, $g_j(x^*) \leq 0$ and the convexity of g_j allow to conclude

$$0 > g_j(z^k) \geq \left(\|z^k\| - 1\right)g_j(x^*) + g_j(z^k) \geq \|z^k\| g_j\left(\left(1 - \frac{1}{\|z^k\|}\right)x^* + \frac{z^k}{\|z^k\|}\right)$$

for all $k > 1$ and $j = 1, \ldots, l$. Therefore, regarding the convergence of $g_j\left(\left(1 - \frac{1}{\|z^k\|}\right)x^* + \frac{z^k}{\|z^k\|}\right)$ to $g_j(x^* + y)$ for all j, there exists a constant $C_0 > 0$ independent of j (because only finitely many

constraints occur) and k with $g_j(z^k) \geq -C_0\|z^k\|$. Using this as well as the monotonicity of the logarithm we obtain

$$\tau \geq f_i(z^k) = f(z^k) - \mu_i \sum_{j=1}^{l} \ln(-g_j(z^k)) \geq f(z^k) - \mu_i l \ln(C_0\|z^k\|).$$

Hence, regarding the convexity of f, one infers

$$f\left(\left(1 - \frac{s}{\|z^k\|}\right)x^* + s\frac{z^k}{\|z^k\|}\right) \leq \left(1 - \frac{s}{\|z^k\|}\right)f(x^*) + s\frac{f(z^k)}{\|z^k\|}$$

$$\leq \left(1 - \frac{s}{\|z^k\|}\right)f(x^*) + s\frac{\tau}{\|z^k\|} + s\mu_i l\frac{\ln(C_0\|z^k\|)}{\|z^k\|}$$

for all $k > s$. Then $k \to \infty$ gives us $f(x^* + sy) \leq f(x^*)$. Consequently $x^* + sy$ is an optimal solution of (2.1) for each $s \geq 0$. As stated above this contradicts the assumption of the compactness of the solution set of (2.1) such that $\mathcal{L}_i(\tau)$ cannot be unbounded.

To show that $\mathcal{L}_i(\tau)$ is closed, we prove that it contains all its accumulation points. Let $\{z^k\}$ be a convergent sequence with $z^k \in \mathcal{L}_i(\tau)$ for all k and $z \in \mathbb{R}^n$ as its limit point. First, from $Az^k = b$ for all $k \in \mathbb{N}$ follows easily that $Az = b$. Further, since the convex functions f and $g_1, \ldots, g_l$ are continuous on $\mathbb{R}^n$ (see, e.g., Rockafellar [45], Corollary 10.1.1), we have $\lim_{k \to \infty} f(z^k) = f(z)$ and $\lim_{k \to \infty} g_j(z^k) = g_j(z)$ for all $j = 1, \ldots, l$. Thus one infers $g_j(z) \leq 0$, $g_j(z^k) \geq C_1$ and $f(x^k) \geq C_1$ with a constant $0 > C_1 > -\infty$ independent of j and k. Therefore, taking $f_i(x^k) \leq \tau$ for all $k \in \mathbb{N}$ and the monotonicity of the logarithm into account, we can conclude

$$-\mu_i \ln\left(-g_\nu(z^k)\right) \leq \tau - f(z^k) + \mu_i \sum_{\substack{j=1 \\ j \neq \nu}}^{l} \ln\left(-g_j(z^k)\right)$$

$$\leq \tau - C_1 + \mu_i(l-1)\ln(-C_1) =: \mu_i C_2$$

for all $\nu = 1, \ldots, l$ with a constant $C_2 < \infty$. Then it follows

$$g_\nu(z^k) \leq -e^{-C_2} < 0$$

and $g_\nu(z) < 0$ for all $\nu = 1, \ldots, l$. Additionally, f_i is obviously continuous on its domain $\mathrm{dom}\,(f_i) = \{x \in \mathbb{R}^n : g_j(x) < 0\,(j = 1, \ldots, l)\}$ so that $f_i(z) \leq \tau$ follows from the inclusion $\{z^k\} \subset \{x \in \mathbb{R}^n : g_j(x) \leq -e^{-C_2}\} \subset \mathrm{dom}\,(f_i)$. Hence, it yields $z \in \mathcal{L}_i(\tau)$ such that the level set is closed.

Finally, we have to show that (2.2) is solvable with a compact solution set. Due to the existing Slater point $\hat{x}$ the level set $\mathcal{L}_i(\tau)$ with $\tau = f_i(\hat{x})$ is nonempty. Moreover, each optimal solution of (2.2) must be an element of $\mathcal{L}_i(f_i(\hat{x}))$. Consequently the optimization problem

$$\text{minimize} \quad f_i(x) \quad \text{s.t.} \quad x \in \mathcal{L}_i(f_i(\hat{x})) \tag{2.4}$$

has the same solution set as problem (2.2). We already know that the level set $\mathcal{L}_i(f_i(\hat{x}))$ is compact and contained in the domain of f_i. In problem (2.4) we have to minimize the continuous function

f_i on a nonempty compact feasible set. Thus there exists at least one optimal solution of (2.4), resp. (2.2). Furthermore the set of optimal solutions coincides with the level set $\mathcal{L}_i(f_i^*)$ where f_i^* is the optimal value of f_i. Thus, the compactness of this set follows from the statements above. $\quad\square$

The following result shows that we can compute an optimal solution of (2.1) with Algorithm 2.1. This theorem corresponds to Theorem 25 in Fiacco, McCormick [9] and Theorem 5 in Wright [62]. Let f^* denote the optimal value of (2.1).

Theorem 2.3 *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g_j : \mathbb{R}^n \to \mathbb{R}$ for $j = 1, \dots, l$ be convex functions. Furthermore, let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ be given. Assume that the set of optimal solutions of (2.1) is nonempty and compact. Moreover, assume that the Slater Constraint Qualification is fulfilled. Let $\{\mu_i\}$ be a positive sequence with $\lim_{i \to \infty} \mu_i = 0$ and let $\{x^i\}$ denote a sequence of arbitrary optimal solutions of (2.2). Then the following is true*

(a) *The functions f_i are convex on their domain.*

(b) *The sequence $\{x^i\}$ is bounded.*

(c) *It holds*
$$0 \leq f(x^i) - f^* \leq \mu_i l \tag{2.5}$$
for all $i \in \mathbb{N}$ and $\lim_{i \to \infty} f(x^i) = f^*$.

(d) *Each accumulation point of $\{x^i\}$ is an optimal solution of (2.1).*

(e) *If $\{\mu_i\}$ is a monotonically decreasing sequence and if $\{x^i\}$ converges, then*
$$\lim_{i \to \infty} f_i^* = f^*.$$

Proof: Let us first remark that the existence of x^i is ensured by Lemma 2.2 for all $i \in \mathbb{N}$. Now the separate propositions are successively proven.

(a) Let $i \in \mathbb{N}$ be fixed. Since f is convex on $\mathbb{R}^n$ and μ_i is positive it remains to prove that the logarithmic part
$$-\sum_{j=1}^{l} \ln\left(-g_j(x)\right) \tag{2.6}$$
is convex on $\operatorname{dom}(f_i) = \{x \in \mathbb{R}^n : g_j(x) < 0 \, (j = 1, \dots, l)\}$. This will be done by showing that each addend of this sum is convex.

The logarithm is a concave increasing function. Consequently, $-\ln(-t)$ is a convex increasing function. Taking the convexity of g_j into account each summand in (2.6) is the post-composition of a convex function with an increasing convex function. Such a composition is also convex after Proposition IV.2.1.8 in Hiriart-Urruty, Lemaréchal [20].

(b) Let $\mu_0 \in \mathbb{R}$ be given such that $\mu_i < \mu_0$ for $i \in \mathbb{N}$ holds. Moreover, let x^0 be an optimal solution of (2.2) with $i = 0$. Then we have
$$f(x^i) - \mu_i \sum_{j=1}^{l} \ln\left(-g_j(x^i)\right) \leq f(x^0) - \mu_i \sum_{j=1}^{l} \ln\left(-g_j(x^0)\right)$$

and

$$f(x^0) - \mu_0 \sum_{j=1}^{l} \ln\left(-g_j(x^0)\right) \le f(x^i) - \mu_0 \sum_{j=1}^{l} \ln\left(-g_j(x^i)\right)$$

for all $i \in \mathbb{N}$. Multiplying the first inequality with μ_0/μ_i and combining the resulting estimate with the second inequality one obtains

$$f(x^0) - f(x^i) \le \mu_0 \sum_{j=1}^{l} \left(\ln\left(-g_j(x^0)\right) - \ln\left(-g_j(x^i)\right)\right) \le \frac{\mu_0}{\mu_i}\left(f(x^0) - f(x^i)\right)$$

for all $i \in \mathbb{N}$. Due to $\mu_0 > \mu_i$ for all $i \in \mathbb{N}$ this can only be true if $f(x^i) \le f(x^0)$ holds for each $i \in \mathbb{N}$. Thus, regarding also $Ax^i = b$ for all $i \in \mathbb{N}$, each x^i is an element of the level set $\{x \in \mathbb{R}^n : f(x) \le f(x^0), Ax = b, g_j(x) \le 0 \, (j = 1, \ldots, l)\}$. Due to the compactness of the solution set of (2.1) these sets are compact which can be proven similarly to Corollary 20 in Fiacco, McCormick [9]. Therefore the sequence $\{x^i\}$ is bounded.

(c) The left inequality in (2.5) is simply true since each x^i is feasible for (2.1). Thus it remains to show $f(x^i) - f^* \le \mu_i l$. In order to prove this let an optimal solution x^* of (2.1) be arbitrarily given. The point x^i is a minimizer of f_i on $\mathcal{M}_0 := \{x \in \mathbb{R}^n : Ax = b, g_j(x) < 0 \, (j = 1, \ldots, l)\}$. Thus, regarding the convexity of $\mathcal{M} := \{x \in \mathbb{R}^n : Ax = b, g_j(x) \le 0 \, (j = 1, \ldots, l)\}$ as well as $\mathcal{M}_0 = \text{ri}\,(\mathcal{M})$, Theorem 6.1 in Rockafellar [45] allows to conclude $x^i + t(x^* - x^i) \in \mathcal{M}_0$ for all $t \in [0, 1)$. Therefore we have

$$0 \le \frac{f_i(x^i + t(x^* - x^i)) - f_i(x^i)}{t}$$

for all $t \in (0, 1)$. Using the existence of the directional derivative $f_i'(x^i; x^* - x^i)$ (cf., e.g., Rockafellar [45], Theorem 23.1) this combined with Theorem 23.4 in Rockafellar [45] leads immediately to

$$0 \le f_i'(x^i; x^* - x^i) = \max_{z \in \partial f_i(x^i)} z^T(x^* - x^i),$$

if we take into account that $x^i \in \text{int}\,(\text{dom}\,(f_i))$ which enforces the compactness of $\partial f_i(x^i)$. So now we have to determine a closed form for the subdifferential of f_i in x^i. From the proof of (a) we know that the functions $-\ln(-g_j(x))$ are convex on $\text{dom}\,(f_i)$ for all $j = 1, \ldots, l$. Thus, regarding Theorem 23.4 in Rockafellar [45], $\partial(-\ln(-g_j(x^i)))$ is nonempty for all i and j. Since $-\ln(-t)$ is an increasing convex function and g_j is convex for all $j = 1, \ldots, l$ we can apply Theorem XI.3.6.1 in Hiriart-Urruty, Lemaréchal [21] so that we infer in combination with Proposition XI.1.3.1 in Hiriart-Urruty, Lemaréchal [21]

$$\partial\left(-\ln\left(-g_j(x^i)\right)\right) = \frac{1}{-g_j(x^i)}\partial g_j(x^i).$$

Consequently, using Theorem 23.8 in Rockafellar [45] and Proposition XI.1.3.1 in Hiriart-Urruty, Lemaréchal [21] and regarding that the intersection

$$\text{ri}\,(\text{dom}\,(f)) \cap \bigcap_{j=1}^{l} \text{ri}\,(\text{dom}\,(-\mu_i \ln\,(-g_j)))$$

is nonempty (since each x^i is an element of it), we have

$$\partial f_i(x^i) = \partial f(x^i) + \mu_i \sum_{j=1}^{l} \partial(-\ln(-g_j(x^i))) = \partial f(x^i) + \mu_i \sum_{j=1}^{l} \frac{1}{-g_j(x^i)} \partial g_j(x^i). \qquad (2.7)$$

Hence, there exist $u \in \partial f(x^i)$ and $v_j \in \partial g_j(x^i)$ with

$$f_i'(x^i; x^* - x^i) = \left(u + \mu_i \sum_{j=1}^{l} \frac{1}{-g_j(x^i)} v_j \right)^T (x^* - x^i).$$

Then, regarding the definition of the subdifferential and $g_j(x^*) \leq 0$ for $j = 1, \ldots, l$, we infer

$$0 \leq u^T(x^* - x^i) + \mu_i \sum_{j=1}^{l} \frac{1}{-g_j(x^i)} v_j^T(x^* - x^i)$$

$$\leq f(x^*) - f(x^i) + \mu_i \sum_{j=1}^{l} \frac{g_j(x^*) - g_j(x^i)}{-g_j(x^i)}$$

$$\leq f^* - f(x^i) + \mu_i l.$$

(d) From (b) we know that $\{x^i\}$ is bounded so that it has an accumulation point $\tilde{x}$. Then it follows $f(\tilde{x}) = f^*$ from (2.5). Furthermore, $\tilde{x}$ is obviously feasible for (2.1). Hence, $\tilde{x}$ is an optimal solution of (2.1).

(e) Let x^* be the limit point of $\{x^i\}$. If we have $g_j(x^*) < 0$ for all $j = 1, \ldots, l$ then one can conclude

$$\lim_{i \to \infty} \mu_i \sum_{j=1}^{l} \ln\left(-g_j(x^i)\right) = 0$$

such that we infer with (c)

$$\lim_{i \to \infty} f_i^* = \lim_{i \to \infty} f(x^i) - \mu_i \sum_{j=1}^{l} \ln\left(-g_j(x^i)\right) = f^*.$$

Thus in the sequel we assume that there exists at least one index $j \in \{1, \ldots, l\}$ with $g_j(x^*) = 0$ which implies

$$\sum_{j=1}^{l} \ln\left(-g_j(x^i)\right) < 0$$

for all i sufficiently large. Then, regarding that we have a nonincreasing sequence $\{\mu_i\}$, one can

conclude

$$f^* \leq f(x^{i+1})$$

$$< f(x^{i+1}) - \mu_{i+1} \sum_{j=1}^{l} \ln\left(-g_j(x^{i+1})\right) = f_{i+1}^*$$

$$\leq f(x^i) - \mu_{i+1} \sum_{j=1}^{l} \ln\left(-g_j(x^i)\right)$$

$$\leq f(x^i) - \mu_i \sum_{j=1}^{l} \ln\left(-g_j(x^i)\right) = f_i^*$$

for all i sufficiently large. Especially the sequence $\{f_i^*\}$ decreases monotonically (at least for large i) and is bounded below which implies the convergence of it. Set $\alpha := \lim_{i \to \infty} f_i^*$. Then we have $\alpha \geq f^*$ from above. We want to show that $\alpha = f^*$ holds. For this purpose we assume $\alpha > f^*$ and set $\delta := (\alpha - f^*)/2 > 0$. Furthermore we choose $\tilde{x} \in \mathbb{R}^n$ with $A\tilde{x} = b$ and $g_j(\tilde{x}) < 0$ for all $j = 1, \ldots, l$ and $f(\tilde{x}) \leq \alpha - \delta$. Such a point has to exist due to (c). Since $\{\mu_i\}$ is a positive sequence with limit point 0 it yields

$$\alpha \leq f_i(x^i) \leq f_i(\tilde{x}) = f(\tilde{x}) - \mu_i \sum_{j=1}^{l} \ln\left(-g_j(\tilde{x})\right) \leq \alpha - \delta + \frac{\delta}{2} = \alpha - \frac{\delta}{2}$$

for all i sufficiently large, which contradicts our assumption. $\qquad \square$

2.2 Transfer to semi-infinite problems

In the sequel we want to transfer the classical logarithmic barrier method analyzed in the previous section to convex semi-infinite problems of the form (1.1). For the sake of simplicity of the presentation we consider (1.1) with $j = 1$ (the index will be dropped) and without linear equality constraints, i.e.

$$\text{minimize} \quad f(x) \quad \text{s.t.} \quad x \in \mathbb{R}^n, \quad g(x, t) \leq 0 \ (t \in T). \tag{2.8}$$

But, in the further course we describe the possibility of the extension to problems of the general form (1.1).

Without specifying any assumptions at this point it turns out that the most difficult question for the transfer of the logarithmic barrier method to semi-infinite problems is how can we choose a suitable barrier function. This is caused by the (possibly) infinitely many constraints.

Considering (2.1) without linear equality constraints we can embed problems of this type into the class of problems described by (2.8) by setting $T := \{1, \ldots, l\}$ and $g(x, t) := g_t(x)$. Thus a natural generalization of the barrier term is given by

$$-\sum_{t \in T} \ln(-g(x, t)).$$

But obviously this leads to serious problems. If T is an uncountable set the definition of this sum is not clear and if T is an infinite countable set, serious numerical problems occur when evaluating the sum. Furthermore, if T is a finite but large set the barrier parameter μ has to be very small to guarantee a certain accuracy by estimate (2.5). But in practice this avoids the machine precision so that a direct transfer of the classical logarithmic barrier in the sense above is inadvisable and we reject this approach.

A first alternative is the method of outer approximation described for instance by Powell [43] in the case of linear problems. Thereby we replace step by step the set T by discretizations which become successively finer. Consequently we have to solve finite problems of type (2.1) without linear equality constraints in each step. These problems can be solved theoretically with the classical logarithmic barrier approach from above, but again several difficulties occur. At first if the relaxed problems are solvable the discretized set and consequently the number of the considered constraints grows such that again the barrier parameter has to be very small to guarantee a good approximate solution. Furthermore, in general the optimal solutions of the relaxed problems are not feasible for the original problem. Consequently if we approximately compute optimal solutions by this method they are typically not feasible for the original problem. Another serious difficulty is that the properties of the original problem do not have to be inherited to the relaxed problems. Especially it is possible that the relaxed problems are not solvable (for examples see, e.g. Kaplan, Tichatschke [24]). Due to these difficulties we look for alternatives.

Sonnevend [54] and Schättler [50, 51] suggest to use the following "Integral Barrier Function"

$$- \int_T \ln(-g(x,t))dt. \tag{2.9}$$

Of course, $\operatorname{meas}(T) > 0$ is assumed in this case so that especially finite sets T are excluded. Nevertheless, let us have a closer look at some important details of the arising method.

Due to the smoothing effect of the integral, (2.9) does not have to possess the barrier property at all. That means it is possible that (2.9) is bounded above if one approaches the boundary of the feasible region. This fact is illustrated by the following example of Jarre [23].

Example 2.4 We consider the linearly bounded feasible set

$$S := \left\{ x \in \mathbb{R}^2 : g(x,t) := -\left(t - \frac{1}{2}\right)^2 x_1 - x_2 \leq 0 \, (t \in [0,1]) \right\}.$$

Now, choosing $x = (1,0)^T$, we have $g(x,t) = -(t-\frac{1}{2})^2 \leq 0$ for all $t \in [0,1]$. Thus $x \in S$ but $g(x,\tilde{t}) = 0$ for $\tilde{t} = \frac{1}{2}$ implies $x \notin \operatorname{int}(S)$. Furthermore, using $T = [0,1]$, we conclude

$$- \int_T \ln(-g(x,t))dt = - \int_0^1 \ln\left(\left(t-\frac{1}{2}\right)^2\right) dt = -4 \int_{1/2}^1 \ln\left(t-\frac{1}{2}\right) dt = 2\ln 2 < \infty.$$

$\square$

Finally, let us have a look at (2.9) from the numerical point of view. Here we have the task to evaluate integrals of the form (2.9) at different points x. If x is not located near the boundary of

the feasible region this could be done with standard formulas for numerical integration. But if we evaluate this integral for a point near the boundary of the feasible region the logarithm will have large absolute values for certain t. Consequently standard formulas for numerical integration do not work very well in this area. But, we have to be able to evaluate the barrier function (and therefore also the integrals) near the boundary of the feasible region, because optimal solutions are typically located on this boundary. Due to these problems Schättler [50, 51] refers to specialized integration rules like Radau's or Lobatto's rule (see, e.g. Davis, Rabinowitz [4]) for evaluating the integrals. In contrast to this Lin et al. [31] use Simpson's method to compute similar integrals arising by transferring the exponential barrier to semi-infinite problems. In order to achieve a suitable accuracy they have to partition the interval $[0, 1]$ into 400000 small parts in one example case. Thus, independent of the formulae, evaluating such integrals requires a high computational effort.

2.3   A conceptual algorithm for semi-infinite problems

Taking all considerations from the previous section into account we decided to look for a more practical variant. In order to do that we consider the following reformulation of the semi-infinite problem (2.8)

$$\text{minimize} \quad f(x) \quad \text{s.t.} \quad x \in \mathbb{R}^n, \quad \max_{t \in T} g(x, t) \leq 0. \tag{2.10}$$

The theoretical properties as well as practical applications of this approach are extensively studied by Polak [37, 38]. The main advantage of the reformulation (2.10) is that we can write it in the form (2.1) with a single constraint by using

$$g_1(x) := \max_{t \in T} g(x, t).$$

Thus we can use the results of our first section for problems of type (2.10). Consequently we deal with the barrier function

$$f(x) - \mu \ln \left(- \max_{t \in T} g(x, t) \right). \tag{2.11}$$

Therefore in contradiction to the approaches mentioned above we have no additional difficulties from the theoretical point of view. But there are two remarkable numerical problems. We now deal with a nondifferentiable barrier function due to the involved *max*-term and we have to solve the global optimization problem

$$\text{maximize} \quad g(x, t) \quad \text{s.t.} \quad t \in T \tag{2.12}$$

in order to evaluate the barrier function at a given point x, which is in general a very hard task. Thus except for special cases we cannot suppose that (2.12) is exactly solvable for any given x with acceptable computational effort. Accordingly there is only an approximate maximizer of (2.12) available such that the barrier function is only approximately evaluable. Consequently we have to use a method for minimizing (2.11) which requires only an approximately computable objective function. Such a method, derived from a bundle method from Kiwiel [28], is presented in the next

chapter. This method requires the feasible sets to be compact. In contradiction to this the barrier function (2.11) has to be minimized on open sets of the form

$$\{x \in \mathbb{R}^n : g(x,t) < 0 \,(t \in T)\}.$$

Nevertheless, in order to use the method proposed we will minimize the (convex) barrier function successively on compact sets like closed boxes or balls. However, we still cannot suppose that we are able to compute an exact minimizer of (2.11) using only approximate values of the objective function. But as it is known from finite problems this do not have to be required (cf., e.g., den Hertog [5]). However, the classical logarithmic barrier method from Algorithm 2.1 has to adapted in the sense that x^k is from now on only an approximate minimizer of the barrier function. Altogether we obtain the following conceptual algorithm for solving (2.10) resp. (2.8).

Algorithm 2.5

- Given $\mu_1 > 0$.

- For $i = 1, 2, \ldots$:

    - For $k = 1, 2, \ldots$:

        * Determine a nonempty compact set $S^{i,k} \subset \{x \in \mathbb{R}^n : \max_{t \in T} g(x,t) < 0\}$.
        * Compute an approximate minimizer $x^{i,k}$ of (2.11) on $S^{i,k}$.
        * If $x^{i,k}$ is an approximate unconstrained minimizer of a certain accuracy of (2.11) set $x^i := x^{i,k}$ and leave the inner loop.

    - Choose $\mu_{i+1} \in (0, \mu_i)$.

In the following chapter we present a numerical method for minimizing the nondifferentiable barrier function (2.11) on $S^{i,k}$ such that in Chapter 4 we can give all necessary details to put this conceptual algorithm into implementable form.

Chapter 3

A bundle method using ε-subgradients

In this chapter we discuss a method for solving the nondifferentiable auxiliary problems which appear in the conceptual algorithm at the end of the previous chapter. In general these problems look like

$$\text{minimize} \quad f(x) \quad \text{s.t.} \quad x \in S \tag{3.1}$$

with a convex function f and a nonempty compact convex set $S \subset \mathbb{R}^n$. Moreover, let (3.1) be solvable and the following assumptions be fulfilled.

Assumption 3.1 *Let $\varepsilon \geq 0$ be given. Then it is assumed that the following holds*

   (a) for any $x \in S$ at least an ε-approximation $\tilde{f}(x)$ of $f(x)$ with

$$f(x) - \varepsilon \leq \tilde{f}(x) \leq f(x) \tag{3.2}$$

   can be computed;

   (b) for any $x \in S$ an ε-subgradient $g_f(x)$ of f can be computed;[1]

   (c) f is Lipschitz continuous on S with Lipschitz constant L_f such that $\|g_f(x)\|_2 \leq L_f$ for all $x \in S$.

These assumptions on (3.1) are generalizations of those of Kiwiel [28] (there we have $\varepsilon = 0$). Therefore we suggest a modification of Kiwiel's proximal level bundle method for solving problem (3.1). In Kiwiel's Algorithm 1 we replace all computations of f by $\tilde{f}$ and all computations of a subgradient by an ε-subgradient. Linearizing f in $x^k \in S$ by

$$f^k(x) := \tilde{f}(x^k) + g_f(x^k)^T (x - x^k)$$

leads to the following algorithm.

---

[1]If $\partial \tilde{f}(x) := \{z \in \mathbb{R}^n : \tilde{f}(y) \geq \tilde{f}(x) + z^T(y-x) \text{ for all } y \in \mathbb{R}^n\} \neq \emptyset$ then (3.2) ensures $\partial \tilde{f}(x) \subset \partial_\varepsilon f(x)$. Thus an ε-subgradient of f in x can be given by an element of $\partial \tilde{f}(x)$. Such a situation will always be given in our applications of the proposed bundle method.

Algorithm 3.2

(S0) Given $x^1 \in S$, the final tolerance $\varepsilon_{opt} \geq 0$, a level parameter $0 < \kappa < 1$ and $\varepsilon \geq 0$. Set $x_c^1 := x^1$, $f_{up}^0 := \infty$, $f_{low}^1 := \min_{x \in S} f^1(x)$, $J^1 := \{1\}$, $k := 1$, $l := 0$, $k(0) := 1$ ($k(l)$ denotes the iteration number of the l-th increase of f_{low}^k).

(S1) Set $f_{up}^k := \min\{\tilde{f}(x^k), f_{up}^{k-1}\}$, $\Delta^k := f_{up}^k - f_{low}^k$. If $f_{up}^k = \tilde{f}(x^k)$ set $x_{rec}^k := x^k$ (x_{rec}^k denotes the "best" known iterate up to the k-th step, i.e. $f_{up}^k = \tilde{f}(x_{rec}^k)$).

(S2) If $\Delta^k \leq \varepsilon_{opt}$ or $g_f(x^k) = 0$ terminate; otherwise continue.

(S3) If the feasible set of

$$
\begin{aligned}
\text{minimize} \quad & \frac{1}{2}\left\|x - x_c^k\right\|_2^2 \\
\text{s.t.} \quad & x \in S, \quad f^j(x) \leq \kappa f_{low}^k + (1 - \kappa)f_{up}^k \quad (j \in J^k)
\end{aligned}
\tag{3.3}
$$

is nonempty, go to (S5); otherwise continue.

(S4) Set $f_{low}^k := \min_{x \in S} \max_{j \in J^k} f^j(x)$. Choose $x_c^k \in \{x^j : j \in J^k\}$ arbitrarily. Set $k(l+1) := k$, increase l by 1 and go to (S1).

(S5) Find the solution x^{k+1} of (3.3) and its multipliers λ_j^k such that $\hat{J}^k := \{j \in J^k : \lambda_j^k > 0\}$ satisfies $|\hat{J}^k| \leq n$.

(S6) Calculate $\tilde{f}(x^{k+1})$ and $g_f(x^{k+1}) \in \partial_\varepsilon f(x^{k+1})$.

(S7) Select $J_s^k \subset J^k$ such that $\hat{J}^k \subset J_s^k$. Set $J^{k+1} := J_s^k \cup \{k+1\}$, $x_c^{k+1} := x_c^k$, $f_{low}^{k+1} := f_{low}^k$. Increase k by 1 and go to (S1).

Let us briefly describe this bundle method. While (S0) and (S1) are initializing steps, (S2) contains the stopping criterion. Then in (S3) a feasibility check of a projection problem with constraints given by the current bundle is done with the consequence of resetting the lower bound of the optimal value of (3.1) in the case of infeasibility in (S4). In the feasible case the projection is in fact done leading to the next iterate in (S5) and new values of the objective function as well as the ε-subgradient in (S6). Finally, in (S7) the bundle update based on the Lagrange multipliers of problem (3.3) is made so that the next iteration step can be done.

The practicability of this method can be easily shown by investigating each step separately. Thereby we regard in (S5) that many QP-methods automatically generate $|\hat{J}^k| \leq n$ since there are n variables involved. In addition let us remark that between two successive updates of the lower bound in (S4) the step (S5) has to reached at least once because the minimum of $\max_{j \in J^k} f^j(x)$ on S is attained at a certain point which is feasible for the following projection (3.3).

Now let us continue with a convergence analysis for the stated method started with a few technical results.

Lemma 3.3 *(cf. Lemma 3.1 in [28])*
For any given $k \in \mathbb{N}$ we have:
   If $k(l) < k < k(l+1)$ for some $l \in \mathbb{N}_0$ then

$$\|x^{k+1} - x^k\|_2 \geq \frac{\kappa \Delta^k}{L_f}$$

otherwise $k = k(l)$ for some $l \in \mathbb{N}_0$ and

$$\|x^{k+1} - x_c^k\|_2 \geq \frac{\kappa \Delta^k}{L_f}.$$

Proof: Taking into account that x^{k+1} is feasible for (3.3) we obtain for any $j \in J^k$

$$f^j(x^{k+1}) = \tilde{f}(x^j) + g_f(x^j)^T(x^{k+1} - x^j) \leq \kappa f_{low}^k + (1-\kappa)f_{up}^k = f_{up}^k - \kappa \Delta^k.$$

Regarding $\tilde{f}(x^j) \geq f_{up}^k$, the Cauchy-Schwarz inequality and the boundedness of the ε-subgradients this leads to

$$\begin{aligned}
\kappa \Delta^k &\leq \tilde{f}(x^j) - f_{up}^k + \kappa \Delta^k \\
&\leq \tilde{f}(x^j) - f^j(x^{k+1}) \\
&= -g_f(x^j)^T(x^{k+1} - x^j) \\
&\leq \|g_f(x^j)\|_2 \|x^{k+1} - x^j\|_2 \\
&\leq L_f \|x^{k+1} - x^j\|_2.
\end{aligned} \tag{3.4}$$

If $k(l+1) > k > k(l)$ it follows $k \in J^k$ from step (S7) so that the choice $j = k$ in (3.4) is possible and the proposition holds in this case. Otherwise if $k = k(l)$ we can find a $j \in J^k$ with $x^j = x_c^k$ due to (S4) so that the proposition follows again from (3.4). $\qquad\square$

Lemma 3.4 *(cf. Lemma 3.2 in [28])*
If $k(l+1) > k > k(l)$ for some $l \geq 0$ then $x_c^k = x_c^{k-1}$ and

$$\|x^{k+1} - x_c^k\|_2^2 \geq \|x^k - x_c^k\|_2^2 + \|x^{k+1} - x^k\|_2^2. \tag{3.5}$$

Proof: Checking (S7) the equation $x_c^k = x_c^{k-1}$ is obvious.
   In order to prove the second proposition consider problem (3.3) in step $k-1$, the orthogonal projection of x_c^{k-1} onto the set described by

$$x \in S, \quad f^j(x) \leq \kappa f_{low}^{k-1} + (1-\kappa)f_{up}^{k-1} \quad (j \in J^{k-1}).$$

Due to (S5) and (S7) x^k is also the projection of x_c^{k-1} onto the enlarged set with J_s^{k-1} instead of J^{k-1}. The projection theorem (cf., e.g., Hiriart-Urruty, Lemaréchal [20]), Theorem III.3.1.1) in combination with $x_c^k = x_c^{k-1}$ gives us

$$(x_c^k - x^k)^T(y - x^k) \leq 0 \quad \text{for all} \quad y \in \left\{ x \in S : f^j(x) \leq \kappa f_{low}^{k-1} + (1-\kappa)f_{up}^{k-1} \quad (j \in J_s^{k-1}) \right\}$$

so that particularly

$$(x_c^k - x^k)^T(x^{k+1} - x^k) \leq 0$$

holds if we additionally regard that (S7) ensures the feasibility of x^{k+1} for the projection onto the enlarged set. Therefore we obtain

$$\|x^{k+1} - x_c^k\|_2^2 = \|x^k - x_c^k\|_2^2 + \|x^{k+1} - x^k\|_2^2 + 2(x^k - x_c^k)^T(x^{k+1} - x^k)$$
$$\geq \|x^k - x_c^k\|_2^2 + \|x^{k+1} - x^k\|_2^2$$

which completes the proof. □

At this point an upper bound for the number of steps with fixed l can be presented.

Lemma 3.5 *(cf. Lemma 3.3 in [28])*
If $k(l) \leq k < k(l+1)$ for some $l \in \mathbb{N}_0$ and $\Delta^k > 0$ then

$$k - k(l) + 1 \leq \left(\frac{\mathrm{diam}\,(S)\,L_f}{\kappa\Delta^k}\right)^2$$

with $\mathrm{diam}\,(S) := \max_{x,y \in S} \|x - y\|_2$.

Proof: If $k = k(l)$ the proposition follows from Lemma 3.3.

But if $k > k(l)$ we have $x_c^{k(l)} = x_c^{k(l)+1} = \ldots = x_c^k$ due to Lemma 3.4. Taking this and the successive application of (3.5) into account one obtains

$$(\mathrm{diam}\,(S))^2 \geq \|x^{k+1} - x_c^k\|_2^2$$
$$\geq \|x^k - x_c^{k-1}\|_2^2 + \|x^{k+1} - x^k\|_2^2$$
$$\vdots$$
$$\geq \|x^{k(l)+1} - x_c^{k(l)}\|_2^2 + \sum_{j=k(l)+1}^{k} \|x^{j+1} - x^j\|_2^2.$$

Note that $f_{up}^j \geq f_{up}^k$ for all $j \leq k$ due to (S1), moreover, that $f_{low}^j = f_{low}^k$ for all $k(l) \leq j \leq k$ due to (S7). Thus $\Delta^j \geq \Delta^k$ for all $k(l) \leq j \leq k$. Therefore, using Lemma 3.3, we can conclude

$$(\mathrm{diam}\,(S))^2 \geq \sum_{j=k(l)}^{k} \left(\frac{\kappa\Delta^j}{L_f}\right)^2 \geq \left(\frac{\kappa\Delta^k}{L_f}\right)^2 (k - k(l) + 1)$$

which leads to our proposition. □

Lemma 3.6 *(cf. Lemma 3.4 in [28])*
If $\Delta^k \geq \varepsilon_{opt}$ for some k then

$$k \leq \left(\frac{\mathrm{diam}\,(S)\,L_f}{\varepsilon_{opt}}\right)^2 \frac{1}{\kappa^2(1 - \kappa^2)}.$$

Proof: Set $K^l := \{k(l), \ldots, k(l+1) - 1\}$ for $l \in \mathbb{N}_0$. The proof of Lemma 3.5 shows that $f_{up}^j \geq f_{up}^i$, $f_{low}^j = f_{low}^k$ and consequently $\Delta^j \geq \Delta^i$ hold for all pairs $i, j \in K^l$ with $j \leq i$. Now, an estimate for combining these separate results is established. Due to (S1) $f_{up}^{k(l+1)} \leq f_{up}^j$ for all $j \in K^l$. Additionally, since (3.3) is not solvable in the $k(l+1)$-th iteration step of the method we have

$$f_{low}^{k(l+1)} \geq \kappa f_{low}^{k(l)} + (1 - \kappa) f_{up}^{k(l+1)}$$

so that altogether

$$\Delta^{k(l+1)} \leq \kappa(f_{up}^{k(l+1)} - f_{low}^j) \leq \kappa(f_{up}^j - f_{low}^j) = \kappa \Delta^j \tag{3.6}$$

follows for all $j \in K^l$.

Let $m \in \mathbb{N}_0$ be given such that $k \in K^m$ holds. Furthermore set $\hat{K} = \{1, \ldots, k\}$. Then $\Delta^k \geq \varepsilon_{opt}$, $\Delta^{j+1} \leq \Delta^j$ for all $j \in K(l) \cap \hat{K}$, $j < k(l+1) - 1$ and (3.6) allow to conclude

$$\Delta^i \geq \frac{\varepsilon_{opt}}{\kappa^{m-l}} \quad \text{for all} \quad i \in K^l \cap \hat{K}, \, l = 0, \ldots, m.$$

Using this and Lemma 3.5 we obtain

$$|K^l \cap \hat{K}| \leq \left(\frac{\text{diam}(S) L_f}{\kappa \varepsilon_{opt}}\right)^2 \kappa^{2(m-l)}$$

for $l = 0, \ldots, m$. Hence,

$$k = \sum_{l=0}^m |K^l \cap \hat{K}| \leq \sum_{l=0}^m \left(\frac{\text{diam}(S) L_f}{\kappa \varepsilon_{opt}}\right)^2 \kappa^{2(m-l)} \leq \left(\frac{\text{diam}(S) L_f}{\kappa \varepsilon_{opt}}\right)^2 \frac{1}{1 - \kappa^2}$$

and the proof is complete. $\qquad\square$

Now we are able to prove the main result of this chapter.

Theorem 3.7 *(cf. Corollary 3.6 in [28])*
If $\varepsilon_{opt} > 0$ then Algorithm 3.2 will terminate in $\hat{k} = 1 + k_{opt}$ iterations where

$$k_{opt} \leq \left(\frac{\text{diam}(S) L_f}{\varepsilon_{opt}}\right)^2 \frac{1}{\kappa^2(1 - \kappa^2)}.$$

Moreover, the inequalities

$$\tilde{f}(x_{rec}^{\hat{k}}) - \min_{x \in S} f(x) \leq \varepsilon_{opt} + \varepsilon \tag{3.7}$$

and

$$f(x_{rec}^{\hat{k}}) - \min_{x \in S} f(x) \leq \varepsilon_{opt} + 2\varepsilon \tag{3.8}$$

are true.

Proof: The first proposition is a consequence of Lemma 3.6, while, using (3.2), inequality (3.8) follows directly from (3.7). Thus it remains to prove (3.7).

If the break in (S2) is caused by $g_f(x^k) = 0$ for any $k \in \mathbb{N}$, Theorem XI.1.1.5 in Hiriart-Urruty, Lemaréchal [21]) gives $f(x^k) \leq \min_{x \in S} f(x) + \varepsilon$. Using this, the definition of x_{rec}^k and (3.2) we can conclude

$$\tilde{f}(x_{rec}^k) \leq \tilde{f}(x^k) \leq f(x^k) \leq \min_{x \in S} f(x) + \varepsilon.$$

Thus (3.7) holds in this case.

In the sequel we assume that the break is caused by $\Delta^k \leq \varepsilon_{opt}$. Then it holds

$$\tilde{f}(x_{rec}^k) = f_{up}^k = f_{low}^k + \Delta^k \leq f_{low}^k + \varepsilon_{opt}.$$

Moreover, due to (3.2) and $g_f(x^j) \in \partial_\varepsilon f(x^j)$, we have

$$
\begin{aligned}
f^j(x) &= \tilde{f}(x^j) + g_f(x^j)^T(x - x^j) \\
&\leq f(x^j) + f(x) - f(x^j) + \varepsilon \\
&= f(x) + \varepsilon
\end{aligned}
$$

for all $j \in \mathbb{N}$. Thus we infer $f_{low}^k \leq \min_{x \in S} f(x) + \varepsilon$ and altogether we obtain (3.7). □

Remark 3.8 If a two-sided approximation $\tilde{f}$ of f is given, i.e.

$$f(x) - \varepsilon \leq \tilde{f}(x) \leq f(x) + \varepsilon$$

for all $x \in S$ instead of the one-sided approximation, the results stated above remain true if we add an additional ε to the right-hand sides of (3.7) and (3.8). □

Chapter 4

A logarithmic barrier method for convex semi-infinite optimization problems

In this chapter we specify the necessary details to put Algorithm 2.5 into implementable form. For that purpose we first consider problems of type (2.10)

$$\text{minimize} \quad f(x) \quad \text{s.t.} \quad x \in \mathbb{R}^n, \quad \max_{t \in T} g(x,t) \le 0,$$

whereby we will denote the feasible set by $\mathcal{M}$ and the optimal value by f^*. Later on, in Section 4.3, the developed algorithm as well as the convergence analysis are transferred to problems of the general form (1.1).

4.1 An implementable algorithm

Assumption 4.1 *Assume the following:*

(1) $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function;

(2) $T \subset \mathbb{R}^p$ is a compact set;

(3) $g(\cdot, t)$ is convex on $\mathbb{R}^n$ for any $t \in T$;

(4) $g(x, \cdot)$ is continuous on T for any $x \in \mathbb{R}^n$;

(5) the set $\mathcal{M}_0 := \{x \in \mathcal{M} : \max_{t \in T} g(x,t) < 0\}$ is nonempty;

(6) the set of optimal solutions

$$\mathcal{M}_{opt} := \{x \in \mathcal{M} : f(x) = f^*\}$$

of (2.10) is nonempty and compact;

(7) in case $h > 0$ the set T_h is a finite h-grid on T (i.e. for each $t \in T$ there exists $t_h \in T_h$ with $\|t - t_h\|_2 \le h$) and in case $h = 0$ the sets T_h, T coincides;

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(8) for each compact set $S \subset \mathbb{R}^n$ there exists a constant L_S^t with

$$|g(x, t_1) - g(x, t_2)| \leq L_S^t \|t_1 - t_2\|_2 \tag{4.1}$$

for all $x \in S$ and all $t_1, t_2 \in T$;

(9) for each compact set $S \subset \mathcal{M}_0$ a constant $C_S < \infty$ with

$$C_S \geq \max_{x \in S} \left| \frac{1}{\max_{t \in T} g(x, t)} \right| \tag{4.2}$$

can be computed such that $S' \subset S \subset \mathcal{M}_0$ implies $C_{S'} \leq C_S$;

(10) for each $x \in \mathbb{R}^n$ and each $t \in T$ an element of the subdifferential of f in x and an element of the subdifferential of $g(\cdot, t)$ in x can be computed.

Regarding (1) and (3) it is ensured that we deal with convex problems of type (2.10). Furthermore, due to (2) and (4) the maximization problems (2.12) are solvable and consequently the barrier functions (2.11) are evaluable at least from the theoretical point of view. Moreover, (5) and (6) are motivated by the theoretical results of the Chapter 2. Then Lemma 2.2 ensures the existence of a minimizer of the barrier function (2.11) for any given $\mu > 0$. Furthermore, the classical logarithmic barrier method with exact minimizers x^k leads to an optimal solution of the semi-infinite problem (2.10) in the sense of Theorem 2.3. But as stated in Section 2.3 we cannot suppose that the maximization problems (2.12) are exactly solvable. Therefore we admitted the next assumptions. (7) allows to compute inexact maximizer while their accuracy can be controlled with (8). The necessity of assumption (9) will become clear in the further course, while by (10) we want to point out that indeed the computation of the subgradients are necessary in the implementation of the method. We now state our method in detail.

Algorithm 4.2

- Given $\mu_1 > 0$ and $x^0 \in \mathcal{M}_0$.

- For $i := 1, 2, \ldots$:

  - Set $x^{i,0} := x^{i-1}$, select $\varepsilon_{i,1} > 0$ and define $f_i : \mathcal{M}_0 \to \mathbb{R}$ by

  $$f_i(x) := f(x) - \mu_i \ln\left(-\max_{t \in T} g(x, t) \right).$$

  - For $k := 1, 2, \ldots$:

    (a) Select $r_{i,k} > 0$ such that

    $$S^{i,k} := \{ x \in \mathbb{R}^n : \|x - x^{i,k-1}\|_\infty \leq r_{i,k} \} \subset \mathcal{M}_0.$$

    (b) Select $h_{i,k} \geq 0$ and define $\tilde{f}_{i,k} : \mathcal{M}_0 \to \mathbb{R}$ by

    $$\tilde{f}_{i,k}(x) := f(x) - \mu_i \ln\left(-\max_{t \in T_{h_{i,k}}} g(x, t) \right)$$

  where $T_{h_{i,k}}$ is a set fulfilling Assumption 4.1(7).

(c) Select $\beta_{i,k} \geq 0$ and compute an approximate solution $x^{i,k}$ of

$$\text{minimize} \quad f_i(x) \quad \text{s.t.} \quad x \in S^{i,k} \tag{4.3}$$

such that

$$\tilde{f}_{i,k}(x^{i,k}) - \min_{x \in S^{i,k}} f_i(x) \leq \frac{\varepsilon_{i,k}}{2} + \beta_{i,k} \tag{4.4}$$

and $\tilde{f}_{i,k}(x^{i,k}) \leq \tilde{f}_{i,k}(x^{i,k-1})$ are true.

(d) ∗ If $\tilde{f}_{i,k}(x^{i,k-1}) - \tilde{f}_{i,k}(x^{i,k}) \leq \varepsilon_{i,k}/2$ and[1] $f(x^{i,k-1}) \leq f(x^0) + 2\mu_i$ then set $x^i := x^{i,k-1}$, $S^i := S^{i,k}$, $r_i := r_{i,k}$, $\varepsilon_i := \varepsilon_{i,k}$, $h_i := h_{i,k}$, stop inner loop;

∗ if $\tilde{f}_{i,k}(x^{i,k-1}) - \tilde{f}_{i,k}(x^{i,k}) \leq \varepsilon_{i,k}/2$ and $f(x^{i,k-1}) > f(x^0) + 2\mu_i$ then set $\varepsilon_{i,k+1} := \varepsilon_{i,k}/2$, continue inner loop;

∗ if $\tilde{f}_{i,k}(x^{i,k-1}) - \tilde{f}_{i,k}(x^{i,k}) > \varepsilon_{i,k}/2$ set $\varepsilon_{i,k+1} := \varepsilon_{i,k}$, continue inner loop.

– Select $0 < \mu_{i+1} < \mu_i$.

The structure of Algorithm 4.2 resembles that of the conceptual Algorithm 2.5, but let us give explanations for each particular step. In (a) we specify the compact set $S^{i,k}$ as a linearly bounded set. This decision is caused by the fact that linearly bounded sets are normally the simplest bounded structures on that minimization can be done. Consequently minimizing the barrier function on the chosen compact set is normally easier than minimizing it on more complex structures like quadratically bounded sets such as balls or ellipsoids. Additionally, having the bundle method from the previous chapter in mind, we point out that the decision to choose linearly bounded sets is important because in consequence of this the auxiliary problems of the bundle method are linear and quadratic problems. Thus each of them should be efficiently solvable by standard approaches. Furthermore, since $\mathcal{M}_0$ is an open set, step (a) is realizable.

In (b) we define the approximation of the barrier function while in (c) the approximate minimization of the barrier function is done with a certain solution accuracy. The condition (4.4) is stimulated by inequality (3.7) in Theorem 3.7, when solving (4.3) with the bundle method proposed in the previous chapter. Finally, in (d) the stopping criterion of the inner loop is given. It is divided into three parts but mainly only two inequalities occur. The first one checks whether there is achieved a sufficient improvement of the approximate solution on the current box with the selected accuracy. The second part of the criterion is motivated by (2.5) so that it checks whether the accuracy parameter and the barrier parameter are in an appropriate order. If this is not the case then the accuracy parameter is readjusted.

The critical point for a realization of the presented method is the question whether there exist approximate solutions of (4.3) which fulfill the demanded criterions. As stated above (4.4) is initiated by the bundle method presented in the previous chapter. Therefore we want to show that we can use this bundle method for solving (4.3). We first notice that the sets $S^{i,k} \subset \mathcal{M}_0$ are convex and compact by construction (for any given $r_{i,k} > 0$). Additionally, they are also nonempty, because $x^{i,k-1} \in \mathcal{M}_0$ holds by construction and due to the open structure of $\mathcal{M}_0$ there exists a radius

---

[1]Alternatively we can use $f(x^{i,k-1}) \leq \min_{j=0,\ldots,i-1} f(x^j) + 2\mu_i$ instead of $f(x^{i,k-1}) \leq f(x^0) + 2\mu_i$ but the latter one suffices to guarantee the boundedness of the computed sequence $\{x^i\}$.

$r_{i,k} > 0$ such that $S^{i,k} \subset \mathcal{M}_0$ is fulfilled. Moreover, we have already remarked that the functions f_i are continuous on $\mathrm{dom}\,(f_i) = \mathcal{M}_0 = \mathrm{int}\,(\mathrm{dom}\,(f_i))$ (this can be proven for instance by using the convexity property of f_i on $\mathrm{dom}\,(f_i)$ in combination with Theorem 2.35 in Rockafellar, Wets [48]). Consequently in (4.3) we have to minimize a continuous function on a compact set, which is obviously solvable. The further requirements contained in Assumption 3.1 are ensured by the following lemma. To formulate this we define for all $x \in \mathbb{R}^n$ and all $h \geq 0$ the set of active constraints

$$T(x) := \left\{ s \in T : g(x, s) = \max_{t \in T} g(x, t) \right\}$$

and its approximation

$$T_h(x) := \left\{ s \in T_h : g(x, s) = \max_{t \in T_h} g(x, t) \right\}.$$

Note that $T(x) \neq \emptyset$ due to Assumption 4.1(2), (4) and $T_h(x) \neq \emptyset$ due to Assumption 4.1(7).

Lemma 4.3 *Let Assumption 4.1 be fulfilled. Let i, k be fixed and $\beta_{i,k} \geq \mu_i L^t_{S^{i,k}} C_{S^{i,k}} h_{i,k}$ be valid. Then Assumption 3.1 is fulfilled for problem* (4.3) *with $\tilde{f}_{i,k}$ as an approximation for f_i and $\varepsilon = \beta_{i,k}$.*

Proof: In order to show Assumption 3.1(a),(b) let $x \in S^{i,k}$ be arbitrarily given. Furthermore, let $t^* \in T(x)$ be fixed. Then, due to Assumption 4.1(7) there exists a $t_h \in T_{h_{i,k}}$ with $\|t^* - t_h\|_2 \leq h_{i,k}$. Thus it holds

$$
\begin{aligned}
0 &\leq \max_{t \in T} g(x, t) - \max_{t \in T_{h_{i,k}}} g(x, t) \\
&\leq g(x, t^*) - g(x, t_h^*) \\
&\leq g(x, t^*) - g(x, t_h) \\
&\leq L^t_{S^{i,k}} \|t^* - t_h\|_2 \\
&\leq L^t_{S^{i,k}} h_{i,k}.
\end{aligned}
\tag{4.5}
$$

The concavity of the logarithm gives $\ln(b) - \ln(a) \leq (b - a)/a$ for all positive $a, b \in \mathbb{R}$. Therefore, regarding (4.5), we can conclude

$$
\begin{aligned}
f_i(x) - \tilde{f}_{i,k}(x) &= \mu_i \left(\ln \left(-\max_{t \in T_{h_{i,k}}} g(x, t) \right) - \ln \left(-\max_{t \in T} g(x, t) \right) \right) \\
&\leq \mu_i \frac{1}{-\max_{t \in T} g(x, t)} \left(\max_{t \in T} g(x, t) - \max_{t \in T_{h_{i,k}}} g(x, t) \right) \\
&\leq \mu_i C_{S^{i,k}} L^t_{S^{i,k}} h_{i,k} \leq \beta_{i,k}.
\end{aligned}
\tag{4.6}
$$

Moreover, taking $\max_{t \in T} g(x, t) \geq \max_{t \in T_{h_{i,k}}} g(x, t)$ and the monotonicity of the logarithm into account, one infers

$$\tilde{f}_{i,k}(x) \leq f_i(x)$$

such that Assumption 3.1(a) is proven.

To show Assumption 3.1(b) let $t_h^* \in T_{h_{i,k}}(x)$ be arbitrarily given. Then, due to Assumption 4.1(10), we can compute $u(x) \in \partial f(x)$ and $v(x) \in \partial g(x, t_h^*)$. Thus the inequality

$$\max_{t \in T} g(z, t) \geq g(z, t_h^*) \geq g(x, t_h^*) + v(x)^T (z - x)$$

is true for all $z \in \mathbb{R}^n$. If $z \in \mathcal{M}_0$ one can conclude

$$\ln\left(-\max_{t \in T} g(z, t)\right) \leq \ln\left(-g(x, t_h^*) - v(x)^T(z - x)\right). \tag{4.7}$$

Regarding (4.6), (4.7), the subgradient property of u as well as the convexity of $-\ln$ we obtain for all $z \in \mathcal{M}_0$ that

$$f_i(z) - f_i(x) \geq f_i(z) - \tilde{f}_{i,k}(x) - \beta_{i,k}$$

$$\geq f(z) - \mu_i \ln\left(-\max_{t \in T} g(z, t)\right) - f(x) + \mu_i \ln\left(-\max_{t \in T_{h_{i,k}}} g(x, t)\right) - \beta_{i,k}$$

$$\geq u(x)^T(z - x) - \mu_i \ln\left(-g(x, t_h^*) - v(x)^T(z - x)\right) + \mu_i \ln\left(-g(x, t_h^*)\right) - \beta_{i,k}$$

$$\geq u(x)^T(z - x) - \frac{\mu_i}{g(x, t_h^*)} v(x)^T(z - x) - \beta_{i,k}$$

$$= \left(u(x) - \frac{\mu_i}{\max_{t \in T_{h_{i,k}}} g(x, t)} v(x)\right)^T (z - x) - \beta_{i,k}.$$

Using $f_i \equiv \infty$ on $\mathbb{R}^n \setminus \mathcal{M}_0$ this inequality is also true for all $z \notin \mathcal{M}_0$ so that

$$u(x) - \frac{\mu_i}{\max_{t \in T_{h_{i,k}}} g(x, t)} v(x) \in \partial_{\beta_{i,k}} f_i(x) \tag{4.8}$$

follows.

Finally, we show that Assumption 3.1(c) is fulfilled. The Lipschitz continuity of f_i on the compact set $S^{i,k} \subset \mathcal{M}_0 = \text{int}(\text{dom}(f_i))$ follows from Rockafellar [45], Theorem 24.7. Moreover, due to the same theorem the subdifferentials of the convex functions f and $\max_{t \in T_{h_{i,k}}} g(\cdot, t)$ are bounded above on $S^{i,k}$ w.r.t. the Euclidean norm by positive constants c_f and c_g. Furthermore, the definition of $v(x)$ combined with Lemma VI.4.4.1 in Hiriart-Urruty, Lemaréchal [20] gives the inclusion $v(x) \in \partial(\max_{t \in T_{h_{i,k}}} g(x, t))$ for all $x \in S^{i,k}$. Thus, regarding Assumption 4.1(9), the Euclidean norm of the $\beta_{i,k}$-subgradients described in (4.8) can be estimated as follows

$$\left\| u(x) - \frac{\mu_i}{\max_{t \in T_{h_{i,k}}} g(x, t)} v(x) \right\|_2 \leq \|u(x)\|_2 + \mu_i \left| \frac{1}{\max_{t \in T_{h_{i,k}}} g(x, t)} \right| \|v(x)\|_2$$

$$\leq c_f + \mu_i C_{S^{i,k}} c_g < \infty$$

for all $x \in S^{i,k}$. Therefore, using the approximate subgradients defined in (4.8) the third part of Assumption 3.1 is also true. $\qquad\square$

Summing up we have shown that we can use the bundle method stated in the previous chapter to determine approximate solutions of the problems (4.3) with $\beta_{i,k} \geq \mu_i L_{S^{i,k}}^t C_{S^{i,k}} h_{i,k}$. Furthermore,

if we use this bundle method we do not have to select $\beta_{i,k}$ explicitly, because $\beta_{i,k} = \mu_i L^t_{S^{i,k}} C_{S^{i,k}} h_{i,k}$ can be used after $h_{i,k}$ is known.

Remark 4.4 If it is possible to determine $\max_{t \in T} g(x, t)$ exactly for each feasible solution x we can set $h_{i,k} = 0$ for all pairs i, k. Consequently $\beta_{i,k} = 0$ is allowed (independent of the values $L^t_{S^{i,k}}, C_{S^{i,k}}$) such that $\tilde{f}_{i,k}$ and f_{μ_i} are identical. This leads to some simplifications in the algorithm above as well as in the following convergence analysis. Furthermore, in this case we can drop the Assumptions 4.1(7) and (8). □

At the end of this section we can summarize that the presented Algorithm 4.2 is practicable. A convergence analysis follows in the next section.

4.2   Convergence analysis

In this section we present conditions on the parameters of Algorithm 4.2 which guarantee that we obtain an optimal solution of (2.10). We start with a characterization of the first part of the stopping criterion in part (d) of the inner loop of Algorithm 4.2.

Lemma 4.5 *Let Assumption 4.1 be fulfilled. Moreover, let i be fixed and $\hat{x}$ be an arbitrary optimal solution of*

$$\text{minimize} \quad f_i(x) \quad \text{s.t.} \quad x \in \mathcal{M}_0. \tag{4.9}$$

f_i^* denotes the minimal value of problem (4.9). Let $x^{i,k-1}, x^{i,k}$ be generated by Algorithm 4.2 and $\beta_{i,k} \geq \mu_i L^t_{S^{i,k}} C_{S^{i,k}} h_{i,k}$ be valid. If the inequality

$$\tilde{f}_{i,k}(x^{i,k-1}) - \tilde{f}_{i,k}(x^{i,k}) \leq \frac{\varepsilon_{i,k}}{2} \tag{4.10}$$

is true, then

$$0 \leq f_i(x^{i,k-1}) - f_i^* \leq \max\left\{1, \frac{\|x^{i,k-1} - \hat{x}\|_\infty}{r_{i,k}}\right\} (\varepsilon_{i,k} + 2\beta_{i,k}) \tag{4.11}$$

holds.

Proof: We first remark that Lemma 2.2 ensures the existence of $\hat{x}$ as optimal solution of (4.9).

The inequality $0 \leq f_i(x^{i,k-1}) - f_i^*$ obviously holds, since $x^{i,k-1}$ is feasible for (4.9) by construction as well as

$$\tilde{f}_{i,k}(x^{i,k}) - \min_{x \in S^{i,k}} f_i(x) \leq \frac{\varepsilon_{i,k}}{2} + \beta_{i,k}$$

by construction. Together with (4.10) this yields

$$\tilde{f}_{i,k}(x^{i,k-1}) - \min_{x \in S^{i,k}} f_i(x) \leq \varepsilon_{i,k} + \beta_{i,k}.$$

Using (4.6) we get

$$f_i(x^{i,k-1}) - \min_{x \in S^{i,k}} f_i(x) \leq \varepsilon_{i,k} + 2\beta_{i,k}. \tag{4.12}$$

At this point we distinguish two cases with regard to the location of $\hat{x}$. We first consider the case $\hat{x} \in S^{i,k}$. Then $\hat{x}$ is also an optimal solution of (4.9) so that

$$f_i(x^{i,k-1}) - f_i^* = f_i(x^{i,k-1}) - f_i(\hat{x}) \leq \varepsilon_{i,k} + 2\beta_{i,k}$$

follows from (4.12). Therefore the proposition (4.11) is true in this case.

Now, let us consider the case $\hat{x} \notin S^{i,k}$ and define the line through $x^{i,k-1}$ and $\hat{x}$ by

$$\gamma(s) := x^{i,k-1} + \frac{s}{\|x^{i,k-1} - \hat{x}\|_\infty}(\hat{x} - x^{i,k-1}).$$

Due to $\hat{x} \notin S^{i,k}$ we have $\|x^{i,k-1} - \hat{x}\|_\infty > r_{i,k} > 0$. Therefore $\gamma(r_{i,k})$ lies between $x^{i,k-1}$ and $\hat{x}$ on that line. Since f_i is convex on $\mathcal{M}_0$ with minimizer $\hat{x}$, one gets $f_i(\gamma(r_{i,k})) \leq f_i(x^{i,k-1})$. Moreover, the equation $\|x^{i,k-1} - \gamma(r_{i,k})\|_\infty = r_{i,k}$ is true so that $\gamma(r_{i,k}) \in S^{i,k}$. Thus, using (4.12), we obtain

$$f_i(x^{i,k-1}) - f_i(\gamma(r_{i,k})) \leq \varepsilon_{i,k} + 2\beta_{i,k}. \tag{4.13}$$

Besides we have

$$f_i(\gamma(r_{i,k})) \leq \frac{r_{i_k}}{\|x^{i,k-1} - \hat{x}\|_\infty} f_i(\hat{x}) + \left(1 - \frac{r_{i,k}}{\|x^{i,k-1} - \hat{x}\|_\infty}\right) f_i(x^{i,k-1}),$$

since f_i is convex on $\mathcal{M}_0$ and $0 < r_{i_k}/\|x^{i,k-1} - \hat{x}\|_\infty < 1$. This leads to

$$f_i(x^{i,k-1}) - f_i(\hat{x}) \leq \frac{\|x^{i,k-1} - \hat{x}\|_\infty}{r_{i,k}}\left(f_i(x^{i,k-1}) - f_i(\gamma(r_{i,k}))\right)$$

such that with (4.13) the estimate

$$f_i(x^{i,k-1}) - f_i^* \leq \frac{\|x^{i,k-1} - \hat{x}\|_\infty}{r_{i,k}}\left(\varepsilon_{i,k} + 2\beta_{i,k}\right)$$

follows. Thus the proposition is also true in the second case $\hat{x} \notin S^{i,k}$ and the proof is complete. $\square$

Now a sufficient termination condition for the inner loop of Algorithm 4.2 can be presented.

Proposition 4.6 *Let Assumption 4.1 be fulfilled. Moreover, let i be fixed, $q_i \in (0,1)$, $\delta_i > 0$ be given and $r_{i,k} \geq \underline{r}_i > 0$ be valid for all k. If*

$$\mu_i L_{S^{i,k}}^t C_{S^{i,k}} h_{i,k} \leq \beta_{i,k} \leq q_i^k \delta_i \tag{4.14}$$

is true for all k, then the inner loop of Algorithm 4.2 terminates after a finite number of steps.

Proof: The inner loop terminates after a finite number of steps if the inequalities (4.10) and

$$f(x^{i,k-1}) \leq f(x^0) + 2\mu_i \tag{4.15}$$

are both true.

In the main part of the proof we assume that both inequalities never hold together. In order to bring this to a contradiction we first exclude that (4.10) never holds.

a) Suppose that the inequality (4.10) never holds.

Then the algorithm generates an infinite sequence $\{x^{i,k}\}_k$ and $\varepsilon_{i,k} = \varepsilon_{i,1}$ for all $k \in \mathbb{N}$. Additionally, the estimate $\tilde{f}_{i,k}(x^{i,k}) \leq \tilde{f}_{i,k}(x^{i,k-1})$ holds by construction so that we infer with (4.6)

$$
\begin{aligned}
0 &\leq \tilde{f}_{i,k}(x^{i,k-1}) - f_i(x^{i,k}) + \beta_{i,k} \\
&\leq \tilde{f}_{i,k}(x^{i,k-1}) - \tilde{f}_{i,k+1}(x^{i,k}) + q_i^k \delta_i \\
&= \left(\tilde{f}_{i,k}(x^{i,k-1}) - \sum_{j=0}^{k-1} q_i^j \delta_i \right) - \left(\tilde{f}_{i,k+1}(x^{i,k}) - \sum_{j=0}^{k} q_i^j \delta_i \right)
\end{aligned}
$$

for all $k \in \mathbb{N}$ and hence

$$
\left\{ \tilde{f}_{i,k}(x^{i,k-1}) - \sum_{j=0}^{k-1} q_i^j \delta_i \right\}_k
\tag{4.16}
$$

is a monotonically nonincreasing sequence. Furthermore, this sequence is bounded below because we have

$$
\tilde{f}_{i,k}(x^{i,k-1}) - \sum_{j=0}^{k-1} q_i^j \delta_i \geq f_i(x^{i,k-1}) - \beta_{i,k} - \sum_{j=0}^{\infty} q_i^j \delta_i \geq f_i^* - \delta_i - \frac{1}{1-q_i}\delta_i
$$

for all k with f_i^* given as in Lemma 4.5. Thus the sequence given in (4.16) converges. Combining this and $q_i^k \delta_i \to 0$ for $k \to \infty$ we can find an index k_0 such that

$$
\tilde{f}_{i,k_0}(x^{i,k_0-1}) - \tilde{f}_{i,k_0+1}(x^{i,k_0}) \leq \tilde{f}_{i,k_0}(x^{i,k_0-1}) - \tilde{f}_{i,k_0+1}(x^{i,k_0}) + q_i^{k_0}\delta_i \leq \frac{\varepsilon_{i,1}}{4}
$$

and

$$
\beta_{i,k_0} \leq q_i^{k_0}\delta_i \leq \frac{\varepsilon_{i,1}}{4}.
$$

Then another use of (4.6) leads to

$$
\begin{aligned}
\tilde{f}_{i,k_0}(x^{i,k_0-1}) - \tilde{f}_{i,k_0}(x^{i,k_0}) &\leq \tilde{f}_{i,k_0}(x^{i,k_0-1}) - f_i(x^{i,k_0}) + \beta_{i,k_0} \\
&\leq \tilde{f}_{i,k_0}(x^{i,k_0-1}) - \tilde{f}_{i,k_0+1}(x^{i,k_0}) + \beta_{i,k_0} \\
&\leq \frac{\varepsilon_{i,1}}{4} + \frac{\varepsilon_{i,1}}{4} = \frac{\varepsilon_{i,1}}{2}.
\end{aligned}
$$

This contradicts our assumption and we have an index k such that (4.10) is fulfilled.

b) Suppose that the inequalities (4.10) and (4.15) never hold together.

As in a) one can show that (4.16) defines a monotonically nonincreasing sequence. Therefore, taking (4.6), (4.14) and $q_i \in (0,1)$ into account, we infer

$$
\begin{aligned}
f_i(x^{i,k-1}) - \sum_{j=0}^{k-1} q_i^j \delta_i &\leq \tilde{f}_{i,k}(x^{i,k-1}) + \beta_{i,k} - \sum_{j=0}^{k-1} q_i^j \delta_i \\
&\leq \tilde{f}_{i,1}(x^{i,0}) + q_i^k \delta_i - \delta_i \\
&\leq f_i(x^{i,0})
\end{aligned}
$$

for all $k \in \mathbb{N}$. Thus

$$
f_i(x^{i,k-1}) \leq f_i(x^{i,0}) + \frac{1}{1-q_i}\delta_i
$$

for all $k \in \mathbb{N}$ implies

$$x^{i,k} \in N_i := \left\{ x \in \mathcal{M}_0 : f_i(x) \leq f_i(x^{i,0}) + \frac{1}{1 - q_i} \delta_i \right\}$$

for all $k \in \mathbb{N}_0$. The set N_i is compact since it is a nonempty level set of f_i (cf. Lemma 2.2). Hence, the sequence $\{\|x^{i,k} - \hat{x}\|_\infty\}_k$ is bounded above by a constant $C \geq \underline{r}_i$, where $\hat{x}$ is an arbitrary optimal solution of (4.9).

From the first part of the proof we know that there exists an index k_1 such that (4.10) holds for $k = k_1$ for the first time. Using the same arguments for all indices greater than k_1 we find an index $k_2 > k_1$ such that (4.10) holds for $k = k_2$ again. Repeating this procedure we get a strictly monotonically increasing sequence $\{k_j\}$ such that (4.10) holds for all $k = k_j$. Then, we deduce from Lemma 4.5

$$
\begin{aligned}
0 \leq f_i(x^{i,k_j-1}) - f_i^* \\
\leq \max \left\{ 1, \frac{\|x^{i,k_j-1} - \hat{x}\|_\infty}{r_{i,k_j}} \right\} (\varepsilon_{i,k_j} + 2\beta_{i,k_j}) \\
\leq \max \left\{ 1, \frac{\|x^{i,k_j-1} - \hat{x}\|_\infty}{r_{i,k_j}} \right\} \left(\varepsilon_{i,k_j} + 2q_i^{k_j} \delta_i \right)
\end{aligned}
$$

for $j \in \mathbb{N}$. It is simple to verify that $k_j \geq j$ and $\varepsilon_{i,k_j} = (1/2)^{j-1} \varepsilon_{i,1}$ are true. Thus, regarding $\|x^{i,k_j-1} - \hat{x}\|_\infty \leq C$ as well as $r_{i,k} \geq \underline{r}_i > 0$ for all k, we have

$$0 \leq f_i(x^{i,k_j-1}) - f_i^* \leq \frac{C}{\underline{r}_i} \left(\left(\frac{1}{2}\right)^{j-1} \varepsilon_{i,0} + 2\delta_i q_i^j \right)$$

for all $j \in \mathbb{N}$. Hence,

$$\lim_{j \to \infty} f_i \left(x^{i,k_j-1} \right) = f_i^*.$$

Since $\{x^{i,k_j}\}_j$ belongs to the compact set N_i there exists an accumulation point $x^* \in N_i$. From the last equation we obtain that x^* solves (4.9). Due to the continuity of f there exists an index $\hat{j} \in \mathbb{N}$ with $f(x^{i,k_{\hat{j}}-1}) \leq f(x^*) + \mu_i$. Combining $f^* \leq f(x^0)$ and (2.5) leads to

$$f(x^{i,k_{\hat{j}}-1}) - f(x^0) \leq f(x^*) + \mu_i - f^* \leq 2\mu_i.$$

This contradicts our assumption, both inequalities (4.10) and (4.15) are true for $k = k_{\hat{j}}$ and the proof is complete. $\qquad\square$

Remark 4.7 The assumption $r_{i,k} \geq \underline{r}_i > 0$ is not used to prove that all iterates belong to the nonempty compact set N_i. Therefore there exists an $r_i^* > 0$ such that the inclusion

$$\left\{ z \in \mathbb{R}^n : \min_{x \in N_i} \|z - x\|_\infty \leq r_i^* \right\} \subset \mathcal{M}_0$$

is valid since $\mathcal{M}_0$ is an open set. Thus, theoretically $r_{i,k} \geq \underline{r}_i > 0$ is no restriction for the algorithm. It still restricts the practical computation of the radii $r_{i,k}$ of course. $\qquad\square$

Remark 4.8 In (4.14) the right-hand side $q_i^k \delta_i$ can be replaced by an arbitrary summable sequence $\delta_{i,k}$ to remain true Proposition 4.6. □

With Proposition 4.6 we are able to control Algorithm 4.2 in order to make sure that each inner loop terminates and a well-defined sequence $\{x^i\}$ is generated. Before we proceed with the main convergence result let us recall the notation of ε_i as final accuracy value at the i-th iteration and of r_i as radius of the finally considered compact box S^i at this iteration.

Theorem 4.9 *Let Assumption 4.1 be fulfilled. Moreover, let $\{\underline{r}_i\}, \{\delta_i\}$ be positive sequences. Additionally, let $R > 0$, $\{q_i\} \subset (0,1)$ be given and assume that (4.14) holds for all $i \in \mathbb{N}$ and all k appearing in the outer step i. Furthermore, assume that*

(i) $\displaystyle \lim_{i \to \infty} \mu_i = 0$;

(ii) $\underline{r}_i \leq r_{i,k} \leq R$ *for all* i, k;

(iii) $\displaystyle \lim_{i \to \infty} \varepsilon_i / r_i = 0$;

(iv) $\displaystyle \lim_{i \to \infty} \delta_i / r_i = 0$.

Then Algorithm 4.2 generates a sequence $\{x^i\}$, which has at least one accumulation point and each accumulation point is an optimal solution of (2.10).

Proof: It is easy to see that the assumptions of Proposition 4.6 are satisfied for each $i \in \mathbb{N}$. Therefore each inner loop terminates after a finite number of steps and the algorithm generates a sequence $\{x^i\}$ which belongs to the level set $\{x \in \mathbb{R}^n : f(x) \leq f(x^0) + 2\mu_1\}$ by construction. Due to Assumption 4.1(6) and Corollary 20 in Fiacco, McCormick [9] this level set is compact. Thus the sequence $\{x^i\}$ has an accumulation point and we have to show that each accumulation point of $\{x^i\}$ is an optimal solution of (2.10).

Let x^* be such an accumulation point of $\{x^i\}$ and let $\{x^{i_j}\}$ be a convergent subsequence of $\{x^i\}$ with $\lim_{j \to \infty} x^{i_j} = x^*$. By x_j^* we denote an optimal solution of problem (4.9) with $i = i_j$. Using Lemma 4.5, (4.14) as well as $q_{i_j} \in (0,1)$ we obtain

$$
\begin{aligned}
0 &\leq f_{i_j}(x^{i_j}) - f_{i_j}^* \\
&\leq \max\left\{1, \frac{\|x^{i_j} - x_j^*\|_\infty}{r_{i_j}}\right\} (\varepsilon_{i_j} + 2\beta_{i_j}) \\
&\leq \max\left\{1, \frac{\|x^{i_j} - x_j^*\|_\infty}{r_{i_j}}\right\} (\varepsilon_{i_j} + 2\delta_{i_j}).
\end{aligned}
\tag{4.17}
$$

Furthermore, applying Theorem 2.3, we know that the sequence $\{x_j^*\}$ has an accumulation point. Without loss of generality we assume that $\{x_j^*\}$ is already convergent to the limit point x^{**}. Applying Theorem 2.3 again we conclude that x^{**} is an optimal solution of (2.10) and

$$
\lim_{j \to \infty} f_{i_j}^* = f^*
\tag{4.18}
$$

is true.

Obviously it is $\|x^{i_j} - x^*_j\|_\infty \leq \|x^{i_j} - x^{**}\|_\infty + \|x^*_j - x^{**}\|_\infty$. Since all x^i belong to the compact level set $\{x \in \mathbb{R}^n : f(x) \leq f(x^0) + 2\mu_1\}$ the first term of the right-hand side is bounded above. The second term is also bounded above due to the convergence of the sequences involved. For this reason there exists a constant C with $\|x^{i_j} - x^*_j\|_\infty \leq C$ for all j. Together with (4.17) we have

$$0 \leq f_{i_j}(x^{i_j}) - f^*_{i_j} \leq \max\left\{1, \frac{C}{r_{i_j}}\right\}(\varepsilon_{i_j} + 2\delta_{i_j}).$$

In view of Assumptions (ii), (iii) and (iv) we obtain

$$0 \leq \lim_{j\to\infty} f_{i_j}(x^{i_j}) - f^*_{i_j} \leq 0$$

and from (4.18) it follows

$$\lim_{j\to\infty} f_{i_j}(x^{i_j}) = f^*. \tag{4.19}$$

In the sequel we show that $\{x^{i_j}\}$ is not only a minimizing sequence but converges to a solution of problem (2.10).

The continuity of f gives $\lim_{j\to\infty} f(x^{i_j}) = f(x^*)$. This combined with (4.19) allows to conclude that the limit point of $\mu_{i_j} \ln(-\max_{t\in T} g(x^{i_j}, t))$ exists and

$$\lim_{j\to\infty} \mu_{i_j} \ln\left(-\max_{t\in T} g(x^{i_j}, t)\right) = \lim_{j\to\infty} f(x^{i_j}) - \lim_{j\to\infty} f_{i_j}(x^{i_j}) = f(x^*) - f^*. \tag{4.20}$$

Now we distinguish the two cases $\max_{t\in T} g(x^*, t) < 0$ and $\max_{t\in T} g(x^*, t) = 0$. One of them must be valid since $\mathcal{M}$ is the closure of $\mathcal{M}_0$ and x^* is an accumulation point of the sequence $\{x^i\}$ with $x^i \in \mathcal{M}_0$ holds for all i.

In the first case we assume $\max_{t\in T} g(x^*, t) < 0$. Then the sequence $\{\ln(-\max_{t\in T} g(x^{i_j}, t))\}$ is bounded and

$$\lim_{j\to\infty} \mu_{i_j} \ln\left(-\max_{t\in T} g(x^{i_j}, t)\right) = 0.$$

Together with (4.20) we obtain $f(x^*) = f^*$. As x^* is feasible for (2.10) it is an optimal solution as well.

Now, in the second case, we assume $\max_{t\in T} g(x^*, t) = 0$. Therefore there exists a constant j_0 so that $\max_{t\in T} g(x^{i_j}) > -1$ is true for all $j \geq j_0$. Thus the inequalities $\ln(-\max_{t\in T} g(x^{i_j})) < 0$ and $\mu_{i_j} \ln(-\max_{t\in T} g(x^{i_j})) < 0$ hold for all $j \geq j_0$. Hence,

$$\lim_{j\to\infty} \mu_{i_j} \ln\left(-\max_{t\in T} g(x^{i_j}, t)\right) \leq 0$$

is true and (4.20) yields $f(x^*) - f^* \leq 0$, proving that x^* solves (2.10) in this case, too. $\qquad\square$

Remark 4.10 The Assumptions (iii), (iv) in Theorem 4.9 are a posteriori criteria since we do not know ε_i and r_i before the inner loop in step i terminates. Relation (iii) can be satisfied, e.g., if we change it into

(iii)$'$ $\lim_{i\to\infty} \varepsilon_{i,1}/\underline{r}_i = 0.$

But, of course this requires an a priori computation of $\underline{r}_i$.

If this is not possible we have to run step i of the algorithm with an arbitrary $\varepsilon_{i,0}$. When the inner loop terminates, we check whether ε_i/r_i and δ_i/r_i satisfy decrease conditions, e.g. geometric decrease. If at least one of them does not do so, we repeat the step with smaller values for $\varepsilon_{i,0}$ and/or δ_i until the decrease conditions are satisfied. This procedure is finite for fixed i if we control the computation of the radii because the values of $r_{i,k}$ can be bounded below (see Remark 4.7). $\square$

4.3   Extension to general convex problems

Up to now we only considered convex semi-infinite problems with a single constraint. As already mentioned before in this section our algorithm as well as the analysis will be transferred to general convex problems of the form (1.1)

$$
\begin{aligned}
\text{minimize} \quad & f(x) \\
\text{s.t.} \quad & x \in \mathbb{R}^n, \quad Ax = b, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \\
& g_i(x,t) \leq 0 \quad \text{for all } t \in T^i \ (i = 1, \ldots, l).
\end{aligned}
$$

We again denote the set of feasible solutions by

$$
\mathcal{M} := \left\{ x \in \mathbb{R}^n : Ax = b, \quad \max_{t \in T^i} g_i(x,t) \leq 0 \, (i = 1, \ldots, l) \right\}.
$$

The required assumptions are:

Assumption 4.11

(1) $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function;

(2) $T^i \subset \mathbb{R}^{p_i}$ is a compact set for each $i \in \{1, \ldots, l\}$;

(3) $g_i(\cdot, t)$ is convex on $\mathbb{R}^n$ for any $t \in T^i$ and each $i \in \{1, \ldots, l\}$;

(4) $g_i(x, \cdot)$ is continuous on T_i for any $x \in \mathbb{R}^n$ and each $i \in \{1, \ldots, l\}$;

(5) the set $\mathcal{M}_0 := \{x \in \mathbb{R}^n : Ax = b, \quad \max_{t \in T^i} g_i(x,t) < 0 \, (i = 1, \ldots, l)\}$ is nonempty;

(6) the set of optimal solutions $\mathcal{M}_{opt}$ of (1.1) is nonempty and compact;

(7) in case $h > 0$, $i \in \{1, \ldots, l\}$ the set T_h^i is a finite h-grid on T^i (i.e. for each $t \in T^i$ there exists $t_h \in T_h^i$ with $\|t - t_h\|_2 \leq h$) and in case $h = 0$, $i \in \{1, \ldots, l\}$ the sets T_h^i, T^i coincides;

(8) for each $i \in \{1, \ldots, l\}$ and each compact set $S \subset \mathbb{R}^n$ there exists a constant $L_{i,S}^t$ with

$$
|g_i(x, t_1) - g_i(x, t_2)| \leq L_{i,S}^t \|t_1 - t_2\|_2
$$

for all $x \in S$ and all $t_1, t_2 \in T^i$;

(9) for each $i \in \{1, \ldots, l\}$ and each compact set

$$S \subset \tilde{\mathcal{M}}_0 := \left\{ x \in \mathbb{R}^n : \max_{t \in T^i} g_i(x, t) < 0 \, (i = 1, \ldots, l) \right\}$$

a constant $C_{i,S} < \infty$ with

$$C_{i,S} \geq \max_{x \in S} \left| \frac{1}{\max\limits_{t \in T^i} g_i(x, t)} \right|$$

can be computed such that $S' \subset S \subset \tilde{\mathcal{M}}_0$ implies $C_{i,S'} \leq C_{i,S}$;

(10) for each $i \in \{1, \ldots, l\}$, each $x \in \mathbb{R}^n$ and each $t \in T^i$ an element of $\partial f(x)$ and an element of the subdifferential of $g_i(\cdot, t)$ in x can be computed.

These are direct generalizations of those in Assumption 4.1. The modified algorithm now reads:

Algorithm 4.12

- Given $\mu_1 > 0$ and $x^0 \in \mathcal{M}_0$.

- For $i := 1, 2, \ldots$:

  - Set $x^{i,0} := x^{i-1}$, select $\varepsilon_{i,1} > 0$ and define $f_i : \tilde{\mathcal{M}}_0 \to \mathbb{R}$ by

  $$f_i(x) := f(x) - \mu_i \sum_{\nu=1}^{l} \ln \left(- \max_{t \in T^\nu} g_\nu(x, t) \right).$$

  - For $k := 1, 2, \ldots$:

    (a) Select $r_{i,k} > 0$ such that

    $$S^{i,k} := \{ x \in \mathbb{R}^n : \| x - x^{i,k-1} \|_\infty \leq r_{i,k} \} \subset \tilde{\mathcal{M}}_0.$$

    (b) Select $h_{i,k}^\nu \geq 0$ for $\nu = 1, \ldots, l$ and define $\tilde{f}_{i,k} : \tilde{\mathcal{M}}_0 \to \mathbb{R}$ by

    $$\tilde{f}_{i,k}(x) := f(x) - \mu_i \sum_{\nu=1}^{l} \ln \left(- \max_{t \in T_{h_{i,k}^\nu}^\nu} g_\nu(x, t) \right)$$

    where $T_{h_{i,k}^\nu}^\nu$ fulfilling Assumption 4.11(7).

    (c) Select $\beta_{i,k} \geq 0$ and compute $x^{i,k}$ as approximate solution of

    $$\text{minimize} \quad f_i(x) \quad \text{s.t.} \quad x \in S^{i,k}, \quad Ax = b$$

    such that

    $$\tilde{f}_{i,k}(x^{i,k}) - \min_{x \in S^{i,k}} f_i(x) \leq \frac{\varepsilon_{i,k}}{2} + l \beta_{i,k}$$

    and $\tilde{f}_{i,k}(x^{i,k}) \leq \tilde{f}_{i,k}(x^{i,k-1})$ are true.

(d)   ∗ If $\tilde{f}_{i,k}(x^{i,k-1}) - \tilde{f}_{i,k}(x^{i,k}) \leq \varepsilon_{i,k}/2$ and $f(x^{i,k-1}) \leq f(x^0) + 2\mu_i$ then set
$x^i := x^{i,k-1}$, $S^i := S^{i,k}$, $r_i := r_{i,k}$, $\varepsilon_i := \varepsilon_{i,k}$, $h_i^\nu := h_{i,k}^\nu$ $(\nu = 1, \ldots, l)$, stop
inner loop;

∗ if $\tilde{f}_{i,k}(x^{i,k-1}) - \tilde{f}_{i,k}(x^{i,k}) \leq \varepsilon_{i,k}/2$ and $f(x^{i,k-1}) > f(x^0) + 2\mu_i$ then set
$\varepsilon_{i,k+1} := \varepsilon_{i,k}/2$, continue inner loop;

∗ if $\tilde{f}_{i,k}(x^{i,k-1}) - \tilde{f}_{i,k}(x^{i,k}) > \varepsilon_{i,k}/2$ set $\varepsilon_{i,k+1} := \varepsilon_{i,k}$, continue inner loop.

- Select $0 < \mu_{i+1} < \mu_i$.

The practicability of this method can be shown analogously to the corresponding part in Section
4.1. There are only some changes based on the different structure of $\mathcal{M}_0$, i.e. we cannot assume
anymore that $\mathcal{M}_0$ is open. But it is still a relatively open set and that suffices to prove the most used
results while in that cases where an open set is required we can replace $\mathcal{M}_0$ by $\tilde{\mathcal{M}}_0$. Further changes
are caused by the fact that we now deal with more than one inequality constraint, for instance we
assume now $\beta_{i,k} \geq \mu_i L_{\nu,S^{i,k}}^t C_{\nu,S^{i,k}} h_{i,k}^\nu$ for each $\nu = 1, \ldots, l$. Then a convergence analysis can be
done analogously to Section 4.2 and we obtain the following main result (cf. Theorem 4.9).

Theorem 4.13 *Let Assumption 4.11 be fulfilled. Moreover, let $\{\underline{r}_i\}, \{\delta_i\}$ be positive sequences.
Additionally, let $R > 0$, $\{q_i\} \subset (0,1)$ be given and assume that*

$$\mu_i L_{\nu,S^{i,k}}^t C_{\nu,S^{i,k}} h_{i,k}^\nu \leq \beta_{i,k} \leq q_i^k \delta_i$$

holds for all $i \in \mathbb{N}$, all k appearing in the outer step i and $\nu = 1, \ldots, l$. Furthermore, assume that

(i) $\lim\limits_{i \to \infty} \mu_i = 0;$

(ii) $\underline{r}_i \leq r_{i,k} \leq R$ *for all i, k;*

(iii) $\lim\limits_{i \to \infty} \varepsilon_i / r_i = 0;$

(iv) $\lim\limits_{i \to \infty} \delta_i / r_i = 0.$

*Then Algorithm 4.12 generates a sequence $\{x^i\}$, which has at least one accumulation point and
each accumulation point is an optimal solution of* (1.1).

Chapter 5

Regularization of the logarithmic barrier approach

In the previous chapter we mainly considered semi-infinite problems of type (2.10)

$$\text{minimize} \quad f(x) \quad \text{s.t.} \quad x \in \mathcal{M} = \left\{ z \in \mathbb{R}^n : \quad \max_{t \in T} g(z, t) \leq 0 \right\}$$

under Assumption 4.1. Particularly we assumed in Assumption 4.1(6) that the solution set of the given problem is compact. But this assumption excludes a lot of problems from being solved with the presented method. Thus the goal of this chapter is to discuss a numerical method even for such problems. This can be combined with an improvement of the convergence quality (e.g. rate of convergence) for problems fulfilling Assumption 4.1.

Let us remark that we consider again problems of type (2.10) for describing and analyzing the method in detail. But, of course, as stated in the final section, it is possible to transfer the approach to general problems of type (1.1).

In the first section the method is introduced, while the following sections contain several convergence results including results on the rate of convergence for the values of the objective function as well as the computed iterates.

5.1 A regularized logarithmic barrier method for convex semi-infinite problems

As stated above we want to drop the assumption of the compactness of the solution set of (2.10). But this compactness (in particular the boundedness) is directly used in the proofs of the basic results Lemma 2.2 and Theorem 2.3 as well as Theorem 4.9. Obviously this assumption is essential for the results of the previous chapter and it turns out that in fact we cannot use the presented Algorithm 4.2 for solving problems of the form (2.10) without the assumption of the compactness. For instance, considering the trivial problem

$$\text{minimize} \quad f(x) \equiv 0 \quad \text{s.t.} \quad x \in \mathbb{R}, \quad xt \leq 0 \ (t \in [0, 1])$$

we will obtain a fast decreasing sequence $\{x^i\}$ by Algorithm 4.2 where convergence of any sub-sequence is not detectable. Therefore we have to look for another method to treat semi-infinite problems under the following weaker assumptions.

Assumption 5.1 *The Assumptions 4.1(1)-(5) and (7)-(10) are assumed to be valid.[1] Moreover, it is assumed that*

(6)$'$ the set of optimal solutions $\mathcal{M}_{opt}$ of (2.10) is nonempty.

One approach to attack problems with an unbounded set of optimal solutions is to use regularization techniques on the original problem. That means the given problem is transformed into a sequence of problems with a bounded set of optimal solutions (or ideally with unique optimal solutions). Several approaches exist for this transformation and are discussed in detail in a couple of papers and monographs mainly in the context of ill-posed problems (see, e.g. Bakushinsky, Goncharsky [3] and Kaplan, Tichatschke [24]). Some promising approaches like the Tichonov-regularization and the Proximal Point method are based on the well-known fact that a strongly convex, continuous function has a unique minimizer on a closed set. Thus the idea is to transform the convex objective function f into a strongly convex function.

As it is already stated, one approach in this context is the Tichonov-regularization, where we consider auxiliary problems of the form

$$\text{minimize} \quad f(x) + \frac{\alpha}{2}\|x\|_2^2 \quad \text{s.t.} \quad x \in \mathcal{M}$$

with positive parameter α. To obtain an optimal solution of the original problem we have to solve a sequence of such auxiliary problems whereby the parameter α has to converge to zero (see, e.g. Theorem 6.4 in Poljak [39]). Thus the regularization effect by means of the added quadratic term is getting smaller and smaller. In fact it vanishes from the numerical point of view if α falls below a certain value depending on the machine precision.

Therefore, in the sequel we consider the proximal point technique which was introduced by Martinet [32, 33] and extensively studied by Rockafellar [46, 47]. In this approach the attempt is made to keep the positive properties (like unique solvability) and remove the described negative properties of the Tichonov-regularization. Both is achieved by applying a different quadratic term in the auxiliary problems such that we now consider the problems

$$\text{minimize} \quad f(x) + \frac{s}{2}\|x - a\|_2^2 \quad \text{s.t.} \quad x \in \mathcal{M} \tag{5.1}$$

with prox-parameter s and so-called proximal point a. In order to obtain an optimal solution of the original problem one has to solve a sequence of auxiliary problems of this kind with the proximal point in each step given by the solution of the previous step. Furthermore, it turns out that the regularization parameter s is not required to converge to zero (see, e.g., Rockafellar [46, 47]).

In the case of convex semi-infinite problems Kaplan, Tichatschke [26] suggest to combine the proximal point technique with the method of outer approximation which is a discretization strategy

---

[1]In the sequel we assume the assumptions to be enumerated as in Assumption 4.1.

of the compact set T. But we want to avoid such an outer approximation and the key for it is the observation that each auxiliary problem of type (5.1) is also a convex semi-infinite problem. These auxiliary problems fulfill slightly differing assumptions as the given problem, namely they fulfill Assumption 4.1 if Assumption 5.1 is valid for the original problem. The Assumptions 4.1(1)-(5) and (7)-(10) can be simply derived from the corresponding parts in Assumption 5.1 keeping in mind that an element of the subdifferential of the objective function in (5.1) in a fixed $x \in \mathbb{R}^n$ is given by the vector $s(x - a)$ added to an arbitrary element of the subdifferential of f in x. Moreover, (6) is enforced by the additional quadratic term in the objective function of (5.1).

Consequently we could solve each auxiliary problem of type (5.1) with Algorithm 4.2 if Assumption 5.1 holds for the given semi-infinite problem. But as Algorithm 4.2 typically terminates with only an approximate solution anyway there is little sense in solving each auxiliary problem of the sequence with an accuracy as high as possible. In particular we suggest to realize only the inner loop of Algorithm 4.2 for each problem of type (5.1) to compute an approximate solution of it with fixed barrier parameter, which is then used as the new proximal point.

A practical realization of such a step requires the predetermination of the barrier and the prox parameter. From the classical logarithmic barrier approach it is known that the barrier parameter has to converge to zero, e.g. by reducing it from step to step. But, due to the fact that the conditioning of the barrier problems is getting worse with decreasing the barrier parameter, it makes sense to keep this parameter fixed for a couple of steps. In order to permit a dynamical control the choice of the barrier parameter is made dependent on the progress of the iterates in the last step. To avoid side effects which can influence this choice we keep the prox-parameter s constant as long as the barrier parameter is not changed. Merely the proximal point is updated more frequently. Altogether we obtain a so-called multi-step-regularization approach (cf., e.g., Kaplan, Tichatschke [24–27]).

Algorithm 5.2

- Given $\mu_1 > 0, x^0 \in \mathcal{M}_0, \sigma_1 > 0$ and s_1 with $0 < \underline{s} \leq s_1 \leq \overline{s}$.

- For $i := 1, 2, \ldots$:

  - Set $x^{i,0} := x^{i-1}$.

  - For $j := 1, 2, \ldots$:

    * Set $x^{i,j,0} := x^{i,j-1}$, select $\varepsilon_{i,j} > 0$ and define $F_{i,j} : \mathcal{M}_0 \rightarrow \mathbb{R}$ by

    $$F_{i,j}(x) := f(x) - \mu_i \ln \left(- \max_{t \in T} g(x,t) \right) + \frac{s_i}{2} \|x - x^{i,j-1}\|_2^2. \qquad (5.2)$$

    * For $k := 1, 2, \ldots$:

    (a) Select $r_{i,j,k} > 0$ such that

    $$S^{i,j,k} := \{x \in \mathbb{R}^n : \|x - x^{i,j,k-1}\|_\infty \leq r_{i,j,k}\} \subset \mathcal{M}_0.$$

    (b) Select $h_{i,j,k} \geq 0$ and define $\tilde{F}_{i,j,k} : \mathcal{M}_0 \rightarrow \mathbb{R}$ by

    $$\tilde{F}_{i,j,k}(x) := f(x) - \mu_i \ln \left(- \max_{t \in T_{h_{i,j,k}}} g(x,t) \right) + \frac{s_i}{2} \|x - x^{i,j-1}\|_2^2$$

where $T_{h_{i,j,k}}$ fulfills Assumption 5.1(7).

(c) Select $\beta_{i,j,k} \geq 0$ and compute an approximate solution $x^{i,j,k}$ of

$$\text{minimize} \quad F_{i,j}(x) \quad \text{s.t.} \quad x \in S^{i,j,k} \tag{5.3}$$

such that

$$\tilde{F}_{i,j,k}(x^{i,j,k}) - \min_{x \in S^{i,j,k}} F_{i,j}(x) \leq \frac{\varepsilon_{i,j}}{2} + \beta_{i,j,k} \tag{5.4}$$

and $\tilde{F}_{i,j,k}(x^{i,j,k}) \leq \tilde{F}_{i,j,k}(x^{i,j,k-1})$ are true.

(d) If

$$\tilde{F}_{i,j,k}(x^{i,j,k-1}) - \tilde{F}_{i,j,k}(x^{i,j,k}) \leq \frac{\varepsilon_{i,j}}{2} \tag{5.5}$$

then set $x^{i,j} := x^{i,j,k-1}$, $S^{i,j} := S^{i,j,k}$, $r_{i,j} := r_{i,j,k}$ and stop the loop in k, otherwise continue with the loop in k.

* If $\|x^{i,j} - x^{i,j-1}\|_2 \leq \sigma_i$ then set $x^i := x^{i,j}, r_i := r_{i,j}, j(i) := j$ and stop the loop in j, otherwise continue with the loop in j.

- Select $0 < \mu_{i+1} < \mu_i$, $0 < \underline{s} \leq s_{i+1} \leq \overline{s}$ and $\sigma_{i+1} > 0$.

Except for the stopping criterion $f(x^{i,k-1}) \leq f(x^0) + 2\mu_i$ the inner loops in k of Algorithms 4.2 and 5.2 compare to each other. The additional rule is needed in Algorithm 4.2 as it does not generate a bounded sequence per se. We will prove later that this behaviour is avoided automatically in Algorithm 5.2 above. To ensure the practicability we have to transfer Lemma 4.3 explicitly to the new situation.

Lemma 5.3 *Let Assumption 5.1 be fulfilled. Let i, j, k be fixed and $\beta_{i,j,k} \geq \mu_i L^t_{S^{i,j,k}} C_{S^{i,j,k}} h_{i,j,k}$ be valid. Then Assumption 3.1 is fulfilled for problem* (5.3) *with $\tilde{F}_{i,j,k}$ as an approximation of $F_{i,j}$ and $\varepsilon = \beta_{i,j,k}$.*

Proof: Due to the fact that Assumption 4.1(6) is not used in the proof of Lemma 4.3 we can apply these results. For that purpose we replace $S^{i,k}$ by $S^{i,j,k}$ and $h_{i,k}$ by $h_{i,j,k}$. Then we obtain

$$0 \leq \max_{t \in T} g(x,t) - \max_{t \in T_{h_{i,j,k}}} g(x,t) \leq L^t_{S^{i,j,k}} h_{i,j,k} \tag{5.6}$$

$$0 \leq F_{i,j}(x) - \tilde{F}_{i,j,k}(x) \leq \mu_i C_{S^{i,j,k}} L^t_{S^{i,j,k}} h_{i,j,k} \leq \beta_{i,j,k} \tag{5.7}$$

and

$$u(x) - \frac{\mu_i}{\max_{t \in T_{h_{i,j,k}}} g(x,t)} v(x) \in \partial_{\beta_{i,j,k}} f_i(x)$$

analogously to (4.5), (4.6) and (4.8) if we use $u(x) \in \partial f(x)$, $v(x) \in \partial \left(\max_{t \in T_{h_{i,j,k}}} g(x,t) \right)$ and the notation f_i as in Section 4.1. Consequently Assumption 3.1(a) is already shown.

In order to show part (b) we regard

$$F_{i,j}(x) = f_i(x) + \frac{s_i}{2} \|x - x^{i,j-1}\|_2^2$$

so that for all $x \in \mathcal{M}_0$

$$\partial_{\beta_{i,j,k}} F_{i,j}(x) \supset \partial_{\beta_{i,j,k}} f_i(x) + \partial \left(\frac{s_i}{2} \|x - x^{i,j-1}\|_2^2 \right) \tag{5.8}$$

follows from Theorem XI.3.1.1 in Hiriart-Urruty, Lemaréchal[21]. Furthermore,

$$\partial \left(\frac{s}{2} \|x - x^{i,j-1}\|_2^2 \right) = \left\{ s_i(x - x^{i,j-1}) \right\} \tag{5.9}$$

since this quadratic function is differentiable in x. Therefore we obtain

$$u(x) - \frac{\mu_i}{\max_{t \in T_{h_{i,j,k}}} g(x,t)} v(x) + s_i(x - x^{i,j-1}) \in \partial_{\beta_{i,j,k}} F_{i,j}(x) \tag{5.10}$$

for all $x \in \mathcal{M}_0$, particularly for all $x \in S^{i,j,k} \subset \mathcal{M}_0$, and Assumption 3.1(b) is fulfilled too.

Finally, Assumption 3.1(c) remains to show. The Lipschitz continuity of $F_{i,j}$ on $S^{i,j,k}$ can be established in the same manner as it is done in the proof of Lemma 4.3. From there we also know that the first term of the subgradient (5.10), coming from f_i, is bounded above on $S^{i,j,k}$. Additionally the second term $s_i(x - x^{i,j-1})$ is simply bounded above by the definition of $S^{i,j,k}$. Therefore all subgradients given by (5.10) are bounded above on $S^{i,j,k}$ which completes the proof. $\square$

In consequence of this lemma we know that the bundle method presented in Chapter 3 can also be used to solve the auxiliary problems arising in Algorithm 5.2. As in the previous chapter we can use $\beta_{i,j,k} = \mu_i C_{S^{i,j,k}} L_{S^{i,j,k}}^t h_{i,j,k}$ with predefined $h_{i,j,k}$ as error level when applying the bundle method.

Remark 5.4 If $\max_{t \in T} g(x,t)$ can be determined exactly for each x we can set $h_{i,j,k} = 0$. Then, as stated in Remark 4.4 for the unregularized algorithm, some simplifications in Algorithm 5.2 as well as in the analysis of it are possible. Particularly, the Assumptions 5.1(7) and (8) are not necessary in that case. $\square$

5.2 Convergence analysis

In this section we want to show that Algorithm 5.2 leads to an optimal solution of problem (2.10) under appropriate assumptions. We start with a closer look at the loop in k. Analogous to the result for the finiteness of the loop in k of Algorithm 4.2 the following result holds.

Lemma 5.5 *Let Assumption 5.1 be fulfilled. Furthermore, let i, j be fixed, $\delta_{i,j} > 0$ and $q_{i,j} \in (0,1)$ be given. If*

$$\mu_i L_{S^{i,j,k}}^t C_{S^{i,j,k}} h_{i,j,k} \leq \beta_{i,j,k} \leq q_{i,j}^k \delta_{i,j} \tag{5.11}$$

is true for all k, then the loop in k of Algorithm 5.2 terminates after a finite number of steps.

Proof: The proof is analogous to part a) of the proof of Proposition 4.6. $\square$

At this point we want to analyze the consequences of the stopping criterion of the loop in k.

Lemma 5.6 *Let Assumption 5.1 be fulfilled. Furthermore, let i, j be fixed and $\hat{x}$ be the unique optimal solution of*

$$\text{minimize} \quad F_{i,j}(x) \quad s.t. \quad x \in \mathcal{M}_0. \tag{5.12}$$

Moreover, let $x^{i,j,k-1}, x^{i,j,k}$ be generated by Algorithm 5.2 and $\beta_{i,j,k} \geq \mu_i L^t_{S^{i,j,k}} C_{S^{i,j,k}} h_{i,j,k}$ be valid. If inequality (5.5) is true, then

$$0 \leq F_{i,j}(x^{i,j,k-1}) - F_{i,j}(\hat{x}) \leq \max\left\{1, \frac{\|x^{i,j,k-1} - \hat{x}\|_\infty}{r_{i,j,k}}\right\}(\varepsilon_{i,j} + 2\beta_{i,j,k}) \tag{5.13}$$

and

$$\left\|x^{i,j,k-1} - \hat{x}\right\|_\infty \leq \left\|x^{i,j,k-1} - \hat{x}\right\|_2 \leq \max\left\{\sqrt{\frac{2(\varepsilon_{i,j} + 2\beta_{i,j,k})}{s_i}}, \frac{2(\varepsilon_{i,j} + 2\beta_{i,j,k})}{s_i r_{i,j,k}}\right\} \tag{5.14}$$

hold.

Proof: First, let us remark that Lemma 2.2 ensures the solvability of (5.12). This theorem can be applied because (5.12) is a barrier problem for a minimization problem of type (5.1). Additionally (5.12) is uniquely solvable since $F_{i,j}$ is strongly convex on $\mathcal{M}_0$.

Inequality (5.13) can be shown analogously as (4.11) in the proof of Lemma 4.5 so only the second inequality needs to be proven.

Due to the strong convexity of $F_{i,j}$ with modulus $s_i/2$ (in the sense of Definition A1.20 in Kaplan, Tichatschke [24]) we have

$$F_{i,j}(\lambda x + (1-\lambda)y) \leq \lambda F_{i,j}(x) + (1-\lambda)F_{i,j}(y) - \frac{s_i}{2}\lambda(1-\lambda)\|x - y\|_2^2$$

for all $\lambda \in [0,1]$ and all $x, y \in \mathcal{M}_0$. Taking into account that

$$F_{i,j}(\hat{x}) = \inf_{z \in \mathcal{M}_0} F_{i,j}(z) \leq F_{i,j}(\lambda x + (1-\lambda)y)$$

for all $\lambda \in [0,1]$ and all $x, y \in \mathcal{M}_0$ it follows that

$$F_{i,j}(\hat{x}) \leq \lambda F_{i,j}(\hat{x}) + (1-\lambda)F_{i,j}(x^{i,j,k-1}) - \frac{s_i}{2}\lambda(1-\lambda)\|\hat{x} - x^{i,j,k-1}\|_2^2$$

and

$$(1-\lambda)F_{i,j}(\hat{x}) \leq (1-\lambda)F_{i,j}(x^{i,j,k-1}) - \frac{s_i}{2}\lambda(1-\lambda)\|\hat{x} - x^{i,j,k-1}\|_2^2$$

are true for all $\lambda \in [0,1]$. Hence,

$$F_{i,j}(\hat{x}) \leq F_{i,j}(x^{i,j,k-1}) - \frac{s_i}{2}\lambda\|\hat{x} - x^{i,j,k-1}\|_2^2$$

for all $\lambda \in [0,1)$ so that

$$\frac{s_i}{2}\|\hat{x} - x^{i,j,k-1}\|_2^2 \leq F_{i,j}(x^{i,j,k-1}) - F_{i,j}(\hat{x}) \tag{5.15}$$

follows with $\lambda \nearrow 1$. Using (5.13) one obtains

$$\frac{s_i}{2}\|\hat{x} - x^{i,j,k-1}\|_2^2 \leq \max\left\{1, \frac{\|x^{i,j,k-1} - \hat{x}\|_\infty}{r_{i,j,k}}\right\}(\varepsilon_{i,j} + 2\beta_{i,j,k}).$$

At this point we distinguish two cases. We first suppose that $1 \geq \|x^{i,j,k-1} - \hat{x}\|_\infty / r_{i,j,k}$. Then it holds

$$\frac{s_i}{2} \|\hat{x} - x^{i,j,k-1}\|_2^2 \leq (\varepsilon_{i,j} + 2\beta_{i,j,k})$$

and

$$\left\| x^{i,j,k-1} - \hat{x} \right\|_\infty \leq \|\hat{x} - x^{i,j,k-1}\|_2 \leq \sqrt{\frac{2(\varepsilon_{i,j} + 2\beta_{i,j,k})}{s_i}}. \tag{5.16}$$

In the second case the inequality $1 < \|x^{i,j,k-1} - \hat{x}\|_\infty / r_{i,j,k}$ is supposed to be true. Then

$$\frac{s_i}{2} \|\hat{x} - x^{i,j,k-1}\|_2^2 \leq \frac{\|x^{i,j,k-1} - \hat{x}\|_\infty}{r_{i,j,k}} (\varepsilon_{i,j} + 2\beta_{i,j,k}) \leq \frac{\|x^{i,j,k-1} - \hat{x}\|_2}{r_{i,j,k}} (\varepsilon_{i,j} + 2\beta_{i,j,k})$$

is valid. We conclude that

$$\|\hat{x} - x^{i,j,k-1}\|_\infty \leq \|\hat{x} - x^{i,j,k-1}\|_2 \leq \frac{2(\varepsilon_{i,j} + 2\beta_{i,j,k})}{s_i r_{i,j,k}}. \tag{5.17}$$

Combining (5.16) and (5.17) completes the proof. □

In the following we denote the Euclidean ball with radius $\tau > 0$ around $x_c \in \mathbb{R}^n$ by $K_\tau(x_c)$, i.e.

$$K_\tau(x_c) := \{x \in \mathbb{R}^n : \|x - x_c\|_2 \leq \tau\}.$$

Theorem 5.7 *Let Assumption 5.1 be fulfilled. Furthermore, let $\tau \geq 1$ and $x_c \in \mathbb{R}^n$ be chosen such that $\mathcal{M}_{opt} \cap K_{\tau/8}(x_c) \neq \emptyset$. Let $x^* \in \mathcal{M}_{opt} \cap K_{\tau/8}(x_c)$, $\tilde{x} \in \mathcal{M}_0 \cap K_\tau(x_c)$ and $x^0 \in \mathcal{M}_0 \cap K_{\tau/4}(x_c)$ be fixed and $\delta_{i,j} > 0$, $q_{i,j} \in (0,1)$, $\alpha_i > 0$, $\tilde{t} \in T(\tilde{x})$, $\tilde{v} \in \partial g(\tilde{x}, \tilde{t})$ as well as*

$$\overline{c} \geq \|\tilde{x} - x^*\|_2 \quad \text{and} \quad c_3 := f(\tilde{x}) - f_- + c_0 + c_1$$

with

$$f_- \leq \min_{x \in \mathcal{M}} f(x), \quad c_0 := \left| \ln\left(-\max_{t \in T} g(\tilde{x}, t) \right) \right|, \quad c_1 := \ln\left(-\max_{t \in T} g(\tilde{x}, t) + 2\|\tilde{v}\|_2 \right)$$

be given. Moreover, assume that (5.11) is true for all $i \in \mathbb{N}$, $1 \leq j \leq j(i)$ and all k occurring in the outer loop (i,j) and that the controlling parameters of Algorithm 5.2 satisfy the following conditions:

$$\max\left\{ \sqrt{\frac{2(\varepsilon_{i,j} + 2\delta_{i,j})}{s_i}}, \frac{2(\varepsilon_{i,j} + 2\delta_{i,j})}{r_{i,j} s_i} \right\} \leq \alpha_i, \tag{5.18}$$

$$0 < \mu_{i+1} \leq \mu_i < 1 \quad \text{for all} \quad i \in \mathbb{N}, \quad \mu_1 \leq e^{-c_3}, \tag{5.19}$$

$$\sum_{i=1}^\infty \left[\left(\frac{2\mu_i}{s_i} (2|\ln \mu_i| + \ln \tau) \right)^{\frac{1}{2}} + 2\overline{c}\mu_i + \alpha_i \right] < \frac{\tau}{2} \tag{5.20}$$

and

$$\sigma_i > \left(\frac{2\mu_i}{s_i} (2|\ln \mu_i| + \ln \tau) + 4\tau\alpha_i \right)^{\frac{1}{2}} + \alpha_i. \tag{5.21}$$

Then it holds

(1) the loop in k is finite for each (i, j);

(2) the loop in j is finite for each i, i.e. $j(i) < \infty$;

(3) $\|x^{i,j} - x_c\|_2 < \tau$ for all pairs (i, j) with $0 \le j \le j(i)$;

(4) the sequence $\{x^{i,j}\} = \{x^{1,0}, \dots, x^{1,j(1)}, x^{2,0}, \dots, x^{2,j(2)}, x^{3,0}, \dots\}$ converges to an element $x^{**} \in \mathcal{M}_{opt} \cap K_\tau(x_c)$.

Proof: Our first proposition follows immediately from Lemma 5.5. The other propositions will be proven similarly to the proof of Theorem 1 in Kaplan, Tichatschke [27].

We first define $z^i = \mu_i \tilde{x} + (1 - \mu_i)x^*$. Due to $0 < \mu_i < 1$, $\tilde{x} \in \mathcal{M}_0 = \mathrm{int}\,(\mathcal{M})$, $x^* \in \mathcal{M}$ and the fact that $\mathcal{M}$ is convex one can infer $z^i \in \mathcal{M}_0$ with Theorem 6.1 in Rockafellar [45]. Then it follows

$$
\begin{aligned}
-\mu_i \ln \left(-\max_{t \in T} g(z^i, t) \right) &= -\mu_i \ln \left(-\max_{t \in T} g(\mu_i \tilde{x} + (1 - \mu_i)x^*, t) \right) \\
&\le -\mu_i \ln \left(-\mu_i \max_{t \in T} g(\tilde{x}, t) - (1 - \mu_i) \max_{t \in T} g(x^*, t) \right) \\
&\le -\mu_i \ln \left(-\mu_i \max_{t \in T} g(\tilde{x}, t) \right) \\
&= -\mu_i \left(\ln \mu_i + \ln \left(-\max_{t \in T} g(\tilde{x}, t) \right) \right) \\
&\le \mu_i \left(|\ln \mu_i| + c_0 \right),
\end{aligned}
\tag{5.22}
$$

because the $\max$ $-$function is convex and the logarithm increases monotonically. Furthermore, the estimates

$$
\|z^i - x^*\|_2 = \mu_i \|\tilde{x} - x^*\|_2 \le \overline{c} \mu_i
\tag{5.23}
$$

and

$$
f(z^i) \le \mu_i f(\tilde{x}) + (1 - \mu_i)f(x^*) \le f(x^*) + \mu_i(f(\tilde{x}) - f_-)
\tag{5.24}
$$

are obviously true. Additionally, this yields

$$
\max_{t \in T} g(x, t) \ge g(x, \tilde{t}) \ge g(\tilde{x}, \tilde{t}) + \tilde{v}^T(x - \tilde{x}) = \max_{t \in T} g(\tilde{x}, t) + \tilde{v}^T(x - \tilde{x})
$$

for all $x \in \mathbb{R}^n$ since $\tilde{v} \in \partial g(\tilde{x}, \tilde{t})$, $\tilde{t} \in T(\tilde{x})$. Consequently, using the Cauchy-Schwarz inequality and $\tilde{x} \in K_\tau(x_c)$, we obtain

$$
0 < -\max_{t \in T} g(x, t) \le -\max_{t \in T} g(\tilde{x}, t) + 2\tau \|\tilde{v}\|_2
$$

for all $x \in K_\tau(x_c) \cap \mathcal{M}_0$. Regarding the monotonicity of the logarithm and $\tau \ge 1$, this leads to

$$
\begin{aligned}
\inf \left\{ -\mu_i \ln \left(-\max_{t \in T} g(x, t) \right) : x \in K_\tau(x_c) \cap \mathcal{M}_0 \right\} &\ge -\mu_i \ln \left(-\max_{t \in T} g(\tilde{x}, t) + 2\|\tilde{v}\|_2 \tau \right) \\
&\ge -\mu_i(c_1 + \ln \tau).
\end{aligned}
\tag{5.25}
$$

As in the previous chapter we introduce

$$f_i(x) = f(x) - \mu_i \ln \left(- \max_{t \in T} g(x, t) \right)$$

for all $x \in \mathcal{M}_0$. The inequalities (5.22), (5.24), (5.25) and the optimality of x^* show that

$$f_i(z^i) \leq f(x) - \mu_i \ln \left(- \max_{t \in T} g(x, t) \right) + \mu_i(c_1 + \ln \tau) + \mu_i(f(\tilde{x}) - f_-) + \mu_i \left(| \ln \mu_i | + c_0 \right)$$

$$= f_i(x) + \mu_i(c_3 + \ln \tau + | \ln \mu_i |) \tag{5.26}$$

for all $x \in \mathcal{M}_0 \cap K_\tau(x_c)$. Combining this with (5.19) leads to

$$f_i(z^i) \leq f_i(x) + \mu_i \left(2 | \ln \mu_i | + \ln \tau \right) \tag{5.27}$$

for all $x \in \mathcal{M}_0 \cap K_\tau(x_c)$.

Using the results above we can prove the second and third proposition of our theorem by induction. For that we assume:

(i) i_0, j_0 are kept fixed with $0 \leq j_0 < j(i_0)$,

(ii) $j(i) < \infty$ if $i < i_0$,

(iii) if we denote

$$\overline{x}^{i,j} := \arg \min_{x \in \mathcal{M}_0} F_{i,j}(x) \quad \text{and} \quad \overline{\overline{x}}^{i,j} := \arg \min_{x \in \mathcal{M}} \left\{ f(x) + \frac{s_i}{2} \| x - x^{i,j-1} \|_2^2 \right\},$$

the relations

$$\overline{\overline{x}}^{i,j} = \arg \min_{x \in \mathcal{M} \cap K_\tau(x_c)} \left\{ f(x) + \frac{s_i}{2} \| x - x^{i,j-1} \|_2^2 \right\}, \tag{5.28}$$

$\| x^{i,j} - x_c \|_2 < \tau$, $\| \overline{x}^{i,j} - x_c \|_2 < \tau$ and $\| \overline{\overline{x}}^{i,j} - x_c \|_2 < \tau$ hold for all pairs of indices

$$(i, j) \in Q_0 := \left\{ (i', j') : \{ i' < i_0, 0 < j' \leq j(i') \} \vee \{ i' = i_0, 0 < j' \leq j_0 \} \right\}.$$

Let us remark that $\overline{x}^{i,j} \in \mathcal{M}_0$ as minimizing point of $F_{i,j}$ exists due to Lemma 2.2. The existence of $\overline{\overline{x}}^{i,j} \in \mathcal{M}$ as minimizer of $f(x) + \frac{s_i}{2} \| x - x^{i,j-1} \|_2^2$ is ensured by the strong convexity and continuity of this function on the nonempty and closed set $\mathcal{M}$.

At this point we have to check (i)-(iii) for the starting values $i_0 = 1$, $j_0 = 0$, but this is easy: By construction $j(1) > 0$ so that $i_0 = 1$, $j_0 = 0$ fulfill the first assumption. The other two assumptions are obvious by construction.

Using the stopping criterion of the loop in k of Algorithm 5.2, (5.11), (5.14), (5.18) as well as the definition of $\overline{x}^{i,j}$ we deduce

$$\left\| \overline{x}^{i,j} - x^{i,j} \right\|_2 \leq \alpha_i. \tag{5.29}$$

Furthermore, taking the definitions of $\overline{x}^{i,j}$ and $\overline{\overline{x}}^{i,j}$, (2.5) into account we can conclude

$$f\left(\overline{x}^{i,j} \right) + \frac{s_i}{2} \left\| \overline{x}^{i,j} - x^{i,j-1} \right\|_2^2 - f\left(\overline{\overline{x}}^{i,j} \right) - \frac{s_i}{2} \left\| \overline{\overline{x}}^{i,j} - x^{i,j-1} \right\|_2^2 \leq \mu_i. \tag{5.30}$$

Additionally one can establish

$$\frac{s_i}{2}\left\|\overline{x}^{i,j}-\overline{\overline{x}}^{i,j}\right\|_2^2 \le f\left(\overline{x}^{i,j}\right) + \frac{s_i}{2}\left\|\overline{x}^{i,j}-x^{i,j-1}\right\|_2^2 - f\left(\overline{\overline{x}}^{i,j}\right) - \frac{s_i}{2}\left\|\overline{\overline{x}}^{i,j}-x^{i,j-1}\right\|_2^2 \qquad (5.31)$$

in the same manner as (5.15). Combining (5.30) and (5.31) we see

$$\frac{s_i}{2}\left\|\overline{x}^{i,j}-\overline{\overline{x}}^{i,j}\right\|_2^2 \le \mu_i,$$

so that

$$\left\|\overline{x}^{i,j}-\overline{\overline{x}}^{i,j}\right\|_2 \le \sqrt{\frac{2\mu_i}{s_i}}. \qquad (5.32)$$

Using this and (5.29) we obtain

$$\left\|\overline{\overline{x}}^{i,j}-x^{i,j}\right\|_2 \le \alpha_i + \sqrt{\frac{2\mu_i}{s_i}}. \qquad (5.33)$$

Due to $\overline{x}^{i,j} \in \mathcal{M}_0 \cap K_\tau(x_c)$ estimate (5.27) implies

$$f_i(z^i) \le f_i(\overline{x}^{i,j}) + \mu_i\left(2|\ln \mu_i| + \ln \tau\right) \qquad (5.34)$$

for all $(i,j) \in Q_0$. In the sequel we distinguish the following cases

a) $i < i_0, 0 \le j < j(i) - 1$ or $i = i_0, 0 \le j < j_0$,

b) $i < i_0, j = j(i) - 1$ and

c) $i = i_0, j = j_0 + 1$.

ad a) In this case we obtain

$$\left\|\overline{x}^{i,j+1}-z^i\right\|_2^2 - \left\|x^{i,j}-z^i\right\|_2^2 \le -\left\|\overline{x}^{i,j+1}-x^{i,j}\right\|_2^2 + \frac{2}{s_i}\left(f_i(z^i) - f_i(\overline{x}^{i,j+1})\right)$$
$$\le -\left\|\overline{x}^{i,j+1}-x^{i,j}\right\|_2^2 + \frac{2\mu_i}{s_i}\left(2|\ln \mu_i| + \ln \tau\right) \qquad (5.35)$$

by using Proposition 8.3 in Kaplan, Tichatschke [24] and (5.34). Taking (5.21), (5.29) and the stopping criterion of the loop in j of Algorithm 5.2 into account we conclude

$$\left\|\overline{x}^{i,j+1}-x^{i,j}\right\|_2 \ge \left\|x^{i,j+1}-x^{i,j}\right\|_2 - \left\|\overline{x}^{i,j+1}-x^{i,j+1}\right\|_2 > \sigma_i - \alpha_i > 0. \qquad (5.36)$$

and we have

$$\left\|\overline{x}^{i,j+1}-z^i\right\|_2^2 - \left\|x^{i,j}-z^i\right\|_2^2 \le -\tilde{\varepsilon}_i^2 + \gamma_i < 0 \qquad (5.37)$$

with $\tilde{\varepsilon}_i = \sigma_i - \alpha_i$ and $\gamma_i = 2\mu_i(2|\ln \mu_i| + \ln \tau)/s_i$.

Moreover, regarding $\|x^{i,j}-x_c\|_2 < \tau$ and $\|z^i - x_c\| < \tau$, the estimate

$$\left\|\overline{x}^{i,j+1}-z^i\right\|_2 - \left\|x^{i,j}-z^i\right\|_2 < \left(2\left\|x^{i,j}-z^i\right\|_2\right)^{-1}\left(-\tilde{\varepsilon}_i^2 + \gamma_i\right) \le \frac{1}{4\tau}\left(-\tilde{\varepsilon}_i^2 + \gamma_i\right) \qquad (5.38)$$

holds. Together with (5.21) and (5.29) we obtain

$$\left\|x^{i,j+1}-z^i\right\|_2 - \left\|x^{i,j}-z^i\right\|_2 < \frac{1}{4\tau}\left(-\tilde{\varepsilon}_i^2 + \gamma_i\right) + \alpha_i < 0. \qquad (5.39)$$

ad b) Now we assume $i < i_0$, $j = j(i) - 1$.

In this case we can combine (5.27), (5.29) and the implications of Proposition 8.3 in Kaplan, Tichatschke [24] to see that

$$
\begin{aligned}
\left\| x^{i,j(i)} - z^i \right\|_2 - \left\| x^{i,j(i)-1} - z^i \right\|_2 &\leq \left\| \overline{x}^{i,j(i)} - x^{i,j(i)-1} \right\|_2 - \left\| x^{i,j(i)-1} - z^i \right\|_2 + \alpha_i \\
&\leq \left(\frac{2}{s_i} \left(f_i(z^i) - f_i(x^{i,j(i)}) \right) \right)^{\frac{1}{2}} + \alpha_i \\
&\leq \left(\frac{2\mu_i}{s_i} (2|\ln \mu_i| + \ln \tau) \right)^{\frac{1}{2}} + \alpha_i \\
&= \sqrt{\gamma_i} + \alpha_i
\end{aligned}
\tag{5.40}
$$

holds. Summing the inequalities (5.39) w.r.t. $j = 0, 1, \ldots, j(i) - 2$ for a fixed $i < i_0$ and adding (5.40) leads to

$$
\left\| x^{i,j(i)} - z^i \right\|_2 - \left\| x^{i,0} - z^i \right\|_2 \leq \sqrt{\gamma_i} + \alpha_i,
\tag{5.41}
$$

and together with (5.23) one has

$$
\left\| x^{i,j(i)} - x^* \right\|_2 - \left\| x^{i,0} - x^* \right\|_2 \leq \sqrt{\gamma_i} + \alpha_i + 2\overline{c}\mu_i.
\tag{5.42}
$$

ad c) Now we assume $i = i_0$, $j = j_0 + 1$.

In this case we consider

$$
\hat{x}^{i_0,j_0+1} := \arg \min_{x \in \mathcal{M} \cap K_\tau(x_c)} \left\{ f(x) + \frac{s_{i_0}}{2} \left\| x - x^{i_0,j_0} \right\|_2^2 \right\}.
$$

The non-expansivity of the prox-mapping (see, e.g. Rockafellar [45]) yields

$$
\left\| \hat{x}^{i_0,j_0+1} - x^* \right\|_2 \leq \left\| x^{i_0,j_0} - x^* \right\|_2.
\tag{5.43}
$$

Using this, (5.23) and (5.39) for $i = i_0, 0 \leq j < j_0$ we obtain

$$
\begin{aligned}
\left\| \hat{x}^{i_0,j_0+1} - x^* \right\|_2 &\leq \left\| x^{i_0,j_0} - z^{i_0} \right\|_2 + \overline{c}\mu_{i_0} \\
&\leq \left\| x^{i_0,0} - z^{i_0} \right\|_2 + \overline{c}\mu_{i_0} \\
&\leq \left\| x^{i_0,0} - x^* \right\|_2 + 2\overline{c}\mu_{i_0}.
\end{aligned}
$$

If $i_0 > 1$ this leads to

$$
\left\| \hat{x}^{i_0,j_0+1} - x^* \right\|_2 \leq \left\| x^{i_0-1,j(i_0-1)} - x^* \right\|_2 + 2\overline{c}\mu_{i_0}.
$$

Now the successive application of (5.42) gives

$$
\left\| \hat{x}^{i_0,j_0+1} - x^* \right\|_2 \leq \left\| x^{1,0} - x^* \right\|_2 + \sum_{k=1}^{i_0-1} \left(\sqrt{\gamma_k} + \alpha_k + 2\overline{c}\mu_k \right) + 2\overline{c}\mu_{i_0}.
\tag{5.44}
$$

As we assumed that $\| x^* - x_c \|_2 \leq \tau/8$ and $\| x^{1,0} - x_c \|_2 < \tau/4$ we can now assemble (5.19), (5.20) and (5.44) to get

$$
\left\| \hat{x}^{i_0,j_0+1} - x_c \right\|_2 < \tau - \alpha_{i_0} - \sqrt{\gamma_{i_0}} < \tau - \alpha_{i_0} - \sqrt{2\frac{\mu_{i_0}}{s_{i_0}}}.
\tag{5.45}
$$

We see that $\left\|\hat{x}^{i_0,j_0+1} - x_c\right\|_2 < \tau$ and due to the strong convexity of $f + \frac{s_{i_0}}{2}\left\|\cdot - x^{i_0,j_0}\right\|_2^2$ we can deduce that $\hat{x}^{i_0,j_0+1}$ must actually be identical to $\overline{\overline{x}}^{i_0,j_0+1}$. But then the estimates (5.32) and (5.33) imply $\left\|\overline{x}^{i_0,j_0+1} - x_c\right\|_2 < \tau$ as well.

So far we have proven that Assumption (iii) is also true for $i = i_0$, $j = j_0 + 1$. It still remains to prove that $j(i_0) < \infty$ holds. In order to do so we sum up the inequalities (5.39) to an arbitrary $\overline{j} \leq j(i_0) - 1$ and obtain

$$\left\|x^{i_0,\overline{j}} - z^{i_0}\right\|_2 < \left\|x^{i_0,0} - z^{i_0}\right\|_2 + \overline{j}\left(\frac{1}{4\tau}\left(-\tilde{\varepsilon}_{i_0}^2 + \gamma_{i_0}\right) + \alpha_{i_0}\right).$$

Dividing by $\frac{1}{4\tau}\left(-\tilde{\varepsilon}_{i_0}^2 + \gamma_{i_0}\right) + \alpha_{i_0}$ we get an upper bound for $\overline{j}$:

$$\overline{j} < -\left\|x^{i_0,0} - z^{i_0}\right\|_2\left(\frac{1}{4\tau}\left(-\tilde{\varepsilon}_{i_0}^2 + \gamma_{i_0}\right) + \alpha_{i_0}\right)^{-1} < \infty. \tag{5.46}$$

Thus we have shown the induction statements to hold for $i_0, j_0 + 1$ if $j_0 < j(i_0)$. But the case $j_0 = j(i_0)$ is equivalent to the case $(i_0 + 1, 0)$ and so the induction holds for all possible indices (i_0, j_0). As a consequence of this the second and third proposition of the theorem are proven.

It remains to prove the convergence of the generated sequence $\{x^{i,j}\}$ to an optimal solution of the given semi-infinite problem (2.10). Let an arbitrary element of $\mathcal{M}_{opt} \cap K_\tau(x_c)$ be given by $\overline{x}$. Defining

$$\overline{z}^i = \tilde{x} + (1 - \mu_i)(\overline{x} - \tilde{x}),$$

we can show $\|\overline{z}^i - \overline{x}\|_2 \leq 2\tau\mu_i$ similar to (5.23) and analogous results to (5.41) and (5.42) with $\overline{z}^i$ instead of z^i and $\overline{x}$ instead of x^*. Additionally, we obtain from (5.20)

$$\sum_{i=1}^{\infty}\sqrt{\gamma_i} < \infty, \quad \sum_{i=1}^{\infty}\mu_i < \infty \quad \text{and} \quad \sum_{i=1}^{\infty}\alpha_i < \infty$$

and the convergence of $\left\{\left\|x^{i,0} - \overline{x}\right\|_2\right\}$ is ensured by Lemma 2.2.2 in Poljak [39]. Moreover, the results (5.26), (5.35), (5.39) and (5.40) remain true if we use $\overline{z}^i$ instead of z^i and

$$\left\|x^{i,j} - \overline{z}^i\right\|_2 < \left\|x^{i,0} - \overline{z}^i\right\|_2, \qquad \left\|x^{i+1,0} - \overline{z}^i\right\|_2 \leq \sqrt{\gamma_i} + \alpha_i + \left\|x^{i,j} - \overline{z}^i\right\|_2$$

for all i and $0 < j < j(i)$. Since $\overline{z}^i, \overline{x} \in K_\tau(x_c)$ this leads to

$$\left\|x^{i+1,0} - \overline{x}\right\|_2 - \sqrt{\gamma_i} - \alpha_i - 4\tau\mu_i \leq \left\|x^{i,j} - \overline{x}\right\|_2 < \left\|x^{i,0} - \overline{x}\right\|_2 + 4\tau\mu_i,$$

hence the sequence $\left\{\left\|x^{i,j} - \overline{x}\right\|_2\right\}$ converges. Furthermore, regarding (5.29) and $\lim_{i\to\infty}\alpha_i = 0$ which is enforced by (5.20), it is clear that $\left\{\left\|\overline{x}^{i,j} - \overline{x}\right\|_2\right\}$ converges to the same limit point.

Due to $\tilde{x}, \overline{x} \in K_\tau(x_c)$ and $0 < \mu_i < 1$ for all $i \in \mathbb{N}$ we have $\overline{z}^i \in K_\tau(x_c)$ for all $i \in \mathbb{N}$ as well as

$$\left\|x^{i,j-1} - \overline{z}^i\right\|_2 \leq \left\|x^{i,j-1} - \overline{x}\right\|_2 + 2\tau\mu_i$$

for all pairs (i, j) with $1 \leq j \leq j(i)$ and

$$\left\|\overline{x}^{i,j} - \overline{z}^i\right\|_2 \geq \left\|\overline{x}^{i,j} - \overline{x}\right\|_2 - 2\tau\mu_i$$

for all pairs (i, j) with $0 \le j \le j(i)$. Consequently we obtain

$$\left\| x^{i,j-1} - \overline{z}^i \right\|_2^2 \le \left\| x^{i,j-1} - \overline{x} \right\|_2^2 + 8\tau^2 \mu_i + 4\tau^2 \mu_i^2$$

for all pairs (i, j) with $1 \le j \le j(i)$ and

$$\left\| \overline{x}^{i,j} - \overline{z}^i \right\|_2 \ge \left\| \overline{x}^{i,j} - \overline{x} \right\|_2^2 - 8\tau^2 \mu_i - 4\tau^2 \mu_i^2$$

for all pairs (i, j) with $0 \le j \le j(i)$. Additionally the modified estimates (5.26)

$$f_i(\overline{z}^i) \le f_i(x) + \mu_i(c_3 + \ln \tau + |\ln \mu_i|)$$

for all $x \in \mathcal{M}_0 \cap K_\tau(x_c)$ and (5.35)

$$\left\| \overline{x}^{i,j} - \overline{z}^i \right\|_2^2 - \left\| x^{i,j-1} - \overline{z}^i \right\|_2^2 \le \frac{2}{s_i} \left(f_i(\overline{z}^i) - f_i(\overline{x}^{i,j}) \right)$$

allow to infer

$$\left\| x^{i,j-1} - \overline{x} \right\|_2^2 - \left\| \overline{x}^{i,j} - \overline{x} \right\|_2^2 \ge \frac{2}{s_i} \left(f_i(\overline{x}^{i,j}) - f_i(x) - \mu_i(c_3 + \ln \tau + |\ln \mu_i|) \right) - 16\tau^2 \mu_i - 8\tau^2 \mu_i^2$$

for all $x \in \mathcal{M}_0 \cap K_\tau(x_c)$. Then, regarding $\overline{x}^{i,j} \in \mathcal{M}_0 \cap K_\tau(x_c)$ and estimate (5.25), we obtain

$$\left\| x^{i,j-1} - \overline{x} \right\|_2^2 - \left\| \overline{x}^{i,j} - \overline{x} \right\|_2^2 \ge \frac{2}{s_i} \left(f(\overline{x}^{i,j}) - f_i(x) - \mu_i(c_1 + \ln \tau) \right) \tag{5.47}$$
$$- \frac{2\mu_i}{s_i}(c_3 + \ln \tau + |\ln \mu_i|) - 8\tau^2 \mu_i(2 + \mu_i).$$

Furthermore, we have $\lim_{i \to \infty} f_i(x) = f(x)$ for each fixed $x \in \mathcal{M}_0$ so $\mu_i \to 0$ and $s_i \le \overline{s}$ give

$$\limsup_{i \to \infty} \left(\max_{1 \le j \le j(i)} \left(f(\overline{x}^{i,j}) - f(x) \right) \right) \le 0 \tag{5.48}$$

for each fixed $x \in \mathcal{M}_0 \cap K_\tau(x_c)$.

Now let x^{**} be an accumulation point of the sequence $\{x^{i,j}\}$. Such an accumulation point exists since $x^{i,j} \in K_\tau(x_c) \cap \mathcal{M}$ for all pairs (i, j). Regarding (5.29) and $\lim_{i \to \infty} \alpha_i = 0$ it follows that x^{**} is also an accumulation point of the sequence $\{\overline{x}^{i,j}\}$. Further we obtain $x^{**} \in \mathcal{M} \cap K_\tau(x_c)$ since the sets $\mathcal{M}$ and $K_\tau(x_c)$ are closed. For each $x \in \mathcal{M}_0 \cap K_\tau(x_c)$ estimate (5.48) establishes $f(x)$ as an upper bound for $f(x^{**})$ so that we deduce

$$f(x^{**}) \le \inf \{f(x) : x \in \mathcal{M}_0 \cap K_\tau(x_c)\}. \tag{5.49}$$

Obviously $\mathcal{M} \cap K_\tau(x_c)$ is the closure of $\mathcal{M}_0 \cap K_\tau(x_c)$. Further $x^* \in \mathcal{M}_{opt} \cap K_\tau(x_c)$ such that (5.49) implies $f(x^{**}) \le f(x^*)$ resp. $x^{**} \in \mathcal{M}_{opt} \cap K_\tau(x_c)$. Consequently, regarding that $\{\left\| x^{i,j} - \overline{x} \right\|_2\}$ converges for each $\overline{x} \in \mathcal{M}_{opt} \cap K_\tau(x_c)$, the sequence $\{\left\| x^{i,j} - x^{**} \right\|_2\}$ converges to zero. Thus the sequence $\{x^{i,j}\}$ converges to $x^{**} \in \mathcal{M}_{opt}$. $\qquad \square$

Remark 5.8 If $\max_{t \in T} g(\cdot, t)$ is bounded below on the feasible set $\mathcal{M}$ of (2.10), i.e. there exists a (nonpositive) constant $d > -\infty$ with $d \le \max_{t \in T} g(x, t)$ for all $x \in \mathcal{M}$, one obtains

$$\inf \left\{ -\mu_i \ln \left(-\max_{t \in T} g(x, t) \right) : x \in K_\tau(x_c) \right\} \ge -\mu_i \ln(-d).$$

Consequently, using this estimate instead of (5.25) in the proof above, the conditions on the parameter of Algorithm 5.2 in Theorem 5.7 can be simplified in the considered case. In particular, c_3 can be changed into $c_3 = f(\check{x}) - f_- + c_0 - \ln(-d)$ and in (5.20) and (5.21) the term $\ln \tau$ can be dropped. Thus the left-hand side of the modified estimate (5.20) does not depend on (the unknown) τ. Therefore one could choose the value of τ after determining $\{\mu_i\}$, $\{\alpha_i\}$ and $\{s_i\}$. Finally, the value of σ_i can be fixed such that (5.21) holds. Altogether the described procedure is much easier than the simultaneous determination of all parameters in the general case. $\square$

Remark 5.9 The conditions on the parameters of the method require their separate adjustment to each example, which can be a very fragile task when applying the multi-step procedure. In case of using the one-step procedure parameters according to Theorem 5.7 are easily chosen. The one-step procedure is given if $j(i) = 1$ for each i, which can be ensured by choosing σ_i sufficiently large[2]. Then (5.21) is automatically satisfied for each fixed $\tau \geq 1$. Furthermore, (5.20) holds for all sufficiently large values of τ if one guarantees that

$$\sum_{i=1}^{\infty} \left(\frac{\mu_i |\ln \mu_i|}{s_i} \right)^{\frac{1}{2}} < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \alpha_i < \infty.$$

Consequently, (5.20) and (5.21) can be replaced by the given conditions above and τ, σ_i need not to be specified explicitly. $\square$

At the end of this section an estimate of the difference between the current value of the objective function f at the end of an outer step and its minimal value f^* on $\mathcal{M}$ is established (cf. Kaplan, Tichatschke [25, 27]).

Lemma 5.10 *Let the assumptions of Theorem 5.7 be satisfied and let f be Lipschitz continuous with modulus L on $K_\tau(x_c)$.[3] Then*

$$f\left(x^i\right) - f^* \leq \left(\frac{9}{8} \tau s_i + L \right) \left(\sqrt{\frac{2\mu_i}{s_i}} + \alpha_i \right) + \frac{9}{8} \tau s_i \sigma_i$$

holds for all $i \in \mathbb{N}$.

Proof: Let i be fixed and $0 \leq j < j(i)$ be arbitrarily given. In the proof of Theorem 5.7 we defined

$$\overline{\overline{x}}^{i,j+1} = \arg\min_{x \in \mathcal{M}} \left\{ f(x) + \frac{s_i}{2} \left\| x - x^{i,j} \right\|_2^2 \right\}.$$

The affiliation $\overline{\overline{x}}^{i,j+1} \in K_\tau(x_c)$ has already been shown. Since $\mathcal{M} \cap K_\tau(x_c)$ is obviously convex $(1 - \lambda)\overline{\overline{x}}^{i,j+1} + \lambda x^* \in \mathcal{M} \cap K_\tau(x_c)$ for all $\lambda \in [0, 1]$. The optimality of $\overline{\overline{x}}^{i,j+1}$ gives

$$f\left((1-\lambda)\overline{\overline{x}}^{i,j+1} + \lambda x^*\right) + \frac{s_i}{2} \left\| (1-\lambda)\overline{\overline{x}}^{i,j+1} + \lambda x^* - x^{i,j} \right\|_2^2 \geq f\left(\overline{\overline{x}}^{i,j+1}\right) + \frac{s_i}{2} \left\| \overline{\overline{x}}^{i,j+1} - x^{i,j} \right\|_2^2$$

so that, regarding the convexity of f, for all $\lambda \in (0, 1]$

$$0 \leq \lambda f(x^*) - \lambda f\left(\overline{\overline{x}}^{i,j+1}\right) + \frac{s_i}{2}\lambda^2 \left\| x^* - \overline{\overline{x}}^{i,j+1} \right\|_2^2 + \lambda s_i \left(x^* - \overline{\overline{x}}^{i,j+1} \right)^T \left(\overline{\overline{x}}^{i,j+1} - x^{i,j} \right)$$

---

[2]Of course, when one actually applies the multi-step approach large values of σ_i must be avoided.

[3]Due to $f(x) \in \mathbb{R}$ for each $x \in K_\tau(x_c)$, the existence of a Lipschitz constant L is ensured by Theorem 24.7 in Rockafellar [45].

and

$$f\left(\overline{\overline{x}}^{i,j+1}\right) - f(x^*) \leq s_i \left(x^* - \overline{\overline{x}}^{i,j+1}\right)^T \left(\overline{\overline{x}}^{i,j+1} - x^{i,j}\right) + \frac{s_i}{2}\lambda \left\|x^* - \overline{\overline{x}}^{i,j+1}\right\|_2^2.$$

Taking the limit $\lambda \searrow 0$ combined with the Cauchy-Schwarz inequality leads to

$$f\left(\overline{\overline{x}}^{i,j+1}\right) - f(x^*) \leq s_i \left(x^* - \overline{\overline{x}}^{i,j+1}\right)^T \left(\overline{\overline{x}}^{i,j+1} - x^{i,j}\right)$$
$$\leq s_i \left\|x^* - \overline{\overline{x}}^{i,j+1}\right\|_2 \left(\left\|\overline{\overline{x}}^{i,j+1} - x^{i,j+1}\right\|_2 + \left\|x^{i,j+1} - x^{i,j}\right\|_2\right).$$

and using the Lipschitz continuity of f

$$f\left(x^{i,j+1}\right) - f(x^*) \leq \left\|\overline{\overline{x}}^{i,j+1} - x^{i,j+1}\right\|_2 \left(L + s_i \left\|x^* - \overline{\overline{x}}^{i,j+1}\right\|_2\right)$$
$$+ s_i \left\|x^* - \overline{\overline{x}}^{i,j+1}\right\|_2 \left\|x^{i,j+1} - x^{i,j}\right\|_2.$$

In view of (5.33) and

$$\left\|x^* - \overline{\overline{x}}^{i,j+1}\right\|_2 \leq \|x^* - x_c\|_2 + \left\|\overline{\overline{x}}^{i,j+1} - x_c\right\|_2 \leq \frac{\tau}{8} + \tau = \frac{9}{8}\tau$$

we obtain

$$f\left(x^{i,j+1}\right) - f(x^*) \leq \left(\frac{9}{8}\tau s_i + L\right)\left(\alpha_i + \sqrt{\frac{2\mu_i}{s_i}}\right) + \frac{9}{8}\tau s_i \left\|x^{i,j+1} - x^{i,j}\right\|_2,$$

so that our proposition follows w.r.t. $\left\|x^{i,j(i)} - x^{i,j(i)-1}\right\|_2 \leq \sigma_i$, $x^i = x^{i,j(i)}$ and $f(x^*) = f^*$. □

5.3   Rate of convergence

In the following sections we analyze further convergence properties of Algorithm 5.2 with regard to the rate of convergence based on results of Kaplan, Tichatschke [25, 27]. For that we consider the sequence $\{\overline{x}^{i,j}\}$ instead of the generated sequence $\{x^{i,j}\}$ whereby

$$\overline{x}^{i,j} = \arg\min_{x \in \mathcal{M}_0} F_{i,j}(x)$$

is defined as in the proof of Theorem 5.7. That means we consider the sequence of the exact minima of $F_{i,j}$ instead of the computed approximate minima. However, based on results for the exact minima we can also achieve results for the approximate minima, e.g. by using (5.29).

First the value of $\max_{t \in T} g(\overline{x}^{i,j}, t)$ is estimated.

Lemma 5.11 *Let the assumptions of Theorem 5.7 be satisfied. Then for each i and $1 \leq j \leq j(i)$*

$$-\max_{t \in T} g(\overline{x}^{i,j}) \geq c_4 \mu_i$$

holds with

$$c_4 := -\left(\mu_1 + f(\tilde{x}) - f^* + 2\tau^2\overline{s}\right)^{-1} \max_{t \in T} g(\tilde{x}, t)$$

and $\tilde{x}$ defined in Theorem 5.7.

Proof: Let i, j be arbitrarily given with $1 \leq j \leq j(i)$. Due to $\overline{x}^{i,j} \in \mathcal{M}_0 = \mathrm{dom}\,(F_{i,j})$ Theorem 23.1 in Rockafellar [45] ensures the existence of the directional derivative $F'_{i,j}(\overline{x}^{i,j}; d)$ for each $d \in \mathbb{R}^n$. Since $F_{i,j}$ attains its minimal value at $\overline{x}^{i,j}$ and since $\mathcal{M}_0$ is open we obtain $F'_{i,j}(\overline{x}^{i,j}; d) \geq 0$ for each $d \in \mathbb{R}^n$ so that $0 \in \partial F_{i,j}\left(\overline{x}^{i,j}\right)$ follows from Theorem 23.2 in Rockafellar [45]. From (5.8) and (5.9) we already know

$$\partial F_{i,j}\left(\overline{x}^{i,j}\right) \supset \partial f_i\left(\overline{x}^{i,j}\right) + \left\{ s_i\left(\overline{x}^{i,j} - x^{i,j-1}\right) \right\}.$$

Regarding

$$\overline{x}^{i,j} \in \mathcal{M}_0 = \mathrm{ri}\,(\mathrm{dom}\,(f_i)) \cap \mathrm{ri}\,\left(\mathrm{dom}\,\left(\frac{s_i}{2}\left\|\cdot - x^{i,j-1}\right\|_2^2\right)\right),$$

Theorem 23.8 in Rockafellar [45] even leads to

$$\partial F_{i,j}\left(\overline{x}^{i,j}\right) = \partial f_i\left(\overline{x}^{i,j}\right) + \left\{ s_i\left(\overline{x}^{i,j} - x^{i,j-1}\right) \right\}.$$

Moreover, analogous to (2.7) in the proof of Theorem 2.3, one obtains

$$\partial f_i\left(\overline{x}^{i,j}\right) = \partial f\left(\overline{x}^{i,j}\right) + \mu_i \frac{1}{-\max_{t \in T} g(\overline{x}^{i,j}, t)} \partial \left(\max_{t \in T} g(\overline{x}^{i,j}, t) \right),$$

hence

$$\partial F_{i,j}\left(\overline{x}^{i,j}\right) = \partial f\left(\overline{x}^{i,j}\right) + \frac{\mu_i}{-\max_{t \in T} g(\overline{x}^{i,j}, t)} \partial \left(\max_{t \in T} g(\overline{x}^{i,j}, t) \right) + \left\{ s_i\left(\overline{x}^{i,j} - x^{i,j-1}\right) \right\},$$

i.e. there exist $u_f \in \partial f(\overline{x}^{i,j})$ and $u_g \in \partial\left(\max_{t \in T} g(\overline{x}^{i,j}, t)\right)$ with

$$u_f - \frac{\mu_i}{\max_{t \in T} g(\overline{x}^{i,j}, t)} u_g + s_i\left(\overline{x}^{i,j} - x^{i,j-1}\right) = 0$$

and multiplication with $(\overline{x}^{i,j} - \tilde{x})$ leads to

$$u_f^T(\overline{x}^{i,j} - \tilde{x}) + \frac{\mu_i}{-\max_{t \in T} g(\overline{x}^{i,j}, t)} u_g^T(\overline{x}^{i,j} - \tilde{x}) + s_i(\overline{x}^{i,j} - x^{i,j-1})^T(\overline{x}^{i,j} - \tilde{x}) = 0.$$

Using the properties of the subgradients u_f and u_g as well as the convexity of the norm we obtain

$$0 \geq f\left(\overline{x}^{i,j}\right) - f\left(\tilde{x}\right) + \frac{\mu_i}{-\max_{t \in T} g(\overline{x}^{i,j}, t)} \left(\max_{t \in T} g(\overline{x}^{i,j}, t) - \max_{t \in T} g(\tilde{x}, t) \right)$$
$$+ \frac{s_i}{2} \left(\left\|\overline{x}^{i,j} - x^{i,j-1}\right\|_2^2 - \left\|\tilde{x} - x^{i,j-1}\right\|_2^2 \right),$$

so that we can conclude

$$\frac{\mu_i}{-\max_{t \in T} g(\overline{x}^{i,j}, t)} \left(\max_{t \in T} g(\overline{x}^{i,j}, t) - \max_{t \in T} g(\tilde{x}, t) \right) \leq f(\tilde{x}) - f^* + \frac{s_i}{2} \left\|\tilde{x} - x^{i,j-1}\right\|_2^2$$

and using $\tilde{x}, x^{i,j-1} \in K_\tau(x_c)$ gives

$$\frac{\mu_i}{-\max_{t \in T} g(\overline{x}^{i,j}, t)} \left(\max_{t \in T} g(\overline{x}^{i,j}, t) - \max_{t \in T} g(\tilde{x}, t) \right) \leq f(\tilde{x}) - f^* + 2\tau^2 \overline{s}.$$

Now, regarding $\mu_i \leq \mu_1$, it is obvious that

$$-\max_{t \in T} g(\overline{x}^{i,j}, t) \geq \mu_i(-\max_{t \in T} g(\tilde{x}, t)) \left(\mu_1 + f(\tilde{x}) - f^* + 2\tau^2 \overline{s}\right)^{-1}$$

holds and the proof is complete. $\qquad\square$

We introduce

$$\Delta_{i,j} := f(\overline{x}^{i,j}) - f^*$$

for each i and $1 \leq j \leq j(i)$. In order to complete this definition for $j = 0$ we set $\overline{x}^{i+1,0} := \overline{x}^{i,j(i)}$ for each $i \in \mathbb{N}$ as well as $\overline{x}^{1,0} := x^{1,0} = x^0$ such that $\Delta_{i,0}$ can be defined as $\Delta_{i,j}$ above.

Theorem 5.12 *Let the assumptions of Theorem 5.7 be satisfied. Moreover, assume that*

$$\mu_1 \leq -\frac{1}{c_4} \max_{t \in T} g(x^0) \tag{5.50}$$

with c_4 given as in Lemma 5.11. Additionally let a positive constant α with

$$\alpha \leq \left(16 \overline{s} \tau^2\right)^{-1}, \quad \alpha \sup_{i,j} \Delta_{i,j} \leq \frac{7}{32} \tag{5.51}$$

be given and assume that for each i the constant

$$\kappa_i := \mu_i(c_1 + \ln \tau) - \mu_i \ln \left(\frac{1}{8} c_4 \mu_i\right) + \frac{s_i}{2} \alpha_i^2 + \frac{7}{4} s_i \alpha_i \tau \tag{5.52}$$

satisfies

$$\kappa_i \leq \alpha \left(\frac{\Delta_{1,0}}{1 + \alpha \sum_{k=1}^{i-1} j^1(k) \Delta_{1,0}}\right)^2, \tag{5.53}$$

$$2\kappa_i \leq \frac{s_i}{2}(\sigma_i - \alpha_i)^2 \tag{5.54}$$

with $j^1(k) := \max\{1, 2j(k) - 2\}$.

Then the estimate

$$\Delta_{i,j} \leq \Delta_{1,0} \left(1 + \alpha \left(2j + \sum_{k=1}^{i-1} j^1(k)\right) \Delta_{1,0}\right)^{-1} \tag{5.55}$$

is true for all $i \in \mathbb{N}, 0 \leq j < j(i)$ if $\Delta_{1,0} > 0$ or $x^0 \notin \mathcal{M}_{opt}$.

Proof: Let us denote

$$\overline{z}^{i,j} := \arg \min_{z \in \mathcal{M}_{opt} \cap K_\tau(x_c)} \left\|\overline{x}^{i,j} - z\right\|_2$$

for each i and $0 \leq j \leq j(i)$. Then $\tilde{z}^{i,j}(\lambda) := \lambda \overline{z}^{i,j} + (1 - \lambda)\overline{x}^{i,j} \in \mathcal{M}_0 \cap K_\tau(x_c)$ for all $\lambda \in [0, 1)$ and $i, 0 \leq j \leq j(i)$ and we have $F_{i,j+1}(\overline{x}^{i,j+1}) \leq F_{i,j+1}(\tilde{z}^{i,j}(\lambda))$ for $\lambda \in [0, 1)$ if $j < j(i)$. Taking

the convexity of f as well as (5.29) into account we obtain

$$
\begin{aligned}
F_{i,j+1}\left(\overline{x}^{i,j+1}\right) &= f\left(\overline{x}^{i,j+1}\right) - \mu_i \ln\left(-\max_{t\in T} g(\overline{x}^{i,j+1}, t)\right) + \frac{s_i}{2}\left\|\overline{x}^{i,j+1} - x^{i,j}\right\|_2^2 \\
&\leq F_{i,j+1}(\tilde{z}^{i,j}(\lambda)) \\
&\leq \lambda f\left(\overline{z}^{i,j}\right) + (1-\lambda)f\left(\overline{x}^{i,j}\right) - \mu_i \ln\left(-\max_{t\in T} g(\tilde{z}^{i,j}(\lambda), t)\right) \\
&\quad + \frac{s_i}{2}\left(\lambda\|\overline{z}^{i,j} - \overline{x}^{i,j})\|_2 + \alpha_i\right)^2.
\end{aligned}
\tag{5.56}
$$

Using the convexity of $\max_{t\in T} g(\cdot, t)$ and the inclusion $\overline{z}^{i,j} \in \mathcal{M}$, we have

$$
\max_{t\in T} g(\tilde{z}^{i,j}(\lambda), t) \leq (1-\lambda)\max_{t\in T} g(\overline{x}^{i,j}, t)
$$

such that we infer

$$
-\mu_i \ln\left(-\max_{t\in T} g(\tilde{z}^{i,j}(\lambda), t)\right) \leq -\mu_i \ln\left(-(1-\lambda)\max_{t\in T} g(\overline{x}^{i,j}, t)\right).
$$

for all pairs (i, j) with $1 \leq j \leq j(i)$ and $\lambda \in [0, 1)$. Applying Lemma 5.11 the inequality

$$
-\mu_i \ln\left(-\max_{t\in T} g(\tilde{z}^{i,j}(\lambda), t)\right) \leq -\mu_i \ln\left((1-\lambda)c_4\mu_i\right)
\tag{5.57}
$$

follows for all pairs (i, j) with $1 \leq j \leq j(i)$. But $\overline{x}^{i,0} = \overline{x}^{i-1,j(i-1)}$ and $j(i-1) > 0$ for each $i > 1$ so that

$$
-\mu_i \ln\left(-\max_{t\in T} g(\tilde{z}^{i,0}(\lambda), t)\right) \leq -\mu_i \ln\left((1-\lambda)c_4\mu_{i-1}\right)
$$

follows w.r.t. Lemma 5.11. The monotonic decrease of $\{\mu_i\}$ leads to (5.57) again and regarding (5.50) the estimate now holds for all i and $0 \leq j \leq j(i)$.

From the proof of Theorem 5.7 we know that $\overline{x}^{i,j+1} \in \mathcal{M}_0 \cap K_\tau(x_c)$ for all i and $0 \leq j < j(i)$. Using (5.25) we therefore conclude

$$
-\mu_i \ln\left(-\max_{t\in T} g(\overline{x}^{i,j+1}, t)\right) \geq -\mu_i(c_1 + \ln\tau).
$$

This together with (5.56), (5.57) and $\overline{z}^{i,j} \in \mathcal{M}_{opt}$ yields

$$
\begin{aligned}
\Delta_{i,j+1} &= f(\overline{x}^{i,j+1}) - f^* \\
&\leq \lambda(f(\overline{z}^{i,j}) - f^*) + (1-\lambda)(f(\overline{x}^{i,j}) - f^*) \\
&\quad - \mu_i \ln\left(-\max_{t\in T} g(\tilde{z}^{i,j}(\lambda), t)\right) + \mu_i \ln\left(-\max_{t\in T} g(\overline{x}^{i,j+1}, t)\right) \\
&\quad + \frac{s_i}{2}\left(\lambda\|\overline{z}^{i,j} - \overline{x}^{i,j})\|_2 + \alpha_i\right)^2 - \frac{s_i}{2}\left\|\overline{x}^{i,j+1} - x^{i,j}\right\|_2^2 \\
&\leq (1-\lambda)\Delta_{i,j} - \mu_i \ln\left((1-\lambda)c_4\mu_i\right) + \mu_i(c_1 + \ln\tau) \\
&\quad + \frac{s_i}{2}\lambda^2\|\overline{z}^{i,j} - \overline{x}^{i,j}\|_2^2 + s_i\alpha_i\lambda\|\overline{z}^{i,j} - \overline{x}^{i,j}\| + \frac{s_i}{2}\alpha_i^2 - \frac{s_i}{2}\left\|\overline{x}^{i,j+1} - x^{i,j}\right\|_2^2
\end{aligned}
\tag{5.58}
$$

for all i and $0 \le j < j(i)$. In view of $\overline{z}^{i,j}, \overline{x}^{i,j} \in K_\tau(x_c)$ as well as the definition of κ_i we obtain

$$0 \le \Delta_{i,j+1} \le (1 - \lambda)\Delta_{i,j} + \kappa_i + \frac{s_i}{2}\lambda^2 \|\overline{z}^{i,j} - \overline{x}^{i,j}\|_2^2 - \frac{s_i}{2}\left\|\overline{x}^{i,j+1} - x^{i,j}\right\|_2^2, \tag{5.59}$$

if $\lambda \in [0, 7/8]$ which can be enforced by setting

$$\lambda = \lambda_{i,j} = \min\left\{ \frac{\Delta_{i,j}}{s_i\|\overline{z}^{i,j} - \overline{x}^{i,j}\|_2^2}, \frac{7}{8} \right\}. \tag{5.60}$$

If $\lambda_{i,j}$ equals $7/8$ (5.60) immediately leads to

$$s_i \left\|\overline{z}^{i,j} - \overline{x}^{i,j}\right\|_2^2 \le \frac{8}{7}\Delta_{i,j}$$

and we can infer

$$\begin{aligned}
\Delta_{i,j+1} &\le \frac{\Delta_{i,j}}{8} + \kappa_i + \frac{7}{16}\Delta_{i,j} - \frac{s_i}{2}\left\|\overline{x}^{i,j+1} - x^{i,j}\right\|_2^2 \\
&= \frac{9}{16}\Delta_{i,j} + \kappa_i - \frac{s_i}{2}\left\|\overline{x}^{i,j+1} - x^{i,j}\right\|_2^2
\end{aligned} \tag{5.61}$$

from (5.59). Due to the second part of (5.51) this allows to conclude

$$\Delta_{i,j+1} \le \Delta_{i,j} - 2\alpha\Delta_{i,j}^2 + \kappa_i - \frac{s_i}{2}\left\|\overline{x}^{i,j+1} - x^{i,j}\right\|_2^2. \tag{5.62}$$

But if $\lambda_{i,j} < 7/8$ holds we obtain

$$\Delta_{i,j+1} \le \Delta_{i,j} - \frac{\Delta_{i,j}^2}{2s_i\|\overline{z}^{i,j} - \overline{x}^{i,j}\|_2^2} + \kappa_i - \frac{s_i}{2}\left\|\overline{x}^{i,j+1} - x^{i,j}\right\|_2^2 \tag{5.63}$$

and now using the first part of (5.51) inequality (5.62) follows again. Consequently this estimate holds for all pairs (i, j) with $0 \le j < j(i)$ making it the basis of the following induction proof.

Let us assume that (5.55) holds for a fixed pair i, j with $j < j(i)$. This is obvious for the starting indices $i = 1, j = 0$. Now we distinguish three cases.

a) We first consider $0 \le j < j(i) - 1$. Then $j + 1 < j(i)$ and due to (5.21) as well as (5.36) we have

$$\frac{s_i}{2}\left\|\overline{x}^{i,j+1} - x^{i,j}\right\|_2^2 > \frac{s_i}{2}(\sigma_i - \alpha_i)^2 \tag{5.64}$$

such that (5.54) leads to

$$\kappa_i - \frac{s_i}{2}\left\|\overline{x}^{i,j+1} - x^{i,j}\right\|_2^2 < 0.$$

Consequently, with (5.62) we obtain

$$\Delta_{i,j+1} \le \Delta_{i,j} - 2\alpha\Delta_{i,j}^2.$$

Moreover, the trivial inequality

$$y - \vartheta y^2 \le \frac{y}{1 + \vartheta y} \tag{5.65}$$

is true for all $y \geq 0$ with fixed $\vartheta > 0$ and the function $\frac{y}{1+\vartheta y}$ increases monotonically for nonnegative y such that one can set $\vartheta = 2\alpha, y = \Delta_{i,j}$. Then, with regard to the induction assumption, we infer

$$\Delta_{i,j+1} < \frac{\Delta_{i,j}}{1 + 2\alpha \Delta_{i,j}}$$

$$\leq \frac{\Delta_{1,0}}{\left(1 + \alpha \left(2j + \sum_{k=1}^{i-1} j^1(k)\right) \Delta_{1,0}\right) \left(1 + 2\alpha \frac{\Delta_{1,0}}{1+\alpha\left(2j+\sum_{k=1}^{i-1} j^1(k)\right)\Delta_{1,0}}\right)}$$

$$= \frac{\Delta_{1,0}}{1 + \alpha \left(2j + 2 + \sum_{k=1}^{i-1} j^1(k)\right) \Delta_{1,0}}$$

and the induction statement holds.

b) In case $j = 0, j(i) = 1$ we have $j^1(i) = 1$ and (5.62) leads to

$$\Delta_{i+1,0} = \Delta_{i,1} \leq \Delta_{i,0} - 2\alpha \Delta_{i,0}^2 + \kappa_i.$$

Furthermore, regarding the second part of (5.51), it holds

$$\frac{\Delta_{1,0}}{1 + \alpha \sum_{k=1}^{i-1} j^1(k)\Delta_{1,0}} \leq \Delta_{1,0} \leq \frac{7}{32\alpha} < \frac{1}{4\alpha}.$$

Since the function $y - 2\alpha y^2$ increases monotonically if $y < 1/(4\alpha)$, the induction assumption allows to conclude

$$\Delta_{i+1,0} \leq \frac{\Delta_{1,0}}{1 + \alpha \sum_{k=1}^{i-1} j^1(k)\Delta_{1,0}} - 2\alpha \left(\frac{\Delta_{1,0}}{1 + \alpha \sum_{k=1}^{i-1} j^1(k)\Delta_{1,0}}\right)^2 + \kappa_i$$

such that

$$\Delta_{i+1,0} \leq \frac{\Delta_{1,0}}{1 + \alpha \sum_{k=1}^{i-1} j^1(k)\Delta_{1,0}} - \alpha \left(\frac{\Delta_{1,0}}{1 + \alpha \sum_{k=1}^{i-1} j^1(k)\Delta_{1,0}}\right)^2 \qquad (5.66)$$

follows from (5.53). Using (5.65) again - this time with $\vartheta = \alpha, y = \frac{\Delta_{1,0}}{1+\alpha\sum_{k=1}^{i-1} j^1(k)\Delta_{1,0}}$ - the combination with (5.66) and $j^1(i) = 1$ leads to

$$\Delta_{i+1,0} \leq \frac{\Delta_{1,0}}{\left(1 + \alpha \sum_{k=1}^{i-1} j^1(k)\Delta_{1,0}\right) \left(1 + \alpha \frac{\Delta_{1,0}}{1+\alpha\sum_{k=1}^{i-1} j^1(k)\Delta_{1,0}}\right)}$$

$$= \frac{\Delta_{1,0}}{1 + \alpha \sum_{k=1}^{i-1} j^1(k)\Delta_{1,0} + \alpha\Delta_{1,0}}$$

$$= \frac{\Delta_{1,0}}{1 + \alpha \sum_{k=1}^{i} j^1(k)\Delta_{1,0}}$$

such that the induction statement holds.

c) Finally let us consider the case $j = j(i) - 1$ with $j(i) > 1$. Then we have the inequalities

$$\Delta_{i,j(i)-1} \leq \Delta_{i,j(i)-2} - 2\alpha \Delta_{i,j(i)-2}^2 + \kappa_i - \frac{s_i}{2} \left\|\overline{x}^{i,j(i)-1} - x^{i,j(i)-2}\right\|_2^2,$$

$$\Delta_{i+1,0} = \Delta_{i,j(i)} \leq \Delta_{i,j(i)-1} - 2\alpha \Delta_{i,j(i)-1}^2 + \kappa_i$$

from (5.62). Coupling these we infer

$$\Delta_{i+1,0} \leq \Delta_{i,j(i)-2} - 2\alpha\Delta_{i,j(i)-2}^2 - 2\alpha\Delta_{i,j(i)-1}^2 + 2\kappa_i - \frac{s_i}{2}\left\|\overline{x}^{i,j(i)-1} - x^{i,j(i)-2}\right\|_2^2,$$

so that together with (5.54), (5.64) and $\alpha\Delta_{i,j(i)-1}^2 \geq 0$ the estimate

$$\Delta_{i,j(i)} < \Delta_{i,j(i)-2} - 2\alpha\Delta_{i,j(i)-2}^2$$

holds. Thus we can conclude in analogy to case a) that

$$\Delta_{i+1,0} < \frac{\Delta_{1,0}}{1 + \alpha\left(2j(i) - 2 + \sum_{k=1}^{i-1} j^1(k)\right)\Delta_{1,0}} = \frac{\Delta_{1,0}}{1 + \alpha\sum_{k=1}^{i} j^1(k)\Delta_{1,0}}$$

since $2j(i) - 2 > 1$ and the induction is complete. $\qquad\square$

5.4 Linear convergence

Theorem 5.12 establishes the important estimate (5.55) which holds for any problem keeping on to Assumption 5.1. If we consider problems adhering to tighter assumptions it is possible to prove linear convergence for the iterates as well as the values of the objective function. The condition to use in our case is the following growth condition

$$\inf_{x\in\hat{\mathcal{M}}} \frac{f(x) - f^*}{\rho^2(x, \mathcal{M}_{opt})} \geq d > 0 \tag{5.67}$$

with

$$\hat{\mathcal{M}} := (\mathcal{M}_0 \cap K_\tau(x_c)) \setminus \mathcal{M}_{opt}, \quad \rho(x, \mathcal{M}_{opt}) := \min_{z\in\mathcal{M}_{opt}\cap K_\tau(x_c)} \|x - z\|_2.$$

This growth condition generalizes that of Rockafellar [47] which occurs in the context of proving linear convergence of the iterates of an inexact proximal point method.

If $\frac{d}{s_i} < \frac{7}{8}$ is true, (5.60) admits $\lambda_{i,j} = 7/8$ as well as $\lambda_{i,j} < 7/8$ for all j with $0 \leq j < j(i)$. If $\lambda_{i,j} = 7/8$ the inequality (5.61) is true, while in the case $\lambda_{i,j} < 7/8$ the estimate (5.63) follows. Then we have

$$\lambda_{i,j} = \frac{\Delta_{i,j}}{s_i\|\overline{z}^{i,j} - \overline{x}^{i,j}\|_2^2}$$

and one can conclude

$$\Delta_{i,j+1} \leq \left(1 - \frac{d}{2\overline{s}}\right)\Delta_{i,j} + \kappa_i - \frac{s_i}{2}\left\|\overline{x}^{i,j+1} - x^{i,j}\right\|_2^2$$

with regard to (5.67) and $s_i \leq \overline{s}$.

But if $\frac{d}{s_i} \geq \frac{7}{8}$ is true (5.60) only admits $\lambda_{i,j} = 7/8$ for all j with $0 \leq j < j(i)$ which immediately leads to (5.61).

Thus $\Delta_{i,j}, \Delta_{i,j+1}$ always fulfill

$$0 \leq \Delta_{i,j+1} \leq (1 - d_1)\Delta_{i,j} + \kappa_i - \frac{s_i}{2}\left\|\overline{x}^{i,j+1} - x^{i,j}\right\|_2^2, \tag{5.68}$$

if $j < j(i)$ where $d_1 = \min\left\{\frac{7}{16}, \frac{d}{2s}\right\}$.

Using these preliminary remarks the linear convergence of the sequence $\{\Delta_{i,j}\}$ can be established under the given growth condition.

Theorem 5.13 *Let the assumptions of Theorem 5.7 be satisfied. Moreover, let* (5.50) *as well as* (5.67) *be satisfied. Additionally assume that*

$$\kappa_i \leq \frac{s_i(1-d_1)}{2(2-d_1)}(\sigma_i - \alpha_i)^2, \quad \kappa_i < \frac{d_1}{2}\Delta_{1,0}q^{p_i} \tag{5.69}$$

with $p_i = \sum\limits_{k=1}^{i-1} j(k), q \in \left[1 - \frac{d_1}{2}, 1\right)$. *Then*

$$\Delta_{i,j} \leq \Delta_{1,0}q^{p_i+j} \tag{5.70}$$

holds.

Proof: The proof is by induction again. The proposition is obviously true for $i = 1$, $j = 0$.

Thus we suppose that a fixed i and $j < j(i)$ are given such that

$$\Delta_{k,j'} \leq \Delta_{1,0}q^{p_k+j'} \tag{5.71}$$

holds for all $k < i, 0 \leq j' \leq j(k)$ and $k = i, 0 \leq j' \leq j$. The condition $j < j(i)$ is not a restriction because in case $j = j(i)$ we consider the equivalent pair $i + 1, j = 0 < j(i+1)$.

We distinguish three cases.

a) Suppose $j + 1 < j(i)$.

Combining $\left\|\overline{x}^{i,j+1} - x^{i,j}\right\|_2 > \sigma_i - \alpha_i$, the first inequality in (5.69) and (5.68) we obtain

$$\Delta_{i,j+1} < (1 - d_1)\Delta_{i,j}$$

and along (5.71) this implies

$$\Delta_{i,j+1} < \Delta_{1,0}q^{p_i+j+1}.$$

b) Suppose $j > 0, j + 1 = j(i)$.

Then the inequalities

$$\Delta_{i,j+1} \leq \Delta_{i,j}(1 - d_1) + \kappa_i - \frac{s_i}{2}\left\|\overline{x}^{i,j+1} - x^{i,j}\right\|_2^2,$$

$$\Delta_{i,j} \leq \Delta_{i,j-1}(1 - d_1) + \kappa_i - \frac{s_i}{2}\left\|\overline{x}^{i,j} - x^{i,j-1}\right\|_2^2$$

and $\left\|\overline{x}^{i,j} - x^{i,j-1}\right\|_2 > \sigma_i - \alpha_i$ hold. Substituting the second in the first gives

$$\Delta_{i,j+1} < \Delta_{i,j-1}(1 - d_1)^2 + \kappa_i(2 - d_1) - (1 - d_1)\frac{s_i}{2}(\sigma_i - \alpha_i)^2,$$

leading to

$$\Delta_{i,j+1} < (1 - d_1)^2\Delta_{i,j-1}$$

if we consider the first inequality in (5.69). Hence,

$$\Delta_{i,j+1} < \Delta_{1,0}q^{p_i+j+1}$$

and the induction statement holds.

c) Suppose $j = 0, j(i) = 1$.

Taking (5.71) and the second inequality in (5.69) into account we obtain

$$\Delta_{i,1} \le \Delta_{1,0} q^{p_i}(1 - d_1) + \frac{d_1}{2}\Delta_{1,0}q^{p_i} < \Delta_{1,0}q^{p_i+1}$$

from (5.68) and the proof is complete. $\qquad\qquad\square$

If the considered problem fulfills the growth condition (5.67) we can additionally prove the linear convergence of the sequence $\{\overline{x}^{i,j}\}$ to an element of $\mathcal{M}_{opt}$.

For this purpose we define $\overline{j}(i) = 16\tau^2(\sigma_i - \alpha_i)^{-2} + 1$ and

$$\zeta_i = \sum_{k=i}^{\infty} \left(\sqrt{\gamma_k} + \alpha_k + 4\tau\mu_k\right), \tag{5.72}$$

for all i where γ_k is given by $\gamma_k = \frac{2\mu_k}{s_k}(2|\ln\mu_k| + \ln\tau)$ as in the proof of Theorem 5.7.

Theorem 5.14 *Let the assumptions of Theorem 5.13 be satisfied. Moreover, assume that*

$$\frac{1}{4\tau}\gamma_i + \alpha_i < \frac{1}{8\tau}(\sigma_i - \alpha_i)^2, \tag{5.73}$$

$$\zeta_i \le \left(\frac{\Delta_{1,0}}{d}\right)^{\frac{1}{2}} q^{\frac{1}{2}(p_i+\overline{j}(i))} \tag{5.74}$$

hold for each i. Then the inequality

$$\left\|\overline{x}^{i,j} - x^{**}\right\|_2 \le 3\left(\frac{\Delta_{1,0}}{d}\right)^{\frac{1}{2}} q^{\frac{1}{2}(p_i+j)} \tag{5.75}$$

is true for each i and $0 \le j < j(i)$, where $x^{**} := \lim_{i\to\infty} x^{i,0}$ is an optimal solution of (2.10).

Proof: Let $\overline{z}^{i,j} = \arg\min_{z\in\mathcal{M}_{opt}\cap K_\tau(x_c)} \left\|\overline{x}^{i,j} - z\right\|_2$ be given as in the proof of Theorem 5.12.

The inequality

$$\left\|\overline{x}^{i,j} - \overline{z}^{i,j}\right\|_2 \le \left(\frac{\Delta_{i,j}}{d}\right)^{\frac{1}{2}} \tag{5.76}$$

is obviously true if $\overline{x}^{i,j} \in \mathcal{M}_{opt}$, otherwise (5.76) holds due to (5.67).

In the sequel let i_0, j_0 be fixed with $0 \le j_0 < j(i_0)$. From the proof of Theorem 5.7 inequality (5.37) is known, i.e.

$$\left\|\overline{x}^{i,j+1} - z^i\right\|_2^2 - \left\|x^{i,j} - z^i\right\|_2^2 \le -(\sigma_i - \alpha_i)^2 + \gamma_i$$

holds for all i, $0 \le j < j(i)-1$ with $z^i = \tilde{x}+(1-\mu_i)(x^*-\tilde{x})$. If we use $\tilde{z}^i = \tilde{x}+(1-\mu_i)(\overline{z}^{i_0,j_0}-\tilde{x})$ instead of z^i we can conclude

$$\left\|\overline{x}^{i,j+1} - \tilde{z}^i\right\|_2^2 - \left\|x^{i,j} - \tilde{z}^i\right\|_2^2 \le -(\sigma_i - \alpha_i)^2 + \gamma_i$$

analogously for all i and $0 \le j < j(i) - 1$. Furthermore it yields (cf. (5.39))

$$\left\|\overline{x}^{i,j+1} - \tilde{z}^i\right\|_2 - \left\|\overline{x}^{i,j} - \tilde{z}^i\right\|_2 < \frac{1}{4\tau}\left(-(\sigma_i - \alpha_i)^2 + \gamma_i\right) + \alpha_i < 0 \tag{5.77}$$

and (cf. (5.40))

$$\left\|\overline{x}^{i,j(i)} - \tilde{z}^i\right\|_2 - \left\|\overline{x}^{i,j(i)-1} - \tilde{z}^i\right\|_2 \le \sqrt{\gamma_i} + \alpha_i \tag{5.78}$$

for all i and $0 \le j < j(i) - 1$. Summing up these inequalities we obtain

$$\left\|\overline{x}^{i,j(i)} - \tilde{z}^i\right\|_2 - \left\|\overline{x}^{i,0} - \tilde{z}^i\right\|_2 \le \sqrt{\gamma_i} + \alpha_i,$$

and together with $\left\|\tilde{z}^i - \overline{z}^{i_0,j_0}\right\|_2 \le 2\mu_i\tau$ and $\overline{x}^{i+1,0} = \overline{x}^{i,j(i)}$ the estimate

$$\left\|\overline{x}^{i+1,0} - \overline{z}^{i_0,j_0}\right\|_2 - \left\|\overline{x}^{i,0} - \overline{z}^{i_0,j_0}\right\|_2 \le \sqrt{\gamma_i} + \alpha_i + 4\mu_i\tau$$

follows. Summing these inequalities for $i = i_0 + 1, \ldots, i' - 1$ with $i' > i_0 + 1$ we get

$$\left\|\overline{x}^{i',0} - \overline{z}^{i_0,j_0}\right\|_2 \le \left\|\overline{x}^{i_0+1,0} - \overline{z}^{i_0,j_0}\right\|_2 + \sum_{i=i_0+1}^{i'-1} \left(\sqrt{\gamma_i} + \alpha_i + 4\mu_i\tau\right).$$

In combination with (5.77) and (5.78) this leads to

$$\left\|\overline{x}^{i',0} - \overline{z}^{i_0,j_0}\right\|_2 \le \left\|\overline{x}^{i_0,j_0} - \overline{z}^{i_0,j_0}\right\|_2 + \sum_{i=i_0}^{i'-1} \left(\sqrt{\gamma_i} + \alpha_i + 4\mu_i\tau\right).$$

Moreover, $\lim_{i\to\infty} x^{i,0} = \lim_{i\to\infty} \overline{x}^{i,0}$ follows from (5.29) and $\lim_{i\to\infty} \alpha_i = 0$ is enforced by (5.20). Thus, taking the limit $i' \to \infty$ allows to conclude

$$\left\|x^{**} - \overline{z}^{i_0,j_0}\right\|_2 \le \left\|\overline{x}^{i_0,j_0} - \overline{z}^{i_0,j_0}\right\|_2 + \sum_{i=i_0}^{\infty} \left(\sqrt{\gamma_i} + \alpha_i + 4\mu_i\tau\right) = \left\|\overline{x}^{i_0,j_0} - \overline{z}^{i_0,j_0}\right\|_2 + \zeta_{i_0}.$$

Yielding

$$\left\|\overline{x}^{i_0,j_0} - x^{**}\right\|_2 \le \left\|\overline{x}^{i_0,s_0} - \overline{v}^{i_0,j_0}\right\|_2 + \left\|\overline{z}^{i_0,j_0} - x^{**}\right\|_2 \le 2\left\|\overline{x}^{i_0,s_0} - \overline{z}^{i_0,j_0}\right\|_2 + \zeta_{i_0}$$

and we obtain

$$\left\|\overline{x}^{i_0,j_0} - x^{**}\right\|_2 \le 2\left(\frac{\Delta_{i_0,j_0}}{d}\right)^{\frac{1}{2}} + \zeta_{i_0} \tag{5.79}$$

in combination with (5.76). We remember that estimate (5.46) gives

$$j(i_0) < 2\tau \left(\frac{1}{4\tau}((\sigma_{i_0} - \alpha_{i_0})^2 - \gamma_{i_0}) - \alpha_{i_0}\right)^{-1} + 1$$

so that $j(i_0) < \overline{j}(i_0)$ follows in view of (5.73). Then relation (5.74) gives

$$\zeta_{i_0} \le \left(\frac{\Delta_{1,0}}{d}\right)^{\frac{1}{2}} q^{\frac{1}{2}(p_i+j(i_0))}$$

and along with (5.70) and (5.79) we can deduce

$$\left\|\overline{x}^{i_0,j_0} - x^{**}\right\|_2 \le 3\left(\frac{\Delta_{1,0}}{d}\right)^{\frac{1}{2}} q^{\frac{1}{2}(p_i+j_0)}.$$

Because i_0 and j_0 were chosen arbitrarily the proposition is proven. $\qquad\square$

Corollary 5.15 *Let the assumptions of Theorem 5.14 be satisfied. Then*

$$\left\| x^{i,j} - x^{**} \right\|_2 \leq 4 \left(\frac{\Delta_{1,0}}{d} \right)^{\frac{1}{2}} q^{\frac{1}{2}(p_i + j)}$$

holds for all i and $1 \leq j < j(i)$, where $x^{**} = \lim_{i \to \infty} x^{i,0}$ is an optimal solution of (2.10).

Proof: Regarding (5.29) and (5.75) we see that

$$\left\| x^{i,j} - x^{**} \right\|_2 \leq 3 \left(\frac{\Delta_{1,0}}{d} \right)^{\frac{1}{2}} q^{\frac{1}{2}(p_i + j)} + \alpha_i$$

holds for all i and $1 \leq j < j(i)$. Moreover, using the definition of ζ_i, (5.74) and $0 < q < 1$, we infer

$$\alpha_i \leq \left(\frac{\Delta_{1,0}}{d} \right)^{\frac{1}{2}} q^{\frac{1}{2}(p_i + j)}$$

for all i and $0 \leq j \leq j(i)$. Combining both estimates our proposition follows. $\square$

5.5   Extension to general convex problems

The regularized method presented can be easily extended to problems of the more general form (1.1) under Assumption 4.11 - but again without the compactness postulate on the solution set. The same generalizations used in Algorithm 4.12 can be integrated into Algorithm 5.2 and the results of the Sections 5.1 and 5.2 remain true with analogous changes to those of Section 4.3.

In order to extend the results of the Sections 5.3 and 5.4 the basic result of Lemma 5.11 must be transferred. And in the first part of that proof a modification is required which cannot be described by the facts stated in Section 4.3. In particular, we cannot conclude $0 \in \partial F_{i,j}(\overline{x}^{i,j})$ if linear equality constraints occur but we deduce $0 \in \partial_x L_{i,j}(\overline{x}^{i,j}, y^{i,j})$ with Lagrange function $L_{i,j}(x, y) := F_{i,j}(x) + y^T(Ax - b)$ and a certain $y^{i,j} \in \mathbb{R}^m$. Thus we have

$$0 \in \partial_x L_{i,j}(\overline{x}^{i,j}, y^{i,j}) = \partial F_{i,j}(\overline{x}^{i,j}) + A^T y.$$

Using this we can analogously proceed as in the proof of Lemma 5.11 in order to obtain a result like in Lemma 5.11. Now, the further results of the Sections 5.3 and 5.4 can be moved to the general situation of problems of type (1.1) with regard to the remarks in Section 4.3.

Chapter 6

Numerical analysis

In this chapter we discuss several numerical difficulties which occur by the practical application of the Algorithms 4.2 and 5.2 or their extensions. We start with the analysis of the inner loops of the algorithms in the first section while in the second section the determination of a starting point is the point of interest. We choose this order because the inner loop has often to be done while for some problems it can be simple to present a feasible starting point.

Let us remark that we mainly consider the problem (2.10) in detail and therefore we suppose that Assumption 5.1 is fulfilled. Nevertheless the extensions to general problems of type (1.1) are always stated.

6.1 Numerical aspects of the inner loops

The first question which raises in the loops in k of the Algorithms 4.2 and 5.2 is how can we determine the positive radius $r_{i,k}$ resp. $r_{i,j,k}$ such that the box $S^{i,k}$ or $S^{i,j,k}$ is completely contained in $\mathcal{M}_0$. The simplest way to find such a radius is a trial-and-error strategy, whereby only the edges of the considered box have to be checked for their feasibility. In particular this fact requires that we can decide whether a given point x fulfills $g(x,t) < 0$ for all $t \in T$. Such a decision procedure can be very costly, especially if the exact evaluation of the constraint values is not possible. Therefore we offer another method in the following lemma. In order to formulate this let L_S^x denote a constant for a given nonempty set $S \subset \mathbb{R}^n$ with

$$\sup_{z \in S} \sup_{t \in T} \sup_{v \in \partial g(z,t)} \|v\|_1 \leq L_S^x \tag{6.1}$$

and the additional property that $S' \subset S$ implies $L_{S'}^x \leq L_S^x$. Furthermore, let us define

$$B_r(V) := \left\{ z \in \mathbb{R}^n : \min_{v \in V} \|z - v\|_\infty \leq r \right\}$$

for $r > 0$ and nonempty compact sets $V \subset \mathbb{R}^n$.

Lemma 6.1 *Let $\hat{x} \in \mathcal{M}_0$ and $\hat{r} > 0$ be given. Moreover, let $h \geq 0$ be given such that*

$$-\max_{t \in T_h} g(\hat{x}, t) - L_{\{\hat{x}\}}^t h > 0$$

holds with $L^t_{\{\hat{x}\}}$ fulfilling (4.1), i.e. $|g(\hat{x}, t_1) - g(\hat{x}, t_2)| \le L^t_{\{\hat{x}\}} \|t_1 - t_2\|_2$ for all $t_1, t_2 \in T$. Then the inclusion $B_r(\{\hat{x}\}) \subset \mathcal{M}_0$ is valid if

$$0 < r < \min \left\{ \hat{r}, \frac{-\max_{t \in T_h} g(\hat{x}, t) - L^t_{\{\hat{x}\}} h}{L^x_{B_{\hat{r}}(\{\hat{x}\})}} \right\}. \tag{6.2}$$

Proof: Let $z \notin \mathcal{M}_0$ be given. Then one has to show $\|z - \hat{x}\|_\infty > r$. If $\|z - \hat{x}\|_\infty \ge \hat{r}$ this follows immediately. Thus in the sequel we assume that $\|z - \hat{x}\|_\infty < \hat{r}$ holds. Let $t^* \in T(z)$ be given, i.e. $g(z, t^*) = \max_{t \in T} g(z, t) \ge 0$. Then we conclude

$$0 > \max_{t \in T_h} g(\hat{x}, t) + L^t_{\{\hat{x}\}} h \ge \max_{t \in T} g(\hat{x}, t) \ge g(\hat{x}, t^*) - g(z, t^*) \ge v^T(\hat{x} - z)$$

with $v \in \partial g(z, t^*)$. Due to $\|z - \hat{x}\|_\infty < \hat{r}$ the estimate

$$-\max_{t \in T_h} g(\hat{x}, t) - L^t_{\{\hat{x}\}} h \le |v^T(\hat{x} - z)| \le L^x_{B_{\hat{r}}(\{\hat{x}\})} \|\hat{x} - z\|_\infty$$

follows. Hence, we obtain $\|z - \hat{x}\|_\infty > r$ and the proof is complete. $\qquad\square$

Remark 6.2 In case of more than one inequality constraint (i.e. $l > 1$ in (1.1)) or in case of occurring linear equality constraints one has to replace $\mathcal{M}_0$ by $\{x \in \mathbb{R}^n : g_\nu(x, t) < 0 \, (t \in T^\nu)\}$ for each inequality constraint in the proposition of the lemma. In this way a feasible radius for each inequality constraint can be separately determined by Lemma 6.1. Then the smallest value of these radii can be used for fixing the box. $\qquad\square$

Lemma 6.1 determines the required boxes in our algorithms if the constants L^t_S and L^x_S are computable. Of course we cannot present a general way for computing these values but they are stated explicitly for each numerical example in the following chapters.

A corollary of Lemma 6.1 establishes admissible values for r^*_i in Remark 4.7 which could be used as lower bounds for all radii $r_{i,k}$ in step i.

Corollary 6.3 *Let $\tau \in \mathbb{R}$, $\mu > 0$ and $\hat{r} > 0$ be given. Moreover, define*

$$N_* := \left\{ x \in \mathcal{M}_0 : f(x) - \mu \ln \left(-\max_{t \in T} g(x, t) \right) \le \tau \right\}$$

and let f_{low} be a lower bound of f on N_*. Then the inclusion $B_r(N_*) \subset \mathcal{M}_0$ is true for all r with

$$r < \min \left\{ \hat{r}, \frac{e^{\frac{1}{\mu}(f_{low} - \tau)}}{L^x_{B_{\hat{r}}(N_*)}} \right\} \tag{6.3}$$

if $N_* \ne \emptyset$.

Proof: Let $x \in N_*$ be given. Then a short calculation shows that

$$-\max_{t \in T} g(x, t) \ge e^{\frac{1}{\mu}(f(x) - \tau)} \ge e^{\frac{1}{\mu}(f_{low} - \tau)}.$$

Using this our proposition follows with Lemma 6.1 (setting $h = 0$). $\qquad\square$

After the determination the boxes we have to select values for $h_{i,k}$ resp. $h_{i,j,k}$. Normally they influence directly the costs of the maximization processes such that we want to choose them as large as possible. Upper bounds for $h_{i,k}$ resp. $h_{i,j,k}$ are given by (4.14) resp. (5.11). But in order to use these upper bounds the constant C_S fulfilling part (9) of the Assumptions 4.1 or 5.1 is needed.

Lemma 6.4 *Let the assumptions of Lemma 6.1 be fulfilled. Furthermore, let $r > 0$ be given such that* (6.2) *is valid. Then*

$$C_S := \frac{1}{-\max_{t \in T_h} g(\hat{x}, t) - L^t_{\{\hat{x}\}} h - L^x_S r} \tag{6.4}$$

fulfills (4.2) *with* $S := B_r(\{\hat{x}\})$.

Proof: From Lemma 6.1 it follows that $S \subset \mathcal{M}_0$. Let $x \in S$, $t^* \in T(x)$ and $v(x, t^*) \in \partial g(x, t^*)$ be arbitrarily given. Then we infer with (4.5)

$$-\max_{t \in T} g(x, t) = -g(x, t^*)$$
$$\geq -g(\hat{x}, t^*) + v(x, t^*)^T (\hat{x} - x)$$
$$\geq -\max_{t \in T} g(\hat{x}, t) - L^x_S r$$
$$\geq -\max_{t \in T_h} g(\hat{x}, t) - L^t_{\{\hat{x}\}} h - L^x_S r.$$

Moreover, using (6.2), we deduce

$$-\max_{t \in T_h} g(\hat{x}, t) - L^t_{\{\hat{x}\}} h - L^x_S r > 0$$

so that we have

$$\left| \frac{1}{\max_{t \in T} g(x, t)} \right| = \frac{1}{-\max_{t \in T} g(x, t)}$$
$$\leq \frac{1}{-\max_{t \in T_h} g(\hat{x}, t) - L^t_{\{\hat{x}\}} h - L^x_S r} = C_S.$$

Consequently (4.2) holds since $x \in S$ was chosen arbitrarily. □

Remark 6.5 The monotonicity property of C_S is automatically given if L^x_S possess such a property (as demanded above) and one computes C_S by (6.4). □

Remark 6.6 In case of more than one constraint one can separately compute the constants $C_{\nu,S}$ for each constraint analogous to Lemma 6.4. □

With constant C_S the values $h_{i,k}$ or $h_{i,j,k}$ can be determined as large as possible by (4.14) resp. (5.11). Consequently, regarding the statements after the algorithms, the values $\beta_{i,k}$ and $\beta_{i,j,k}$ are already fixed and the minimization problems (4.3) resp. (5.3) can be solved with the bundle method stated in Chapter 3.

Finally, let us have a closer look at the inexact maximization of g. Such a maximization procedure can be very costly depending on the grid width and has to be done for each evaluation of $\tilde{f}_{i,k}$ and $\tilde{F}_{i,j,k}$. Thus we want to look for an acceleration of this procedure. For that purpose a typical situation is considered in the following lemma.

Lemma 6.7 *Let $\tilde{x} \in \mathcal{M}_0$, $r > 0$, $h > 0$, $T_h \subset T$ and $S := \{z \in \mathbb{R}^n : \|z - \tilde{x}\|_\infty \leq r\} \subset \mathcal{M}_0$ be given. Moreover, assume that $\max_{t \in T_{\tilde{h}}} g(\tilde{x}, t)$ is known for a certain $\tilde{h} \geq 0$. Then*

$$\bigcup_{x \in S} T_h(x) \subset \tilde{T}_h^S := \left\{ \hat{t} \in T_h : g(\tilde{x}, \hat{t}) \geq \max_{t \in T_{\tilde{h}}} g(\tilde{x}, t) - (L_{\{\tilde{x}\}}^x + L_S^x)r - L_S^t h \right\} \qquad (6.5)$$

holds.

Proof: Let $\tilde{t} \in T_{\tilde{h}}(\tilde{x})$ and $\tilde{v} \in \partial g(\tilde{x}, \tilde{t})$ be given. Then we have for all $z \in S$

$$\max_{t \in T} g(z, t) \geq g(z, \tilde{t}) \geq g(\tilde{x}, \tilde{t}) + \tilde{v}^T(z - \tilde{x}) \geq \max_{t \in T_{\tilde{h}}} g(\tilde{x}, t) - L_{\{\tilde{x}\}}^x r.$$

This combined with (4.5) leads to

$$\max_{t \in T_h} g(z, t) \geq \max_{t \in T} g(z, t) - L_S^t h \geq \max_{t \in T_{\tilde{h}}} g(\tilde{x}, t) - L_{\{\tilde{x}\}}^x r - L_S^t h \qquad (6.6)$$

for all $z \in S$.

Additionally, $g(\tilde{x}, t) \geq g(z, t) + v(z, t)^T(\tilde{x} - z)$ for all $z \in S$, $t \in T$ with $v(z, t) \in \partial g(z, t)$. Thus one infers

$$g(z, t) \leq g(\tilde{x}, t) + L_S^x r \qquad (6.7)$$

for all $z \in S$, $t \in T$. Combining (6.6) and (6.7), for all $\hat{t} \in T_h(z)$, $z \in S$ the inequality

$$g(\tilde{x}, \hat{t}) \geq \max_{t \in T_{\tilde{h}}} g(\tilde{x}, t) - (L_{\{\tilde{x}\}}^x + L_S^x)r - L_S^t h$$

follows and the proof is complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Remark 6.8 In the same way one can determine a (mostly infinite) subset $\tilde{T}^S \subset T$ with

$$\bigcup_{z \in S} T(z) \subset \tilde{T}^S. \qquad (6.8)$$

Then, by investigating the use of L_S^t in the analysis of the chapters before, we observe that it is only used to estimate the error of the inexact maximization. Therefore the assumptions on this constant can be weaken. Namely, since all $t \in T \setminus \tilde{T}^S$ cannot be maxima points of g on S, it suffices to demand that (4.1) holds for all $t_1, t_2 \in \tilde{T}^S$ and all $x \in S$. Thus in fact one considers the constraint $\max_{t \in \tilde{T}^S} g(x, t) \leq 0$ on S. In many cases this will lead to a smaller value of L_S^t.

Furthermore, the results of this section remain true with

$$L_S^x \geq \sup_{z \in S} \sup_{t \in \tilde{T}^S} \sup_{v \in \partial g(z, t)} \|v\|_1$$

which especially makes larger radii values possible.

However, the described changes are only applicable after the determination of $\tilde{T}^S \subset T$. And this requires previously computed constants L_S^x, L_S^t fulfilling the whole conditions. Nevertheless, if the determination of a subset $\tilde{T}^S \subset T$ is successful one could use the new (possibly smaller)

constants to repeat the deletion process with the constants. Particularly one can use $\tilde{T}^S$ as basis for determining $\tilde{T}_h^S$. □

Remark 6.9 In the case of considering general problems of type (1.1) we can use a deletion rule as given in the previous lemma for each inequality constraint. But then we have to replace $\mathcal{M}_0$ by the set $\{z \in \mathbb{R}^n : \max_{t \in T^\nu} g_\nu(z) < 0\}$ in order to determine a subset of T^ν. □

6.2 Feasible starting points

In this section we discuss the determination of a feasible starting point for the Algorithms 4.2 and 5.2 or their extensions. For some examples such a point can be easily given. If this is not the case we first consider semi-infinite problems of type (2.10)

$$\text{minimize} \quad f(x) \quad \text{s.t.} \quad x \in \mathbb{R}^n, \quad \max_{t \in T} g(x, t) \le 0$$

under Assumption 5.1. Then the simplest way to find a feasible starting point is to consider the problem

$$\text{minimize} \quad \max_{t \in T} g(x, t) \quad \text{s.t.} \quad x \in \mathbb{R}^n$$

which can be formulated as convex semi-infinite problem as follows

$$\text{minimize} \quad c \quad \text{s.t.} \quad (x, c) \in \mathbb{R}^n \times \mathbb{R}, \quad g(x, t) - c \le 0 \, (t \in T). \tag{6.9}$$

Consequently we can solve it with Algorithm 5.2 if Assumption 5.1 holds for this problem. But it turns out that the solvability does not have to hold in each case. Caused by this fact we change the objective function into $(c - c_0)^2$ and consider

$$\text{minimize} \quad (c - c_0)^2 \quad \text{s.t.} \quad (x, c) \in \mathbb{R}^n \times \mathbb{R}, \quad g(x, t) - c \le 0 \, (t \in T) \tag{6.10}$$

with any fixed $c_0 \in \mathbb{R}$. Then the solvability is enforced by the quadratic objective function and the continuity of g. The further assumptions demanded by Assumption 5.1 can be simply transferred from the properties of the given problem (2.10). Thus Algorithm 5.2 is possible to use for solving this problem, whereby we remark that a feasible starting point for problem (6.10) can be given by fixing any $x^0 \in \mathbb{R}^n$ and choosing $c > \max_{t \in T} g(x^0, t)$. Furthermore, in case of $c_0 < 0$ and $\mathcal{M}_0 \ne \emptyset$, a solution of (6.10) must be a point of $\mathcal{M}_0$ because the optimal value of c has to be as close as possible to c_0. Of course, if $\mathcal{M}_0$ is nonempty we do not have to solve (6.10) exactly because we can stop the used method when the current iterate is located in $\mathcal{M}_0$.

Now let us consider the general case where we have given problems of type (1.1)

$$\begin{aligned}
\text{minimize} \quad & f(x) \\
\text{s.t.} \quad & x \in \mathbb{R}^n, \quad Ax = b, \quad A \in \mathbb{R}^{m \times n}, \quad b \in \mathbb{R}^m, \\
& g_i(x, t) \le 0 \quad \text{for all } t \in T^i \, (i = 1, \dots, l).
\end{aligned}$$

We assume that the generalization of Assumption 5.1 holds. Then we obtain analogously to (6.10) the optimization problem

$$
\begin{aligned}
\text{minimize} \quad & (c - c_0)^2 \\
\text{s.t.} \quad & (x, c) \in \mathbb{R}^n \times \mathbb{R}, \quad Ax = b, \\
& g_i(x, t) - c \leq 0 \quad \text{for all } t \in T^i \ (i = 1, \ldots, l)
\end{aligned}
\tag{6.11}
$$

which fulfills the generalization of Assumption 5.1 for each fixed $c_0 \in \mathbb{R}^n$. Consequently the extension of Algorithm 5.2 can be used to solve this problem. Thereby we can state again that in the case $c_0 < 0$ and $\mathcal{M}_0 \neq \emptyset$ each exact solution of (6.11) is located in $\mathcal{M}_0$ so that it is a feasible starting point for computing a solution of (1.1) by Algorithm 4.12 or the extension of Algorithm 5.2.

Chapter 7

Application to model examples

In the following chapters we present numerical results computed by the proposed methods. For that purpose the algorithms were implemented in the programming language C by using version 2.7.2.3 of the gcc-compiler on a Pentium III/800-computer with the operating system Suse Linux 6.2. The included linear programs are solved by the Simplex-method while the quadratic problems are solved by a finite algorithm of Fletcher [10].

Before we have a closer look at the examples some general settings are given. The sets $T_h \subset T$ are always determined as equidistant discretizations of T with step size $2h$. Furthermore, the radii of the considered boxes are always computed as $9/10$ of the maximal value allowed by Lemma 6.1. But the values of $\hat{r}$ in the formula for this maximal value have to be adapted to each example. Particularly they are adapted to each step of the chosen algorithm. Finally, the values of C_S are always computed as suggested in (6.4).

Let us finish our general statements with a remark on the application of the several convergence results stated before. Each of them says that the algorithms generate sequences which converge to an optimal solution (in case of Algorithm 5.2 or its generalization) or which have at least an accumulation point as optimal solution (in case of Algorithm 4.2 or 4.12). Moreover, in each presented convergence theorem it is required that some positive sequences converge to zero. But, caused by the fact that we cannot generate complete sequences, in practice it is impossible to check these assumptions. Nevertheless, they are the basis of the practical parameter setting in the following sense: We choose the occurring finite values of each sequence which has to converge in such a way that they fulfill a geometric decrease condition.

7.1   The unregularized case

We start with considering two examples which can be solved with Algorithm 4.2.

Example 7.1 For fixed $n \in \mathbb{N}$ we consider the problem (cf. Example 3 in Voetmann [61])

$$
\begin{aligned}
\text{minimize} \quad & f(x) := x_n \\
\text{s.t.} \quad & g(x,t) := \left| \phi(t) - \sum_{m=0}^{n-1} x_m t^m \right| - x_n \leq 0 \quad \text{for all } t \in T := [-1, 2]
\end{aligned}
\tag{7.1}
$$

with

$$
\phi(t) := \begin{cases} t^n & \text{if } t \in [-1, 1] \\ \max\{1, t^n - P_n(t)\} & \text{if } t \in (1, 2] \end{cases}
$$

and the normalized Chebyshev polynomial (cf., e.g., Hackbusch [16])

$$
P_n(t) := \begin{cases} 2^{1-n} \cos(n \arccos(t)) & \text{if } t \in [-1, 1] \\ 2^{1-n} \cosh(n \operatorname{arcosh}(t)) & \text{if } |t| > 1 \end{cases}
$$

of degree n. That means we want to approximate ϕ on the compact interval $[-1, 2]$ by a polynomial based on the functions $1, t, \ldots, t^{n-1}$. Voetmann [61] shows that this problem is uniquely solvable with optimal solution x^* characterized by

$$
\sum_{m=0}^{n-1} x_m^* t^m = t^n - P_n(t) \quad \text{for all } t \in \mathbb{R} \quad \text{and} \quad x_n^* = 2^{1-n}.
$$

Consequently Assumption 4.1(6) is fulfilled. Furthermore, since part (3) of this assumption holds due to Theorem 5.7 in Rockafellar [45] the validity of the parts (1)-(4) for the considered problem is obvious. Then, setting $x_0 = \ldots = x_{n-1} = 0$ and x_n sufficiently large, we find an interior point of the feasible set so that the fifth part holds as well. Regarding the introductory remarks of the chapter part (7) is simply fulfilled with equidistant grids T_h with grid widths $2h$. Furthermore, the constants C_S should be computed by (6.4) which requires computable values for L_S^x for nonempty compact sets $S \subset \mathbb{R}^{n+1}$. They can be given by

$$
L_S^x = \max_{t \in T} \sum_{m=0}^{n-1} |t^m| + 1 = 2^n
$$

which is an upper bound of all possible slopes of g w.r.t. x. But since 2^n increases very fast with parameter n this constant is chosen by

$$
L_S^x = \max_{t \in \tilde{T}^S} \sum_{m=0}^{n-1} |t^m| + 1
$$

if Remark 6.8 is regarded and a subset $\tilde{T}^S \subset T$ with (6.8) is known. Further the constants L_S^t for nonempty compact boxes $S \subset \mathbb{R}^{n+1}$ have to be known in order to fulfill (8). But the computation of these constants is much more difficult than that of L_S^x. Therefore we divide the interval $[-1, 2]$ into the parts $[-1, 1]$ and $(1, 2]$ as it is done by the definition of ϕ. On $[-1, 1]$ one can use the differentiability of $t^n - \sum_{m=0}^{n-1} x_m t^m$ w.r.t. t as well as the linear structure in x to find an upper bound of

$$
\sup_{t \in [-1,1]} \sup_{x \in S} \left| n t^{n-1} - \sum_{m=1}^{n-1} m x_m t^{m-1} \right|
$$

which is used as L_S^t on $[-1, 1]$. Considering the interval $(1, 2]$ instead of $[-1, 2]$ one has to determine two constants, one for each possible constraint function. In the first case $\max\{1, t^n - P_n(t)\} = 1$ the constraint is obviously polynomial in t such that it can be treated as it is done on $[-1, 1]$. In

case $\max\{1, t^n - P_n(t)\} > 1$ one can use the well-known recurrence scheme of the Chebyshev polynomials (cf., e.g., Hackbusch [16]) so that in fact we deal with a (more complicated) polynomial again. Altogether, the larger of the both computed constants is used as L_S^t on $(1, 2]$ and then the sum of the constants of both interval parts is used as L_S^t on $[-1, 2]$. Pointing to Remark 6.8 again this constant can also be getting smaller if $\tilde{T}^S \subset T$ with (6.8) is known. Then the described procedure above must be adapted in an easy way.

Finally, part (10) of Assumption 4.1 requires the computation of a subgradient of f and $g(\cdot, t)$ in x for each t. But due to the linear structure of f and g in x this can be easily done so that Assumption 4.1 is completely fulfilled and we can use Algorithm 4.2 for solving (7.1). For that the standard parameter setting is given in Table 7.1. Additionally it must be remarked that the values

parameter	start value	decreasing factor	lower bound
μ_i	1	0.2	10^{-5}
$\varepsilon_{i,0}$	0.001	0.15	–
δ_i	10	0.15	–
q_i	0.999	–	–

Table 7.1: Example 7.1 - standard parameter

of $\varepsilon_{i,0}$ and δ_i were automatically adapted in the sense of Remark 4.10. That means, in accordance to our introductory convention, if $\varepsilon_i/r_i \geq 0.99\varepsilon_{i-1}/r_{i-1}$ was detected we halved $\varepsilon_{i,0}$ (and δ_i) and restarted the i-th step.

Furthermore, the starting vector was chosen as $x_0 = \ldots = x_{n-1} = 0$, $x_n = \lceil P_n(2) + 1 \rceil$ while, regarding Lemma 6.1 as well as the introductory remarks of the chapter, the radii were computed by

$$r_{i,k} = 0.9 \min\left\{ \hat{r}, \frac{-\max_{t \in T_h} g(x^{i,k-1}, t) - L_{\{x^{i,k-1}\}}^t h}{L_{B_{\hat{r}}(\{x^{i,k-1}\})}^x} \right\} \tag{7.2}$$

with $\hat{r} = \min\{1, 2r_{i,k-1}\}$, $h = h_{i,k-1}$ if $k > 1$ and $\hat{r} = 1$, $h = 0.003$ if $k = 1$. Thereby the improvement of the constants $L_{\{x^{i,k-1}\}}^t$ and $L_{B_{\hat{r}}(\{x^{i,k-1}\})}^x$ in the sense of Remark 6.8 was regarded. Additionally, all values $h_{i,k}$ were computed as minimum of 0.003 and the maximal value fulfilling (4.14). With these settings we obtained for $n = 1, \ldots, 9$ the results stated in Appendix A.

Let us pick out the case $n = 5$ for a detailed discussion. The starting vector was $(0, 0, 0, 0, 0, 24)$ and we obtained the iteration process presented in the Table 7.2. From this table we observe that our computed solution approximates the exact optimal solution $x^* = (0, -0.3125, 0, 1.25, 0, 0.625)$ very well.

Furthermore, Table 7.3 contains information which allow an analysis of the iteration process. But for reading this table a short explanation of the columns is needed. While the first columns should be clear the column titled "restarts" gives the number of restarts during the several steps which are caused by insufficient accuracy values ε_i. The column "$h_{\min}$" contains the minimal computed value of all $h_{i,k}$ for fixed i and the next column gives the average of all $h_{i,k}$ for the fixed i. Both columns show that these values decrease during the iteration process which is not surprising

i	μ_i	x_0^i	x_1^i	x_2^i	x_3^i	x_4^i	x_5^i
1	1.00E+00	-0.000054	-0.311187	0.001868	1.247626	-0.003320	1.061415
2	2.00E-01	-0.000261	-0.311517	0.002854	1.248081	-0.003753	0.260689
3	4.00E-02	-0.000030	-0.312379	0.000362	1.249759	-0.000482	0.102581
4	8.00E-03	-0.000035	-0.312358	0.000425	1.249717	-0.000566	0.070454
5	1.60E-03	-0.000001	-0.312497	0.000008	1.249995	-0.000011	0.064105
6	3.20E-04	-0.000001	-0.312497	0.000009	1.249994	-0.000013	0.062819
7	6.40E-05	-0.000000	-0.312500	0.000001	1.250000	-0.000001	0.062564
8	1.28E-05	-0.000000	-0.312500	0.000000	1.250000	-0.000000	0.062513

Table 7.2: Example 7.1 - iteration process for $n = 5$

since the computational accuracy is improved from step to step. Then the column "$\varnothing \frac{|\tilde{T}_h|}{|T_h|}$" contains the average ratio of the values $|\tilde{T}_{h_{i,k}}|/|T_{h_{i,k}}|$ which shows that our deletion rule, stated in Lemma 6.7, works very effective. The next column contains the average mightiness of the grids and we observe that the number of elements of these sets increases in spite of the deletion rule. Finally, the last four columns contain information about the computational effort of the method. While in the #LP-column the number of considered linear problems is stated, the #QP-column gives the number of considered quadratic problems which equals the number of inexact maximizations. The linear and quadratic problems originates from the used bundle method for solving the successive box-constrained minimizations, whereby the number of investigated boxes is stated in the #BP-column. The last column "Time" contains the total time in seconds from starting the algorithm until finishing step i. Thus the last value of this column is the total running time in seconds of the algorithm for generating the presented approximate solution.

As stated above we regarded Remark 6.8 for the computation of the constants L_S^x and L_S^t. To demonstrate the effect of the improved constants we also computed results for $n = 5$ with the same parameter values but without using the (possibly) smaller values for L_S^x and L_S^t. First of all we remark that the computed approximate solution differs only slightly from that given in Table 7.2. Thus we proceed without showing a table containing these iterates. However, the parameter values are much more interesting and they are given in Table 7.4. Now, comparing the Tables 7.3 and 7.4 we recognize that for smaller barrier parameter the radii values r_i as well as the averages $\varnothing r_{i,k}$ are larger in the case where Remark 6.8 is regarded. This is especially caused by smaller values for $L_{B_{\hat{r}}(\{x^{i,k-1}\})}^x$ which are possible since the deletion process for generating $\tilde{T}^S$ in the sense of Remark 6.8 detects that there cannot exist a maximum of $g(x, \cdot)$ near $t = 2$ for any $x \in B_{\hat{r}}(\{x^{i,k-1}\})$. Consequently $L_{B_{\hat{r}}(\{x^{i,k-1}\})}^x$ is much less than 2^n. For instance, in the last inner step of the fifth outer iteration we can previously detect

$$\tilde{T}^S = [-1.000, -0.100] \cup [0.110, 1.022]$$

so that $L_{B_{\hat{r}}(\{x^{i,k-1}\})}^x = 6.224893$ instead of $L_{B_{\hat{r}}(\{x^{i,k-1}\})}^x = 32$ is chosen.

A further consequence of the larger radii is the faster decrease of ε_i/r_i. Also directly influenced

i	μ_i	r_i	$\varnothing r_{i,k}$	$\frac{\varepsilon_i}{r_i}$	restarts	$h_{\min}$	$\varnothing h_{i,k}$	$\varnothing\frac{|\tilde{T}_h|}{|T_h|}$	$\varnothing|\tilde{T}_h|$	#LP	#QP	#BP	Time
1	1.00E+00	2.8E-02	2.0E-01	3.6E-02	0	3.00E-03	3.00E-03	1.00	501	174	974	113	0.79
2	2.00E-01	5.0E-03	1.3E-02	3.0E-02	0	3.00E-03	3.00E-03	1.00	501	102	440	61	1.16
3	4.00E-02	9.0E-04	1.9E-03	1.2E-02	1	1.67E-04	2.56E-03	0.99	583	126	550	83	1.66
4	8.00E-03	9.7E-04	1.2E-03	1.7E-03	0	9.93E-05	4.44E-04	0.67	2504	67	328	29	2.10
5	1.60E-03	2.0E-04	3.9E-04	1.3E-03	0	2.75E-05	7.87E-05	0.36	6854	45	223	17	2.55
6	3.20E-04	4.2E-05	8.6E-05	9.0E-04	0	4.29E-06	1.14E-05	0.18	23078	43	218	16	3.59
7	6.40E-05	8.7E-06	1.6E-05	6.6E-04	0	7.37E-07	1.91E-06	0.10	81934	46	244	17	5.60
8	1.28E-05	1.7E-06	3.3E-06	5.0E-04	0	1.46E-07	4.18E-07	0.10	364346	44	220	16	7.97

Table 7.3: Example 7.1 - parameter values for $n = 5$ with regard to Remark 6.8

i	μ_i	r_i	$\varnothing r_{i,k}$	$\frac{\varepsilon_i}{r_i}$	restarts	$h_{\min}$	$\varnothing h_{i,k}$	$\varnothing\frac{|\tilde{T}_h|}{|T_h|}$	$\varnothing|\tilde{T}_h|$	#LP	#QP	#BP	Time
1	1.00E+00	2.8E-02	2.0E-01	3.6E-02	0	3.00E-03	3.00E-03	1.00	501	174	974	113	0.78
2	2.00E-01	5.0E-03	1.3E-02	3.0E-02	0	3.00E-03	3.00E-03	1.00	501	102	440	61	1.17
3	4.00E-02	9.0E-04	1.9E-03	1.2E-02	1	7.77E-04	2.49E-03	1.00	602	132	570	89	1.66
4	8.00E-03	1.9E-04	4.6E-04	9.0E-03	0	1.49E-04	3.83E-04	0.63	2453	116	595	71	2.50
5	1.60E-03	3.9E-05	9.4E-05	6.5E-03	0	2.35E-05	6.05E-05	0.39	9720	124	589	68	4.00
6	3.20E-04	8.0E-06	2.0E-05	4.7E-03	0	3.64E-06	9.35E-06	0.19	31019	125	605	65	7.56
7	6.40E-05	1.7E-06	4.0E-06	3.5E-03	0	6.29E-07	1.59E-06	0.13	120482	130	634	65	13.11
8	1.28E-05	3.3E-07	7.9E-07	2.6E-03	0	1.24E-07	3.20E-07	0.13	619496	130	653	66	21.20

Table 7.4: Example 7.1 - parameter values for $n = 5$ without regard to Remark 6.8

by the radii values is the number of considered boxes in each step. Here we realize that this number is much less if the improved constants are used and this leads also to a smaller number of solved linear and quadratic problems. Moreover, since the constants L_S^t also profit from detecting $\tilde{T}^S$ we obtain larger grid constants $h_{i,k}$ (at least in the average) if the deletion process successfully works. Additionally in these cases we have a better deletion so that, combining both, there occur much smaller grids.

Altogether the use of improved constants L_S^x, L_S^t in the sense of Remark 6.8 often leads to much less effort and thus to a faster algorithm. For the considered example this effect is intensified when the dimension grows up. $\qquad\qquad\square$

Example 7.2 We consider for $n \geq 3$ the problem (cf. Example 6 in Voetmann [61])

$$\text{minimize} \quad f(x) := -\sum_{i=1}^{n} x_i$$

$$\text{s.t.} \quad g(x,t) := \rho_1(t)x_1 + \rho_2(t)x_2 + \sum_{i=3}^{n} \frac{i}{i+1}x_i^2 - 1 \leq 0 \quad \text{for all } t \in T := [0,1] \tag{7.3}$$

with

$$\rho_1(t) := 1 - \left|t - \frac{\sqrt{2}}{2}\right|, \quad \rho_2(t) := \begin{cases} 1 - \dfrac{\sqrt{2}}{2}\left|t - \dfrac{\sqrt{2}}{2}\right| & \text{if } t < \dfrac{\sqrt{2}}{2} \\[3mm] 1 - \sqrt{2}\left|t - \dfrac{\sqrt{2}}{2}\right| & \text{if } t \geq \dfrac{\sqrt{2}}{2} \end{cases}.$$

In order to decide whether the unregularized or the regularized algorithm has to be taken into account the solution set of (7.3) is first stated (which can be found by investigating the possibly restrictive constraints). We have

- for $n = 3, 4, 5$

$$\mathcal{M}_{opt} = \left\{ x \in \mathbb{R}^n : \begin{array}{l} x_1 + x_2 = 1 - \sum_{i=3}^{n} \frac{i+1}{4i}, \ x_1 + \min\left\{\sqrt{2}x_2, \frac{\sqrt{2}}{2}x_2\right\} \geq 0, \\[2mm] x_i = \frac{i+1}{2i} \ (i \geq 3) \end{array} \right\},$$

- for $6 \leq n \leq 13$

$$\mathcal{M}_{opt} = \left\{ x \in \mathbb{R}^n : x_1 = x_2 = 0, \ x_i = \frac{i+1}{i\sqrt{\sum_{k=3}^{n} \frac{k+1}{k}}} \ (i \geq 3) \right\}$$

- and for $n \geq 14$

$$\mathcal{M}_{opt} = \left\{ x \in \mathbb{R}^n : x_1 = -\frac{\sqrt{2}}{3}c_n, \ x_2 = \frac{4 + 2\sqrt{2}}{3}c_n, \ x_i = \frac{1}{\lambda + \kappa}\frac{i+1}{2i} \ (i \geq 3) \right\}$$

  with

$$\lambda = \frac{2}{3}, \quad \kappa = \frac{2 + \sqrt{2}}{3} \quad \text{and} \quad c_n = 1 - \frac{1}{(\lambda + \kappa)^2}\sum_{i=3}^{n} \frac{i+1}{4i}.$$

Thus the solution set is nonempty and compact in each case but its structure depends on the dimension. Consequently Assumption 4.1(6) is fulfilled. Furthermore the parts (1)-(4) of this assumption and, regarding $0 \in \mathcal{M}_0$, also (5) hold. The Assumptions 4.1(7) and (9) are fulfilled by the standard procedures of choosing finite grids for the first one and proceeding as suggested in (6.4) for the second one. The needed subgradients in Assumption 4.1(10) are simply calculable as derivatives such that it remains to determine the constants L_S^t enforced by Assumption 4.1(8) and L_S^x required for the computation of C_S and the radii of the boxes. These constants can be given by

$$L_S^t := \max_{x \in S} \max \left\{ \left| x_1 + \frac{\sqrt{2}}{2} x_2 \right|, \left| x_1 + \sqrt{2} x_2 \right| \right\}$$

and

$$L_S^x := \max_{t \in T} \rho_1(t) + \max_{t \in T} \rho_2(t) + 2 \max_{x \in S} \sum_{i=3}^{n} \frac{i}{i+1} |x_i| = 2 + 2 \max_{x \in S} \sum_{i=3}^{n} \frac{i}{i+1} |x_i|$$

by estimating all possible slopes. The formula of L_S^x allows the improvement of this constant in the sense of Remark 6.8, while such a consideration is not possible for L_S^t. However, Assumption 4.1 is completely fulfilled so that we can use Algorithm 4.2 for solving (7.3).

The standard parameters were chosen as in Example 7.1 (cf. Table 7.1) and the starting vector $x^0 = 0 \in \mathbb{R}^n$ was used in each case. Moreover, the radii were computed by (7.2) with the setting $\hat{r} = \min\{1, 2r_{i,k-1}\}$, $h = h_{i,k-1}$ if $k > 1$ and $\hat{r} = 1$, $h = 0.0005$ if $k = 1$. Of course, the improvement of L_S^x in the sense of Remark 6.8 was regarded by this computation. The grid constants $h_{i,k}$ were given as minimum of 0.0005 and the maximal value fulfilling (4.14). With these settings we obtained for $n = 3, \ldots, 15$ the results stated in Appendix A.

Let us now investigate the influence of the starting point in detail. For that we consider the case $n = 5$. Due to the structure of the constraint it is simple to choose different feasible starting points by choosing arbitrary negative values for the first two components of x^0 and setting $x_i^0 := 0$ for $i = 3, 4, 5$. In that case we observed the results contained in Table 7.5 and, looking at the column

start vector		restarts	$d_2(x^8, \mathcal{M}_{opt})$	$f(x^8)$	effort		
x_1^0	x_2^0				#LP	#QP	#Box
0	0	0	1.80E-04	-1.945821	104	214	40
-100	0	0	1.69E-04	-1.945821	837	6130	749
-1000	0	0	1.08E-04	-1.945822	6754	58068	6769
0	-100	0	1.50E-04	-1.945821	495	4725	672
0	-1000	0	1.17E-04	-1.945821	3716	40156	6192
-100	-100	0	8.01E-05	-1.945821	643	3305	453
-1000	-100	2	1.05E-04	-1.945821	6415	54529	6463

Table 7.5: Example 7.2 - The influence of the starting point

titled "$f(x^8)$" (note that there are 8 outer steps), it turns out that the obtained approximate solutions are of the same accuracy w.r.t. the value of the objective function (as predicted by (2.5) for the exact

logarithmic barrier method - the optimal value is -1.945833). Within these bounds differences in the final distance to the solution set are allowed and occur. However, it is not surprising that there are big differences in the computational effort caused by the chosen starting points. We can roughly state that a larger distance of x^0 to the solution set leads to a higher computational effort. In some examples it may be possible to adjust the computational effort by a larger maximal value for the radii (here we used $\hat{r} = 1$), but in the considered example such larger values will have no performance effect since the radii are bounded above by the restrictive bounds for the iterate components $3, 4$ and 5. $\square$

7.2   The regularized case

Now let us consider examples where we have unbounded solution sets. We start with a small example with two variables. It is presented in order to show that the unregularized algorithm does not have to work in the case of an unbounded solution set.

Example 7.3 We consider the problem

$$\begin{aligned} \text{minimize} \quad & f(x) := (x_1 - x_2)^2 \\ \text{s.t.} \quad & g(x, t) := x_1 \cos t + x_2 \sin t - 1 \leq 0 \quad \text{for all } t \in T := [0, 1] \end{aligned}$$

The feasible (the grayed area) and the solution set (the dotted line) of this problem are illustrated in
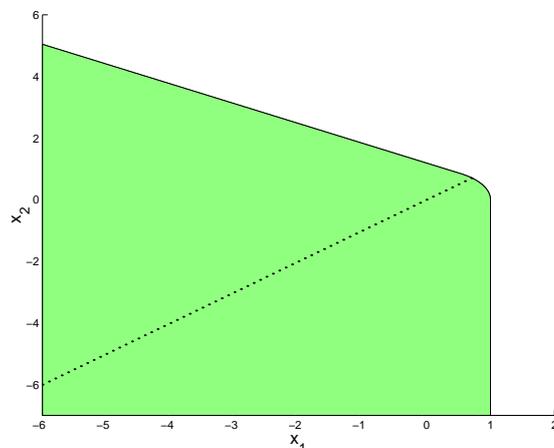
Figure 7.1: Example 7.3 - The feasible and the solution set

Figure 7.1 (both sets are shrunk to the presented clipped area). The solution set is easily given by

$$\mathcal{M}_{opt} = \left\{ x \in \mathbb{R}^2 : x_1 = x_2 \leq \frac{1}{2}\sqrt{2} \right\}.$$

Thus we deal with an unbounded feasible and an unbounded solution set so that Algorithm 4.2 cannot be used to solve the problem in the sense that we cannot expect convergence as specified in Theorem 4.9. Therefore we want to show that Assumption 5.1 is fulfilled but excepting for (8) this

is obvious if we regard our introductory remarks for (7) and (9). Part (8) is fulfilled with constants L_S^t defined by

$$\max_{x \in S} \max_{t \in [0,1]} \left| \frac{\partial g}{\partial t}(x,t) \right| = \max_{x \in S} \max_{t \in [0,1]} |-x_1 \sin t + x_2 \cos t|$$

$$\leq \max_{x \in S} \|x\|_\infty \max_{t \in [0,1]} |\cos t - \sin t|$$

$$= \max_{x \in S} \|x\|_\infty =: L_S^t.$$

Furthermore, for computing the values C_S and the radii, the constants L_S^x are needed and can be given by

$$L_S^x := \max_{t \in [0,1]} \left\| \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \right\|_1 = \sqrt{2}.$$

Thus both constants are given in a form so that they cannot be improved in the sense of Remark 6.8. But for this small example it is not essential. However, Assumption 5.1 is completely fulfilled and we can use the regularized method for solving the problem.

Before we do this it is shown that the assumption of the boundedness of the solution set is essential for a meaningful use of Algorithm 4.2. Since Assumption 4.1 is fulfilled except for the part concerning the boundedness of the solution set we started Algorithm 4.2 anyhow with the standard parameters given in Table 7.6. Furthermore, $x^0 = (-5, 0)$ while the radii were computed by (7.2)

parameter	start value	decreasing factor	lower bound
$\varepsilon_{i,0}$	0.001	0.06	–
δ_i	0.001	0.06	–
q_i	0.999	–	–

Table 7.6: Example 7.3 - standard parameter

with $\hat{r} = \min\{1, 2r_{i,k-1}\}$, $h = h_{i,k-1}$ if $k > 1$ and $\hat{r} = 1$, $h = 0.001$ if $k = 1$. The grid constants $h_{i,k}$ were given as minimum of 0.001 and the maximal value fulfilling (4.14) and, finally, C_S was computed by (6.4) as already stated above. With these values the method was started with $\mu_1 = 1$ and we obtained the iterates given in Table 7.7. From there we observe that no convergence behaviour of the iterates is cognizable which already shows that the assumption of the boundedness of the solution set is essential for a meaningful use of Algorithm 4.2. However, the values of the objective function at the iterates converge to the minimal possible value zero.

After this experiment we now use Algorithm 5.2 for solving the given problem. More precisely we intend to use all features of this method including the multi-step technique. Having in mind Theorem 5.7 we have to specify a few more constants than in the case of using the unregularized method. Nevertheless, we can first state that the standard parameters contained in Table 7.6 as well as the given computations of the radii and the grid constants were used again. Furthermore, we set $s_1 := 1$, $s_{i+1} := \max\{0.01, 0.2s_i\}$, $\tau := 12$, $x_c := (-3.3, -1.7)$, $x^* := (-2.5, 2.5)$, $\tilde{x} := (0,0)$ and $x^0 := (-5,0)$ in order to fulfill $x^* \in \mathcal{M}_{opt} \cap K_{\tau/8}(x_c)$, $\tilde{x} \in \mathcal{M}_0 \cap K_\tau(x_c)$ and

i	x_1^i	x_2^i	$f(x^i)$
1	-1271.73	-1271.73	1.72E-07
2	-1817.85	-1817.85	7.98E-09
3	-2596.74	-2596.74	1.41E-10
4	-3710.59	-3710.59	1.56E-11
5	-5299.64	-5299.64	1.85E-12
6	-7572.42	-7572.42	1.85E-14

Table 7.7: Example 7.3 - Iterates computed by Algorithm 4.2

$x^0 \in \mathcal{M}_0 \cap K_{\tau/4}(x_c)$. This led to $\overline{c} = \|\tilde{x} - x^*\|_2 = \sqrt{12.5}$, $f(\tilde{x}) = 0$, $f_- = 0$, $c_0 = |\ln(1)| = 0$, $\tilde{t} = 0$, $\tilde{v} = (1,0)$ and $c_1 = \ln(1+2) = \ln(3)$ so that we had $c_3 = \ln(3)$ and $\mu_1 = 0.1 \leq e^{-c_3}$ could be used. Setting the lower bound of the barrier parameter to 10^{-6} and σ_i as small as possible by (5.21) all assumptions of Theorem 5.7 are fulfilled and we obtained the iteration process given in Table 7.8. We must remark that there was not required any restart procedure for adapting the

i,j	x_1^i	x_2^i	$d_2(x^i, M_{opt})$	#LP	#QP	#BP	Time
$1,1$	-3.013535	-2.008399	7.11E-01	15	31	4	0.01
$1,2$	-2.622698	-2.414308	1.47E-01	11	33	2	0.02
$2,1$	-2.531565	-2.519300	8.67E-03	13	36	2	0.03
$3,1$	-2.529324	-2.528754	4.03E-04	12	24	2	0.04
$4,1$	-2.529324	-2.528754	4.03E-04	6	10	2	0.06
$5,1$	-2.525245	-2.524960	2.01E-04	10	33	2	0.12
$6,1$	-2.525539	-2.525535	2.69E-06	14	40	2	0.22

Table 7.8: Example 7.3 - Iterates computed by Algorithm 5.2 with multi-step

accuracy parameter since it turned out that the radius was always equal the maximal possible value 0.9. In contrast to Algorithm 4.2 we observe a convergence-like behaviour of the iterates from Table 7.8. Furthermore, the multi-step technique was in fact used in the first outer step.   □

Example 7.4 Now we consider for fixed $n \in \mathbb{N}$ and $k \in \{1, \ldots, n-2\}$ the following perturbed version of Example 7.1

minimize $f(x) := x_{n+1}$

s.t. $g(x,t) := \left| \phi(t) - \sum_{m=0}^{n-1} x_m t^m - x_n(t^k + t^{k+1}) \right| - x_{n+1} \leq 0$ for all $t \in T := [-1, 2]$

with the same function ϕ as in Example 7.1. The perturbation reflects a typical situation in the numerical approximation where we have to approximate a given function by linearly dependent basis functions. In the given situation we have to approximate ϕ on $[-1, 2]$ by linearly dependent polynomials. In consequence of this the solution set is unbounded so that Assumption 4.1 cannot

hold and Algorithm 4.2 is outside the further considerations. Particularly the complete solution set is given by

$$\mathcal{M}_{opt} = \left\{ (x_0, \ldots, x_{n+1}) : \begin{array}{l} (y_0, \ldots, y_n) \text{ solves (7.1) with } y_m = x_m \text{ if } m \neq k, k+1, n \text{ and} \\ y_k = x_k - x_n, y_{k+1} = x_{k+1} - x_n, y_n = x_{n+1} \end{array} \right\}.$$

Then the weaker Assumption 5.1 is fulfilled if we taking into account that the parts (1)-(5), (7)-(10) can transferred from the analysis of Example 7.1. But of course the calculation of the constants L_S^x and L_S^t requires some changes caused by the additional summand $x_n(t^k + t^{k+1})$. In the case of L_S^x this leads to

$$L_S^x = \max_{t \in \tilde{T}^S} \sum_{m=0}^{n-1} |t^m| + \max_{t \in \tilde{T}^S} |t^k + t^{k+1}| + 1$$

while in the case of L_S^t one can summarize x_k and x_n to one variable as well as x_{k+1} and x_n to another variable. Then the same procedure as for Example 7.1 is usable with the additional fact that the combined variables have a range of $\pm 2r$ instead of $\pm r$.

Altogether Algorithm 5.2 can be used to solve the given problem. The standard parameter setting is given in Table 7.9 while the choice of a starting point and a barrier parameter fulfilling

parameter	start value	decreasing factor	lower bound
s_i	0.01	0.8	10^{-5}
$\varepsilon_{i,0}$	0.01	0.15	−
δ_i	10	0.15	−
q_i	0.999	−	−

Table 7.9: Example 7.4 - standard parameter

(5.19) is much more complicated than in the unregularized case. In the given situation we can easily determine feasible starting points by using information of Example 7.1. But then, using the starting point also as $\tilde{x}$, the constant c_3 defined as in Theorem 5.7 is typically large which implies that the starting barrier parameter has to be very small. Having in mind the notice on avoiding too small barrier parameters for Example 7.1 we should find a better vector $\tilde{x}$ in the sense that the resulting c_3 is less than before. For that purpose Algorithm 5.2 can be also applied since Lemma 5.5 ensures the finiteness of each inner loop under much weaker conditions. Consequently we started Algorithm 5.2 with fixed barrier parameter $\mu = 1$ while all other parameters were set to the starting values given above. Then we ran exactly one outer step of Algorithm 5.2 and calculated new values for c_3 resp. μ_1 based on the computed iterate. If μ_1 is too small again we repeated the step with the previously computed iterate as starting point and a slightly lower accuracy parameter. Of course, this procedure can be repeated more often and in the example cases it was stopped when $\mu_1 = 0.05$ was possible as starting barrier parameter. For that there were at most 23 steps needed, but excepting the first step they considered only a few boxes.

Then we used the final iterate of these pre-steps as starting point and $\mu_{i+1} = 0.2\mu_i$ as standard update as in Example 7.1. Furthermore, the algorithm was stopped when the barrier parameter fell

below 10^{-5}. The values of $r_{i,j,k}$ and $h_{i,j,k}$ were computed analogously to the values of $r_{i,k}$, $h_{i,k}$ in Example 7.1 and the deletion rule as well as the improvement of the constants L_S^x, L_S^x in the sense of Remark 6.8 were regarded again. Moreover a restart procedure similar to that of the unregularized algorithm was used. But, now the prox parameter was reset in addition to the accuracy values. The effect of the restart procedure should be that we achieve a geometric decrease of the values of α_i since these values have to be summable. Nevertheless, in order to avoid too many restarts at the beginning of the iteration process the restart condition is not directly correlated to the values α_i, it only depends again on the ratio ε_i/r_i. If there is a geometric decrease then the definition of α_i implies also a geometric decrease of these values (at least from a certain index) in combination with the boundedness of s_i and r_i.

Then we obtained the results summarized in the appendix for $n \leq 6$ and $k = 1, 2, 3$. At this point we want to investigate the influence of the choice of the prox parameter. For that we have a closer look at the case $n = 5$, $k = 3$ where we successively set the starting prox parameter to 10, 1, 0.01 and 0.0001. The other settings are given as above. Furthermore, since the pre-steps to generate the starting point depends on the prox parameter we excluded this phase from the investigation and used the standard parameters for it. Thus Algorithm 5.2 was always started with $x^0 = (-0.000605, -0.312555, 0.002830, 0.832642, -0.419964, 0.416176, 1.199804)$. Then the detailed results contained in Table 7.10 were obtained. Therein the given time values include 0.72 seconds in each case which was needed for the computation of the pre-steps.

First of all we remark that all final approximate solutions have nearly the same distance to the solution set. But, of course there are some differences in the iteration process. So the results of the first steps document the several starting prox parameters by the value of the distance to the solution set: large prox parameter allow a short step and small prox parameter allow a long step. This is also made clear by the number of considered boxes during each first step, whereby this number increases if the starting prox parameter decreases. Additionally it can be observed that in each case an insufficient accuracy value is detected in a certain step. It is remarkable that this detection occurs earlier if the prox parameter is lower. The reason for this behaviour is that lower prox parameters allow larger steps inside the given boxes so that we faster go to the solution set on the boundary of the feasible set. But this leads to smaller radii and consequently to larger ratios ε_i/r_i so that this ratio can increase at this point. Thus, if the starting prox parameter is chosen too large the restarts occur for very small barrier parameters with bad conditioning so that one typically observes a slow walk along the boundary of the feasible region documented by many considered boxes as in case $s_1 = 10$ in the example. On the other hand we have to avoid too small (starting) prox parameters in order to guarantee the regularization effect. But this is supported by our method since the step sizes are automatically restricted by the radii of the boxes so that the iteration processes approach to each other if the starting prox parameters are sufficiently small. This can be observed by looking at the cases $s_1 = 0.01$ and $s_1 = 0.0001$. $\qquad\square$

s_1	i	μ_i	$d_2(x^i, M_{opt})$	r_i	$\frac{\varepsilon_i}{r_i}$	restarts	$h_{\min}$	$\varnothing h_{i,j,k}$	$\varnothing\frac{|\tilde{T}_h|}{|T_h|}$	#LP	#QP	#BP	Time
10	1	5.00E-02	1.07E+00	1.4E-02	7.3E-01	0	3.00E-03	3.00E-03	1.00	6	6	6	0.75
	2	1.00E-02	8.83E-01	1.1E-02	1.4E-01	0	3.00E-03	3.00E-03	1.00	31	34	16	0.78
	3	2.00E-03	4.88E-01	5.4E-03	4.2E-02	0	3.00E-03	3.00E-03	1.00	86	154	51	0.86
	4	4.00E-04	1.03E-03	1.2E-05	1.6E-02	2	1.80E-05	1.49E-03	0.28	1455	4227	867	13.58
	5	8.00E-05	8.02E-05	6.5E-07	3.3E-03	1	2.72E-08	1.10E-05	0.00	524	1563	245	27.96
	6	1.60E-05	1.60E-05	1.5E-06	2.2E-04	0	1.68E-07	4.51E-07	0.06	88	358	23	36.24
1	1	5.00E-02	8.30E-01	1.0E-02	9.8E-01	0	3.00E-03	3.00E-03	1.00	34	34	34	0.76
	2	1.00E-02	2.21E-01	1.6E-03	9.5E-01	0	3.00E-03	3.00E-03	1.00	162	249	158	0.92
	3	2.00E-03	2.17E-03	2.3E-05	5.4E-02	2	3.43E-05	1.39E-03	0.34	1132	3198	683	7.83
	4	4.00E-04	4.03E-04	4.1E-06	4.7E-02	0	1.37E-05	3.84E-05	0.12	327	869	176	11.75
	5	8.00E-05	8.04E-05	8.7E-07	3.3E-02	0	2.25E-06	5.99E-06	0.05	327	905	154	20.67
	6	1.60E-05	1.61E-05	1.3E-06	3.4E-03	0	3.85E-07	1.32E-06	0.03	95	336	37	25.30
0.01	1	5.00E-02	6.55E-01	7.7E-03	1.3E+00	0	3.00E-03	3.00E-03	1.00	62	62	62	0.78
	2	1.00E-02	1.02E-02	9.9E-05	8.6E-02	2	4.12E-05	1.94E-03	0.35	921	2105	616	4.77
	3	2.00E-03	2.08E-03	1.9E-05	6.6E-02	0	8.61E-05	2.30E-04	0.28	331	835	181	6.71
	4	4.00E-04	4.06E-04	4.1E-06	4.6E-02	0	1.39E-05	3.79E-05	0.12	332	903	169	10.71
	5	8.00E-05	8.06E-05	8.7E-07	3.3E-02	0	2.26E-06	6.05E-06	0.05	325	905	153	19.60
	6	1.60E-05	1.58E-05	1.2E-06	3.4E-03	0	3.80E-07	1.31E-06	0.03	101	335	36	24.28
0.0001	1	5.00E-02	6.45E-01	7.6E-03	1.3E+00	0	3.00E-03	3.00E-03	1.00	64	64	64	0.76
	2	1.00E-02	1.01E-02	9.8E-05	8.6E-02	2	3.79E-05	1.94E-03	0.35	912	2091	616	4.74
	3	2.00E-03	2.01E-03	1.9E-05	6.8E-02	0	8.26E-05	2.24E-04	0.28	334	846	185	6.71
	4	4.00E-04	4.04E-04	4.1E-06	4.7E-02	0	1.39E-05	3.70E-05	0.12	336	880	167	10.61
	5	8.00E-05	7.98E-05	8.7E-07	3.3E-02	0	2.24E-06	6.01E-06	0.05	340	930	154	19.55
	6	1.60E-05	1.60E-05	1.3E-06	3.3E-03	0	3.72E-07	1.25E-06	0.03	105	369	38	24.73

Table 7.10: Example 7.4 - The influence of the prox parameter

Example 7.5 We consider for fixed $n \in \mathbb{N}$ and $1 \leq \kappa < n$ the problem

$$\text{minimize} \quad f(x) := - \sum_{l=\kappa+1}^{n} x_l$$

$$\text{s.t.} \quad g_\nu(x,t) := - \left| t - \frac{\sqrt{2}}{\nu+1} \right| x_\nu + \sum_{l=\kappa+1}^{n} \cos^2 \left(\pi l \left(t - \frac{\sqrt{2}}{\nu+1} \right) \right) x_l^2 - 1 \leq 0 \quad (7.4)$$

$$\text{for all } t \in T^\nu := [0,1], \ \nu = 1, \dots, \kappa.$$

For an extensive investigation of this problem we refer to Voetmann [61], where the problem above is considered as Example 2. In comparison to the previously considered problems we are now confronted with an additional difficulty, namely there can occur more than one constraint by choosing $\kappa > 1$. Consequently we have to take Algorithm 4.12 and the analogous extension of Algorithm 5.2 into account.

First of all the solution set of (7.4) is given by

$$\mathcal{M}_{opt} = \left\{ x \in \mathbb{R}^n : x_i \geq 0 \, (i = 1, \dots, \kappa), \quad x_i = \frac{1}{\sqrt{n-\kappa}} \, (i = \kappa+1, \dots, n) \right\}.$$

which is unbounded so that Assumption 4.1 resp. 4.11 cannot be fulfilled. Nevertheless, since $\mathcal{M}_{opt}$ is nonempty the parts (1)-(5), (6)' and (7) of Assumption 5.1 or their generalizations are obviously fulfilled. Additionally, the subgradients of f and $g_\nu(\cdot, t)$ required by part (10) of these assumptions are in fact gradients since f, $g_\nu(\cdot, t)$ are differentiable w.r.t. x. Consequently the needed (sub)gradients are easily calculable. The constants $L^t_{\nu,S}$ can be computed by

$$L^t_{\nu,S} := \max_{x \in S} |x_\nu| + \sum_{l=\kappa+1}^{n} \max_{t \in T^\nu} \left| 2\pi l \sin \left(\pi l \left(t - \frac{\sqrt{2}}{\nu+1} \right) \right) \cos \left(\pi l \left(t - \frac{\sqrt{2}}{\nu+1} \right) \right) \right| \max_{x \in S} x_l^2$$

$$= \max_{x \in S} |x_\nu| + \sum_{l=\kappa+1}^{n} \pi l \max_{t \in T^\nu} \left| \sin \left(2\pi l \left(t - \frac{\sqrt{2}}{\nu+1} \right) \right) \right| \max_{x \in S} x_l^2$$

$$\leq \max_{x \in S} |x_\nu| + \sum_{l=\kappa+1}^{n} \pi l \max_{x \in S} x_l^2$$

which estimates all possible slopes of g_ν w.r.t. t. This description of $L^t_{\nu,S}$ allows the improvement of these constants in the sense of Remark 6.8. Finally, $C_{\nu,S}$ is separately calculated by (6.4) for each ν. For that we also require the constants $L^x_{\nu,S}$ which can be given by

$$L^x_{\nu,S} := \max_{t \in T^\nu} \left| t - \frac{\sqrt{2}}{\nu+1} \right| + 2 \sum_{l=\kappa+1}^{n} \max_{t \in T^\nu} \left| \cos^2 \left(\pi l \left(t - \frac{\sqrt{2}}{\nu+1} \right) \right) \right| \max_{x \in S} |x_l|.$$

Summing up Assumption 5.1 or its generalization for more than one constraint is fulfilled. Thus Algorithm 5.2 or its extension in the sense of Remark 5.5 can be used to solve (7.4). This was done with the standard parameters given in Table 7.11. For the determination of the starting barrier parameter we used again the procedure described in Example 7.4 with $\mu = \kappa$ and the origin as feasible starting point now. But it was not possible to give a standard starting barrier parameter

parameter	start value	decreasing factor	lower bound
s_i	0.01	0.5	10^{-5}
$\varepsilon_{i,0}$	0.01	0.15	–
δ_i	1	0.15	–
q_i	0.999	–	–

Table 7.11: Example 7.5 - standard parameter

like 0.05 in the example before. Instead of this we stopped the determination of a starting barrier parameter after exactly one step. Then the maximal possible barrier parameter, determined by (5.19), was used as starting value, whereby it turned out that the starting barrier parameter decreased if the number of variables and constraints increase. Nevertheless, the standard update for the barrier parameter was $\mu_{i+1} = 0.2\mu_i$ while the algorithm stopped when the barrier parameter fell below 10^{-5}. In addition there was also used a restart procedure as described for Example 7.4 and for each $\nu \in \{1, \ldots, l\}$ a radius was computed by applying (7.2) with $\hat{r} = \min\{1, 2r_{i,k-1}\}$, $h = h_{i,k-1}^\nu$ if $k > 1$ and $\hat{r} = 1$, $h = 0.001$ if $k = 1$. Then the minimal of these radii was used as radius of the box which had to be determined.

Regarding all the stated facts we obtained the results presented in the appendix for several values of n and k. At this point we want to have a detailed look at the influence of the starting accuracy value. For that purpose we consider the case $n = 12, \kappa = 1$ with results given in Table 7.12. Thereby

$\varepsilon_{1,0}$	restarts	ε_6	$d_2(x^6, \mathcal{M}_{opt})$	$f(x^6)$
10	4	2.40E-08	1.44E-04	-3.3166088
1	3	3.20E-08	1.47E-04	-3.3166094
0.01	2	4.27E-09	1.46E-04	-3.3166088
0.001	1	5.69E-09	1.55E-04	-3.3166088
0.0001	0	7.59E-09	1.41E-04	-3.3166087

Table 7.12: Example 7.5 - The influence of the accuracy parameter

all parameters except for $\varepsilon_{1,0}$ were given as described above. In particular we ran the algorithm with the same starting point and the same starting barrier parameter $\mu_1 = 0.05$ in each case. Thus we had 6 outer steps and the final barrier parameter 1.6E-05.

From Table 7.12 we observe the remarkable fact that the final distances to the solution set and the final values of the objective function at the approximate solutions are comparable to each other. This and the fact that the accuracy parameters ε_i approach each other are caused by the restart procedure. Of course each restart causes an additional computational effort so that the starting accuracy value should not be chosen too large in order to avoid too many restarts.

Furthermore, investigating the values of the objective function at the approximate solutions we notice that they are all in a range near the optimal value $-\sqrt{11} \approx -3.3166248$ - as predicted by

(2.5) for the classical logarithmic barrier methods with exact minimizers. This fact is especially remarkable since the iterates were computed as approximate minimizers of the regularized function whereas (2.5) holds in the unregularized case. Nevertheless, the values of the objective function at the approximate solutions of the previous examples are also mostly in the range around the optimal value predicted by (2.5). This can be especially observed very well by reinvestigating the approximation problems before where the last component of the final iterates equals the final objective values and approves again the prognose by (2.5).

Additionally we want to have a look at the influence of the number of considered barrier problems which can be controlled by the barrier parameter update. For that we also observed results with non-standard updates for the barrier parameter. In order to ensure the same conditions in each case it was necessary to adapt the updates for the accuracy and the prox parameter. They were chosen in such way that the predicted final values (without regarding of possible restarts) were nearly the same. Considering the case $n = 12, \kappa = 1$ again we obtained the results summarized in Table 7.13 with chosen starting values as in the standard case. Reading this table we can first state that the final

decreasing factors			outer steps	restarts	$d_2(x, \mathcal{M}_{opt})$	$f(x)$	effort		
μ_i	$\varepsilon_{i,0}$	s_i					#LP	#QP	#Box
0.2	0.15	0.5	6	2	1.46E-04	-3.3166088	93	118	37
0.41	0.34	0.68	10	3	1.15E-04	-3.3166083	120	156	54
0.605	0.52	0.805	17	4	4.11E-05	-3.3166087	186	260	69

Table 7.13: Example 7.5 - The influence of the barrier update

values of the objective function are nearly the same in all cases. Consequently from that point of view there is no remarkable influence of the number of outer steps on the final result. Of course the distance of the final iterate to the solution set differs from case to case and it decreases if the number of outer steps increases. Additionally, the computational effort increases with the number of outer steps since we solve more barrier problems. This effect is intensified by the increasing number of restarts (more planned outer steps lead to more checks of the restart condition). Nevertheless, the computational effort for each separate outer step decreases if we use larger decreasing factors. Especially this second observation is typical for our methods and can lead to the possibly surprising fact that the total computational effort can decrease if more outer steps are done.

Such a behaviour was observed for Example 7.2 with $n = 14, 15$. If we use the standard updates $\mu_{i+1} = 0.2\mu_i$ for the barrier parameter and $\varepsilon_{i+1,0} = 0.15\varepsilon_i$ for the accuracy parameter the computational effort is much higher than using the updates $\mu_{i+1} = 0.4\mu_i$ and $\varepsilon_{i+1,0} = 0.33\varepsilon_i$ (which leads to a similar final accuracy). For instance the computation of the approximate solution in the case $n = 15$ with the standard parameters tooks about 135 seconds against about 18 seconds with the changed update. But unfortunately we have no general rule to obtain an "optimal" update strategy in order to minimize the computational effort. □

Chapter 8

An application to the financial market

8.1 The mathematical model

In this chapter we present the application of our algorithms to a problem occurring in the field of finance. Our goal is to approximate the yield curve of an underlying asset. For that purpose we follow the considerations by Tichatschke et al. [56] which are based on the model of Vasicek [60].

Let $y(t)$ be a given yield curve fulfilling the stochastic differential equation

$$dy = (\alpha + \beta y)dt + \sigma dZ$$

with a Brownian motion Z and parameters α, β, σ. Now this yield curve y should be approximated on an interval $T = [0, \bar{t}]$ by a function r. Then r has to fulfill the initial value problem

$$\dot{r} = \beta r + \alpha + \sigma w(t), \quad r(0) = r_0 \in [\underline{r}, \overline{r}],$$
$$\underline{w} \le w(t) \le \overline{w}, \quad t \in T \tag{8.1}$$

which can be derived from the stochastic differential equation stated above. Therein the Brownian motion Z is modeled by a piecewise continuous function w with bounds $\underline{w}, \overline{w}$. Additionally the initial value r_0 is variable so far. The solution of (8.1) is given by

$$r(t) = -\frac{\alpha}{\beta}(1 - e^{\beta t}) + r_0 e^{\beta t} + \sigma \int_0^t e^{\beta(t-\tau)} w(\tau) d\tau \tag{8.2}$$

so that the approximation error can be minimized by solving the problem

$$
\begin{aligned}
\text{minimize} \quad & \max_{t \in T} |y(t) - r(t)| \\
\text{s.t.} \quad & r_0 \in [\underline{r}, \overline{r}], \\
& \underline{w} \le w(t) \le \overline{w}, \quad \text{for all } t \in T.
\end{aligned}
\tag{8.3}
$$

Since the feasible set is mainly described by the function space $\{w : \underline{w} \le w(t) \le \overline{w}, t \in T\}$ we deal with an infinite problem. In order to simplify this the available functions of w are restricted to piecewise constant functions, i.e. we set

$$w(t) := w_i \quad \text{for all } t \in T_i, \quad i = 1, \dots, N$$

with $0 =: t_0 < t_1 < \ldots < t_{N-1} < t_N := \bar{t}$, $T^i := [t_{i-1}, t_i)$ for $i \in \{1, \ldots, N-1\}$ and $T^N := [t_{N-1}, t_N]$. Then the integral term in (8.2) becomes

$$\sigma \int_0^t e^{\beta(t-\tau)} w(\tau) d\tau = -\sum_{j=1}^{i-1} B_j e^{\beta t} w_j - \frac{\sigma}{\beta} \left(1 - e^{\beta(t-t_{i-1})} \right) w_i$$

for all $t \in T^i$ with

$$B_j := \frac{\sigma}{\beta} \left(e^{-\beta t_j} - e^{-\beta t_{j-1}} \right)$$

for all $j = 1, \ldots, N-1$. Consequently, using

$$f_i(r_0, w, t) := -\frac{\alpha}{\beta}(1 - e^{\beta t}) + r_0 e^{\beta t} - \sum_{j=1}^{i-1} B_j e^{\beta t} w_j - \frac{\sigma}{\beta} \left(1 - e^{\beta(t-t_{i-1})} \right) w_i$$

for all $t \in T^i$, (8.3) can be rewritten as

$$
\begin{aligned}
\text{minimize} \quad & f(r_0, w, \vartheta) := \vartheta \\
\text{s.t.} \quad & g_i(r_0, w, \vartheta, t) := |\hat{y}(t) - f_i(r_0, w, t)| - \vartheta \le 0 \quad \text{for all } t \in T^i \ (i = 1, \ldots, N) \\
& g_{N+1}(r_0, w, \vartheta) := \max \left\{ \max_{i=1,\ldots,N} \{ w_i - \underline{w}, \overline{w} - w_i \}, r_0 - \underline{r}, \overline{r} - r_0 \right\} \le 0
\end{aligned}
$$
(8.4)

with an approximation $\hat{y}$ of y constructed by observable values. Thus we now deal with a linear semi-infinite problem with $N+1$ constraints. The number of constraints in (8.4) is much smaller than in the formulation by Tichatschke et al. [56] which is caused by the fact that we can treat nondifferentiable constraint functions. Consequently, motivated by (2.5), we can use larger barrier parameter in order to expect similar accuracies.

8.2   Numerical results

We want to show that Algorithm 4.12 can be used for solving (8.4) approximately. For that purpose we have to check Assumption 4.11. We first notice that some parts of this assumptions does not have to hold for the last constraint g_{N+1} since it does not depend on t. However, in practice g_{N+1} is treated as constraint of type $g(x,t) \le 0$, $t \in T$ with single-valued T. Additionally we have to know something more about $\hat{y}$ if we want to show some parts of Assumption 4.11. Therefore we only consider the special case $\hat{y} = y_i$ is constant on each interval T^i as it was done by Voetmann [61].

We observe that the assumptions of the convexity of f and all g_i are fulfilled since r_0, w, ϑ occur at most linearly in each constraint and the absolute values of linear functions as well as the maxima of finitely many linear functions are convex. Part (2) of Assumption 4.11 is not fulfilled since $T^1, \ldots, T^{N-1}$ are not closed. But it is possible to consider the closures of all sets T^i and the continuous extensions of all g_i in (8.4) without changing the feasible set. Then the continuity of all constraints w.r.t. t is obvious. Furthermore, part (5) of Assumption 4.11 is fulfilled if $\underline{w} < \overline{w}$ and $\underline{r} < \overline{r}$ are true. Moreover, we observe that lower level sets of our considered semi-infinite problem

are bounded since w, r_0 are bounded by the $(N + 1)$-th constraint and ϑ is bounded below by 0 and bounded above by the given level. Consequently, since we only deal with continuous functions, these level sets are compact. Thus, regarding that the given problem is feasible, the solution set $\mathcal{M}_{opt}$ has to be nonempty and compact. Part (7) is simply fulfilled if we choose equidistant finite grids for each constraint as it is done in the chapter before. Regarding Lemma 6.4 part (9) of Assumption 4.11 can be fulfilled while subgradients of f and $g_i(\cdot, t)$ can be easily given if one regards that, excluding the absolute value or the maxima, the functions therein are differentiable. Thus it remains to determine the constants $L_{i,S}^t$ enforced by part (8) of our assumption and L_S^x for determining $C_{i,S}$ and the radii. Regarding the differentiability of the function inside the absolute value in g_i we can set

$$L_{i,S}^t := \max_{(r_0,w,\vartheta) \in S} \sup_{t \in T^i} \left| \alpha e^{\beta t} + r_0 \beta e^{\beta t} - \sum_{j=1}^{i-1} B_j \beta e^{\beta t} w_j + \sigma e^{\beta(t-t_{i-1})} w_i \right|$$

for all $i = 1, \ldots, N$. In addition to this

$$L_{i,S}^x := \max_{t \in T^i} \left(e^{\beta t} + \sum_{j=1}^{i-1} |B_j| e^{\beta t} + \left| \frac{\sigma}{\beta} \left(1 - e^{\beta(t-t_{i-1})} \right) \right| \right) + 1$$

for all $i = 1, \ldots, N$ and $L_{N+1,S}^x := 1$ are used. Consequently Assumption 4.11 is completely fulfilled so that Algorithm 4.12 can be used for solving (8.4).

Demonstrating this we want to approximate the German stock index DAX in two time periods of each 30 days. The required data, consisting of the daily opening DAX prices, are given in Table 8.1. The first period represents a quite stable but slowly growing DAX while the second period covers a big fluctuation in a short time interval. In addition there are needed values for α, β and σ. Voetmann [61] uses the setting

$$\alpha = 0.0154, \quad \beta = -0.1779 \quad \text{and} \quad \sigma = 0.02$$

which is derived from the observation of US interests for government bonds within the years 1964 to 1989. Since there was not made a similar investigation of the German stock exchange we use the values stated above too. Moreover, the German stock exchange tends to follow the US stock exchange so that this choice is not a bad choice.

Due to the fact that we only consider two different scenarios we do not give a standard parameter setting. Rather we have a separate look at both situations. Nevertheless, there were some common settings. So in both situations the considered time period was uniformly mapped to the interval $[0, 1]$ which implies that each trading day was represented by a subinterval of length $1/30$ of $[0, 1]$. Additionally we set $q_i = 0.999$ in each case and, regarding Remark 6.2, the radii were computed by Lemma 6.1. In fact we used (7.2) to determine a radius for each constraint with $\hat{r} = \min\{1000, 2r_{i,k-1}\}$ if $k > 1$ and $\hat{r} = 1000$ if $k = 1$ for all constraints and $h = h_{i,k-1}^\nu$ if $k > 1$, $h = 0.001$ if $k = 1$ for $\nu = 1, \ldots, 30$. In consequence of this it was possible to compute $C_{\nu,S^{i,k}}$ by Lemma 6.4 for each constraint.

Now considering only the first example data DAX1 the starting point

$$r_0 := 1600, \quad w_1 = \ldots = w_{30} := 0, \quad \vartheta := 3000$$

Example DAX1			Example DAX2		
Bounds	Trading Day t_i^1	Price y_i^1	Bounds	Trading Day t_i^2	Price y_i^2
	04.01.1993	1533.06		05.03.1998	4642.79
	05.01.1993	1547.99		06.03.1998	4686.24
	06.01.1993	1560.27		09.03.1998	4775.83
	07.01.1993	1546.33		10.03.1998	4807.92
	08.01.1993	1540.56		11.03.1998	4855.22
	11.01.1993	1526.66		12.03.1998	4822.78
	12.01.1993	1527.33		13.03.1998	4863.44
	13.01.1993	1529.61		16.03.1998	4891.85
	14.01.1993	1521.03		17.03.1998	4932.42
	15.01.1993	1542.91		18.03.1998	4936.17
$\underline{w} = -10^5$	18.01.1993	1559.83	$\underline{w} = -10^6$	19.03.1998	4923.51
	19.01.1993	1576.13		20.03.1998	4993.53
$\overline{w} = \quad 10^5$	20.01.1993	1586.94	$\overline{w} = \quad 10^6$	23.03.1998	5017.48
	21.01.1993	1577.62		24.03.1998	5014.62
$\underline{r} = 1000$	22.01.1993	1587.95	$\underline{r} = 4000$	25.03.1998	5058.54
	25.01.1993	1582.21		26.03.1998	5093.52
$\overline{r} = 2000$	26.01.1993	1566.83	$\overline{r} = 6000$	27.03.1998	5041.84
	27.01.1993	1570.96		30.03.1998	5069.98
	28.01.1993	1561.02		31.03.1998	5070.81
	29.01.1993	1571.28		01.04.1998	5093.52
	01.02.1993	1582.35		02.04.1998	5163.11
	02.02.1993	1587.20		03.04.1998	5203.58
	03.02.1993	1595.08		06.04.1998	5256.69
	04.02.1993	1605.07		07.04.1998	5276.79
	05.02.1993	1635.67		08.04.1998	5282.94
	08.02.1993	1643.83		09.04.1998	5270.35
	09.02.1993	1642.32		14.04.1998	5378.91
	10.02.1993	1649.79		15.04.1998	5379.99
	11.02.1993	1651.22		16.04.1998	5362.26
	12.02.1993	1655.13		17.04.1998	5266.34

Table 8.1: DAX data

was used and we set

$$\varepsilon_{1,0} := 0.001, \quad \varepsilon_{i+1,0} := 0.7\varepsilon_i, \quad \delta_1 := 10, \quad \delta_{i+1} := 0.7\delta_i, \quad \mu_1 := 10, \quad \mu_{i+1} := 0.8\mu_i.$$

But as in all examples before the restart procedure proposed by Remark 4.10 was applied to adapt automatically the accuracy parameter. Then the algorithm was stopped when the barrier parameter fell below 0.1. Although this stopping criterion seems to be very bad, Figure 8.1 shows that the tendency of the DAX curve is correctly reconstructed by our final approximate solution which can be found in the appendix. Furthermore the final approximation error of 15.65 is better than 16.78 achieved by Voetmann [61] and close to the correct minimal value 15.30 given by the half of the maximal gap between two successive observed DAX values.

For the second example data DAX2 we used

$$r_0 := 5000, \quad w_1 = \ldots = w_{30} := 30000, \quad \vartheta := 5000$$

as starting point,

$$\varepsilon_{1,0} := 0.005, \quad \varepsilon_{i+1,0} := 0.6\varepsilon_i, \quad \delta_1 := 50, \quad \delta_{i+1} := 0.6\delta_i, \quad \mu_1 := 100, \quad \mu_{i+1} := 0.7\mu_i$$

and, again, the restart procedure. The stopping criterion was now fulfilled if the barrier parameter reached 0.01. The resulting approximate solution is stated in the appendix too, while our final approximation error given by 54.30 is comparable with the result achieved by Voetmann [61]. The optimal value is 54.28 so that the more accurate stopping criterion leads also to a more accurate final solution in comparison to the first situation. Figure 8.2 shows again that the complete curve is correctly reconstructed, but it can be observed as in the first case that there is not detected each particular fluctuation.
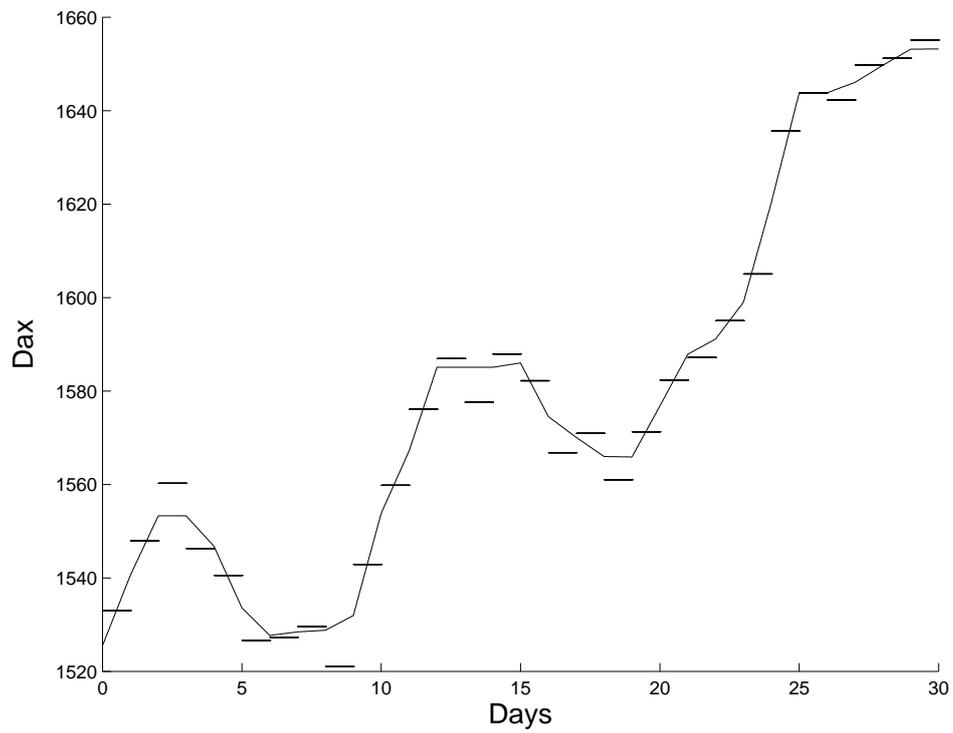
Figure 8.1: Example DAX1 - trajectory of the solution
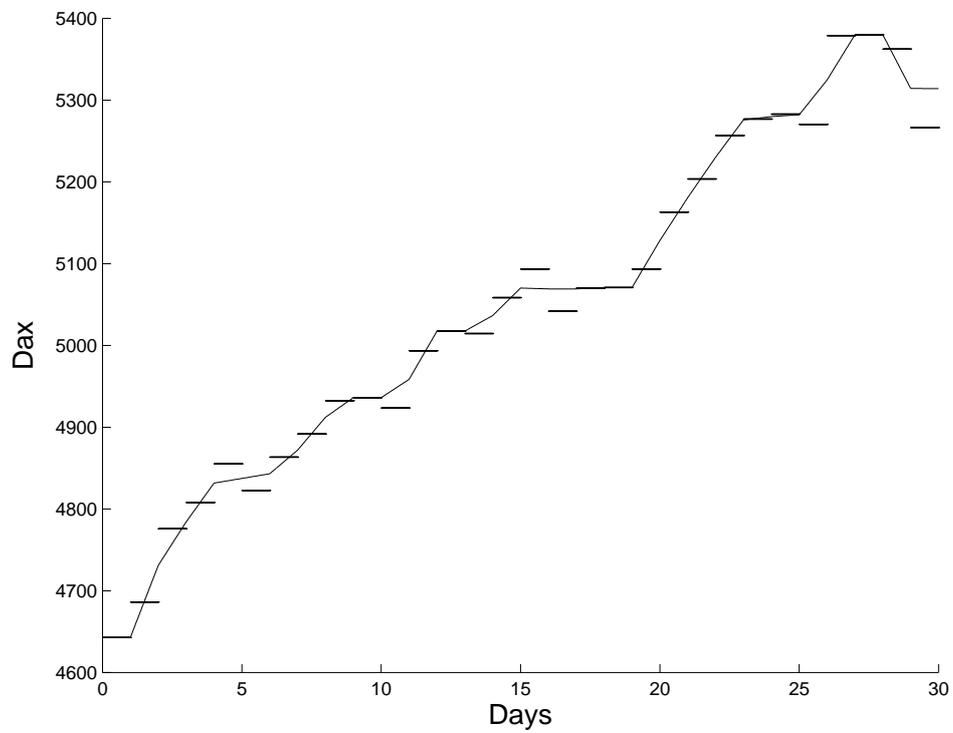
Figure 8.2: Example DAX2 - trajectory of the solution

Chapter 9

Perfect reconstruction filter bank design

9.1 The mathematical model

In this chapter we want to analyze the design of so-called perfect reconstruction filter banks. Let us first give a short introduction into the filter theory from the mathematical point of view (we follow Kortanek, Moulin [30]). We refer to Antoniou [2], Meyer [34] or generally to the references in Kortanek, Moulin [30] for much more details than here are presented.

A discrete input signal x is given by an arbitrary infinite sequence $\{x(n)\}_{n \in \mathbb{Z}}$ which is square summable, i.e. $x \in l^2(\mathbb{Z})$. Generally a *filter* is a linear operator that acts on an input signal x through convolution. Thus, identifying the filter with $\{h(n)\}_{n \in \mathbb{Z}}$, the output vector y of a *linear time-invariant* system (LTI) is given by

$$y(n) = \sum_{i=-\infty}^{\infty} h(i)x(n-i). \tag{9.1}$$

Moreover, we assume that the filter coefficients $h(i)$ are real-valued since data converters work with real-valued signals only (cf., e.g., Potchinkov [41]).

A very simple filter is described by the decimation operator $\downarrow 2$ which picks out only the terms of $x(n)$ with even index and corresponds to down-sampling. The adjoint operator $\uparrow 2$ corresponds to up-sampling and fills in zeros at the odd indices.

For filter h the *transfer function* in the complex domain is

$$H(z) := \sum_{n=-\infty}^{\infty} h(n)z^{-n}.$$

Then (9.1) is equivalent to multiplying the corresponding transfer functions, i.e. $Y(z) = H(z)X(z)$. Similarly, the *frequency response* is given by

$$H(\omega) := \sum_{n=-\infty}^{\infty} h(n)e^{-jn\omega}$$

with j as imaginary unit.

When the filter has only finitely many nonzero components, it is called *finite impulse response* (FIR-)filter. Such FIR-filters with length $2N$ will be only considered in the sequel.

The combination of at least two filters is called *filter bank*. As stated above we intend to design perfect reconstruction filter banks. Thus let us describe the simplest case of such a filter bank, a two-band PR filter bank. Generally, it is divided into two sections, the analysis section and the synthesis section. The analysis section consists of a *lowpass* and a *highpass* filter which decomposes the input

Analysis section   Synthesis section
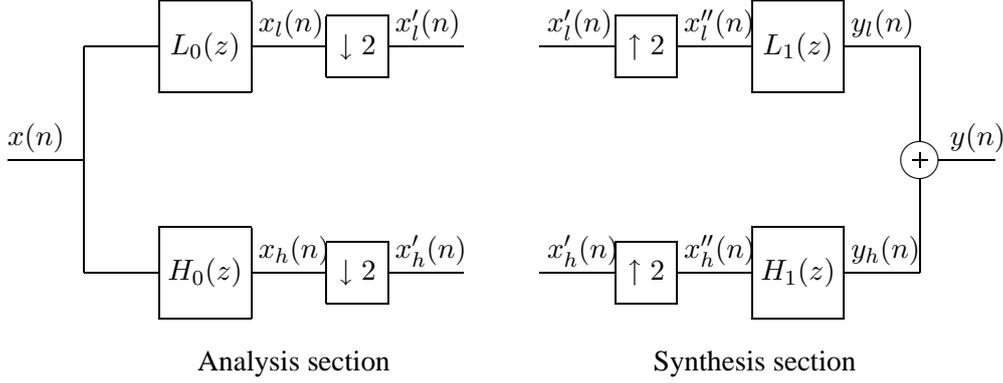
Figure 9.1: Two-band perfect reconstruction filter bank

signal $x(n)$ into two components $x_l(n)$ and $x_h(n)$. The lowpass filter takes averages to smooth out variations while the highpass picks out the high frequencies in the signal. After these filters there is a down-sampling part to shorten the signal. The synthesis section also consists of a lowpass and a highpass filter. Furthermore, there is an up-sampling part. The task of the synthesis section is to reconstruct (thus reconstruction filter bank) a signal y from the two signals x'_h and x'_l. Our design goal is to construct the filters in such a way that the output signal $y(n)$ is identical with the input signal $x(n)$.

For the mathematical description let the transfer functions of the lowpass filters L_0, L_1 be given by $L_m(z) = \sum_{k=2p_m}^{2p_m+2N-1} h_m^l(k)z^{-k}$ with $m = 0, 1$ and $p_0, p_1 \in \mathbb{Z}$. For simplicity we set $p_0 := 0$ (otherwise we only have a delayed version of the resulting signal). Furthermore, $H_m(z) = \sum_{k=2q_m}^{2q_m+2N-1} h_m^h(k)z^{-k}$ with $q_0, q_1 \in \mathbb{Z}$ are the transfer functions of the highpass filters. Then one has

$$x''_l(2n) = x'_l(n) = \sum_{k=0}^{2N-1} h_0^l(k)x(2n-k) \quad \text{and} \quad x''_l(2n+1) = 0 \tag{9.2}$$

after a short calculation so that we obtain

$$X''_l(z) = \frac{1}{2}\left(L_0(z)X(z) + L_0(-z)X(-z)\right)$$

as transfer function of the lowpass band up to the second lowpass filter. Hence,

$$Y_l(z) = L_1(z)X''_l(z) = \frac{1}{2}\left(L_0(z)L_1(z)X(z) + L_0(-z)L_1(z)X(-z)\right)$$

is the complete transfer function of the lowpass band. Analogously we get

$$Y_h(z) = \frac{1}{2} \left(H_0(z)H_1(z)X(z) + H_0(-z)H_1(z)X(-z) \right)$$

as transfer function of the highpass band. Adding both we have

$$Y(z) = \frac{1}{2} \left(L_0(z)L_1(z) + H_0(z)H_1(z) \right) X(z) + \frac{1}{2} \left(L_0(-z)L_1(z) + H_0(-z)H_1(z) \right) X(-z)$$

as transfer function for the complete output signal. The first term of this expression is called the *distortion* transfer function, while the second term is the *aliasing* transfer function. The perfect reconstruction condition requires that $Y(z) = X(z)z^{-m}$ with some odd integer m (cf., e.g., Goswami, Chan [14]), i.e. the output signal can only be a delayed version of the input signal. Consequently the filters have to be chosen such that the aliasing part is eliminated. This can be achieved by

$$L_1(z) = \pm H_0(-z) \quad \text{and} \quad H_1(z) = \mp L_0(-z).$$

Choosing the upper sign and defining product filters for each band we have

$$P_l(z) := L_0(z)L_1(z) = L_0(z)H_0(-z)$$
$$P_h(z) := H_0(z)H_1(z) = -L_0(-z)H_0(z) = -P_l(-z).$$

Then the perfect reconstruction condition becomes

$$P_l(z) - P_l(-z) = 2z^{-m}.$$

At this point there exist two basic approaches for determining the filters L_0, H_0 (cf., e.g., Goswami, Chan [14]). The first one is the quadrature mirror approach, i.e. $H_0(z) = L_0(-z)$ while the second one is the half-band filter approach. In the sequel we assume $H_0(z) = -z^{-m}L_0(-z^{-1})$ in correspondence to the second approach. Then we have

$$P_l(z) = -(-z)^{-m}L_0(z)L_0(-(-z)^{-1}) = z^{-m}L_0(z)L_0(z^{-1}).$$

Setting $P(z) = z^m P_l(z) = L_0(z)L_0(z^{-1})$ the perfect reconstruction condition is transformed into

$$P(z) + P(-z) = 2. \tag{9.3}$$

Our goal will be the design of this product filter P. Then the underlying lowpass filter L_0 comes from the spectral factorization of P (cf., e.g., Smith, Barnwell [52]).

To analyze the structure of P we simply expand $L_0(z)L_0(z^{-1})$:

$$P(z) = \sum_{i=0}^{2N-1} (h_0^l(i))^2 + \sum_{i=0}^{N-1} \sum_{k=0}^{2N-1-2i} h_0^l(k)h_0^l(k+2i)(z^{2i} + z^{-2i})$$
$$+ \sum_{i=0}^{N-1} \sum_{k=0}^{2N-2-2i} h_0^l(k)h_0^l(k+2i+1)(z^{2i+1} + z^{-2i-1}).$$

Taking (9.3) into account this leads to the following conditions

$$\sum_{i=0}^{2N-1-2k} h_0^l(i)h_0^l(i+2k) = \delta_{k0}, \quad 0 \leq k < N. \tag{9.4}$$

Thus, setting

$$a_i := \sum_{k=0}^{2N-2i-2} h_0^l(k)h_0^l(k+2i+1), \quad 0 \leq i < N, \tag{9.5}$$

we have

$$P(z) = 1 + \sum_{k=0}^{N-1} a_k(z^{-2k-1} + z^{2k+1}).$$

Moreover, under the change of variable, $z = e^{2\pi j\omega}, 0 \leq \omega \leq 0.5$, we obtain

$$P(\omega) = 1 + 2\sum_{k=0}^{N-1} a_k \cos(2(2k+1)\pi\omega)$$

as well as $P(\omega) = |L_0(\omega)|^2$. Therefore $P(\omega) \geq 0$ has to hold such that the feasible product filters are described by

$$1 + 2\sum_{k=0}^{N-1} a_k \cos(2(2k+1)\pi\omega) \geq 0, \quad 0 \leq \omega \leq 0.5. \tag{9.6}$$

Now let us look for a design goal or an objective function. This will come from the application of subband coding which is illustrated in Figure 9.2.
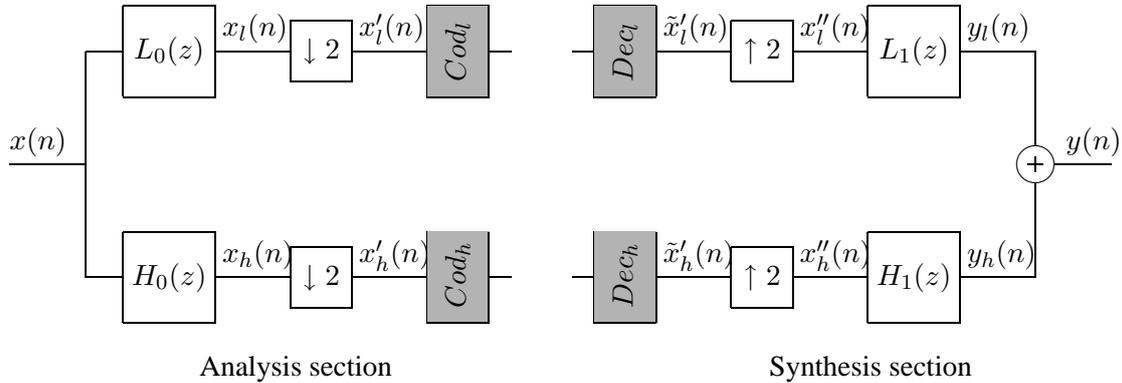
Figure 9.2: Two-band coding system

Our digital input signal x has length $2P$ which is split correctly into x'_l and x'_h (alternatively we can consider an analog input signal which is split correctly into two analog signals). But now these signals are transmitted and for this transmission they are shorten (or digitalized) by the coder to additional length $2p$. Decoding these coded signals it leads to input signals $\tilde{x}'_l, \tilde{x}'_h$ of the synthesis section which are typically different from x'_l, x'_h. Consequently the output signal y of the whole

coding system is different from the input signal x. The signal $y(n) - x(n)$ is termed the reconstruction error and is due to the subband quantization errors $\tilde{x}'_l(n) - x'_l(n)$ and $\tilde{x}'_h(n) - x'_h(n)$. For the mathematical model these quantization errors are modeled as random processes as follows:

Let a signal v be given in the binary representation, i.e. $v = \sum_{k=-\infty}^{l} v_k 2^k$. Then the quantization $Q(v)$ is the $b-$bit representation of v (b is called the *transmission rate*), i.e. $Q(v)$ can be given by $Q(v) = \sum_{k=l-b+1}^{l} \tilde{v}_k 2^k$. Thus, setting $\Delta = 2^{-(b-1)} 2^l$, we have

$$-\frac{\Delta}{2} \leq q = Q(v) - v \leq \frac{\Delta}{2},$$

where the division by 2 occurs from the action of round off. Now, assuming that the quantization error q has a uniform probability over $[-\Delta/2, \Delta/2]$ we obtain

$$\sigma_q^2 = \int\limits_{-\Delta/2}^{\Delta/2} q^2 \frac{1}{\Delta} dq = \frac{\Delta^2}{12} = \frac{2^{-2b} 2^{2l}}{3}$$

Moreover, we assume that the quantization errors are statistically independent and statistically independent of the signal v. These assumptions are valid in the limit as Δ tends to zero if v is itself a random variable with variance σ_v^2. But for a given bit budget Δ and hence σ_q^2 are related to the input variance σ_v^2 in a special way that depends only on the statistical properties of the input signal, i.e.

$$\sigma_q^2 = c 2^{-2b} \sigma_v^2, \tag{9.7}$$

where c is a constant which includes the informations on the statistical properties of the input signal.

Now, let us come back to the two-band case. We assume that the input signal $x(n)$ is a random variable for each n which is *Wide Sense Stationary*. This means that $m_x = Ex(n)$ and the autocorrelations $R_{xx}(k) = E[x(n)x(n-k)]$ are independent of n. These assumptions are standard for noise analysis in LTI systems. Then the signals $x'_l(n)$ and $x'_h(n)$ are also random variables which are independent of n in the sense above. Thus they have variances $\sigma_{l'}^2, \sigma_{h'}^2$ independent of n. Consequently, using (9.7) we obtain the variances

$$\sigma_l^2 = c 2^{-2\rho_l} \sigma_{l'}^2 \quad \text{and} \quad \sigma_h^2 = c 2^{-2\rho_h} \sigma_{h'}^2$$

in the lowpass and the highpass band with the same constant c for both bands (because both signals x'_l, x'_h coming from the input signal x), but possibly different transmission rates in each band. Due to the independence assumption the variance of the whole system is

$$\sigma_{SBC}^2(\rho_l, \rho_h) = c 2^{-2\rho_l} \sigma_{l'}^2 + c 2^{-2\rho_h} \sigma_{h'}^2.$$

Now, the design goal is to minimize this variance under the additional constraint $\rho_l + \rho_h = 2\rho$. This leads directly to

$$\rho_l = 2\rho + \frac{1}{2} \log_2 \left(\frac{\sigma_{l'}^2}{\sqrt{\sigma_{l'}^2 \sigma_{h'}^2}} \right) \quad \text{and} \quad \rho_h = 2\rho + \frac{1}{2} \log_2 \left(\frac{\sigma_{h'}^2}{\sqrt{\sigma_{l'}^2 \sigma_{h'}^2}} \right)$$

such that the minimal variance of the whole system is

$$\sigma_{SBC}^2 = 2c2^{-2\rho}\sqrt{\sigma_{l'}^2\sigma_{h'}^2}.$$

Comparing this result with a *Pulse Code Modulation* (equals a one-band coding system) with transmission rate ρ we have

$$\sigma_{PCM}^2 = c2^{-2\rho}\sigma_x^2$$

which leads to the *coding gain*

$$G_{SBC} := \frac{\sigma_{PCM}^2}{\sigma_{SBC}^2} = \frac{\sigma_x^2}{2\sqrt{\sigma_{l'}^2\sigma_{h'}^2}}.$$

Moreover, due to our assumptions of the independence and the perfect reconstruction, we have $\sigma_x^2 = \sigma_{l'}^2 + \sigma_{h'}^2$ and the coding gain becomes

$$G_{SBC} = \frac{(\sigma_{l'}^2 + \sigma_{h'}^2)/2}{\sqrt{\sigma_{l'}^2\sigma_{h'}^2}}, \tag{9.8}$$

which has to maximized for the best filter. Due to the fixed sum $\sigma_{l'}^2 + \sigma_{h'}^2$ this maximization is equivalent to the maximization of $\sigma_{l'}^2$ if we assume $\rho_l > \rho_h$ in accordance with the lowpass/highpass interpretation.

Now there is only the question how can we calculate the variances $\sigma_{l'}^2, \sigma_{h'}^2$. For this we assume without loss of generality that all inputs are zero-mean random processes (cf., e.g., Usevitch, Orchard [59]) so that we have $m_x = E(x(n)) = 0$ for all n. Furthermore, $\sigma_{l'}^2$ is given by $\sigma_{l'}^2 = E((x_l'(n))^2) - (E(x_l'(n)))^2$. Determining both components we recall the definition

$$x_l'(n) = \sum_{k=0}^{2N-1} h_0^l(k)x(2n-k)$$

from (9.2) and, regarding $m_x = E(x(n)) = 0$, we infer

$$E(x_l'(n)) = m_x \sum_{k=0}^{2N-1} h_0^l(k) = 0$$

such that $\sigma_{l'}^2 = E((x_l'(n))^2)$. Additionally

$$(x_l'(n))^2 = \sum_{k=0}^{2N-1}\sum_{m=0}^{2N-1} h_0^l(k)h_0^l(m)x(2n-k)x(2n-m)$$

and consequently

$$E((x_l'(n))^2) = \sum_{k=0}^{2N-1}\sum_{m=0}^{2N-1} h_0^l(k)h_0^l(m)E(x(2n-k)x(2n-m)).$$

Due to the independence of $E(x(n)x(n-m))$ of n as well as $E(x(k)) = 0$ for all k we can set

$$r_m := R_{xx}(m) = E(x(n)x(n-m)) = \mathrm{Cov}(x(n), x(n-m))$$

so that we obtain

$$
\begin{aligned}
\sigma_{l'}^2 &= \sum_{k=0}^{2N-1} \sum_{m=0}^{2N-1} h_0^l(k) h_0^l(m) r_{|k-m|} \\
&= \sum_{k=0}^{2N-1} (h_0^l(k))^2 r_0 + 2 \sum_{k=1}^{2N-1} \sum_{m=0}^{2N-1-k} h_0^l(m) h_0^l(m+k) r_k.
\end{aligned}
$$

Using (9.4) and (9.5) we conclude

$$\sigma_{l'}^2 = r_0 + 2 \sum_{k=0}^{N-1} a_k r_{2k+1} \tag{9.9}$$

and, regarding $H_0(z) = -z^{-m} L_0(-z^{-1})$,

$$\sigma_{h'}^2 = r_0 - 2 \sum_{k=0}^{N-1} a_k r_{2k+1}.$$

Then, summing up the statements above and combining the objective function (9.9) with the feasible set described by (9.6), our design problem is

$$
\begin{aligned}
\text{maximize} \quad & r_0 + 2 \sum_{k=0}^{N-1} a_k r_{2k+1} \\
\text{s.t.} \quad & 1 + 2 \sum_{k=0}^{N-1} a_k \cos(2(2k+1)\pi\omega) \geq 0, \quad 0 \leq \omega \leq 0.5
\end{aligned}
$$

or, written as convex minimization problem,

$$
\begin{aligned}
\text{minimize} \quad & f(a) := -r_0 - 2 \sum_{k=0}^{N-1} a_k r_{2k+1} \\
\text{s.t.} \quad & g(a, \omega) := -1 - 2 \sum_{k=0}^{N-1} a_k \cos(2(2k+1)\pi\omega) \leq 0, \quad \omega \in T := [0, 0.5].
\end{aligned}
\tag{9.10}
$$

9.2 Numerical results

Before we consider several numerical examples let us check Assumption 4.1. The parts (1)-(4) are obviously fulfilled. Furthermore, the origin of $\mathbb{R}^N$ is an interior point of the feasible region since $g(0, \omega) = -1$ for all $\omega \in T$. Thus (5) is fulfilled. In order to show that (6) holds we first prove that the feasible region is bounded. For that let a feasible $a \in \mathbb{R}^N$ be fixed and consider the function

$g_a : \mathbb{R} \to \mathbb{R}$ defined by $g_a(\omega) := g(a, \omega)$. Then g_a is a periodic even function with period 1 and $g_a(\omega) \leq 0$ for all $\omega \in \mathbb{R}$ which follows from the feasibility of a for (9.10). Therefore the identity

$$g_a(0.5 - \omega) = -1 + 2 \sum_{k=0}^{N-1} a_k \cos(2(2k+1)\pi\omega) = -2 - g_a(\omega)$$

leads immediately to $g_a(\omega) \geq -2$ for all $\omega \in \mathbb{R}$ such that $|g_a(\omega)| \leq 2$ holds for all $\omega \in \mathbb{R}$. Now, considering the $(2N-1)$-th partial sum

$$\frac{c_0}{2} + \sum_{\nu=1}^{2N-1} c_\nu \cos(2\pi\nu\omega) + \sum_{\nu=1}^{2N-1} b_\nu \sin(2\pi\nu\omega)$$

of the Fourier series of g_a we have particularly (cf., e.g., Pinkus, Zafrany [36])

$$c_\nu = 2 \int_0^1 g_a(\omega) \cos(2\pi\nu\omega) d\omega$$

for $\nu = 0, \ldots, 2N - 1$. Thus, using the orthogonality property of the Cosinus-function, we obtain

$$c_{2k+1} = -4 \int_0^1 a_k \cos^2(2(2k+1)\pi\omega) d\omega = -2a_k$$

for all $k \in \{0, \ldots, N - 1\}$. Hence,

$$|a_k| \leq \int_0^1 |g_a(\omega)| d\omega \leq 2$$

for all $k \in \{0, \ldots, N - 1\}$ such that $\|a\|_\infty \leq 2$ follows. Consequently the feasible set of (9.10) is bounded. Further this set is closed since the involved functions are continuous. Therefore, in (9.10), we deal with a continuous objective function on a nonempty compact set so that the solution set is also nonempty and compact, i.e. part (6) holds. Moreover, due to the differentiability of $g(a, \omega)$, we can set

$$L_S^t := 4\pi \max_{a \in S} \sum_{m=1}^{N} (2m+1)|a_m| \geq \sup_{a \in S} \sup_{\omega \in T} \left| \frac{\partial g}{\partial \omega}(a, \omega) \right|$$

for each given compact set $S \subset \mathbb{R}^N$ so that (8) holds. Then the constants C_S can be computed by (6.4) if the constants L_S^x are given too. But these constants are simply given by

$$L_S^x := 2N \geq \sup_{x \in S} \sup_{\omega \in T} \sum_{m=0}^{N-1} \left| \frac{\partial g}{\partial x_m}(x, \omega) \right|.$$

Let us remark that both constants L_S^t and L_S^x can be improved in the sense of Remark 6.8. For that we make use of the fact that each summand of the respective gradient can be estimated more exact on subsets of T by estimating the appropriate Sinus- or Cosinus-term more exact. Additionally,

(10) holds since the involved functions f, g are differentiable and the required subgradients can be computed by differentiation. Altogether, regarding also the general remarks in the beginning of the chapter, Assumption 4.1 holds such that Algorithm 4.2 can be used for the design of a perfect reconstruction two-band filter bank.

Thus let us have a look at numerical examples. We consider the examples given in Kortanek, Moulin [30] and Moulin et al. [35] so that we deal with the following three cases:

1. AR(1)-process with $r_n = \rho^n$, $\rho = 0.95$;

2. AR(2)-process $r_n = 2\rho \cos \theta r_{n-1} - \rho^2 r_{n-2}$ with $r_0 = 1$, $r_1 = \frac{2\rho \cos \theta}{1+\rho^2}$, $\rho = 0.975$ and $\theta = \pi/3$;

3. lowpass process with box spectrum with $r_n = \frac{\sin(2\pi f_s n)}{2\pi f_s n}$, $f_s = 0.225$.

For the purpose of comparing the results with those of Kortanek, Moulin [30] and Moulin et al. [35] we also consider the cases $N = 4$ and $N = 10$. Additionally we consider $N = 14$ and the standard parameters are contained in Table 9.1. But, the restart procedure for $\varepsilon_{i,0}$ and δ_i described

parameter	start value	decreasing factor	lower bound
μ_i	1	0.2	10^{-5}
$\varepsilon_{i,0}$	0.001	0.15	–
δ_i	100	0.15	–
q_i	0.999	–	–

Table 9.1: filter design - standard parameter

for Example 7.1 was used again if insufficient accuracy values were detected. Furthermore we set $x^0 := 0 \in \mathbb{R}^N$ and the radii were computed by (7.2) with $h = h_{i,k-1}$ if $k > 1$ and $h = 0.0005$ if $k = 1$. Additionally all values $h_{i,k}$ were given as minimum of 0.0005 and the maximal value which fulfills (4.14).

Then we obtained for $N = 4$ the approximate solutions given in Table 9.2 which also includes the results of Kortanek, Moulin [30] (in the lower row of each process).

Thus we obtained similar results as presented by Kortanek, Moulin [30]. For the purpose of evaluating these results the last column of Table 9.2, containing the values of the coding gain, is of special interest since these values represent the improvement achieved by the application of the constructed two-band filter banks instead of transferring a single signal. The given values in the table are different from those calculated by (9.8) since they are now given in a logarithmic scale (*decibel*) as usual in the field of filter design, i.e. there is written down $10 \log_{10} G_{SBC}$. Nevertheless, the values computed by our algorithm are comparable with the results of Kortanek, Moulin [30] and Moulin et al. [35], whereby we should state that the slight difference is only caused by the fact that we stopped the algorithm with the barrier parameter 1.28E-5. If one computes approximate solutions for smaller barriers better results are the consequence. For instance, in the AR(2)-case we also computed a coding gain of 6.070 with barrier parameter 2.56E-6, but the additional step of

Process	Algorithm	a_0	a_1	a_2	a_3	Coding gain (in dB)
AR(1)	4.2	0.612048	-0.149279	0.045733	-0.008533	5.860
	cf. [30, 35]	0.612104	-0.149404	0.045859	-0.008577	5.862
AR(2)	4.2	0.594990	-0.193611	0.059889	-0.042127	6.069
	cf. [30, 35]	0.595198	-0.193416	0.060023	-0.042055	6.070
lowpass with	4.2	0.613735	-0.169685	0.072194	-0.026933	4.884
box-spectrum	cf. [30, 35]	0.613755	-0.169685	0.072184	-0.026923	4.885

Table 9.2: approximate solutions for $N = 4$

Algorithm 4.2 was disproportional costly due to too large grids T_h and small radii. Thus we did not compute results for this parameter in general.

In case $N = 10$ we obtained the coding gains 5.942 (instead of 5.945 by Kortanek, Moulin [30] and Moulin et al. [35]) for the AR(1)-process, 6.833 (6.835) for the AR(2)-process and 9.869 (9.879) for the lowpass process with box spectrum. Thus we can state that the quality of the filter designed by our algorithm for $N = 10$ is comparable to those of Kortanek, Moulin [30] and Moulin et al. [35].

In case $N = 14$ we obtained the coding gains 5.951 for the AR(1)-process, 6.920 for the AR(2)-process and 12.868 for the lowpass process with box spectrum so that the values of the coding gain increase when the number of variables or equivalently the signal length increases.

More details of all iteration processess are given in the appendix. But, finally, the frequency response in dB of the computed filters is plotted in the Figures 9.3, 9.4 and 9.5 for $N = 4$ (left-hand side) and $N = 10$ (right-hand side).
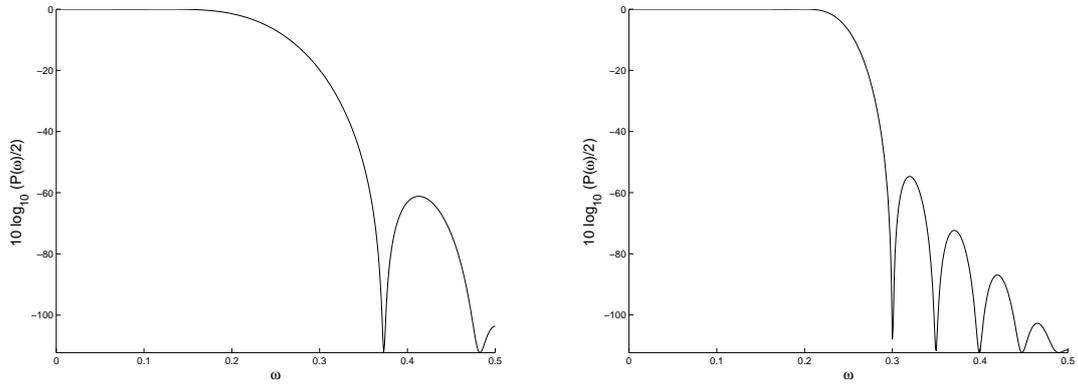
Figure 9.3: AR(1)-process
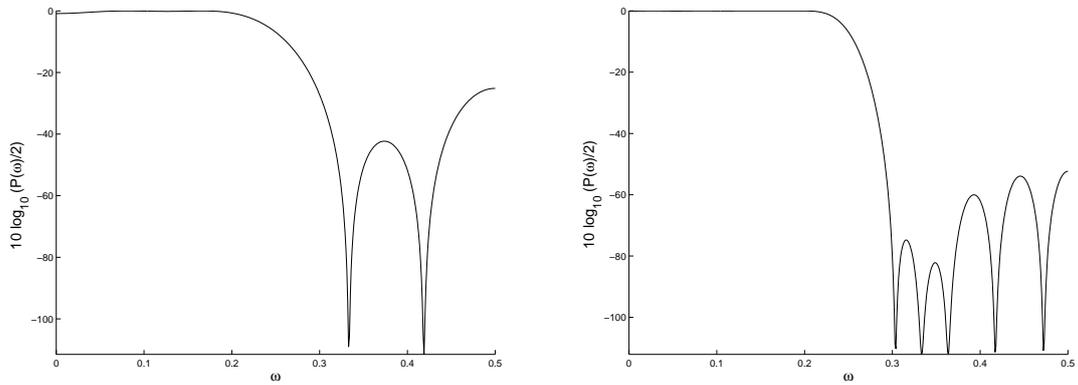
Figure 9.4: AR(2)-process
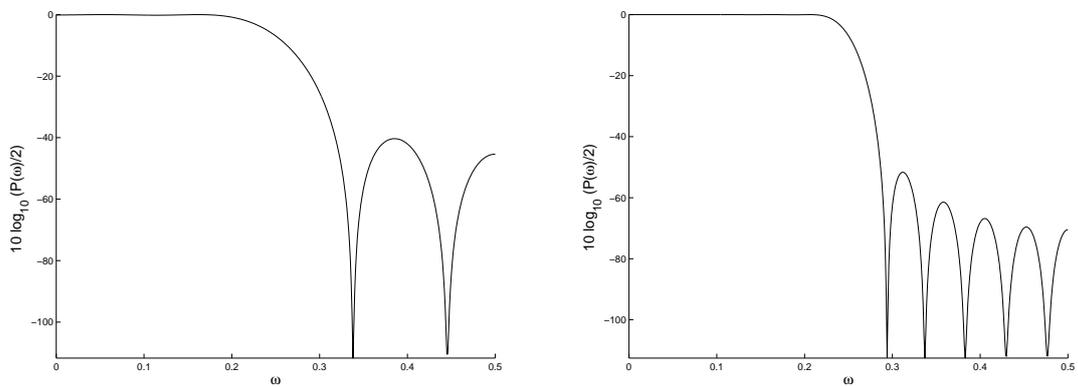
Figure 9.5: lowpass with box spectrum

Appendix A

Numerical results

In this appendix we state numerical results of the examples considered in the Chapters 7, 8 and 9 in tabularized form. In order to make the understanding of the tables easier they have of a certain structure. So the first column contains the dimension of the problem in the sense of its occurrence in the previous chapters. Then in the second column the used feasible starting vector is given followed by the exact optimal solution and the optimal value as far as they known. The next column contains the computed final approximate solution including its value of the objective function and in the case of a known solution set the distance to this set measured by the Euclidean norm is given in the "accuracy"-column. Apart from the column containing non-standard parameter values (emanated from the parameter settings given in the Chapters 7, 8 or 9) there occur two columns titled "effort" and "final values" which have to be specified in a more detailed way. So the "effort"-column has the general structure

<table>
<tr><td rowspan="4">restarts: #RES
#LP/#QP/#Box
$t_{LP}/t_{QP}/t_{MAX}$
t_{Total}</td><td rowspan="4">with</td><td>#RES : number of restarts of inner loops</td></tr>
<tr><td>#LP : number of solved linear programs</td></tr>
<tr><td>#QP : number of solved quadratic programs</td></tr>
<tr><td>#Box : number of considered boxes</td></tr>
</table>

#RES : number of restarts of inner loops

#LP : number of solved linear programs

#QP : number of solved quadratic programs

#Box : number of considered boxes

t_{LP} : time in seconds for solving all LP

t_{QP} : time in seconds for solving all QP

t_{MAX} : time in seconds for all maximizations

t_{Total} : total time in seconds for the complete iteration process

while the "final values"-column contains the final barrier parameter μ, the final radius r (which is mostly an indication for the smallest radius), the final prox-parameter s (if the regularized method was used), the average h_{av} of the grid constants of the final outer step, the minimal grid constant h_{min} and the average ratio of the the values $|\tilde{T}_{h_{i,k}}|/|T_{h_{i,k}}|$ or $|\tilde{T}_{h_{i,j,k}}|/|T_{h_{i,j,k}}|$ during the final outer step as measure for the final effectivity of the deletion rule.

n	start vector	exact solution	approximate solution	accuracy	effort	final values
1	$x_0 = 0$ $x_1 = 3$	$x_0 = 0.000000$ $x_1 = 1.000000$ $f(x) = 1$	$x_0 = -0.000000$ $x_1 = 1.000013$ $f(x) = 1.000013$	1.28E-05	restarts: 0 129/340/44 0.01 s/0.01 s/0.18 s 0.27 s	$\mu = 1.28\text{E-}05$ $r = 4.99\text{E-}06$ $h_{av} = 1.67\text{E-}06$ $h_{min} = 7.36\text{E-}07$ $|\tilde{T}_h|/|T_h| = 0.88$
2	$x_0 = 0$ $x_1 = 0$ $x_2 = 5$	$x_0 = 0.500000$ $x_1 = 0.000000$ $x_2 = 0.500000$ $f(x) = 0.5$	$x_0 = 0.500000$ $x_1 = -0.000000$ $x_2 = 0.500013$ $f(x) = 0.500013$	1.28E-05	restarts: 0 214/673/76 0.05 s/0.03 s/10.30 s 12.41 s	$\mu = 1.28\text{E-}05$ $r = 2.47\text{E-}06$ $h_{av} = 1.76\text{E-}06$ $h_{min} = 7.28\text{E-}07$ $|\tilde{T}_h|/|T_h| = 0.81$
3	$x_0 = 0$ $x_1 = 0$ $x_2 = 0$ $x_3 = 8$	$x_0 = 0.000000$ $x_1 = 0.750000$ $x_2 = 0.000000$ $x_3 = 0.250000$ $f(x) = 0.25$	$x_0 = 0.000000$ $x_1 = 0.750000$ $x_2 = -0.000000$ $x_3 = 0.250013$ $f(x) = 0.250013$	1.28E-05	restarts: 0 290/1054/106 0.13 s/0.02 s/0.52 s 0.85 s	$\mu = 1.28\text{E-}05$ $r = 2.54\text{E-}06$ $h_{av} = 1.42\text{E-}06$ $h_{min} = 5.89\text{E-}07$ $|\tilde{T}_h|/|T_h| = 0.08$
4	$x_0 = 0$ $x_1 = 0$ $x_2 = 0$ $x_3 = 0$ $x_4 = 14$	$x_0 = -0.125000$ $x_1 = 0.000000$ $x_2 = 1.000000$ $x_3 = 0.000000$ $x_4 = 0.125000$ $f(x) = 0.125$	$x_0 = -0.125000$ $x_1 = 0.000000$ $x_2 = 1.000000$ $x_3 = -0.000000$ $x_4 = 0.125013$ $f(x) = 0.125013$	1.27E-05	restarts: 0 445/1834/188 0.58 s/0.14 s/0.81 s 1.82 s	$\mu = 1.28\text{E-}05$ $r = 2.04\text{E-}06$ $h_{av} = 1.85\text{E-}06$ $h_{min} = 6.97\text{E-}07$ $|\tilde{T}_h|/|T_h| = 0.06$

Table A.1: Example 7.1 - $n = 1, 2, 3, 4$

n	start vector	exact solution	approximate solution	accuracy	effort	final values
5	$x_0 = 0$ $x_1 = 0$ $x_2 = 0$ $x_3 = 0$ $x_4 = 0$ $x_5 = 24$	$x_0 = 0.000000$ $x_1 = -0.312500$ $x_2 = 0.000000$ $x_3 = 1.250000$ $x_4 = 0.000000$ $x_5 = 0.062500$ $f(x) = 0.0625$	$x_0 = -0.000000$ $x_1 = -0.312500$ $x_2 = 0.000000$ $x_3 = 1.250000$ $x_4 = -0.000000$ $x_5 = 0.062513$ $f(x) = 0.062513$	1.28E-05	restarts: 1 647/3197/352 1.34 s/0.45 s/5.46 s 7.97 s	$\mu = 1.28\text{E-}05$ $r = 1.73\text{E-}06$ $h_{av} = 4.18\text{E-}07$ $h_{\min} = 1.46\text{E-}07$ $|\tilde{T}_h|/|T_h| = 0.10$
6	$x_0 = 0$ $x_1 = 0$ $x_2 = 0$ $x_3 = 0$ $x_4 = 0$ $x_5 = 0$ $x_6 = 49$	$x_0 = 0.031250$ $x_1 = 0.000000$ $x_2 = -0.562500$ $x_3 = 0.000000$ $x_4 = 1.500000$ $x_5 = 0.000000$ $x_6 = 0.031250$ $f(x) = 0.03125$	$x_0 = 0.031250$ $x_1 = -0.000000$ $x_2 = -0.562500$ $x_3 = 0.000000$ $x_4 = 1.500000$ $x_5 = -0.000000$ $x_6 = 0.031263$ $f(x) = 0.031263$	1.28E-05	restarts: 0 1316/8456/1180 5.23 s/2.08 s/13.68 s 27.11 s	$\mu = 1.28\text{E-}05$ $r = 1.44\text{E-}06$ $h_{av} = 8.41\text{E-}07$ $h_{\min} = 2.95\text{E-}07$ $|\tilde{T}_h|/|T_h| = 0.06$
7	$x_0 = 0$ $x_1 = 0$ $x_2 = 0$ $x_3 = 0$ $x_4 = 0$ $x_5 = 0$ $x_6 = 0$ $x_7 = 80$	$x_0 = 0.000000$ $x_1 = 0.109375$ $x_2 = 0.000000$ $x_3 = -0.875000$ $x_4 = 0.000000$ $x_5 = 1.750000$ $x_6 = 0.000000$ $x_7 = 0.015625$ $f(x) = 0.015625$	$x_0 = 0.000000$ $x_1 = 0.109375$ $x_2 = -0.000000$ $x_3 = -0.875000$ $x_4 = 0.000000$ $x_5 = 1.750000$ $x_6 = -0.000000$ $x_7 = 0.015638$ $f(x) = 0.015638$	1.28E-05	restarts: 0 1913/16474/2357 12.11 s/4.24 s/63.03 s 105.00 s	$\mu = 1.28\text{E-}05$ $r = 1.28\text{E-}06$ $h_{av} = 2.83\text{E-}07$ $h_{\min} = 9.38\text{E-}08$ $|\tilde{T}_h|/|T_h| = 0.09$

Table A.2: Example 7.1 - $n = 5, 6, 7$

n	start vector	exact solution	approximate solution	accuracy	effort	non-standard parameter	final values
8	$x_0 = 0$ $x_1 = 0$ $x_2 = 0$ $x_3 = 0$ $x_4 = 0$ $x_5 = 0$ $x_6 = 0$ $x_7 = 0$ $x_8 = 149$	$x_0 = -0.007813$ $x_1 = 0.000000$ $x_2 = 0.250000$ $x_3 = 0.000000$ $x_4 = -1.250000$ $x_5 = 0.000000$ $x_6 = 2.000000$ $x_7 = 0.000000$ $x_8 = 0.007813$ $f(x) = 0.007813$	$x_0 = -0.007812$ $x_1 = 0.000000$ $x_2 = 0.250000$ $x_3 = -0.000001$ $x_4 = -1.249999$ $x_5 = 0.000002$ $x_6 = 1.999999$ $x_7 = -0.000001$ $x_8 = 0.007832$ $f(x) = 0.007832$	1.97E-05	restarts: 1 3764/31661/5205 27.88 s/8.45 s/164.68 s 309.69 s	$\mu_{i+1} = 0.3\mu_i$ $\varepsilon_{i+1,1} = 0.27\varepsilon$	$\mu = 1.97\text{E-}05$ $r = 1.70\text{E-}06$ $h_{av} = 5.64\text{E-}07$ $h_{min} = 2.10\text{E-}07$ $|\tilde{T}_h|/|T_h| = 0.06$
9	$x_0 = 0$ $x_1 = 0$ $x_2 = 0$ $x_3 = 0$ $x_4 = 0$ $x_5 = 0$ $x_6 = 0$ $x_7 = 0$ $x_8 = 0$ $x_9 = 276$	$x_0 = 0.000000$ $x_1 = -0.035156$ $x_2 = 0.000000$ $x_3 = 0.468750$ $x_4 = 0.000000$ $x_5 = -1.687500$ $x_6 = 0.000000$ $x_7 = 2.250000$ $x_8 = 0.000000$ $x_9 = 0.003906$ $f(x) = 0.003906$	$x_0 = -0.000000$ $x_1 = -0.035156$ $x_2 = 0.000000$ $x_3 = 0.468750$ $x_4 = -0.000000$ $x_5 = -1.687500$ $x_6 = 0.000000$ $x_7 = 2.250000$ $x_8 = -0.000000$ $x_9 = 0.003923$ $f(x) = 0.003923$	1.67E-05	restarts: 1 3753/448436/11466 39.32 s/11.81 s/642.86 s 1333.23 s	$\mu_{i+1} = 0.4\mu_i$ $\varepsilon_{i+1,1} = 0.38\varepsilon$	$\mu = 1.68\text{E-}05$ $r = 1.30\text{E-}06$ $h_{av} = 2.24\text{E-}07$ $h_{min} = 8.64\text{E-}08$ $|\tilde{T}_h|/|T_h| = 0.07$

Table A.3: Example 7.1 - $n = 8, 9$.

n	start vector	exact solution	approximate solution	accuracy	effort	final values
3	$x^0 = 0$	$x_1 + \min\left\{\sqrt{2}x_2 + \frac{\sqrt{2}}{2}x_2\right\} \geq 0$ $x_1 + x_2 = 0.666667$ $x_3 = 0.666667$ $f^* = -1.333333$	$x_1 = 0.356345$ $x_2 = 0.308748$ $x_1 + x_2 = 0.665093$ $x_3 = 0.668226$ $f(x) = -1.333319$	1.92E-03	restarts: 1 116/261/42 0.01 s/0.00 s/0.02 s 0.08 s	$\mu = 1.28E-05$ $r = 3.60E-06$ $h_{av} = 2.46E-06$ $h_{\min} = 1.12E-06$ $|\tilde{T}_h|/|T_h| = 0.01$
4	$x^0 = 0$	$x_1 + \min\left\{\sqrt{2}x_2 + \frac{\sqrt{2}}{2}x_2\right\} \geq 0$ $x_1 + x_2 = 0.354167$ $x_3 = 0.666667$ $x_4 = 0.625000$ $f^* = -1.645833$	$x_1 = 0.185472$ $x_2 = 0.167095$ $x_1 + x_2 = 0.352567$ $x_3 = 0.667412$ $x_4 = 0.625841$ $f(x) = -1.645820$	1.59E-03	restarts: 0 107/221/36 0.02 s/0.01 s/0.02 s 0.07 s	$\mu = 1.28E-05$ $r = 2.26E-06$ $h_{av} = 7.09E-06$ $h_{\min} = 3.17E-06$ $|\tilde{T}_h|/|T_h| = 0.02$
5	$x^0 = 0$	$x_1 + \min\left\{\sqrt{2}x_2 + \frac{\sqrt{2}}{2}x_2\right\} \geq 0$ $x_1 + x_2 = 0.054167$ $x_3 = 0.666667$ $x_4 = 0.625000$ $x_5 = 0.600000$ $f^* = -1.945833$	$x_1 = 0.028449$ $x_2 = 0.025910$ $x_1 + x_2 = 0.054359$ $x_3 = 0.666585$ $x_4 = 0.624934$ $x_5 = 0.599944$ $f(x) = -1.945821$	1.80E-04	restarts: 0 104/214/40 0.05 s/0.01 s/0.02 s 0.10 s	$\mu = 1.28E-05$ $r = 2.03E-06$ $h_{av} = 4.96E-05$ $h_{\min} = 2.28E-05$ $|\tilde{T}_h|/|T_h| = 0.10$

Table A.4: Example 7.2 - $n = 3, 4, 5$

n	start vector	exact solution	approximate solution	accuracy	effort	final values
6	$x^0 = 0$	$x_1 = 0.000000$ $x_2 = 0.000000$ $x_3 = 0.599289$ $x_4 = 0.561833$ $x_5 = 0.539360$ $x_6 = 0.524378$ $f^* = -2.224860$	$x_1 = 0.000000$ $x_2 = 0.000000$ $x_3 = 0.599307$ $x_4 = 0.561800$ $x_5 = 0.539335$ $x_6 = 0.524405$ $f(x) = -2.224847$	5.30E-05	restarts: 0 138/419/45 0.13 s/0.01 s/0.11 s 0.29 s	$\mu = 1.28$E-05 $r = 1.86$E-06 $h_{av} = 1.00$E-03 $h_{min} = 1.00$E-03 $\lvert \tilde{T}_h\rvert/\lvert T_h\rvert = 1.00$
8	$x^0 = 0$	$x_1 = 0.000000$ $x_2 = 0.000000$ $x_3 = 0.496289$ $x_4 = 0.465271$ $x_5 = 0.446660$ $x_6 = 0.434253$ $x_7 = 0.425391$ $x_8 = 0.418744$ $f^* = -2.686607$	$x_1 = 0.000000$ $x_2 = 0.000000$ $x_3 = 0.496274$ $x_4 = 0.465293$ $x_5 = 0.446681$ $x_6 = 0.434272$ $x_7 = 0.425348$ $x_8 = 0.418726$ $f(x) = -2.686594$	6.04E-05	restarts: 0 130/414/44 0.10 s/0.04 s/0.18 s 0.38 s	$\mu = 1.28$E-05 $r = 1.33$E-06 $h_{av} = 1.00$E-03 $h_{min} = 1.00$E-03 $\lvert \tilde{T}_h\rvert/\lvert T_h\rvert = 1.00$
10	$x^0 = 0$	$x_1 = 0.000000$ $x_2 = 0.000000$ $x_3 = 0.434217$ $x_4 = 0.407078$ $x_5 = 0.390795$ $x_6 = 0.379940$ $x_7 = 0.372186$ $x_8 = 0.366370$ $x_9 = 0.361847$ $x_{10} = 0.358229$ $f^* = -3.070663$	$x_1 = 0.000000$ $x_2 = 0.000000$ $x_3 = 0.434155$ $x_4 = 0.407066$ $x_5 = 0.390782$ $x_6 = 0.379961$ $x_7 = 0.372187$ $x_8 = 0.366369$ $x_9 = 0.361875$ $x_{10} = 0.358255$ $f(x) = -3.070650$	7.77E-05	restarts: 0 176/634/72 0.47 s/0.05 s/0.19 s 0.83 s	$\mu = 1.28$E-05 $r = 1.04$E-06 $h_{av} = 1.00$E-03 $h_{min} = 1.00$E-03 $\lvert \tilde{T}_h\rvert/\lvert T_h\rvert = 1.00$
12	$x^0 = 0$	$x_1 = 0.000000$ $x_2 = 0.000000$ $x_3 = 0.391426$ $x_4 = 0.366962$ $x_5 = 0.352283$ $x_6 = 0.342498$ $x_7 = 0.335508$ $x_8 = 0.330266$ $x_9 = 0.326188$ $x_{10} = 0.322926$ $x_{11} = 0.320258$ $x_{12} = 0.318034$ $f^* = -3.406349$	$x_1 = 0.000000$ $x_2 = 0.000000$ $x_3 = 0.391268$ $x_4 = 0.366976$ $x_5 = 0.352305$ $x_6 = 0.342520$ $x_7 = 0.335527$ $x_8 = 0.330283$ $x_9 = 0.326204$ $x_{10} = 0.322940$ $x_{11} = 0.320269$ $x_{12} = 0.318043$ $f(x) = -3.406336$	1.65E-04	restarts: 0 228/989/120 0.88 s/0.21 s/0.28 s 1.53 s	$\mu = 1.28$E-05 $r = 8.64$E-07 $h_{av} = 1.00$E-03 $h_{min} = 1.00$E-03 $\lvert \tilde{T}_h\rvert/\lvert T_h\rvert = 1.00$

Table A.5: Example 7.2 - $n = 6, 8, 10, 12$

n	start vector	exact solution	approximate solution	accuracy	effort	non-stand. param.	final values
14	$x^0 = 0$	$x_1 = 0.026169$ $x_2 = -0.126357$ $x_3 = 0.369398$ $x_4 = 0.346311$ $x_5 = 0.332458$ $x_6 = 0.323223$ $x_7 = 0.316627$ $x_8 = 0.311680$ $x_9 = 0.307832$ $x_{10} = 0.304753$ $x_{11} = 0.302235$ $x_{12} = 0.300136$ $x_{13} = 0.298360$ $x_{14} = 0.296838$ $f^* = -3.709663$	$x_1 = 0.027115$ $x_2 = -0.130925$ $x_3 = 0.369748$ $x_4 = 0.346639$ $x_5 = 0.332773$ $x_6 = 0.323529$ $x_7 = 0.316927$ $x_8 = 0.311975$ $x_9 = 0.308123$ $x_{10} = 0.305041$ $x_{11} = 0.302520$ $x_{12} = 0.300419$ $x_{13} = 0.298642$ $x_{14} = 0.297118$ $f(x) = -3.709644$	4.78E-03	restarts: 1 1293/8027/3143 7.42 s/0.95 s/0.65 s 12.70 s	$\mu_{i+1} = 0.4\mu_i$ $\varepsilon_{i+1,0} = 0.33\varepsilon_i$ $\delta_{i+1} = 0.33\delta_i$	$\mu = 1.68\text{E-}05$ $r = 1.03\text{E-}06$ $h_{av} = 2.76\text{E-}06$ $h_{\min} = 1.16\text{E-}06$ $\lvert\tilde{T}_h\rvert/\lvert T_h\rvert = 0.05$
15	$x^0 = 0$	$x_1 = 0.064765$ $x_2 = -0.312711$ $x_3 = 0.369398$ $x_4 = 0.346311$ $x_5 = 0.332458$ $x_6 = 0.323223$ $x_7 = 0.316627$ $x_8 = 0.311680$ $x_9 = 0.307832$ $x_{10} = 0.304753$ $x_{11} = 0.302235$ $x_{12} = 0.300136$ $x_{13} = 0.298360$ $x_{14} = 0.296838$ $x_{15} = 0.295518$ $f^* = -3.857422$	$x_1 = 0.065839$ $x_2 = -0.317899$ $x_3 = 0.369768$ $x_4 = 0.346657$ $x_5 = 0.332790$ $x_6 = 0.323546$ $x_7 = 0.316943$ $x_8 = 0.311990$ $x_9 = 0.308139$ $x_{10} = 0.305057$ $x_{11} = 0.302536$ $x_{12} = 0.300435$ $x_{13} = 0.298657$ $x_{14} = 0.297133$ $x_{15} = 0.295813$ $f(x) = -3.857404$	5.42E-03	restarts: 1 1635/9942/3885 10.19 s/1.41 s/1.04 s 17.80 s	$\mu_{i+1} = 0.4\mu_i$ $\varepsilon_{i+1,0} = 0.33\varepsilon_i$ $\delta_{i+1} = 0.33\delta_i$	$\mu = 1.68\text{E-}05$ $r = 9.74\text{E-}07$ $h_{av} = 9.14\text{E-}07$ $h_{\min} = 2.54\text{E-}07$ $\lvert\tilde{T}_h\rvert/\lvert T_h\rvert = 0.03$

Table A.6: Example 7.2 - $n = 14, 15$

n	start vector	exact solution	approximate solution	accuracy	effort	final values
3	$x_0 = 0$ $x_1 = 0$ $x_2 = 0$ $x_3 = 0$ $x_4 = 8$	$x_0 = 0.000000$ $x_1 =$ $x_1 + x_3 = 0.750000$ $x_2 + x_3 = 0.000000$ $x_3 \in \mathbb{R}$ $x_4 = 0.250000$ $f(x) = 0.25$	$x_0 = 0.000000$ $x_1 = 0.492192$ $x_1 + x_3 = 0.750000$ $x_2 + x_3 = -0.257808$ $x_3 = 0.257808$ $x_4 = 0.250016$ $f(x) = 0.250016$	1.61E-05	restarts: 2 435/978/217 0.23 s/0.08 s/0.59 s 1.17 s	$\mu = 1.60E\text{-}05$ $r = 1.65E\text{-}06$ $s_i = 3.13E\text{-}04$ $h_{av} = 1.71E\text{-}06$ $h_{min} = 6.79E\text{-}07$ $|\tilde{T_h}|/|T_h| = 0.08$
4	$x_0 = 0$ $x_1 = 0$ $x_2 = 0$ $x_3 = 0$ $x_4 = 0$ $x_5 = 14$	$x_0 = -0.125000$ $x_1 = -0.338274$ $x_1 + x_4 = 0.000000$ $x_2 + x_4 = 1.000000$ $x_3 = 0.000000$ $x_4 \in \mathbb{R}$ $x_5 = 0.125000$ $f(x) = 0.125$	$x_0 = -0.125000$ $x_1 = -0.338274$ $x_1 + x_4 = 0.000001$ $x_2 = 0.661725$ $x_2 + x_4 = 1.000000$ $x_3 = -0.000000$ $x_4 = 0.338275$ $x_5 = 0.125016$ $f(x) = 0.125016$	1.59E-05	restarts: 2 712/1660/384 0.69 s/0.19 s/1.72 s 3.16 s	$\mu = 1.60E\text{-}05$ $r = 1.43E\text{-}06$ $s_i = 3.13E\text{-}04$ $h_{av} = 1.61E\text{-}06$ $h_{min} = 6.47E\text{-}07$ $|\tilde{T_h}|/|T_h| = 0.07$

Table A.7: Example 7.4 - $n = 3, 4$, $k = 1$

n	start vector	exact solution	approximate solution	accuracy	effort	final values
5	$x_0 = 0$ $x_1 = 0$ $x_2 = 0$ $x_3 = 0$ $x_4 = 0$ $x_5 = 0$ $x_6 = 24$	$x_0 = 0.000000$ $x_1 + x_5 = -0.312500$ $x_2 + x_5 = 0.000000$ $x_3 = 1.250000$ $x_4 = 0.000000$ $x_5 \in \mathbb{R}$ $x_6 = 0.062500$ $f(x) = 0.0625$	$x_0 = -0.000000$ $x_1 = -0.214367$ $x_1 + x_5 = -0.312500$ $x_2 = 0.098133$ $x_2 + x_5 = 0.000000$ $x_3 = 1.250000$ $x_4 = -0.000000$ $x_5 = -0.098133$ $x_6 = 0.062516$ $f(x) = 0.062516$	1.60E-05	restarts: 2 1445/3488/946 1.85 s/0.45 s/4.02 s 7.71 s	$\mu = 1.60\text{E-05}$ $r = 1.27\text{E-06}$ $s_i = 3.13\text{E-04}$ $h_{av} = 1.84\text{E-06}$ $h_{\min} = 7.06\text{E-07}$ $|\tilde{T}_h|/|T_h| = 0.04$
6	$x_0 = 0$ $x_1 = 0$ $x_2 = 0$ $x_3 = 0$ $x_4 = 0$ $x_5 = 0$ $x_6 = 0$ $x_7 = 49$	$x_0 = 0.031250$ $x_1 + x_6 = 0.000000$ $x_2 + x_6 = -0.562500$ $x_3 = 0.000000$ $x_4 = 1.500000$ $x_5 = 0.000000$ $x_6 \in \mathbb{R}$ $x_7 = 0.031250$ $f(x) = 0.03125$	$x_0 = 0.031250$ $x_1 = 0.185106$ $x_1 + x_6 = 0.000000$ $x_2 = -0.377394$ $x_2 + x_6 = -0.562500$ $x_3 = 0.000000$ $x_4 = 1.500000$ $x_5 = -0.000000$ $x_6 = -0.185106$ $x_7 = 0.031266$ $f(x) = 0.031266$	1.59E-05	restarts: 2 4202/12002/3115 8.67 s/1.86 s/42.21 s 88.89 s	$\mu = 1.60\text{E-05}$ $r = 1.18\text{E-06}$ $s_i = 3.13\text{E-04}$ $h_{av} = 1.37\text{E-06}$ $h_{\min} = 4.16\text{E-07}$ $|\tilde{T}_h|/|T_h| = 0.04$

Table A.8: Example 7.4 - $n = 5, 6, k = 1$

n	start vector	exact solution	approximate solution	accuracy	effort	final values
5	$x_0 = 0$ $x_1 = 0$ $x_2 = 0$ $x_3 = 0$ $x_4 = 0$ $x_5 = 0$ $x_6 = 24$	$x_0 = 0.000000$ $x_1 = -0.312500$ $x_2 + x_5 = 0.000000$ $x_3 + x_5 = 1.250000$ $x_4 = 0.000000$ $x_5 \in \mathbb{R}$ $x_6 = 0.062500$ $f(x) = 0.0625$	$x_0 = -0.000000$ $x_1 = -0.312500$ $x_2 + x_5 = -0.421595$ $x_3 + x_5 = 1.250000$ $x_4 = -0.000000$ $x_5 = 0.421595$ $x_6 = 0.062516$ $f(x) = 0.062516$	1.60E-05	restarts: 2 1400/3502/919 1.54 s/0.42 s/4.83 s 8.50 s	$\mu = 1.60\text{E-}05$ $r = 1.27\text{E-}06$ $s_i = 3.13\text{E-}04$ $h_{av} = 1.28\text{E-}04$ $h_{\min} = 4.77\text{E-}07$ $|\tilde{T}_h|/|T_h| = 0.05$
6	$x_0 = 0$ $x_1 = 0$ $x_2 = 0$ $x_3 = 0$ $x_4 = 0$ $x_5 = 0$ $x_6 = 0$ $x_7 = 49$	$x_0 = 0.031250$ $x_1 = 0.000000$ $x_2 + x_6 = -0.562500$ $x_3 + x_6 = 0.000000$ $x_4 = 1.500000$ $x_5 = 0.000000$ $x_6 \in \mathbb{R}$ $x_7 = 0.031250$ $f(x) = 0.03125$	$x_0 = 0.031250$ $x_1 = -0.000000$ $x_2 = -0.378013$ $x_3 = 0.184488$ $x_4 = 1.499999$ $x_5 = -0.000001$ $x_6 = -0.184487$ $x_7 = 0.031266$ $f(x) = 0.031266$	1.61E-05	restarts: 2 4383/12708/3156 8.80 s/1.91 s/40.96 s 86.20 s	$\mu = 1.60\text{E-}05$ $r = 1.20\text{E-}06$ $s_i = 3.13\text{E-}04$ $h_{av} = 1.34\text{E-}04$ $h_{\min} = 3.58\text{E-}07$ $|\tilde{T}_h|/|T_h| = 0.04$

Table A.9: Example 7.4 - $n = 5, 6$, $k = 2$

n	start vector	exact solution	approximate solution	accuracy	effort	final values
5	$x_0 = 0$ $x_1 = 0$ $x_2 = 0$ $x_3 = 0$ $x_4 = 0$ $x_5 = 0$ $x_6 = 24$	$x_0 = 0.000000$ $x_1 = -0.312500$ $x_2 = 0.000000$ $x_3 + x_5 = 1.250000$ $x_4 + x_5 = 0.000000$ $x_5 \in \mathbb{R}$ $x_6 = 0.062500$ $f(x) = 0.0625$	$x_0 = -0.000000$ $x_1 = -0.312500$ $x_2 = 0.000000$ $x_3 = 0.832594$ $x_3 + x_5 = 1.250000$ $x_4 = -0.417406$ $x_4 + x_5 = 0.000000$ $x_5 = 0.417406$ $x_6 = 0.062516$ $f(x) = 0.062516$	1.58E-05	restarts: 2 2410/5891/1455 3.15 s/0.78 s/11.94 s 24.28 s	$\mu = 1.60\text{E-}05$ $r = 1.24\text{E-}06$ $s_i = 3.13\text{E-}04$ $h_{av} = 1.31\text{E-}06$ $h_{\min} = 3.80\text{E-}07$ $|\tilde{T}_h|/|T_h| = 0.03$
6	$x_0 = 0$ $x_1 = 0$ $x_2 = 0$ $x_3 = 0$ $x_4 = 0$ $x_5 = 0$ $x_6 = 0$ $x_7 = 49$	$x_0 = 0.031250$ $x_1 = 0.000000$ $x_2 = -0.562500$ $x_3 + x_6 = 0.000000$ $x_4 + x6 = 1.500000$ $x_5 = 0.000000$ $x_6 \in \mathbb{R}$ $x_7 = 0.031250$ $f(x) = 0.03125$	$x_0 = 0.031250$ $x_1 = 0.000000$ $x_2 = -0.562500$ $x_3 = -0.501085$ $x_3 + x_6 = 0.000001$ $x_4 = 0.998914$ $x_4 + x_6 = 1.500000$ $x_5 = -0.000001$ $x_6 = 0.501086$ $x_7 = 0.031266$ $f(x) = 0.031266$	1.63E-05	restarts: 2 4530/12498/3096 8.52 s/2.13 s/50.45 s 100.20 s	$\mu = 1.60\text{E-}05$ $r = 1.16\text{E-}06$ $s_i = 3.13\text{E-}04$ $h_{av} = 1.11\text{E-}06$ $h_{\min} = 3.16\text{E-}07$ $|\tilde{T}_h|/|T_h| = 0.04$

Table A.10: Example 7.4 - $n = 5, 6, k = 3$

n	start vector	exact solution	approximate solution	accuracy	effort	non-standard parameters	final values
5	$x^0 = 0$	$x_1 \geq 0$ $x_i = 0.500000$ $(i = 2,\ldots,5)$ $f^* = -2$	$x_1 = 0.056616$ $x_2 = 0.301456$ $x_3 = 0.499985$ $x_4 = 0.500003$ $x_5 = 0.500027$ $f(x) = -1.999984$	4.38E-05	restarts: 3 82/114/34 0.03 s/0.01 s/0.03 s 0.20 s	$\mu_1 = 0.05$	$\mu = 1.60E\text{-}05$ $r = 3.34E\text{-}06$ $s_i = 3.13E\text{-}04$ $h_{av} = 1.48E\text{-}05$ $h_{min} = 6.88E\text{-}06$ $|\tilde{T}_h|/|T_h| = 0.05$
12	$x^0 = 0$	$x_1 \geq 0$ $x_i = 0.301511$ $(i = 2,\ldots,12)$ $f^* = -3.316625$	$x_1 = 0.000516$ $x_2 = 0.301456$ $x_3 = 0.301461$ $x_4 = 0.301468$ $x_5 = 0.301477$ $x_6 = 0.301488$ $x_7 = 0.301500$ $x_8 = 0.301515$ $x_9 = 0.301531$ $x_{10} = 0.301550$ $x_{11} = 0.301570$ $x_{12} = 0.301593$ $f(x) = -3.316609$	1.46E-04	restarts: 2 93/118/37 0.19 s/0.00 s/0.04 s 0.55 s	$\mu_1 = 0.05$	$\mu = 1.28E\text{-}05$ $r = 3.05E\text{-}07$ $s_i = 3.13E\text{-}04$ $h_{av} = 5.49E\text{-}06$ $h_{min} = 2.14E\text{-}06$ $|\tilde{T}_h|/|T_h| = 0.03$
20	$x^0 = 0$	$x_1 \geq 0$ $x_i = 0.229416$ $(i = 2,\ldots,20)$ $f^* = -4.358899$	$x_1 = 0.004671$ $x_2 = 0.229202$ $x_3 = 0.229209$ $x_4 = 0.229220$ $x_5 = 0.229233$ $x_6 = 0.229248$ $x_7 = 0.229267$ $x_8 = 0.229289$ $x_9 = 0.229314$ $x_{10} = 0.229341$ $x_{11} = 0.229371$ $x_{12} = 0.229405$ $x_{13} = 0.229441$ $x_{14} = 0.229480$ $x_{15} = 0.229522$ $x_{16} = 0.229567$ $x_{17} = 0.229615$ $x_{18} = 0.229666$ $x_{19} = 0.229719$ $x_{20} = 0.229776$ $f(x) = -4.358885$	7.79E-04	restarts: 2 83/107/41 0.46 s/0.00 s/0.10 s 1.32 s	$\mu_1 = 0.04$	$\mu = 1.28E\text{-}05$ $r = 3.05E\text{-}07$ $s_i = 3.13E\text{-}04$ $h_{av} = 2.37E\text{-}05$ $h_{min} = 8.88E\text{-}05$ $|\tilde{T}_h|/|T_h| = 0.00$

Table A.11: Example 7.5 - $n = 5, 12, 20, k = 1$

n	start vector	exact solution	approximate solution	accuracy	effort	non-standard parameters	final values
6	$x^0 = 0$	$x_i \geq 0$ $(i = 1,2)$ $x_i = 0.500000$ $(i = 3,\ldots,6)$ $f^* = -2$	$x_1 = 0.097618$ $x_2 = 0.065079$ $x_3 = 0.499974$ $x_4 = 0.499980$ $x_5 = 0.499987$ $x_6 = 0.499996$ $f(x) = -1.999936$	3.56E-05	restarts: 2 74/99/35 0.05 s/0.00 s/0.02 s 0.23 s	$\mu_1 = 0.02$	$\mu = 3.20E-05$ $r = 1.22E-05$ $s_i = 6.25E-04$ $h_{av} = 2.37E-05$ $h_{\min} = 1.23E-05$ $|\tilde{T}_h|/|T_h| = 0.07$
12	$x^0 = 0$	$x_i \geq 0$ $(i = 1,2)$ $x_i = 0.316228$ $(i = 3,\ldots,12)$ $f^* = -3.162278$	$x_1 = 0.042213$ $x_2 = 0.028142$ $x_3 = 0.316211$ $x_4 = 0.316213$ $x_5 = 0.316215$ $x_6 = 0.316218$ $x_7 = 0.316221$ $x_8 = 0.316225$ $x_9 = 0.316229$ $x_{10} = 0.316234$ $x_{11} = 0.316239$ $x_{12} = 0.316245$ $f(x) = -3.162252$	3.56E-05	restarts: 3 72/104/37 0.16 s/0.00 s/0.10 s 0.81 s	$\mu_1 = 0.008$	$\mu = 1.28E-05$ $r = 5.95E-07$ $s_i = 6.25E-04$ $h_{av} = 7.52E-06$ $h_{\min} = 1.19E-07$ $|\tilde{T}_h|/|T_h| = 0.01$
20	$x^0 = 0$	$x_i \geq 0$ $(i = 1,2)$ $x_i = 0.235702$ $(i = 3,\ldots,20)$ $f^* = -4.242641$	$x_1 = 0.024927$ $x_2 = 0.016618$ $x_3 = 0.235618$ $x_4 = 0.235621$ $x_5 = 0.235626$ $x_6 = 0.235632$ $x_7 = 0.235639$ $x_8 = 0.235647$ $x_9 = 0.235656$ $x_{10} = 0.235667$ $x_{11} = 0.235678$ $x_{12} = 0.235691$ $x_{13} = 0.235704$ $x_{14} = 0.235719$ $x_{15} = 0.235734$ $x_{16} = 0.235751$ $x_{17} = 0.235769$ $x_{18} = 0.235788$ $x_{19} = 0.235808$ $x_{20} = 0.235829$ $f(x) = -4.242578$	2.80E-04	restarts: 1 82/126/57 0.56 s/0.00 s/0.15 s 1.44 s	$\mu_1 = 0.004$	$\mu = 3.20E-05$ $r = 7.55E-07$ $s_i = 1.25E-03$ $h_{av} = 9.30E-05$ $h_{\min} = 1.54E-05$ $|\tilde{T}_h|/|T_h| = 0.12$

Table A.12: Example 7.5 - $n = 6, 12, 20, k = 2$

n	start vector	exact solution	approximate solution	accuracy	effort	non-standard parameters	final values
7	$x^0 = 0$	$x_i \geq 0$ $(i = 1, 2, 3)$ $x_i = 0.500000$ $(i = 4, \ldots, 7)$ $f^* = -2$	$x_1 = 0.104093 \quad x_2 = 0.080503$ $x_3 = 0.060378 \quad x_4 = 0.499976$ $x_5 = 0.499983 \quad x_6 = 0.499992$ $x_7 = 0.500002$ $f(x) = -1.999953$	3.02E-05	restarts: 2 $82/115/36$ 0.05 s/0.00 s/0.07 s 0.40 s	$\mu_1 = 0.01$	$\mu = 1.60\text{E-}05$ $\tau = 9.71\text{E-}06$ $s_i = 6.25\text{E-}04$ $h_{av} = 2.12\text{E-}05$ $h_{min} = 7.12\text{E-}06$ $\lvert \tilde{T}_h \rvert / \lvert T_h \rvert = 0.06$
12	$x^0 = 0$	$x_i \geq 0$ $(i = 1, 2, 3)$ $x_i = 0.333333$ $(i = 4, \ldots, 12)$ $f^* = -3$	$x_1 = 0.049793 \quad x_2 = 0.043036$ $x_3 = 0.032277 \quad x_4 = 0.333305$ $x_5 = 0.333307 \quad x_6 = 0.333310$ $x_7 = 0.333314 \quad x_8 = 0.333318$ $x_9 = 0.333323 \quad x_{10} = 0.333328$ $x_{11} = 0.333334 \quad x_{12} = 0.333341$ $f(x) = -2.999880$	5.32E-05	restarts: 2 $75/107/36$ 0.13 s/0.00 s/0.13 s 0.88 s	$\mu_1 = 0.005$	$\mu = 4.00\text{E-}05$ $\tau = 9.03\text{E-}06$ $s_i = 1.25\text{E-}03$ $h_{av} = 3.15\text{E-}05$ $h_{min} = 1.43\text{E-}05$ $\lvert \tilde{T}_h \rvert / \lvert T_h \rvert = 0.08$
20	$x^0 = 0$	$x_i \geq 0$ $(i = 1, 2, 3)$ $x_i = 0.242536$ $(i = 4, \ldots, 20)$ $f^* = -4.123106$	$x_1 = 0.026088 \quad x_2 = 0.024121$ $x_3 = 0.018091 \quad x_4 = 0.242529$ $x_5 = 0.242529 \quad x_6 = 0.242529$ $x_7 = 0.242530 \quad x_8 = 0.242530$ $x_9 = 0.242530 \quad x_{10} = 0.242531$ $x_{11} = 0.242531 \quad x_{12} = 0.242532$ $x_{13} = 0.242533 \quad x_{14} = 0.242534$ $x_{15} = 0.242534 \quad x_{16} = 0.242535$ $x_{17} = 0.242536 \quad x_{18} = 0.242537$ $x_{19} = 0.242538 \quad x_{20} = 0.242539$ $f(x) = -4.123058$	1.79E-05	restarts: 3 $102/169/68$ 0.69 s/0.02 s/0.56 s 2.76 s	$\mu_1 = 0.001$	$\mu = 1.60\text{E-}05$ $\tau = 1.70\text{E-}06$ $s_i = 1.25\text{E-}03$ $h_{av} = 3.49\text{E-}06$ $h_{min} = 7.29\text{E-}07$ $\lvert \tilde{T}_h \rvert / \lvert T_h \rvert = 0.04$

Table A.13: Example 7.5 - $n = 7, 12, 20, k = 3$

n	start vector	exact solution	approximate solution	accuracy	effort	non-standard parameters	final values
10	$x^0 = 0$	$x_i \geq 0$ ($i = 1, \ldots, 5$) $x_i = 0.447214$ ($i = 6, \ldots, 10$) $f^* = -2.236068$	$x_1 = 0.085728$ $x_2 = 0.085539$ $x_3 = 0.064262$ $x_4 = 0.051764$ $x_5 = 0.042843$ $x_6 = 0.447187$ $x_7 = 0.447192$ $x_8 = 0.447197$ $x_9 = 0.447203$ $x_{10} = 0.447209$ $f(x) = -2.235988$	3.98E-05	restarts: 3 92/136/35 0.07 s/0.01 s/0.22 s 0.96 s	$\mu_1 = 0.01$	$\mu = 1.60E\text{-}05$ $r = 9.40E\text{-}06$ $s_i = 6.25E\text{-}04$ $h_{av} = 1.41E\text{-}05$ $h_{\min} = 1.02E\text{-}06$ $|\tilde{T}_h|/|T_h| = 0.01$
20	$x^0 = 0$	$x_i \geq 0$ ($i = 1, \ldots, 5$) $x_i = 0.258199$ ($i = 6, \ldots, 20$) $f^* = -3.872983$	$x_1 = 0.029509$ $x_2 = 0.029509$ $x_3 = 0.029061$ $x_4 = 0.023408$ $x_5 = 0.019374$ $x_6 = 0.258165$ $x_7 = 0.258166$ $x_8 = 0.258169$ $x_9 = 0.258171$ $x_{10} = 0.258174$ $x_{11} = 0.258176$ $x_{12} = 0.258180$ $x_{13} = 0.258183$ $x_{14} = 0.258187$ $x_{15} = 0.258191$ $x_{16} = 0.258195$ $x_{17} = 0.258200$ $x_{18} = 0.258205$ $x_{19} = 0.258210$ $x_{20} = 0.258215$ $f(x) = -3.872786$	7.95E-05	restarts: 2 99/177/75 0.64 s/0.01 s/0.55 s 2.75 s	$\mu_1 = 0.001$	$\mu = 4.00E\text{-}05$ $r = 5.19E\text{-}06$ $s_i = 2.50E\text{-}03$ $h_{av} = 4.81E\text{-}04$ $h_{\min} = 6.17E\text{-}07$ $|\tilde{T}_h|/|T_h| = 0.22$

Table A.14: Example 7.5 - $n = 10, 20$, $k = 5$

n	start vector	approximate solution		effort	final values
32	$r_0 = 1600$ $w_1 = \ldots = w_{30} = 0$ $\vartheta = 3000$	$r_0 =\ \ 1525.52$	$w_1 = 36242.75$	restarts: 8 $946/39337/7508$ 633.58 s $/\,137.58$ s $/\,395.34$ s 1429.63 s	$\mu = 1.15\text{E-}01$ $r = 1.59\text{E-}01$ $h_{av} = 3.20\text{E-}05$ $h_{\min} = 2.13\text{E-}05$ $|\bar{T}_h|/|T_h| = 0.90$
		$w_2 = 32815.99$	$w_3 = 13815.77$		
		$w_4 =\ \ 4020.24$	$w_5 = -6069.15$		
		$w_6 =\ \ 4758.85$	$w_7 = 14737.18$		
		$w_8 = 14112.17$	$w_9 = 18346.01$		
		$w_{10} = 46544.21$	$w_{11} = 33868.20$		
		$w_{12} = 40881.96$	$w_{13} = 14098.55$		
		$w_{14} = 14098.55$	$w_{15} = 15506.86$		
		$w_{16} = -3195.49$	$w_{17} =\ \ 7344.09$		
		$w_{18} =\ \ 7793.20$	$w_{19} = 13793.60$		
		$w_{20} = 30348.69$	$w_{21} = 30674.80$		
		$w_{22} = 19024.95$	$w_{23} = 26004.27$		
		$w_{24} = 46345.79$	$w_{25} = 49706.85$		
		$w_{26} = 14621.09$	$w_{27} = 17968.50$		
		$w_{28} = 20163.07$	$w_{29} = 19863.05$		
		$w_{30} = 14764.52$	$\vartheta =\ \ \ \ 15.65$		

Table A.15: Example DAX1

n	start vector	approximate solution		effort	final values
32	$r_0 = 5000$ $w_1 = \ldots = w_{30} = 10000$ $\vartheta = 5000$	$r_0 =$ 4642.79 $w_2 =$174057.85 $w_4 =$113992.06 $w_6 =$ 51645.83 $w_8 =$103728.91 $w_{10}=$ 43906.46 $w_{12}=$132807.74 $w_{14}=$ 73364.74 $w_{16}=$ 43553.69 $w_{18}=$ 47586.70 $w_{20}=$131617.59 $w_{22}=$120279.34 $w_{24}=$ 52892.84 $w_{26}=$110867.23 $w_{28}=$ 47854.24 $w_{30}=$ 46863.05	$w_1 =$ 41296.84 $w_3 =$ 122032.14 $w_5 =$ 51575.18 $w_7 =$ 86537.16 $w_9 =$ 79851.57 $w_{11}=$ 77530.96 $w_{13}=$ 44629.71 $w_{15}=$ 95344.36 $w_{17}=$ 45089.31 $w_{19}=$ 45104.08 $w_{21}=$ 124602.13 $w_{23}=$ 115373.19 $w_{25}=$ 50427.28 $w_{27}=$ 130648.27 $w_{29}=$−50973.20 $\vartheta =$ 54.30	restarts: 7 1643/35399/3366 848.61 s/153.91 s/528.86 s 1711.77 s	$\mu = 1.34E\text{-}02$ $r = 1.17E\text{-}02$ $h_{av} = 3.02E\text{-}05$ $h_{\min} = 1.01E\text{-}07$ $|\tilde{T}_h|/|T_h| = 0.78$

Table A.16: Example DAX2

n	start vector	approximate solution	effort	non-standard parameters	final values
4	$x^0 = 0$	$a_0 = 0.612048$ $a_1 = -0.149279$ $a_2 = 0.045733$ $a_3 = -0.008533$ Coding gain: 5.860	restarts: 2 614/1816/258 0.34 s/0.10 s/12.64 s 17.91 s	-	$\mu = 1.28\text{E-}05$ $r = 2.26\text{E-}06$ $h_{av} = 1.12\text{E-}06$ $h_{min} = 6.53\text{E-}09$ $|\tilde{T}_h|/|T_h| = 0.06$
10	$x^0 = 0$	$a_0 = 0.631895$ $a_1 = -0.198906$ $a_2 = 0.107007$ $a_3 = -0.065041$ $a_4 = 0.040580$ $a_5 = -0.024743$ $a_6 = 0.014092$ $a_7 = -0.007013$ $a_8 = 0.002653$ $a_9 = -0.000539$ Coding gain: 5.943	restarts: 2 1897/8420/923 8.13 s/3.80 s/286.47 s 360.29 s	-	$\mu = 1.28\text{E-}05$ $r = 9.17\text{E-}07$ $h_{av} = 2.07\text{E-}07$ $h_{min} = 1.77\text{E-}07$ $|\tilde{T}_h|/|T_h| = 0.14$
14	$x^0 = 0$	$a_0 = 0.634045$ $a_1 = -0.204803$ $a_2 = 0.115607$ $a_3 = -0.075458$ $a_4 = 0.052038$ $a_5 = -0.036541$ $a_6 = 0.025579$ $a_7 = -0.017563$ $a_8 = 0.011636$ $a_9 = -0.007286$ $a_{10} = 0.004175$ $a_{11} = -0.002063$ $a_{12} = 0.000770$ $a_{13} = -0.000154$ Coding gain: 5.951	restarts: 3 2260/13069/911 35.33 s/14.91 s/380.55 s 509.43 s	$\mu_{i+1} = 0.4\mu_i$ $\varepsilon_{i+1,1} = 0.38\varepsilon_i$	$\mu = 1.68\text{E-}05$ $r = 6.90\text{E-}07$ $h_{av} = 2.88\text{E-}07$ $h_{min} = 2.63\text{E-}07$ $|\tilde{T}_h|/|T_h| = 0.20$

Table A.17: Design of perfect reconstruction filter banks - AR(1)-process

n	start vector	approximate solution	effort	non-standard parameters	final values
4	$x^0 = 0$	$a_0 = 0.594990$ $a_1 = -0.193611$ $a_2 = 0.059889$ $a_3 = -0.042127$ Coding gain: 6.069	restarts: 2 540/1132/302 0.10 s/0.05 s/0.70 s 2.01 s	-	$\mu = 1.28E\text{-}05$ $r = 2.30E\text{-}06$ $h_{av} = 4.52E\text{-}07$ $h_{\min} = 4.05E\text{-}07$ $|\tilde{T}_h|/|T_h| = 0.01$
10	$x^0 = 0$	$a_0 = 0.630535 \quad a_1 = -0.200938$ $a_2 = 0.108131 \quad a_3 = -0.065681$ $a_4 = 0.041014 \quad a_5 = -0.025251$ $a_6 = 0.014921 \quad a_7 = -0.008523$ $a_8 = 0.003052 \quad a_9 = -0.002626$ Coding gain: 6.833	restarts: 2 1465/6426/585 5.45 s/2.89 s/45.40 s 65.57 s	-	$\mu = 1.28E\text{-}05$ $r = 8.96E\text{-}07$ $h_{av} = 2.10E\text{-}07$ $h_{\min} = 1.65E\text{-}07$ $|\tilde{T}_h|/|T_h| = 0.03$
14	$x^0 = 0$	$a_0 = 0.633611 \quad a_1 = -0.205747$ $a_2 = 0.116444 \quad a_3 = -0.076323$ $a_4 = 0.052979 \quad a_5 = -0.037556$ $a_6 = 0.026631 \quad a_7 = -0.018599$ $a_8 = 0.012613 \quad a_9 = -0.008184$ $a_{10} = 0.005015 \quad a_{11} = -0.002939$ $a_{12} = 0.001063 \quad a_{13} = -0.000938$ Coding gain: 6.920	restarts: 4 1806/12624/487 29.46 s/13.66 s/299.24 s 374.66 s	$\mu_{i+1} = 0.3\mu_i$ $\varepsilon_{i+1,1} = 0.27\varepsilon_i$	$\mu = 1.97E\text{-}05$ $r = 1.05E\text{-}06$ $h_{av} = 1.75E\text{-}07$ $h_{\min} = 1.14E\text{-}07$ $|\tilde{T}_h|/|T_h| = 0.08$

Table A.18: Design of perfect reconstruction filter banks - AR(2)-process

n	start vector	approximate solution	effort	non-standard parameters	final values
4	$x^0 = 0$	$a_0 = 0.613735$ $a_1 = -0.169685$ $a_2 = 0.072194$ $a_3 = -0.026933$ Coding gain: 4.884	restarts: 1 590/1307/223 0.24 s/0.05 s/0.68 s 1.45 s	-	$\mu = 1.28\text{E-}05$ $r = 1.85\text{E-}06$ $h_{min} = 1.04\text{E-}06$ $|\tilde{T}_h|/|T_h| = 0.01$
10	$x^0 = 0$	$a_0 = 0.632184$ $a_1 = -0.201212$ $a_2 = 0.110762$ $a_3 = -0.069688$ $a_4 = 0.045660$ $a_5 = -0.029877$ $a_6 = 0.018947$ $a_7 = -0.011254$ $a_8 = 0.005893$ $a_9 = -0.002283$ Coding gain: 9.869	restarts: 2 1581/7906/376 8.59 s/3.27 s/54.26 s 75.95 s	-	$\mu = 1.28\text{E-}05$ $r = 8.86\text{E-}07$ $h_{av} = 2.78\text{E-}07$ $h_{min} = 1.88\text{E-}07$ $|\tilde{T}_h|/|T_h| = 0.04$
14	$x^0 = 0$	$a_0 = 0.634205$ $a_1 = -0.205603$ $a_2 = 0.116862$ $a_3 = -0.077002$ $a_4 = 0.053742$ $a_5 = -0.038311$ $a_6 = 0.027350$ $a_7 = -0.019287$ $a_8 = 0.013277$ $a_9 = -0.008806$ $a_{10} = 0.005531$ $a_{11} = -0.003198$ $a_{12} = 0.001605$ $a_{13} = -0.000586$ Coding gain: 12.868	restarts: 3 2022/16278/473 45.76 s/20.03 s/264.72 s 360.55 s	$\mu_{i+1} = 0.3\mu_i$ $\varepsilon_{i+1,1} = 0.27\varepsilon_i$	$\mu = 1.97\text{E-}05$ $r = 8.74\text{E-}07$ $h_{av} = 3.18\text{E-}07$ $h_{min} = 1.76\text{E-}07$ $|\tilde{T}_h|/|T_h| = 0.09$

Table A.19: Design of perfect reconstruction filter banks - lowpass process with box spectrum

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