Hadamard convolution operators on spaces of holomorphic functions

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Contents

Preface

The main topic of this thesis is the examination of the Hadamard product of two holomorphic functions which can be interpreted as some kind of convolution. There has been a long line of research devoted to this topic which goes back to Jacques Hadamard. In his paper "Théorème sur les séries entières" (see [Ha]) he considered the following question: Given two power series $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ and $\sum_{\nu=0}^{\infty} b_\nu z^\nu$, what information concerning the singularities of the *Hadamard product series* $\sum_{\nu=0}^{\infty} a_\nu b_\nu z^\nu$ can be obtained from the information about the singularities of the original series? In the subsequent decades, various famous mathematicians dedicated theirselves to the improvement and development of this theory (see for example [Bo], [Fa], [Pol33], for a summary see [Scho]). It turned out to be highly complicated to formulate sufficient conditions for a point $\alpha \in \mathbb{C}$ to be a singular point of the Hadamard product series.

In course of history, the following question moved to the center of the research: given two functions φ and f holomorphic in open sets Ω and U containing the origin with $\varphi(z) = \sum_{\nu=0}^{\infty} \varphi_\nu z^\nu$ and $f(z) = \sum_{\nu=0}^{\infty} f_\nu z^\nu$ near zero, what is a domain of holomorphy of the function $\varphi * f$ which is in a neighbourhood of the origin defined by $(\varphi * f)(z) := \sum_{\nu=0}^{\infty} \varphi_\nu f_\nu z^\nu$ and what properties do the function $\varphi * f$ and the operation "$*$" have? The original result of Hadamard can be interpreted as a first step in this direction for the case that both Ω and U are subsets of the complex plane and starlike with respect to the origin. In case that the closed unit disc is contained in $\Omega \cap U$, the Taylor coefficients φ_ν and f_ν coincide with the Fourier coefficients of the mappings $t \mapsto \varphi(e^{it})$ and $t \mapsto f(e^{it})$ $(t \in [0, 2\pi])$ respectively. Therefore the idea is to define the function $\varphi * f$, under suitable assumptions, by a certain convolution integral. Müller and in a subsequent paper Grosse-Erdmann showed that the *Hadamard product*

$$(\varphi * f)(z) := (\varphi *_{\Omega,U} f)(z) := \frac{1}{2\pi i} \int_\Gamma f(\zeta)\varphi(\frac{z}{\zeta})\frac{d\zeta}{\zeta} \quad (z \in \Omega * U) \tag{1}$$

(where $\Gamma = \Gamma_z$ is a suitable integration cycle) is an analytic continuation of $\sum_{\nu=0}^{\infty} \varphi_\nu f_\nu z^\nu$ into the set $\mathbb{C} \setminus \big((\mathbb{C} \setminus \Omega) \cdot (\mathbb{C} \setminus U)\big)$ (see [Mue92] and [GE]). This assertion is called the Hadamard multiplication theorem. However, Müller and Pohlen have shown that under quite general assumptions on the sets Ω and U being subsets of the Riemann sphere $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ (holomorphic functions are assumed to vanish at ∞ if this point belongs to their domain of holomorphy), the function $\varphi * f$ defined via the convolution integral (1) is holomorphic in

$$\Omega * U := \mathbb{C}_\infty \setminus \big((\mathbb{C}_\infty \setminus \Omega) \cdot (\mathbb{C}_\infty \setminus U)\big)$$

and, if $0 \in \Omega \cap U$, the Hadamard multiplication theorem also holds in this context (see [MP]). It turns out that the sticking point for the transition from plane sets Ω, U containing the origin to subsets of $\mathbb{C}_\infty$ not necessarily containing the origin is to find a concept for a suitable generalization of the integration cycle Γ_z.

From an alternative point of view, the Hadamard product of two holomorphic functions can be regarded as a generalization of the well-known Cauchy integral formula. Indeed, the function $\Theta(z) := 1/(1-z)$ is holomorphic in $\mathbb{C}_\infty \setminus \{1\}$ and the Cauchy integral formula yields for an arbitrary open set $U \subset \mathbb{C}$ and for all functions f holomorphic in U

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_z} \frac{f(\zeta)}{\zeta - z}\, d\zeta = (\Theta * f)(z) \quad (z \in U)\,.$$

Hence, the Hadamard product can be interpreted as a generalization of this formula to more general sets $\Omega \subset \mathbb{C}_\infty$ and more general functions $\varphi \in H(\Omega)$.

After a few preliminary remarks, we are going to repeat some properties of the Hadamard product in Chapter 2. In addition, it will be shown that, under suitable conditions, the Hadamard product is associative, which is a desirable property but nevertheless, its proof requires some technical difficulties to be resolved. Based on the classical Köthe duality, we prove that the dual space $H'(D)$ (where $D \subset \mathbb{C}$ is open) is topologically isomorphic to the space of germs of holomorphic functions on $1/D^C$ in such a way that to every functional $u \in H'(D)$ there exists a unique germ $[(g,U)]_{1/D^C} \in H(1/D^C)$ such that $u(f) = (f *_{D,U} g)(1)$ holds for all $f \in H(D)$.

The main part of this thesis is devoted to the examination of the *Hadamard convolution operator* (or *Hadamard operator*)

$$T_\varphi = T_{\varphi,U} : H(U) \to H(\Omega * U),\ f \mapsto \varphi * f$$

for different sets $\Omega, U \subset \mathbb{C}_\infty$ and different functions $\varphi \in H(\Omega)$. Müller and Pohlen have shown that this is a linear and continuous operator (where the Fréchet spaces $H(U)$ and $H(\Omega * U)$ are equipped with the topology of locally uniform convergence; see [MP]). We are going to introduce that operator in Chapter 3 and the following eigenvalue property will be shown: If K is a compact and convex subset of the strip $\{z \in \mathbb{C} : |\mathrm{Im}\, z| < \pi\}$ and if φ is a non-vanishing function which is holomorphic in $\Omega := \mathbb{C}_\infty \setminus e^{-K}$ and if the open set $U \subset \mathbb{C} \setminus \{0\}$ has connected complement, then the generalized monomials $z \mapsto \exp(\alpha \log_U z)$ are eigenfunctions of the Hadamard operator $T_{\varphi,U}$. Furthermore, the transposed operator $T'_{\varphi,U}$ will be computed for rather general sets Ω and U. It will turn out that $T'_{\varphi,U}$ is basically again a Hadamard operator induced by the same function φ but, of course, mapping between the spaces of germs of holomorphic functions on $1/(\Omega * U)^C$ and $1/U^C$.

This gives rise to the idea to examine the kernel and the range of Hadamard operators simultaneously because some classical functional analysis yields that the operator $T_{\varphi,U}$ has dense range if and only if its transpose is injective (see Chapter 4). Results in this direction have been obtained by Frerick who examined Hadamard operators for the case $\varphi \in H(\mathbb{C}_\infty \setminus \{1\})$ and gave characterizations for the surjectivity of $T_{\varphi,U}$ for the case that $U \subset \mathbb{C}$ is a domain containing the origin (see [Fre]). Throughout the whole examination of the range of Hadamard operators, we are going to concentrate on the case that U neither contains the origin nor the point at infinity. For example, we are going to show that for a non-vanishing function φ which is holomorphic in $\Omega := \mathbb{C}_\infty \setminus e^{-K}$, the induced Hadamard operator $T_{\varphi,U} : H(U) \to H(\Omega * U)$ has dense range for all open sets $U \subset \mathbb{C}\setminus\{0\}$ having connected complement (if $\Omega * U$ is non-empty). If, more general, Ω is a domain containing the origin and the point at infinity, then $T_{\varphi,U}$ has dense range if both $U \subset \mathbb{C} \setminus \{0\}$ is a domain and $\Omega * U$ is "small enough". "Small" will be specified by a criterion which is based on the maximal density of the non-vanishing coefficients in the power series expansion of φ about zero and infinity. Finally, Section 4.4 contains sufficient conditions for $T_{\varphi,U}$ to be surjective which read for the special case $0 \not\equiv \varphi \in H(\mathbb{C}_\infty \setminus \{1\})$ as follows: $T_{\varphi,U} : H(U) \to H(U)$ is surjective for all open sets $U \subset \{z \in \mathbb{C} : \mathrm{Im}\, z \neq 0 \text{ if } \mathrm{Re}\, z \leq 0\}$ such that $\log U$ is convex.

In Chapter 5 we are going to elaborate on the connection of Hadamard operators and operators defined via the convolution with analytic functionals. If Ω is again of the form $\mathbb{C}_\infty \setminus e^{-K}$, then those two types of operators coincide modulo a composition with the exponential function. If the convex support of the analytic functional is $\{0\}$, then the convolution operator coincides with a corresponding infinite order differential operator with constant coefficients. We are going to put the surjectivity result for Hadamard operators mentioned above into the research context of those convolution operators and we are going to give a new proof of the following result which goes back to Korobeĭnik (see [Kor69]): If $0 \not\equiv \Phi$ is an entire function of exponential type zero and if $G \subset \mathbb{C}$ is a convex domain, then firstly, the kernel of the infinite order differential operator $\Phi(D)$ acting on $H(G)$ is the closure (in $H(G)$) of the linear span of the functions $z \mapsto z^k e^{\alpha z}$ where α is an $m-$fold zero of Φ and $k \leq m-1$ and secondly, $\Phi(D)$ is surjective. This result, in turn, allows to improve the surjectivity criterion for Hadamard operators mentioned above for the special case $\varphi \in H(\mathbb{C}_\infty \setminus \{1\})$: If $0 \not\equiv \varphi \in H(\mathbb{C}_\infty \setminus \{1\})$, then $T_{\varphi,U} : H(U) \to H(U)$ is surjective for all simply connected sets $U \subset \mathbb{C} \setminus \{0\}$ such that $\log_U U$ is convex.

In Chapter 6 we consider a second application for the Hadamard product: the locally uniform approximation of holomorphic functions on an open set $D \subset \mathbb{C}$ by polynomials. The celebrated approximation theorem of Runge states that it is possible to approximate every function which is holomorphic in D by polynomials if and only if $\mathbb{C}_\infty \setminus D$ is connected.

We are going to consider the following question: Given a set $\Lambda \subset \mathbb{N}_0$, under which conditions on the open set $D \subset \mathbb{C}$ are we able to locally uniformly approximate every function $g \in H(D)$ by polynomials whose powers belong to the set Λ, i. e. by *lacunary polynomials*? Results of this kind can be regarded as a generalization of the approximation theorem of Runge. The well-known theorem of Müntz for real intervals can be considered as a starting point for lacunary approximation. In the literature, one can find several results concerning this question, see for example [An], [AM], [DK], [GLM], [LMM98], [LMM02] and [MR] (for further references see [GLM]). The proofs of these results are typically based on duality, that is, on the theorem of Hahn-Banach, which is without any doubt an elegant method of proof but is intrinsincly non-constructive.

The approach pursued in Chapter 6 relying on a suitable application of the information about the range of Hadamard operators yields new proofs of results of this kind. For example, we give a short proof of a result of Arakelian and Martirosian (see [AM]) stating that for $\Lambda \subset \mathbb{N}_0$ having unit density, it is possible to approximate every function holomorphic in D by polynomials with powers belonging to Λ for every set $D \subset \mathbb{C} \setminus \{0\}$ with connected complement.

In addition, the approach presented here bears the advantage of establishing the possibility of obtaining some information about the (geometric) rate of approximation of a holomorphic function on compact subsets of D by lacunary polynomials. It will turn out that the definition of the Hadamard product via the convolution integral (1) allows to derive some information about the rate of approximation by lacunary polynomials from the rate of approximation by arbitrary polynomials. The latter quantity has been examined in the literature (see for example [Gai]). We are going to obtain an upper bound for the geometric rate of approximation by lacunary polynomials which, in general, can not be improved.

Chapter 1

Notations and preliminaries

In this chapter we want to collect some notations and repeat some well-known facts. The complex plane $\mathbb{C}$ shall be equipped with the euclidian metric and the extended complex plane (or the *Riemann sphere*) shall be denoted by $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ and shall be equipped with the chordal metric. For a set $A \subset \mathbb{C}_\infty$, the complement with respect to the extended complex plane shall be denoted by A^C and the complement with respect to the complex numbers by $\mathbb{C} \setminus A$. A^* shall denote the set $1/A^C := \{1/a : \ a \in A^C\}$ (where as usual $1/0 := \infty$ and $1/\infty := 0$). If $\xi \in \{0, \infty\}$ and $\xi \in A$, we denote by A_ξ the component of A that contains the point ξ.

For a point $z \in \mathbb{C}$ let $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real and imaginary part of z. Furthermore, if $z \neq 0$, there exist unique numbers $r > 0$ and $t \in [-\pi, \pi)$ such that $z = re^{it}$. We write $\arg z := t$.

A set $\Omega \subset \mathbb{C}_\infty$ is open if and only if either Ω is open in $\mathbb{C}$ or $\infty \in \Omega$ and Ω^C is compact in $\mathbb{C}$. If the point at infinity belongs to an open set $\Omega \subset \mathbb{C}_\infty$, then there exists a number $R \geq 0$ such that $\{z \in \mathbb{C} : \ |z| > R\} \subset \Omega$.

A non-empty, open and connected set $G \subset \mathbb{C}_\infty$ is called a *domain*. We note that $G \subset \mathbb{C}_\infty$ is a domain if and only if $G \cap \mathbb{C}$ is connected in $\mathbb{C}$. Furthermore, we call a set $A \subset \mathbb{C}_\infty$ *simply connected*, if both A and A^C are connected.

For the sake of abbreviation we introduce some notations for special sets. We set for $0 \le r < R \le \infty$ and $z_0 \in \mathbb{C}$

$$\begin{aligned} \mathbb{D}_r &:= \{z \in \mathbb{C} : \ |z| < r\}, \\ \mathbb{T}_r &:= \{z \in \mathbb{C} : \ |z| = r\}, \\ U_r(z_0) &:= \{z \in \mathbb{C} : \ |z - z_0| < r\}, \\ U_r(\infty) &:= \{z \in \mathbb{C} : \ |z| > r\} \cup \{\infty\}, \\ V_{r,R} &:= \{z \in \mathbb{C} : \ r < |z| < R\} \end{aligned}$$

and especially $\mathbb{D} := \mathbb{D}_1$ and $\mathbb{T} := \mathbb{T}_1$.

Finally we set for a continuous function f on a compact set $K \subset \mathbb{C}$

$$\|f\|_K := \|f\|_{K,\infty} := \max_{z \in K} |f(z)| .$$

This is a norm on the space $C(K) := \{f : K \to \mathbb{C} : \ f \text{ continuous}\}$ and $C(K)$ is a Banach space with respect to this norm.

In the following, we will often use enumerations $\Lambda = \{z_n : \ n \in \mathbb{N}\}$ of countable sets $\Lambda \subset \mathbb{C}$ without finite accumulation point (for example the set of zeros of an entire function). If not stated otherwise, we assume that the elements z_n are numbered in order of increasing modulus (if there are finitely many elements of the same modulus, the numbering among them is of no relevance). The same shall be assumed while considering sequences $(z_n)_{n \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ without finite accumulation point.

1.1 Cauchy cycles and the Cauchy integral formula

In this section we introduce the concept of integration over cycles and formulate a version of the Cauchy integral formula. We follow the presentation in [Ru].

Definition 1.1 :

1. Let $a, b \in \mathbb{R}$ with $a < b$. A piecewise continuously differentiable map $\gamma : [a, b] \to \mathbb{C}$ is called *path*. Its range $\gamma([a, b])$ will be denoted by $|\gamma|$ and is called *trace*. γ is called *closed*, if $\gamma(a) = \gamma(b)$. Finally we set $\gamma^{-}(t) := \gamma(a + b - t)$ $(t \in [a, b])$.

Let $n \in \mathbb{N}$ and $a_j, b_j \in \mathbb{R}$ with $a_j < b_j$, $\gamma_j : [a_j, b_j] \to \mathbb{C}$ be closed paths ($j \in \{1, \ldots, n\}$) and $\Gamma := (\gamma_1, \ldots, \gamma_n)$.

2. The tuple Γ is called a *cycle* and the *trace* of Γ is the union of all traces $|\gamma_j|$, that means $|\Gamma| := \bigcup_{j=1}^{n} |\gamma_j|$. The cycle $(\gamma_1^-, \ldots, \gamma_n^-)$ is denoted by Γ^-.

3. For a continuous function $f : |\Gamma| \to \mathbb{C}$, the integral of f over Γ is defined by
$$\int_\Gamma f(\zeta)\, d\zeta := \sum_{j=1}^{n} \int_{\gamma_j} f(\zeta)\, d\zeta\,.$$

4. The number
$$L(\Gamma) := \sum_{j=1}^{n} \int_{a_j}^{b_j} |\gamma_j'(t)|\, dt$$
is called the *length* of Γ.

5. The number
$$\mathrm{ind}_\Gamma(z) := \frac{1}{2\pi i} \int_\Gamma \frac{1}{\zeta - z}\, d\zeta \quad (z \in \mathbb{C} \setminus |\Gamma|)$$
is called *index* of z with respect to Γ. Furthermore, we define
$$\mathrm{ind}_\Gamma(\infty) := 0\,.$$

Remark 1.2 :
Let $\Gamma = (\gamma_1, \ldots, \gamma_n)$ be a cycle.

1. For every $f \in C(|\Gamma|)$ we have
$$\int_{\Gamma^-} f(\zeta)\, d\zeta = -\int_\Gamma f(\zeta)\, d\zeta\,.$$

2. (a) For all $z \in \mathbb{C}_\infty \setminus |\Gamma|$ we have $\mathrm{ind}_\Gamma(z) \in \mathbb{Z}$.
   (b) If G is a component of $\mathbb{C}_\infty \setminus |\Gamma|$, then ind_Γ is constant on G.

3. If $0 \notin |\Gamma|$, then $1/\Gamma := (1/\gamma_1, \ldots, 1/\gamma_n)$ is also a cycle with $(1/\Gamma)^- = 1/(\Gamma^-)$ and $|1/\Gamma| = 1/|\Gamma|$.

   Furthermore we obtain for all $z \in \mathbb{C} \setminus \big((1/|\Gamma|) \cup \{0\}\big)$
$$\begin{aligned} \mathrm{ind}_{1/\Gamma}(z) &= -\frac{1}{2\pi i} \int_\Gamma \frac{1}{1/\zeta - z} \frac{1}{\zeta^2}\, d\zeta \\ &= \frac{1}{2\pi i} \int_\Gamma \frac{z}{\zeta z - 1}\, d\zeta - \frac{1}{2\pi i} \int_\Gamma \frac{d\zeta}{\zeta} \\ &= \mathrm{ind}_\Gamma(1/z) - \mathrm{ind}_\Gamma(0) \end{aligned}$$

and

$$\mathrm{ind}_{1/\Gamma}(0) = -\mathrm{ind}_{\Gamma}(0)\,.$$[1]

4. The fact that $|\Gamma| \subset U$ for some set $U \subset \mathbb{C}_\infty$ shall briefly be referred to as Γ *is a cycle in* U.

In the following, we will sometimes be concerned with integrals over circles and introduce the following notation: for $z_0 \in \mathbb{C}$ and $r > 0$ we define

$$\tau_r(z_0) : [0, 2\pi] \to \mathbb{C},\ t \mapsto z_0 + re^{it}$$

and especially $\tau_r := \tau_r(0)$.

The following result is a general version of Cauchy's integral formula (see [Ru, Th. 10.35]).

Theorem 1.3 :
Let $U \subset \mathbb{C}$ be open and let f be holomorphic in U. If Γ is a cycle in U such that

$$\mathrm{ind}_\Gamma(z) = 0 \quad (z \in \mathbb{C} \setminus U)\,,$$

then the following version of Cauchy's integral formula is valid:

$$f(z) \cdot \mathrm{ind}_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \quad (z \in U \setminus |\Gamma|)$$

and

$$\int_\Gamma f(\zeta)\, d\zeta = 0\,.$$

If Γ_1 and Γ_2 are cycles in U such that

$$\mathrm{ind}_{\Gamma_1}(z) = \mathrm{ind}_{\Gamma_2}(z) \quad (z \in \mathbb{C} \setminus U)\,,$$

then

$$\int_{\Gamma_1} f(\zeta)\, d\zeta = \int_{\Gamma_2} f(\zeta)\, d\zeta\,.$$

In the following definition, we introduce a special kind of cycles which will be of high relevance in the following (see [MP, Def. 2.2]). The concept of Cauchy cycles is suitable to assure concise formulation in various occasions.

[1] See also [Po, L. 2.2.2].

Definition 1.4 :
Let $U \subset \mathbb{C}_\infty$ be open and let $K \subset U$ be compact with $\infty \notin K$. A cycle Γ in $U \setminus (K \cup \{0, \infty\})$ with

$$\mathrm{ind}_\Gamma(z) = \begin{cases} 1 & , \text{ for all } z \in K \\ 0 & , \text{ for all } z \in \mathbb{C}_\infty \setminus U \end{cases},$$

is called a *Cauchy cycle for* K in U.

The following result ensures the existence of Cauchy cycles (see [Ru, Th. 13.5][2]).

Proposition 1.5 :
Let $U \subset \mathbb{C}_\infty$ be open and let $K \subset U$ be compact with $\infty \notin K$. Then there exists a Cauchy cycle Γ for K in U and Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

holds for all $f \in H(U)$ and for all $z \in K$.

1.2 The space $H(\Omega)$ and its dual space

The purpose of this section is to get familiar with the space $H(\Omega)$ of functions which are holomorphic in an open set $\Omega \subset \mathbb{C}_\infty$ and its dual space $(H(\Omega))' =: H'(\Omega)$. Since we are interested in the representation of $H'(\Omega)$ established by Köthe, we also introduce the concept of germs of holomorphic functions (see [Koe, Ch. 27], [Mori, Ch. 1.5]).

Definition 1.6 :
Let $\Omega \subset \mathbb{C}_\infty$ be open.

1. We denote by $H(\Omega)$ the space of all functions holomorphic in Ω and vanishing at ∞ (if $\infty \in \Omega$), endowed with the topology of locally uniform convergence. In doing so, a function $f(z)$ is called holomorphic at the point at infinity if $f(z^{-1})$ is holomorphic at the origin. $H(\Omega)$ is a Fréchet space.

[2] In the textbook of Rudin, this result is shown for sets $U \subset \mathbb{C}$. However, it can easily be generalized to the case $U \subset \mathbb{C}_\infty$.

2. We denote by $H^\infty(\Omega)$ the space of all functions $f \in H(\Omega)$ which are bounded on Ω, endowed with the norm $\|f\|_\Omega := \sup_{z\in\Omega} |f(z)|$. $(H^\infty(\Omega), \|\cdot\|_\Omega)$ is a Banach space.

3. Let $K \subset \mathbb{C}$ be compact. We denote by $\mathcal{H}(K)$ the space of all functions $f \in C(K)$ such that there exists an open set $O \supset K$ and a function $F \in H(O)$ such that $F|_K = f$.

4. Let $B \subset \mathbb{C}_\infty$. We consider the set

$$M_B := \{(f, U) : \ U \supset B \text{ open}, \ f \in H(U)\}$$

and define the following relation on it: $(f, U) \sim_B (\tilde{f}, \tilde{U})$ if and only if $f = \tilde{f}$ on an open set W with $B \subset W \subset U \cap \tilde{U}$. This relation is clearly an equivalence relation and the corresponding quotient space $H(B) := M_B / \sim_B$ is called the space of *germs of holomorphic functions on* B. The notation for an element of this space is $[(f, U)]_{\sim_B} =: [(f, U)]_B$.[3]

Remark 1.7 :
Let $\Omega \subset \mathbb{C}_\infty$ be open. If $0 \in \Omega$, then it is well-known that $\varphi \in H(\Omega)$ can for all z of small modulus be represented by the power series expansion $\varphi(z) = \sum_{\nu=0}^\infty \varphi_\nu z^\nu$ where

$$\varphi_\nu := \frac{\varphi^{(\nu)}(0)}{\nu!} = \frac{1}{2\pi i} \int_{\tau_r} \frac{\varphi(\zeta)}{\zeta^{\nu+1}} \, d\zeta \quad (\nu \in \mathbb{N}_0)$$

with an arbitrary $0 < r < \sup\{t > 0 : \ \mathbb{D}_t \subset \Omega\}$. We set

$$\Lambda_\varphi^+ := \{\nu \in \mathbb{N}_0 : \ \varphi_\nu \neq 0\}$$

and for a given set $\Lambda \subset \mathbb{N}_0$

$$H_\Lambda(\Omega) := \{f \in H(\Omega) : \ \Lambda_f^+ \subset \Lambda\}.$$

If $\infty \in \Omega$, we already stated that there is an $R \geq 0$ such that $U_R(\infty) \subset \Omega$. Then the Laurent series expansion of $\varphi \in H(\Omega)$ in $U_R(\infty)$ implies that for all z of large modulus φ can be represented by the series expansion $\varphi(z) = \sum_{\nu=1}^\infty \varphi_{-\nu} z^{-\nu}$ where

$$\varphi_{-\nu} := \frac{1}{2\pi i} \int_{\tau_r} \varphi(\zeta) \cdot \zeta^{\nu-1} \, d\zeta \quad (\nu \in \mathbb{N})$$

with an arbitrary $R < r < \infty$. We call the above series expansion the *power series expansion of* φ *about infinity*. We set

$$\Lambda_\varphi^- := \{\nu \in \mathbb{N} : \ \varphi_{-\nu} \neq 0\}$$

[3] We will adress the issue of the topology on $H(B)$ in Remark 1.11.

and for a given set $\Lambda' \subset \mathbb{N}$

$$H_{\infty,\Lambda'}(\Omega) := \{f \in H(\Omega) : \Lambda_f^- \subset \Lambda'\}.$$

If both 0 and ∞ belong to Ω, then we set $\Lambda_\varphi := \Lambda_\varphi^+ \cup (-\Lambda_\varphi^-)$ $(\subset \mathbb{Z})$.

Before we examine spaces of germs of holomorphic functions in some more detail, we introduce the notion of the hull of a set with respect to another set and some simple properties according to [LR, p. 84]:

Definition 1.8 :
Let $\Omega \subset \mathbb{C}_\infty$ and $M \subset \Omega$. The set $h_\Omega(M)$ which is defined as the union of M with all those components of M^C which lie entirely in Ω is called the *hull of M with respect to Ω*.

Remark 1.9 :
Let $\Omega \subset \mathbb{C}_\infty$ and $M \subset \Omega$.

1. It is clear that $h_\Omega(M)$ is a subset of Ω.
2. If Ω^C is connected, then $h_\Omega(M)$ is the smallest superset of M with connected complement.
3. If M is closed, then $h_\Omega(M)$ is closed (and hence compact in $\mathbb{C}_\infty$) because it is the complement in $\mathbb{C}_\infty$ of the open set consisting of all those components of M^C which intersect Ω^C. It is clear that if $M \subset \Omega \subset \mathbb{C}$ and M is compact in $\mathbb{C}$, then $h_\Omega(M)$ is also compact in $\mathbb{C}$.

Definition 1.10 :
Let $\Omega \subset \mathbb{C}_\infty$ be open. We call a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of Ω a *standard exhaustion*, if the following conditions are satisfied:

1. $K_1^\circ \neq \emptyset$ and $\xi \in K_1^\circ$ if $\xi \in \Omega$ $(\xi \in \{0, \infty\})$.
2. $K_n \subset K_{n+1}^\circ$ $(n \in \mathbb{N})$ and $\Omega = \bigcup_{n\in\mathbb{N}} K_n$.
3. $h_\Omega(K_n) = K_n$ $(n \in \mathbb{N})$.

Such a standard exhaustion always exists and can be constructed using [Ru, Th. 13.3].

Remark 1.11 :
Let $\Omega \subset \mathbb{C}_\infty$ be open and let $(K_n)_{n\in\mathbb{N}}$ be a standard exhaustion of Ω. We now turn towards the space $H(\Omega^C)$ of germs of holomorphic functions on Ω^C.

For $n \in \mathbb{N}$, the map $r_n : H^\infty(K_n^C) \to H(\Omega^C),\ f \mapsto [(f, K_n^C)]_{\Omega^C}$ is injective. To see this, let $f, g \in H^\infty(K_n^C)$ such that $[(f, K_n^C)]_{\Omega^C} = [(g, K_n^C)]_{\Omega^C}$. This means that $(f, K_n^C) \sim_{\Omega^C} (g, K_n^C)$ and there exists an open set W with $\Omega^C \subset W \subset K_n^C$ and $f = g$ on W. According to condition 3 for standard exhaustions, every component D of K_n^C contains at least one point of Ω^C so that $W \cap D \neq \emptyset$ (and open). By the identity theorem (applied to each component D of K_n^C separately), we obtain $f = g$.

Hence, we can identify each $f \in H^\infty(K_n^C)$ with the corresponding germ $[(f, K_n^C)]_{\Omega^C} \in H(\Omega^C)$ and consider the Banach space $H^\infty(K_n^C)$ as a subspace of $H(\Omega^C)$. Together with condition 2 for standard exhaustions we obtain

$$H(\Omega^C) = \bigcup_{n\in\mathbb{N}} H^\infty(K_n^C)\,.$$ [4]

Since $(1/K_n)_{n\in\mathbb{N}}$ is a standard exhaustion of $1/\Omega$, by the same arguments as above we obtain

$$H(\Omega^*) = \bigcup_{n\in\mathbb{N}} H^\infty(K_n^*)\,.$$ [5]

By means of a certain space of germs of holomorphic functions, one can give a useful representation for the dual of the space of holomorphic functions in an open set which goes back to Köthe (see [Koe, Ch. 27]). We recall the general definition of dual spaces and subsequently, we formulate a version of the Köthe duality which can be found in [GE, p. 107].

Definition 1.12 :
Let E and F be locally convex spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We define

$$\begin{aligned} L(E,F) &:= \{T : E \to F :\ T \text{ is linear and continuous}\}\,, \\ E' &:= L(E,\mathbb{K})\,. \end{aligned}$$

[4] This equality does not only hold in an algebraical sense. From a topological point of view, the union of the Banach spaces $H^\infty(K_n^C)$ shall be interpreted as an abbreviation for the inductive limit of the Banach spaces $H^\infty(K_n^C)$. Therefore $H(\Omega^C)$ is considered to be equipped with the resulting inductive limit topology. This topology is independent of the choice of the standard exhaustion K_n (see [Koe, Ch. 27.4]).

[5] Again, $H(\Omega^*)$ is considered to carry the inductive limit topology of the Banach spaces $H^\infty(K_n^*)$.

E' is called the *dual space* of E and shall be equipped with the strong topology.[6]

Theorem 1.13 :
Let $D \subset \mathbb{C}$ be open. The dual of the space $H(D)$ is topologically isomorphic to the space $H(D^C)$ in such a way that to every functional $u \in H'(D)$ there exists a unique germ $[(g,V)]_{D^C} \in H(D^C)$ such that

$$u(f) = \frac{1}{2\pi i} \int_\Gamma f(\zeta) g(\zeta)\, d\zeta \quad (f \in H(D))\,, \tag{1.1}$$

where Γ is a Cauchy cycle for V^C in D.

Remark 1.14 :
The Cauchy integral formula (see Theorem 1.3) assures that the value of the integral in (1.1) does not change while using another representative $(\tilde{g}, \tilde{V})$ of $[(g,V)]_{D^C}$ or integrating over another Cauchy cycle $\tilde{\Gamma}$ for V^C in D.

[6]The strong topology is the topology of uniform convergence on bounded subsets of E, i. e. $x'_n \to x'$ $(n \to \infty)$ in E' with respect to the strong topology if and only if $\sup_{x\in B}|x'_n(x) - x'(x)| \to 0$ $(n \to \infty)$ for all bounded sets $B \subset E$. The strong topology is thus induced by the system of seminorms $\{p_B :\ B \subset E \text{ bounded}\}$ where $p_B(x') = \sup\limits_{x\in B}|x'(x)|$ $(x' \in E')$ (see [Yos, p. 110 f.]).

Chapter 2

The Hadamard product of holomorphic functions

In this chapter we introduce the Hadamard product of holomorphic functions whose examination is the main purpose of this thesis. In Section 2.1 we are going to deal with the star product of sets which is important for the definition of the Hadamard product in Section 2.2. The main result formulated in Section 2.3 is an associative law which is not only of interest by itself but which will also be required in the following. We conclude the chapter by giving an alternative representation of the dual space of $H(D)$ (where $D \subset \mathbb{C}$ is open) by means of the Hadamard product.

However, we start this chapter by introducing some further notations. In the following we will often be concerned with the multiplication of numbers in $\mathbb{C}_\infty$. Therefore, we agree upon some arithmetic operations with the point at infinity by setting

$$\begin{aligned} z \cdot \infty := \infty \cdot z &:= \infty \ (z \in \mathbb{C}_\infty \setminus \{0\})\,, \\ z/\infty &:= 0 \quad (z \in \mathbb{C})\,, \\ z/0 &:= \infty \ (z \in \mathbb{C}_\infty \setminus \{0\})\,. \end{aligned}$$

All other arithmetic operations remain undefined (especially the term $0 \cdot \infty$ which is why we have to pay special attention to this term).

For non-empty sets $A, B \subset \mathbb{C}_\infty$ we consider the algebraic product $A \cdot B := \{a \cdot b : \ a \in A,\ b \in B\}$. According to the conventions concerning the point at infinity mentioned above, the algebraic product is defined if $(0,\infty), (\infty, 0) \notin A \times B$. For the empty set we define $A \cdot \emptyset := \emptyset \cdot A := \emptyset \cdot \emptyset := \emptyset$. Finally we set $1/A := A^{-1} := \{1/a : a \in A\}$ and $1/\emptyset := \emptyset$. If $A \cdot B$ is defined, it is compact if both A and B are compact.

For $-1 \leq \gamma < \delta \leq 1$ we define the set

$$S_{\gamma,\delta} := \{z \in \mathbb{C} \setminus \{0\} : \gamma\pi < \arg z < \delta\pi\}$$

and for $0 < \alpha \leq 1$ we define $S_\alpha := S_{-\alpha,\alpha}$. Finally, we set $\mathbb{S} := S_1$ and $\mathbb{V} := \log \mathbb{S} = \{z \in \mathbb{C} : |\operatorname{Im} z| < \pi\}$. Then $\mathbb{S}$ is the standard cut plane $\mathbb{C} \setminus (-\infty, 0]$ and if not stated otherwise, throughout this thesis, log shall denote the principal branch of the logarithm in $\mathbb{S}$.

A *sector of opening* $\alpha \in (0, 1]$ shall be a set which can be transformed via rotation into the sector S_α. A *cone of opening* $\alpha \in (0, 1]$ shall be a set which can be transformed via rotation into the set $\overline{S}_\alpha$.

Finally, for $K \subset \mathbb{C}$ we set $D_K := \mathbb{C}_\infty \setminus e^{-K}$ and for $\delta \geq 0$ we set $K_\delta := i\pi[-\delta, \delta]$.

2.1 The star product of sets

As already mentioned above, the investigation of the Hadamard product of holomorphic functions is one of the key subjects examined in this thesis. First of all, we are going to define the star product of sets $A_1, A_2 \subset \mathbb{C}_\infty$ which will occur throughout the whole framework (see [MP, p. 259]).

Definition 2.1 :
The *star product* of two sets $A_1, A_2 \subset \mathbb{C}_\infty$ is given by

$$A_1 * A_2 := \left(A_1^C \cdot A_2^C\right)^C.$$

The sets A_1, A_2 are called *star-eligible* if $A_1 * A_2$ is defined (that means $0 \cdot \infty$ should not appear) and non-empty. Furthermore, for $A \subset \mathbb{C}_\infty$ we set $A^* := 1/A^C$.

Remark 2.2 :
It is clear that as long as both the origin and the point at infinity belong to one of the sets A_1 or A_2, the star product is at least defined. Star-eligibility then means that $A_1 * A_2$ is not the empty set.

Example 2.3 :

1. Let $A_1, A_2 \subset \mathbb{C}_\infty$ be star-eligible. If $1 \notin A_1$, then we have
$$A_1 * A_2 \subset A_2$$
and especially
$$(\mathbb{C}_\infty \setminus \{1\}) * A = A$$
for an arbitrary set $A \subset \mathbb{C}_\infty$.

2. Let $A_1, A_2, A_3 \subset \mathbb{C}_\infty$. Then
$$A_1 * (A_2 * A_3) = (A_1 * A_2) * A_3 =: A_1 * A_2 * A_3$$
(if all occuring sets are defined).

3. For $-1 < \gamma \leq \delta < 1$ we consider the set $B_{\gamma,\delta} := \{z \in \mathbb{T} : \; \gamma\pi \leq \arg z \leq \delta\pi\}$.

   Let now $0 < \alpha \leq 1$ and $-1 < \gamma \leq 0 \leq \delta < 1$ be given. Firstly we note that the star product $(\mathbb{C}_\infty \setminus B_{\gamma,\delta}) * S_\alpha$ is defined and the assumption $0 \in [\gamma, \delta]$ implies that $1 \in B_{\gamma,\delta}$ so that part 1. yields that $(\mathbb{C}_\infty \setminus B_{\gamma,\delta}) * S_\alpha$ is a subset of S_α.

   Furthermore, the set $\mathbb{C}_\infty \setminus B_{\gamma,\delta}$ and the sector S_α are star-eligible if and only if $2\alpha > \delta - \gamma$ and in this case we have
$$(\mathbb{C}_\infty \setminus B_{\gamma,\delta}) * S_\alpha = S_{\delta-\alpha,\alpha+\gamma} \,.$$
   Hence, $(\mathbb{C}_\infty \setminus B_{\gamma,\delta}) * S_\alpha$ is a sector of opening $\alpha + (\gamma - \delta)/2 > 0$.

   If in particular, $\gamma = -\delta$, then $\mathbb{C}_\infty \setminus B_{-\delta,\delta} = D_{K_\delta}$ and therefore
$$D_{K_\delta} * S_\alpha = S_{\delta-\alpha,\alpha-\delta} = S_{\alpha-\delta} \,.$$
   If $0 \notin [\gamma, \delta]$, then the geometry remains the same except for some rotation and it is clear that still, $\mathbb{C}_\infty \setminus B_{\gamma,\delta}$ and S_α are star-eligible if and only if $2\alpha > \delta - \gamma$ and in this case the star product is a sector of opening $\alpha + (\gamma - \delta)/2 > 0$.

4. More general, we consider a compact and connected set $K \subset \mathbb{V}$ intersecting the real axis and set
$$\gamma := (\min_{z \in -K} \operatorname{Im} z)/\pi \quad \text{and} \quad \delta := (\max_{z \in -K} \operatorname{Im} z)/\pi \,.$$
   Then we have $-1 < \gamma \leq 0 \leq \delta < 1$ and for a given number $0 < \alpha \leq 1$ with $2\alpha > \delta - \gamma$ we obtain
$$D_K * S_\alpha = (\mathbb{C}_\infty \setminus B_{\gamma,\delta}) * S_\alpha = S_{\delta-\alpha,\alpha+\gamma}$$

and especially

$$D_K * \mathbb{S} = S_{\delta-1,1+\gamma}\,.$$

If in particular, $K = I \times i\pi[-1+\eta, 1-\eta]$ for some compact real interval I (which may reduce to a point) and $0 < \eta \leq 1$, i. e. $\gamma = -1+\eta,\ \delta = 1-\eta$, we have

$$D_K * \mathbb{S} = S_\eta\,.$$

Again, if K does not intersect the real axis, the geometry remains the same except for some rotation and it is clear that still, $D_K * S_\alpha$ is a sector of opening $\alpha + (\gamma - \delta)/2 > 0$.

5. For $K, L \subset \mathbb{C}$ we obtain

$$D_K = (e^K)^*, \quad D_K * D_L = D_{K+L}\,.$$

6. Let $\Omega \subsetneq \mathbb{C}_\infty$ be open with $\{0, \infty\} \subset \Omega$. Let furthermore $0 \leq r < R \leq \infty$. Then

$$\begin{aligned} \mathbb{D}_R * \Omega &= \mathbb{D}_{R\cdot\min\{|w|:\ w\in\Omega^C\}}\,, \\ U_r(\infty) * \Omega &= U_{r\cdot\max\{|w|:\ w\in\Omega^C\}}(\infty)\,. \end{aligned}$$

Consequently, $V_{r,R}$ and Ω are star-eligible if and only if

$$\frac{r}{R} < \frac{\min\{|w|:\ w\in\Omega^C\}}{\max\{|w|:\ w\in\Omega^C\}}$$

and in this case we obtain

$$V_{r,R} * \Omega = V_{r\cdot\max\{|w|:\ w\in\Omega^C\},R\cdot\min\{|w|:\ w\in\Omega^C\}}\,.$$

Remark 2.4 :
It is obvious that for sets $A, A_1, A_2 \subset \mathbb{C}_\infty$ the following is valid:

$0 \in A^*$ if and only if $\infty \notin A$ and $\infty \in A^*$ if and only if $0 \notin A$. Additionally, we have $A_1 \subset A_2^*$ if and only if $A_1^* \supset A_2$.

Furthermore, A_1 and A_2 are star-eligible if and only if $1/A_1$ and $1/A_2$ are star-eligible and

$$\frac{1}{A_1} * \frac{1}{A_2} = \frac{1}{A_1 * A_2}\,.$$

In the next proposition, we note some further properties of the star product (see [MP, Th. 2.1], [Po, Prop. 1.3.15]).

Proposition 2.5 :
Let $\Omega, U \subset \mathbb{C}_\infty$ be star-eligible, $\xi \in \{0, \infty\}$ and $S \subset \mathbb{C}_\infty$ such that $S \cdot \Omega^*$ is defined. Then the following are valid:

1. $\Omega * U = U * \Omega$.
2. $\xi \in \Omega * U$ if and only if $\xi \in \Omega \cap U$.
3. $S \subset \Omega * U$ if and only if $S \cdot \Omega^* \subset U$.
4. If in addition Ω and U are open, then so is $\Omega * U$.
5. If in addition Ω is open and $S \subset \Omega * U$ is compact, then $S \cdot \Omega^*$ is a compact subset of U.
6. If $\xi \notin U$ and U^C is connected, then $(\Omega * U)^C$ is connected, too.

Remark 2.6 :
We have a closer look at parts 3 and 6 of Proposition 2.5.

1. By setting

   $$\Omega := D_{K_{1/4}}, \quad S := S_{1/4} \setminus \{1\}, \quad , U := S \cdot \Omega^* = S_{1/2}$$

   we obtain $\Omega * U = S_{1/4}$. This shows, with regard to Proposition 2.5.3, that in general, S is a proper subset of $\Omega * U$ even if $S \cdot \Omega^*$ equals U.

   However, there are cases in which $\Omega * U = S$ as Proposition 4.32 will show.

2. By setting

   $$\Omega := \mathbb{C}_\infty \setminus [1, 2], \quad U := \mathbb{S} \cup (-2, -1)$$

   we obtain $\Omega * U = S_1$. This shows that the inversion of Proposition 2.5.6 is in general false.

Remark 2.7 :

1. The star product of two connected sets does not need to be connected again:

   By setting

   $$\Omega := U := D_{K_{1/2}}$$

   we obtain with Example 2.3.5 that $\Omega * U = D_{K_1} = \mathbb{C}_\infty \setminus \mathbb{T}$ which is not connected (see also [Po, Ex. 1.3.14]).

2. Let $\Omega, U \subset \mathbb{C}_\infty$ be open and star-eligible and $\xi \in \{0, \infty\}$. If $\xi \in \Omega \cap U$, then $(\Omega * U)_\xi = (\Omega_\xi * U_\xi)_\xi$.[1]

   To show this, we note that the set
$$(\Omega * U)_\xi \cdot \Omega^* = \bigcup_{\omega \in \Omega^*} (\Omega * U)_\xi \cdot \omega$$
   is as a union over connected sets which contain ξ itself connected and containing ξ. Furthermore, it is a subset of U (see Proposition 2.5.3) and therefore also of U_ξ. Applying again Proposition 2.5.3, we obtain $(\Omega * U)_\xi \subset \Omega * U_\xi$ and hence $(\Omega * U)_\xi \subset (\Omega * U_\xi)_\xi$. The opposite inclusion is clear so that $(\Omega * U)_\xi = (\Omega * U_\xi)_\xi$. By interchanging the roles of Ω and U we obtain $(\Omega * U)_\xi = (\Omega_\xi * U_\xi)_\xi$.

   A first consequence is that if $\xi \in \Omega \cap U$ and we assume $\Omega * U$ to be connected, then we can without loss of generality also assume Ω and U to be connected, since $\Omega * U = (\Omega * U)_\xi = (\Omega_\xi * U_\xi)_\xi$.

   A second consequence is that if $\{0, \infty\} \subset \Omega \cap U$ and if we assume that 0 and ∞ belong to different components of one of the sets Ω or U, then they belong to different components of $\Omega * U$. This is clear since (note that $\infty \in \Omega_0^C$ or U_0^C respectively)
$$\infty \in \left(\left((\Omega_0^C \cdot U_0^C)^C \right)_0 \right)^C = \left(\left((\Omega_0 * U_0)_0 \right)^C = \left((\Omega * U)_0 \right)^C .$$

2.2 Definition and basic properties of the Hadamard product

Before we can define the Hadamard product of holomorphic functions, we return again to cycles and introduce new types of cycles that serve, besides Cauchy cycles, as paths of integration in the definition of the Hadamard product (see [MP, Def. 2.2]).

Definition 2.8 :
Let $U \subset \mathbb{C}_\infty$ be open with $\infty \in U$ and let $K \subset U$ be compact. A cycle Γ in $U \setminus (K \cup \{0, \infty\})$ with
$$\operatorname{ind}_\Gamma(z) = \begin{cases} 0 & , \text{ for all } z \in K \\ -1 & , \text{ for all } z \in \mathbb{C}_\infty \setminus U \end{cases} ,$$

[1] See also [GE, p. 104] for the case $\Omega, U \subset \mathbb{C}$ and $\xi = 0$.

is called an *anti-Cauchy cycle for* K in U.

The concept of Hadamard cycles constitutes a synthesis of Cauchy and anti-Cauchy cycles (see [MP, Def. 2.3]).

Definition 2.9 :
Let $\Omega, U \subset \mathbb{C}_\infty$ be open and star-eligible, and let $z \in \Omega * U$. Furthermore, let Γ be a cycle in $U \setminus ((z \cdot \Omega^*) \cup \{0, \infty\})$ with the following property:

1. If $0 \in \Omega \cap U$ and $z = 0$, let Γ be a Cauchy cycle for $\{0\}$ in U.
2. If $\infty \in \Omega \cap U$ and $z = \infty$, let Γ be an anti-Cauchy cycle for $\{\infty\}$ in U.
3. If $z \neq 0$ and $z \neq \infty$, let Γ be
   (a) a Cauchy cycle for $z \cdot \Omega^*$ in U with $\text{ind}_\Gamma(0) = 1$ if $0 \in \Omega \cap U$ and $\infty \notin \Omega \cap U$,
   (b) an anti-Cauchy cycle for $z \cdot \Omega^*$ in U with $\text{ind}_\Gamma(0) = -1$ if $0 \notin \Omega \cap U$ and $\infty \in \Omega \cap U$,
   (c) a Cauchy cycle for $z \cdot \Omega^*$ in U with $\text{ind}_\Gamma(0) = 1$ or an anti-Cauchy cycle for $z \cdot \Omega^*$ in U with $\text{ind}_\Gamma(0) = -1$ if $\{0, \infty\} \subset \Omega \cap U$,
   (d) a Cauchy cycle for $z \cdot \Omega^*$ in U if $\{0, \infty\} \subset \Omega \setminus U$,
   (e) an anti-Cauchy cycle for $z \cdot \Omega^*$ in U if $\{0, \infty\} \subset U \setminus \Omega$.

Then Γ is called a *Hadamard cycle for* $z \cdot \Omega^*$ in U. If this holds for all $z \in K$, where K is a compact subset of $\Omega * U$, then Γ is called a *Hadamard cycle for* $K \cdot \Omega^*$ in U.

Analogously to the case of Cauchy cycles, we have to ensure the existence of anti-Cauchy and Hadamard cycles (see [MP, L. 3.1.2, Th. 2.4]).

Proposition 2.10 :
Let $\Omega, U \subset \mathbb{C}_\infty$ be open and star-eligible and let $K \subset \Omega * U$ be compact. Then the following are valid:

1. If $\infty \in U$, then there exists an anti-Cauchy cycle for $K \cdot \Omega^*$ in U.
2. If $K \cap \{0, \infty\} \neq \{0, \infty\}$, then there exists a Hadamard cycle for $K \cdot \Omega^*$ in U.

The following definition introduces the notion of the Hadamard product of two holomorphic functions as in [MP, Def. 2.6].

Definition 2.11 :
Let $\Omega, U \subset \mathbb{C}_\infty$ be open and star-eligible, $\varphi \in H(\Omega)$ and $f \in H(U)$. The function $\varphi * f : \Omega * U \to \mathbb{C}$ defined by

$$(\varphi * f)(z) := \frac{1}{2\pi i} \int_\Gamma f(\zeta)\varphi(\frac{z}{\zeta})\frac{d\zeta}{\zeta} \quad (z \in \Omega * U)$$

where $\Gamma = \Gamma_z$ is a Hadamard cycle for $z \cdot \Omega^*$ in U, is called the *Hadamard product* of φ and f.

Remark 2.12 :

1. Concerning the choice of the integration cycle Γ in Definition 2.11, it is important to note two things: even if it is suppressed in the notation, Γ depends on z and the value of the integral does not change when choosing another Hadamard cycle $\tilde{\Gamma}$ for $z \cdot \Omega^*$ in U (see [MP, Th. 2.5]). Hence, the function $\varphi * f$ is well defined.

2. The Hadamard product is some kind of convolution of the functions φ and f. Later it will become clear why the name Hadamard product is appropriate for this convolution (see Remark 2.15).

3. Müller and Pohlen use in [MP] the notation $\varphi *_{\Omega,U} f$ in order to specify the underlying sets. Because we don not want to overload the notation, we only use this notation if we want to stress the sets. The following so-called compatibility theorem from [MP, Th. 2.7] admits this procedure.

Theorem 2.13 :
Let $\Omega, U \subset \mathbb{C}_\infty$ as well as $D_1 \subset \Omega,\ D_2 \subset U$ be open and star-eligible, $\varphi \in H(\Omega)$ and $f \in H(U)$. Then

$$(\varphi *_{\Omega,U} f)|_{D_1 * D_2} = (\varphi|_{D_1}) *_{D_1,D_2} (f|_{D_2})\,.$$

In the next theorem, we list some further properties of the Hadamard product (see [MP, Th. 2.9, Th. 2.10]).

Theorem 2.14 :
Let $\Omega, U \subset \mathbb{C}_\infty$ be open and star-eligible, $\varphi \in H(\Omega)$ and $f \in H(U)$. Then the following assertions hold:

1. *$\varphi * f \in H(\Omega * U)$.*
2. *$\varphi * f = f * \varphi$.*
3. *(Hadamard multiplication theorem, I)*
   If $0 \in \Omega \cap U$ and $\varphi(z) = \sum_{\nu=0}^{\infty} \varphi_\nu z^\nu$, $f(z) = \sum_{\nu=0}^{\infty} f_\nu z^\nu$ near the origin, then we have
   $$(\varphi * f)(z) = \sum_{\nu=0}^{\infty} \varphi_\nu f_\nu z^\nu \tag{2.1}$$
   for all $z \in \mathbb{D}_r$ with $r := \sup\{t > 0 : \ \mathbb{D}_t \subset \Omega * U\}$.
4. *(Hadamard multiplication theorem, II)*
   If $\infty \in \Omega \cap U$ and $\varphi(z) = \sum_{\nu=1}^{\infty} \varphi_{-\nu} z^{-\nu}$, $f(z) = \sum_{\nu=1}^{\infty} f_{-\nu} z^{-\nu}$ near the point at infinity, then we have
   $$(\varphi * f)(z) = -\sum_{\nu=1}^{\infty} \varphi_{-\nu} f_{-\nu} z^{-\nu}$$
   for all $z \in U_d(\infty)$ with $d := \inf\{t > 0 : U_t(\infty) \subset \Omega * U\}$.

Remark 2.15 :
As we already stated in the introduction, the right-hand side of Equation (2.1) is called the Hadamard product of the power series expansions of φ and f. The Hadamard multiplication theorem then states that the power series expansion of $\varphi * f$ about zero coincides with the Hadamard product of the power series expansions of φ and f about zero. This means that $\varphi * f$ is the analytic continuation of $\sum_{\nu=0}^{\infty} \varphi_\nu f_\nu z^\nu$ to the set $\Omega * U$. Therefore the convolution $\varphi * f$ is called Hadamard product.

In the following example we reveal the relation between the Hadamard product and the Cauchy integral formula.

Example 2.16 :
Let $\Theta(z) := 1/(1-z)$ $(z \in \mathbb{C}_\infty \setminus \{1\})$ and let $U \subset \mathbb{C}$ be open and $f \in H(U)$. Then we have for all $z \in U \setminus \{0\}$

$$\begin{aligned} \Theta * f(z) &= \frac{1}{2\pi i} \int_\Gamma \frac{1}{1 - z/\zeta} f(\zeta) \frac{d\zeta}{\zeta} \\ &= f(z) \end{aligned}$$

where according to Definition 2.9 the cycle Γ_z can be chosen as

$$\Gamma_z = \begin{cases} \tau_r(z) & , \text{ if } 0 \notin U \\ (\tau_r(z), \tau_{r'}) & , \text{ if } 0 \in U \end{cases} ,$$

with $0 < r, r'$ sufficiently small. The last identity above follows from the Cauchy integral formula noting that in case $0 \in U$, the cycle $\tau_{r'}$ does not change the value of the integral because the integrand in holomorphic in a neighbourhood of the origin and $\tau_{r'}$ is a closed path. Obviously, the above identity also holds for $z = 0$ (if $0 \in U$).

Hence, the Hadamard product can be regarded as a generalization of the Cauchy integral formula from $\Theta \in H(\mathbb{C}_\infty \setminus \{1\})$ to $\varphi \in H(\Omega)$.

In the next example we show what the Hadamard product of an arbitrary function with some very specific kind of function looks like (see [Po, Ex. 3.4.6, Ex. 3.4.7]). This example will be crucial for the application of the Hadamard product in approximation theory.

Example 2.17 :
Let $\Omega \subset \mathbb{C}_\infty$ be open with $\{0, \infty\} \subset \Omega$ and let $\varphi \in H(\Omega)$. Let furthermore $\nu \in \mathbb{Z}$ and $U \subset \mathbb{C}_\infty$ be open and star-eligible to Ω with $\infty \notin U$ in case $\nu \geq 0$ and $0 \notin U$ in case $\nu < 0$. Then the functions $p_{\nu,U}(z) := z^\nu$ $(z \in U)$ are holomorphic in U and

$$\varphi * p_{\nu,U} = \operatorname{sign}(\nu)\varphi_\nu \cdot p_{\nu,\Omega * U} .$$

In Theorem 3.7 we are going to transfer this property on the one hand to a more specific class of sets Ω and U but on the other hand to more general monomials $p_{\alpha,U}$.

Remark 2.18 :
The assertion of Example 2.17 could, at first glance, seem as a triviality. That is, if $\nu \geq 0$ and $U \subset \mathbb{C}$ is a domain containing the origin, then the assertion is indeed clear by means of the respective power series expansions about 0 and Theorem 2.14.3. However, this statement is also true if $0 \notin U$ or, if U is not connected, for those components of U that do not contain the origin even though we do not have a power series expansion about zero of the function $p_{\nu,U}$ in that case.

A similar remark applies to the case $\nu < 0$ and $U \subset \mathbb{C}_\infty \setminus \{0\}$.

2.3 Further properties of the Hadamard product

In the following, we list some further properties of the Hadamard product which will be useful in the rest of the thesis.

Remark 2.19 :
Let $\Omega, U \subset \mathbb{C}_\infty$ be open and star-eligible. Then for every $a \in \mathbb{C} \setminus \{0\}$ the following is valid:

$$\Omega * (a \cdot U) = (a \cdot \Omega) * U = a \cdot (\Omega * U), \ (a \cdot \Omega)^* = a^{-1} \cdot \Omega^* .$$

This follows immediately from Definition 2.1 and the fact that $(a \cdot B)^C = a \cdot (B^C)$ for an arbitrary subset B of $\mathbb{C}_\infty$.

Furthermore, for an open set $D \subset \mathbb{C}_\infty$, the transformation

$$H(D) \ni h \mapsto \tilde{h} \in H(a \cdot D) \text{ with } \tilde{h}(w) = h(w/a) \ (w \in a \cdot D)$$

is an isomorphism (and $h(z) = \tilde{h}(az)$ $(z \in D)$).

If we are given functions $\varphi \in H(\Omega)$ and $f \in H(U)$ we obtain for all $w \in a \cdot \Omega * U$

$$\varphi *_{\Omega,U} f(w/a) = \tilde{\varphi} *_{a\cdot\Omega,U} f(w) = \varphi *_{\Omega,a\cdot U} \tilde{f}(w) . \tag{2.2}$$

Furthermore, for $f_n \in H(U)$ $(n \in \mathbb{N})$ and $g \in H(\Omega * U)$ the following is valid:

If $\varphi *_{\Omega,a\cdot U} \tilde{f}_n \to \tilde{g}$ $(n \to \infty)$ or $\tilde{\varphi} *_{a\cdot\Omega,U} f_n \to \tilde{g}$ $(n \to \infty)$ locally uniformly on $a \cdot \Omega * U$, then $\varphi *_{\Omega,U} f_n \to g$ $(n \to \infty)$ locally uniformly on $\Omega * U$.

Proposition 2.20 :
Let $\Omega, U \subset \mathbb{C}_\infty$ be open and star-eligible, $\varphi \in H(\Omega)$ and $f \in H(U)$. If $\Omega = -\Omega$, then $\Omega * U = -\Omega * U$ and if φ is an even (odd) function, then so is $\varphi * f$.

Proof: Remark 2.19 yields $-\Omega * U = \Omega * U$.

Furthermore, we obtain with Equation (2.2) $(a = -1)$ in the case that φ is even $\varphi * f(-z) = \varphi * f(z)$ $(z \in \Omega * U)$. In the case that φ is odd we also use Equation (2.2) $(a = -1)$ and the obvious fact that $(\alpha \cdot \varphi) * f = \alpha \cdot (\varphi * f)$ for all $\alpha \in \mathbb{C}$ to obtain that $\varphi * f(-z) = -\varphi * f(z)$ $(z \in \Omega * U)$. $\square$

Remark 2.21 :

1. Let $\Omega \subset \mathbb{C}_\infty$ be open. We consider the map

$$\mathcal{U} : H(\Omega) \to H(1/\Omega),\ \mathcal{U}\varphi(z) := \begin{cases} -\frac{1}{z}\varphi(\frac{1}{z}) & , z \in (1/\Omega) \setminus \{0\} \\ -\varphi_{-1} & , z = 0 \quad (\text{if } 0 \in 1/\Omega) \end{cases} .$$

   Then one easily verifies that this map is well defined and even a topological isomorphism.

   If $\infty \in \Omega$, then $0 \in 1/\Omega$ and for all z with small modulus we obtain

$$\mathcal{U}\varphi(z) = -\sum_{\nu=0}^{\infty} \varphi_{-\nu-1} z^{\nu}$$

   and therefore $(\mathcal{U}\varphi)_\nu = -\varphi_{-\nu-1}$ ($\nu \in \mathbb{N}_0$) and $\Lambda^+_{\mathcal{U}\varphi} = \Lambda^-_\varphi - 1$.

   If $0 \in \Omega$, then $\infty \in 1/\Omega$ and for all z with large modulus we obtain

$$\mathcal{U}\varphi(z) = -\sum_{\nu=1}^{\infty} \varphi_{\nu-1} z^{-\nu}$$

   and therefore $(\mathcal{U}\varphi)_{-\nu} = -\varphi_{\nu-1}$ ($\nu \in \mathbb{N}$) and $\Lambda^-_{\mathcal{U}\varphi} = \Lambda^+_\varphi + 1$.

2. Let $U \subset \mathbb{C}$ be open. Then the map

$$[\mathcal{U}] : H(U^C) \to H(U^*),\ [(\varphi, V)]_{U^C} \mapsto [(\mathcal{U}\varphi, 1/V)]_{U^*}$$

   is a topological isomorphism, too.

Proposition 2.22 :
Let $\Omega, U \subset \mathbb{C}_\infty$ be open and star-eligible with $\{0,\infty\} \subset \Omega$, $\varphi \in H(\Omega)$ and $f \in H(U)$. Then

$$(\mathcal{U}\varphi) * (\mathcal{U}f) = \mathcal{U}(\varphi * f) .$$

Proof: First of all we note that according to Remark 2.4 the sets $1/\Omega$ and $1/U$ are star-eligible and on both sides of the asserted identity occur functions belonging to the space $H(1/(\Omega * U))$.

Let now $z \in 1/(\Omega * U)$. Without loss of generality we can assume that $z \neq \infty$. If $z = 0$ we observe with Remark 2.21.1 and Theorem 2.14.3 that

$$\big((\mathcal{U}\varphi) * (\mathcal{U}f)\big)(0) = \varphi_{-1} \cdot f_{-1} = -(\varphi * f)_{-1} = \big(\mathcal{U}(\varphi * f)\big)(0) .$$

For $z \neq 0, \infty$ we obtain

$$\begin{aligned}\big((\mathcal{U}\varphi) * (\mathcal{U}f)\big)(z) &= \frac{1}{2\pi i}\int_\Gamma \mathcal{U}\varphi(\frac{z}{\zeta})\mathcal{U}f(\zeta)\frac{d\zeta}{\zeta} \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{\zeta}{z}\varphi(\frac{\zeta}{z})f(\frac{1}{\zeta})\frac{d\zeta}{\zeta^2} \\ &= -\frac{1}{z}\frac{1}{2\pi i}\int_{1/\Gamma}\varphi(\frac{1}{z\zeta})f(\zeta)\frac{d\zeta}{\zeta} \\ &= -\frac{1}{z}\cdot(\varphi * f)(\frac{1}{z}) = \mathcal{U}(\varphi * f)(z)\,,\end{aligned}$$

where Γ is a Hadamard cycle for $z \cdot \Omega^C$ in $1/U$ (using Remark 1.2.3 one verifies that $1/\Gamma$ is a Hadamard cycle for $(1/z)\cdot\Omega^*$ in U). □

As a next step, we want to formulate an associative law for the Hadamard product which will be of high relevance in the following.

Theorem 2.23 :
Let $\Omega \subset \mathbb{C}_\infty$ as well as $U \subset \mathbb{C}$ be open and star-eligible with $\{0,\infty\} \subset \Omega$ and let $V \subset \mathbb{C}_\infty$ be open, star-eligible to $\Omega * U$ and such that

1. *$\{0,\infty\} \subset V$ in case $0 \notin U$,*
2. *$0 \in V, \infty \notin V$ in case $0 \in U$.*

For $g \in H(V)$, $\varphi \in H(\Omega)$ and $f \in H(U)$ we have

$$g * (\varphi * f) = (g * \varphi) * f\,.$$

Proof: By assumption, all occuring star products are defined and non-empty and according to Example 2.3.2 we obtain $V * (\Omega * U) = (V * \Omega) * U$.

Let $w \in V * \Omega * U$. If $w = 0$, Theorem 2.14.3 implies that

$$g * (\varphi * f)(0) = g_0 \cdot \varphi_0 \cdot f_0 = (g * \varphi) * f(0)\,.$$

For $w \neq 0$ we obtain with Theorem 2.14.2

$$\begin{aligned}
(g *_{V,\Omega*U} (\varphi *_{\Omega,U} f))(w) &= ((\varphi *_{\Omega,U} f) *_{\Omega*U,V} g)(w) \\
&= \frac{1}{2\pi i}\int_{\Gamma_1} (\varphi *_{\Omega,U} f)(\frac{w}{t})g(t)\frac{dt}{t} && (2.3) \\
&= \frac{1}{2\pi i}\int_{\Gamma_1} \frac{g(t)}{t}\frac{1}{2\pi i}\int_{\Gamma_2} \varphi(\frac{w}{t\zeta})f(\zeta)\frac{d\zeta}{\zeta}dt && (2.4) \\
&= \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(\zeta)}{\zeta}\frac{1}{2\pi i}\int_{\Gamma_1} g(t)\varphi(\frac{w}{\zeta t})\frac{dt}{t}d\zeta \\
&= \frac{1}{2\pi i}\int_{\Gamma_2} \frac{f(\zeta)}{\zeta}\frac{1}{2\pi i}\int_{\tilde{\Gamma}} g(t)\varphi(\frac{w}{\zeta t})\frac{dt}{t}d\zeta && (2.5) \\
&= \frac{1}{2\pi i}\int_{\Gamma_2} (\varphi *_{\Omega,V} g)(\frac{w}{\zeta})f(\zeta)\frac{d\zeta}{\zeta} && (2.6) \\
&= ((\varphi *_{\Omega,V} g) *_{\Omega*V,U} f)(w) && (2.7) \\
&= ((g *_{V,\Omega} \varphi) *_{V*\Omega,U} f)(w)\,.
\end{aligned}$$

To (2.3):

Γ_1 is chosen as a Hadamard cycle for $w \cdot (\Omega * U)^*$ in V. We note that by definition of the hull it is clear that $|\Gamma_1| \cap (h_{(\Omega*U)/w}(1/|\Gamma_1|))^* = \emptyset$.

Hence, according to Definition 2.9 we choose Γ_1 to be

1. an anti-Cauchy cycle for $w \cdot (\Omega * U)^*$ in V and therefore

$$\mathrm{ind}_{\Gamma_1}(z) = 0 \quad (z \in w \cdot (\Omega * U)^*), \quad \mathrm{ind}_{\Gamma_1}(z) = -1 \quad (z \in V^C)\,. \tag{2.8}$$

   Then we obtain additionally

$$\mathrm{ind}_{\Gamma_1}(z) = 0 \quad \big(z \in (h_{(\Omega*U)/w}(1/|\Gamma_1|))^*\big)\,. \tag{2.9}$$

   Indeed, by definition of the hull, each component of $(h_{(\Omega*U)/w}(1/|\Gamma_1|))^*$ meets a component of $w \cdot (\Omega * U)^*$. Therefore (2.9) is a direct consequence of (2.8) and Remark 1.2.2.

2. a Cauchy cycle for $w \cdot (\Omega * U)^*$ in V and therefore

$$\mathrm{ind}_{\Gamma_1}(z) = 1 \quad (z \in w \cdot (\Omega * U)^*), \quad \mathrm{ind}_{\Gamma_1}(z) = 0 \quad (z \in V^C)\,. \tag{2.10}$$

   (note that the supplementary index condition $\mathrm{ind}_{\Gamma_1}(0) = 1$ is automatically fulfilled since $0 \in (\Omega * U)^*$). Then we obtain additionally

$$\mathrm{ind}_{\Gamma_1}(z) = 1 \quad \big(z \in (h_{(\Omega*U)/w}(1/|\Gamma_1|))^*\big)\,. \tag{2.11}$$

We remark that in both cases, the respective two index properties imply

$$V^* \subset h_{(\Omega * U)/w}(1/|\Gamma_1|)$$

and we conclude

$$(w \cdot (1/|\Gamma_1|) \cdot \Omega^*) \cup (w \cdot V^* \cdot \Omega^*) \subset w \cdot h_{(\Omega * U)/w}(1/|\Gamma_1|) \cdot \Omega^* . \tag{2.12}$$

To (2.4):

Γ_2 has to be chosen as a Hadamard cycle for $(w/|\Gamma_1|) \cdot \Omega^*$ in U. We impose a somewhat stronger condition and require Γ_2 to be a Hadamard cycle for $w \cdot h_{(\Omega * U)/w}(1/|\Gamma_1|) \cdot \Omega^*$ in U. This is possible since the choice of Γ_1 ensures that the relation $|\Gamma_1| \cap w \cdot (\Omega * U)^* = \emptyset$ holds. Hence, $(1/|\Gamma_1|)$ and consequently $h_{(\Omega * U)/w}(1/|\Gamma_1|)$ is a compact subset of $(\Omega * U)/w$ (see Remark 1.9.3). This, in turn, implies that $w \cdot h_{(\Omega * U)/w}(1/|\Gamma_1|) \cdot \Omega^*$ is a compact subset of U (see Proposition 2.5.5) with $\infty \notin w \cdot h_{(\Omega * U)/w}(1/|\Gamma_1|)$. Proposition 2.10.2 ensures the existence of a Hadamard cycle for $w \cdot h_{(\Omega * U)/w}(1/|\Gamma_1|) \cdot \Omega^*$ in U. Relation (2.12) yields that the so chosen cycle is suitable.

Hence, according to Definition 2.9 we choose Γ_2 to be

1. a Cauchy cycle for $w \cdot h_{(\Omega * U)/w}(1/|\Gamma_1|) \cdot \Omega^*$ in U.
2. a Cauchy cycle for $w \cdot h_{(\Omega * U)/w}(1/|\Gamma_1|) \cdot \Omega^*$ in U (note that (2.12) implies that the supplementary index condition $\text{ind}_{\Gamma_2}(0) = 1$ is automatically fulfilled since $0 \in V^*$).

To (2.5):

The cycle $\tilde{\Gamma}$ is chosen as

1. $\tilde{\Gamma} := (\Gamma_1, \tau_r^-)$ with $0 < r < \min\{\text{dist}(0, V^C), \text{dist}(0, (w/|\Gamma_2|) \cdot \Omega^*)\}$. Note that the addition of τ_r^- does not change the value of the integral because the inner integrand is holomorphic in a neighbourhood of the origin and τ_r^- is a closed curve.
2. $\tilde{\Gamma} := \Gamma_1$.

To (2.6):

$\tilde{\Gamma}$ should be a Hadamard cycle for $(w/|\Gamma_2|) \cdot \Omega^*$ in V. We are going to check that now:

1. (a) $|\Gamma_1| \subset V$ according to the choice of Γ_1 and $|\tau_r^-| \subset V$ according to the choice of r.

   (b) We have
   $$w \cdot h_{(\Omega * U)/w}(1/|\Gamma_1|) \cap (|\Gamma_2| \cdot \Omega^C) = \emptyset\,. \tag{2.13}$$
   Otherwise, this would contradict the fact that $|\Gamma_2| \cap w \cdot h_{(\Omega*U)/w}(1/|\Gamma_1|) \cdot \Omega^* = \emptyset$. A first consequence is that $|\Gamma_1| \cap (w/|\Gamma_2|) \cdot \Omega^* = \emptyset$ and obviously, $|\tau_r^-| \cap (w/|\Gamma_2|) \cdot \Omega^* = \emptyset$ according to the choice of r.

   (c) A second consequence of (2.13) is that $(w/|\Gamma_2|) \cdot \Omega^* \subset (h_{(\Omega*U)/w}(1/|\Gamma_1|))^*$ and therefore (see (2.9) and (2.8))
   $$\begin{aligned} \mathrm{ind}_{\Gamma_1}(z) = \mathrm{ind}_{\tilde{\Gamma}}(z) = &\quad 0 \qquad (z \in (w/|\Gamma_2|) \cdot \Omega^*), \\ \mathrm{ind}_{\Gamma_1}(z) = \mathrm{ind}_{\tilde{\Gamma}}(z) = &\ -1 \qquad (z \in V^C)\,. \end{aligned}$$
   Furthermore it is exactly the addition of τ_r^- to the integration cycle which ensures
   $$\mathrm{ind}_{\tilde{\Gamma}}(0) = -1$$
   (note that $0 \in (\Omega * U)^*$ and therefore $\mathrm{ind}_{\Gamma_1}(0) = 0$). Hence, $\tilde{\Gamma}$ is as an anti-Cauchy cycle for $(w/|\Gamma_2|) \cdot \Omega^*$ in V with $\mathrm{ind}_{\tilde{\Gamma}}(0) = -1$ a suitable Hadamard cycle.

2. It can be checked analogously to above that $|\Gamma_1| \subset V$ and that (2.13) holds. Again, we obtain $|\Gamma_1| \cap (w/|\Gamma_2|) \cdot \Omega^* = \emptyset$ and $(w/|\Gamma_2|) \cdot \Omega^* \subset (h_{(\Omega*U)/w}(1/|\Gamma_1|))^*$ and therefore (see (2.11), (2.10))
   $$\begin{aligned} \mathrm{ind}_{\Gamma_1}(z) = \mathrm{ind}_{\tilde{\Gamma}}(z) = &\ 1 \qquad (z \in (w/|\Gamma_2|) \cdot \Omega^*)\,, \\ \mathrm{ind}_{\Gamma_1}(z) = \mathrm{ind}_{\tilde{\Gamma}}(z) = &\ 0 \qquad (z \in V^C)\,, \\ \mathrm{ind}_{\Gamma_1}(0) = \mathrm{ind}_{\tilde{\Gamma}}(0) = &\ 1 \qquad . \end{aligned}$$
   Hence, $\tilde{\Gamma}$ is as a Cauchy cycle for $(w/|\Gamma_2|) \cdot \Omega^*$ in V with $\mathrm{ind}_{\tilde{\Gamma}}(0) = 1$ a suitable Hadamard cycle.

To (2.7):

Γ_2 should be a Hadamard cycle for $w \cdot (\Omega * V)^*$ in U. We are going to check that now:

1. (a) $|\Gamma_2| \subset U$ according to the choice of Γ_2.

   (b) $|\Gamma_2| \cap w \cdot (\Omega * V)^* = |\Gamma_2| \cap w \cdot \Omega^* \cdot V^* = \emptyset$ due to (2.12) and the choice of Γ_2.

(c) The second consequence of (2.12) and the choice of Γ_2 is

$$\begin{aligned} \mathrm{ind}_{\Gamma_2}(z) = &\ 1 \quad (z \in w \cdot \Omega^* \cdot V^*)\,, \\ \mathrm{ind}_{\Gamma_2}(z) = &\ 0 \quad (z \in U^C)\,. \end{aligned}$$

Hence, Γ_2 is as a Cauchy cycle for $w \cdot (\Omega * V)^*$ in U a suitable Hadamard cycle.

2. Analogously we obtain

   (a) $|\Gamma_2| \subset U$ according to the choice of Γ_2.

   (b) $|\Gamma_2| \cap w \cdot (\Omega * V)^* = |\Gamma_2| \cap w \cdot \Omega^* \cdot V^* = \emptyset$ due to (2.12) and the choice of Γ_2.

   (c) The second consequence of (2.12) and the choice of Γ_2 is

$$\begin{aligned} \mathrm{ind}_{\Gamma_2}(z) = &\ 1 \quad (z \in w \cdot \Omega^* \cdot V^*)\,, \\ \mathrm{ind}_{\Gamma_2}(z) = &\ 0 \quad (z \in U^C)\,, \\ \mathrm{ind}_{\Gamma_2}(0) = &\ 1 \quad . \end{aligned}$$

   Hence, Γ_2 is as a Cauchy cycle for $w \cdot (\Omega * V)^*$ in U with $\mathrm{ind}_{\Gamma_2}(0) = 1$ a suitable Hadamard cycle. □

We conclude this chapter by giving an alternative representation of the dual space $H'(D)$ (for $D \subset \mathbb{C}$ open) by means of the Hadamard product. The theorem is an extension of a result obtained by Grosse-Erdmann for the case $0 \in D$ (see [GE, Th. 3.6]).

Theorem 2.24 :
Let $D \subset \mathbb{C}$ be open. The dual of the space $H(D)$ is topologically isomorphic to the space $H(D^*)$ in such a way that to every functional $u \in H'(D)$ there exists a unique germ $[(g,U)]_{D^*} \in H(D^*)$ such that

$$u(f) = (f *_{D,U} g)(1) \quad (f \in H(D))\,.$$

Proof: Theorem 1.13 and Remark 2.21.2 yield that $H'(D)$ and $H(D^*)$ are topologically isomorphic. Let $u \in H'(D)$ and let $[(\tilde{g}, V)]_{D^C} \in H(D^C)$ be the unique germ such that

$$u(f) = \frac{1}{2\pi i} \int_\Gamma f(\zeta)\tilde{g}(\zeta)\ d\zeta$$

holds for all $f \in H(D)$, where Γ is a Cauchy cycle for V^C in D.

Then a representative of $-[\mathcal{U}][(\tilde{g}, V)]_{D^C} \in H(D^*)$ is given by $-\mathcal{U}\tilde{g} =: g$ being holomorphic in $1/V =: U$ (see Remark 2.21.2). Then the identity

$$\begin{aligned} u(f) &= \frac{1}{2\pi i}\int_\Gamma f(\zeta)\tilde{g}(\zeta)\, d\zeta \\ &= \frac{1}{2\pi i}\int_\Gamma f(\zeta)g(\frac{1}{\zeta})\frac{1}{\zeta}\, d\zeta \end{aligned}$$

holds for all $f \in H(D)$. In order that the latter integral is equal to $(g *_{U,D} f)(1)$ (and hence to $(f *_{D,U} g)(1)$, note that $D * U$ is defined and $1 \in D * U$ is always ensured) we have to check that Γ is a Hadamard cycle for $1 \cdot U^* = V^C$ in D. We consider the cases $0 \notin D$ and $0 \in D$ separately.

1. If $0 \notin D$, Definition 2.9.(d) requires Γ to be a Cauchy cycle for V^C in D which is just what Γ is.
2. If $0 \in D$, Definition 2.9.(a) requires Γ to be a Cauchy cycle for V^C in D with $\mathrm{ind}_\Gamma(0) = 1$. Since we can without loss of generality choose V so small that $0 \notin V$ and hence $0 \in V^C$, this additional condition does not pose a problem.

Hence, in both cases we obtain that $[(g, U)]_{D^*}$ is a germ of holomorphic functions on D^* with

$$u(f) = (f *_{D,U} g)(1) \quad (f \in H(D))\,.$$

Furthermore, it is clear that $[(g, U)]_{D^*}$ is the only element in $H(D^*)$ such that $u(f) = (f *_{D,U} g)(1)$ holds for all $f \in H(D)$.

Finally, we show that every element $[(g, U)]_{D^*} \in H(D^*)$ defines a functional $u \in H'(D)$ in this way. We have to show that the mapping $H(D) \ni f \mapsto (f *_{D,U} g)(1)$ is linear and continuous. The linearity is an immediate consequence of the linearity of the Hadamard product and we obtain

$$|(f *_{D,U} g)(1)| \le M \cdot \|f\|_{|\Gamma|}$$

with $M := L(\Gamma) \cdot \|\tilde{g}\|_{|\Gamma|}/2\pi$ which implies the continuity of the mapping. □

Remark 2.25 :
The independence of the Hadamard product from the choice of the specific Hadamard cycle (see [MP, Th. 2.5]) assures that the value of $(f *_{D,U} g)(1)$ in Theorem 2.24 does not change while using another representative $(\tilde{g}, V)$ of $[(g, U)]_{D^*}$.

Chapter 3

The Hadamard operator T_φ as a convolution operator on spaces of holomorphic functions

In the following, we will mainly consider the Hadamard product of holomorphic functions as an operator between Fréchet spaces of the type $H(\Omega)$, i. e. for open and star-eligible sets $\Omega, U \subset \mathbb{C}_\infty$ we consider the operator

$$T : H(\Omega) \times H(U) \to H(\Omega * U),\ (\varphi, f) \mapsto \varphi * f$$

or with a fixed set Ω and a fixed function $\varphi \in H(\Omega)$ the *Hadamard operator*

$$T_{\varphi,\Omega,U} : H(U) \to H(\Omega * U),\ f \mapsto \varphi * f\,.$$

Since the Hadamard multiplication of φ and f means convolving φ and f in an appropriate way, Hadamard operators can be considered as convolution operators on $H(U)$.

It is important to keep in mind that while speaking of the Hadamard operator $T_{\varphi,\Omega,U}$, it is crucial to consider the underlying sets Ω and U, i. e. the operator is only well specified by naming both the function φ and the sets Ω, U. However, in order to keep the notation as short as possible, depending on what degree of exactness the context requires we also use the notations $T_\varphi := T_{\varphi,U} := T_{\varphi,\Omega,U}$.

The main purpose of this thesis is the examination of the class of Hadamard operators with respect to their kernel and their range. For the application of the Hadamard product in approximation theory (see Chapter 6) it is in particular interesting under which conditions we can expect Hadamard operators to have dense range or even to be surjective. In other words, we shall try to find conditions on the sets Ω and U and on the function $\varphi \in H(\Omega)$ such that we can deduce some information about the induced operator $T_\varphi : H(U) \to H(\Omega * U)$.

Müller and Pohlen showed a first essential property of Hadamard operators (see [MP, Th. 2.9]).

Theorem 3.1 :
Let $\Omega, U \subset \mathbb{C}_\infty$ be open and star-eligible. The operator $T : H(\Omega) \times H(U) \to H(\Omega * U), (\varphi, f) \mapsto \varphi * f$ is bilinear and continuous (where $H(\Omega) \times H(U)$ carries the product topology).

Theorem 3.1 especially implies that Hadamard operators $T_\varphi : H(U) \to H(\Omega * U)$ are linear and continuous.

We begin the more detailed examination of Hadamard operators by a simple and illustrative example.

Example 3.2 :
We consider the *Koebe function*

$$\kappa(z) := \frac{z}{(1-z)^2} \quad (z \in \mathbb{C}_\infty \setminus \{1\}) .$$

The function κ is obviously holomorphic in $\mathbb{C}_\infty \setminus \{1\}$ and its power series expansion about zero is given by

$$\kappa(z) = \sum_{\nu=1}^{\infty} \nu z^\nu \quad (z \in \mathbb{D})$$

and hence $\Lambda_\kappa^+ = \mathbb{N}$. For a given open set $U \subset \mathbb{C}$ we easily compute what the operator $T_\kappa : H(U) \to H(U)$ looks like. Indeed, for $f \in H(U)$ and $z \in U \setminus \{0\}$ we have

$$\begin{aligned} T_\kappa f(z) &= \frac{1}{2\pi i} \int_{\Gamma_z} f(\zeta) \frac{z/\zeta}{(1 - z/\zeta)^2} \frac{d\zeta}{\zeta} \\ &= z \cdot \frac{1}{2\pi i} \int_{\Gamma_z} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \\ &= z \cdot f'(z) \end{aligned}$$

where according to Definition 2.9 the cycle Γ_z can be chosen as

$$\Gamma_z = \begin{cases} \tau_r(z) & , \text{ if } 0 \notin U \\ (\tau_r(z), \tau_{r'}) & , \text{ if } 0 \in U \end{cases} ,$$

with $0 < r, r'$ sufficiently small. The last identity above follows from the Cauchy integral formula noting that in case $0 \in U$, the cycle $\tau_{r'}$ does not change the value of the integral because the integrand in holomorphic in a neighbourhood of the origin and $\tau_{r'}$ is a closed path. Obviously, the above identity also holds for $z = 0$ (if $0 \in U$).

It is clear that the functions being constant on each component of U belong to the kernel of T_κ. To reveal some information concerning its range, we assume that U is a simply connected domain not containing the origin. We shall prove that for such sets U, the operator $T_\kappa : H(U) \to H(U)$ is surjective.

Let $g \in H(U)$ be given. We are supposed to find a function $f \in H(U)$ such that

$$f'(z) = \frac{g(z)}{z} \quad (z \in U)\,.$$

The right-hand side of this equation is holomorphic in U and since holomorphic functions on simply connected domains have a primitive, we obtain the existence of a function $f \in H(U)$ with $T_\kappa f = g$.

In Proposition 4.30 we are going to generalize this result.

Remark 3.3 :
In Example 3.2, the considered function κ and the underlying sets $\mathbb{C}_\infty \setminus \{1\}$ and U have been comparatively easy or rather exhibited some geometric structure which allowed to obtain results concerning the kernel and the range of the induced Hadamard operator (for example U was simply connected and we were able to use the well-known fact that holomorphic functions on simply connected domains have a primitive). In a more general context this will of course not be possible and it is not surprising that the less restrictive the assumptions are, for example the more general the sets Ω and U are chosen, the less properties the corresponding Hadamard operator will have or at least, the more difficult it will be to prove them. If, on the contrary, we restrict ourselves to more specific sets, we can hope to profit from the geometric structure and prove stronger results. It will turn out that one possible shape of the set Ω which is both of high relevance and allows to exploit the geometric structure is $\Omega = D_K = (e^K)^*$ for some compact and convex set $K \subset \mathbb{V}$. Indeed, it is the existence of a one-to-one correspondence between the space $H(D_K)$ and a space of certain entire functions which makes the examination of the Hadamard operator $T_{\varphi,D_K,U}$ more easily accessible. The required concepts are introduced in Appendix B.

In order that the Hadamard operator $T_\varphi : H(U) \to H(\Omega * U)$ is meaningful, we first of all have to impose the conditions that $\Omega, U \subset \mathbb{C}_\infty$ are open and star-eligible

(i. e. $\Omega * U$ is defined and non-empty) and that the function $\varphi \in H(\Omega)$ does not vanish identically. However, in the following we are mostly going to assume that Ω is a domain containing both the origin and the point at infinity. This procedure is motivated by the following observations:

1. While examining the Hadamard product it already turned out that the integration cycle Γ in Definition 2.11 is of essential significance. In the definition of Hadamard cycles (see Definition 2.9) we had to distinguish multiple cases with respect to the respective position of the origin and the point at infinity. Requiring Ω to contain both the origin and the point at infinity bears two advantages: Firstly, the condition of star-eligibility is reduced to the condition $\Omega * U \neq \emptyset$ so that U may contain the origin or the point at infinity or not and secondly, the amount of different cases that have to be considered concerning the Hadamard cycles is reduced considerably.

2. While examining the kernel of Hadamard operators, we will often encounter the assumption that $\Omega * U$ shall be connected. Furthermore we will pay most of our attention to the case $\{0, \infty\} \subset U$. As we have seen in Remark 2.7.2, if the origin or the point at infinity belongs to both sets Ω and U, then it is meaningful to assume that Ω and U are connected, too.

3. While examining the range of Hadamard operators, we will often be faced with the assumption that $(\Omega * U)^*$ (or equivalently $(\Omega * U)^C$) is connected. In addition, we will mostly be concerned with the case $U \subset \mathbb{C} \setminus \{0\}$. A typical situation would be that $U \subset \mathbb{C} \setminus \{0\}$ has connected complement (see Proposition 2.5.6). If we take the star product of an open set Ω with $\Omega_0 \neq \Omega_\infty$ with a set $U \subset \mathbb{C} \setminus \{0\}$ having connected complement, then a consequence of the definition of the star product is that $\Omega * U$ will typically be the empty set. Hence, the condition of star-eligibility will at least require Ω to contain zero and infinity in the same component.

4. As already mentioned above, we will often be concerned with the case that Ω is of the form D_K for some compact and convex set $K \subset \mathbb{V}$. Since D_K is obviously a domain containing zero and infinity and since this is the most important case for practical applications, it does not seem like being too much of a loss of generality to require a more general set Ω to be connected and to contain zero and infinity, too.

If Ω is a domain containing the origin and the point at infinity, then the identity theorem yields that all the information about a function $\varphi \in H(\Omega)$ is already contained in its power series expansions about zero or infinity introduced in Remark 1.7. Especially the sets Λ_φ^+ and Λ_φ^- of non-vanishing coefficients will play an important role in the following. There is a close interplay between the set Ω and the "size" of the sets Λ_φ^+ and Λ_φ^-. As we already stated above, it is desirable that the function φ does not vanish identically, otherwise we would obtain a trivial operator. This requirement already imposes necessary conditions on Λ_φ^+ and Λ_φ^-, that means those sets may not be "too small". We are going to state that interdependency more precisely in Section 3.1. In order to measure this size accurately, different notions of density for subsets of the non-negative real numbers are introduced in Appendix A.

In Section 3.2, we are going to present some kind of "eigenvalue" property for Hadamard operators which will be of high relevance in the examination of their range.

We shall continue the examination of Hadamard operators by computing the transposed operator of $T_\varphi : H(U) \to H(\Omega * U)$ for rather general sets Ω, U and functions $\varphi \in H(\Omega)$. This is not only of interest by itself but will be particularly useful for the investigation of the kernel and the range of T_φ.

3.1 Preliminary information about the operator T_φ

First of all the question should be answered under what assumptions on the sets Λ_φ^+ or Λ_φ^- the function φ being holomorphic in a domain Ω containing zero and infinity can be expected not to vanish identically and if so, what further information can be obtained concerning the geometry of Ω (we recall that we assumed holomorphic functions to vanish at infinity). This information is needed to know under which circumstances we obtain a meaningful and non-trivial operator $T_\varphi : H(U) \to H(\Omega * U)$.

Proposition 3.4 :
Let $\Omega \subset \mathbb{C}_\infty$ be a domain with $\{0, \infty\} \subset \Omega$. In order that $\varphi \in H(\Omega)$ does not vanish identically, it is necessary that $\underline{d}(\Lambda_\varphi^+) > 0$ and $\underline{d}(\Lambda_\varphi^-) > 0$.

Proof: A result of Pólya states that the domain of existence of a power series whose non-vanishing coefficients have lower density zero is a simply connected part of the plane (see [Pol33, p. 737]). The assumption that φ is holomorphic in the domain Ω which contains the origin and the point at infinity implies that the power series expansion of φ about zero can be analytically continued up to infinity.

If $\underline{d}(\Lambda_\varphi^+) = 0$, then the above mentioned result of Pólya only leaves the possibility that φ vanishes identically.

Since $\mathcal{U}\varphi$ is holomorphic in the domain $1/\Omega$ which contains the origin and the point at infinity, too, the power series expansion of $\mathcal{U}\varphi$ about zero can be analytically continued up to infinity. If $\underline{d}(\Lambda_\varphi^-) = 0$, then Remark 2.21.1 implies that $\underline{d}(\Lambda_{\mathcal{U}\varphi}^+) = 0$ and Pólya's result only leaves the possibility that $\mathcal{U}\varphi$ vanishes identically. Since $\mathcal{U}$ is an isomorphism, we obtain $\varphi \equiv 0$. □

Remark 3.5 :
In Proposition 3.4 we stated some necessary conditions on the density of Λ_φ^+ and Λ_φ^- for the function φ being holomorphic in a domain containing zero and infinity not to vanish identically. The question arises whether such a density condition is also sufficient to obtain a domain Ω with $\{0, \infty\} \subset \Omega$ and a non-vanishing function $\varphi \in H(\Omega)$ with the desired gaps in its power series expansion about zero. The answer is yes as the following result of Pólya shows (see [Pol42]). We only give a short idea of the proof:

Let $\Lambda \subset \mathbb{N}_0$ with $\underline{d}(\Lambda) > 0$. Then there exists a domain Ω with $\{0, \infty\} \subset \Omega$ and a non-vanishing function $\varphi \in H(\Omega)$ with $\Lambda_\varphi^+ = \Lambda$.

Indeed, assuming without loss of generality $0 \notin \Lambda =: \{\lambda_n :\ n \in \mathbb{N}\}$, the function

$$G(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right) \quad (z \in \mathbb{C})$$

is an entire function with simple zeros at each point in $\pm\Lambda$ and we set

$$\Phi(z) := \begin{cases} \dfrac{\sin \pi z}{G(z)} & ,\ z \in \mathbb{C} \setminus \{\pm\Lambda\} \\ \dfrac{\pi(-1)^z}{G'(z)} & ,\ z \in \pm\Lambda \end{cases} .$$

Then we obtain $\Phi(n) = 0$ $(n \in \mathbb{N}_0 \setminus \Lambda)$ and $\Phi(n) \neq 0$ $(n \in \Lambda)$. Furthermore it can be shown that Φ is an entire function of exponential type with $K(\Phi) \subset \mathbb{V}$ so that Theorem B.16 yields that $\varphi := \mathfrak{M}^{-1}\Phi \in H(D_{K(\Phi)})$ fulfills all the requirements (where $K(\Phi)$ denotes the conjugate indicator diagram of Φ and $\mathfrak{M}$ denotes the Mellin transformation, see Appendix B).

Remark 3.6 :
After examining under which conditions on the set Λ_φ^+ we can expect the function $\varphi \in H(\Omega)$ not to vanish identically, we give further results which reveal a connection between the size of the set Λ_φ^+ and the geometry of the set Ω:

1. Let $\Lambda \subset \mathbb{N}_0$ with $d(\mathbb{N}_0 \setminus \Lambda) = \delta \in [0, 1)$. Then there exists a non-vanishing function $\varphi \in H(D_{K_\delta})$ with $\Lambda_\varphi^+ = \Lambda$.[1]
2. Let $\Lambda \subset \mathbb{N}_0$ with $d_L(\mathbb{N}_0 \setminus \Lambda) < b \in (0, 1)$. Then there exists a domain Ω with $\{0, \infty\} \subset \Omega$ and containing a sector of opening $(1 - b)$ and a non-vanishing function $\varphi \in H(\Omega)$ with $\Lambda_\varphi^+ = \Lambda$.[2]

3.2 An eigenvalue property of the operator T_φ

In Example 2.17 we have seen that for $\Omega, U \subset \mathbb{C}_\infty$ open and star-eligible with $\{0, \infty\} \subset \Omega \setminus U$ and $\varphi \in H(\Omega)$ we have

$$T_\varphi p_{\nu,U} = \operatorname{sign}(\nu)\varphi_\nu \cdot p_{\nu,\Omega * U} \quad (\nu \in \mathbb{Z})\,.$$

This identity can be interpreted in such a way that the monomials $p_{\nu,U}$ are eigenfunctions of the operator $T_\varphi : H(U) \to H(\Omega * U)$ with eigenvalue $\operatorname{sign}(\nu)\varphi_\nu$.[3]

In case that Ω is of the form D_K for some compact and convex set $K \subset \mathbb{V}$ (so that we can dispose of the Mellin transform of the function $\varphi \in H(\Omega)$) and $U \subset \mathbb{C} \setminus \{0\}$ has connected complement (so that we can define a logarithm on U), we can extend this eigenvalue property to generalized monomials. Therefore, we introduce the following notation: for a set $A \subset \mathbb{C}$ and a (subset of an) open set $B \subset \mathbb{C} \setminus \{0\}$ with connected complement, $k \in \mathbb{N}_0$, $\alpha \in \mathbb{C}$ we set

$$\begin{array}{rcll} h_{k,\alpha,A}(z) & := & z^k e^{\alpha z} & (z \in A) \\ q_{k,\alpha,B} & := & h_{k,\alpha,\log_B B} \circ \log_B & \\ p_{\alpha,B} & := & q_{0,\alpha,B} & \end{array}$$

where $\log_B$ is a branch of the logarithm on every component of B. Since $p_{\alpha,B}(z) = \exp(\alpha \log_B z) =: z^\alpha$, we call $p_{\alpha,B}$ a *generalized monomial* and the definition of $p_{\alpha,B}$ coincides for the case $\alpha = \nu \in \mathbb{Z}$ with the definition of $p_{\nu,B}$ in Example 2.17. Even if it is suppressed in the notation, the function $q_{k,\alpha,B}$ depends on the choice of the branch of the underlying logarithm on B.

Finally we define for non-empty sets $A, B \subset \mathbb{C}$ the *Minkowski sum* $A + B := \{a + b : \ a \in A,\ b \in B\}$ and $A + \emptyset := \emptyset + A := \emptyset + \emptyset := \emptyset$.

---

[1] This is a consequence of Example B.19 and Theorem B.16.

[2] See [MR, L. 3.3, Th. 9.2].

[3] Strictly speaking, T_φ is in general not a self-mapping and does therefore not have eigenfunctions and eigenvalues in a narrow sense. However, speaking of these terms in this context should not provoke notional confusion.

Theorem 3.7 :
Let $K \subset \mathbb{V}$ be compact and convex and let $U \subset \mathbb{C} \setminus \{0\}$ be open and star-eligible to D_K with connected complement. Let furthermore $\varphi \in H(D_K)$ and $\Phi = \mathfrak{M}\varphi$. For every branch of the logarithm on U exists a branch of the logarithm on $D_K * U$ such that

$$T_\varphi q_{k,\alpha,U} = p_{\alpha,D_K*U} \sum_{l=0}^{k} \binom{k}{l} (\log_{D_K*U})^{k-l} \Phi^{(l)}(\alpha) \tag{3.1}$$

holds for all $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$ and especially

$$T_\varphi p_{\alpha,U} = \Phi(\alpha) \cdot p_{\alpha,D_K*U} \quad (\alpha \in \mathbb{C})\,. \tag{3.2}$$

Proof: The assumption that U^C is connected implies that U itself has simply connected components and Proposition 2.5.6 ensures that the same is true for $D_K * U$. We fix a branch of the logarithm on each component of U denoting it by $\log_U$ and show what branch of the logarithm on $D_K * U$ shall be chosen such that the asserted identity holds.

It is clear that there exists a number $a \in K$ such that the set $K - a := K + (-a)$ contains the origin (if K itself contains the origin choose $a = 0$). This implies that $1 \notin D_{K-a} = e^a \cdot D_K$ and therefore $D_{K-a} * U = e^a \cdot (D_K * U) \subset U$. Especially, every component of $e^a \cdot (D_K * U)$ is a subset of a component of U. Therefore it is meaningful to set $\log_{e^a\cdot(D_K*U)}(e^a z) := \log_U(e^a z)$ $(z \in D_K * U)$.

Obviously, every branch of the logarithm on $D_K * U$ fulfills the following equation for all $z \in D_K * U$:

$$\log_{D_K*U}(z) = \log_U(e^a z) - a + 2\pi i k(z)$$

for some $k(z) \in \mathbb{Z}$. The map

$$D_K * U \to \mathbb{C},\ z \mapsto \log_{D_K*U}(z) - (\log_U(e^a z) - a)$$

is continuous and its range is a discrete subset of $\mathbb{C}$. Therefore it must be constant on every component of $D_K * U$. Hence, the branch of the logarithm on every component of $D_K * U$ shall be chosen such that

$$\log_{D_K*U}(z) = \log_U(e^a z) - a \quad (z \in D_K * U)\,. \tag{3.3}$$

Let now $z \in D_K * U$ be given. The set $z \cdot U^*$ is a compact subset of the open set D_K (see Proposition 2.5.5) and therefore we can find a number $\delta_1 = \delta_1(z) > 0$ such that $(e^{-K} + U_{\delta_1}(0)) \cap z \cdot U^* = \emptyset$. On the other hand, e^{-K} is a compact subset of the open set $\mathbb{S}$ and therefore we can find a number $\delta_2 > 0$ such that $e^{-K} + U_{\delta_2}(0) \subset \mathbb{S}$. We set $\delta := \min\{\delta_1, \delta_2\}$ and $V_\delta := e^{-K} + U_\delta(0)$.

The following functional equation is essential for the rest of the proof: For all $\zeta \in V_\delta$ we have

$$\log_U(\frac{z}{\zeta}) = \log_{D_K * U}(z) - \log(\zeta) \tag{3.4}$$

where log denotes the principal branch of the logarithm on $\mathbb{S}$.

In order to prove (3.4) we first of all note that the left-hand side of (3.4) is defined since $z/\zeta \in U$ for all $\zeta \in V_\delta$. Indeed, assuming the existence of a number $w \in U^C$ with $z/\zeta = w$ would imply $z \cdot U^* \ni z/w = \zeta \in V_\delta$ which contradicts the choice of δ.

Obviously we have

$$g_z(\zeta) := \log_U(\frac{z}{\zeta}) - (\log_{D_K*U}(z) - \log(\zeta)) = 2k_z(\zeta)\pi i \quad (\zeta \in V_\delta)$$

for some $k_z(\zeta) \in \mathbb{Z}$. The same argument as above yields that g_z is constant on V_δ (note that V_δ is connected) and inserting $\zeta_0 = e^{-a} \in V_\delta$ yields (with Equation (3.3) and noting that $a \in K$ and therefore $\log(e^{-a}) = -a$)

$$g_z(\zeta) = g_z(\zeta_0) = \log_U(e^a z) - (\log_U(e^a z) - a - \log(e^{-a})) = 0 \quad (\zeta \in V_\delta)$$

which completes the proof of the asserted functional equation.

Since e^{-K} is a compact subset of the open set V_δ there exists a Cauchy cycle $\tilde{\Gamma}$ for e^{-K} in V_δ (see Proposition 1.5). The choice of δ ensures that

1. $\tilde{\Gamma}$ is a Cauchy cycle for e^{-K} in $\mathbb{S}$,
2. $\Gamma := \tilde{\Gamma}^-$ is an anti-Cauchy cycle for $z \cdot U^*$ in D_K.

Finally we obtain with Equation (3.4)

$$\begin{aligned}
T_\varphi q_{k,\alpha,U}(z) &= q_{k,\alpha,U} * \varphi(z) \\
&= \frac{1}{2\pi i}\int_\Gamma \varphi(\zeta)\big(\log_U(\frac{z}{\zeta})\big)^k \exp(\alpha \log_U(\frac{z}{\zeta}))\,\frac{d\zeta}{\zeta} \qquad (3.5)\\
&= p_{\alpha,D_K*U}(z)\sum_{l=0}^{k}\binom{k}{l}(\log_{D_K*U} z)^{k-l}\frac{(-1)^{l+1}}{2\pi i}\int_{\tilde{\Gamma}} \frac{\varphi(\zeta)(\log\zeta)^l}{\exp((\alpha+1)\log\zeta)}\,d\zeta \\
&= p_{\alpha,D_K*U}(z)\sum_{l=0}^{k}\binom{k}{l}(\log_{D_K*U} z)^{k-l}\Phi^{(l)}(\alpha)\,. \qquad (3.6)
\end{aligned}$$

To (3.5): As an anti-Cauchy cycle for $z \cdot U^*$ in D_K, the cycle Γ is a suitable integration cycle (see Definition 2.9.(e)).

To (3.6): $\tilde{\Gamma}$ is a Cauchy cycle for e^{-K} in $\mathbb{S}$ so that (3.6) follows from Remark B.17.2. □

3.3 The transpose of the operator T_φ

First of all, we shortly recall the notion of the transpose of linear operators and introduce some further notation.

Definition 3.8 :
Let E, F be locally convex spaces and $T \in L(E, F)$. The map

$$T' : F' \to E', \quad u \mapsto u \circ T$$

is called the *dual map* or *transpose* of T.

Theorem 3.9 :
Let $\Omega \subset \mathbb{C}_\infty$ with $\{0, \infty\} \subset \Omega$ as well as $U \subset \mathbb{C}$ be open and star-eligible. Let furthermore $\varphi \in H(\Omega)$.

Then the transpose of the Hadamard operator $T_\varphi : H(U) \to H(\Omega * U),\ f \mapsto \varphi * f$, i. e. the operator $T'_\varphi : H'(\Omega * U) \to H'(U),\ u \mapsto T'_\varphi u$ can be represented by the operator

$$[T]_\varphi : H((\Omega * U)^*) \to H(U^*),\ [(g, V)]_{(\Omega * U)^*} \mapsto [(\varphi * g, \Omega * V)]_{U^*} .$$

Proof: Let $u \in H'(\Omega * U)$ and $[(g, V)]_{(\Omega * U)^*} \in H((\Omega * U)^*)$ be the unique corresponding germ according to the dual space representation presented in Theorem 2.24. We note that for an open superset D of $(\Omega * U)^*$, Remark 2.4 implies that $D^* \subset \Omega * U$ and Proposition 2.5 yields $(\Omega * D)^* = \Omega^* \cdot D^* \subset U$ and therefore $\Omega * D \supset U^*$. Hence we have shown that $[(\varphi * g, \Omega * V)]_{U^*}$ belongs to the space $H(U^*)$.

To show that the operator $[T]_\varphi$ is well defined it is important to know that the value of $[T]_\varphi[(g, V)]_{(\Omega * U)^*}$ is independent of the choice of the representative of the equivalence class $[(g, V)]_{(\Omega * U)^*}$: To see this, let $(g, V) \sim_{(\Omega * U)^*} (\tilde{g}, \tilde{V})$. Then we have to show that $(\varphi *_{\Omega, V} g, \Omega * V) \sim_{U^*} (\varphi *_{\Omega, \tilde{V}} \tilde{g}, \Omega * \tilde{V})$. Hence, we have to find an open set W with $U^* \subset W \subset \Omega * V \cap \Omega * \tilde{V}$ and such that $\varphi *_{\Omega,V} g = \varphi *_{\Omega,\tilde{V}} \tilde{g}$ on W. By assumption there exists an open set D with $(\Omega * U)^* \subset D \subset V \cap \tilde{V}$ and such that $g = \tilde{g}$ on D. We set $W := \Omega * D$. Then we have $U^* \subset W \subset \Omega * V \cap \Omega * \tilde{V}$ and obtain with Theorem 2.13

$$(\varphi *_{\Omega,V} g)|_W = \varphi *_{\Omega,D} g|_D = \varphi *_{\Omega,D} \tilde{g}|_D = (\varphi *_{\Omega,\tilde{V}} \tilde{g})|_W \,.$$

Hence, the operator $[T]_\varphi$ is well defined.

Now we have to check that the unique germ corresponding to $T'_\varphi u \in H'(U)$ is given by $[(\varphi * g, \Omega * V)]_{U^*} \in H(U^*)$. The idea of this proof is to apply the associative law for the Hadamard product formulated in Theorem 2.23. There it has been important to know whether the origin and the point at infinity belong to the respective sets Ω, U, V or not. The set V depends as a superset of $(\Omega * U)^*$ on the set U. Indeed, if $0 \notin U$ we obtain $\{0, \infty\} \subset V$ and in case $0 \in U$ we obtain $0 \in V$ and choose without loss of generality V so small that $\infty \notin V$. Then the set $V * (\Omega * U)$ is defined and $1 \in V * (\Omega * U)$. Hence, V and $\Omega * U$ are star-eligible.

We obtain with Theorems 2.24, 2.14.2 and 2.23 for all $f \in H(U)$

$$\begin{aligned} T'_\varphi u(f) &= u(T_\varphi f) \\ &= ((T_\varphi f) *_{\Omega * U, V} g)(1) \\ &= (g *_{V, \Omega * U} (\varphi *_{\Omega, U} f))(1) \\ &= ((g *_{V,\Omega} \varphi) *_{V * \Omega, U} f)(1) \\ &= (f *_{U, \Omega * V} (\varphi *_{\Omega, V} g))(1) \,. \end{aligned}$$

Applying again Theorem 2.24 finishes the proof. □

Remark 3.10 :
Using Remark 2.21.2 and Proposition 2.22, one can give another representation of the transposed operator. Setting $\tilde{g} := \mathcal{U} g$ and $\tilde{V} := 1/V$ we can summarize the situation in the following diagram

$$\begin{array}{cccccccc}
T'_\varphi & : & H'(\Omega * U) & \longrightarrow & H'(U), & u & \longmapsto & T'_\varphi u \\
 & & \updownarrow & & \updownarrow & \updownarrow & & \updownarrow \\
[T]_\varphi & : & H((\Omega * U)^*) & \longrightarrow & H(U^*), & [(g, V)]_{(\Omega * U)^*} & \longmapsto & [(\varphi * g, \Omega * V)]_{U^*} \\
 & & \updownarrow & & \updownarrow & \updownarrow & & \updownarrow \\
[T]_{\mathcal{U}\varphi} & : & H((\Omega * U)^C) & \longrightarrow & H(U^C), & [(\tilde{g}, \tilde{V})]_{(\Omega * U)^C} & \longmapsto & [((\mathcal{U}\varphi) * \tilde{g}, (1/\Omega) * \tilde{V})]_{U^C}
\end{array}$$

We remarked earlier that a more accurate notation for the operator T_φ is $T_{\varphi,U}$ or $T_{\varphi,\Omega,U}$. By using an analogue notation for $[T]_\varphi$ and $[T]_{\mathcal{U}\varphi}$ we roughly summarize

$$(T_{\varphi,U})' = [T]_{\varphi,(\Omega*U)^*} = [T]_{\mathcal{U}\varphi,(\Omega*U)^C}$$

or even more precise

$$(T_{\varphi,\Omega,U})' = [T]_{\varphi,\Omega,(\Omega*U)^*} = [T]_{\mathcal{U}\varphi,1/\Omega,(\Omega*U)^C} .$$

The bottom line of Theorem 3.9 is that the transpose of $T_{\varphi,U}$ is -in simple terms- again a Hadamard operator which is induced by the same function φ but by the "dual" set $(\Omega * U)^*$. This insight will be important for the examination of the kernel and the range of Hadamard operators in the following chapter.

Furthermore we note that according to [MV, Prop. 23.30], the maps $[T]_\varphi : H((\Omega * U)^*) \to H(U^*)$ and $[T]_{\mathcal{U}\varphi} : H((\Omega * U)^C) \to H(U^C)$ are continuous.

Chapter 4

The kernel and the range of the operator T_φ

The main purpose of this chapter is the examination of the kernel and the range of Hadamard operators

$$T_{\varphi,U} : H(U) \to H(\Omega * U)$$

for different sets Ω and U and functions $\varphi \in H(\Omega)$. For the examination of the range of $T_{\varphi,U}$ we refer the reader to the criteria for linear and continuous operators to have dense range or to be surjective which are formulated in Appendix C. For an operator $T : E \to F$ between vector spaces we denote by $N(T)$ and $R(T)$ the kernel and the range of T respectively.

In Section 4.1 we describe the kernel of Hadamard operators. Subsequently we formulate conditions under which Hadamard operators can be expected to be injective or to have dense range (see Section 4.2). In Section 4.3 the different approaches which are used in Section 4.2 are compared. We are going to formulate a variety of clarifying and distinguishing examples. Section 4.4 tackles the question when $T_{\varphi,U}$ can even expected to be surjective.

For the examination of the kernel and the range of Hadamard operators, we assume once and for all that the inducing function $\varphi \in H(\Omega)$ does not vanish identically.

4.1 Description of the kernel of T_φ

Theorem 4.1 :
Let $\Omega \subset \mathbb{C}_\infty$ be a domain with $\{0, \infty\} \subset \Omega$ and let $\varphi \in H(\Omega)$. Let furthermore

$U \subset \mathbb{C}_\infty$ be open and star-eligible to Ω. Then the following are valid for the operator $T_\varphi : H(U) \to H(\Omega * U)$:

1. (a) *If $0 \notin U$, then* $\operatorname{clspan}\{p_{-\nu,U} :\ \nu \in \mathbb{N} \setminus \Lambda_\varphi^-\} \subset N(T_\varphi)$.[1]
   If $\infty \notin U$, then $\operatorname{clspan}\{p_{\nu,U} :\ \nu \in \mathbb{N}_0 \setminus \Lambda_\varphi^+\} \subset N(T_\varphi)$.
   (b) *If U is a ring domain $V_{r,R}$ with $0 \le r < R \le \infty$, then* $\operatorname{clspan}\{p_{\nu,U} :\ \nu \in \mathbb{Z} \setminus \Lambda_\varphi\} = N(T_\varphi)$.
2. *If $0 \in U$, then we have:*
   (a) *Each function $f \in H(U)$ such that $T_\varphi f$ vanishes in a neighbourhood of the origin belongs to the space $H_{\mathbb{N}_0\setminus\Lambda_\varphi^+}(U)$ and $N(T_\varphi) \subset H_{\mathbb{N}_0\setminus\Lambda_\varphi^+}(U)$.*
   (b) *If, in addition, $\Omega * U$ is connected, then $N(T_\varphi) = H_{\mathbb{N}_0\setminus\Lambda_\varphi^+}(U)$.*
3. *If $\infty \in U$, then we have:*
   (a) *Each function $f \in H(U)$ such that $T_\varphi f$ vanishes in a neighbourhood of the point at infinity belongs to the space $H_{\infty,\mathbb{N}\setminus\Lambda_\varphi^-}(U)$ and $N(T_\varphi) \subset H_{\infty,\mathbb{N}\setminus\Lambda_\varphi^-}(U)$.*
   (b) *If, in addition, $\Omega * U$ is connected, then $N(T_\varphi) = H_{\infty,\mathbb{N}\setminus\Lambda_\varphi^-}(U)$.*

Proof:

1. (a) These assertions are a direct consequence of Example 2.17 and the linearity and continuity of T_φ.
   (b) Part (a) implies that $\operatorname{clspan}\{p_{\nu,U} :\ \nu \in \mathbb{Z} \setminus \Lambda_\varphi\} \subset N(T_\varphi)$.
   If $U = V_{r,R}$, then every function $f \in H(U)$ allows a Laurent series expansion $f = \sum_{\nu=-\infty}^{\infty} f_\nu p_{\nu,U}$ and therefore belongs to the space $\operatorname{clspan}\{p_{\nu,U} :\ \nu \in \mathbb{Z}\}$. However, for $f \in N(T_\varphi)$, the continuity and linearity of the operator T_φ together with Example 2.17 yield

$$
\begin{aligned}
0 &= T_\varphi f = T_\varphi\Big(\sum_{\nu=-\infty}^{\infty} f_\nu p_{\nu,U}\Big) = \lim_{n\to\infty}\Big(\sum_{\nu=-n}^{n} f_\nu T_\varphi p_{\nu,U}\Big) \\
&= \sum_{\nu=-\infty}^{\infty} \operatorname{sign}(\nu) f_\nu \varphi_\nu p_{\nu,\Omega*U} = \sum_{\nu\in\Lambda_\varphi} \operatorname{sign}(\nu) f_\nu \varphi_\nu p_{\nu,\Omega*U}
\end{aligned}
$$

[1] For $D \subset \mathbb{C}_\infty$ open and $\emptyset \neq A \subset H(D)$, $\operatorname{clspan}(A)$ means the closure in $H(D)$ of the linear span of A and $\operatorname{clspan}\emptyset := \{0\}$.

and therefore $f_\nu = 0$ for all $\nu \in \Lambda_\varphi$. Hence, $f \in \operatorname{clspan}\{p_{\nu,U} : \nu \in \mathbb{Z} \setminus \Lambda_\varphi\}$.

2. (a) According to the Hadamard multiplication theorem 2.14.3 we obtain for all $f \in H(U)$ such that $T_\varphi f$ vanishes in a neighbourhood of the origin and all z with small modulus

$$0 = T_\varphi f(z) = \sum_{\nu=0}^{\infty} \varphi_\nu f_\nu z^\nu = \sum_{\nu \in \Lambda_\varphi^+} \varphi_\nu f_\nu z^\nu \tag{4.1}$$

and therefore $f_\nu = 0$ for all $\nu \in \Lambda_\varphi^+$. Hence $f \in H_{\mathbb{N}_0 \setminus \Lambda_\varphi^+}(U)$.

(b) Let $f \in H_{\mathbb{N}_0 \setminus \Lambda_\varphi^+}(U)$. Then the Hadamard multiplication theorem 2.14.3 yields that $T_\varphi f$ vanishes in a neighbourhood of the origin. Since $\Omega * U$ is connected, f belongs to the kernel of T_φ.

3. (a) According to the Hadamard multiplication theorem 2.14.4 we obtain for all $f \in H(U)$ such that $T_\varphi f$ vanishes in a neighbourhood of the point at infinity and all z with large modulus

$$0 = T_\varphi f(z) = -\sum_{\nu=1}^{\infty} \varphi_{-\nu} f_{-\nu} z^{-\nu} = -\sum_{\nu \in \Lambda_\varphi^-} \varphi_{-\nu} f_{-\nu} z^{-\nu} \tag{4.2}$$

and therefore $f_{-\nu} = 0$ for all $\nu \in \Lambda_\varphi^-$. Hence $f \in H_{\infty, \mathbb{N} \setminus \Lambda_\varphi^-}(U)$.

(b) Let $f \in H_{\infty, \mathbb{N} \setminus \Lambda_\varphi^-}(U)$. Then the Hadamard multiplication theorem 2.14.4 yields that $T_\varphi f$ vanishes in a neighbourhood of the point at infinity. Since $(\Omega * U) \cap \mathbb{C}$ is connected, f belongs to the kernel of T_φ.

□

Remark 4.2 :
Let $\Omega \subset \mathbb{C}_\infty$ be a domain and $U \subset \mathbb{C}_\infty$ be open with $\{0, \infty\} \subset \Omega \cap U$ and such that $\Omega * U$ is connected and let $\varphi \in H(\Omega)$. Theorem 4.1 especially implies

$$N(T_\varphi) = H_{\mathbb{N}_0 \setminus \Lambda_\varphi^+}(U) = H_{\infty, \mathbb{N} \setminus \Lambda_\varphi^-}(U)\,. \tag{4.3}$$

This observation will be important to examine under which conditions T_φ can be expected to be injective.

We want to formulate an interpretation of the second equality in (4.3) and have a closer look at the assumption that $\Omega * U$ shall be connected:

1. For the following interpretation of the second equality in (4.3) we note that the assumption that $\Omega * U$ shall be connected implies that 0 and ∞ belong to the same component of U (see Remark 2.7.2).

   For $f \in H(U)$ we have $\Lambda_f^+ \subset \mathbb{N}_0 \setminus \Lambda_\varphi^+$ if and only if $\Lambda_f^- \subset \mathbb{N} \setminus \Lambda_\varphi^-$. That means that whenever the gaps of the power series expansion of a function $f \in H(U)$ about 0 are influenced by those of the function $\varphi \in H(\Omega)$, then the same is true for the gaps of the power series expansion of f about ∞ and vice versa.

2. It may happen that the assumption of the connectedness of $\Omega * U$ has influence on the set of non-vanishing coefficients Λ_φ^+ and Λ_φ^-: if the set U^C contains a closed arc of angular length $2\pi\delta$ for some $\delta \geq 0$ (for example $U = D_{K_\delta}$), then the definition of the star product implies that the following condition is necessary for $\Omega * U$ to be connected:

   For all $R > 0$ the set $\mathbb{T}_R \cap \Omega$ contains an open arc of length larger than $2\pi R\delta$. This, in turn, implies according to the Pólya gap theorem that $d^*(\Lambda_\varphi^+) > \delta$ (see Remark A.8). Applying this argument to the function $\mathcal{U}\varphi \in H(1/\Omega)$ we obtain that also $d^*(\Lambda_\varphi^-) > \delta$.

Remark 4.3 :
Without the assumption of the connectedness of $\Omega * U$ in Theorem 4.1.2/3, the kernel of T_φ is in general a proper subset of $H_{\mathbb{N}_0 \setminus \Lambda_\varphi^+}(U)$ or $H_{\infty, \mathbb{N} \setminus \Lambda_\varphi^-}(U)$ respectively. The following example illustrates this:

The functions

$$\Phi(z) := z \cdot \prod_{\substack{\lambda=2 \\ \lambda \text{ even}}}^{\infty} \big(1 - \frac{z^2}{\lambda^2}\big), \; F(z) := (z-1) \cdot \prod_{\substack{\lambda=3 \\ \lambda \text{ odd}}}^{\infty} \big(1 - \frac{z^2}{\lambda^2}\big), \; \tilde{F}(z) := F(-z) \;\; (z \in \mathbb{C})$$

belong to the space $\operatorname{Exp}(K_{1/2})$ (see Example B.19.2 and Corollary B.13.2).

Hence, $\varphi := \mathfrak{M}^{-1}\Phi$, $f := \mathfrak{M}^{-1}F$ and $\tilde{f} := \mathfrak{M}^{-1}\tilde{F}$ are holomorphic in $\Omega := U := D_{K_{1/2}}$. Obviously,

$$\Lambda_f^+ = 2\mathbb{N}_0 = \mathbb{N}_0 \setminus \Lambda_\varphi^+, \quad \Lambda_{\tilde{f}}^- = 2\mathbb{N} = \mathbb{N} \setminus \Lambda_\varphi^- .$$

Hence, $f \in H_{\mathbb{N}_0 \setminus \Lambda_\varphi^+}(U)$ and $\tilde{f} \in H_{\infty, \mathbb{N} \setminus \Lambda_\varphi^-}(U)$ while $\Omega * U = \mathbb{D} \cup \overline{\mathbb{D}}^C$ is not connected.

However, Theorems 2.14 and B.16 yield for all $z \in \mathbb{D}$

$$T_\varphi \tilde{f}(z) = \sum_{\nu=0}^{\infty} \Phi(\nu)\tilde{F}(\nu) z^\nu = \tilde{F}(1) z \neq 0$$

and for all $z \in \overline{\mathbb{D}}^C$

$$T_\varphi f(z) = -\sum_{\nu=1}^{\infty} \Phi(-\nu)F(-\nu)z^{-\nu} = -F(-1)z^{-1} \neq 0\,.$$

Therefore, f and $\tilde{f}$ do not belong to the kernel of T_φ.

If the geometry of the set Ω allows a Mellin transform of the function $\varphi \in H(\Omega)$ and if the set $U \subset \mathbb{C} \setminus \{0\}$ has connected complement so that we can dispose of the generalized monomials, we obtain the following theorem which is a direct consequence of Theorem 3.7 and the linearity and continuity of T_φ.

Theorem 4.4 :
Let $K \subset \mathbb{V}$ be compact and convex and let $U \subset \mathbb{C} \setminus \{0\}$ be open and star-eligible to D_K with connected complement. Let furthermore $\varphi \in H(D_K)$, $\Phi = \mathfrak{M}\varphi$, $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}_0$.

Then the function $q_{k,\alpha,U}$ belongs to kernel of the operator $T_\varphi : H(U) \to H(D_K * U)$ if and only if α is an $m-$fold zero of Φ and $k \leq m-1$. Especially

$$\text{clspan}\{q_{k,\alpha,U} : \ \alpha \ m\text{-fold zero of } \Phi,\ k \leq m-1\} \subset N(T_\varphi)\,.^2 \tag{4.4}$$

4.2 Criteria for T_φ to be injective or to have dense range

The aim of this section is to disclose under which conditions the Hadamard operator $T_{\varphi,U}$ is injective or has dense range. The main idea pursued here is the following: Combining Theorem C.5.1 and Theorem 3.9 it follows that $T_\varphi : H(U) \to H(\Omega * U)$ has dense range if and only if

$$[T]_\varphi : H((\Omega * U)^*) \to H(U^*),\ [(g,V)]_{(\Omega * U)^*} \mapsto [(\varphi * g, \Omega * V)]_{U^*}$$

is injective. As we already pointed out in Remark 3.10, the transposed operator is basically again a Hadamard operator induced by the same function φ but by the

[2] The functions $q_{k,\alpha,U}$ depend on the branch of the logarithm which is chosen on every component of U. Theorem 3.7 implies that (4.4) holds for any of these branches. However, one easily verifies that the set clspan$\{q_{k,\alpha,U} : \ \alpha \ m-$fold zero of $\Phi,\ k \leq m-1\}$ does not change while altering the underlying branch of the logarithm. Therefore it is legitimate not to go into any detail on what branch of the logarithm is chosen if it is not, for some reason, convenient to choose a special one.

"dual" set $(\Omega * U)^*$. Neglecting the fact that the transposed operator $[T]_\varphi$ maps between spaces of *germs* of holomorphic functions it is not surprising that given that we know under which conditions a Hadamard operator $T_{\varphi,U}$ is injective, we can hope to obtain a result concerning the range of a corresponding operator using duality and vice versa.

We want to highlight two things: Firstly we note that 0 or ∞ belong to the set $(\Omega * U)^*$ if and only if ∞ or 0 do not belong to the set U. Hence, the respective position of the origin and the point at infinity for the "dual" operator is in some sense contrary to the situation for the "primal" operator. For example, if we know that $T_{\varphi,U}$ has dense range for some $U \subset \mathbb{C} \setminus \{0\}$, we can hope to derive some information about the injectivity of a Hadamard operator $T_{\varphi,\tilde{U}}$ where $\tilde{U} \supset \{0, \infty\}$ is suitable. Secondly, assumptions about the connectedness of U and $\Omega * U$ are going to be transferred to assumptions about the connectedness of $(\Omega * U)^*$ and U^*(i. e. of $(\Omega * U)^C$ and U^C).

To put it in a nutshell, the results concerning the injectivity and the range of $T_{\varphi,U}$ listed in this section can be interpreted as the "dual version" of each other.

We will see that sometimes it will be convenient to prove some results concerning the injectivity of a Hadamard operator and transfer them to the respective "dual" operator while sometimes it is more appropriate to prove a result for the range of $T_{\varphi,U}$ in a first step. The structure of this section is chosen in such a way that we will always present the "primal" results first even if that means hopping a little bit between properties of the kernel and the range of $T_{\varphi,U}$.

Theorem 4.5 :
Let $\Omega \subset \mathbb{C}_\infty$ be a domain with $\{0, \infty\} \subset \Omega$ and $\varphi \in H(\Omega)$. Let furthermore $U \subset \mathbb{C}_\infty$ be open and star-eligible to Ω. Then the following are valid for the operator $T_\varphi : H(U) \to H(\Omega * U)$:

1. *Let $0, \infty \notin U$. If T_φ is injective, then $\Lambda_\varphi = \mathbb{Z}$. If, in addition, U is connected and contains a ring domain $V_{r,R}$ with $0 \leq r < R \leq \infty$ and $\dfrac{r}{R} < \dfrac{\min\{|w| : w \in \Omega^C\}}{\max\{|w| : w \in \Omega^C\}}$, then this condition is also sufficient.*
2. *Let $0 \in U$ and $\infty \notin U$. If T_φ is injective, then $\Lambda_\varphi^+ = \mathbb{N}_0$. If, in addition, U is connected, then this condition is also sufficient.*
3. *Let $0 \notin U$ and $\infty \in U$. If T_φ is injective, then $\Lambda_\varphi^- = \mathbb{N}$. If, in addition, U is connected, then this condition is also sufficient.*

Proof:

1. The first part of the assertion is a consequence of Theorem 4.1.1.(a). To show the second part let $\Lambda_\varphi = \mathbb{Z}$ and $f \in N(T_{\varphi,U})$. Since U contains a ring domain $V_{r,R}$ and $f|_{V_{r,R}} \in N(T_{\varphi,V_{r,R}})$ (see Theorem 2.13), Theorem 4.1.1.(b) yields that $f|_{V_{r,R}}$ vanishes (note that Example 2.3.6 ensures that $\Omega * V_{r,R} \neq \emptyset$). Since U is connected, f vanishes on the whole set U and therefore, T_φ is injective.

2. The first part of the assertion is a consequence of Theorem 4.1.1.(a) and the second one is a consequence of Theorem 4.1.2.(a) noting that the connectedness of U implies $H_\emptyset(U) = \{0\}$.

3. The first part of the assertion is a consequence of Theorem 4.1.1.(a) and the second one is a consequence of Theorem 4.1.3.(a) noting that the connectedness of U implies $H_{\infty,\emptyset}(U) = \{0\}$. □

Having a look at Theorem 4.5.1 it is important to remark that the condition $\Lambda_\varphi = \mathbb{Z}$ is in general not sufficient for the injectivity of T_φ. Indeed, if the geometry of Ω allows a Mellin transform of the function $\varphi \in H(\Omega)$ and $U \subset \mathbb{C} \setminus \{0\}$ is simply connected, which is in some sense the "opposite" of containing a ring domain, the following theorem holds.

Theorem 4.6 :
Let $K \subset \mathbb{V}$ be compact and convex and let $U \subset \mathbb{C} \setminus \{0\}$ be a simply connected domain and star-eligible to D_K. Let furthermore $\varphi \in H(D_K)$ and $\Phi = \mathfrak{M}\varphi$. Then the following are equivalent:

1. *$T_\varphi : H(U) \to H(D_K * U)$ is injective.*

2. *Φ has no zeros.*

3. *$\varphi(z) = \lambda/(1 - \beta z)$ $(z \in D_K)$ for some $\lambda \neq 0$ and some $\beta \in e^K$.*

Proof: Theorem 4.4 yields that 1. implies 2.

In order to show that 2. implies 3. we assume that $\Phi \in \mathrm{Exp}(K)$ has no zeros. Then according to the Hadamard factorization theorem (see [Boa, Th. 2.7.1]), there are numbers $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $\Phi(z) = \exp(\alpha_1 z + \alpha_2)$ $(z \in \mathbb{C})$. In order that the condition $\Phi \in \mathrm{Exp}(K)$ is satisfied, α_1 must belong to the set K (see Example B.11). Then the power series expansion of φ about zero yields that

$$\varphi(z) = \frac{e^{\alpha_2}}{1 - e^{\alpha_1} z} \quad (z \in D_K)\,.$$

Setting $\lambda := e^{\alpha_2} \neq 0$ and $\beta := e^{\alpha_1} \in e^K$ completes this part of the proof.

To prove that 3. implies 1. we examine how T_φ acts on a function $f \in H(U)$. For all $z \in D_K * U$ we obtain

$$\begin{aligned} T_\varphi f(z) &= \frac{1}{2\pi i}\int_\Gamma \varphi(\frac{z}{\zeta}) f(\zeta)\frac{d\zeta}{\zeta} \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{\lambda}{1-\beta z/\zeta} f(\zeta)\frac{d\zeta}{\zeta} \\ &= \lambda f(\beta z) \end{aligned}$$

where Γ is a Cauchy cycle for $z \cdot e^K$ in U and the last identity follows from the Cauchy integral formula noting that $z \cdot \beta \in z \cdot e^K$.

Since U is connected, this operator T_φ is injective. □

The "dual version" of Theorem 4.5 reads as follows:

Theorem 4.7 :
Let $\Omega \subset \mathbb{C}_\infty$ be a domain with $\{0, \infty\} \subset \Omega$ and $\varphi \in H(\Omega)$. Let furthermore $U \subset \mathbb{C}_\infty$ be open and star-eligible to Ω. Then the following are valid for the operator $T_\varphi : H(U) \to H(\Omega * U)$:

1. *Let $\{0, \infty\} \subset U$. If T_φ has dense range, then $\Lambda_\varphi = \mathbb{Z}$.*
2. *Let $0 \in U$ and $\infty \notin U$. If T_φ has dense range, then $\Lambda_\varphi^+ = \mathbb{N}_0$. If, in addition, $(\Omega * U)^*$ is connected, then this condition is also sufficient.*
3. *Let $0 \notin U$ and $\infty \in U$. If T_φ has dense range, then $\Lambda_\varphi^- = \mathbb{N}$. If, in addition, $(\Omega * U)^*$ is connected, then this condition is also sufficient.*

Proof:

1. Since $0 \in U$ the Hadamard multiplication theorem 2.14.3 yields that $T_\varphi(H(U)) \subset H_{\Lambda_\varphi^+}(\Omega * U)$ where the right-hand side is a closed subspace

of $H(\Omega * U)$. If T_φ has dense range, then this implies $H_{\Lambda_\varphi^+}(\Omega * U) = H(\Omega * U)$ which obviously requires Λ_φ^+ to equal $\mathbb{N}_0$.

Since $\infty \in U$ the Hadamard multiplication theorem 2.14.4 yields that $T_\varphi(H(U)) \subset H_{\infty,\Lambda_\varphi^-}(\Omega * U)$ where the right-hand side is again a closed subspace of $H(\Omega * U)$. If T_φ has dense range, then this implies $H_{\infty,\Lambda_\varphi^-}(\Omega * U) = H(\Omega * U)$ which requires Λ_φ^- to equal $\mathbb{N}$.

Hence, we obtain $\Lambda_\varphi = \mathbb{Z}$.

2. The first part of the assertion follows analogously to part 1.

   To prove the second part, Theorems C.5.1 and 3.9 yield that it is enough to show that

$$[T]_\varphi : H((\Omega * U)^*) \to H(U^*),\ [(g, V)]_{(\Omega * U)^*} \mapsto [(\varphi * g, \Omega * V)]_{U^*}$$

   is injective.

   Let $[(g, V)]_{(\Omega*U)^*} \in H((\Omega * U)^*)$ with $[(\varphi * g, \Omega * V)]_{U^*} = [0]_{U^*}$. Since $(\Omega * U)^*$ is connected, without loss of generality we can choose V to be connected. Furthermore, V contains the origin and V shall without loss of generality be chosen so small that $\infty \notin V$.

   The fact that $0 \in U^*$ implies that $\varphi * g$ vanishes in an open neighbourhood of the origin and Theorem 4.1.2.(a) yields

$$g \in H_{\mathbb{N}_0 \setminus \Lambda_\varphi^+}(V) = H_\emptyset(V) = \{0\}\,.$$

   Hence, $[(g, V)]_{(\Omega*U)^*} = [0]_{(\Omega*U)^*}$.

3. The first part of the assertion follows analogously to part 1.

   To prove the second part, let $g \in H(\Omega * U)$. By part 2 of this theorem, the operator $T_{\mathcal{U}\varphi} : H(1/U) \to H(1/(\Omega * U))$ has dense range (note that $\mathcal{U}\varphi \in H(1/\Omega)$ where $\{0, \infty\} \subset 1/\Omega$ and $0 \in 1/U$, $\infty \notin 1/U$). Therefore there exists a sequence $(\tilde{f}_n)$ in $H(1/U)$ with $T_{\mathcal{U}\varphi}\tilde{f}_n \to \mathcal{U}g$ in $H(1/(\Omega * U))$. Because $\mathcal{U} : H(U) \to H(1/U)$ is bijective (see Remark 2.21.1) there are functions $f_n \in H(U)$ with $\mathcal{U}f_n = \tilde{f}_n$ $(n \in \mathbb{N})$. Proposition 2.22 yields $\mathcal{U}(T_\varphi f_n) \to \mathcal{U}g$ in $H(1/(\Omega * U))$ and since $\mathcal{U}$ is a topological isomorphism we obtain $T_\varphi f_n \to g$ in $H(\Omega * U)$. □

Remark 4.8 :
No assumptions on the connectedness of U (or $(\Omega * U)^*$ respectively) are needed in order to obtain necessary conditions for T_φ to be injective or to have dense range. This is not surprising because connectedness assumptions are mostly needed to

retrieve global information from local one. However, the necessary conditions formulated above work the other way around: Knowing that T_φ is injective or has dense range is some kind of global information whereas the deduced properties of Λ_φ^+ and Λ_φ^- are some kind of local information because they concern the power series expansions of φ about zero or infinity.

We want to continue the examination of the range of $T_{\varphi,U}$ in case that neither the origin nor the point at infinity belong to the set U. If the geometry of the set Ω allows a Mellin transform of the function $\varphi \in H(\Omega)$, we get the following sufficient condition for the resulting Hadamard operator to have dense:

Theorem 4.9 (Approach (R.I)) **:**
Let $K \subset \mathbb{V}$ be compact and convex and let $U \subset \mathbb{C} \setminus \{0\}$ be open and star-eligible to D_K with connected complement. Let furthermore $\varphi \in H(D_K)$. Then $T_\varphi : H(U) \to H(D_K * U)$ has dense range.

Proof: As usual, we write $\Phi = \mathfrak{M}\varphi \in \operatorname{Exp}(K)$. Then we obtain with Theorem 3.7

$$T_\varphi p_{\alpha,U} = \Phi(\alpha) \cdot p_{\alpha,D_K*U} \quad (\alpha \in \mathbb{C})\,.$$ [3]

Denoting the set of zeros of Φ by $Z(\Phi)$, this identity implies that for all $\alpha \notin Z(\Phi)$ the monomials p_{α,D_K*U} belong to the range of T_φ. Since $Z(\Phi)$ is countable, there exists a number $c \in [0,1)$ such that $(\mathbb{N}_0 + c) \cap Z(\Phi) = \emptyset$.

Let now $g \in H(D_K*U)$. Then the function $p_{-c,D_K*U} \cdot g$ is also holomorphic in $D_K * U$. Since $(D_K * U)^C$ is connected (see Proposition 2.5.6), Runge's approximation theorem implies that there exists a sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ converging locally uniformly to $p_{-c,D_K*U} \cdot g$ on $D_K * U$. Hence, $(p_{c,D_K*U} \cdot P_n)_{n\in\mathbb{N}}$ converges locally uniformly on $D_K * U$ to the function g. This shows that the linear span of $\{p_{\alpha,D_K*U} : \ \alpha \in \mathbb{C} \setminus Z(\Phi)\}$ is dense in $H(D_K * U)$ and therefore $T_\varphi : H(U) \to H(D_K * U)$ has dense range. □

Example 4.10 :
Let $\Lambda = \{\lambda_n : \ n \in \mathbb{N}\} \subset \mathbb{N}$ with $d(\Lambda) = \delta < 1$ and

$$\Phi(z) := \prod_{n=1}^{\infty} \big(1 - \frac{z^2}{\lambda_n^2}\big) \quad (z \in \mathbb{C})\,.$$

[3] The functions $p_{\alpha,U}$ are assumed to be induced by an arbitrary branch of the logarithm on each component of U.

Example B.19.2 implies that $\varphi = \mathfrak{M}^{-1}\Phi$ is holomorphic in D_{K_δ} and Theorem 4.9 yields that the operator $T_\varphi : H(\mathbb{S}) \to H(S_{1-\delta})$ has dense range (note that $D_{K_\delta} * \mathbb{S} = S_{1-\delta}$ according to Example 2.3.4).

It is obvious that the larger δ, i. e. the larger K_δ, the smaller the set $D_{K_\delta} * \mathbb{S} = S_{1-\delta}$ gets. Applying this concept to questions of approximation theory in Chapter 6 of this thesis, this phenomenon becomes quite natural.

Remark 4.11 :
As we will see in Proposition 4.22.2, the assertion of Theorem 4.9 is in general false if U^C is not connected.

Theorem 4.12 (Approach (K.I)) **:**
Let $K \subset \mathbb{V}$ be compact and convex and let $U \subset \mathbb{C}_\infty$ be a domain with $\{0, \infty\} \subset U$ and such that $D_K * U$ is connected. Let furthermore $\varphi \in H(D_K)$. Then $T_\varphi : H(U) \to H(D_K * U)$ is injective.

Proof: If $U = \mathbb{C}_\infty$ the assertion is trivial. Therefore we assume $U \subsetneq \mathbb{C}_\infty$ and note that since $D_K * U$ is connected there exists a compact and connected set $L \subset D_K * U$ with $\{0, \infty\} \subset L^\circ$. We set $W := L^*$ and obtain an open set containing neither the origin nor the point at infinity and having connected complement. Furthermore, we have $W^* = L \subset D_K * U$ and Proposition 2.5.5 yields that $(D_K * W)^* = W^* \cdot D_K^*$ is a compact subset of U (because $U \neq \mathbb{C}_\infty$, this implies also that $D_K * W \neq \emptyset$, i. e. D_K and W are star-eligible).

Theorem 4.9 yields that the operator $T_{\varphi,W} : H(W) \to H(D_K * W)$ has dense range. Hence, Theorem C.5.1 implies that

$$[T]_{\varphi,(D_K*W)^*} : H((D_K * W)^*) \to H(W^*),\ [(g, V)]_{(D_K*W)^*} \mapsto [(\varphi * g, D_K * V)]_{W^*}$$

is injective.

Let now $f \in N(T_{\varphi,U})$ be given. Then $[(f, U)]_{(D_K*W)^*} \in H((D_K * W)^*)$ and $[T]_{\varphi,(D_K*W)^*}[(f, U)]_{(D_K*W)^*} = [(\varphi*f, D_K*U)]_{W^*} = [0]_{W^*}$. Hence, $[(f, U)]_{(D_K*W)^*} = [0]_{(D_K*W)^*}$ which means that f vanishes in an open neighbourhood O of $(D_K*W)^*$. Since $O \cap U \neq \emptyset$ and since U is connected, f vanishes on U which completes the proof. □

Remark 4.13 :
The assertion of Theorem 4.12 is in general false if U or $D_K * U$ is not connected:

1. Let $K := K_{1/2}$ and let $U := \mathbb{C}_\infty \setminus e^{-\partial L}$ where $L = I \times i\pi[-1/4, 1/4]$ for some compact real interval I with $I^\circ \neq \emptyset$.

   Then U is not connected, but $D_K * U = D_{K_{1/2}+L}$ is connected. Setting
$$f(z) = \begin{cases} 0 & , \; z \in D_L \\ 1 & , \; z \in e^{-L^\circ} \end{cases}$$
   we obtain a function $0 \not\equiv f \in H(U)$ with $f_\nu = 0$ ($\nu \in \mathbb{N}_0$) and the Hadamard multiplication theorem 2.14.3 yields for all z with small modulus
$$T_\varphi f(z) = \sum_{\nu=0}^{\infty} \varphi_\nu f_\nu z^\nu = 0\,.$$
   Since $D_K * U$ is connected, we obtain $f \in N(T_\varphi)$, i. e. T_φ is not injective.

2. According to Example B.19.2 the function
$$\Phi(z) = \prod_{\substack{\lambda=1 \\ \lambda \text{ odd}}}^{\infty} \Big(1 - \frac{z^2}{\lambda^2}\Big) \quad (z \in \mathbb{C})$$
   belongs to the space $\operatorname{Exp}(K_{1/2})$ and $\varphi := \mathfrak{M}^{-1}\Phi$ is holomorphic in $D_{K_{1/2}}$ with $\Lambda_\varphi^+ = 2\mathbb{N}_0$ and $\Lambda_\varphi^- = 2\mathbb{N}$. We set $U := \mathbb{D}_r \cup U_R(\infty) \cup S_{1/4} \cup (-S_{1/4})$ (where $0 < r < R < \infty$). Then U is connected but $D_{K_{1/2}} * U = \mathbb{D}_r \cup U_R(\infty)$ is not connected.

   Let $f \in H(U)$ be a non-even function, i. e. $\tilde{f}(z) := f(-z) \in H(U)$ is not equal to f. The Hadamard multiplication theorem 2.14.3 and Equation (2.2) yield for all $z \in \mathbb{D}_r$
$$T_\varphi \tilde{f}(z) = T_\varphi f(-z) = \sum_{\nu \in 2\mathbb{N}_0} \varphi_\nu f_\nu (-z)^\nu = \sum_{\nu \in 2\mathbb{N}_0} \varphi_\nu f_\nu z^\nu = T_\varphi f(z)$$
   and for all $z \in U_R(\infty)$
$$T_\varphi \tilde{f}(z) = -\sum_{\nu \in 2\mathbb{N}} \varphi_{-\nu} f_{-\nu} (-z)^{-\nu} = T_\varphi f(z)\,.$$
   Hence, T_φ is not injective.

Remark 4.14 :
At first glance, it might seem asymmetric that Theorem 4.12 requires a connectedness assumption for both U and $D_K * U$ whereas in Theorem 4.9 we only need a connectedness assumption for U (or to be more precise for U^C). The corresponding second assumption that would have to be made is that $(D_K * U)^C$ is connected, too. However, this is already guaranteed by Proposition 2.5.6 and does not have to be expressed specifically.

We want to formulate a second condition under which Hadamard operators $T_{\varphi,\Omega,U}$ are injective in case that both the origin and the point at infinity belong to the set U. So far we only considered the case that Ω is of the form D_K. If we want to generalize this result to a domain Ω containing 0 and ∞, it is natural that we have to impose other assumptions. It will turn out that the "number" of non-vanishing coefficients in the power series expansions of φ measured by means of the densities introduced in Appendix A provides an appropriate concept to do that. However, the idea of the proof of this theorem will be inspired by the proof of Theorem 4.5. The main idea was to make use of Theorem 4.1 and to exploit the obvious identity $H_\emptyset(U) = \{0\}$ for the connected set U. Also in case $\{0,\infty\} \subset U$ Theorem 4.1 will play an important role but in addition, we need the following consequence of Theorem 4.12.

Proposition 4.15 :
Let $M_1 \subset \mathbb{N}_0$ and $M_2 \subset \mathbb{N}$ with $d^*(M_1) = d^*(M_2) =: \delta \in [0,1)$. Let $U \subset \mathbb{C}_\infty$ be a domain with $\{0,\infty\} \subset U$ and such that

$$D_{K_\delta} * W \text{ is connected for some domain } W \subset U \text{ with } \{0,\infty\} \subset W\,. \tag{4.5}$$

Then we have $H_{M_1}(U) = \{0\}$ and $H_{\infty,M_2}(U) = \{0\}$.

Proof: For the time being, we assume that $0 \notin M_1$. Remark A.2 implies the existence of a (countable) set $\tilde{M}$ with $M_1 \subset \tilde{M} \subset (0,\infty)$ and $d(\tilde{M}) = \delta$. We write $\tilde{M} = M_1 \cup \tilde{D}$ where M_1 and $\tilde{D}$ are assumed to be disjoint. Since $\tilde{D}$ is countable, there exists a number $\sigma \in (0,1)$ such that the set $D := \tilde{D}+\sigma$ does not intersect the non-negative integers. Then $M := \{\mu_n :\ n \in \mathbb{N}\} := M_1 \cup D$ is still a measurable superset of M_1 with $d(M) = \delta$ and $M \cap \mathbb{N}_0 = M_1$.

We set

$$\Psi(z) := \prod_{n=1}^{\infty} \big(1 - \frac{z^2}{\mu_n^2}\big) \quad (z \in \mathbb{C})$$

and obtain according to Example B.19.2 an entire function of exponential type belonging to the space $\operatorname{Exp}(K_\delta)$ with $Z(\Psi) = \pm M$. Hence, $\psi := \mathfrak{M}^{-1}\Psi$ is holomorphic in D_{K_δ} with $\mathbb{N}_0 \setminus \Lambda_\psi^+ = M_1$.

If $0 \in M_1$, we consider the function $\tilde{\Psi}(z) := z\cdot\Psi(z)$ $(z \in \mathbb{C})$ (where Ψ is constructed as above based on the set $M_1 \setminus \{0\}$). Then $\tilde{\Psi} \in \operatorname{Exp}(K_\delta)$ (see Corollary B.13) and $\psi := \mathfrak{M}^{-1}\tilde{\Psi}$ is also holomorphic in $H(D_{K_\delta})$ with $\mathbb{N}_0 \setminus \Lambda_\psi^+ = M_1$.

By assumption, there exists a domain $W \subset U$ with $\{0,\infty\} \subset W$ and such that $D_{K_\delta} * W$ is connected. We obtain with Remark 4.2 and Theorem 4.12

$$H_{M_1}(W) = H_{\mathbb{N}_0\setminus\Lambda_\psi^+}(W) = N(T_{\psi,D_{K_\delta},W}) = \{0\}\,.$$

Because U is a domain, the restriction map $\rho : H(U) \to H(W),\ f \mapsto f|_W$ is injective. Hence we can consider $H(U)$ as a subspace of $H(W)$ and obtain $H_{M_1}(U) \subset H_{M_1}(W) = \{0\}$.

The second assertion follows analogously. □

Remark 4.16 :

1. If $d^*(M_j) = d(M_j) = 0$ $(j = 1, 2)$, then Condition (4.5) is superfluous because U itself is assumed to be connected.
2. Condition (4.5) is fulfilled if U contains a sector of opening larger than δ: Indeed, in this case we can without loss of generality assume the existence of numbers $0 < r < R < \infty$ and $\delta < \alpha \le 1$ such that
$$W := U_r(0) \cup U_R(\infty) \cup S_\alpha \subset U$$
(otherwise take a rotated set W). Then Example 2.3.3 implies that
$$D_{K_\delta} * W = U_r(0) \cup U_R(\infty) \cup S_{\alpha-\delta}$$
which is a connected set because $\alpha - \delta > 0$.
3. The first assertion of Proposition 4.15 can be interpreted in the following way:

   Whenever a power series about zero whose non-vanishing coefficients have density zero can be analytically continued up to infinity, then the function represented by that power series expansion must vanish identically.

   Whenever a power series about zero whose non-vanishing coefficients have maximal density larger than zero can be analytically continued into a so-called "keyhole domain" where the actual "keyhole" is large enough in the sense formulated in Condition (4.5), then the function represented by that power series must vanish identically.

   Interpreted this way, these assertions are special cases of the Fabry and Pólya gap theorems (see Appendix A).

Proposition 4.15 allows to formulate the following sufficient condition for T_φ to be injective.

Theorem 4.17 (Approach (K.II)) **:**
Let $\Omega \subset \mathbb{C}_\infty$ be a domain and let $U \subset \mathbb{C}_\infty$ be open with $\{0, \infty\} \subset \Omega \cap U$, $\varphi \in H(\Omega)$ and
$$\Delta_\varphi := \min\{d^*(\mathbb{N}_0 \setminus \Lambda_\varphi^+), d^*(\mathbb{N} \setminus \Lambda_\varphi^-)\}\,.$$
Then $T_\varphi : H(U) \to H(\Omega * U)$ is injective if the following two conditions hold:

(K.a) U is connected.

(K.b) $D_{K_{\Delta_\varphi}} * W$ is connected for some domain $W \subset U$ with $\{0,\infty\} \subset W$.

Proof: In case $\Delta_\varphi = d^*(\mathbb{N}_0 \setminus \Lambda_\varphi^+)$ we obtain with Theorem 4.1.2.(a) and Proposition 4.15

$$N(T_\varphi) \subset H_{\mathbb{N}_0 \setminus \Lambda_\varphi^+}(U) = \{0\}\,,$$

and in case $\Delta_\varphi = d^*(\mathbb{N} \setminus \Lambda_\varphi^-)$ we obtain with Theorem 4.1.3.(a) and Proposition 4.15

$$N(T_\varphi) \subset H_{\infty,\mathbb{N} \setminus \Lambda_\varphi^-}(U) = \{0\}\,.$$

□

Theorem 4.18 (Approach (R.II)) **:**
Let $\Omega \subset \mathbb{C}_\infty$ be a domain and let $U \subset \mathbb{C}_\infty$ be open and star-eligible to Ω with $\{0,\infty\} \subset \Omega \setminus U$, $\varphi \in H(\Omega)$ and

$$\Delta_\varphi := \min\{d^*(\mathbb{N}_0 \setminus \Lambda_\varphi^+), d^*(\mathbb{N} \setminus \Lambda_\varphi^-)\}\,.$$

Then $T_\varphi : H(U) \to H(\Omega * U)$ has dense range if the following two conditions hold:

(R.a) $(\Omega * U)^*$ is connected.

(R.b) Every open set $V \supset (\Omega * U)^*$ contains a domain $W \supset \{0,\infty\}$ such that $D_{K_{\Delta_\varphi}} * W$ is connected.

Proof: We show again that the transposed operator is injective.

Let $[(g,V)]_{(\Omega*U)^*} \in H((\Omega * U)^*)$ with $[(\varphi * g, \Omega * V)]_{U^*} = [0]_{U^*}$. Since $(\Omega * U)^*$ is connected, without loss of generality we can choose V to be connected, too. Furthermore we have $\{0,\infty\} \subset (\Omega * U)^* \cap U^*$. Therefore, the function $\varphi * g$ vanishes in an open neighbourhood of the origin and in an open neighbourhood of the point at infinity. Condition (R.b) implies that $D_{K_{\Delta_\varphi}} * W$ is connected for some domain W with $\{0,\infty\} \subset W \subset V$. Theorem 2.13 implies that $\varphi *_{\Omega,W} (g|_W)$ vanishes in an open neighbourhood of the origin and in an open neighbourhood of the point at infinity.

In case $\Delta_\varphi = d^*(\mathbb{N}_0 \setminus \Lambda_\varphi^+)$ we obtain with Theorem 4.1.2.(a) and Proposition 4.15 (noting that W is connected)

$$g|_W \in H_{\mathbb{N}_0 \setminus \Lambda_\varphi^+}(W) = \{0\}\,.$$

Since V is connected, g vanishes in the open superset V of $(\Omega * U)^*$ and therefore, $[(g,V)]_{(\Omega*U)^*} = [0]_{(\Omega*U)^*}$.

In case $\Delta_\varphi = d^*(\mathbb{N} \setminus \Lambda_\varphi^-)$, we obtain with Theorem 4.1.3.(a) and Proposition 4.15

$$g|_W \in H_{\infty,\mathbb{N}\setminus\Lambda_\varphi^-}(W) = \{0\}\,.$$

Since V is connected, g vanishes in the open superset V of $(\Omega * U)^*$ and therefore, $[(g,V)]_{(\Omega*U)^*} = [0]_{(\Omega*U)^*}$. □

Putting the results of this section together we obtain the following corollary.

Corollary 4.19 :
Let $\Omega \subset \mathbb{C}_\infty$ be a domain with $\{0,\infty\} \subset \Omega$ and let $U \subset \mathbb{C}_\infty$ be open and star-eligible to Ω. Let furthermore $\varphi \in H(\Omega)$. Then the following are valid for the operator $T_\varphi : H(U) \to H(\Omega * U)$:

1. If U is connected and if $U \cap \{0,\infty\} \neq \emptyset$, then T_φ having dense range implies the injectivity.
2. If $(\Omega * U)^*$ is connected and if $U \cap \{0,\infty\} \neq \{0,\infty\}$, then the injectivity of T_φ implies that it has dense range.

Proof: Everything is a consequence of Theorems 4.5/4.17 and 4.7/4.18. □

Remark 4.20 :
Corollary 4.19 together with Proposition 2.5.6 implies that for all simply connected domains $U \subset \mathbb{C}$ containing the origin, the operator $T_{\varphi,U}$ is injective if and only if it has dense range. Going back to the Koebe function presented in Example 3.2 we observe that as soon as the origin does not belong to U this is no longer true: $T_{\kappa,U}$ is even surjective for each simply connected domain $U \subset \mathbb{C} \setminus \{0\}$ but not injective.

This shows how severely the situation changes depending on whether the origin or the point at infinity belong to the set U or not.

Remark 4.21 :

1. Obviously, Conditions (K.b) and (R.b) rely on an interplay between Δ_φ and the geometry of the set U or $\Omega * U$ respectively. In Section 4.3 we are going to elaborate on that interplay and in addition, we are going to show that in general, only one of the conditions (K.a), (K.b) or (R.a), (R.b) is not enough for $T_{\varphi,U}$ to be injective or to have dense range. This shows in particular that

$T_{\varphi,U}$ is by no means injective for every domain U containing 0 and ∞ and $T_{\varphi,U}$ does by no means have dense range for every open set $U \subset \mathbb{C} \setminus \{0\}$ with $(\Omega * U)^*$ connected.

2. In order to show that T_φ has dense range (see Theorems 4.9/4.18) or that T_φ is injective (see Theorems 4.12/4.17) we used in each case two different approaches. We are going to compare these approaches in Section 4.3.

At the end of this section, we formulate the following necessary and sufficient conditions for T_φ to be injective or to have dense range in case that φ is holomorphic in $\mathbb{C}_\infty \setminus \{1\}$.

Proposition 4.22 :
Let $\varphi \in H(\mathbb{C}_\infty \setminus \{1\})$, $\Phi = \mathfrak{M}\varphi$ and let $U_1 \subset \mathbb{C}_\infty$ with $\{0, \infty\} \subset U_1$ and $U_2 \subset \mathbb{C} \setminus \{0\}$ be open.

1. (a) If U_1 is connected, then T_{φ,U_1} is injective.
   If U_2^C is connected, then T_{φ,U_2} has dense range.

   (b) If $Z(\Phi) = \emptyset$, then T_{φ,U_1} is injective and T_{φ,U_2} is surjective.

2. (a) If $Z(\Phi) \cap \mathbb{Z} \neq \emptyset$, then it is necessary
   - for T_{φ,U_1} to be injective that U_1 is connected,
   - for T_{φ,U_2} to have dense range that U_2^C is connected.

   (b) If $Z(\Phi) \neq \emptyset$ but $Z(\Phi) \cap \mathbb{Z} = \emptyset$, it is necessary
   - for T_{φ,U_1} to be injective that U_1 does not have a simply connected component containing neither 0 nor ∞,
   - for T_{φ,U_2} to have dense range that U_2^C does not have a simply connected component containing neither 0 nor ∞.

Proof:

1. (a) These assertion are a direct consequence of Theorem 4.12 and Theorem 4.9.

   (b) The proof of Theorem 4.6 shows that the conditions $\Phi \in \text{Exp}(\{0\})$ and $Z(\Phi) = \emptyset$ require that $\Phi \equiv \lambda$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. Furthermore, it can be shown analogously that $T_{\varphi,U_j} f(z) = \lambda f(z)$ $(z \in U_j,\ j = 1, 2)$. Clearly, this operator exhibits the desired properties.

2. We just show the assertions concerning the injectivity of T_{φ,U_1}. The corresponding assertions concerning the range of T_{φ,U_2} follow, as usual, using the transposed operator.

   (a) Let U_1 be non-connected and $\nu \in Z(\Phi) \cap \mathbb{Z}$. If $\nu \geq 0$ we set

   $$g(z) = \begin{cases} 0 & , \ z \in (U_1)_\infty \\ z^\nu & , \ z \in U_1 \setminus (U_1)_\infty \end{cases} ,$$

   and if $\nu < 0$ we set

   $$g(z) = \begin{cases} 0 & , \ z \in (U_1)_0 \\ z^\nu & , \ z \in U_1 \setminus (U_1)_0 \end{cases} .$$

   We noted in Example 2.17 that $T_{\varphi,U_1} p_{\nu,U_1} = 0$ and hence, T_{φ,U_1} is not injective.

   (b) If U_1 has a simply connected component $V \subset \mathbb{C} \setminus \{0\}$, then Theorem 4.6 shows that $T_{\varphi,V}$ is not injective which clearly implies that T_{φ,U_1} is not injective. □

Remark 4.23 :
We consider again the situation in Proposition 4.22. If $Z(\Phi) \neq \emptyset$ but $Z(\Phi) \cap \mathbb{Z} = \emptyset$, then it is not necessary for T_{φ,U_1} to be injective that U_1 is connected. For example, if U_1 consists of two components, one of them containing zero and the other containing the point at infinity, then T_{φ,U_1} is injective if and only if both $T_{\varphi,(U_1)_0}$ and $T_{\varphi,(U_1)_\infty}$ are injective. However, Theorem 4.5 ensures this latter condition.

A similar remark applies to the range of T_{φ,U_2}.

4.3 Remarks and distinguishing examples

In Section 4.2 we gathered several conditions for $T_{\varphi,U}$ to be injective or to have dense range. We always had to pay attention to the fact whether the origin and the point at infinity belong to the set U or not. It turned out that the respective properties of the resulting operator $T_{\varphi,U}$ changed heavily.

While examining the injectivity, the case $\{0,\infty\} \subset U \subset \mathbb{C}_\infty$ turned out to be most delicate (throughout this section, U_1 shall denote an open subset of $\mathbb{C}_\infty$ containing zero and infinity) whereas while studying the range we paid most of our attention to the case that U does neither contain the origin nor the point at infinity (throughout this section, U_2 shall denote an open subset of $\mathbb{C} \setminus \{0\}$). This is not surprising because the Hadamard operators T_{φ,U_1} and T_{φ,U_2} are linked with each other via duality.

In this section we want to shed more light on the situations mentioned above.

It is obvious that while examining the kernel as well as the range of $T_{\varphi,U}$ we were engaged with two different approaches (K.I)/(R.I) and (K.II)/(R.II). They have in common that they rely on an interplay between density conditions on the sets Λ_φ^+ or Λ_φ^- on the one hand and the geometry of the sets Ω and U on the other hand. It is a question of some kind of tradeoff: The more restrictive the sets Ω and U are chosen, the less assumptions on the sets Λ_φ^+ or Λ_φ^- are needed and vice versa.

The first approach (K.I)/(R.I) required the domain of holomorphy of the function φ (that means the set Ω) to be of the specific form D_K and imposed a geometrical condition on the set U which was independent of the "size" of the gaps of φ. It is the geometric context of the Mellin transformation which allows to renounce all kind of density conditions. We managed to transfer the argumentation to the level of the Mellin transform, which is an entire function. On this level, we were able to exploit the eigenvalue property formulated in Theorem 3.7 and Runge's approximation theorem. To put it in a nutshell, the first approach requires rather specific geometric situations and imposes connectedness assumptions on the sets U or $\Omega * U$ but does not have to impose conditions on the density of Λ_φ^+ or Λ_φ^-.

The respective second approaches (K.II)/(R.II) arose from a result which could be interpreted as a special case of the Fabry and Pólya gap theorems (see Proposition 4.15). This theorem in turn was a consequence of Approach (K.I). Pursueing this idea, Ω is not required to be of the form D_K but we also deal with connectedness assumptions. However, there are two kinds of them: Conditions (K.a) and (R.a) are independent of the function $\varphi \in H(\Omega)$ whereas Conditions (K.b) and (R.b) depend on the "size" of the gaps in the power series expansions of φ about zero or infinity measured by the quantity Δ_φ. We can observe a direct correlation between the "size" of those gaps and the size of the set U (or $\Omega * U$): The more gaps are allowed (i. e. the larger Δ_φ), the more restrictive Condition (K.b)/(R.b) becomes and hence, the less general the set U can be chosen.

This section shall serve the purpose to have a closer look at the theorems mentioned above: In a first step we are going to scrutinize Approach (K.II)/(R.II) and examine the correlation between Δ_φ and the geometry of the set U (or $\Omega * U$) formulated in Condition (K.b)/(R.b) (see Remark 4.24). In a second step we are going to show that in general, only one of the conditions (K.a)/(R.a) or (K.b)/(R.b) is not enough to get the desired result of injectivity or dense range of $T_{\varphi,U}$ (see Remarks 4.25 and 4.26). In a third step we compare the different approaches (K.I)/(R.I) and (K.II)/(R.II) among one another and are going to point out examples in which one of the approaches is superior to the other (see Remaks 4.28 and 4.29).

Because we want to discuss the results concerning the kernel and the range of the considered operators $T_{\varphi,U}$ simultaneously, we agree upon the following notation: in the bullet points (α) we collect the remarks concerning the injectivity of T_{φ,U_1} (i. e. concerning (K.I) or (K.II)) and in the bullet points (β) we collect the remarks concerning the range of T_{φ,U_2} (i. e. concerning (R.I) or (R.II)).

Remark 4.24 :
Let $\Omega \subset \mathbb{C}_\infty$ be a domain with $\{0,\infty\} \subset \Omega$, $\varphi \in H(\Omega)$ and

$$\Delta_\varphi = \min\{d^*(\mathbb{N}_0 \setminus \Lambda_\varphi^+), d^*(\mathbb{N} \setminus \Lambda_\varphi^-)\}\,.$$

Let furthermore $U_1 \subset \mathbb{C}_\infty$ be open with $\{0,\infty\} \subset U_1$ and let $U_2 \subset \mathbb{C}\setminus\{0\}$ be open and star-eligible to Ω.

1. Δ_φ vanishes if and only if $d(\Lambda_\varphi^+) = 1$ or $d(\Lambda_\varphi^-) = 1$.
2. (α) If Condition (K.b) is valid, then the origin and the point at infinity belong to the same component of U_1.

   (β) If Condition (R.b) is valid, then the origin and the point at infinity belong to the same component of $(\Omega * U_2)^*$.

   (Indeed, if this is not the case, it is clear that there are open supersets V of $(\Omega * U_2)^*$ which contain the origin and the point at infinity in different components. Hence, these sets V can not contain a domain $W \supset \{0,\infty\}$).
3. If $\Delta_\varphi = 0$, then the following holds:

   (α) Condition (K.b) is valid if and only if the origin and the point at infinity belong to the same component of U_1 (Indeed, since $D_{K_0} = \mathbb{C}_\infty \setminus \{1\}$ we can take W as the component of U_1 containing zero and infinity).

   (β) Condition (R.b) is valid if and only if the origin and the point at infinity belong to the same component of $(\Omega * U_2)^*$.

   (Indeed, if the origin and the point at infinity belong to the same component of $(\Omega * U)^*$, then the same applies to every open superset V of $(\Omega * U)^*$. Hence we can choose W as being the component of V containing zero and infinity).

Especially, if Condition (K.a)/(R.a) holds, then Condition (K.b)/(R.b) holds, too (for $\Delta_\varphi > 0$, this is not at all true (see Remark 4.26)!).

4. If $0 < \Delta_\varphi < 1$, then the following holds:

   (α) Condition (K.b) is valid if U_1 contains a sector of opening larger than Δ_φ (see Remark 4.16.2).

   (β) Condition (R.b) is valid if $(\Omega * U_2)^*$ contains a cone of opening at least Δ_φ.
   (Indeed, in this case every open superset V of $(\Omega * U_2)^*$ contains zero and infinity and therefore also a sector of opening larger than Δ_φ. Then the assertion follows with Remark 4.16.2).

5. If $\Delta_\varphi = 1$, then Conditions (K.b) and (R.b) can not be met (at least in the non-trivial case $U_1 \neq \mathbb{C}_\infty$). That means in case that $d_*(\Lambda_\varphi^+) = d_*(\Lambda_\varphi^-) = 0$, Approaches (K.II) and (R.II) do not make an assertion about the injectivity or the range of T_φ.

   In Proposition 3.4 we have shown that in case $\underline{d}(\Lambda_\varphi^+) = 0$ or $\underline{d}(\Lambda_\varphi^-) = 0$, the induced operator $T_{\varphi,U}$ is trivial for all star-eligible sets U and hence of course neither injective nor having dense range. Hence, the only case where Approaches (K.II) and (R.II) do not yield a result is when $d_*(\Lambda_\varphi^+)$ and $d_*(\Lambda_\varphi^-)$ vanish but $\underline{d}(\Lambda_\varphi^+)$ and $\underline{d}(\Lambda_\varphi^-)$ are positive.

Remark 4.25 :
We consider the Koebe function $\kappa \in H(\mathbb{C}_\infty \setminus \{1\})$ introduced in Example 3.2. Since $\mathfrak{M}\kappa = \mathrm{id}_\mathbb{C}$, Proposition 4.22 yields:

(α) If U_1 contains the origin and the point at infinity in the same component (i. e. Condition (K.b) is fulfilled, see Remark 4.24.3) but U_1 is not connected (i. e. Condition (K.a) is not fulfilled) then T_{κ,U_1} is not injective.

(β) If U_2 contains the origin and the point at infinity in the same component of its complement (i. e. Condition (R.b) is fulfilled, see Remark 4.24.3) but U_2^C is not connected (i. e. Condition (R.a) is not fulfilled) then T_{κ,U_2} does not have dense range.

These observations show that in general, Condition (K.b) or (R.b) alone is not sufficient for T_φ to be injective or to have dense range.

Remark 4.26 :
If $\Delta_\varphi = 0$, then we have seen in Remark 4.24 that Condition (K.b)/(R.b) is fulfilled whenever Condition (K.a)/(R.a) holds. However, we are going to show that this is not true if $\Delta_\varphi > 0$ and that in this case, Condition (K.a)/(R.a) alone is not sufficient to obtain that the corresponding Hadamard operator is injective or has dense range. That means, we can not weaken the assumptions on the density of the non-vanishing coefficients in the power series expansions of φ (i. e. enlarging Δ_φ) without imposing stronger geometrical conditions on the set U (or $\Omega * U$) such as (K.b)/(R.b).

We consider the function

$$\varphi(z) := \frac{1}{1-z^2} \quad (z \in \mathbb{C}_\infty \setminus \{\pm 1\}) .$$

Then φ is an even function and holomorphic in $\Omega := \mathbb{C}_\infty \setminus \{\pm 1\}$ (i. e. $\tilde{\varphi}(z) := \varphi(-z) = \varphi(z)$ $(z \in \Omega)$). The power series expansions of φ read as follows

$$\begin{aligned} \varphi(z) &= \sum_{\nu=0}^{\infty} z^{2\nu} && (z \in \mathbb{D}) , \\ \varphi(z) &= -\sum_{\nu=1}^{\infty} z^{-2\nu} && (z \in \overline{\mathbb{D}}^C) . \end{aligned}$$

Consequently, we have $d(\Lambda_\varphi^+) = d(\Lambda_\varphi^-) = \Delta_\varphi = 1/2$. We consider the domain

$$D := \mathbb{D}_r \cup U_R(\infty) \cup S_{1/4} \cup (-S_{1/4})$$

(where $0 < r < R < \infty$). Then for every domain $W \subset D$ with $\{0, \infty\} \subset W$ we obtain

$$\{0, \infty\} \subset D_{K_{1/2}} * W \subset D_{K_{1/2}} * D = \mathbb{D}_r \cup U_R(\infty) \tag{4.6}$$

and therefore, the set $D_{K_{1/2}} * W$ is not connected.

(α) We set $U_1 := D$. Then U_1 is clearly connected (i. e. Condition (K.a) is fulfilled) but (4.6) shows that Condition (K.b) is not fulfilled. As a matter of fact, $T_\varphi : H(U_1) \to H(U_1)$ is not injective, because for every non-even function $f \in H(U_1)$, the function $\tilde{f}(z) := f(-z)$ $(z \in U_1)$ is holomorphic in U_1 and not equal to f but Equation (2.2) implies

$$T_\varphi \tilde{f} = T_{\tilde{\varphi}} f = T_\varphi f .$$

(β) We set $U_2 := \mathbb{S}$ and obtain $\Omega * U_2 = \mathbb{C} \setminus \mathbb{R}$ and $(\Omega * U_2)^* = \mathbb{R} \cup \{\infty\}$ (i. e. Condition (R.a) is fulfilled). (4.6) implies that Condition (R.b) is not fulfilled (choose $V = D$). As a matter of fact, $T_\varphi : H(U_2) \to H(\mathbb{C} \setminus \mathbb{R})$ does not have dense range because Proposition 2.20 yields that $T_\varphi f$ is an even function for all $f \in H(U_2)$.

These observations show that in general, Condition (K.a) or (R.a) alone is not sufficient for T_φ to be injective or to have dense range.

Now we turn towards a comparison of the Approaches (K.I)/(R.I) and (K.II)/(R.II).

Remark 4.27 :
Of course there are situations in which Ω is not of the form D_K and therefore we only have a chance to apply Approach (K.II)/(R.II). For example, the function

$$\varphi(z) := \frac{1}{1-z} + \frac{1}{2+z} \quad (z \in \mathbb{C}_\infty \setminus \{-2, 1\})$$

is holomorphic in $\Omega := \mathbb{C}_\infty \setminus \{-2, 1\}$ and $\Lambda_\varphi^+ = \mathbb{N}_0$ and therefore $\Delta_\varphi = 0$.

(α) By setting $U_1 := \mathbb{C}_\infty \setminus \{1\}$, Approach (K.II) yields that

$$T_\varphi : H(\mathbb{C}_\infty \setminus \{1\}) \to H(\Omega)$$

is injective while Approach (K.I) can not be applied.

(β) By setting $U_2 := \mathbb{S}$, we obtain $(\Omega * U_2)^* = \mathbb{R} \cup \{\infty\}$ and Approach (R.II) yields that

$$T_\varphi : H(\mathbb{S}) \to H(\mathbb{C} \setminus \mathbb{R})$$

has dense range while Approach (R.I) can not be applied.

Now we want to have a look at what happens if Ω is indeed of the form D_K. Which approach should then be preferred? There is no clear-cut answer to this question. The proofs of Approaches (K.II) and (R.II) both rely on an application of Proposition 4.15. This proposition in turn is a consequence of Approach (K.I)/(R.I). This observation could give the impression that Approach (K.I)/(R.I) is superior to the second approach. However, this is not at all true. As we will show in the following remarks, there are sets $\Omega = D_K$ and functions $\varphi \in H(\Omega)$ such that it is not difficult to specify sets U_1, U_2 for which Approach (K.I)/(R.I) can be applied while Approach (K.II)/(R.II) can not and vice versa.

Remark 4.28 :
We consider the set

$$\Lambda_{33} := \bigcup_{k\in\mathbb{N}_0} \{n \in \mathbb{N} :\ 33^k < n \le 2\cdot 33^k\}\,.$$

We show in Lemma A.4 that

$$\overline{d}(\Lambda_{33}) = 33/64 > 1/2, \quad d_L(\Lambda_{33}) = \ln 2/\ln 33 < 1/5\,.$$

According to Remark 3.6.2 there exist real numbers $a \le b$ and a function $0 \not\equiv \psi \in H(D_M)$ (where $M = [a,b]\times i\pi[-1/5,1/5]$) with $\Lambda_\psi^+ = \mathbb{N}_0 \setminus \Lambda_{33}$. Then $\Psi := \mathfrak{M}\psi$ belongs to the space $\mathrm{Exp}(M)$ and we set

$$\Phi(z) := \Psi(z)\cdot\Psi(-z)\ (z\in\mathbb{C}), \quad \varphi := \mathfrak{M}^{-1}\Phi\,.$$

Then Corollary B.13.2 and Proposition B.10.2 imply that Φ belongs to the space $\mathrm{Exp}(K)$ (with $K := [a-b, b-a]\times i\pi[-2/5,2/5]$) and therefore $\varphi \in H(D_K)$. Furthermore we have

$$\Delta_\varphi = \min(d^*(\mathbb{N}_0\setminus\Lambda_\varphi^+), d^*(\mathbb{N}\setminus\Lambda_\varphi^-)) \ge d^*(\Lambda_{33}) \ge \overline{d}(\Lambda_{33}) > 1/2\,.$$

(α) We set $U_1 := D_{K_{1/2}}$ and observe that for every domain $W \subset U_1$ with $\{0,\infty\}\subset W$ we have

$$\{0,\infty\} \subset D_{K_{\Delta_\varphi}} * W \subset D_{K_{\Delta_\varphi}} * U_1 = \mathbb{C}_\infty \setminus \mathbb{T}$$

and therefore the set $D_{K_{\Delta_\varphi}} * W$ is not connected. Hence, Condition (K.b) is not fulfilled and Approach (K.II) can not be applied. However, Approach (K.I) yields that T_{φ,U_1} has dense range (note that $D_K * U_1 = D_{K+K_{1/2}}$ is connected).

(β) We set $U_2 := \mathbb{S}$. Example 2.3.4 implies that $D_K * U_2 = S_{3/5}$ and therefore, the set

$$V := \mathbb{D}_r \cup U_R(\infty) \cup (-S_{1/2})$$

(where $0 < r < R < \infty$) is an open superset of $(D_K * U_2)^*$ but for every domain $W\subset V$ with $\{0,\infty\}\subset W$ we obtain

$$\{0,\infty\} \subset D_{K_{\Delta_\varphi}} * W \subset D_{K_{\Delta_\varphi}} * V = \mathbb{D}_r \cup U_R(\infty)$$

and therefore the set $D_{K_{\Delta_\varphi}} * W$ is not connected. Hence, Condition (R.b) is not fulfilled and therefore, Approach (R.II) can not be applied. However, Approach (R.I) yields that T_{φ,U_2} has dense range.

Remark 4.29 :
Let $K \subset \mathbb{V}$ be compact and convex and $\varphi \in H(D_K)$. If $d(\Lambda_\varphi^+)$ exists and if $\{z \in K : \operatorname{Re} z = \max_{w \in K} \operatorname{Re} w\}$ is a singleton, then Remark A.8 implies that $d(\Lambda_\varphi^+) = 1$ and therefore $\Delta_\varphi = 0$.

(α) Approach (K.I) yields that

$$T_{\varphi,U_1} \text{ is injective for all connected sets } U_1 \text{ such that } D_K * U_1 \text{ is connected}$$

whereas Approach (K.II) implies that

$$T_{\varphi,U_1} \text{ is injective for all connected sets } U_1\,.$$

Obviously, Approach (K.II) can be applied to a broader class of sets U_1.

(β) Approach (R.I) yields that

$$T_{\varphi,U_2} \text{ has dense range for all sets } U_2 \text{ such that } U_2^C \text{ is connected}$$

whereas Approach (R.II) implies that

$$T_{\varphi,U_2} \text{ has dense range for all sets } U_2 \text{ such that } (D_K * U_2)^C \text{ is connected}\,.$$

Having a look at Proposition 2.5.6 and Example 2.6.2 we observe that Approach (R.II) can be applied to a broader class of sets U_2.

4.4 Surjectivity of T_φ

We now turn towards the question under which conditions the operator $T_{\varphi,U}$ is even surjective. In Example 3.2 we showed that the operator $T_{\kappa,U} : H(U) \to H(U)$ is surjective for every simply connected domain $U \subset \mathbb{C} \setminus \{0\}$ (where κ is the Koebe function). Since $\mathfrak{M}\kappa = \mathrm{id}_{\mathbb{C}}$, the following proposition constitutes a generalization of this simple example.

Proposition 4.30 :
Let Φ be a non-vanishing polynomial and $\varphi = \mathfrak{M}^{-1}\Phi$. Then $T_\varphi : H(U) \to H(U)$ is surjective for every set $U \subset \mathbb{C} \setminus \{0\}$ with connected complement.

Proof: Without loss of generality we can assume U to be simply connected (otherwise apply the subsequent argumentation to every component of U separately).

If Φ is constant, then the assertion follows from Proposition 4.22.1.

If Φ is of degree $n \in \mathbb{N}$ and $\alpha \in \mathbb{C} \setminus \{0\}$ is the leading coefficient, we set $Z(\Phi) =: \{\lambda_j : \ j = 1, \dots, n\}$, $\Phi_j(z) := (z - \lambda_j)$ $(z \in \mathbb{C}, \ j \in \{1, \dots, n\})$ and we obtain

$$\Phi(z) = \alpha \prod_{j=1}^{n} \Phi_j(z) \quad (z \in \mathbb{C})\,.$$

Then for every $j \in \{1, \dots, n\}$ we have $\varphi_j := \mathfrak{M}^{-1}\Phi_j = \kappa - \lambda_j \Theta$, where κ is the Koebe function and $\Theta(z) = 1/(1-z)$ $(z \in \mathbb{C}_\infty \setminus \{1\})$.

In a first step we show that every operator $T_{\varphi_j} : H(U) \to H(U)$, $T_{\varphi_j} f(z) = z \cdot f'(z) - \lambda_j f(z)$ is surjective.

Let $g \in H(U)$ be given. We have to find a function $f \in H(U)$ with $f'(z) = \lambda_j f(z)/z + g(z)/z$ $(z \in U)$. We fix a number $z_0 \in U$ and set

$$f(z) := \exp(\lambda_j \log_U z) \cdot \Big(\int_{\gamma_{z_0,z}} \exp(-\lambda_j \log_U \zeta) \frac{g(\zeta)}{\zeta}\, d\zeta \Big)$$

where $\gamma_{z_0,z}$ is a path in U joining z_0 and z. Since U is simply connected, this function is well defined (i. e. independent of the choice of $\gamma_{z_0,z}$), holomorphic in U and solves the differential equation formulated above.

In a second step we show that the surjectivity of each T_{φ_j} implies the surjectivity of T_φ:

Remark B.17.3 yields that $\varphi = \alpha \cdot \big(\dots \big((\varphi_1 * \varphi_2) * \varphi_3 \big) \dots * \varphi_{n-1} \big) * \varphi_n$. The associative law formulated in Theorem 2.23 ensures that $T_\varphi = \alpha \cdot T_{\varphi_1} \circ \dots \circ T_{\varphi_n}$ and hence, T_φ is surjective as a composition of surjective operators. □

Remark 4.31 :
The assertion of Proposition 4.30 can be considered as a consequence of a corresponding result for a certain (finite order) differential operator with constant coefficients (see Remark 5.10). In Chapter 5 we are going to elaborate on the relation between Hadamard operators and certain differential operators.

Before we come to a more general surjectivity criterion we need the following result for the star product:

Proposition 4.32 :
Let $M, W \subset \mathbb{V}$ be convex, where M is compact and W is open. Let furthermore $W + M \subset \mathbb{V}$ and $U := e^{W+M}$. Then $D_M * U = e^W$.[4]

Proof: Since $U = e^{W+M} = e^W \cdot D_M^*$, Proposition 2.5.3 implies that $D_M * U$ is a superset of e^W.

To obtain the reverse inclusion, we show $(e^W)^C \subset (D_M * U)^C$. We have

$$\begin{aligned}(D_M * U)^C &= D_M^C \cdot U^C = e^{-M} \cdot (e^{W+M})^C \\ &= e^{-M} \cdot \{e^{(O^C)} \cup \{0\}\}\end{aligned}$$

where $O := \bigcup_{k \in \mathbb{Z}} (W + M + 2k\pi i)$ and $e^\infty := \infty$ (note that the sets $W + M + 2k\pi i$ ($k \in \mathbb{Z}$) are pairwise disjoint).

Now let $z \in (e^W)^C$ (since $\{0, \infty\} \subset (D_M * U)^C$ we assume $z \neq 0, \infty$). Then there is a point $v \in \overline{N}_1 \setminus W$ such that for all $m \in M$ we have

$$z = e^v = e^{-m} \cdot e^{v+m}.$$

If $m \in M$ can be chosen in such a way that $v + m$ belongs to O^C we are done.

Assume that this is not the case, i. e. $v + M \subset O$. Since $v + M$ is connected, it has to lie entirely in one component of O and that component shall without loss of generality be the set $W + M$ itself. Since W is convex, without loss of generality we can require a standard exhaustion $(L_n)_{n \in \mathbb{N}}$ of W to consist of convex sets. Then $L_n + M$ is a standard exhaustion of $W + M$ consisting of convex sets. Since $v + M$ is compact, the definition of standard exhaustions implies the existence of an integer n_0 such that $v + M \subset L_{n_0} + M$. Since in the latter inclusion, all occuring sets are compact and convex, we can deduce $v \in L_{n_0} \subset W$ (see Proposition B.7) which contradicts the choice of v. □

Remark 4.33 :
In the situation of Proposition 4.32, the assumption $W + M \subset \mathbb{V}$ is crucial:

Let $M = K_{1/2}$ and

$$\begin{aligned}W_1 &= (0,1) \times i\pi(-1/4, 1/4), \\ W_2 &= (0,1) \times i\pi(-3/4, 3/4).\end{aligned}$$

Then $W_1 + M \subset \mathbb{V}$ but $W_2 + M \not\subset \mathbb{V}$. As a matter of fact, $D_M * e^{W_1+M} = e^{W_1}$ but $D_M * e^{W_2+M} = e^{W_2+M} \supsetneq e^{W_2}$.

[4]This result may be compared to Remark 2.6.1.

We obtain the following sufficient criterion for a Hadamard operator T_φ to be surjective.

Theorem 4.34 :
Let Φ be an entire function of exponential type with $K(\Phi) =: M \subset \mathbb{V}$. Let furthermore $W \subset \mathbb{V}$ be open and convex and such that $W + M \subset \mathbb{V}$. Let $U := e^{W+M}$ and $\varphi := \mathfrak{M}^{-1}\Phi$.

If Φ is of completely regular growth, then the following are valid for the operator $T_\varphi : H(U) \to H(D_M * U)$:

$$N(T_\varphi) = \text{clspan}\{q_{k,\alpha,U} :\ \alpha\ m\text{−fold zero of } \Phi,\ k \leq m-1\} \tag{4.7}$$

and

$$R(T_\varphi) = H(D_M * U)\,. \tag{4.8}$$

Proof: First of all we note that according to Proposition 4.32 we have $D_M * U = e^W$.

In order to prove (4.7) we set $Q_{\Phi,U} := \text{span}\{q_{k,\alpha,U} :\ \alpha\ m\text{−fold zero of } \Phi,\ k \leq m-1\}$ and without loss of generality we assume that the functions $q_{k,\alpha,U}$ are induced by the principal branch of the logarithm on $\mathbb{S}$.

Theorem 4.4 states that the left-hand side is a superset of the right-hand side (note that U is simply connected).

In order to show the opposite inclusion, according to the theorem of Hahn-Banach it is enough to prove that for all $f \in N(T_\varphi)$ the following holds: For all $u \in Q^\perp_{\Phi,U} := \{u \in H'(U) :\ u(q) = 0 \text{ for all } q \in Q_{\Phi,U}\}$ we have $u(f) = 0$.

Let now $u \in Q^\perp_{\Phi,U}$ be given. According to Theorem 2.24 there exists a unique germ $[(g, V)]_{U^*}$ such that

$$u(f) = (f *_{U,V} g)(1) \quad (f \in H(U))\,.$$

Since $U^* = \mathbb{C}_\infty \setminus e^{-(W+M)}$ and $W + M$ is convex, without loss of generality we can choose V to be of the form $V = D_{L+M}$ for some convex and compact set $L \subset W$. Therefore we can consider the Mellin transform $G := \mathfrak{M}g \in \text{Exp}(L + M)$ of g.

If $\alpha \in \mathbb{C}$ is an m−fold zero of Φ and $k \leq m - 1$, we obtain with Theorem 2.24 and Theorem 3.7

$$\begin{aligned} 0 &= u(q_{k,\alpha,U}) = (g *_{D_{L+M},U} q_{k,\alpha,U})(1) \\ &= p_{\alpha,D_{L+M}*U}(1) \sum_{l=0}^{k} \binom{k}{l} (\log_{D_{L+M}*U} 1)^{k-l} G^{(l)}(\alpha) \\ &= G^{(k)}(\alpha) \end{aligned}$$

because $\log_{D_{L+M}*U} 1 = 0$ (note that the proof of Theorem 3.7 reveals that the branch of the logarithm on $D_{L+M} * U$ that fits to the principal branch of the logarithm on $\mathbb{S}$ fulfills $\log_{D_{L+M}*U} 1 = 0$).

Hence, α is a zero of G with multiplicity at least m which implies that the function $\Psi := G/\Phi$ is an entire function (or to be more precise: can be continued to an entire function) and Theorem B.20.2 yields that Ψ is of exponential type. Together with Theorem B.12 and Proposition B.7 we obtain for all $t \in [-\pi, \pi)$

$$\begin{aligned} H_{K(\Psi)}(e^{it}) &= h_\Psi(t) = h_G(t) - h_\Phi(t) \\ &= H_{K(G)}(e^{it}) - H_{K(\Phi)}(e^{it}) \\ &\leq H_{L+M}(e^{it}) - H_M(e^{it}) \\ &= H_L(e^{it}) \,. \end{aligned}$$

Applying again Proposition B.7.2 we obtain $K(\Psi) \subset L$ which results in $\Psi \in \operatorname{Exp}(L)$ and $\psi := \mathfrak{M}^{-1}\Psi \in H(D_L)$. Remark B.17.3 yields the identity $\mathfrak{M}(\varphi * \psi) = \Phi \cdot \Psi = G$ and since $\mathfrak{M}$ is bijective it follows that $\varphi * \psi = g$.

For a given $f \in N(T_\varphi)$ we obtain with Theorem 2.23

$$\begin{aligned} u(f) &= (g *_{D_{M+L},U} f)(1) = ((\varphi *_{D_M,D_L} \psi) *_{D_{M+L},U} f)(1) \\ &= (\psi *_{D_L,D_M*U} (\varphi *_{D_M,U} f))(1) \\ &= (\psi *_{D_L,D_M*U} 0)(1) = 0 \,. \end{aligned}$$

This completes the proof of (4.7).

In order to prove (4.8) we note that Theorem 4.9 yields that $R(T_\varphi)$ is dense in $H(D_M * U)$. If we manage to show that $R(T_\varphi)$ is closed in $H(D_M * U)$, the proof will be complete. In order to do that, the Closed Range Theorem C.4 ensures that it is enough to show that $R(T'_\varphi) = Q^\perp_{\Phi,U}$.[5]

Since $Q_{\Phi,U} \subset N(T_\varphi)$ it is clear that the left-hand side is a subset of the right-hand side.

If, on the other hand, $u \in Q^\perp_{\Phi,U}$, then we have shown above that the corresponding germ $[(g, V)]_{U^*}$ can be written as

$$[(g, V)]_{U^*} = [(\varphi *_{D_M,D_L} \psi, D_{M+L})]_{U^*}$$

for some suitable $[(\psi, D_L)]_{(D_M*U)^*} \in H((D_M * U)^*)$. Hence, Theorem 3.9 yields that the corresponding functional $v \in H'(D_M * U)$ fulfills $T'_\varphi v = u$ and we obtain $u \in R(T'_\varphi)$. □

[5]Note that $Q^\perp_{\Phi,U} = N(T_\varphi)^\perp$.

Corollary 4.35 :
Let $\varphi \in H(\mathbb{C}_\infty \setminus \{1\})$, $\Phi = \mathfrak{M}\varphi$ and let $U \subset \mathbb{S}$ be a simply connected domain with $\log U$ convex. Then the following are valid for the operator $T_\varphi : H(U) \to H(U)$:

$$N(T_\varphi) = \text{clspan}\{q_{k,\alpha,U} : \ \alpha \ m\text{-fold zero of } \Phi,\ k \leq m-1\}$$

and

$$R(T_\varphi) = H(U)\,.$$

Proof: Example B.19.1 yields that $\Phi \in \text{Exp}(\{0\})$ is of completely regular growth and Remark B.9.1 implies $K(\Phi) = \{0\}$ (note that we excluded the case $\varphi \equiv 0$). The assertions are a direct consequence of Theorem 4.34. □

Remark 4.36 :

1. The assertion of Theorem 4.34 can be considered as a special case of results concerning the surjectivity of operators which are defined via a convolution of an analytic functional with a holomorphic function formulated in [BG, Prop. 1.5.12]. The exact relation between Hadamard operators and those convolution operators will be revealed in Chapter 5. The idea of the proof of Theorem 4.34 is motivated by the proof of [BG, Prop. 1.5.12] but we were able to express everything by means of the Hadamard product and we were able to use the related results of the preceding chapters.

   The surjectivity result formulated in Theorem 4.34 can also be shown using a different approach not needing the representation of the kernel. However, the main idea of exploiting the complete regular growth of Φ still remains the same:

   Let L_n be a standard exhaustion of W consisting of convex sets. Then $C_n := L_n + M$ is a standard exhaustion of $W + M$ consisting of convex sets and e^{L_n} and e^{C_n} are standard exhaustions of $D_M * U$ and U respectively. Therefore we obtain with Remark 1.11

$$\begin{aligned} H((D_M * U)^*) &= \bigcup_{n\in\mathbb{N}} H^\infty(D_{L_n})\,, \\ H(U^*) &= \bigcup_{n\in\mathbb{N}} H^\infty(D_{C_n})\,. \end{aligned}$$

   To prove the surjectivity of T_φ we apply Theorem C.6. $T_\varphi : H(U) \to H(D_M * U)$ is a linear and continuous operator (see Theorem 3.1) having dense range (see Theorem 4.9) and as an FS-space, $H(D_M * U)$ is a barrelled Schwartz space (see Remark C.2), so that all the conditions of Theorem C.6 are fulfilled. We therefore have to show that for each bounded subset $\mathcal{B} \subset H'(U)$ there is a bounded subset $\mathcal{A} \subset H'(D_M * U)$ such that $(T'_\varphi)^{-1}(\mathcal{B}) \subset [\mathcal{A}]$ (where $[\mathcal{A}]$ denotes the linear span of $[\mathcal{A}]$).

Let $\mathcal{B} \subset H'(U) \cong H(U^*)$ be bounded. According to Remark C.2, $\mathcal{B}$ is equicontinuous and there is an integer $n \in \mathbb{N}$ such that $\mathcal{B}$ is bounded in the Banach space $H^\infty(D_{C_n})$. At this point we identify a functional $u \in \mathcal{B} \subset H'(U)$ with the corresponding germ $[(h, Y)]_{U^*} \in \mathcal{B} \subset H(U^*)$ and this in turn with the representative $h \in \mathcal{B} \subset H^\infty(D_{C_n})$. In the rest of the proof, we use this identification tacitly.

Let $\mathcal{A}$ be the unit ball of $H^\infty(D_{L_{n+1}})$. Then $\mathcal{A}$ is bounded in $H((D_M * U)^*)$ with $[\mathcal{A}] = H^\infty(D_{L_{n+1}})$ and applying Theorem 3.9 we have to make sure that

$$([T]_\varphi)^{-1}(\mathcal{B}) = \{[(g, V)]_{(D_M * U)^*} \in H((D_M * U)^*) : \ \varphi * g \in \mathcal{B}\} \subset H^\infty(D_{L_{n+1}})\,.$$

We will show that each $[(g, V)]_{(D_M * U)^*} \in H((D_M * U)^*)$ with $\varphi * g \in H^\infty(D_{C_n})$ belongs to $H^\infty(D_{L_{n+1}})$ which will complete the proof.

Let $[(g, V)]_{(D_M * U)^*} \in H((D_M * U)^*)$ be given. Without loss of generality V can be chosen to be of the form $V = D_{L_j}$ for some $j \in \mathbb{N}$. Therefore we can consider the Mellin transform $G := \mathfrak{M}g$ of g.

If $\varphi * g \in H^\infty(D_{C_n})$, then on the one hand we obtain

$$K(\mathfrak{M}(\varphi * g)) \subset C_n = L_n + M \tag{4.9}$$

and on the other hand, applying Remark B.17.3 and Remark B.21 we obtain

$$K(\mathfrak{M}(\varphi * g)) = K(\Phi \cdot G) = K(\Phi) + K(G) = M + K(G)\,. \tag{4.10}$$

Since all occuring sets on the respective right-hand sides of (4.9) and (4.10) are compact and convex, Proposition B.7 yields $K(G) \subset L_n$ and hence $G \in \operatorname{Exp}(L_n)$. This yields $g \in H(D_{L_n})$ and especially $g \in H^\infty(D_{L_{n+1}})$.

2. Proposition 4.30 shows that the assumption that the set W shall be convex is in general not necessary.

Example 4.37 :
We consider again the situation in Example 4.10. There we stated that $T_\varphi : H(\mathbb{S}) \to H(S_{1-\delta})$ has dense range. However, Example B.19.2 ensures that Φ is of completely regular growth with $K(\Phi) = K_\delta$. Hence, Theorem 4.34 yields that this operator is even surjective.

Remark 4.38 :
Let $\varphi \in H(\mathbb{C}_\infty \setminus \{1\})$. After collecting some information about the range of the

operator $T_{\varphi,U}$ in case that U does neither contain the origin nor the point at infinity, we shall point out that in [Fre, Th. 10, Th. 11], Frerick gives a characterization of the surjectivity of $T_\varphi : H(U) \to H(U)$ for certain sets U containing the origin:

$T_\varphi : H(U) \to H(U)$ is surjective for all simply connected domains $U \subset \mathbb{C}$ with $0 \in U$, if and only if on the one hand $\Lambda_\varphi^+ = \mathbb{N}_0$ and on the other hand $\mathfrak{M}\varphi$ is a polynomial or $\lim_{z\to\infty,\ \mathfrak{M}\varphi(z)=0} z/|z| = -1$.

$T_\varphi : H(U) \to H(U)$ is surjective for all domains which are starlike with respect to the origin, if and only if on the one hand $\Lambda_\varphi^+ = \mathbb{N}_0$ and on the other hand $\mathfrak{M}\varphi$ is a polynomial or $\limsup_{z\to\infty,\ \mathfrak{M}\varphi(z)=0} \mathrm{Re}(z/|z|) \leq 0$.

Chapter 5

The relationship between Hadamard operators, other convolution operators and infinite order differential operators

In this chapter we want to have a closer look at the case where Ω is of the form $D_M = \mathbb{C}_\infty \setminus e^{-M}$ for some compact and convex set $M \subset \mathbb{V}$. We are going to see that if $\varphi \in H(D_M)$ and if the set U (which is assumed to be star-eligible to D_M) exhibits an appropriate structure, the induced Hadamard operator

$$T_\varphi : H(U) \to H(D_M * U)$$

reveals an intimate connection to another type of convolution operators and to certain infinite order differential operators.

The latter operators have been examined during the last decades (see for example [Kor69], [Kor1969], [Ep1974] and [MM]) and we are going to give an alternative proof of a well-known surjectivity result for infinite order differential operators. This proof is based on the corresponding result for Hadamard operators formulated in Corollary 4.35. At the same time, we are going to improve (for the special case $\Omega = \mathbb{C}_\infty \setminus \{1\}$) the assertions concerning the kernel and the range of Hadamard operators obtained in the preceding chapter.

Remark and Definition 5.1 :
Let $U, G \subset \mathbb{C}$ be open and $\Phi \in \operatorname{Exp}(\{0\})$ with $\Phi(z) = \sum_{\nu=0}^{\infty} \Phi_\nu z^\nu$ $(z \in \mathbb{C})$.

1. $\vartheta := \vartheta_U : H(U) \to H(U),\ (\vartheta f)\ (z) := z \cdot f'(z)\ (z \in U)$ and $D := D_G : H(G) \to H(G),\ (Df)(w) := f'(w)\ (w \in G)$ define linear and continuous operators.

   The iterates are as usual defined by

$$\begin{aligned} \vartheta^0 &:= \mathrm{id}_{H(U)}, \quad \vartheta^\nu := \vartheta \circ \vartheta^{\nu-1} \qquad (\nu \geq 2)\,, \\ D^0 &:= \mathrm{id}_{H(G)}, \quad D^\nu := D \circ D^{\nu-1} \qquad (\nu \geq 2)\,. \end{aligned}$$

2. In [Hi, Th. 11.2.3] it is shown that the operator

$$\Phi(\vartheta) := \Phi_U(\vartheta) : H(U) \to H(U),\ \Phi(\vartheta)f := \sum_{\nu=0}^{\infty} \Phi_\nu \vartheta^\nu f$$

   is well defined, linear and continuous.[1]

3. In [BG, Prop. 6.4.2] it is shown that the operator

$$\Phi(D) := \Phi_G(D) : H(G) \to H(G),\ \Phi(D)f := \sum_{\nu=0}^{\infty} \Phi_\nu D^\nu f$$

   is well defined, linear and continuous.

   If $V \subset W \subset \mathbb{C}$ are open sets, it is clear that for $f \in H(W)$ the following is valid:

$$\Phi_V(D)(f|_V) = (\Phi_W(D)f)|_V\,. \tag{5.1}$$

Let $M \subset \mathbb{C}$ be compact and convex and let $W \subset \mathbb{C}$ be open and convex.

4. An element $\mathcal{T} \in H'(M)$ is called *analytic functional carried by M*. We call the smallest compact and convex set L such that $\mathcal{T} \in H'(L)$ (which exists according to [BG, Ch. 1.3]) the *convex support* of $\mathcal{T}$. There exists a bijective map $\mathfrak{F} : H'(M) \to \mathrm{Exp}(M)$ where for a given $\mathcal{T} \in H'(M)$ the entire function $\mathfrak{F}\mathcal{T}$ is given by

$$\mathfrak{F}\mathcal{T}(z) := \mathcal{T}\big([(\exp(z\cdot), \mathbb{C})]_M\big) \quad (z \in \mathbb{C})\,.$$ [2]

[1] As Pohlen already remarks in [Po], the result in [Hi] is only shown for domains instead of open sets. But a closer look into the proof reveals that only local arguments are used so that one can generalize the theorem for open sets.

[2] See [BG, Ch. 1.3], [Mori, Th. 2.5.2].

5. For $\mathcal{T} \in H'(M)$ we consider the operator $\mathcal{T}\circledast = \mathcal{T}\circledast_W$ defined by

$$\mathcal{T}\circledast : H(W+M) \to H(W), \quad (\mathcal{T} \circledast f)(z) = \mathcal{T}\big([(f(z+\cdot), V_z)]_M\big) \quad (z \in W)$$

with $V_z = M + U_{\delta(z)}(0)$ where $\delta(z) > 0$ is chosen such that $z + U_{\delta(z)}(0) \subset W$. In [BG, Ch. 1.5] it is shown that the operator $\mathcal{T}\circledast_W$ is well defined, linear and continuous.

If, in particular, $M = \{0\}$, then the operators $\mathcal{T}\circledast : H(W) \to H(W)$ and $(\mathfrak{F}\mathcal{T})(D) : H(W) \to H(W)$ coincide (see [BG, p. 90]).

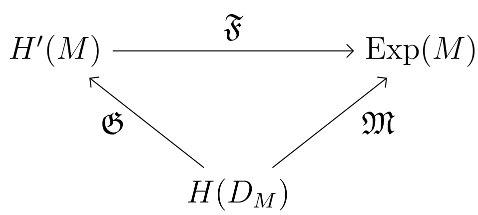
Hence, the differential operators introduced in 3. can be considered as a special case of these convolution operators.

6. If $M \subset \mathbb{V}$ is compact and convex, it is shown in [BG, Ch. 4.1] that there exists a bijective map $\mathfrak{G} : H(D_M) \to H'(M)$ and for a given $\varphi \in H(D_M)$ the analytic functional $\mathcal{T}_\varphi := \mathfrak{G}\varphi \in H'(M)$ is given by

$$\mathcal{T}_\varphi\big([(h,V)]_M\big) = -\frac{1}{2\pi i}\int_\Gamma \varphi(\zeta) h(-\log\zeta)\,\frac{d\zeta}{\zeta} \quad ([(h,V)]_M \in H(M))$$

where without loss of generality V is required to be a subset of $\mathbb{V}$ and Γ is a Cauchy cycle for e^{-M} in e^{-V}.

Furthermore, it is shown that the following diagram commutes:



As a consequence, in case $M = \{0\}$ (and if we denote $\mathfrak{M}\varphi$ by Φ) the operators $\mathcal{T}_\varphi\circledast : H(W) \to H(W)$ and $\Phi(D) : H(W) \to H(W)$ coincide.

Remark 5.2 :
There is a close relation between the Hadamard product and the convolution operators introduced above:

Let $M \subset \mathbb{V}$ be compact and convex and let $W \subset \mathbb{C}$ be open and convex. If $\varphi \in H(D_M)$ we obtain for all $f \in H(e^{W+M})$

$$(T_{\varphi, e^{W+M}} f) \circ \exp|_W = \mathcal{T}_\varphi \circledast_W (f \circ \exp|_{W+M}) \,. \tag{5.2}$$

In order to prove (5.2), let $z \in W$. Since $D_M * e^{W+M} \supset e^W$ we obtain that $e^z \cdot (e^{W+M})^*$ is a compact subset of D_M (see Proposition 2.5.5). Hence, there exists a number $\delta_1(z) > 0$ such that $e^{-(M+U_{\delta_1(z)}(0))} \cap e^z \cdot (e^{W+M})^* = \emptyset$. Since e^{-M} is a compact subset of $\mathbb{S}$ there exists a number $\delta_2 > 0$ such that $e^{-(M+U_{\delta_2}(0))} \subset \mathbb{S}$. Since W is open, there exists a number $\delta_3(z) > 0$ such that $z + U_{\delta_3(z)}(0) \subset W$. We set $\delta(z) := \min\{\delta_1(z), \delta_2, \delta_3(z)\}$, $V_z := M + U_{\delta(z)}(0)$ and choose Γ_z to be a Cauchy cycle for e^{-M} in e^{-V_z}.

The choice of $\delta(z)$ ensures that Γ^- is an anti-Cauchy cycle for $e^z \cdot (e^{W+M})^*$ in D_M and we obtain

$$\begin{aligned}
(T_{\varphi,e^{W+M}} f) \circ \exp|_W(z) &= (f *_{e^{W+M},D_M} \varphi)(e^z) \\
&= \frac{1}{2\pi i} \int_{\Gamma^-} \varphi(\zeta) f(\frac{e^z}{\zeta}) \frac{d\zeta}{\zeta} \\
&= -\frac{1}{2\pi i} \int_{\Gamma} \varphi(\zeta) f \circ \exp(z - \log\zeta) \frac{d\zeta}{\zeta} \\
&= \mathcal{T}_\varphi\big([(f \circ \exp|_{W+M}(z + \cdot), V_z)]_M\big) \\
&= \mathcal{T}_\varphi \circledast_W (f \circ \exp|_{W+M})(z)\,.
\end{aligned}$$

This means that the following diagram commutes:

$$\begin{array}{ccc}
H(W+M) & \xrightarrow{\mathcal{T}_\varphi \circledast_W} & H(W) \\
\uparrow {\scriptstyle \circ \exp|_{W+M}} & & \uparrow {\scriptstyle \circ \exp|_W} \\
H(e^{W+M}) & \xrightarrow[T_{\varphi,e^{W+M}}]{} & H(D_M * e^{W+M})
\end{array}$$

where Proposition 2.5 and Remark 4.33 show that in general, $D_M * e^{W+M}$ is a proper superset of e^W.

If $W \subset \mathbb{V}$ and $M + W \subset \mathbb{V}$, then Proposition 4.32 yields $D_M * e^{W+M} = e^W$ and in this case we have $\log_{e^{W+M}} \circ \exp|_{W+M} = \mathrm{id}_{W+M}$ and $\log_{e^W} \circ \exp|_W = \mathrm{id}_W$ which implies that the following diagram commutes:

$$\begin{array}{ccc}
H(W+M) & \xrightarrow{\mathcal{T}_\varphi \circledast_W} & H(W) \\
{\scriptstyle \circ \exp|_{W+M}} \updownarrow {\scriptstyle \circ \log_{e^{W+M}}} & & {\scriptstyle \circ \exp|_W} \updownarrow {\scriptstyle \circ \log_{e^W}} \\
H(e^{W+M}) & \xrightarrow[T_{\varphi,e^{W+M}}]{} & H(e^W)
\end{array}$$

Remark 5.3 :
Let $W \subset \mathbb{C}$ be open and convex, $\varphi \in H(\mathbb{C}_\infty \setminus \{1\})$ and $\Phi := \mathfrak{M}\varphi$. Remark and Definition 5.1 yields that Equation (5.2) reads

$$\Phi_W(D)(f \circ \exp|_W) = (T_{\varphi,e^W} f) \circ \exp|_W \quad (f \in H(e^W))\,. \tag{5.3}$$

Considering a result of Müller and Pohlen stating that for an arbitrary open set $U \subset \mathbb{C}$, the operators $T_\varphi : H(U) \to H(U)$ and $\Phi(\vartheta) : H(U) \to H(U)$ coincide (see [MP, Th. 2.12]) and taking into account that $\Phi_G(D)(f \circ \exp|_G) = (\Phi(\vartheta)f) \circ \exp|_G$ for an arbitrary open set $G \subset \mathbb{C}$, we obtain that Equation (5.3) also holds for non-convex open sets $G \subset \mathbb{C}$.

Hence, if $G \subset \mathbb{C}$ is open and $U := e^G$, the following diagram commutes:

$$\begin{array}{ccc} H(G) & \xrightarrow{\Phi_G(D)} & H(G) \\ {\scriptstyle \circ\exp|_G}\uparrow & & \uparrow{\scriptstyle \circ\exp|_G} \\ H(U) & \xrightarrow[T_{\varphi,U}]{} & H(U) \end{array}$$

If

- $G \subset \mathbb{V}$ and $U := e^G$ $(\subset \mathbb{S})$ (and $\log_U$ is induced by the principal branch of the logarithm on $\mathbb{S}$) or if
- $U \subset \mathbb{C} \setminus \{0\}$ is a simply connected domain and $G := \log_U U$,

then we have $\log_U \circ \exp|_G = \mathrm{id}_G$ and (5.3) is equivalent to

$$(\Phi_G(D)h) \circ \log_U = T_{\varphi,U}(h \circ \log_U) \quad (h \in H(G)) \tag{5.4}$$

which means that the following diagram commutes:

$$\begin{array}{ccc} H(G) & \xrightarrow{\Phi_G(D)} & H(G) \\ {\scriptstyle \circ\exp|_G}\updownarrow{\scriptstyle \circ\log_U} & & {\scriptstyle \circ\exp|_G}\updownarrow{\scriptstyle \circ\log_U} \\ H(U) & \xrightarrow[T_{\varphi,U}]{} & H(U) \end{array}$$

Remarks 5.2 and 5.3 show that the Hadamard product is closely linked to operators of the kind $\Phi(D)$ or $\mathcal{T}\circledast$. As far as the latter operators are concerned, several surjectivity results are known. In Remark 5.10 we are going to further elaborate on that. For the time being, we cite the following two results which can for example be found in [BG, Prop. 1.5.12, Th. 6.4.4].

Remark 5.4 :
Let $W \subset \mathbb{C}$ be open and convex.

1. Let $M \subset \mathbb{C}$ be compact and convex and let $0 \not\equiv \mathcal{T} \in H'(M)$ have convex support M. If $\mathfrak{F}\mathcal{T}$ has completely regular growth, then the following are valid for the operator $\mathcal{T}\circledast : H(W+M) \to H(W)$:
   (a) $N(\mathcal{T}\circledast) = \text{clspan}\{h_{k,\alpha,W+M} :\ \alpha\ m\text{−fold zero of } \mathfrak{F}\mathcal{T},\ k \leq m-1\}$.
   (b) $\mathcal{T}\circledast$ is surjective.
2. If $0 \not\equiv \Phi \in \text{Exp}(\{0\})$, then the following are valid for the operator $\Phi(D) : H(W) \to H(W)$:
   (a) $N(\Phi(D)) = \text{clspan}\{h_{k,\alpha,W} :\ \alpha\ m\text{−fold zero of } \Phi,\ k \leq m-1\}$.
   (b) $\Phi(D)$ is surjective.[3]

Remark 5.5 :
Remark 5.2 shows that there is a close interplay between surjectivity results for Hadamard operators and for the convolution operators $\mathcal{T}\circledast$. In this context, the surjectivity criterion for Hadamard operators formulated in Theorem 4.34 can be obtained as a consequence of the result for the corresponding operator $\mathcal{T}_\varphi\circledast$ formulated in Remark 5.4.1. However, an unproblematic mutual transfer of the results is only possible if $M, W, W+M \subset \mathbb{V}$ (see Remark 5.2, Proposition 4.32 and Remark 4.33): If, for example, Φ is an entire function of completely regular growth with $K(\Phi) =: M = K_{1/2}$, $\varphi = \mathfrak{M}^{-1}\Phi$, $\mathcal{T}_\varphi = \mathfrak{F}^{-1}\Phi$ and

$$\begin{aligned} W_1 &= (0,1) \times i\pi(-1/4, 1/4)\,, \\ W_2 &= (0,1) \times i\pi(-3/4, 3/4)\,, \end{aligned}$$

then $\mathcal{T}_\varphi\circledast_{W_1}$ and $\mathcal{T}_\varphi\circledast_{W_2}$ are surjective (the sets $W_1 + M$ and $W_2 + M$ do not differ essentially). However, a direct transfer to the corresponding Hadamard operators is not possible. This is because the sets e^{W_1+M} (which is simply connected) and e^{W_2+M} (which is not simply connected) do differ essentially. This shows that a direct transfer of results requires caution and may require some additional techniques.

However, it is also possible to prove the results formulated in Remark 5.4 in their full generality using Theorem 4.34. The required methods are of rather technical nature and can in full be illustrated in the special case $M = \{0\}$. This is why we give an alternative proof of the result formulated in Remark 5.4.2.

[3] These assertions are a direct consequence of 1. ($M = \{0\}$) paying attention to Remark and Definition 5.1 and observing that functions belonging to $\text{Exp}(\{0\})$ are of completely regular growth (see Example B.19.1).

For the sake of abbreviation we set for a non-vanishing entire function of exponential type Φ

$$\begin{aligned} H_{\Phi,G} &:= \operatorname{span}\{h_{k,\alpha,G} :\ \alpha\ m\text{−fold zero of } \Phi,\ k \leq m-1\}\,, \\ Q_{\Phi,U} &:= \operatorname{span}\{q_{k,\alpha,U} :\ \alpha\ m\text{−fold zero of } \Phi,\ k \leq m-1\}\,. \end{aligned}$$

where $G \subset \mathbb{C}$ is open, $U \subset \mathbb{C}\setminus\{0\}$ is a simply connected domain and the functions $q_{k,\alpha,U}$ are induced by an arbitrary branch of the logarithm on U.[4]

Theorem 5.6 :
Let $0 \not\equiv \Phi \in \operatorname{Exp}(\{0\})$ *and let* $W \subset \mathbb{C}$ *be open and convex. Then*

$$N(\Phi_W(D)) = \overline{H_{\Phi,W}} \text{ and } R(\Phi_W(D)) = H(W)\,.$$

Proof: One easily verifies that for all $k \in \mathbb{N}_0,\ \alpha \in \mathbb{C}$ we have

$$\Phi_W(D)(h_{k,\alpha,W}) = h_{0,\alpha,W} \sum_{l=0}^{k} \binom{k}{l} \mathrm{id}_W^{k-l} \Phi^{(l)}(\alpha)\,.$$

The linearity and continuity of $\Phi_W(D)$ imply that

$$N(\Phi_W(D)) \supset \overline{H_{\Phi,W}}\,.$$

The proof of the opposite inclusion and of the surjectivity is divided into several steps.

1. Let $W \subset \mathbb{V}$ be open and convex and $f \in N(\Phi_W(D))$. We set $\varphi := \mathfrak{M}^{-1}\Phi \in H(\mathbb{C}_\infty \setminus \{1\})$ and $U := e^W (\subset \mathbb{S})$.

   We define $\tilde{f} := f \circ \log|_U$ using the principal branch of the logarithm on $\mathbb{S}$. Then $\tilde{f} \in H(U)$ and Equation (5.4) implies

   $$T_{\varphi,U}\tilde{f} = (\Phi_W(D)f) \circ \log_U = 0$$

   and therefore $\tilde{f} \in N(T_{\varphi,U})$. According to Corollary 4.35 there exists a sequence $(\tilde{f}_n)_{n\in\mathbb{N}}$ in $Q_{\Phi,U}$ converging locally uniformly on U to $\tilde{f}$. Since $q_{k,\alpha,U} \circ \exp|_W = h_{k,\alpha,W}$, by defining $f_n := \tilde{f}_n \circ \exp|_W$ $(n \in \mathbb{N})$ we obtain a sequence in $H_{\Phi,W}$ converging locally uniformly on W to $\tilde{f} \circ \exp|_W = f$.

   Hence, $f \in \overline{H_{\Phi,W}}$.

   The assertion about the surjectivity is a direct consequence of Corollary 4.35 and Remark 5.3.

[4] We already noted earlier that the branch of the logarithm on U which is chosen to induce the functions $q_{k,\alpha,U}$ has no influence on the set $Q_{\Phi,U}$ anyway.

2. Let $W \subset \mathbb{C}$ be open, convex and bounded.

   There exists a number $a \in \mathbb{C} \setminus \{0\}$ such that $aW \subset \mathbb{V}$. The assertion follows by applying 1. to the operator $\Psi_{aW}(D)$ where $\Psi(z) := \Phi(az)$ $(z \in \mathbb{C})$ belongs to the space $\mathrm{Exp}(\{0\})$ (see Corollary B.13.2).

3. Let $W \subset \mathbb{C}$ be open and convex.

   Let $f \in N(\Phi_W(D))$ be given. For $K \subset W$ compact and convex and $0 < \varepsilon < \mathrm{dist}(K, \partial W)$, the set $G_\varepsilon := K + U_\varepsilon(0)$ is an open, convex and bounded subset of W and $f|_{G_\varepsilon} \in N(\Phi_{G_\varepsilon}(D))$ (see (5.1)). According to 2., there exists a sequence $(\tilde{f}_n)_{n\in\mathbb{N}}$ in H_{Φ,G_ε} with $\|\tilde{f}_n - f|_{G_\varepsilon}\|_K \to 0$ $(n \to \infty)$. Since $h_{k,\alpha,W}$ is an analytic continuation of $h_{k,\alpha,G_\varepsilon}$ to the set W, we can consider the corresponding continuation of $\tilde{f}_n$ to W denoting the resulting function by f_n. Hence $(f_n)_{n\in\mathbb{N}}$ is a sequence in $H_{\Phi,W}$ and

   $$\|f_n - f\|_K = \|\tilde{f}_n - f|_{G_\varepsilon}\|_K \to 0 \ (n \to \infty)\,.$$

   This completes the proof of the assertion concerning the kernel of $\Phi_W(D)$.

   By the same argument as above we obtain that if $V \subset W$ is convex, the linear and continuous operator

   $$\rho_W^V : N(\Phi_W(D)) \to N(\Phi_V(D)),\ f \mapsto f|_V$$

   has dense range.

   It remains to show that $\Phi_W(D)$ is surjective. Let $g \in H(W)$ be given.

   We consider a standard exhaustion $(K_n)_{n\in\mathbb{N}}$ of W consisting of convex sets. Then the sets $W_n := K_n^\circ$ $(n \in \mathbb{N})$ are open, non-empty, convex and bounded. We set $g_n := g|_{W_n}$, $\Phi_n(D) := \Phi_{W_n}(D)$ and $X_n := N(\Phi_n(D))$ $(n \in \mathbb{N})$.

   According to 2., for all $n \in \mathbb{N}$ a function $f_n \in H(W_n)$ exists with $\Phi_n(D)f_n = g_n$. Then we obtain with Equation (5.1) and the linearity of $\Phi_n(D)$

   $$\Phi_n(D)(f_{n+1}|_{W_n} - f_n) = (\Phi_{n+1}(D)f_{n+1})|_{W_n} - \Phi_n(D)f_n = g_{n+1}|_{W_n} - g_n = 0$$

   and hence $f_{n+1}|_{W_n} - f_n \in X_n$.

   We set $x_1 := 0$. Since for all $n \in \mathbb{N}$ the map $\rho_{n+1}^n : X_{n+1} \to X_n,\ \mu \mapsto \mu|_{W_n}$ has dense range, we can inductively choose functions $x_{n+1} \in X_{n+1}$ with

   $$\|f_{n+1}|_{W_n} - f_n + x_n - x_{n+1}|_{W_n}\|_{K_{n-1}} < \frac{1}{2^n}\,.$$

   We set $h_n := (f_{n+1} - x_{n+1})|_{W_n} - (f_n - x_n)$ and observe that $h_n \in X_n$.

Let now $\nu \in \mathbb{N}$ be fixed. The series $\sum\limits_{l=\nu+1}^{\infty} h_l|_{W_\nu}$ converges locally uniformly on W_ν. Indeed, if $M \subset W_\nu$ is compact, then

$$\sum_{l=\nu+1}^{\infty} \|h_l|_{W_\nu}\|_M \leq \sum_{l=\nu+1}^{\infty} \|h_l\|_{K_\nu} < \sum_{l=\nu+1}^{\infty} \frac{1}{2^l} .$$

According to the Weierstrass comparison test, the series $\sum\limits_{l=\nu+1}^{\infty} h_l|_{W_\nu}$ converges uniformly on M.
Furthermore, we obtain for $n \geq \nu + 1$

$$(f_{n+1} - x_{n+1})|_{W_\nu} = \Big(\sum_{l=\nu+1}^{n} h_l|_{W_\nu} \Big) + (f_{\nu+1} - x_{\nu+1})|_{W_\nu} .$$

Hence, for all $\nu \in \mathbb{N}$ the following limit exists

$$r_\nu := \lim_{n\to\infty} (f_n - x_n)|_{W_\nu} \in H(W_\nu) .$$

In the remaining part of the proof we are going to show that the function $f : W \to \mathbb{C},\ f(z) := r_\nu(z)$ in case $z \in W_\nu$ is well defined, holomorphic in W and satisfies $\Phi_W(D)f = g$.

(a) f is well defined because we obtain with the continuity of the restriction map for all $\nu \in \mathbb{N}$

$$r_{\nu+1}|_{W_\nu} - r_\nu = \big(\lim_{n\to\infty} (f_n - x_n)|_{W_{\nu+1}}\big)|_{W_\nu} - \lim_{n\to\infty} (f_n - x_n)|_{W_\nu} = 0 .$$

(b) f is holomorphic in W since r_ν is holomorphic in W_ν for all $\nu \in \mathbb{N}$.

(c) We obtain for all $z \in W$ and $\nu \in \mathbb{N}$ such that $z \in W_\nu$ with equation (5.1) and the continuity of $\Phi_\nu(D)$ observing that $x_n \in X_n$ $(n \in \mathbb{N})$

$$\begin{aligned}
(\Phi_W(D)f)(z) &= (\Phi_\nu(D)r_\nu)(z) \\
&= \Big(\lim_{n\to\infty} \big(\Phi_\nu(D)(f_n|_{W_\nu})\big)\Big)(z) \\
&= \Big(\lim_{n\to\infty} \big((\Phi_n(D)f_n)|_{W\nu}\big)\Big)(z) \\
&= \big(\lim_{n\to\infty} (g_n|_{W_\nu})\big)(z) \\
&= g_\nu(z) = g(z) .
\end{aligned}$$

□

Remark 5.7 :
In the third part of the proof of Theorem 5.6 we have implicitly shown via the Mittag-Leffler procedure (see for example [Bour, Ch. 2 §3]) that the first right-derivative of the projective limit functor in the sense of Palamodov vanishes for the projective spectrum (X_n, ρ^n_{n+1}). Showing this is enough to make surjectivity on the steps sufficient for surjectivity on the limits (see for example [Weng, Prop. 3.1.8]). However, in order to remain self-contained, we did not use this abstract theory and carried out this procedure using only arguments from function theory.

Corollary 5.8 :
Let $0 \not\equiv \varphi \in H(\mathbb{C}_\infty \setminus \{1\})$, $\Phi = \mathfrak{M}\varphi$ and let $U \subset \mathbb{C} \setminus \{0\}$ be a simply connected domain with $\log_U U$ convex. Then the following are valid for the operator $T_\varphi : H(U) \to H(U)$:

$$N(T_{\varphi,U}) = \overline{Q_{\Phi,U}} \quad \text{and} \quad R(T_{\varphi,U}) = H(U)\,.$$

Proof: Theorem 4.4 states that $N(T_{\varphi,U}) \supset \overline{Q_{\Phi,U}}$.

In order to prove the opposite inclusion we set $W := \log_U U$ and take a function $f \in N(T_{\varphi,U})$.

Due to Equation (5.3), the function $\tilde{f} := f \circ \exp|_W$ belongs to $N(\Phi_W(D))$. Since W is convex, Theorem 5.6 implies the existence of a sequence $\tilde{f}_n$ in $H_{\Phi,W}$ converging locally uniformly on W to $\tilde{f}$. Since $q_{k,\alpha,U} = h_{k,\alpha,W} \circ \log_U$, by defining $f_n := \tilde{f}_n \circ \log_U$ $(n \in \mathbb{N})$ we obtain a sequence in $Q_{\Phi,U}$ converging locally uniformly on U to the function $\tilde{f} \circ \log_U = f$. Hence, $f \in \overline{Q_{\Phi,U}}$.

The second assertion is a direct consequence of Theorem 5.6 and Remark 5.3. □

Example 5.9 :
Let $U \subset \mathbb{C} \setminus \{0\}$ be a simply connected domain with $\log_U U$ convex and let $\Lambda \subset \mathbb{N}$ with $\sum_{\lambda \in \Lambda} 1/\lambda < \infty$. Then [Boa, L. 2.10.13] implies that the function

$$\Phi(z) = \prod_{\lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right) \quad (z \in \mathbb{C})$$

belongs to the space $\operatorname{Exp}(\{0\})$. We consider the operator $T_\varphi : H(U) \to H(U)$ (where $\varphi := \mathfrak{M}^{-1}\Phi$). Corollary 5.8 together with the fact that Φ has only simple zeros at the points in Λ and no other zeros implies that the kernel of T_φ is given by

$$N(T_\varphi) = \operatorname{clspan}\{p_{\lambda,U} : \lambda \in \Lambda\}\,.$$

Hence, the kernel of T_φ consists of all functions $f \in H(U)$ which can be locally uniformly approximated by polynomials having powers only in the set Λ.

Using a result of Müller about convergence properties of lacunary polynomials (see [Mue90, S. 2.4.4]) we obtain

$$N(T_\varphi) = \{f|_U\colon\ f \in H_\Lambda(\mathbb{D}_{\sup\limits_{z\in U}|z|})\}\,.$$

As already mentioned earlier, there are several results known concerning the surjectivity of operators of the type $\Phi(D)$ or $\mathcal{T}\circledast$. Before we come to the end of this chapter, we summarize some of these results in the next Remark.

Remark 5.10 :
In [Kor69], Korobeĭnik gave a proof of the result already formulated in Remark 5.4.2. Furthermore the following are valid (where $W \subset \mathbb{C}$ is open and $\Phi \in \mathrm{Exp}(\{0\})$ is non-constant):

- If W^C is connected, then $R(\Phi_W(D)) = H(W)$ if Φ is a polynomial.
- If W^C is not connected, then $R(\Phi_W(D)) \subsetneq H(W)$ for all $\Phi \in \mathrm{Exp}(\{0\})$, to be more precise, there exists a function $f \in H(W)$ such that $f \notin R(\Phi_W(D))$ for all $\Phi \in \mathrm{Exp}(\{0\})$ (see [Kor69, Th. 1.3]).
- If W is not convex, then there exists a function $\Phi \in \mathrm{Exp}(\{0\})$ such that $R(\Phi_W(D)) \subsetneq H(W)$ (see [Kor69, Th. 4.1]). However, in general we do not have $R(\Phi_W(D)) \subsetneq H(W)$ for all $\Phi \in \mathrm{Exp}(\{0\})$ (for further references see [Kor69, p. 64]).
- In Chapter 5 and 6 of his paper, Korobeĭnik gives necessary and sufficient conditions for a function $f \in H(W)$ to belong to the range of $\Phi_W(D)$ or for $\Phi_W(D)$ to be surjective for certain non-convex domains W.

With an additional result due to Znamenskiĭ (see [Zna]) one can give a characterization of the surjectivity of $\Phi_W(D)$ for non-convex domains W depending on the zeros of Φ and the shape of the set W. In addition, there are also several necessary and sufficient conditions for the existence of a continuous linear right inverse of the operator $\Phi_W(D)$ (see [Schwe] and [Tay] for the case $W = \mathbb{C}$, [Mo90] for the case of open discs and [Mo92] for arbitrary convex domains $W \subsetneq \mathbb{C}$).

Korobeĭnik gave also a proof of the result already formulated in Remark 5.4.1 (see [Kor68] and [Kor1969]) : Let $M \subset \mathbb{C}$ be compact and convex and let $0 \not\equiv \mathcal{T} \in$

$H'(M)$ with convex support M. Then the operator $\mathcal{T}\circledast_W : H(W+M) \to H(W)$ is surjective for all convex sets $W \subset \mathbb{C}$ if $\mathfrak{F}\mathcal{T}$ is of completely regular growth. If $\mathfrak{F}\mathcal{T}$ is not of completely regular growth but W is convex, Epifanov formulated a characterization for the surjectivity of $\mathcal{T}\circledast_W$ depending on an interplay between the rays on which $\mathfrak{F}\mathcal{T}$ does have completely regular growth and the geometry of the set W (see [Ep74] and [Ep1974]; for a definition of completely regular growth on single rays we refer to [Le62, p. 137 ff.]). Taking these results together one can note that $\mathcal{T}\circledast_W$ is surjective *for all* convex domains W if and only if $\mathfrak{F}\mathcal{T}$ is of completely regular growth (in the whole complex plane). If W is neither convex nor $\mathfrak{F}\mathcal{T}$ of completely regular growth, Epifanov gave a characterization of the surjectivity of a closely related operator in [Ep82, p. 355]. Also for the operator $\mathcal{T}\circledast_W$ there are several results known concerning the existence of a continuous linear right inverse (see [KM, Ch. 5] for bounded convex sets W and [MM, Ch. 4]).

Both operators $\Phi(D)$ and $\mathcal{T}\circledast$ have been studied as operators between spaces of analytic functions on subsets of $\mathbb{C}^d$ (for some $d \in \mathbb{N}$, $d > 1$) (see [Mar] for a surjectivity result of $\Phi(D)$ and [MM] for the existence of a continuous linear right inverse; see [Morz], [Kri] and [MM] for the operator $\mathcal{T}\circledast$).

Chapter 6

Lacunary approximation by means of the Hadamard product

In this chapter we are going to apply the concept of the Hadamard product to the approximation of holomorphic functions by polynomials, especially by polynomials with gaps. While doing so we are in particular going to benefit from the results concerning the range of the operator T_φ obtained in Section 4.2.

In Section 6.1 we consider the first problem which can be tackled using the Hadamard product: Under which conditions on the set $\Lambda \subset \mathbb{N}_0$ and the open set $D \subset \mathbb{C}$ is it possible to approximate every function which is holomorphic in D by polynomials whose exponents belong to the set Λ?

Section 6.2 is concerned with the second main question which can be examined by means of the Hadamard product: Given a compact set $K \subset D$, how "fast" can a function which is holomorphic in D be approximated on K by those polynomials?

6.1 Locally uniform approximation by lacunary polynomials: Runge type results

Definition 6.1 :
Let $\Lambda \subset \mathbb{N}_0$ and $n \in \mathbb{N}_0$. We call a function

$$p \in P_{\Lambda,n} := \operatorname{span}\{p_{\nu,\mathbb{C}} : \ \nu \in \Lambda, \ \nu \leq n\}$$

a *lacunary polynomial* of degree less or equal to n with respect to Λ. We denote the set of all lacunary polynomials with respect to Λ by

$$P_\Lambda := \bigcup_{n\in\mathbb{N}_0} P_{\Lambda,n} \, .$$

In the special case $\Lambda = \mathbb{N}_0$, we denote by $P_n := P_{\mathbb{N}_0,n}$ the space of all polynomials of degree less or equal to n.

Finally, for an open set $D \subset \mathbb{C}$ we set

$$P_\Lambda(D) := \operatorname{clspan}\{p_{\nu,D} : \ \nu \in \Lambda\} \, .$$

The first problem that shall be considered here reads as follows: For a given set $\Lambda \subset \mathbb{N}_0$ and an open set $D \subset \mathbb{C}$ we are interested in the question whether a given function $g \in H(D)$ can be locally uniformly approximated on D by polynomials whose powers belong to Λ, i. e. by polynomials in P_Λ. In particular, if we are given a set $\Lambda \subset \mathbb{N}_0$, we want to know under which conditions on the open set $D \subset \mathbb{C}$ we are able to approximate *every* function $g \in H(D)$ by lacunary polynomials with respect to Λ, i. e. under which conditions on D we have $P_\Lambda(D) = H(D)$. In the literature we can find several results of this kind, for example in [An], [AM], [LMM98], [LMM02], [MR] and [GLM]. They all have in common that their proofs rely on an application of the theorem of Hahn-Banach. This is an elegant method of proof but is intrinsincly non-constructive. We are going to present a new approach to prove results of this kind which relies on a suitable application of the Hadamard product. However, in the following remark we list two obvious necessary conditions.

Remark 6.2 :
Let $\Lambda \subset \mathbb{N}_0$ and $D \subset \mathbb{C}$ be open.

1. The maximum principle implies that it is necessary for $P_\Lambda(D)$ to equal $H(D)$ that D^C is connected. If $\Lambda = \mathbb{N}_0$, then Runge's approximation theorem yields that this is also sufficient.

2. If $\Lambda \subsetneq \mathbb{N}_0$ it is necessary for $P_\Lambda(D)$ to equal $H(D)$ that $0 \notin D$: Indeed, if $0 \in D$ then the locally uniform convergence of all derivatives at 0 implies that $P_\Lambda(D) \subset H_\Lambda(D) \subsetneq H(D)$. Under certain additional conditions on D (with $0 \in D$) and Λ, it can be shown that $P_\Lambda(D) = H_\Lambda(D)$ (see for example [DK]).

If the origin does not belong to the set D, we can hope that $P_\Lambda(D)$ equals $H(D)$ also for sets $\Lambda \subsetneq \mathbb{N}_0$. In the following remark we repeat some properties of the Hadamard operator T_φ which will be crucial to prove results of this kind.

Remark 6.3 :
Let $\Omega \subset \mathbb{C}_\infty$ be a domain with $\{0,\infty\} \subset \Omega$, $\varphi \in H(\Omega)$ and let $U \subset \mathbb{C}$ be open and star-eligible to Ω.

1. Theorem 3.1 implies that the operator $T_\varphi : H(U) \to H(\Omega * U),\ f \mapsto \varphi * f$ is linear and continuous. Therefore, if there is a sequence of polynomials $(p_n)_{n\in\mathbb{N}_0}$ converging to f in $H(U)$, then $(\varphi * p_n)_{n\in\mathbb{N}_0}$ converges to $\varphi * f$ in $H(\Omega * U)$.

2. Example 2.17 implies that for $p \in P_n$ $(n \in \mathbb{N}_0)$ we have

$$\varphi * p \in P_{\Lambda_\varphi^+, n} .$$

   Thus, the operator T_φ, i. e. the Hadamard multiplication by φ, converts arbitrary polynomials into lacunary polynomials with respect to Λ_φ^+ of no higher degree. Therefore, if $\Lambda_\varphi^+ \subsetneq \mathbb{N}_0$, the function $\varphi \in H(\Omega)$ can be called *gap producing function.*

These observations together with Runge's theorem show that in case that U^C is connected, every function $g \in R(T_\varphi)$ can be locally uniformly approximated on $\Omega * U$ by lacunary polynomials with respect to Λ_φ^+, i. e. $T_\varphi(H(U)) \subset P_{\Lambda_\varphi^+}(\Omega * U)$.

The following proposition states the connection between properties of a certain Hadamard operator $T_{\varphi,U}$ and the possible approximation of every function $g \in H(\Omega * U)$ by lacunary polynomials.

Proposition 6.4 :
Let $\Lambda \subset \mathbb{N}_0$ and let $\Omega \subset \mathbb{C}_\infty$ be a domain with $\{0,\infty\} \subset \Omega$ and $\varphi \in H(\Omega)$ with $\Lambda_\varphi^+ \subset \Lambda$. Let furthermore $U \subset \mathbb{C}$ be open and star-eligible to Ω with connected complement. If $T_\varphi : H(U) \to H(\Omega*U)$ has dense range, then $P_\Lambda(\Omega*U) = H(\Omega*U)$.

Proof: If $T_\varphi : H(U) \to H(\Omega * U)$ has dense range, we obtain with Remark 6.3

$$H(\Omega * U) = \overline{T_\varphi(H(U))} \subset P_{\Lambda_\varphi^+}(\Omega * U) \subset P_\Lambda(\Omega * U) \subset H(\Omega * U) .$$

□

Theorem 6.5 :
Let $\Lambda \subset \mathbb{N}_0$ have unit density. Then $P_\Lambda(D) = H(D)$ for all open sets $D \subset \mathbb{C} \setminus \{0\}$ with connected complement.

Proof: We write $\mathbb{N}_0 \setminus (\Lambda \cup \{0\}) =: \{\mu_n : \ n \in \mathbb{N}\}$. Example B.19.2 yields that

$$\Phi(z) := z \cdot \prod_{n=1}^{\infty} \big(1 - \frac{z^2}{\mu_n^2}\big) \quad (z \in \mathbb{C})$$

is an entire function of exponential type with $K(\Phi) = \{0\}$ and $Z(\Phi) = \pm(\mathbb{N}_0 \setminus \Lambda) \cup \{0\}$.

Theorem B.16 implies that the function $\varphi := \mathfrak{M}^{-1}\Phi$ is holomorphic in $\mathbb{C}_\infty \setminus \{1\}$ with $\Lambda_\varphi^+ = \Lambda \setminus \{0\} \subset \Lambda$. Hence, Proposition 4.22 yields that the operator $T_\varphi : H(D) \to H(D)$ has dense range so that the assertion follows from Proposition 6.4.

□

Remark 6.6 :

1. Remark 6.2 shows that Theorem 6.5 is false for every set $D \subset \mathbb{C}$ which contains the origin (if $\Lambda \subsetneq \mathbb{N}_0$) or which does not have connected complement.
2. The assertion of Theorem 6.5 already follows from a result of Arakelian and Martirosian (see [AM]) on uniform approximation on compact sets which is proved using the classical approach via the Riesz Representation Theorem and the theorem of Hahn-Banach. Based on a result of Pólya (which we used in the proof of Proposition 3.4), in [LMM02] it is shown that the same is true even with $\bar{d}(\Lambda) = 1$ instead of $d(\Lambda) = 1$.

Theorem 6.7 :
Let $\Lambda \subset \mathbb{N}_0$. Then $P_\Lambda(D) = H(D)$ for all simply connected domains $D \subset \mathbb{C} \setminus \{0\}$ which are contained in a sector of opening $\alpha \le 1 - d_L(\mathbb{N}_0 \setminus \Lambda)$.

Proof: Let $\alpha \le 1 - d_L(\mathbb{N}_0 \setminus \Lambda)$.

1. Let $D = S_\alpha$.

   Since the sector S_α is exhausted by the sectors $S_{\alpha'}$ $(\alpha' < \alpha)$ and since we are dealing with locally uniform convergence, we can assume that $d_L(\mathbb{N}_0 \setminus \Lambda) < 1 - \alpha$, without loss of generality. According to Remark 3.6.2 there exist numbers $a \leq b$ and a non-vanishing function $\varphi \in H(D_K)$ with $\Lambda_\varphi^+ = \Lambda$ (where $K = [a,b] \times i\pi[\alpha-1, 1-\alpha]$). Example 2.3.4 implies that $D_K * \mathbb{S} = S_\alpha$ and Theorem 4.9 yields that the operator $T_\varphi : H(\mathbb{S}) \to H(S_\alpha)$ has dense range so that the assertion follows from Proposition 6.4.

2. Let $D \subset S_\alpha$ be a simply connected domain.

   According to Runge's theorem, every function $g \in H(D)$ can be locally uniformly on D approximated by polynomials. These polynomials are in particular holomorphic in S_α so that they can, according to 1., in turn locally uniformly on S_α and hence on D be approximated by lacunary polynomials with respect to Λ. Hence, g itself can be approximated by lacunary polynomials.

3. In case that $D \subset \mathbb{C} \setminus \{0\}$ is simply connected and contained in a sector of opening α the assertion follows from part 2. by rotation.

□

Remark 6.8 :
Let $\Lambda \subset \mathbb{N}_0$.

1. According to Remark A.3 we have $\underline{d}(\Lambda) = 1 - \overline{d}(\mathbb{N}_0 \setminus \Lambda) \leq 1 - d_L(\mathbb{N}_0 \setminus \Lambda)$ and therefore, Theorem 6.7 especially holds for all $\alpha \leq \underline{d}(\Lambda)$.

2. From [MR, Th. 9.1] follows that $P_\Lambda(D) = H(D)$ for all simply connected domains $D \subset \mathbb{C} \setminus \{0\}$ which are contained in a sector of opening $\alpha \leq d_L(\Lambda)$. While the density condition in Theorem 6.7 is on the set $\mathbb{N}_0 \setminus \Lambda$ of gaps, the condition in [MR] in on the set Λ itself.

Remark 6.9 :

1. Obviously it is desirable to be able to carry out approximation on a set D which is as large as possible. Proposition 6.4 reveals that the inherent shape of the set D using the approach via the Hadamard product is $\Omega * U$. From the definition of the star product it is clear that the larger the sets Ω and U are chosen, the larger the set $\Omega * U$ is. Therefore, the "best case" is that the complement of the set Ω consists only of one point, for example $\Omega = \mathbb{C}_\infty \setminus \{1\}$.

   In the proof of Theorem 6.5 we discovered that for $\Lambda \subset \mathbb{N}_0$ the condition $d(\Lambda) = 1$ is sufficient to obtain a non-vanishing function $\varphi \in H(\mathbb{C}_\infty \setminus \{1\})$ with $\Lambda_\varphi^+ \subset \Lambda$. However, this condition is also necessary: Indeed, if we are given a non-vanishing function $\varphi \in H(\mathbb{C}_\infty \setminus \{1\})$ with $\Lambda_\varphi^+ \subset \Lambda$, then

$Z(\Phi) := \{z \in \mathbb{C} : \Phi(z) = 0\} \supset \mathbb{N}_0 \setminus \Lambda$ (where $\Phi := \mathfrak{M}\varphi$). Hence, if $\underline{d}(\Lambda) < 1$, then

$$\limsup_{r\to\infty} \frac{n_\Phi(r)}{r} \geq \overline{d}(\mathbb{N}_0 \setminus \Lambda) > 0\,.$$

Since $\Phi \in \mathrm{Exp}(\{0\})$, Theorem B.3 together with Remark B.9.4 implies that Φ and hence φ vanish identically which is a contradiction.

2. In general, it is not a loss of generality to consider only sets of the form $D = \Omega * U$, i. e. we can in general not expect to be able to approximate on larger sets as the following example shows:

   For $\Lambda = 2\mathbb{N}_0$, Theorem 6.7 yields especially $P_{2\mathbb{N}_0}(D) = H(D)$ for every half-plane D with $0 \in \partial D$. That means every function which is holomorphic in this half-plane can be locally uniformly approximated by even polynomials. However, we can not expect the set D to exceed a half-plane since then there would be points $z \in D$ with $-z \in D$ and we could obviously not approximate every holomorphic function by even polynomials.

Remark 6.10 :
In the theorems formulated above we were able to specify sets D on which we can approximate every holomorphic function by lacunary polynomials with respect to a given set $\Lambda \subset \mathbb{N}_0$ up to the case $d_L(\mathbb{N}_0 \setminus \Lambda) < 1$. If $d_L(\mathbb{N}_0 \setminus \Lambda) = 1$ (i. e. $\overline{d}(\mathbb{N}_0 \setminus \Lambda) = 1$ and $\underline{d}(\Lambda) = 0$, see Remark A.3.4), Theorem 6.7 can not be applied. However, it is not possible to pursue the approach described in this section to obtain some information for which sets D we have $P_\Lambda(D) = H(D)$: As we have shown in Proposition 3.4 it is not possible to specify a domain $\Omega \subset \mathbb{C}_\infty$ with $\{0, \infty\} \subset \Omega$ and a function $0 \not\equiv \varphi \in H(\Omega)$ with $\underline{d}(\Lambda_\varphi^+) = 0$.

If Λ is measurable and $d(\Lambda) = 0$, then $P_\Lambda(D)$ is indeed a proper subspace of $H(D)$ for every open set $D \subset \mathbb{C}$: If the origin belongs to D, then this is an obvious consequence of Remark 6.2. If the origin does not belong to D we apply a result of Müller about convergence properties of lacunary polynomials which states $P_\Lambda(D) = \{f|_D\colon\ f \in H_\Lambda(\mathbb{D}_{\sup_{z\in D}|z|})\}$ (see [Mue90, S 2.4.4]). Obviously, the function $f(z) = 1/z$ ($z \in D$) is holomorphic in D but does not belong to $P_\Lambda(D)$.

Hence, the only case where the approach described here does not yield a result is when $\overline{d}(\mathbb{N}_0 \setminus \Lambda) = 1$ but $\underline{d}(\mathbb{N}_0 \setminus \Lambda) < 1$.

6.2 The rate of approximation by lacunary polynomials

In particular, the Hadamard product can be useful to obtain results concerning the rate of approximation of a holomorphic function by lacunary polynomials. We want to concretize how this rate shall be measured and introduce the required notation in the following definition.

Remark and Definition 6.11 :
Let $K \subset \mathbb{C}$ be compact, $f \in \mathcal{H}(K),\ \Lambda \subset \mathbb{N}_0$ and $n \in \mathbb{N}_0$.

Then there exists a polynomial $p^*_{\Lambda,n} \in P_{\Lambda,n}$ with

$$\|f - p^*_{\Lambda,n}\|_K = \min_{p \in P_{\Lambda,n}} \|f - p\|_K =: E_n(f, K, \Lambda)\,.$$[1]

In particular, we set $E_n(f,K) := E_n(f,K,\mathbb{N}_0)$.

To measure how fast the function f can be uniformly approximated on K by (lacunary) polynomials we apply the geometric rate ($\limsup_{n\to\infty} E_n(f,K,\Lambda)^{1/n}$ or) $\limsup_{n\to\infty} E_n(f,K)^{1/n}$. In the following, we will call this geometric rate the *rate of approximation.*

Remark and Definition 6.12 :
Let $K \subset \mathbb{C}$ be compact and such that both K and K^C are connected and K contains more than one point. According to the Riemann mapping theorem, there exists a unique conformal mapping (normed at the point at infinity) $\alpha : \mathbb{C}_\infty \setminus K \to \mathbb{C}_\infty \setminus \overline{\mathbb{D}}$ such that we get for all z with sufficiently large modulus

$$\alpha(z) = \frac{1}{c_K} z + c_0 + \sum_{\nu=1}^{\infty} c_\nu z^{-\nu}\,,$$

where $c_K > 0$ is the *logarithmic capacity* of K. The inverse mapping (which is also conformal) shall be denoted by $\beta := \alpha^{-1} : \mathbb{C}_\infty \setminus \overline{\mathbb{D}} \to \mathbb{C}_\infty \setminus K$ and is given by

$$\beta(w) = c_K w + d_0 + \sum_{\nu=1}^{\infty} d_\nu w^{-\nu} \quad (|w| > 1)\,.$$

For $R > 1$ we set $\gamma_R := \beta \circ \tau_R$. Hence $|\gamma_R|$ is a closed Jordan curve and according to the Jordan curve theorem, we obtain

$$\mathbb{C} \setminus |\gamma_R| = \mathrm{Int}(|\gamma_R|) \cup \mathrm{Ext}(|\gamma_R|)$$

[1] See [Schoe, S. 1.1].

where $\mathrm{Int}(|\gamma_R|)$ is a bounded domain with $\mathrm{ind}_{\gamma_R}(z) = 1$ $(z \in \mathrm{Int}(|\gamma_R|))$ and $\mathrm{Ext}(|\gamma_R|)$ is an unbounded domain with $\mathrm{ind}_{\gamma_R}(z) = 0$ $(z \in \mathrm{Ext}(|\gamma_R|))$. The traces $|\gamma_R|$ will be called *level curves* and the sets $\mathrm{Int}(|\gamma_R|)$ will be called *contour domains* (see for example [Gai, p. 64 f.]).

For $f \in \mathcal{H}(K)$ we set

$$R_{f,K} := \sup\{R > 1 :\ \exists\, F \in H(\mathrm{Int}(|\gamma_R|)),\ F|_K = f\} \in (1, \infty]\,.$$

Remark 6.13 :
Let $K \subset \mathbb{C}$ be compact and such that both K and K^C are connected and K contains more than one point and let $f \in \mathcal{H}(K)$.

Using the Bernstein Lemma which enables to estimate the growth of a polynomial on the level curves $|\gamma_R|$ by the growth of the polynomial on the compact set K, one can show that a sufficiently fast convergence of a sequence of polynomials on the set K to f implies the locally uniform convergence of this sequence in the corresponding contour domain (see [Gai, p. 33 f.]). This, in turn, implies that f can be analytically continued into this domain. To be more precise:

For $\Lambda \subset \mathbb{N}_0$ and $R > 1$ let $p_{\Lambda,n} \in P_{\Lambda,n}$ $(n \in \mathbb{N}_0)$ such that

$$\limsup_{n\to\infty} \|f - p_{\Lambda,n}\|_K^{1/n} = 1/R\,.$$

Then $p_{\Lambda,n}$ converges locally uniformly on $\mathrm{Int}(|\gamma_R|)$ to a function $F \in H(\mathrm{Int}(|\gamma_R|))$ with $F|_K = f$.

This result and the definition of $R_{f,K}$ immediately imply

$$\limsup_{n\to\infty} (E_n(f, K))^{1/n} \geq 1/R_{f,K}\,.$$

However, in [Gai, p. 67] it is shown that there exists always a sequence of polynomials which realizes the fastest possible rate of approximation so that we obtain

$$\limsup_{n\to\infty} (E_n(f, K))^{1/n} = \frac{1}{R_{f,K}}\,.$$

That means, the further the function f can be analytically continued over K, the faster it can be approximated by polynomials.

Remark 6.14 :
We want to consider the following question: For $\Lambda \subset \mathbb{N}_0$ and an open set $D \subset \mathbb{C}$, a given function $g \in H(D)$ and a compact subset $M \subset D$, one is interested in

the behaviour of the sequence of errors $E_n(g, M, \Lambda)$. As already mentioned above, while applying the concept of the Hadamard product, the sets D that can be considered are always of the form $\Omega * U$ and we are going to see now that it is natural to consider sets M of the form $\Omega * K$ (where K is a compact subset of U; note that in this case the set $\Omega * K$ is a compact subset of $\Omega * U$). If we know that $P_\Lambda(\Omega * U) = H(\Omega * U)$ we obtain

$$E_n(g, \Omega * K, \Lambda) \to 0 \ (n \to \infty)$$

for all $g \in H(\Omega * U)$. However, we are interested in the speed of this decreasing.

Again, according to the given set $\Lambda \subset \mathbb{N}_0$, we assume to have found a domain $\Omega \subset \mathbb{C}_\infty$ with $\{0, \infty\} \subset \Omega$ and a function $\varphi \in H(\Omega)$ with $\Lambda_\varphi^+ \subset \Lambda$. Let $U \subset \mathbb{C}$ be open and star-eligible to Ω and $K \subset U$ be compact and such that both K and K^C are connected and K contains more than one point. Then there exists a compact set L with $K \subset L^\circ \subset U$ with the same properties. Proposition 2.10.2 ensures the existence of a Hadamard cycle Γ for $(\Omega * K) \cdot \Omega^*$ in L° (note that $\infty \notin \Omega * K$). Then Γ is obviously also a Hadamard cycle for $(\Omega * K) \cdot \Omega^*$ in U and for a function $f \in H(U)$ Definition 2.11 yields

$$(\varphi *_{\Omega,U} f)(z) = \frac{1}{2\pi i} \int_\Gamma f(\zeta) \varphi(\frac{z}{\zeta}) \frac{d\zeta}{\zeta} \quad (z \in \Omega * K)\,.$$

Hence, for all $z \in \Omega * K$ we obtain

$$|(\varphi *_{\Omega,U} f)(z)| \leq \frac{1}{2\pi} \int_\Gamma \frac{|f(\zeta)|}{|\zeta|} |\varphi(\frac{z}{\zeta})| \, |d\zeta| \leq c \cdot \|f\|_{|\Gamma|}$$

where $c := \frac{1}{2\pi} L(\Gamma) \cdot \max_{\zeta \in |\Gamma|} \frac{1}{|\zeta|} \cdot \max_{\omega \in (\Omega * K)/|\Gamma|} |\varphi(\omega)|$ (note that $(\Omega * K)/|\Gamma| := \{a/b : \ a \in \Omega * K, \ b \in |\Gamma|\}$ is a compact subset of Ω). Since Γ is a cycle in L we finally get

$$\|\varphi *_{\Omega,U} f\|_{\Omega * K} \leq c \cdot \|f\|_L \,.$$

If the function $g \in H(\Omega * U)$ belongs to the range of the operator $T_{\varphi,U}$, i. e. there is a function $f \in H(U)$ with $\varphi *_{\Omega,U} f = g$, then for each polynomial $p \in P_n$ we have

$$\|g - \varphi *_{\Omega,U} p|_U\|_{\Omega * K} \leq c \cdot \|f - p|_U\|_L$$

and especially (note that $\varphi *_{\Omega,\mathbb{C}} p \in P_{\Lambda_\varphi^+,n} \subset P_{\Lambda,n}$)

$$\limsup_{n\to\infty} E_n(g, \Omega * K, \Lambda)^{1/n} \leq \limsup_{n\to\infty} E_n(f, L)^{1/n} = \frac{1}{R_{f,L}} \,.$$

Since we can choose L arbitrarily close to K, the definition of $R_{f,L}$ yields that the inequality remains true for K instead of L on the right-hand side and we obtain

$$\limsup_{n\to\infty} E_n(g,\Omega * K,\Lambda)^{1/n} \le \frac{1}{R_{f,K}}\,.$$

This inequality means that we can give an upper bound for the rate of approximation of g by lacunary polynomials by the rate of approximation of a preimage of g under the operator $T_{\varphi,U}$, namely $f \in H(U)$, by arbitrary polynomials. Furthermore, we observe that the domain into which f can be analytically continued is relevant for the rate of approximation of g by lacunary polynomials.

We summarize the situation in the following theorem.

Theorem 6.15 :
Let $\Lambda \subset \mathbb{N}_0$ and $\Omega \subset \mathbb{C}_\infty$ be a domain with $\{0,\infty\} \subset \Omega$ and let $\varphi \in H(\Omega)$ with $\Lambda_\varphi^+ \subset \Lambda$. Let furthermore $U \subset \mathbb{C}$ be open and star-eligible to Ω and let $K \subset U$ be compact and such that both K and K^C are connected and K contains more than one point. Let $g \in H(\Omega * U)$ be given and let $f \in H(U)$ such that $T_{\varphi,U} f = g$. Then we have

$$\limsup_{n\to\infty} E_n(g,\Omega * K,\Lambda)^{1/n} \le \frac{1}{R_{f,K}}\,. \tag{6.1}$$

Remark 6.16 :
In order to obtain inequality (6.1) we assumed the function $g \in H(\Omega * U)$ to belong to the range of a suitable Hadamard operator $T_{\varphi,U}$. Hence, in order to apply the approach described in Remark 6.14 it is desirable to know that the Hadamard operator $T_{\varphi,U}$ is surjective (see Section 4.4 and Corollary 5.8).

If $\Lambda \subset \mathbb{N}_0$ has unit density, then the proof of Theorem 6.5 shows that there exists a function $0 \not\equiv \varphi \in H(\mathbb{C}_\infty \setminus \{1\})$ with $\Lambda_\varphi^+ \subset \Lambda$. If $D \subset \mathbb{C} \setminus \{0\}$ is a simply connected domain with $\log_D D$ convex, then Corollary 5.8 states that $T_{\varphi,D} : H(D) \to H(D)$ is surjective. Hence, in order to exploit inequality (6.1), we have to find for a given function $g \in H(D)$ a function $f \in H(D)$ with $T_{\varphi,D} f = g$ and determine $R_{f,K}$ (for a suitable compact set $K \subset D$). This is what we are going to do in the following example.

Example 6.17 :
Let $\Lambda = \mathbb{N}$, i. e. we require the approximating polynomials to vanish at the origin.

Then obviously the Koebe function κ introduced in Example 3.2 is a suitable gap producing function becasue $\kappa \in H(\mathbb{C}_\infty \setminus \{1\})$ and $\Lambda_\kappa^+ = \mathbb{N}$ (and $\mathfrak{M}\kappa = \mathrm{id}_{\mathbb{C}}$).

In Example 3.2 it has been shown that $T_{\kappa,U}$ is surjective for every simply connected domain $U \subset \mathbb{C} \setminus \{0\}$ (we did not require $\log_U U$ to be convex; see also Proposition 4.30) so that we can hope to get quantitative results for the approximation of a function $g \in H(U)$ by polynomials which vanish at the origin.

Let $U := \mathbb{S}$ and $g \equiv 1$. Then Example 3.2 reveals that $f(z) := \log z$ $(z \in \mathbb{S})$ fulfills $T_{\kappa,\mathbb{S}} f = g$. We want to compute $R_{\log,K}$ for different compact sets $K \subset \mathbb{S}$. In order to do so, let $x_0 \in (0, \infty)$ and $0 < \delta < x_0$.

1. For $K_1 = \overline{U_\delta(x_0)}$ it can easily be seen that the conformal mapping β_1 mentioned in Remark and Definition 6.12 is given by

$$\beta_1(w) = \delta w + x_0 \quad (w \in \mathbb{C} \setminus \overline{\mathbb{D}})\,.$$

   Hence, the conformal mapping β_1 transforms a circle around the origin with radius R into a circle around x_0 with radius δR.

2. For $K_2 := [x_0 - \delta, x_0 + \delta]$ one can (based on the standard case $K = [-1, 1]$ and the *Joukowski mapping*) easily verify that the conformal mapping β_2 is given by

$$\beta_2(w) = \frac{\delta}{2}(w + \frac{1}{w}) + x_0 \quad (w \in \mathbb{C} \setminus \overline{\mathbb{D}})\,.$$

   Hence, the conformal mapping β_2 transforms a circle around the origin with radius R into an ellipse around x_0 with semiaxis $\frac{\delta}{2}(R + \frac{1}{R})$ and $\frac{\delta}{2}(R - \frac{1}{R})$.

By means of the mappings β_1 and β_2 we can calculate $R_{\log,K_1}$ and $R_{\log,K_2}$. Since the logarithm can not be continued into a domain that contains the origin, the numbers $R_{\log,K_1}$ and $R_{\log,K_2}$ obviously coincide with the parameter of the respective level curves $|\gamma_R|$ that touch the origin. A short calculation shows that

$$R_{\log,K_1} = \frac{x_0}{\delta} \quad \text{and} \quad R_{\log,K_2} = \frac{x_0}{\delta} + \sqrt{\frac{x_0^2}{\delta^2} - 1}\,.$$

Hence, inequality (6.1) reads

$$\limsup_{n\to\infty} E_n(1, \overline{U_\delta(x_0)}, \mathbb{N})^{1/n} \le \delta / x_0$$

or

$$\limsup_{n\to\infty} E_n(1, [x_0 - \delta, x_0 + \delta], \mathbb{N})^{1/n} \le \Big(\frac{x_0}{\delta} + \sqrt{\frac{x_0^2}{\delta^2} - 1}\Big)^{-1}$$

respectively.

The question may arise if these upper bounds for the rate of approximation can be improved. However, in general they can not. Indeed, if we assume that

$$\limsup_{n\to\infty} E_n(1, \overline{U_\delta(x_0)}, \mathbb{N})^{1/n} = 1/\tilde{R}$$

with $\tilde{R} > x_0/\delta$, then Remark 6.13 implies that a sequence of best approximating polynomials $(p^*_{\Lambda,n})_{n\in\mathbb{N}_0}$ converges to a function $F \in H(\mathrm{Int}(|\gamma_{\tilde{R}}|))$. By construction, $p^*_{\Lambda,n}$ converges to 1 on K so that $F|_K = 1$ and the identity theorem implies that $F \equiv 1$ on $\mathrm{Int}(|\gamma_{\tilde{R}}|)$. However, since $\tilde{R} > x_0/\delta$, the origin belongs to $\mathrm{Int}(|\gamma_{\tilde{R}}|)$ and $0 = p^*_{\Lambda,n}(0) \to F(0)$ $(n \to \infty)$ which is a contradiction.

A similar consideration with the same result can be performed for the set K_2.

At the end of this example, we want to examine the results obtained above with respect to geometric considerations. The approximating polynomials are determined to approximate the function $g \equiv 1$ on K_j $(j = 1, 2)$ and are restricted to vanish at the origin. Therefore it is natural that the rate of approximation is better, the farther K_j is away from the origin, i. e. the greater x_0 and/or the smaller δ and vice versa. This relation is accurately reflected through the fraction δ/x_0 or the number $\left(\frac{x_0}{\delta} + \sqrt{\frac{x_0^2}{\delta^2} - 1}\right)^{-1}$.

Example 6.18 :
Let $\Lambda = 2\mathbb{N}_0 + 1$, i. e. we want to approximate by odd polynomials.

Then the function

$$\varphi(z) = \frac{z}{1+z^2} \quad (z \in \mathbb{C} \setminus \{\pm i\})$$

is holomorphic in $\Omega := \mathbb{C}_\infty \setminus \{\pm i\}$ with $\Lambda_\varphi^+ = \Lambda$ and therefore a suitable gap producing function.

Let $U \subset \mathbb{C} \setminus \{0\}$ be open with $U \cap (-U) \neq \emptyset$. Then U is star-eligible to Ω and we obtain for a function $f \in H(U)$ and $z \in \Omega * U = iU \cap (-iU)$

$$\begin{aligned} T_{\varphi,\Omega,U} f(z) &= \frac{1}{2\pi i} \int_\Gamma \frac{z/\zeta}{1 + z^2/\zeta^2}\, f(\zeta)\, \frac{d\zeta}{\zeta} \\ &= \frac{1}{2i}\Big(\frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - iz}\, d\zeta - \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta + iz}\, d\zeta\Big) \\ &= \frac{1}{2i}(f(iz) - f(-iz)) \end{aligned}$$

where $\Gamma = (\tau_r(iz), \tau_r(-iz))$ for some sufficiently small r and the last identity follows with Cauchy's integral theorem and Cauchy's integral formula.

Let in the following $U = \mathbb{S}$ which implies $\Omega * U = \mathbb{C} \setminus i\mathbb{R}$. We show that

$$R(T_{\varphi,\Omega,\mathbb{S}}) = \{g \in H(\mathbb{C} \setminus i\mathbb{R}) : \; g \text{ odd}\}.$$

Proposition 2.20 yields that the left-hand side is a subset of the right-hand side. In order to show the other inclusion, we set $N := \{z \in \mathbb{C} : \; \operatorname{Re} z > 0\}$ and prove in a first step that

$$T_{\tilde{\varphi},D_{K_{1/2}},\mathbb{S}} : H(\mathbb{S}) \to H(N)$$

(where $\tilde{\varphi} := \varphi|_{D_{K_{1/2}}}$) is surjective. The power series expansion of $\tilde{\varphi}$ about zero reads as follows

$$\tilde{\varphi}(z) = \sum_{\nu=0}^{\infty} (-1)^{\nu} z^{2\nu+1} = \sum_{n=0}^{\infty} a_n z^n \quad (z \in \mathbb{D})$$

with

$$a_n = \begin{cases} 0 & , \; n \in 2\mathbb{N}_0 \\ (-1)^{(n-1)/2} & , \; n \in 2\mathbb{N}_0 + 1 \end{cases}.$$

The function

$$\Phi(z) = \sin(\frac{\pi}{2} z) \quad (z \in \mathbb{C})$$

is an entire function of exponential type with $K(\Phi) = K_{1/2}$ and $\Phi(n) = a_n$ ($n \in \mathbb{N}_0$). Carlson's Theorem (see Theorem B.14) yields that $\mathfrak{M}\tilde{\varphi} = \Phi$. Since Φ is of completely regular growth, Theorem 4.34 implies that $T_{\tilde{\varphi},D_{K_{1/2}},\mathbb{S}} : H(\mathbb{S}) \to H(N)$ is surjective.

Let now an odd function $g \in H(\mathbb{C} \setminus i\mathbb{R})$ be given. Then there exists a function $f \in H(\mathbb{S})$ such that $T_{\tilde{\varphi},D_{K_{1/2}},\mathbb{S}} f = g|_N$. Theorem 2.13 yields for all $z \in N$

$$T_{\tilde{\varphi},D_{K_{1/2}},\mathbb{S}} f(z) = g(z) = T_{\varphi,\Omega,\mathbb{S}} f(z).$$

Since $T_{\varphi,\Omega,\mathbb{S}} f$ is an odd function we obtain for all $z \in -N$

$$T_{\varphi,\Omega,\mathbb{S}} f(z) = -T_{\varphi,\Omega,\mathbb{S}} f(-z) = -g(-z) = g(z)$$

which completes the proof.

Especially the function

$$g(z) = \begin{cases} 1 & , \; z \in N \\ -1 & , z \in -N \end{cases}$$

belongs to the range of $T_{\varphi,\Omega,\mathbb{S}}$.

The principal branch of the logarithm on $\mathbb{S}$ fulfills

$$\begin{aligned} \log(iz) - \log(-iz) &= i\pi, \quad z \in N\,, \\ \log(iz) - \log(-iz) &= -i\pi, \quad z \in -N\,. \end{aligned}$$

Therefore, by setting $f(z) := 2\log z/\pi$ $(z \in \mathbb{S})$ we obtain a function $f \in H(\mathbb{S})$ with $T_{\varphi,\Omega,\mathbb{S}} f = g$.

If we want to get some information about the rate of approximation of g by odd polynomials on a set of the form $\Omega * K$ using the approach described in this section, we would have to find a suitable compact set $K \subset \mathbb{S}$ and compute $R_{f,K} = R_{\log,K}$. However, $\Omega * K = iK \cap (-iK)$ is empty if K is a disc or an interval. The next type of sets which could be considered are annular sectors of the form

$$K_{r,\alpha} = \{z \in \mathbb{C} : \; r \leq |z| \leq 1, \; |\arg z| \leq \alpha\}$$

for some $0 < r < 1$ and $0 \leq \alpha < \pi$. If α is large enough, then $\Omega * K_{r,\alpha}$ is a non-empty, symmetric set and we can ask for the rate of approximation of g on the set $\Omega * K_{r,\alpha}$ by odd polynomials. Since $\Omega * K_{r,\alpha}$ is symmetric and we want to approximate the odd function g by odd polynomials, it is enough to ask how fast the function 1 can be approximated on the set $(\Omega * K_{r,\alpha}) \cap N$ by odd polynomials.

As already mentioned, to give a satisfactory answer we would have to know $R_{\log,K_{r,\alpha}}$. A first step is to know the conformal mapping $\beta : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K_{r,\alpha}$. This map is given by Coleman and Myers (see [CM, Th. 1]) but it is highly complicated and we can not expect to compute the level curves $|\gamma_R|$ and hence $R_{\log,K_{r,\alpha}}$ manually.

If we choose $\alpha = 1/2$, then $\Omega * K_{r,1/2} = [-1,-r] \cup [r,1]$. We end up with the question about the rate of approximation of $\operatorname{sign}(x)$ on the union of intervals $[-1,-r] \cup [r,1]$ by odd polynomials. This question has been investigated by Eremenko and Yuditskii in [EY07] (see also [EY11]).

Remark 6.19 :
There are very few types of sets K for which the conformal mapping $\beta : \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K$ is known. Besides the case of discs, intervals and annular sectors it is known for circular sectors of the form $K = \{z \in \mathbb{C} : \; |z| \leq 1, \; |\arg z| \leq \alpha\}$ for some $\alpha < \pi$ (see [CS, Th. 1]). However, even in these cases of relatively simple compact sets K the conformal mapping β is rather complicated and $R_{f,K}$ can not be expected to be computed easily. In addition, it is very challanging to compute a sequence of polynomials which realizes the fastest possible rate of approximation (see for example [GHO]).

Remark 6.20 :
We come back to two important tasks in estimating the rate of approximation of a function $g \in H(\Omega * U)$ by lacunary polynomials by means of the Hadamard product (see Remark 6.14): We have to know that g belongs to the range of the corresponding Hadamard operator $T_{\varphi,U}$ and if so, we have to find a function $f \in H(U)$ with $T_{\varphi,U} f = g$. In the preceding examples we were able to answer both questions by computing what impact the operator $T_{\varphi,U}$ has on the function f. This is far from being possible in a general situation. Therefore, in general, some information about the range of the Hadamard operator is needed and additionally, it would be useful to know a right-inverse of the operator $T_{\varphi,U}$. This may be subject to future research.

Open Problem 6.21 :

1. Given that the operator $T_{\varphi,U}$ is not (known to be) surjective. What functions $g \in H(\Omega * U)$ belong to its range?

2. Given that we know that a function $g \in H(\Omega * U)$ belongs to the range of the operator $T_{\varphi,U}$, how can we determine a function $f \in H(U)$ with $T_{\varphi,U} f = g$?

Appendix A

Notions of density for subsets of the non-negative real numbers

In this part of the appendix we introduce different notions of density for subsets of the non-negative real numbers which are needed troughout this thesis (see for example [Pol29] and [MR]). In the following, the logarithm on the positive real axis shall be denoted by $\ln : (0,\infty) \to \mathbb{R},\ x \mapsto \ln x$ (and $\ln 0 := -\infty$).

Definition A.1 :
Let for $\Lambda \subset [0,\infty)$ without finite accumulation point $n(r) = n_\Lambda(r)$ be the number of $\lambda \in \Lambda$ with $\lambda \le r$ ($r \ge 0$).

1. The *upper* and *lower density* of Λ are defined by

$$\overline{d}(\Lambda) := \limsup_{r\to\infty} \frac{n(r)}{r}, \quad \underline{d}(\Lambda) := \liminf_{r\to\infty} \frac{n(r)}{r} .$$

   In case of equality, the common value $d(\Lambda)$ is the *density* of Λ and Λ is called *measurable.*

2. According to [Pol29, p. 559], the limits

$$\begin{aligned} d^*(\Lambda) &:= \lim_{\xi\to 1^-} \limsup_{r\to\infty} \frac{n(r)-n(r\xi)}{r-r\xi} , \\ d_*(\Lambda) &:= \lim_{\xi\to 1^-} \liminf_{r\to\infty} \frac{n(r)-n(r\xi)}{r-r\xi} \end{aligned}$$

   exist and are called the *maximal* and *minimal density* of Λ.

3. The *logarithmic block density* of Λ is defined by

$$d_L(\Lambda) := \inf_{a>1} \frac{1}{\ln a} \limsup_{t\to\infty} \sum_{\lambda\in\Lambda\cap(t,at]} \frac{1}{\lambda}\,.$$

Remark A.2 :
Let $\Lambda \subset [0,\infty)$ without finite accumulation point. In [Pol29, p. 562 f.] it is shown that

$$\begin{aligned} d^*(\Lambda) &= \min\{d(\overline{\Lambda}) : \ \overline{\Lambda} \text{ measurable and } \Lambda \subset \overline{\Lambda}\}\,, \\ d_*(\Lambda) &= \max\{d(\underline{\Lambda}) : \ \underline{\Lambda} \text{ measurable and } \Lambda \supset \underline{\Lambda}\}\,. \end{aligned}$$

Remark A.3 :
Let $\Lambda = \{\lambda_k : k \in \mathbb{N}\} \subset [0,\infty)$ without finite accumulation point (in this notation we assume $\lambda_{k+1} \geq \lambda_k$ $(k \in \mathbb{N})$).

1. Even if Λ is not measurable, the minimal and maximal, the upper and lower and the logarithmic block density exist (and may be infinity) and the following inequalities hold (see [Pol29, p. 559] and [Rub, Th. 6])

$$0 \leq d_*(\Lambda) \leq \underline{d}(\Lambda) \leq \overline{d}(\Lambda) \leq d^*(\Lambda)\,,$$

$$d_L(\Lambda) \leq \overline{d}(\Lambda)\,.$$

   If Λ is measurable, then all occuring densities coincide with $d(\Lambda)$.

2. Assuming that Λ is infinite, we can express the density (if existing) and the upper and lower density by (see [Pol29, p. 557])

$$\lim_{k\to\infty} k/\lambda_k, \quad \limsup_{k\to\infty} k/\lambda_k, \quad \liminf_{k\to\infty} k/\lambda_k\,.$$

3. If $d(\Lambda) < \infty$, then $\sum\limits_{k=2}^{\infty} 1/\lambda_k^\alpha$ converges for all $\alpha > 1$.

   Indeed, according to 2., given a number $\varepsilon > 0$ there exists a natural number N such that for all $k \geq N$ we have $k/\lambda_k < d(\Lambda) + \varepsilon$.

4. In the special case $\Lambda \subset \mathbb{N}_0$, it is clear that $0 \leq d_*(\Lambda) \leq d^*(\Lambda) \leq 1$ and

$$\begin{aligned} \underline{d}(\Lambda) + \overline{d}(\mathbb{N}_0 \setminus \Lambda) &= 1\,, \\ d_*(\Lambda) + d^*(\mathbb{N}_0 \setminus \Lambda) &= 1 \end{aligned}$$

   and $d(\Lambda)$ exists if and only if $d(\mathbb{N}_0 \setminus \Lambda)$ exists.

   Furthermore, $d_L(\Lambda) = 1$ if and only if $\overline{d}(\Lambda) = 1$ (see [Rub, Th. 4]).

The following lemma gives an example of a set of non-negative integers for which the various densities introduced above do not coincide.

Lemma A.4 :
Let $q \in \mathbb{N},\ q \geq 3$ *and*

$$\Lambda_q := \bigcup_{k \in \mathbb{N}_0} \{n \in \mathbb{N} :\ q^k < n \leq 2 \cdot q^k\}\,.$$

Then

$$\overline{d}(\Lambda_q) = \frac{q}{2(q-1)}, \quad d_L(\Lambda_q) = \frac{\ln 2}{\ln q}, \quad \underline{d}(\Lambda_q) = \frac{1}{q-1}\,.$$ [1]

Proof: Obviously, the set Λ_q is composed of "blocks" consisting of q^k numbers. Hence, counting the elements of Λ_q we observe that up to the end of the $(l+1)$-th block, $\sum_{\nu=0}^{l} q^\nu$ elements belong to the set Λ_q. Given an enumeration $\{\lambda_{q,j} :\ j \in \mathbb{N}\}$ of Λ_q we obtain with Remark A.3.2

$$\overline{d}(\Lambda_q) \;=\; \limsup_{j\to\infty} \frac{j}{\lambda_{q,j}} = \lim_{l\to\infty} \frac{\sum\limits_{\nu=0}^{l} q^\nu}{2q^l} = \lim_{l\to\infty} \frac{1}{2} \sum_{\nu=0}^{l} \Big(\frac{1}{q}\Big)^\nu = \frac{q}{2(q-1)}$$

and

$$\underline{d}(\Lambda_q) \;=\; \liminf_{j\to\infty} \frac{j}{\lambda_{q,j}} = \lim_{l\to\infty} \frac{\sum\limits_{\nu=0}^{l-1} q^\nu + 1}{q^l + 1} = \lim_{l\to\infty} \sum_{\nu=1}^{l} \Big(\frac{1}{q}\Big)^\nu = \frac{1}{q-1}\,.$$

Now we are going to compute the logarithmic block density of Λ_q.

For $a > 1$ we set $k_0 := \max\{\nu \in \mathbb{N}_0 :\ a \geq q^\nu\}$, i. e. $aq^{-k_0} \in [1, q)$. We set $\tilde{a} := \min\{aq^{-k_0}, 2\}$ and obtain

$$\begin{aligned}
\frac{1}{\ln a} \sum_{\lambda \in \Lambda_q \cap (q^k, aq^k]} \frac{1}{\lambda} \;&=\; \frac{1}{\ln a}\Big(\Big(\sum_{\mu=0}^{k_0-1} \sum_{\lambda \in \mathbb{N} \cap (q^{k+\mu}, 2\cdot q^{k+\mu}]} \frac{1}{\lambda}\Big) + \sum_{\lambda \in \mathbb{N} \cap (q^{k+k_0}, \tilde{a} q^{k+k_0}]} \frac{1}{\lambda}\Big) \\
&=\; \frac{1}{\ln a}\big(k_0 \ln 2 + \ln \tilde{a} + o(1)\big) \\
&\to\; \frac{k_0 \ln 2 + \ln \tilde{a}}{k_0 \ln q + \ln(aq^{-k_0})} \geq \frac{\ln 2}{\ln q} \quad (k \to \infty)
\end{aligned}$$

---

[1] See [Rub, p. 421 ff.] for the case q=4.

where the latter inequality can easily be verified separately for the case $\tilde{a} = aq^{-k_0}$ and $\tilde{a} = 2$ and also holds for the case $k_0 = 0$.

An immediate consequence is that $d_L(\Lambda_q) \geq \ln 2 / \ln q$.

To show the opposite inequality, we consider the case $a = q$.

(i) For every $t > 1$ such that there exists a number $k(t) \in \mathbb{N}_0$ such that $t \in (2q^{k(t)}, q^{k(t)+1}]$ we have

$$\sum_{\lambda \in \Lambda_q \cap (t, qt]} \frac{1}{\lambda} = \sum_{\lambda \in \mathbb{N} \cap (q^{k(t)+1}, 2q^{k(t)+1}]} \frac{1}{\lambda} = \ln 2 + o(1)$$

which results in

$$\sum_{\lambda \in \Lambda_q \cap (t, qt]} \frac{1}{\lambda} \longrightarrow \ln 2$$

as t tends to infinity in $\bigcup_{n \in \mathbb{N}_0} (2q^n, q^{n+1}]$.

(ii) For every $t > 1$ such that there exists a number $k(t) \in \mathbb{N}_0$ such that $t \in (q^{k(t)}, 2q^{k(t)}]$ we have

$$\begin{aligned} \sum_{\lambda \in \Lambda_q \cap (t, qt]} \frac{1}{\lambda} &= \sum_{\lambda \in \mathbb{N} \cap (t, 2q^{k(t)}]} \frac{1}{\lambda} + \sum_{\lambda \in \mathbb{N} \cap (q^{k(t)+1}, qt]} \frac{1}{\lambda} \\ &= \ln(2q^{k(t)}/t) + \ln(qt/q^{k(t)+1}) + o(1) \\ &= \ln 2 + o(1) \end{aligned}$$

which results in

$$\sum_{\lambda \in \Lambda_q \cap (t, qt]} \frac{1}{\lambda} \longrightarrow \ln 2$$

as t tends to infinity in $\bigcup_{n \in \mathbb{N}_0} (q^n, 2q^n]$.

Hence,

$$\sum_{\lambda \in \Lambda_q \cap (t, qt]} \frac{1}{\lambda} \longrightarrow \ln 2 \quad (t \to \infty)$$

and therefore

$$d_L(\Lambda_q) \leq \limsup_{t \to \infty} \frac{1}{\ln q} \sum_{\lambda \in \Lambda_q \cap (t, qt]} \frac{1}{\lambda} = \frac{\ln 2}{\ln q}. \qquad \square$$

In the following we state two gap theorems which reveal a connection between the density of the non-vanishing coefficients in the power series expansion of a function f about zero and the analytic continuability of f (see for example [Hi, p. 89] and [Pol29, p. 626]).

Theorem A.5 (Fabry gap theorem) **:**
Let $\Lambda \subset \mathbb{N}_0$ *with* $d(\Lambda) = 0$, $a_\lambda \in \mathbb{C}$ $(\lambda \in \Lambda)$. *Either the series*

$$\sum_{\lambda \in \Lambda} a_\lambda z^\lambda$$

represents an entire function or it can not be continued beyond its (finite) circle of convergence.

Remark A.6 :
The above cited (negative) version of Fabry's gap theorem is equivalent to the following (positive) formulation:

Let $G \subset \mathbb{C}$ be a domain with $0 \in G$ and $f \in H(G)$ with $d(\Lambda_f^+) = 0$. Then the series $\sum\limits_{\nu \in \Lambda_f^+} f_\nu z^\nu$ converges in $\mathbb{D}_{\sup\limits_{u \in G} |u|}$.

Theorem A.7 (Pólya gap theorem) **:**
Let $\Lambda \subset \mathbb{N}_0$ *with* $d^*(\Lambda) = \delta$, $a_\lambda \in \mathbb{C}$ $(\lambda \in \Lambda)$. *Let the power series*

$$f(z) = \sum_{\lambda \in \Lambda} a_\lambda z^\lambda \qquad \text{(A.1)}$$

have radius of convergence $R < \infty$. *Then on every open subarc of* $\mathbb{T}_R$ *with length larger than* $2\pi R\delta$, f *has at least one singularity.*

Remark A.8 :
The Pólya gap theorem yields the following density condition for the set Λ_φ^+ of a function $\varphi \in H(\Omega)$:

Let $\Omega \subset \mathbb{C}_\infty$ be open with $0 \in \Omega$ and let $R < \infty$ be the radius of convergence of the power series expansion of $\varphi \in H(\Omega)$ about zero. If $\mathbb{T}_R \cap \Omega$ contains an open arc of length $2\pi R\delta$, then the Pólya gap theorem implies that $d^*(\Lambda_\varphi^+) \geq \delta$.

If, in particular, Λ_φ^+ is measurable, then $d(\Lambda_\varphi^+) \geq \delta$.

Appendix B

Entire functions of exponential type

B.1 Definition and basic properties

Since in this thesis we make repeatedly use of properties of certain entire functions, this section introduces the required concepts according to [Boa]. We agree upon the following notation: for an entire function Φ the set of zeros shall be denoted by $Z(\Phi) := \{z \in \mathbb{C} : \Phi(z) = 0\}$. If Φ does not vanish identically, then we denote by $n_\Phi(r)$ the number of zeros (counted according to multiplicity) of Φ in $\overline{\mathbb{D}}_r$ $(r \geq 0)$.

Definition B.1 :
Let Φ be an entire function.

1. We denote by $M_\Phi(r) := \max\{|\Phi(z)| : |z| = r\}$ $(r \geq 0)$ the *maximum modulus* of Φ. The maximum principle implies that this function increases monotonically.

2. Φ is called an *entire function of exponential type* if

$$\tau(\Phi) := \limsup_{r\to\infty} \frac{\ln M_\Phi(r)}{r} < \infty\,. \tag{B.1}$$

Remark B.2 :
Let Φ be an entire function of exponential type. Then $\tau(\Phi) \geq 0$ if and only if $\Phi \not\equiv 0$ and $\tau(0) = -\infty$.

Theorem B.3 :
Let $\Phi \not\equiv 0$ be an entire function of exponential type. Then

$$\limsup_{r \longrightarrow \infty} \frac{n_\Phi(r)}{r} \leq e \cdot \tau(\Phi) \quad \text{and} \quad \liminf_{r \longrightarrow \infty} \frac{n_\Phi(r)}{r} \leq \tau(\Phi)\,.^1$$

In order to describe the growth of entire functions on different rays, we introduce the Phragmén-Lindelöf indicator function:

Definition B.4 :
Let Φ be an entire function of exponential type. Then the function

$$h_\Phi : [-\pi, \pi) \to [-\infty, \infty),\ h_\Phi(t) = \limsup_{r \to \infty} \frac{\ln|\Phi(re^{it})|}{r}$$

is called the *Phragmén-Lindelöf indicator function* of f.

Proposition B.5 :
Let Φ, Ψ be entire functions of exponential type.

1. If $\Phi \not\equiv 0$, then h_Φ can be continued to a real valued, 2π-periodic, continuous function on $\mathbb{R}$ with values in $[-\tau(\Phi), \tau(\Phi)]$. If $\Phi \equiv 0$, then $h_\Phi \equiv -\infty$.
2. $h_{\Phi\Psi} \leq h_\Phi + h_\Psi$ and $h_{\Phi+\Psi} \leq \max\{h_\Phi, h_\Psi\}$.

There is a class of entire functions of exponential type for which equality holds in the first inequality in Proposition B.5.2 (see Theorem B.20.1).

Definition B.6 :
Let $K \subset \mathbb{C}$ be non-empty, compact and convex. Then the function

$$H_K : \mathbb{C} \longrightarrow \mathbb{C},\ H_K(z) := \sup_{u \in K} \operatorname{Re}(zu)$$

is called the *support function* of K.[2]

We list some properties of the support function (see [BG, Prop. 1.3.14] and [Mori, Cor. 1.8.2]).

[1] See [Boa, Th. 2.5.13].
[2] See [BG, p. 64].

Proposition B.7 :
Let $K, L \subset \mathbb{C}$ be non-empty, compact and convex. Then the following assertions hold:

1. $H_{K+L} = H_K + H_L$.
2. K is a subset of L if and only if $H_K \leq H_L$.
3. If $K = i[-a, a]$ for some $a > 0$, then $H_K(z) = a|z||\sin(\arg z)|$.

In the following, we want to associate a certain convex set with an entire function of exponential type which reflects in some sense its growth properties.

Let Φ be an entire function of exponential type and for $\zeta \in \mathbb{T}$ we set $W(\zeta) := \{z \in \mathbb{C} : \ \mathrm{Re}(z\zeta) > h_\Phi(\arg \zeta)\}$. We consider the *Laplace transform*

$$\mathcal{B}\Phi(z;\zeta) := \zeta \int_0^\infty \Phi(t\zeta) e^{-zt\zeta} \, dt \quad (z \in W(\zeta))$$

and list the properties of $\mathcal{B}\Phi(\cdot;\zeta)$ according to [Mori, p. 36 f.]:

1. The function $z \mapsto \mathcal{B}\Phi(z;\zeta)$ is holomorphic in the open half-plane $W(\zeta)$.
2. For $\zeta, \zeta' \in \mathbb{T}$ we have
$$\mathcal{B}\Phi(z;\zeta) = \mathcal{B}\Phi(z;\zeta') \quad (z \in W(\zeta) \cap W(\zeta')) \, .$$
3. The above mentioned property allows to paste the functions $\mathcal{B}\Phi(\cdot;\zeta)$ $(\zeta \in \mathbb{T})$ together ending up with a function $\mathfrak{B}\Phi$ being holomorphic in $\bigcup_{\zeta\in\mathbb{T}} W(\zeta)$ and vanishing at infinity.

Definition B.8 :
For a given entire function of exponential type Φ, the function $\mathfrak{B}\Phi : \bigcup_{\zeta\in\mathbb{T}} W(\zeta) \to \mathbb{C}$ constructed as above is called the *Borel transform* of Φ. The set $\mathbb{C} \setminus \bigcup_{\zeta\in\mathbb{T}} W(\zeta)$ is compact and convex and is called the *conjugate indicator diagram* of Φ. We will denote it by $K(\Phi)$.

For a given compact and convex set $K \subset \mathbb{C}$ we denote by $\mathrm{Exp}(K)$ the space of all entire functions of exponential type whose conjugate indicator diagram is contained in K.

Remark B.9 :
Let Φ be an entire function of exponential type.

1. An immediate consequence of the definition of the Borel transform is that $K(\Phi) = \emptyset$ if and only if $\Phi \equiv 0$.

2. According to [Boa, p. 74], the conjugate indicator diagram of Φ if the smallest closed convex set outside which $\mathcal{B}\Phi$ is holomorphic.

3. According to [BG, p. 64] the following condition is necessary and sufficient for Φ to belong to the space $\mathrm{Exp}(K)$: For each $\varepsilon > 0$ there is a constant C_ε such that
$$\sup_{z\in\mathbb{C}} |\Phi(z)|e^{-H_K(z)-\varepsilon|z|} \leq C_\varepsilon\,.$$

4. The space $\mathrm{Exp}(\{0\})$ consists of all entire functions of exponential type Φ with $\tau(\Phi) \leq 0$.

We list some properties of the conjugate indicator diagram in the following proposition (see [Boa, p. 75-77]).

Proposition B.10 :
Let Φ, Ψ be entire functions of exponential type. Then the following are valid:

1. $\tau(\Phi) = \max\{h_\Phi(t) : t \in [-\pi, \pi)\} = \max\{|u| : u \in K(\Phi)\}$.[3]

2. $K(\Phi + \Psi) \subset \mathrm{conv}(K(\Phi) \cup K(\Psi))$ and $K(\Phi\Psi) \subset K(\Phi) + K(\Psi)$.[4] If $K(\Phi)$ or $K(\Psi)$ is a singleton, then $K(\Phi\Psi) = K(\Phi) + K(\Psi)$.

Example B.11 :
The conjugate indicator diagram of the function $e_\alpha : \mathbb{C} \to \mathbb{C},\ z \mapsto e^{\alpha z}$ is the set $K(e_\alpha) = \{\alpha\}$ $(\alpha \in \mathbb{C})$.

The following theorem establishes an important relation between the indicator function and the support function of the conjugate indicator diagram (see [BG, Th. 1.3.21]).

[3] We set $\max_\emptyset := -\infty$.
[4] For $A \subset \mathbb{C}$, the smallest convex superset of A is denoted by $\mathrm{conv}(A)$.

Theorem B.12 :
Let $\Phi \not\equiv 0$ be an entire function of exponential type. Then we have

$$h_\Phi(t) = H_{K(\Phi)}(e^{it}) \quad (t \in [-\pi, \pi))\,.$$

Combining Proposition B.7.2 and Theorem B.12 we obtain the following corollary:

Corollary B.13 :
Let Φ be an entire function of exponential type.

1. If Ψ is an entire function of exponential type, then $K(\Phi) = K(\Psi)$ if and only if $h_\Phi(t) = h_\Psi(t)$ $(t \in [-\pi, \pi))$.
2. If $c \in \mathbb{C} \setminus \{0\}$ and $\tilde{\Phi}(z) := \Phi(cz)$ $(z \in \mathbb{C})$, then $K(\tilde{\Phi}) = c \cdot K(\Phi)$.

Theorem B.14 (Carlson's theorem[5]) **:**
Let Φ be an entire function of exponential type with

$$\max_{z \in K(\Phi)} \operatorname{Im} z - \min_{z \in K(\Phi)} \operatorname{Im} z < 2\pi\,.^{6}$$

If $Z(\Phi) \supset \mathbb{N}_0$, then $\Phi \equiv 0$.

B.2 The Mellin transformation

In this section we introduce the Mellin transformation according to [BG, Ch. 4.1].

Definition B.15 :
Let $K \subset \mathbb{V}$ be compact and convex and $D_K := \mathbb{C}_\infty \setminus e^{-K}$. For $\varphi \in H(D_K)$, the *Mellin transform* $\mathfrak{M}\varphi$ of φ is given by

$$\mathfrak{M}\varphi(z) := -\frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\zeta)}{\zeta^{z+1}}\,d\zeta \quad (z \in \mathbb{C})\,,$$

where Γ is a Cauchy cycle for e^{-K} in $\mathbb{S}$ and $\zeta^c := \exp(c \log \zeta)$. The map $\mathfrak{M} : H(D_K) \to \operatorname{Exp}(K)$ is called *Mellin transformation.*

[5] See [Boa, Th. 9.2.1].

[6] We set $-\infty + (-\infty) := -\infty$.

Theorem B.16 :
Let $K \subset \mathbb{V}$ be compact and convex. The Mellin transformation $\mathfrak{M} : H(D_K) \to \mathrm{Exp}(K)$ is a linear and bijective operator. Furthermore, for a given function $\Phi \in \mathrm{Exp}(K)$ we obtain for all z with small modulus

$$\mathfrak{M}^{-1}\Phi(z) = \sum_{\nu=0}^{\infty} \Phi(\nu) z^{\nu} \tag{B.2}$$

and for all z with large modulus

$$\mathfrak{M}^{-1}\Phi(z) = -\sum_{\nu=1}^{\infty} \Phi(-\nu) z^{-\nu}\,. \tag{B.3}$$

Remark B.17 :
Let $K \subset \mathbb{V}$ be compact and convex, $\varphi \in H(D_K)$ and $\Phi = \mathfrak{M}\varphi$.

1. The situation of Theorem B.16 in the special case $K = \{0\}$ is known as the Wigert-Leau Theorem (see for example [Le96, p. 72]).

   In this case, Theorem B.3 yields for a non-vanishing function $\varphi \in H(\mathbb{C}_\infty \setminus \{1\})$ that Λ_φ^+ and Λ_φ^- are measurable and $d(\Lambda_\varphi^+) = d(\Lambda_\varphi^-) = 1$.

2. By differentiation of the defining parameter integral one can verify that the derivatives of Φ are given by

$$\Phi^{(k)}(z) := -\frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\zeta)}{\zeta^{z+1}}\,(-1)^{(k)}(\log\zeta)^k\,d\zeta \quad (z \in \mathbb{C},\ k \in \mathbb{N}_0)\,.$$

3. Let $K, L \subset \mathbb{V}$ be compact and convex and such that $K + L \subset \mathbb{V}$. For $\varphi \in H(D_K)$ and $f \in H(D_L)$ the following is valid:

$$\mathfrak{M}(\varphi * f) = \mathfrak{M}\varphi \cdot \mathfrak{M}f\,. \tag{B.4}$$

   Indeed, according to Example 2.3.5, we obtain $D_K * D_L = D_{K+L}$ and therefore we can consider the Mellin transform $\mathfrak{M}(\varphi * f) \in \mathrm{Exp}(K + L)$ of $\varphi * f$.

   Furthermore, the Hadamard multiplication theorem 2.14.3 together with Theorem B.16 implies that for all z with small modulus we obtain

$$\varphi * f(z) = \sum_{\nu=0}^{\infty} \varphi_\nu \cdot f_\nu \cdot z^\nu = \sum_{\nu=0}^{\infty} \mathfrak{M}\varphi(\nu) \cdot \mathfrak{M}f(\nu) \cdot z^\nu\,.$$

   According to Proposition B.10.2, $\mathfrak{M}\varphi \cdot \mathfrak{M}f$ belongs to the space $\mathrm{Exp}(K + L)$ and the assertion follows from Carlson's Theorem.

B.3 Functions of completely regular growth

In this section we introduce a class of entire functions of exponential type which fulfill stronger conditions on their growth (see [BG, D. 1.5.9] and [Le62, Ch. 3]).

Definition B.18 :

1. Let $E \subset [0,\infty)$ be a Lebesgue-measurable set. Then E is of *relative zero measure* if
$$\lim_{r\to\infty} \frac{\lambda(E\cap[0,r])}{r} = 0\,,$$
where λ denotes the Lebesgue measure on $\mathbb{R}$.

2. Let Φ be an entire function of exponential type. Then Φ is of *completely regular growth* if there exists a set E of relative zero measure such that
$$h_\Phi(t) = \lim_{\substack{r\to\infty\\ r\notin E}} \frac{\ln|\Phi(re^{it})|}{r}$$
holds for all $t\in[-\pi,\pi)$ and these limits are uniform in t.[7]

Example B.19 :

1. Every function belonging to the space $\operatorname{Exp}(\{0\})$ is of completely regular growth.

2. Let $\Lambda = \{\lambda_n : \ n\in\mathbb{N}\} \subset (0,\infty)$ without finite accumulation point and $d(\Lambda)=\delta$. Then [Le62, p. 205] yields that
$$\Phi(z) := \prod_{n=1}^{\infty}\big(1-\frac{z^2}{\lambda_n^2}\big)$$
is of completely regular growth with $Z(\Phi) = \pm\Lambda$ and conjugate indicator diagram being the line segment K_δ.

   Theorem B.12 together with Proposition B.7.3 yields
$$h_\Phi(t) = \pi\delta|\sin t| \quad (t\in[-\pi,\pi))\,.$$

The following theorem contains an important property of functions of completely regular growth concerning their indicator functions.

[7] Note that the set E is independent of t.

Theorem B.20 :

1. *Let* Ψ *be a function of completely regular growth and let* Φ *be an arbitrary entire function of exponential type. Then*

$$h_{\Phi\Psi} = h_\Phi + h_\Psi \,.$$

2. *Let* Ψ *be a function of completely regular growth and let* Φ *be an entire function of exponential type such that* Φ/Ψ *is entire.*[8] *Then* Φ/Ψ *is of exponential type and*

$$h_{\Phi/\Psi} = h_\Phi - h_\Psi \,.$$

Remark B.21 :
Combining Theorem B.20, Proposition B.7 and Theorem B.12 we obtain:

Let Ψ be a function of completely regular growth and Φ be an arbitrary entire function of exponential type. Then

$$K(\Phi\Psi) = K(\Phi) + K(\Psi)\,.$$

[8] Or to be more precise, Φ/Ψ can be continued to an entire function.

Appendix C

Density and surjectivity criteria

In this chapter we present some criteria for a linear and continuous operator to have dense range or to be surjective. They are used for the examination of the range of Hadamard operators in Chapter 4. We begin by summarizing some information about various types of locally convex spaces and their interdependence. The presentation in this chapter follows [MV].

Definition C.1 :
Let E be a locally convex space.

1. A set $M \subset E$ is called a *barrel*, if M is absolutely convex, closed and absorbing (i. e. $E = \bigcup_{n \in \mathbb{N}} nM$).

2. E is called *barrelled*, if each barrel in E is a zero neighbourhood.

3. E is called a *Schwartz space*, if for each absolutely convex zero neighbourhood U in E there exists a zero neighbourhood V so that for each $\varepsilon > 0$, there exist $x_1, \ldots, x_n \in V$ such that $V \subset \bigcup_{j=1}^{n}(x_j + \varepsilon U)$.

4. E is called an *FS-space*, if it is at the same time a Fréchet and a Schwartz space.

5. A set $\mathcal{B} \subset E'$ is called *(pointwise) bounded*, if it is bounded with respect to the weak*-topology, i. e. $\sup\{|x'(x)| : x' \in \mathcal{B}\} < \infty$ for all $x \in E$.

Remark C.2 :

1. A locally convex space E is barrelled, if every bounded set $\mathcal{B} \subset E'$ is equicontinuous.

2. If E is a Fréchet space, then a set $\mathcal{B} \subset E'$ is bounded if and only if it is equicontinuous (see [Koe, p. 169]). Hence, every Fréchet space is barrelled.

3. For an open set $\Omega \subset \mathbb{C}_\infty$, the space $H(\Omega)$ is an FS-space (see [Mori, Th. 1.4.1, Th. A.4.5]). With the notation of Remark 1.11, we note that every equicontinuous set $\mathcal{B} \subset H'(\Omega)$ is bounded in some of the Banach spaces $H^\infty(K_n^*)$ (see [Koe, Ch. 27.4]).[1]

Definition C.3 :
Let E be a locally convex space. For a set $M \subset E$ the *annihilator* of M is defined by
$$M^\perp := \{x' \in E' : \ x'(x) = 0 \text{ for all } x \in M\}.$$

Theorem C.4 (Closed range theorem) **:**
Let E, F be Fréchet spaces and $T \in L(E,F)$. Then the following are equivalent:

1. *$R(T)$ is closed.*
2. *$R(T')$ is closed.*
3. *$R(T) = N(T')^\perp$.*
4. *$R(T') = N(T)^\perp$.*

Now we turn towards surjectivity criteria.

Theorem C.5 :
Let E, F be locally convex spaces and $T \in L(E,F)$. Then the following assertions hold:

1. *T has dense range if and only if T' is injective.*
2. *T is surjective if and only if T' is injective and $R(T)$ is closed.*
3. *If, in addition, E and F are Fréchet spaces, then the condition in 2. is equivalent to the condition: T' is injective and $R(T')$ is closed.*

We conclude this chapter by citing the following surjectivity criterion of Frerick, Müller and Wengenroth presented in [FMW, Th. 3]:

[1] Here, we identify a functional u with the representative g of the corresponding germ $[(g,V)]_{\Omega^*}$.

Theorem C.6 :
Let E be a Fréchet space, F be a barrelled Schwartz space and let $T \in L(E,F)$ have dense range. Suppose that for each bounded subset $\mathcal{B}$ of E' there is a bounded subset $\mathcal{A}$ of F' such that $(T')^{-1}(\mathcal{B})$ is contained in the linear span $[\mathcal{A}]$ of $\mathcal{A}$. Then T is surjective and open.

List of symbols

$\mathbb{C}_\infty$	the Riemann sphere, see p. 5
A^C	$\mathbb{C}_\infty \setminus A$, see p. 5
$1/A$	$\{1/a : \ a \in A\}$, see p. 5
A^*	$1/A^C$, see p. 5
A_ξ	the component of A containing $\xi \in \{0, \infty\}$, see p. 5
$\arg z$	the argument of z, see p. 5
$\mathbb{D}_r$	$\{z \in \mathbb{C} : \ \|z\| < r\}$, see p. 6
$\mathbb{D}$	$\{z \in \mathbb{C} : \ \|z\| < 1\}$, see p. 6
$\mathbb{T}_r$	$\{z \in \mathbb{C} : \ \|z\| = r\}$, see p. 6
$\mathbb{T}$	$\{z \in \mathbb{C} : \ \|z\| = 1\}$, see p. 6
$U_r(z_0)$	$\{z \in \mathbb{C} : \ \|z - z_0\| < r\}$, see p. 6
$U_r(\infty)$	$\{z \in \mathbb{C} : \ \|z\| > r\} \cup \{\infty\}$, see p. 6
$V_{r,R}$	$\{z \in \mathbb{C} : \ r < \|z\| < R\}$, see p. 6
$\\|f\\|_K$	$\max_{z \in K} \|f(z)\|$, see p. 6
$C(K)$	$\{f : K \to \mathbb{C} : \ f \text{ continuous}\}$, see p. 6
γ^-	$\gamma^-(t) := \gamma(a + b - t) \ (t \in [a, b])$, see p. 6
$\|\Gamma\|$	the trace of Γ, see p. 7
Γ^-	$(\gamma_1^-, \ldots, \gamma_n^-)$, see p. 7
$L(\Gamma)$	the length of Γ, see p. 7
$\mathrm{ind}_\Gamma(z)$	the index of z with respect to Γ, see p. 7
$\tau_r(z_0)$	$\tau_r(z_0) : [0, 2\pi] \to \mathbb{C}, \ t \mapsto z_0 + re^{it}$, see p. 8
$H(\Omega)$	space of all functions holomorphic in Ω and vanishing at ∞ (if $\infty \in \Omega$), see p. 9
$H^\infty(\Omega)$	space of all functions $f \in H(\Omega)$ which are bounded on Ω, see p. 10
$\mathcal{H}(K)$	space of all functions $f \in C(K)$ such that there exists an open set $O \supset K$ and a function $F \in H(O)$ such that $F\|_K = f$, see p. 10
$H(B)$	space of germs of holomorphic functions on B, see p. 10
$[(f, U)]_B$	germ of holomorphic functions on B, see p. 10

φ_ν	ν-th coefficient in the power series expansion of φ about zero, see p. 10
$\varphi_{-\nu}$	ν-th coefficient in the power series expansion of φ about infinity, see p. 10
Λ_φ^+	$\{\nu \in \mathbb{N}_0 : \ \varphi_\nu \neq 0\}$, see p. 10
Λ_φ^-	$\{\nu \in \mathbb{N} : \ \varphi_{-\nu} \neq 0\}$, see p. 10
Λ_φ	$\Lambda_\varphi^+ \cup (-\Lambda_\varphi^-)$, see p. 11
$H_\Lambda(\Omega)$	$\{f \in H(\Omega) : \ \Lambda_f^+ \subset \Lambda\}$, see p. 10
$H_{\infty,\Lambda'}(\Omega)$	$\{f \in H(\Omega) : \ \Lambda_f^- \subset \Lambda'\}$, see p. 11
$h_\Omega(M)$	hull of M with respect to Ω, see p. 11
$\mathbb{K}$	$\mathbb{R}$ or $\mathbb{C}$, see p. 12
$L(E,F)$	$\{T : E \to F : \ T$ is linear and continuous$\}$, see p. 12
E'	$L(E,\mathbb{K})$, see p. 12
$A \cdot B$	$\{a \cdot b : \ a \in A, \ b \in B\}$, see p. 14
$S_{\gamma,\delta}$	$\{z \in \mathbb{C} \setminus \{0\} : \ \gamma\pi < \arg z < \delta\pi\}$, see p. 15
S_α	$S_{-\alpha,\alpha}$, see p. 15
$\mathbb{S}$	$\mathbb{S} := S_1 = \mathbb{C} \setminus (-\infty, 0]$, see p. 15
$\mathbb{V}$	$\log \mathbb{S} = \{z \in \mathbb{C} : \ \lvert \operatorname{Im} z \rvert < \pi\}$, see p. 15
log	principal branch of the logarithm on $\mathbb{S}$, see p. 15
D_K	$\mathbb{C}_\infty \setminus e^{-K}$, see p. 15
K_δ	$i\pi[-\delta,\delta]$, see p. 15
$A * B$	the star product of A and B, see p. 15
Θ	$\Theta(z) := 1/(1-z)$ $(z \in \mathbb{C}_\infty \setminus \{1\})$, see p. 22
$p_{\nu,U}$	$p_{\nu,U}(z) := z^\nu$ $(z \in U)$, see p. 23
$\mathcal{U}$	see p. 25
$[\mathcal{U}]$	see p. 25
T_φ	also $T_{\varphi,U}, \ T_{\varphi,\Omega,U}$: Hadamard operator, see p. 32
κ	Koebe function ($\kappa(z) = z/(1-z^2)$ $(z \in \mathbb{C}_\infty \setminus \{1\})$), see p. 33
$\log_B$	branch of the logarithm on B, see p. 38
$A + B$	$\{a + b : \ a \in A, \ b \in B\}$, see p. 38
$h_{k,\alpha,A}$	$h_{k,\alpha,A}(z) := z^k e^{\alpha z}(z \in A)$, see p. 38
$q_{k,\alpha,B}$	$q_{k,\alpha,B} := h_{k,\alpha,\log_B B} \circ \log_B$, see p. 38
$p_{\alpha,B}$	$p_{\alpha,B} := q_{0,\alpha,B}$, see p. 38
$N(T)$	the kernel of T, see p. 44
$R(T)$	the range of T, see p. 44
clspan	the closure of the linear span, see p. 45
$Z(\Phi)$	the set of zeros of Φ, see p. 53
Δ_φ	$\Delta_\varphi := \min\{d^*(\mathbb{N}_0 \setminus \Lambda_\varphi^+), d^*(\mathbb{N} \setminus \Lambda_\varphi^-)\}$, see p. 57
ϑ_U	$(\vartheta_U f)\ (z) := z \cdot f'(z)$ $(z \in U)$, see p. 77
D_G	$(D_G f)(w) := f'(w)$ $(w \in G)$, see p. 77
$\Phi_U(\vartheta)$	$\Phi_U(\vartheta) f := \sum_{\nu=0}^\infty \Phi_\nu \vartheta^\nu f$, see p. 77
$\Phi_G(D)$	$\Phi_G(D) f := \sum_{\nu=0}^\infty \Phi_\nu D^\nu f$, see p. 77

$\mathfrak{F}$	bijective map from $H'(M)$ to $\mathrm{Exp}(M)$, see p. 77
$\mathcal{T}\circledast$	convolution operator from $H(W+M)$ to $H(W)$, see p. 78
$\mathfrak{G}$	bijective map from $H(D_M)$ to $H'(M)$, see p. 78
$H_{\Phi,G}$	$\mathrm{span}\{h_{k,\alpha,G}:\ \alpha\ m-$fold zero of $\Phi,\ k \le m-1\}$, see p. 82
$Q_{\Phi,U}$	$\mathrm{span}\{q_{k,\alpha,U}:\ \alpha\ m-$fold zero of $\Phi,\ k \le m-1\}$, see p. 82
$P_{\Lambda,n}$	$\mathrm{span}\{p_{\nu,\mathbb{C}}:\ \nu \in \Lambda,\ \nu \le n\}$, see p. 88
P_Λ	$\bigcup_{n\in\mathbb{N}_0} P_{\Lambda,n}$, see p. 89
P_n	$P_{\mathbb{N}_0,n}$, see p. 89
$P_\Lambda(D)$	$\mathrm{clspan}\{p_{\nu,D}:\ \nu \in \Lambda\}$, see p. 89
$p^*_{\Lambda,n}(f)$	a best approximating lacunary polynomial of degree less or equal to n, see p. 94
$E_n(f,K,\Lambda)$	$\min_{p\in P_{\Lambda,n}} \Vert f-p\Vert_K$, see p. 94
$E_n(f,K)$	$E_n(f,K,\mathbb{N}_0)$, see p. 94
$\alpha(z)$	conformal mapping from $\mathbb{C}\setminus K$ to $\mathbb{C}\setminus\overline{\mathbb{D}}$, see p. 94
c_K	logarithmic capacity of K, see p. 94
$\beta(w)$	inverse mapping of α, see p. 94
$\mathrm{Int}(\lvert\gamma_R\rvert)$	interior of $\lvert\gamma_R\rvert$, see p. 95
$\mathrm{Ext}(\lvert\gamma_R\rvert)$	exterior of $\lvert\gamma_R\rvert$, see p. 95
$R_{f,K}$	$R_{f,K} := \sup\{R>1:\ \exists F \in H(\mathrm{Int}(\lvert\gamma_R\rvert)),\ F\vert_K = f\}$, see p. 95
$\ln$	logarithm on the positive real axis, see p. 103
$\overline{d}(\Lambda)$	upper density of Λ, see p. 103
$\underline{d}(\Lambda)$	lower density of Λ, see p. 103
$d^*(\Lambda)$	maximal density of Λ, see p. 103
$d_*(\Lambda)$	minimal density of Λ, see p. 103
$d_L(\Lambda)$	logarithmic block density of Λ, see p. 104
Λ_q	$\bigcup_{k\in\mathbb{N}_0}\{n\in\mathbb{N}:\ q^k < n \le 2\cdot q^k\}$, see p. 105
$n_\Phi(r)$	the number of zeros of Φ in $\overline{\mathbb{D}}_r$ $(r \ge 0)$, see p. 108
$M_\Phi(r)$	the maximum modulus of Φ, see p. 108
$\tau(\Phi)$	$\tau(\Phi) := \limsup_{r\to\infty}\ (\ln M_\Phi(r))/r$, see p. 108
h_Φ	the Phragmén-Lindelöf indicator function of Φ, see p. 109
H_K	the support function of K, see p. 109
$\mathfrak{B}$	the Borel transformation, see p. 110
$K(\Phi)$	the conjugate indicator diagram of Φ, see p. 110
$\mathrm{Exp}(K)$	the space of all entire functions of exponential type whose conjugate indicator diagram is contained in K, see p. 110
$\mathfrak{M}$	the Mellin transformation, see p. 112
$M^\perp$	the annihilator of M, see p. 117

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Zusammenfassung

Der Hauptgegenstand der vorliegenden Arbeit ist die Untersuchung des Hadamardproduktes zweier holomorpher Funktionen, welches als eine gewisse Faltung dieser Funktionen interpretiert werden kann. Die lange Forschungsgeschichte zu diesem Thema geht zurück auf Jacques Hadamard. In seinem Artikel "Théorème sur les séries entières" (vgl. [Ha]) betrachtete er die folgende Frage: Gegeben sind zwei Potenzreihen $\sum_{\nu=0}^{\infty} a_\nu z^\nu$ und $\sum_{\nu=0}^{\infty} b_\nu z^\nu$. Welche Informationen über die Singularitäten der *Hadamardschen Produktreihe* $\sum_{\nu=0}^{\infty} a_\nu b_\nu z^\nu$ kann man aus der Kenntnis der Singularitäten der Ausgangsreihen ableiten? In den darauffolgenden Jahrzehnten trugen namhafte Mathematiker zur Weiterentwicklung dieser Theorie bei (vgl. beispielsweise [Bo], [Fa], [Pol33], eine Zusammenfassung bietet [Scho]). Es stellte sich als hoch kompliziert heraus, hinreichende Bedingungen dafür anzugeben, dass ein Punkt $\alpha \in \mathbb{C}$ tatsächlich singulärer Punkt der Hadamardschen Produktreihe ist.

Im geschichtlichen Verlauf rückte die folgende Frage in den Mittelpunkt der einschlägigen Forschung: Gegeben seien zwei Funktionen φ und f, die holomorph sind in offenen Mengen Ω und U, welche den Ursprung enthalten. Die Potenzreihenentwicklungen um den Nullpunkt seien gegeben durch $\varphi(z) = \sum_{\nu=0}^{\infty} \varphi_\nu z^\nu$ und $f(z) = \sum_{\nu=0}^{\infty} f_\nu z^\nu$. Was ist ein mögliches Holomorphiegebiet für die Funktion $\varphi * f$, die in der Nähe des Ursprungs definiert ist durch $(\varphi * f)(z) := \sum_{\nu=0}^{\infty} \varphi_\nu f_\nu z^\nu$? Die originalen Arbeiten von Hadamard können als erste Ergebnisse in dieser Richtung angesehen werden für den Fall, dass sowohl Ω als auch U bezüglich des Ursprungs sternförmige Teilmengen der komplexen Ebene sind. Ist die abgeschlossene Einheitskreisscheibe in beiden Mengen enthalten, so stimmen die Taylorkoeffizienten φ_ν und f_ν mit den Fourierkoeffizienten der Abbildungen $t \mapsto \varphi(e^{it})$ und $t \mapsto f(e^{it})$ $(t \in [0, 2\pi])$ überein. Daher ist die Idee, die Funktion $\varphi * f$ unter geeigneten Voraussetzungen als ein gewisses Faltungsintegral zu definieren. Müller und in einem darauf folgenden Artikel Grosse-Erdmann zeigten, dass das *Hadamardprodukt*

$$(\varphi * f)(z) := (\varphi *_{\Omega,U} f)(z) := \frac{1}{2\pi i} \int_\Gamma f(\zeta)\varphi(\frac{z}{\zeta})\frac{d\zeta}{\zeta} \quad (z \in \Omega * U)$$

(wobei $\Gamma = \Gamma_z$ ein geeigneter Integrationszyklus ist) die analytische Fortsetzung der Reihe $\sum_{\nu=0}^{\infty} \varphi_\nu f_\nu z^\nu$ in die Menge $\mathbb{C} \setminus \big((\mathbb{C} \setminus \Omega) \cdot (\mathbb{C} \setminus U)\big)$ ist (vgl. [Mue92] und [GE]). Diese Aussage wird auch Hadamardscher Multiplikationssatz genannt. Müller und Pohlen zeigten, dass unter recht allgemeinen Voraussetzungen an die Mengen Ω und U als Teilmengen der Riemannschen Sphäre $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ (wobei wir stets annehmen, dass holomorphe Funktionen im Punkt ∞ verschwinden, falls dieser zum Holomorphiegebiet gehört) die Funktion $\varphi * f$ definiert über obiges Faltungsintegral holomorph ist in der Menge

$$\Omega * U := \mathbb{C}_\infty \setminus \big((\mathbb{C}_\infty \setminus \Omega) \cdot (\mathbb{C}_\infty \setminus U)\big)$$

und, falls $0 \in \Omega \cap U$, der Hadamardsche Multiplikationssatz auch in diesem Kontext gültig bleibt (vgl. [MP]). Es stellt sich heraus, dass der springende Punkt für den Übergang von ebenen Mengen Ω, U, die den Ursprung enthalten, zu Teilmengen von $\mathbb{C}_\infty$, die nicht notwendigerweise den Ursprung enthalten, eine geeignete Verallgemeinerung des Integrationszyklus Γ_z ist.

Alternativ kann das Hadamardprodukt zweier holomorpher Funktionen auch als Verallgemeinerung der Cauchyschen Integralformel aufgefasst werden. Die Funktion $\Theta(z) := 1/(1-z)$ ist holomorph in $\mathbb{C}_\infty \setminus \{1\}$ und die Cauchysche Integralformel besagt, dass für eine beliebige offene Menge $U \subset \mathbb{C}$ und für alle in U holomorphen Funktionen f gilt

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_z} \frac{f(\zeta)}{\zeta - z}\, d\zeta = (\Theta * f)(z) \quad (z \in U)\,.$$

Also kann das Hadamardprodukt als eine Verallgemeinerung dieser Formel im Hinblick auf die zugrunde liegende Menge Ω und die dort definierte Funktion φ angesehen werden.

Nach einigen einführenden Bemerkungen werden in Kapitel 2 gewisse bereits bekannte Eigenschaften des Hadamardproduktes aufgelistet. Darüber hinaus wird gezeigt, dass das Hadamardprodukt unter geeigneten Voraussetzungen assoziativ ist. Dies ist eine wünschenswerte Eigenschaft, jedoch müssen für ihren Beweis eine Reihe technischer Schwierigkeiten überwunden werden. Ausgehend von der klassischen Köthedualität wird gezeigt, dass der Dualraum $H'(D)$ (wobei $D \subset \mathbb{C}$ offen ist) topologisch isomorph ist zum Raum der Keime holomorpher Funktionen auf $1/D^C$, das heißt zu jedem Funktional $u \in H'(D)$ existiert genau ein Keim $[(g, U)]_{1/D^C} \in H(1/D^C)$ so, dass für alle $f \in H(D)$ gilt $u(f) = (f *_{D,U} g)(1)$.

Der Hauptteil dieser Arbeit ist der Untersuchung des *Hadamardschen Faltungsoperators* (oder kurz: des *Hadamardoperators*)

$$T_\varphi = T_{\varphi,U} : H(U) \to H(\Omega * U),\ f \mapsto \varphi * f$$

für unterschiedliche Mengen $\Omega, U \subset \mathbb{C}_\infty$ und unterschiedliche Funktionen $\varphi \in H(\Omega)$ gewidmet. Müller und Pohlen haben bewiesen, dass dies ein linearer und stetiger Operator ist (wobei die Frécheträume $H(U)$ und $H(\Omega * U)$ mit der Topologie der lokal gleichmäßigen Konvergenz versehen sind; vgl. [MP]). In Kapitel 3 wird die folgende Eigenwerteigenschaft gezeigt: Ist K eine kompakte und konvexe Teilmenge des Streifens $\{z \in \mathbb{C} : \; |\operatorname{Im} z| < \pi\}$ und ist φ eine nichtverschwindende Funktion, die holomorph ist in $\Omega := \mathbb{C}_\infty \backslash e^{-K}$ und ist $U \subset \mathbb{C}\backslash\{0\}$ eine offene Menge mit zusammenhängendem Komplement, dann sind die verallgemeinerten Monome $z \mapsto \exp(\alpha \log_U z)$ Eigenfunktionen des Operators $T_{\varphi,U}$ (falls $\Omega * U$ nichtleer ist). Darüber hinaus wird der transponierte Operator $T'_{\varphi,U}$ für relativ allgemeine Mengen Ω und U berechnet. Es stellt sich heraus, dass $T'_{\varphi,U}$ im Wesentlichen wieder ein Hadamardoperator ist, der von derselben Funktion φ induziert wird, aber zwischen den Räumen der Keime holomorpher Funktionen auf $1/(\Omega * U)^C$ und $1/U^C$ abbildet.

Dies motiviert die Idee, den Kern und das Bild von Hadamardoperatoren gleichzeitig zu untersuchen (vgl. Kapitel 4), denn ein klassisches Ergebnis aus der Funktionalanalysis besagt, dass $T_{\varphi,U}$ dichtes Bild hat genau dann, wenn der transponierte Operator injektiv ist. Ergebnisse in dieser Richtung wurden bereits von Frerick erzielt, der Hadamardoperatoren im Fall $\varphi \in H(\mathbb{C}_\infty \setminus \{1\})$ untersuchte und Charakterisierungen für die Surjektivität von $T_{\varphi,U}$ formulierte für den Fall, dass $U \subset \mathbb{C}$ den Ursprung enthält (vgl. [Fre]). Während der Untersuchungen des Bildes von Hadamardoperatoren konzentrieren wir uns auf den Fall, dass U weder den Ursprung noch den Punkt ∞ enthält. Beispielsweise wird gezeigt, dass für eine nichtverschwindende Funktion φ, die in $\Omega := \mathbb{C}_\infty \setminus e^{-K}$ holomorph ist, der zugehörige Hadamardoperator $T_{\varphi,U} : H(U) \to H(\Omega * U)$ dichtes Bild hat für alle offenen Mengen $U \subset \mathbb{C}\backslash\{0\}$, die zusammenhängendes Komplement haben und für die $\Omega * U$ nichtleer ist. Ist allgemeiner Ω ein Gebiet, welches 0 und ∞ enthält, dann hat $T_{\varphi,U}$ dichtes Bild, falls sowohl $U \subset \mathbb{C} \setminus \{0\}$ ein Gebiet ist als auch $\Omega * U$ "klein genug" ist. "Klein" wird dabei durch eine Bedingung, die auf der Maximaldichte der nichtverschwindenden Koeffizienten in den Potenzreihenentwicklungen von φ um 0 und ∞ beruht, ausgedrückt. Schließlich beinhaltet Abschnitt 4.4 ein Surjektivitätskriterium für $T_{\varphi,U}$, welches im Spezialfall $0 \not\equiv \varphi \in H(\mathbb{C}_\infty \setminus \{1\})$ wie folgt lautet: $T_{\varphi,U}$ ist surjektiv für alle offenen Mengen $U \subset \{z \in \mathbb{C} : \; \operatorname{Im} z \neq 0 \text{ falls } \operatorname{Re} z \leq 0\}$, für die $\log U$ konvex ist.

In Kapitel 5 wird der Zusammenhang zwischen Hadamardoperatoren und Faltungsoperatoren mit analytischen Funktionalen erläutert. Ist Ω von der Form $\mathbb{C}_\infty \setminus e^{-K}$, dann stimmen diese beiden Typen von Operatoren modulo einer Verknüpfung mit der Exponentialfunktion überein. Ist der konvexe Träger des analytischen Funktionals die Menge $\{0\}$, so entspricht der zugehörige Faltungsoperator wiederum einem Differenzialoperator unendlicher Ordnung mit konstanten Koeffizienten. Das oben erwähnte Surjektivitätsergebnis für Hadamardoperatoren wird in den Forschungshintergrund der Faltungsoperatoren eingeordnet und es wird ein neuer Beweis zu dem folgenden Resultat gegeben, welches auf Korobeĭnik zurückgeht (vgl. [Kor69]): Ist $0 \not\equiv \Phi$ eine ganze Funktion vom Exponentialtyp null und $G \subset \mathbb{C}$ ein konvexes Gebiet, dann ist erstens der Kern des Differentialoperators unendlicher Ordnung $\Phi(D)$ definiert auf $H(G)$ gegeben durch den Abschluss (in $H(G)$) des linearen Spanns der Funktionen $z \mapsto z^k e^{\alpha z}$, wobei α eine $m-$fache Nullstelle von Φ ist und $k \leq m-1$ und zweitens ist $\Phi(D)$ surjektiv. Dieses Ergebnis wiederum erlaubt es, das Surjektivitätsresultat für Hadamardoperatoren für den Spezialfall $\varphi \in H(\mathbb{C}_\infty \setminus \{1\})$ zu verbessern: Ist $0 \not\equiv \varphi \in H(\mathbb{C}_\infty \setminus \{1\})$, dann ist $T_{\varphi,U} : H(U) \to H(U)$ surjektiv für alle einfach zusammenhängenden Gebiete $U \subset \mathbb{C} \setminus \{0\}$, für die $\log_U U$ konvex ist.

In Kapitel 6 wird eine zweite Anwendung des Hadamardproduktes besprochen: die lokal gleichmäßige Approximation von holomorphen Funktionen auf offenen Mengen $D \subset \mathbb{C}$ durch Polynome. Der bekannte Rungesche Approximationssatz besagt, dass es genau dann möglich ist, jede in D holomorphe Funktion durch Polynome zu approximieren, wenn $\mathbb{C}_\infty \setminus D$ zusammenhängend ist.

Wir betrachten die folgende Frage: Gegeben sei eine Menge $\Lambda \subset \mathbb{N}_0$; unter welchen Voraussetzungen an die offene Menge $D \subset \mathbb{C}$ kann jede Funktion $g \in H(D)$ lokal gleichmäßig durch Polynome approximiert werden, deren Exponenten ausschließlich der Menge Λ angehören, das heißt durch *Lückenpolynome*? Ergebnisse dieser Art können als Verallgemeinerung des Rungeschen Approximationssatzes angesehen werden. Der bekannte Satz von Müntz für reelle Intervalle kann als Ausgangspunkt für die Lückenapproximation angesehen werden. In der Literatur findet sich eine Reihe von Ergebnissen zur oben gestellten Frage, vgl. beispielsweise [An], [AM], [DK], [GLM], [LMM98], [LMM02] und [MR] (für weitere Literaturangaben siehe [GLM]). Die Beweise stützen sich typischerweise auf Dualität, genauer gesagt auf den Satz von Hahn-Banach. Dies ist zweifeslsfrei eine elegante Beweismethode, jedoch hat sie einen eher nicht-konstruktiven Charakter.

Der Ansatz, der in Kapitel 6 vorgestellt wird, beruht auf einer geeigneten Anwendung der erzielten Ergebnisse über das Bild von Hadamardoperatoren und liefert neue Beweise zu Sätzen über Lückenapproximation. So wird beispielsweise ein kurzer Beweis des folgenden Resultates von Arakelian und Martirosian gegeben (vgl. [AM]): Hat $\Lambda \subset \mathbb{N}_0$ Dichte eins und hat $D \subset \mathbb{C} \setminus \{0\}$ zusammenhängendes Komplement, dann kann jede in D holomorphe Funktion lokal gleichmäßig auf D durch Polynome approximiert werden, deren Exponenten der Menge Λ angehören.

Darüber hinaus erlaubt der hier vorgestellte Ansatz, auch Aussagen über die

Approximationsgüte auf kompakten Teilmengen von D bei der Approximation mit Lückenpolynomen zu erzielen. Es stellt sich heraus, dass die Definition des Hadamardproduktes als Faltungsintegral erlaubt, Informationen über die geometrische Approximationsrate bei Lückenapproximation aus derjenigen bei der Verwendung beliebiger Polynome abzuleiten. Letztere Größe wurde eingehend untersucht (so z. B. in [Gai]). Wir erhalten eine obere Schranke für die geometrische Approximationsrate durch Lückenpolynome, welche im Allgemeinen nicht verbessert werden kann.