Splitting theory for PLH spaces

Dissertation

zur Erlangung des akademischen Grades eines Doktors der
Naturwissenschaften (Dr. rer. nat.)

Dem Fachbereich IV der Universität Trier
vorgelegt von

Bernhard Dierolf

Gutachter:
Prof. Dr. L. Frerick
Prof. Dr. J. Wengenroth
Prof. Dr. D. Vogt

März 2014

Abstract

In splitting theory of locally convex spaces we investigate evaluable characterizations of the pairs (E, X) of locally convex spaces such that each exact sequence

$$0 \longrightarrow X \overset{f}{\longrightarrow} G \overset{g}{\longrightarrow} E \longrightarrow 0 \qquad\qquad (\star)$$

of locally convex spaces *splits*, i.e. either f has a continuous linear left inverse or g has a continuous linear right inverse. In the thesis at hand we deal with splitting of short exact sequences of so-called PLH spaces, which are defined as projective limits of strongly reduced spectra of strong duals of Fréchet-Hilbert spaces. This class of locally convex spaces contains most of the spaces of interest for application in the theory of partial differential operators as the space of Schwartz distributions $\mathscr{D}'(\Omega)$, the space of real analytic functions $\mathscr{A}(\Omega)$ and various spaces of ultradifferentiable functions and ultradistributions, cf. [Dom04]. It also contains non-Schwartz spaces as $\mathscr{D}_{L_2}$ and the $\mathrm{B}_{2,k}^{\mathrm{loc}}(\Omega)$ spaces that are not covered by the current theory for PLS spaces, cf. e.g. [BD06, BD08, Dom10]. We prove a complete characterizations of the above problem in the case of X being a PLH space and E either being a Fréchet-Hilbert space or a strong dual of one by conditions of type (T) respectively (T_ε), cf. [Dom10], which gives an answer to a variation of [BD06, problem (9.9)]. To this end, we establish the full homological toolbox of Yoneda Ext functors in exact categories for the category of PLH spaces including the long exact sequence, which in particular involves a thorough discussion of the proper concept of exactness. Furthermore, we exhibit the connection to the parameter dependence problem via the Hilbert tensor product for hilbertizable locally convex spaces. We show that the Hilbert tensor product of two PLH spaces is again a PLH space which in particular proves the positive answer to Grothendieck's *problème des topologies*, cf. [Gro55]. In addition to that we give a complete characterization of the vanishing of proj[1] for tensorized PLH spectra if one of the PLH spaces E and X meets some nuclearity assumptions. To apply our results to concrete cases we establish sufficient conditions of (DN)-(Ω) type due to [BD07], and apply them to the parameter dependence problem for partial differential operators with constant coefficients on $\mathrm{B}_{2,k}^{\mathrm{loc}}(\Omega)$ spaces as well as to the parameter dependence problem for $\mathscr{D}_{L_2}$. Concluding we give a complete solution of all the problems under consideration for PLH spaces of Köthe type.

Zusammenfassung

Motiviert durch die Frage nach der Existenz von linearen und stetigen Lösungsoperatoren zu partiellen Differentialoperatoren mit konstanten Koeffizienten, beschäftigt sich die Splitting Theorie lokalkonvexer Räume mit der Charakterisierung der Raumpaare (E, X) lokalkonvexer Räume, sodass jede kurze exakte Sequenz

$$0 \longrightarrow X \xrightarrow{f} G \xrightarrow{g} E \longrightarrow 0 \qquad\qquad (\star)$$

lokalkonvexer Räume *zerfällt*, d.h. entweder f eine lineare und stetige Linksinverse besitzt oder, äquivalent, die Abbildung g eine lineare und stetige Rechtsinverse. In der vorliegenden Arbeit wird diese Fragestellung für die Kategorie sogenannter PLH Räume untersucht. Alle für die Anwendung in der Lösungstheorie von partiellen Differentialgleichungen nach Hörmander relevanten Räume der Funktionalanalysis, wie der Raum der Schwartz'schen Distributionen $\mathscr{D}'(\Omega)$, der Raum der reell analytischen Funktionen $\mathscr{A}(\Omega)$ und viele Räume ultradifferenzierbarer Funktionen und Ultradistributionen sind dieser Kategorie zuzuordnen; erlauben also eine Darstellung als projektiver Limes eines stark reduzierten Spektrums von starken Dualen von Fréchet-Hilbert Räumen. Außerdem enthält diese Kategorie auch viele Beispiele von Funktionen- und Distibutionenräumen, wie den $\mathscr{D}_{L_2}$ und die $B_{2,k}^{\mathrm{loc}}(\Omega)$ Räume, die nicht Schwartz sind, also nicht von der aktuellen Theorie für PLS Räume, man vergleiche beispielsweise [BD06, BD08, Dom10], erfasst werden. Wir geben eine vollständige Charakterisierung mittels einer Bedingung vom Typ (T_ε), vgl. [Dom10], für das Zerfallen aller Sequenzen $(\star)$ in PLH für Fréchet-Hilbert oder LH Räume E und PLH Räume X, was eine Antwort auf eine Variation von [BD06, Problem (9.9)] gibt. Dazu werden alle homologischen Methoden, insbesondere die lange exakte Kohomologie-Sequenz und die Verbindung von $\mathrm{Ext}_{\mathrm{PLH}}^1$ und proj^1 für die Yoneda Ext Gruppen in Analogie zum PLS Raum Fall [Sie10] bereitgestellt. Insbesondere wird die Wahl der richtigen Exaktheitstruktur in PLH diskutiert. Über das Hilbert Tensor Produkt für hilbertisierbare lokalkonvexe Räume stellen wir den Zusammenhang zu dem Problem der Parameterabhängigkeit von Lösungen her und weisen nach, dass das Hilbert Tensor Produkt von zwei PLH Räumen wieder ein PLH Raum ist. Dazu lösen wir insbesondere Grothendieck's *problème des topologies*, vgl. [Gro55], positiv für Fréchet-Hilbert Räume und das Hilbert Tensor Produkt. Schließlich geben wir eine vollständige Charakterisierung des Verschwindens von proj^1 für tensorierte PLH Spektren wieder mittels einer Bedingung vom Typ (T_ε) im Falle, dass einer der beteiligten PLH Räume gewissen Nuklearitätsbedingungen genügt. Um die charakterisierenden Bedingungen auf konkrete Fälle anwenden zu können, weisen wir hinreichende Bedingungen vom Typ (DN)-(Ω) für PLH Räume nach und wenden diese an sowohl auf das Problem der Parameterabhängigkeit von Lösungen für partielle Differentialoperatoren mit konstanten Koeffizienten in $B_{2,k}^{\mathrm{loc}}(\Omega)$ Räumen, als auch auf das Problem der $\mathscr{D}_{L_2}$ Parameterabhängigkeit. Schließlich geben wir eine vollständige Charakterisierung aller behandelten Fragestellungen für das Beispiel der Köthe PLH Räume.

Acknowledgments

Contents

1 Introduction

In splitting theory of locally convex spaces we investigate evaluable characterizations of the pairs (E, X) of locally convex spaces such that each exact sequence

$$0 \longrightarrow X \overset{f}{\longrightarrow} G \overset{g}{\longrightarrow} E \longrightarrow 0 \qquad\qquad (\star)$$

of locally convex spaces *splits*, i.e. either f has a continuous linear left inverse or g has a continuous linear right inverse. The notion of exactness is one of the aspects considered in chapter 2, for now the reader may picture a linear continuous and surjective operator g between locally convex spaces G and E with the continuous inclusion of $X = g^{-1}(\{0\})$ via f into the domain G.

Since the seventies of the last century this problem has been studied extensively in the context of Fréchet and locally convex spaces and can be considered complete for these categories, a short survey is given at the beginning of chapter 3. The problem that motivates investigating splitting theory in other categories than those two occurs, when considering many classical spaces of functional analysis that are important for the theory of partial differential equations. This concerns particularly the space of Schwartz distributions $\mathscr{D}'(\Omega)$, the space of real analytic functions $\mathscr{A}(\Omega)$ and various spaces of ultradifferentiable functions and ultradistributions, cf. [Dom04]. Those are far from metrizable, hence the classical theory for Fréchet spaces is not applicable. Furthermore, the locally convex theory does not yield satisfactory results: We simply do not know, whether each exact sequence

$$0 \longrightarrow \mathscr{D}'(\Omega) \overset{f}{\longrightarrow} G \overset{g}{\longrightarrow} \mathscr{D}'(\Omega) \longrightarrow 0$$

of locally convex spaces splits. This leads us to the quest for a suitable subcategory of the locally convex spaces, i.e. a class that is on the one hand big enough to contain the function and distribution spaces mentioned above but on the other hand small enough to yield reasonable splitting results. The natural structure of the examples mentioned above – they all are projective limits of (strongly reduced) spectra of complete separated LB spaces – provides suggestions: Although it contains all of the non-metrizable examples, the category PLN of projective limits of (strongly reduced) spectra of strong duals of nuclear Fréchet spaces does not seem appropriate as it only contains strongly nuclear Fréchet spaces, leaving important examples such as the space s of rapidly decreasing sequences unattended. This led to the implementation of the category of so-called PLS spaces, i.e. projective limits of (strongly reduced) spectra of strong duals of Fréchet Schwartz spaces. Splitting theory in this category has been subject to recent investigation and considerable results have been achieved under certain assumptions about nuclearity and being Köthe PLS spaces, cf. e.g. [BD06, BD08, Dom10]. A more detailed survey will be given in the introductions of chapters 3 and 4. Nevertheless, several spaces are not covered by the PLS space approach: In particular, the Fréchet-Hilbert space $\mathscr{D}_{L_2}$ of all $\mathscr{C}^\infty$ functions all the derivatives of which are in L_2 and the $B_{2,k}^{\mathrm{loc}}(\Omega)$ spaces, cf. [Hör05, chap. 10], as well as the PLH spaces of Köthe type. Furthermore, the question arises whether the nuclearity assumptions or those about compactness can be omitted or softened,

cf. [BD06, problem (9.9)].

This leads us to the category PLH of projective limits of strongly reduced spectra of strong duals of Fréchet-Hilbert spaces, a class of locally convex spaces that not only contains all the spaces of interest for applications covered so far but also the new ones mentioned before. We address the problem in four steps:

In chapter 2 we will establish the complete homological toolbox, in particular the long exact sequence we had access to in splitting theory for Fréchet spaces via Retakh's and Palamodov's approach, cf. [Pal72, Ret70, Pal71], also in the category of PLH spaces via the Yoneda Ext functors. To this end we will follow Sieg's approach [Sie10] using the theory of *exact structures*, cf. [Büh10]. Essential to this approach is the determination of a certain class of kernel-cokernel pairs, which has to fulfill the technical requirements of an exact structure to facilitate the homological toolbox. Here the main difference to PLS emerges: While Sieg proved [Sie10] that the class of *short topologically exact sequences* is the maximal exact structure in PLS, the category PLH is not subspacestable, which leads us to a significantly larger maximal exact structure involving well-located subspaces in (2.2.17) and several delimiting examples as the category of LB spaces, which we pick up alongside. This allows us to prove the isomorphy of $\mathrm{Ext}^1_{\mathrm{PLH}}$ and proj^1 in a certain spectrum of operator spaces under mild assumptions as local splitting in (2.3.6) in analogy to [Sie10, ((5.2))]. Furthermore, we will prove in (2.2.18) that being a PLH space is not a three-space property in the category of locally convex spaces, a fact that remains unknown for PLS spaces. Finally we will establish the splitting theory for the space of distributions in (2.3.7) in analogy to [Sie10, (5.3)], cf. [DV00b, Wen01] , i.e. that $\mathrm{Ext}^1_{\mathrm{PLH}}(E, X)$ vanishes for subspaces E and complete Hausdorff quotients X of $\mathscr{D}'(\Omega)$. Note that we will show the results for the even larger class of PLS_w spaces, where we replace "Hilbert space" in the definition of PLH by "reflexive Banach space".

In chapter 3 we will prove a characterization of the vanishing of the first derivative of the projective limit functor in the spectrum of operator spaces $\mathscr{L} := \left(\mathrm{L}\left(E, X_N\right), X_M^{N\,*} \right)$ for Fréchet-Hilbert (3.4.5) or LH spaces E (3.5.4) and PLH spaces X if all spaces are deeply reduced. The characterization is given in terms of some inequality preceded by a long sequence of quantifiers, see conditions (T) and (T_ε) in (3.2.1) and is achieved without any nuclearity assumptions, thus giving an answer to a variation of [BD06, problem (9.9)]. The long and technical proof relies on duality: We will in fact characterize weak acyclicity for some LF spectrum of projective tensor products that is pre-dual to $\mathscr{L}$. This characterization will use interpolation for nuclear operators due to Domański and Mastyło in [DM07] in the variant (3.2.6) due to Vogt [Vog11], as well as a slightly enhanced characterization of condition (M) of Retakh essentially due to Wengenroth [Wen95, Wen96]. The connection between $\mathrm{Ext}^1_{\mathrm{PLH}}(E, X)$ and $\mathrm{proj}^1 \mathscr{L}$ in the above situation that we established in 2 will allow us to use the characterizations obtained to the vanishing of $\mathrm{Ext}^1_{\mathrm{PLH}}(E, X)$. This generalizes Domański's and Mastyło's splitting result for Fréchet-Hilbert spaces [DM07] and constitutes the PLH analogons of the splitting results for PLS spaces due to Bonet and Domański [BD06, BD08].

Chapter 4 is motivated by the following parameter dependence problem: Given a linear partial differential equation with constant coefficients

$$P(D)u_\lambda = f_\lambda,$$

where the family $(f_\lambda)_{\lambda \in I}$ of distributions, ultradistributions, real analytic functions or ultra-differentiable functions depends in a certain sense "nicely" on the parameter $\lambda \in I$, is there a family of solutions $(u_\lambda)_{\lambda \in I}$ depending equally "nice" on the parameter λ?

This research has its roots in an even more general setting, where for instance also $P(D)$ depends on $\lambda \in I$, cf. [Bro62, Trè62a, Trè62b, Man90, Man91, Man92], cf. [BD06, p. 563] respectively [Dom10, p. 1 & 2] for details. The tensor product representation of various function spaces, cf. [Sch58, Gro55, DF93, Var02, Var07] connects this problem to the problem of surjectivity of tensorized maps:

Given a locally convex space E and a quotient map $q \colon Y \longrightarrow Z$ between locally convex spaces Y and Z, under which conditions will the induced operator $\mathrm{id}_E \tilde{\otimes}_\alpha q \colon E \tilde{\otimes}_\alpha Y \longrightarrow E \tilde{\otimes}_\alpha Z$ be surjective for $\alpha = \pi$ or $\alpha = \varepsilon$?

In the setting of projective spectra of LB spaces this question is closely connected to the vanishing of proj^1 for tensorized spectra, which is subject to investigation in our PLH setting. As even $\ell_2 \tilde{\otimes}_\alpha \ell_2$ is not reflexive for $\alpha \in \{\varepsilon, \pi\}$, cf. [MV97, (16.27)]. Hence, contrary to the PLS situation, we can not expect the local tensor products $E_N \tilde{\otimes}_\alpha X_N, \alpha \in \{\varepsilon, \pi\}$ for two PLH spaces $E = \mathrm{proj}\left(E_T, E_S^T\right)$ and $X = \mathrm{proj}\left(X_N, X_M^N\right)$ to be ultrabornological, reflexive or even an LH space. Thus we introduce the Hilbert tensor topology for hilbertizable locally convex spaces. We start by defining the Hilbert tensor product not only on the tensor product of two Hilbert spaces but directly on the tensor product of vector spaces that are equipped with Hilbert seminorms, so-called *semi-unitary* spaces. This will allow for rather elegant proofs when verifying in the second part of the chapter that the induced tensor topology $\mathcal{T}_\sigma$ commutes with the LH space structure, i.e. that the completed σ tensor product of two LH spaces is the LH space arising from the σ tensor product of the two LH space spectra giving rise to the original LH spaces in (4.2.9). This is the crucial step in demonstrating that $\mathcal{T}_\sigma$ shows the same compatibility concerning the PLH space structure as do the ε and π topology towards the PLS space structure, cf. [Pis10]. Other by-products will be the positive solution of Grothendieck's *problème des topologies*, cf. [Gro55], for $\mathcal{T}_\sigma$ and Fréchet-Hilbert spaces and the exactness of the induced tensor product functor on the category of Fréchet-Hilbert spaces and on the category of LH spaces. In the last part we will characterize the vanishing of proj^1 for the tensorized spectrum of both a nuclear Fréchet space and a PLN space E with a PLH space X under mild assumptions with our condition (T_ε), thus proving Bonet and Domański's result [BD06, BD08, Dom10] in the category of PLH spaces. Note that since we can not interpolate Hilbert-Schmidt operators in the spirit of (3.2.6), we can not forgo the nuclearity assumptions.

In the final chapter 5 of this thesis we want to consider concrete applications of the structure theory established in the preceding chapters. In the first part we divide the condition (T_ε) for the pair (E, X) into two conditions of (DN)-(Ω) type for PLH spaces on E and X respectively and prove sufficiency in complete analogy to [Dom10]. These variants of the *dual interpolation estimate* (DIE) are due to Bonet and Domański [BD07]. In the rest we consider the examples mentioned at the beginning: We prove a result about parameter dependence of linear partial differential operators with constant coefficients $P(D)$ on $\mathrm{B}_{2,\kappa}^{\mathrm{loc}}(\Omega)$ and show that for elliptic partial differential operators $P(D)$ on $\mathrm{B}_{2,\kappa}^{\mathrm{loc}}(\Omega)$ admitting a continuous linear right inverse is a lot more than having a parameter dependence for a PLN space E with (DIE) for big θ. Considering $\mathscr{D}_{L_2}$ and its strong dual, we prove that partial differential operators with constant

coefficients on $\mathscr{D}'(\Omega)$ for open and convex subsets $\Omega \subset \mathbb{R}^d$ as well as surjective convolution operators T_μ on $\mathscr{D}'_{(\omega)}\left(\mathbb{R}^d\right)$ have the corresponding parameter dependence. We conclude by giving a complete characterization of all the problems under consideration for Köthe PLH spaces in terms of the matrices without any nuclearity assumptions.

Concerning functional analysis, we apply the standard notation, as used for example in [Köt69, Köt79, MV97], in particular for tensor norms and topological tensor products we apply notation as in [DF93] and concerning homological algebra, we refer to [Wen03, Sie10]. Presenting a self-contained thesis is rather out of the question considering the variety of tools we utilize. Be that as it may, we try to limit the premises to reasonable fields: Chapter 2 requires a good acquaintance with homological methods as presented in [Wei94, chap. 1] concerning the fundamentals, [Bou07, §7.5] regarding the Yoneda Ext groups, [Büh10] as to exact structures and [Wen03, chap. 3] referring the functor proj and its derivatives. Chapters 3 and 4 assume insight in Retakh's and Palamodov's theory of regularity of LF spaces concerning the conditions (M) and (M_0) respectively acyclicity and weak acyclicity, cf. [Pal72, Ret70, Pal71] or [Vog92, Wen95] for a survey. Furthermore, a good understanding of the ε and π tensor product of locally convex spaces, cf. [Köt79], and a rudimentary acquaintance with nuclearity, cf. [Pie72], is of advantage. Considering the last chapter, we refer to [Hör03, Hör05] for background and context for the theory of partial differential equations.

2 Homological methods

From the 70's to the 90's of the 20th century splitting theory was primarily considered in the context of Fréchet or locally convex spaces, i.e. in *quasiabelian* categories with many injective objects. In a number of articles [Pal72, Ret70, Pal71] Retakh and Palamodov were able to connect the splitting problem to the vanishing of the first derived functor $\mathrm{Ext}^1(E, \_)$ of $\mathrm{Hom}(E, \_)$. The latter in turn they linked to the vanishing of the first derivative of the projective limit functor in certain spectra of operator spaces via the long exact sequence. As Sieg elaborated in [Sie10, (3.1.5) & (3.1.6)], Palamodov's approach to the derived functors of Hom, cf. [Pal71] respectively [Wen03] for a survey, does not work in the category of PLS spaces, since it is neither semi-abelian nor is it known, whether it does have enough injective objects. Until rectified by Sieg [Sie10] in 2010 this circumstance lead to an *ad-hoc* definition of the vanishing of $\mathrm{Ext}^1_{\mathrm{PLS}}$ via the splitting of short exact sequences, cf. e.g. [Wen03, (5.3)]. Hence the connection between the vanishing of $\mathrm{Ext}^1_{\mathrm{PLS}}$ and proj^1 – which allows for applicable characterizations of splitting – had to be implemented manually, cf. [BD06, section 3] and [BD08, (3.4)]. In this chapter we will prove that Sieg's methods to establish the long exact sequence for the Yoneda Ext functor in exact categories can be applied to our categories of interest – PLH and the even larger variant $\mathrm{PLS_w}$ – as well, although both are not stable with respect to the formation of closed subspaces. We will start by recalling the categorical setting and prove some auxiliary results in the category of Hausdorff locally convex spaces in section 2.1. Using the concept of *well-located* subspaces and some lifting properties for pushouts in the category of Hausdorff locally convex spaces $\overline{(\mathrm{LCS})}$, we will determine the maximal exact structure in the category of LB spaces in section 2.2. This leads us to the maximal exact structures in $\mathrm{PLS_w}$ and PLH via the dual version of Grothendieck's factorization theorem due to Vogt [Vog89]. Based on the results in 2.2 we construct the Yoneda Ext groups in the categories LH and PLH as well as in $\mathrm{LS_w}$ and $\mathrm{PLS_w}$ in 2.3 and establish in (2.3.6) the splitting results in these categories in complete analogy to the PLS case, cf. [Sie10, chap. 5]: We prove that $\mathrm{Ext}^1(E, X) = \mathrm{proj}^1\left(\mathrm{L}(E, X_N), X_M^{N\,*}\right)$ in $\mathrm{PLS_w}$ and PLH under the premise of local splitting and the vanishing of $\mathrm{proj}^1 X$. This result will be the key to apply the results of section 3 to splitting theory. Concluding we will give a first application showing in (2.3.7) that the space of Schwartz distributions $\mathscr{D}'(\Omega)$ plays the same role in $\mathrm{PLS_w}$ and PLH as does the space s of rapidly decreasing sequences in the category of Fréchet(-Hilbert) spaces. More precise we will transfer Sieg's new proof with minor adaptions for a splitting theorem due to Wengenroth [Wen01] for the space of distributions, which was itself an improvement of a result due to Domański and Vogt [DV00b, (2.19) & (3.1)], to our categories.

2.1 Exact categories – notions and auxiliary results

Definition and Remark 2.1.1. Our vantage point is a *pre-additive* category, i.e. a category with a zero object and an addition on each set of morphisms, endowing the sets with abelian group structures, that makes the composition of morphisms biadditive. A pre-additive category $\mathcal{C}$ is called *additive* if all finite biproducts exist, i.e. for every finite family of objects $(X_1, \ldots, X_n)$ in $\mathcal{C}$ there is a an object $\prod_{1 \le j \le n} X_j = \bigoplus_{1 \le j \le n} X_j$ and there are morphisms

$\omega_{X_k} \colon X_k \longrightarrow \bigoplus_{1 \le j \le n} X_j$ and $\pi_{X_k} \colon \prod_{1 \le j \le n} X_j \longrightarrow X_k, 1 \le k \le n$ such that $\left(\prod_{1 \le j \le n} X_j, (\pi_{X_k})_{1 \le k \le n} \right)$ is a *product* of $(X_1, \ldots, X_n)$ and $\left(\bigoplus_{1 \le j \le n} X_j, (\omega_{X_k})_{1 \le k \le n} \right)$ is a *coproduct* of $(X_1, \ldots, X_n)$. Such an additive category C is called *pre-abelian* if every morphism in C has a kernel and a cokernel. An important property of pre-abelian categories is not only the existence of pullbacks and pushouts but that we can even determine a convenient version of them:

- The pullback of a diagram $\begin{array}{c} S \\ \downarrow s \\ Y \xrightarrow{g} Z \end{array}$ is given by the triple (P, p_Y, p_S) which consists of the kernel object P of the kernel k of the morphism $(g, -s) \colon Y \times S \longrightarrow Z$ together with the two morphisms $p_Y = \pi_Y \circ \mathrm{k}$ and $p_S = \pi_S \circ \mathrm{k}$.

- The pushout of a diagram $\begin{array}{c} X \xrightarrow{f} Y \\ {\scriptstyle s}\downarrow \\ S \end{array}$ is given by the triple (Q, q_Y, q_S) which consists of the cokernel object Q of the cokernel ck of the morphism $\left(\begin{smallmatrix} f \\ -s \end{smallmatrix} \right) \colon X \longrightarrow Y \times S$ together with the two morphisms $q_Y = \mathrm{ck} \circ \omega_Y$ and $q_S = \mathrm{ck} \circ \omega_S$.

As all of the previous notions are defined by universal properties, they are only unique except for canonical isomorphisms. Be that as it may, we will forgo long-winded formulations and *par abus* we will speak about *the* kernel, *the* product, etc.

Definition and Remark 2.1.2. Let C be an additive category. A pair (f, g) of composable morphisms in C is called a *kernel-cokernel-pair* if f is the kernel of g and g is the cokernel of f. If a class $\mathcal{E}$ of kernel-cokernel pairs in C is fixed, then a morphism $f \colon X \longrightarrow Y$ is called an *admissible kernel* if there is a morphism $g \colon Y \longrightarrow Z$ in C such that (f, g) is in $\mathcal{E}$. Dually a morphism $g \colon Y \longrightarrow Z$ in C is called an *admissible cokernel* if there is a morphism $f \colon X \longrightarrow Y$ in C such that (f, g) is in $\mathcal{E}$. A class $\mathcal{E}$ of kernel-cokernel-pairs in C is an *exact structure* if it is closed under isomorphisms and the following hold:

- (*E*0) id_X is an admissible cokernel for all $X \in C$.
- (*E*0)$^{\mathrm{op}}$ id_X is an admissible kernel for all $X \in C$.
- (*E*1)   If allowed, the composition of two admissible cokernels is an admissible cokernel.
- (*E*1)$^{\mathrm{op}}$ If allowed, the composition of two admissible kernels is an admissible kernel.
- (*E*2)   For all $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{E}$ the pushout $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle q_Y} \\ S & \xrightarrow{q_S} & Q \end{array}$ exists for all morphisms s

  and q_S is an admissible kernel.

$(E2)^{\text{op}}$ For all $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{E}$ the pullback $\begin{array}{ccc} P & \xrightarrow{p_S} & S \\ {\scriptstyle p_Y}\downarrow & & \downarrow{\scriptstyle s} \\ Y & \xrightarrow{g} & Z \end{array}$ exists for all morphisms s

and p_S is an admissible cokernel.

An *exact category* $(C, \mathcal{E})$ is an additive category C together with an exact structure $\mathcal{E}$. The kernel-cokernel pairs in $\mathcal{E}$ are called *short exact sequences*. A pre-abelian category C is *quasi-abelian* if and only if the maximal exact structure coincides with the class of all kernel-cokernel pairs (see [SW11, (3.3)]). The definition of an exact category is self-dual, i.e. $(C, \mathcal{E})$ is an exact category if and only if $(C^{\text{op}}, \mathcal{E}^{\text{op}})$ is an exact category.

Given a pre-abelian category C we always have a minimal exact structure, i.e. an exact structure that is contained in any other exact structure in C, the class of short exact sequences that are *split-exact*, i.e. the class of kernel-cokernel pairs (f, g) in C such that g has a right inverse (see [Büh10, Lemma (2.7) & Remark (2.8)]). In [SW11] Sieg and Wegner proved that every pre-abelian category C admits an exact structure $\mathcal{E}_{\text{max}}^C$ that is maximal in the sense that it contains every other exact structure in C. To formulate their description of this maximal exact structure we need the following notation:

Definition 2.1.3. Let C be a pre-abelian category.

i) A cokernel $g \colon Y \longrightarrow Z$ in C is said to be *semi-stable*, if for every morphism $s \colon S \longrightarrow Z$ in C the morphism p_S in the pullback diagram $\begin{array}{ccc} P & \xrightarrow{p_S} & S \\ {\scriptstyle p_Y}\downarrow & & \downarrow{\scriptstyle s} \\ Y & \xrightarrow{g} & Z \end{array}$ is also a cokernel in C.

ii) A kernel $f \colon X \longrightarrow Y$ in C is said to be *semi-stable* if for every morphism $s \colon X \longrightarrow S$ in C the morphism q_S in the pushout diagram $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle q_Y} \\ S & \xrightarrow{q_S} & Q \end{array}$ is also a kernel in C.

Theorem 2.1.4. *If C is a pre-abelian category, then the class $\mathcal{E}_{\text{max}}^C$, which consists of all kernel-cokernel pairs (f, g) in C such that f is a semi-stable kernel and g is a semi-stable cokernel, is an exact structure in C. Moreover, it is maximal in the sense that it contains any other exact structure in C.*

Proof. See [SW11, (3.3)]. □

Remark 2.1.5. With (2.1.2) it is easy to verify that

$$\left(\mathcal{E}_{\text{max}}^C \right)^{\text{op}} = \mathcal{E}_{\text{max}}^{C^{\text{op}}}$$

holds in any pre-abelian category C.

Theorem (2.1.4) allows us to fix a distinguished exact structure in every pre-abelian category:

Definition 2.1.6. We endow every pre-abelian category C with its maximal exact structure $\mathcal{E}_{\max}^C$ obtaining the exact category $\left(C, \mathcal{E}_{\max}^C\right)$. We call an element (f, g) $\left(\text{or } X \xrightarrow{f} Y \xrightarrow{g} Z\right.$ or $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0\left.\right)$ of $\mathcal{E}_{\max}^C$ a *short exact sequence of C objects* or a *sequence exact in C*.

Let us close this first part about exact categories with the following well-known pullback and pushout constructions, which we will use frequently throughout the rest of this chapter. The proof makes use of [RW77, Theorem 5]) and the axioms defining an exact structure $(E2)$ respectively $(E2)^{\mathrm{op}}$, cf. (2.1.2), also see [Sie10, (2.1.2)].

Lemma 2.1.7. *Let $(C, \mathcal{E})$ be an exact category.*

  i) *If $g\colon Y \longrightarrow Z$ and $s\colon S \longrightarrow Z$ are morphisms in C and (P, p_Y, p_S) is their pullback, then there is a morphism $\varphi\colon \ker(g) \longrightarrow P$ making the diagram*

$$
\begin{array}{ccccc}
\ker(g) & \xrightarrow{\ \varphi\ } & P & \xrightarrow{\ p_S\ } & S \\
\Big\| & & \Big\downarrow{\scriptstyle p_Y} & & \Big\downarrow{\scriptstyle s} \\
\ker(g) & \xrightarrow[\ k_g\ ]{} & Y & \xrightarrow[\ g\ ]{} & Z
\end{array}
$$

  commutative and being a kernel of p_S.

  If $g\colon Y \longrightarrow Z$ is an admissible cokernel, then (φ, p_S) is an element of $\mathcal{E}$.

  ii) *If $f\colon X \longrightarrow Y$ and $s\colon X \longrightarrow S$ are morphisms in C and (Q, q_Y, q_S) is their pushout, then there is a morphism $\psi\colon Q \longrightarrow \operatorname{coker}(f)$ making the diagram*

$$
\begin{array}{ccccc}
X & \xrightarrow{\ f\ } & Y & \xrightarrow{\ \mathrm{ck}_f\ } & \operatorname{coker}(f) \\
\Big\downarrow{\scriptstyle s} & & \Big\downarrow{\scriptstyle q_Y} & & \Big\| \\
S & \xrightarrow[\ q_S\ ]{} & Q & \xrightarrow[\ \psi\ ]{} & \operatorname{coker}(f)
\end{array}
$$

  commutative and being a cokernel of q_S.

  If $f\colon X \longrightarrow Y$ is an admissible kernel, then (q_S, ψ) is an element of $\mathcal{E}$.

As a transition to the functional analytic part of this section let us consider Palamodov's notion of exactness, cf. [Pal71]:

Remark 2.1.8. Palamodov calls a sequence $\to X \xrightarrow{f} Y \xrightarrow{g} Z \to$ of objects and morphisms in a semi-abelian category C exact at Y if the image of f is a kernel of g and the coimage of g is a cokernel of f and both morphisms are homomorphisms. In quasi-abelian categories

this notion coincides with our notion of exactness, cf. (2.3.1) i), as in those categories the maximal exact structure coincides with the class of all kernel-cokernel pairs by definition. Thus we call a sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of vector spaces and linear maps a short algebraic exact sequence if it is exact in the category VS of vector spaces, i.e. if f is injective, g is surjective and $f(X) = g^{-1}(\{0\})$. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of locally convex spaces (lcs) and linear and continuous maps is called *topologically exact* if it is exact in the category LCS of lcs and linear and continuous maps, i.e. if it is algebraically exact and all the maps involved are continuous and open onto their ranges.

In 2.2 we will see on the one hand the well-known result, cf. [Wen03, (5.3)], that the maximal exact structure in the category of LS spaces coincides with the class of short topologically exact sequences which in turn coincides with the class of short topologically exact sequences, whereas on the other hand in the category of LS_w spaces it coincides with the class of short algebraically exact sequences, but contains the class of short topologically exact sequences as a proper subclass. In the category of PLS_w spaces as well as in the category of LB spaces it is both a proper subset of the short algebraically exact sequences and contains the short topologically exact sequences as a proper subset. Furthermore, various examples of subcategories of LCS where the class of short topologically exact sequences is not even an exact structure can be found in [DS12, section 5].

We start the second part of this section by recalling the basic well-known facts about pullbacks and pushouts in $\overline{(\mathrm{LCS})}$ as they will be of great use when determining the maximal exact structure in the category of LB spaces.

Remark 2.1.9. Given two vector spaces (or lcs or Hausdorff lcs) E and F, the biproduct of E and F in VS (or LCS or $\overline{(\mathrm{LCS})}$) is given by the product $E \times F$ of E and F in VS (or LCS or $\overline{(\mathrm{LCS})}$) together with the canonical projections π_E, π_F and the canonical embeddings ω_E, ω_F. Thus we can use (2.1.1) to determine the pullback and pushout in $\overline{(\mathrm{LCS})}$, cf. (2.1.7):

i) The pullback of a diagram $\begin{array}{c} S \\ \downarrow s \\ Y \xrightarrow{g} Z \end{array}$ in $\overline{(\mathrm{LCS})}$ is given by the triple (P, p_Y, p_S), where $P := \{(y, r) \in Y \times S : g(y) = s(r)\}$ is endowed with the relative product topology and p_E and p_G are the restrictions of the canonical projections. Furthermore, we have:

   a) If g is surjective, then so is p_S.

   b) The kernel of p_S in $\overline{(\mathrm{LCS})}$ is given by

$$\varphi \colon g^{-1}(\{0\}) \longrightarrow P, x \longmapsto (x, 0).$$

ii) The pushout of a diagram $X \xrightarrow{f} Y$ in $\overline{\text{(LCS)}}$ is given by the triple (Q, q_Y, q_S), where

$$X \xrightarrow{\;\;f\;\;} Y$$
$$\downarrow s$$
$$S$$

Q is the topological quotient $Y \times S \big/ \overline{(f, -s)(X)}$ and q_Y and q_S are the compositions of the quotient map $q \colon Y \times S \longrightarrow Q$ with the respective inclusions ω_Y and ω_S. Furthermore, we have:

    a) If f has closed range, then so does q_S.

    b) The cokernel of q_S in $\overline{\text{(LCS)}}$ is given by

$$\psi \colon Q \longrightarrow Y \big/ \overline{f(X)}, (y, r) + \overline{(f, -s)(X)} \longmapsto y + \overline{f(X)}.$$

.

As LB and all its subcategories we consider in this thesis are stable with respect to the formation of Hausdorff quotients, they inherit cokernels from $\overline{\text{(LCS)}}$. As they are also stable with respect to the formation of biproducts, only the kernels need special consideration. We will have to endow the algebraic kernel of a linear and continuous map between LB spaces with its associated ultrabornological topology to obtain the kernel in LB (see (2.2.3)). Thus LB kernels carry a topology that is potentially finer than the relative topology. It turns out that the "right" kernels to consider for the maximal exact structure are the well-located subspaces, hence we provide convenient characterizations of being a weak isomorphism onto:

Proposition 2.1.10. *Given a linear, continuous, and injective map $f \colon (X, \mathcal{T}) \longrightarrow (Y, \mathcal{R})$ between lcs, the following are equivalent:*

    i) *The map f is a weak isomorphism onto its range.*

    ii) *The transposed $f^t \colon Y' \longrightarrow X'$ is surjective.*

    iii) *For any lcs S and any linear and continuous map $s \colon (X, \mathcal{T}) \longrightarrow S$ the graph of s is closed in $\left(X, f^{-1}(\mathcal{R})\right) \times S$.*

    iv) *iii) holds for $S = \mathbb{K}$.*

Proof.

"*i) $\Leftrightarrow$ ii)*" Is a consequence of the Hahn-Banach Theorem.

"*i) $\Rightarrow$ iii)*" The graph of s is closed in $(X, \mathcal{T}) \times S$, hence it is closed in $\sigma(X, (X, \mathcal{T})') \times \sigma(S, S')$. Now, if i) holds, we have iii), since the graph of s is absolutely convex and thus its closure coincides with its weak closure.

"*iii) $\Rightarrow$ iv)*" is trivial.

"*iv*) $\Rightarrow$ *i*)" Assuming that iv) holds but not i), we obtain a Φ in $(X, \mathcal{T})'$ that is not continuous with respect to $f^{-1}(\mathcal{R})$. Since Φ is not the zero map, there is an $x \in X$ with $\Phi(x) = 1$, and as the kernel of Φ can not be closed in $(X, f^{-1}(\mathcal{R}))$, it is dense in $(X, f^{-1}(\mathcal{R}))$. Hence we obtain a net $(x_\iota)_{\iota \in I}$ in X with $x_\iota \xrightarrow[\iota \in I]{} x$ in $(X, f^{-1}(\mathcal{R}))$ and $\Phi(x_\iota) = 0$ for all $\iota \in I$, a contradiction to iv).

$\square$

Now we are ready to prove the necessary lifting results for pushout diagrams:

Lemma 2.1.11. *Let* $(X, \mathcal{T}) \xrightarrow{f} (Y, \mathcal{R})$ *be a pushout diagram in* $\overline{\text{(LCS)}}$*, where f is injective*

$$\begin{array}{ccc} (X,\mathcal{T}) & \xrightarrow{f} & (Y,\mathcal{R}) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle q_Y} \\ (S,\mathcal{S}) & \xrightarrow[q_S]{} & Q \end{array}$$

with closed range. Then the following are equivalent:

  i) The map q_S is injective.

  ii) The graph of s is closed in $(X, f^{-1}(\mathcal{R})) \times S$.

  iii) The image $(f, -s)(X)$ is closed in $Y \times S$.

Proof.

"*i*) $\Rightarrow$ *ii*)" Let $(x_\iota)_{\iota \in I}$ be a net in X such that $x_\iota \xrightarrow[\iota \in I]{} 0$ in $(X, f^{-1}(\mathcal{R}))$ and $s(x_\iota) \xrightarrow[\iota \in I]{} r$ in S. Assuming that $r \neq 0 = s(0)$, we obtain that $(0, -r) = \lim_{\iota \in I}(f(x_\iota), -s(x_\iota)) \in \overline{(f, -s)(X)}$, hence $q_S(r) = 0$, but $r \neq 0$, which is not possible since q_S is injective by assumption. Hence $r = 0$ and we have ii).

"*ii*) $\Rightarrow$ *iii*)" Let $(x_\iota)_{\iota \in I}$ be a net in X such that $f(x_\iota) \xrightarrow[\iota \in I]{} y$ in Y and $-s(x_\iota) \xrightarrow[\iota \in I]{} r$ in S. Since the range of f is closed there is an x in X with $y = f(x)$ and thus $x_\iota \xrightarrow[\iota \in I]{} x$ in $(X, f^{-1}(\mathcal{R}))$ and ii) yields $-s(x_\iota) \xrightarrow[\iota \in I]{} -s(x) = r$.

"*iii*) $\Rightarrow$ *i*)" Let $r \in S$ with $q_S(r) = 0$. Since then $(0, r) \in \overline{(f, -s)(X)} = (f, -s)(X)$ holds, there is an x in X with $0 = f(x)$ and $r = -s(x)$. As f is injective we have $x = 0$, hence $r = 0$, which yields i).

$\square$

2.2 The categories of PLS_w and PLH spaces

In this section we will provide the basic homological properties of the two subcategories PLS_w and PLH of the category PLB of projective limits of strongly reduced spectra of LB spaces. The two subcategories are generated by demanding the Banach spaces giving rise to the LB space steps to be reflexive or even Hilbert spaces. We will show that they are pre-abelian

categories – i.e. they are additive with kernels and cokernels – and compute the respective maximal exact structure. Both will turn out to be significantly larger than the class of short topologically exact sequences, which is the largest exact structure in the subcategories PLS and PLN, that were under investigation until now (see [Sie10, (3.2.1)]). We will show that the respective maximal exact structures consist of all kernel-cokernel pairs with surjective cokernel. The discrepancy to the known examples of maximal exact structures originates from the fact that both categories under consideration are not stable with respect to closed subspaces, i.e. closed subspaces of PLS_w (or PLH) spaces endowed with the relative topology do not need to be PLS_w (or PLH) spaces. Thus, to obtain the kernel of a linear and continuous map between PLS_w (or PLH) spaces in the respective category, its algebraic kernel has to be endowed with its associated PLS_w (or PLH) space topology. Having to deal with this problem is what is new about the categories PLS_w and PLH. The fact that both categories are not stable with respect to the formation of (Hausdorff) quotients can be dealt with in the same way as in the PLS case – by taking the Hausdorff completion of the quotient.

The key to the solution of the continuity problems created by the association process is connecting morphisms between PLS_w (or PLH) spaces to spectra of morphisms between the spectra giving rise to the spaces involved, cf. (2.2.9), to be able to apply deWilde's closed graph theorem [MV97, (24.31)]. Hence we start by investigating the categories of the steps:

Definition 2.2.1. A locally convex Hausdorff space $(X, \mathcal{T})$ is an LB space if there is an embedding spectrum of Banach spaces, i.e. a sequence $(X_n)_{n \in \mathbb{N}}$ of Banach spaces with linear and norm-decreasing embeddings $X_n \hookrightarrow X_{n+1}$ $(n \in \mathbb{N})$, such that $(X, \mathcal{T})$ is the inductive limit of the embedding spectra, i.e. $X = \bigcup_{n \in \mathbb{N}} X_n$ endowed with the finest locally convex topology $\mathcal{T}$ such that all inclusions $X_n \hookrightarrow (X, \mathcal{T})$ $(n \in \mathbb{N})$ are continuous. Furthermore, X is called an

  i) LS_w space if the spaces X_n can be chosen to be reflexive,

  ii) LH space if the spaces X_n can be chosen as Hilbert spaces,

  iii) LS space if the embeddings $X_n \hookrightarrow X_{n+1}$ can be chosen compact,

  iv) LN space if the embeddings $X_n \hookrightarrow X_{n+1}$ can be chosen nuclear $(n \in \mathbb{N})$.

We collect the well-known basic properties of the respective categories in the following remark:

Remark 2.2.2. i) Naming the full subcategories of $\overline{(\text{LCS})}$ that correspond to the previous definition accordingly, we obtain the following inclusions:

$$
\begin{array}{ccccc}
\text{LH} & \subset & \text{LS}_w & \subset & \text{LB} \\
\cup & & \cup & & \\
\text{LN} & \subset & \text{LS} & &
\end{array},
$$

since the category LS_w can also be described as the category of all LB spaces that arise from weakly compact embedding spectra, which we obtain from the fact that weakly compact operators between locally convex spaces factorize over reflexive Banach spaces, see [DFJP74, chap. 2].

12

ii) Two embedding spectra $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ of Banach spaces are *equivalent* if there are increasing sequences $(k(n))_{n\in\mathbb{N}}$ and $(l(n))_{n\in\mathbb{N}}$ of natural numbers with $n \le k(n) \le l(n) \le k(n+1)$ such that $X_{k(n)}$ is continuously embedded into $Y_{l(n)}$, which in turn is continuously embedded into $X_{k(n+1)}$. Grothendieck's factorization theorem [MV97, (24.33)] yields that two embedding spectra giving rise to the same LB (LS$_w$, LH, ...) space are equivalent, cf. [MV97, (24.35)]. Hence whenever considering properties of embedding spectra that are passed on to equivalent embedding spectra we may and will identify an LB space with its equivalence class of embedding spectra and write $X = \text{ind}_{n\in\mathbb{N}} X_n$, where $(X_n)_{n\in\mathbb{N}}$ is one representative. If we deal with an LS$_w$ space, we choose a representative consisting of reflexive Banach spaces (respectively Hilbert spaces, when considering an LH or an LN space).

iii) The factorization theorem and the universal property of locally convex inductive limits [MV97, (24.7)] yields that we may assume that any linear and continuous operator f between LB spaces $E = \text{ind}_{n\in\mathbb{N}} E_n$ and $F = \text{ind}_{n\in\mathbb{N}} F_n$ is induced by operators f_n between the Banach steps, i.e. $f_n \in L(E_n, F_n)$ and $f|_{E_n} = f_n$, $n \in \mathbb{N}$. This property will be of use when constructing kernels and cokernels in LB in (2.2.3).

iv) By [MV97, (25.19)] LS$_w$ spaces are regular. In complete analogy to [FW68, 26. 2.] this does not only imply mere reflexivity of LS$_w$ spaces but even that the following dualities hold. Note that the notion of a projective spectrum is defined in (2.2.7) and the duality extends that of products and direct sums, cf. [MV97, (24.3)], when treating LB spaces as quotients of direct sums, cf. [MV97, (24.8)].

 a) Given an LS$_w$ space $X = \text{ind}_{n\in\mathbb{N}} X_n$, the strong dual of X is given by the Fréchet space arising from the projective spectrum of the strong duals X'_n, $n \in \mathbb{N}$.

 b) Given a Fréchet space Y that arises as the projective limit of a reduced projective spectrum of reflexive Banach spaces Y_n, $n \in \mathbb{N}$, a so-called FS$_w$ space, the strong dual of Y is given by the LS$_w$ space created by the strong duals Y'_n, $n \in \mathbb{N}$.

As the transposed of a compact respectively nuclear operator between Banach spaces is compact (nuclear) by Schauder's theorem [MV97, (15.3)] respectively by [MV97, (16.7) 2.], the dualities in a) and b) instantiate in the dual pairs LS$_w$ / FS$_w$, LH / FH, LS / FS, LN / FN, where FH (FS, FN) denotes the category of Fréchet-Hilbert (Fréchet-Schwartz, nuclear Fréchet) spaces.

v) By [MV97, (27.23) b)] there is an LH space X containing a closed subspace L such that L endowed with the relative topology is not an LH space. But at least we have the following: It is a matter of mere calculation to show that LS$_w$ spaces satisfy condition (M_0) of Retakh, cf. [Ret70]. Thus, if L is a closed subspace of an LS$_w$ space $X = \text{ind}_{n\in\mathbb{N}} X_n$, then the quotient spectrum $\left(X_n/L_n\right)_{n\in\mathbb{N}}$, where $L_n := \overline{L \cap X_n}^{X_n}$, is an LS$_w$ spectrum as well, hence it satisfies (M_0), therefore it is weakly acyclic by [Pal71], from which we

obtain by [Ret70] that L is well-located in X, i.e. that L endowed with the associated ultrabornological topology has the same dual as L endowed with the relative topology.

The determination of the kernels and cokernels in LB and the mentioned subcategories is straightforward:

Lemma 2.2.3. *Let* $C \in \{\mathrm{LN}, \mathrm{LS}, \mathrm{LH}, \mathrm{LS_w}, \mathrm{LB}\}$ *and* $t: E \longrightarrow F$ *be a linear and continuous map between the* C *spaces* $E = \mathrm{ind}_{n \in \mathbb{N}} E_n$ *and* $F = \mathrm{ind}_{n \in \mathbb{N}} F_n$. *As* C *is stable under Hausdorff quotients, the cokernel of* t *in* C *is given by the quotient map*

$$F \longrightarrow F \big/ \overline{t(E)} = \mathrm{ind}_{n \in \mathbb{N}} F_n \big/ \overline{\left(\overline{t(E)} \cap F_n\right)},$$

i.e. C *inherits cokernels from* $\overline{(\mathrm{LCS})}$.

With the closed graph theorem we obtain that the kernel of t *in* C *is given by the continuous inclusion of the algebraic kernel of* t *endowed with its* associated ultrabornological *topology:*

$$\left(t^{-1}(\{0\})\right)^{ub} = \mathrm{ind}_{n \in \mathbb{N}} \left(t^{-1}(\{0\}) \cap E_n\right) \hookrightarrow E.$$

In C *the product of* E *and* F *endowed with the product topology and the four standard maps* $\pi_E, \pi_F, \omega_E, \omega_F$ *(cf. (2.1.1) and (2.1.6)) is a biproduct of* E *and* F*, as* $E \times F = \mathrm{ind}_{n \in \mathbb{N}} (E_n \times F_n)$ *holds topologically. Thus* C *is pre-abelian.*

Furthermore, it is easy to verify that t *is a kernel in* C *iff it is injective with closed range and a cokernel in* C *iff it is surjective.*

Now we can determine the maximal exact structure in the category of LB spaces with help of the description of the maximal exact structure of a pre-abelian category in (2.1.4) due to Sieg and Wegner [SW11, Theorem (3.3)] and the inheritance properties in (2.1.11):

Proposition 2.2.4. *The maximal exact structure* $\mathcal{E}_{max}^{\mathrm{LB}}$ *in the category* LB *consists of all kernel-cokernel pairs* (f, g) *such that* f *is well-located. In other words:* $\mathcal{E}_{max}^{\mathrm{LB}}$ *consists of all short algebraically exact sequences* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *of* LB *spaces such that* f *is a weak isomorphism onto its range.*

Proof. With (2.1.4) it is sufficient to prove that a kernel in LB is semi-stable iff it is well-located and each cokernel is semi-stable.

Let $f: X \longrightarrow Y$ be a kernel in LB, $s: X \longrightarrow S$ be a linear and continuous map from X into an LB space S and $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle q_Y} \\ S & \xrightarrow[q_S]{} & Q \end{array}$ be their pushout diagram in LB. The map q_S is a kernel in LB iff it is injective with closed range. By (2.1.9) ii) a) q_S has closed range for any kernel f. By (2.1.11) the injectivity of q_S is equivalent to the graph of s being closed in the product of X endowed with the topology of Y via f and S. As S is an arbitrary LB space, this is equivalent to f being well-located by (2.1.10) iv).

As a morphism in LB is a cokernel iff it is surjective, (2.1.9) i) a) yields that in every pullback diagram $\begin{array}{ccc} P & \xrightarrow{p_S} & S \\ {\scriptstyle p_Y}\downarrow & & \downarrow{\scriptstyle s} \\ Y & \xrightarrow{g} & Z \end{array}$ in LB the morphism p_S is a cokernel if g is a cokernel. Hence all cokernels in LB are semi-stable and the proof is complete. □

As the categories LS_w and LH inherit kernels, cokernels and biproducts from LB and the semi-stability of kernels and cokernels is not changed by the restriction to these two sub-categories and closed subspaces of LS_w spaces are well-located by (2.2.2) v), the previous proposition yields the following corollary:

Corollary 2.2.5. *The restriction of $\mathcal{E}_{max}^{LB}$ to LS_w respectively LH coincides with the respective maximal exact structure $\mathcal{E}_{max}^{LS_w}$ respectively $\mathcal{E}_{max}^{LH}$. Hence the categories LS_w and LH are quasi-abelian, i.e. both maximal exact structures consist of all kernel-cokernel pairs – in this particular case even of all short algebraically exact sequences.*

We have two exact structures in the category of LB spaces and the four mentioned subcategories: the class of short topologically exact sequences (see [DS12, (3.3)]) and the maximal exact structure. As Sieg proved in [Sie10], these two classes coincide in the category of LS spaces, hence as well in the category of LN spaces. In the following, we will show that the class of short topologically exact sequences is a proper subclass of the maximal exact structure in the categories LH, LS_w and LB and discuss the "right" choice of the exact structure to work with in LH and LS_w.

Remark 2.2.6. In order to be able to use our knowledge about splitting theory in the category of Fréchet spaces respectively the four subcategories under consideration to control splitting in the four subcategories of LB, cf. [VW80, Vog87, Fre96, FW96, Wen03, DM07], we need to extend the dualities of (2.2.2) iv) to the exact structures.

On the Fréchet side of the duality the exact structure is determined to be the class of short topologically exact sequences by Palamodov's approach, cf. (2.1.8). The quasinormability of Fréchet Schwartz spaces (see e.g. [MV97, p. 313]) yields by [MV97, (26.12) & (26.13)] that the dual sequences of short topologically exact sequences of Fréchet-Schwartz (nuclear Fréchet) spaces are short topologically exact sequences of LS (LN) spaces. This duality does not hold in the dual pairs LS_w / FS_w and LH / FH: According to [MV97, (27.23) c)] for any $1 \leq p < \infty$ there is a Köthe matrix A and a linear and surjective map Q from the Köthe sequence space $\lambda^p(A)$ to ℓ^p such that the dual sequence of the short topologically exact sequence

$$0 \longrightarrow Q^{-1}(\{0\}) \xhookrightarrow{I} \lambda^p(A) \xrightarrow{Q} \ell^p \longrightarrow 0$$

of Fréchet spaces is not topologically exact. Choosing $p = 2$ yields the claim and thus that the class of short topologically exact sequences is a proper subclass of the maximal exact structure in LH and LS_w, hence it is not suitable for our investigation, unlike the maximal exact structure:

The "perfect" duality (2.2.2) iv) can be expressed with the duality functor D, which assigns each locally convex space X its strong dual X' and each linear and continuous operator f between lcs its transposed operator f^t between the strong duals, cf. [Wen03, chap. 7]. (2.2.2) iv) signifies that D coincides on each of the four subcategories of the Fréchet spaces and on each of the four subcategories of the LB spaces with the $\cdot^{\text{op}}$-functor. Thus (2.1.5), the fact that $\left(\mathcal{E}_{\text{max}}^C\right)^{\text{op}} = \mathcal{E}_{\text{max}}^{C^{\text{op}}}$ holds in a pre-abelian category C, yields the desired duality of the exact structures. We even could have used this fact to determine the four maximal exact structures. We have chosen to give a direct description, as our proof is valid for LB and LF spaces as well, in particular our proof is independent of the mentioned dualities.

Having established all the necessary notions and results in the categories of the steps, we may start investigating the arising subcategories of PLB by recalling the concept of the locally convex projective limit of a locally convex projective spectrum, cf. [Wen03, chap. 3]. This part may seem exceptionally detailed compared to the discussion of LB spaces in (2.2.2), which is due to the fact that the projective situation is more complicated than the inductive one when considering equivalence of spectra; a dual variant of Grothendieck's factorization theorem, cf. (2.2.9), requires additional premises, hence more detailed investigation.

Definition and Remark 2.2.7. A locally convex projective spectrum $\mathscr{X} = \left(X_N, X_M^N\right)$ consists of a sequence $(X_N)_{N \in \mathbb{N}}$ of locally convex spaces and a family of continuous linear maps $X_M^N \colon X_M \longrightarrow X_N$ for $M \geq N$ such that

   i) $X_N^N = \text{id}_{X_N}$ $(N \in \mathbb{N})$ and

   ii) $X_M^N \circ X_K^M = X_K^N$ for all $K \geq M \geq N$.

A morphism $(f_N)_{N \in \mathbb{N}}$ between two locally convex projective spectra $\mathscr{X} = \left(X_N, X_M^N\right)$ and $\mathscr{Y} = \left(Y_N, Y_M^N\right)$ consists of a sequence of linear and continuous maps $f_N \colon X_N \longrightarrow Y_N$ such that $f_N \circ X_M^N = Y_M^N \circ f_M$, i.e. the diagram

$$
\begin{array}{ccc}
X_N & \xrightarrow{\;f_N\;} & Y_N \\
{\scriptstyle X_M^N}\Big\uparrow & & \Big\uparrow{\scriptstyle Y_M^N} \\
X_M & \xrightarrow[\;f_M\;]{} & Y_M
\end{array}
$$

is commutative, for all $M \geq N$.

The locally convex projective limit of a locally convex projective spectrum $\mathscr{X} = \left(X_N, X_M^N\right)$ is defined as

$$
X_\infty := \text{proj}\,\mathscr{X} := \text{proj}\left(X_N, X_M^N\right) := \left\{(x_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} X_N \colon X_{N+1}^N\,(x_{N+1}) = x_N \text{ for all } N \in \mathbb{N}\right\},
$$

endowed with the relative product topology, i.e. the initial topology induced by the restrictions of the canonical projections $X_\infty^N \colon X_\infty \longrightarrow X_N$ $(N \in \mathbb{N})$. A morphism $(f_N)_{N \in \mathbb{N}}$ between two

locally convex projective spectra $\mathscr{X} = \left(X_N, X_M^N\right)$ and $\mathscr{Y} = \left(Y_N, Y_M^N\right)$ induces a linear and continuous map between the respective locally convex projective limits:

$$f_\infty := \operatorname{proj} (f_N)_{N \in \mathbb{N}} : \operatorname{proj} \mathscr{X} \longrightarrow \operatorname{proj} \mathscr{Y}, (x_N) \longmapsto (f_N (x_N))_{N \in \mathbb{N}}.$$

It is easy to verify that the rule, which assigns to each locally convex projective spectrum its locally convex projective limit and each morphism between two locally convex projective spectra the induced linear and continuous map between the respective locally convex projective limits, is a covariant functor acting on the category of locally convex projective spectra and taking values in the category of locally convex spaces. Two locally convex projective spectra $\mathscr{X} = \left(X_N, X_M^N\right)$ and $\mathscr{Y} = \left(Y_N, Y_M^N\right)$ are called *equivalent* if there are increasing sequences $(K(N))_{N \in \mathbb{N}}$ and $(L(N))_{N \in \mathbb{N}}$ of natural numbers with $N \le L(N) \le K(N) \le L(N+1)$ and continuous linear maps $\alpha_N \colon X_{K(N)} \longrightarrow Y_{L(N)}$ and $\beta_N \colon Y_{L(N)} \longrightarrow X_{K(N-1)}$ such that $\beta_N \circ \alpha_N = X_{K(N)}^{K(N-1)}$ and $\alpha_N \circ \beta_{N+1} = Y_{L(N+1)}^{L(N)}$, i.e. the diagram

commutes for all $N \ge 2$. If two locally convex projective spectra $\mathscr{X} = \left(X_N, X_M^N\right)$ and $\mathscr{Y} = \left(Y_N, Y_M^N\right)$ are equivalent, then their locally convex projective limits are topologically isomorphic.

Quite a variety of reducedness conditions on locally convex projective spectra has been established. Here, we only state the two basic forms needed for our purpose, later on, cf. (3.2.3), we will consider another one. A locally convex projective spectrum $\mathscr{X} = \left(X_N, X_M^N\right)$ is called

i) *reduced* if

$$\underset{N \in \mathbb{N}}{\forall} \ \underset{M \ge N}{\exists} \ \underset{K \ge M}{\forall} \ X_M^N (X_M) \subset \overline{X_K^N (X_K)}^{X_N},$$

ii) *strongly reduced* if

$$\underset{N \in \mathbb{N}}{\forall} \ \underset{M \ge N}{\exists} \ X_M^N (X_M) \subset \overline{X_\infty^N (\operatorname{proj} (\mathscr{X}))}^{X_N}.$$

By dropping all topologies and continuity demands we can consider the mere algebraic functor proj, which is sufficient to calculate its right derivative in the sense of Palamodov (cf. [Wen03, (3.1.4)]). As we will deal mainly with locally convex projective spectra and projective limits of

those, we will just refer to projective spectra and projective limits meaning the locally convex variant and mention explicitly when talking about merely algebraic projective constructions.

Finally it is possible to consider (locally convex) projective spectra that are not countable (cf. [Wen03, chap. 4]), where we essentially replace sequences by families indexed over directed sets. As we will need the uncountable projective limit functor only once – in (4.2.1) – we only introduce the projective limit functor for countable spectra.

Now we are ready to define the category of PLB spaces and its subcategories:

Definition 2.2.8. A locally convex Hausdorff space $(X, \mathcal{T})$ is called a PLB space if there is a strongly reduced projective spectrum $\left(X_N, X_M^N\right)_{N \in \mathbb{N}}$ of complete separated LB spaces – where each X_N arises as the inductive limit of an embedding spectrum $(X_{N,n})_{n \in \mathbb{N}}$ of Banach spaces $(N \in \mathbb{N})$ – such that

$$X = X_\infty =: \operatorname*{proj}_{N \in \mathbb{N}} \operatorname*{ind}_{n \in \mathbb{N}} X_{N,n}.$$

A PLB space E is called a

   i) $\mathrm{PLS_w}$ space, if the spaces X_N can be chosen as $\mathrm{LS_w}$ spaces,

  ii) PLH space, if the spaces X_N can be chosen as LH spaces,

 iii) PLS space, if the spaces X_N can be chosen as LS spaces,

 iv) PLN space, if the spaces X_N can be chosen as LN spaces $(N \in \mathbb{N})$.

Naming the corresponding full subcategories of the locally convex Hausdorff spaces accordingly, we obtain by the following inclusions (2.2.2) i):

$$
\begin{array}{ccccc}
\mathrm{PLH} & \subset & \mathrm{PLS_w} & \subset & \mathrm{PLB} \\
\cup & & \cup & & \\
\mathrm{PLN} & \subset & \mathrm{PLS} & &
\end{array} .
$$

As mentioned we need a dual variant of the factorization theorem that is due to Vogt [Vog89] to expedite the investigation:

Proposition 2.2.9. *Let $\left(E_N, E_M^N\right)$ and $\left(F_N, F_M^N\right)$ be two strongly reduced projective spectra of complete separated* LB *spaces and* $t \colon \operatorname{proj}\left(E_N, E_M^N\right) \longrightarrow \operatorname{proj}\left(F_N, F_M^N\right)$ *be a linear and continuous map. Then there is a subsequence* $(K(N))_{N \in \mathbb{N}}$ *and a morphism of spectra* $(t_{K(N)})_{N \in \mathbb{N}}$ *from* $\left(E_{K(N)}, E_{K(M)}^{K(N)}\right)$ *to* $\left(F_N, F_M^N\right)$ *such that* $t = \operatorname{proj}(t_{K(N)})_{N \in \mathbb{N}}$.

We will state and prove the following results in the category of $\mathrm{PLS_w}$ spaces. All of them remain valid with the verbatim proofs if we replace $\mathrm{PLS_w}$ by PLH and $\mathrm{LS_w}$ by LH.

We start the analysis by determining the connection between $\mathrm{PLS_w}$ spaces and strongly reduced projective spectra of $\mathrm{LS_w}$ § spaces giving rise to them:

Remark 2.2.10.   i) By (2.2.9) strongly reduced projective spectra of LS_w spaces, which we call from now on PLS_w spectra, giving rise to the same PLS_w space are equivalent. Thus we may and will identify a PLS_w space X with the equivalence class of PLS_w spectra giving rise to it whenever we consider properties of projective spectra that are invariant under passing to equivalent spectra. In particular this involves the projective limit and all its derivatives by [Wen03, (3.1.7)].

ii) Given a PLS_w space $X = \text{proj}\left(X_N, X_M^N\right)$, we may choose a PLS_w spectrum $\left(\tilde{X}_N, \tilde{X}_M^N\right)$ giving rise to X such that the projections of X onto $\tilde{X}_N$ have dense range:

By passing to a subsequence we may assume that $X_{N+1}^N\left(X_{N+1}\right) \subset \overline{X_\infty^N\left(X\right)}^{X_N}$ holds for each $N \in \mathbb{N}$. We define

$$\tilde{X}_N := \left(\overline{X_{N+1}^N\left(X_{N+1}\right)}^{X_N}\right)^{\text{ub}} \text{ and } \tilde{X}_{N+1}^N := X_{N+1}^N|_{\tilde{X}_{N+1}},$$

for every $N \in \mathbb{N}$. As passing to the associated ultrabornological topologies preserves continuity of linear maps – i.e. it is functorial – the inclusion $\alpha_N \colon \tilde{X}_N \hookrightarrow X_N$ and the linear map $\beta_N \colon X_N \longrightarrow \tilde{X}_{N-1}, x \mapsto X_N^{N-1}\left(x\right)$ are continuous and we have $\beta_N \circ \alpha_N = \tilde{X}_N^{N-1}$ and $\alpha_N \circ \beta_{N+1} = X_{N+1}^N$ for each $N \geq 2$. Thus $\left(X_N, X_M^N\right)$ and $\left(\tilde{X}_N, \tilde{X}_M^N\right)$ are equivalent spectra of LS_w spaces and the projections from X to $\tilde{X}_N$ have dense range as $\tilde{X}_N$ is a well-located subspace of X_N by (2.2.2) v). From now on, whenever working with PLS_w spaces $X = \text{proj}\left(X_N, X_M^N\right) = \text{proj}_{N\in\mathbb{N}} \text{ind}_{n\in\mathbb{N}} X_{N,n}$, we will always assume that the projections X_∞^N have dense range.

iii) Given a projective spectrum $\left(X_N, X_M^N\right)$ of LS spaces that is not necessarily strongly reduced with projective limit X, we can always pass on to the *associated strongly reduced spectrum* $\left(\tilde{X}_N, \tilde{X}_M^N\right)$, where $\tilde{X}_N$ is the closure of the projection of X onto X_N in X_N endowed with the relative topology and the corresponding restrictions of the linking maps $\tilde{X}_M^N$. Although the two spectra are generally not equivalent, the projective limits are topologically the same, as we can work with the relative topologies. If the steps X_N are LS_w spaces we have to endow the closure of the projection of X to X_N in X_N with its associated ultrabornological topology to obtain LS_w spaces $\tilde{X}_N$. This is possibly strictly finer than the relative topology. As closed subspaces of LS_w spaces are well-located by (2.2.5), we obtain that the projective limits are algebraically the same but $\text{proj}\left(\tilde{X}_N, \tilde{X}_M^N\right)$ carries a potentially finer topology. As we need the factorization of operators over the steps (2.2.9), we need to include the technical feature of strong reducedness in the definition of PLS_w spaces.

Now we can construct the kernel and cokernel in PLS_w:

Proposition 2.2.11. *Let $t \colon E \longrightarrow F$ be a linear and continuous map between* PLS_w *spaces* $(E, \mathcal{T}) = \text{proj}_{N\in\mathbb{N}} \text{ind}_{n\in\mathbb{N}} E_{N,n}$ *and* $(F, \mathcal{S}) = \text{proj}_{N\in\mathbb{N}} \text{ind}_{n\in\mathbb{N}} F_{N,n}$.

i) *The kernel of t in* $\mathrm{PLS_w}$ *is given by the continuous inclusion of the algebraic kernel of t, endowed with its associated* $\mathrm{PLS_w}$ *topology, into E. We denote this object by* $\left(t^{-1}\left(\{0\}\right)\right)_{\mathrm{PLS_w}}$. *By definition, it is the* $\mathrm{PLS_w}$ *space arising from the* $\mathrm{PLS_w}$ *spectrum consisting of the closures of the projections of the algebraic kernel of t to* E_N ($N \in \mathbb{N}$) *endowed with their associated ultrabornological topology – the* $\mathrm{LS_w}$ *spaces* $\left(\overline{E_\infty^N\left(t^{-1}\left(\{0\}\right)\right)}^{E_N}\right)^{ub}$ *– and the corresponding restrictions of the linking maps* E_M^N.

ii) *As in the category of* PLS *spaces, the cokernel of t in the category of* $\mathrm{PLS_w}$ *spaces is given by the Hausdorff completion of the quotient* $F/_{t(E)}$, *which arises as the projective limit of the strongly reduced spectrum consisting of the* $\mathrm{LS_w}$ *spaces* $F_N/\overline{F_\infty^N(t(E))}^{F_N}$ *together with the maps induced by the linking maps* F_M^N.

Proof. i) The kernel of t in the category of locally convex spaces is given by the inclusion of $\left(t^{-1}\left(\{0\}\right), \mathcal{T} \cap t^{-1}\left(\{0\}\right)\right)$ into E and it is clear that it arises as the projective limit of the strongly reduced projective spectrum generated by the restrictions of the spectral maps E_{N+1}^N to the closure of $E_\infty^{N+1}\left(t^{-1}\left(\{0\}\right)\right)$ in E_{N+1} endowed with the respective relative topology induced by E_{N+1}. We obtain our spectrum by passing to the associated $\mathrm{LS_w}$ topologies. Thus the closed graph theorem implies that it is a well-defined projective spectrum of $\mathrm{LS_w}$ spaces. As closed subspaces of $\mathrm{LS_w}$ spaces are well-located by (2.2.5), it is strongly reduced because the closure of $E_\infty^N\left(t^{-1}\left(\{0\}\right)\right)$ in E_N is the same as its weak closure which in turn remains unchanged by passing to the associated ultrabornological topologies. Furthermore, we have the continuous identity map from $\left(t^{-1}\left(\{0\}\right)\right)_{\mathrm{PLS_w}}$ to $\left(t^{-1}\left(\{0\}\right), \mathcal{T} \cap t^{-1}\left(\{0\}\right)\right)$. Let now $s\colon S \longrightarrow E$ be a linear and continuous map from a $\mathrm{PLS_w}$ space $S = \mathrm{proj}_{N \in \mathbb{N}}\left(S_N, S_{N+1}^N\right)$ into E with $t \circ s = 0$. Then we have the unique linear and continuous map $\lambda\colon S \longrightarrow \left(t^{-1}\left(\{0\}\right), \mathcal{T} \cap t^{-1}\left(\{0\}\right)\right)$ induced by the locally convex kernel of t and it remains to show that $\lambda\colon S \longrightarrow \left(t^{-1}\left(\{0\}\right)\right)_{\mathrm{PLS_w}}$ is continuous. As $\lambda\colon S \longrightarrow E$ is continuous, by passing to a subsequence, we may assume by (2.2.9) that it arises as a morphism of spectra $(\lambda_N)_{N \in \mathbb{N}}\colon \left(S_N, S_{N+1}^N\right) \longrightarrow \left(E_N, E_{N+1}^N\right)$. As $\lambda(S) \subset t^{-1}\left(\{0\}\right)$, we obtain by construction that $\lambda_N(S_N)$ is a subset of the closure of $E_\infty^N\left(t^{-1}\left(\{0\}\right)\right)$ in E_N. Thus again with the closed graph theorem λ_N remains continuous when passing to the associated ultrabornological topologies and $\lambda = \mathrm{proj}\,(\lambda_N)_{N \in \mathbb{N}}$ is continuous even from S to $\left(t^{-1}\left(\{0\}\right)\right)_{\mathrm{PLS_w}}$.

ii) Here we can essentially follow the proof for the category of PLS spaces in [Sie10, (3.1.3)]: For $N \in \mathbb{N}$ we define A_N as the closure of $F_\infty^N(t(E))$ in F_N, here endowed with the relative topology, and A_{N+1}^N as the restriction of F_{N+1}^N to A_{N+1} and the topological inclusion i_N of A_N into F_N. Furthermore, we define Y_N as the topological quotient $F_N/_{A_N}$, Y_{N+1}^N as the map from Y_{N+1} to Y_N induced by F_{N+1}^N and the quotient map q_N from F_N to Y_N ($N \in \mathbb{N}$). Then $\left(A_N, A_{N+1}^N\right)$ is a strongly reduced spectrum of locally convex

spaces, $\left(Y_N, Y_{N+1}^N\right)$ is a strongly reduced spectrum of LS_w spaces and we have a short topologically exact sequence of projective spectra

$$0 \longrightarrow \left(A_N, A_{N+1}^N\right) \xrightarrow{(i_N)_{N\in\mathbb{N}}} \left(F_N, F_{N+1}^N\right) \xrightarrow{(q_N)_{N\in\mathbb{N}}} \left(Y_N, Y_{N+1}^N\right) \longrightarrow 0.$$

The map proj $(q_N)_{N\in\mathbb{N}}$ is open onto its range by [Wen03, (3.3.1)], as $\left(A_N, A_{N+1}^N\right)$ is strongly reduced, hence the map

$$j\colon Y\big/\overline{t\,(E)}^F \longrightarrow \operatorname*{proj}_{N\in\mathbb{N}}\left(Y_N, Y_{N+1}^N\right), y + \overline{t\,(E)}^F \mapsto \left(Y_\infty^N\,(y) + A_N\right)_{N\in\mathbb{N}}$$

is open onto its range as well. The space $\operatorname{proj}_{N\in\mathbb{N}}\left(Y_N, Y_{N+1}^N\right)$ is a complete Hausdorff space, since each Y_N $(N \in \mathbb{N})$ is complete and Hausdorff. As j has dense range, it is a version of the Hausdorff completion of $Y\big/\overline{t\,(E)}^F$. Thus it is an easy exercise to check that it satisfies the universal property of the cokernel of f in PLS_w.

$\square$

Corollary 2.2.12. *The category* PLS_w *is pre-abelian.*

Proof. As the product of two PLS_w spaces $X = \operatorname{proj}_{N\in\mathbb{N}} \operatorname{ind}_{n\in\mathbb{N}} X_{N,n}$ and $Y = \operatorname{proj}_{N\in\mathbb{N}} \operatorname{ind}_{n\in\mathbb{N}} Y_{N,n}$ is the PLS_w space $X \times Y = \operatorname{proj}_{N\in\mathbb{N}} \operatorname{ind}_{n\in\mathbb{N}} X_{N,n} \times Y_{N,n}$, (2.2.11) yields the assertion. $\square$

Remark 2.2.13. In (2.1.11) and (2.2.4) we discovered that the attribute "well-located" is crucial concerning the lifting properties in pushout diagrams of pre-abelian categories. Although we will not need this property in PLS_w to determine the maximal exact structure, we want to remark, that the embeddings of closed subspaces of PLS_w spaces, endowed with their associated PLS_w topology, into the original PLS_w space are weak isomorphisms onto, as every linear and continuous functional on a PLS_w space factorizes over a step and closed subspaces of LS_w spaces are well-located by (2.2.2) v).

Furthermore, we obtain with (2.2.11) ii) that complete Hausdorff quotients of PLS_w spaces are again PLS_w spaces, a fact that is well-known for PLS and PLN spaces, see [DV00b, (1.2)].

Before we can determine the maximal exact structure of PLS_w, we need to apprehend the possibilities of localization and its limits:

Remark 2.2.14. Let again $t\colon E \longrightarrow F$ be a linear and continuous map between PLS_w spaces $E = \operatorname{proj}_{N\in\mathbb{N}} \operatorname{ind}_{n\in\mathbb{N}} E_{N,n}$ and $F = \operatorname{proj}_{N\in\mathbb{N}} \operatorname{ind}_{n\in\mathbb{N}} F_{N,n}$. By passing to a subsequence we may assume by (2.2.9) that there is a morphism of spectra $(t_N)_{N\in\mathbb{N}} : \left(E_N, E_{N+1}^N\right) \longrightarrow \left(F_N, F_{N+1}^N\right)$, called localization of t, such that $t = \operatorname{proj}(t_N)_{N\in\mathbb{N}}$. We investigate the relation between the projective limits of the spectra of local kernels respectively cokernels and the PLS_w kernels respectively cokernels:

i) For each $N \in \mathbb{N}$ we have the kernel $\left(t_N^{-1}(\{0\})\right)^{ub}$ of t_N in LS_w. Together with the corresponding restrictions of the linking maps E_{N+1}^N these spaces generate the projective

spectrum of *local kernels* of t. The projection of the kernel of t onto E_N is always a subspace of the kernel of t_N. Thus we obtain continuous inclusions of the steps giving rise to the $\mathrm{PLS_w}$ kernel of t into the local kernels. These induce a continuous inclusion of the $\mathrm{PLS_w}$ kernel of t into the projective limit of the local kernels. That inclusion is even surjective as each sequence of elements of the kernels of t_N which is an element of E is already an element of the kernel of t. However, we do not know if this inclusion is open, as we do not even know if the local inclusions are open onto their ranges. Thus the projective limit of the local kernels induces a topology on the algebraic kernel of t that is potentially finer than the associated $\mathrm{PLS_w}$ topology.

However if the spectrum of local kernels happens to be strongly reduced, then the two topologies on the kernel of t coincide, since then the construction (2.2.10) ii) of the equivalent spectrum with dense projection images leads to the spectrum giving rise to the $\mathrm{PLS_w}$ kernel of t, which is thus equivalent to the spectrum of local kernels.

ii) In contrast to that, a localization of the cokernel of t in $\mathrm{PLS_w}$ is possible: For each $N \in \mathbb{N}$ we have the cokernel $F_N / \overline{t_N(E_N)}^{F_N}$ of t_N in $\mathrm{LS_w}$. Together with the maps induced by F_{N+1}^N these spaces generate the projective spectrum of *local cokernels* of t. As we may assume dense projection images for E by (2.2.10) ii), the closures of the images of t_N coincide with the closures of the projection of the image of t to F_N for all $N \in \mathbb{N}$. Thus the spectrum of local cokernels and the spectrum giving rise to the $\mathrm{PLS_w}$ cokernel of t, see (2.2.11) ii), are identical.

However bad the situation concerning the localization of kernels may be, in complete analogy to the PLS case, cf. [DV00b, p. 64], a localization of kernel-cokernel pairs is possible in $\mathrm{PLS_w}$, as the category of the steps, $\mathrm{LS_w}$, is quasi-abelian by (2.2.5):

Proposition 2.2.15. *Let (f, g) be a kernel-cokernel pair in $\mathrm{PLS_w}$. Then there are strongly reduced spectra $\left(X_N, X_M^N\right), \left(Y_N, Y_M^N\right), \left(Z_N, Z_M^N\right)$ of $\mathrm{LS_w}$ spaces as well as morphisms of spectra $(f_N)_{N \in \mathbb{N}} : \left(X_N, X_M^N\right) \longrightarrow \left(Y_N, Y_M^N\right)$ and $(g_N)_{N \in \mathbb{N}} : \left(Y_N, Y_M^N\right) \longrightarrow \left(Z_N, Z_M^N\right)$ such that each (f_N, g_N) is a kernel-cokernel pair in $\mathrm{LS_w}$, i.e. $(f_N, g_N) \in \mathcal{E}_{max}^{\mathrm{LS_w}}$ $(N \in \mathbb{N})$, and (f, g) arises as the projective limit of the sequence of spectra*

$$0 \longrightarrow \left(X_N, X_M^N\right) \overset{(f_N)_{N \in \mathbb{N}}}{\longrightarrow} \left(Y_N, Y_M^N\right) \overset{(g_N)_{N \in \mathbb{N}}}{\longrightarrow} \left(Z_N, Z_M^N\right) \longrightarrow 0.$$

Moreover, if $\left(\tilde{X}_N, \tilde{X}_M^N\right), \left(\tilde{Y}_N, \tilde{Y}_M^N\right)$ and $\left(\tilde{Z}_N, \tilde{Z}_M^N\right)$ are strongly reduced spectra of $\mathrm{LS_w}$ spaces giving rise to X, Y and Z then we can either take $\left(Y_N, Y_M^N\right) = \left(\tilde{Y}_N, \tilde{Y}_M^N\right)$ or $\left(X_N, X_M^N\right)$ and $\left(Z_N, Z_M^N\right)$ as subsequences of $\left(\tilde{X}_N, \tilde{X}_M^N\right)$ and $\left(\tilde{Z}_N, \tilde{Z}_M^N\right)$.

Proof. Let $\left(Y_N, Y_M^N\right)$ be a strongly reduced spectrum of $\mathrm{LS_w}$ spaces with $\mathrm{proj}\left(Y_N, Y_M^N\right) = Y$. First using that f is a kernel of g and then that g is a cokernel of f we obtain by (2.2.11) and

(2.2.14) ii) that (f, g) arises as the projective limit of the sequences

$$0 \longrightarrow \left(\overline{Y_\infty^N \left(g^{-1}\left(\{0\}\right)\right)}^{Y_N}\right)^{\mathrm{ub}} \overset{f_N}{\hookrightarrow} Y_N \overset{g_N}{\longrightarrow} Y_N\big/_{f_N(X_N)} \longrightarrow 0,$$

where f_N is the inclusion and g_N is the quotient map and the linking maps X_M^N and Z_M^N are the maps induced by Y_M^N ($M \geq N$). Denoting the first space of the sequence X_N and the last Z_N yields the first assertion.

If $\left(\tilde{X}_N, \tilde{X}_M^N\right)$ and $\left(\tilde{Z}_N, \tilde{Z}_M^N\right)$ are strongly reduced spectra of LS_w spaces giving rise to X and Z, these spectra are equivalent to $\left(X_N, X_M^N\right)$ respectively $\left(Z_N, Z_M^N\right)$ by (2.2.9). Thus by (2.2.7) we obtain factorizations

$$
\begin{array}{ccccccccc}
& & & & \tilde{X}_{K(N)} & & & & \\
& & & & \uparrow & & & & \\
& & & & & & & & \\
0 & \longrightarrow & X_N & \overset{f_N}{\longrightarrow} & Y_N & \overset{g_N}{\longrightarrow} & Z_N & \longrightarrow & 0. \\
& & & & & & \uparrow & & \\
& & & & & & \tilde{Z}_{L(N)} & &
\end{array}
$$

Since the category LS_w is quasi-abelian by (2.2.5), the middle row of the above diagram is exact in LS_w for each $N \in \mathbb{N}$. Hence we can use the pullback and pushout constructions (2.1.7), cf. [DV00b, p. 64] or Karidopoulou [Kar06, (1.1.39)] for details, to obtain a sequence of spectra

$$0 \longrightarrow \left(\tilde{X}_{K(N)}, \tilde{X}_{K(M)}^{K(N)}\right) \longrightarrow \left(Y_N, Y_M^N\right) \longrightarrow \left(\tilde{Z}_{L(N)}, \tilde{Z}_{L(M)}^{L(N)}\right) \longrightarrow 0$$

giving rise to (f, g), in which each sequence of steps is exact in LS_w. $\qquad\square$

Now we can determine the semi-stable kernels and cokernels in PLS_w, which leads to the maximal exact structure of PLS_w by (2.1.4):

Proposition 2.2.16.   *i) In* PLS_w *every kernel is semi-stable.*

ii) In PLS_w *a cokernel is semi-stable if and only if it is surjective.*

Proof. Let (f, g) be a kernel-cokernel pair in PLS_w with corresponding localization (2.2.15):

$$0 \longrightarrow \left(X_N, X_M^N\right) \overset{(f_N)_{N\in\mathbb{N}}}{\longrightarrow} \left(Y_N, Y_M^N\right) \overset{(g_N)_{N\in\mathbb{N}}}{\longrightarrow} \left(Z_N, Z_M^N\right) \longrightarrow 0.$$

i) Let $s\colon X \longrightarrow S$ be a linear and continuous map from $X = \mathrm{proj}\left(X_N, X_M^N\right)$ into any PLS_w space $S = \mathrm{proj}\left(S_N, S_M^N\right)$. By passing to a subsequence we may assume by (2.2.9) that there is a morphism of spectra $(s_N)_{N\in\mathbb{N}}$ from $\left(X_N, X_M^N\right)$ to $\left(S_N, S_M^N\right)$ such that $s =$

proj $(s_N)_{N \in \mathbb{N}}$. Denoting $Y := \text{proj}\left(Y_N, Y_M^N\right)$, the pushout of the diagram $\begin{array}{c} X \xrightarrow{f} Y \\ {\scriptstyle s}\downarrow \\ S \end{array}$ in

PLS_w is given by the cokernel of the map $(f, -s) : X \longrightarrow Y \times S, x \mapsto (f(x), -s(x))$ in PLS_w. By (2.2.14) ii) this cokernel is given by the projective limit of the local cokernels, i.e. by the projective limit of the pushouts of the diagrams $\begin{array}{c} X_N \xrightarrow{f_N} Y_N \\ {\scriptstyle s_N}\downarrow \\ S_N \end{array}$ in LS_w. These

pushouts are given by the triples (Q_N, q_{Y_N}, q_{S_N}) from (2.1.9) ii). Then by (2.1.9) ii) b) and (2.2.3) each q_{S_N} is the kernel of

$$\psi_N : Q_N = Y_N \times S_N / \overline{(f_N, -s_N)(X_N)} \longrightarrow Z_N = Y_N / f_N(X_N),$$
$$(y_N, r_N) + \overline{(f_N, -s_N)(X_N)} \mapsto y_N + f_N(X_N).$$

Thus $q_S := \text{proj}(q_{S_N})_{N \in \mathbb{N}}$ is the projective limit of the spectrum of the local kernels $(q_{S_N})_{N \in \mathbb{N}}$ of the map $\psi := \text{proj}(\psi_N)_{N \in \mathbb{N}}$. As the spectrum $\left(S_N, S_M^N\right)$ is strongly reduced, q_S is indeed the PLS_w kernel of ψ by (2.2.14) i), hence f is a semi-stable kernel.

ii) If g is semi-stable, it has to be surjective by [Sie10, (2.2.3)]. Now let g be surjective and $s : S \longrightarrow Z$ be a linear and continuous map from a PLS_w space $S = \text{proj}\left(S_N, S_M^N\right)$ to $Z = \text{proj}\left(Z_N, Z_M^N\right)$. By passing to a subsequence we may assume by (2.2.9) that s arises as the projective limit of a morphism of spectra $(s_N)_{N \in \mathbb{N}}$ from $\left(S_N, S_M^N\right)$ to $\left(Z_N, Z_M^N\right)$. Then the pullback of the diagram $\begin{array}{c} S \\ {\scriptstyle s}\downarrow \\ Y \xrightarrow{g} Z \end{array}$ in PLS_w is given by the triple (P, p_Y, p_S), where P is the kernel of the map $Y \times S \longrightarrow Z, (y, r) \mapsto g(y) - s(r)$ in PLS_w, and p_Y and p_S are the restrictions of the respective projections to P. We have to show that p_S is a cokernel.

To prove (2.2.16) i) we have used the localized pushouts to show that q_S is the projective limit of local kernels. Then the strong reducedness of the spectrum of local kernels yielded that q_S was indeed a kernel. Now we proceed inversely: We first show that the spectrum of local pullbacks is strongly reduced, hence it gives rise to the PLS_w pullback by (2.2.14) i). Then p_S arises as the projective limit of some local cokernels which makes it a cokernel.

Let now (P_N, p_{Y_N}, p_{S_N}) be the pullback of the local diagram $\begin{array}{c} S_N \\ {\scriptstyle s_N}\downarrow \\ Y_N \xrightarrow{g_N} Z_N \end{array}$ in LS_w, i.e.

$P_N = (\{(y_N, r_N) : g_N(y_N) = s_N(r_N)\})^{\text{ub}}$ and p_{Y_N} and p_{S_N} are the restrictions of the canonical projections to P_N. We show that for each $N \in \mathbb{N}$ the restriction of the projection $P_\infty^N \colon P \longrightarrow P_N$ has dense range:

Let (y_N, r_N) be an element of P_N and let U_N and V_N be 0-neighbourhoods in Y_N respectively S_N. We define $\tilde{U}_N := \frac{1}{2} U_N$ and $\tilde{V}_N := V_N \cap s_N^{-1}\left(g_N\left(\tilde{U}_N\right)\right)$, which is a 0-neighbourhood in S_N as g_N is a quotient map, hence open. As we may assume that S_∞^N has dense range by (2.2.10) ii), we can choose an element r in S with $S_\infty^N(r) - r_N \in \tilde{V}_N$. As g is surjective, there is an element y in Y with $g(y) = s(r)$. Then we have by construction that $g_N\left(Y_\infty^N(y) - y_N\right) = s_N\left(S_\infty^N(r) - r_N\right) \in g_N\left(\tilde{U}_N\right)$ holds. Thus $Y_\infty^N(y) - y_N$ is an element of $g_N^{-1}\left(g_N\left(\tilde{U}_N\right)\right) = \tilde{U}_N + g_N^{-1}(\{0\}) = \tilde{U}_N + X_N$. Since by construction we have $X_N = \overline{Y_\infty^N(X)}^{Y_N} \subset Y_\infty^N(X) + \tilde{U}_N$, there is an $x \in X$ with $Y_\infty^N(y) - y_N - Y_\infty^N(x) \in 2\tilde{U}_N = U_N$. Now we define $z := y - x$ and obtain that (z, r) is an element of P, because $g(z) = g(y) - g(x) = s(r)$ as $x \in X = g^{-1}(\{0\})$. Furthermore, we calculate $P_\infty^N((z, r)) - (y_N, r_N) = \left(Y_\infty^N(z) - y_N, S_\infty^N(r) - r_N\right) = \left(Y_\infty^N(y) - Y_\infty^N(x) - y_N, S_\infty^N(r) - r_N\right) \in U_N \times V_N$. Thus the spectrum of local pullbacks is indeed strongly reduced, hence it gives rise to the $\mathrm{PLS_w}$ pullback of

$$\begin{array}{ccc} & & S \\ & & \downarrow{\scriptstyle s} \\ Y & \xrightarrow{\;g\;} & Z \end{array}$$

by (2.2.14) i). By (2.1.9) i) b) and (2.2.3) each p_{S_N} is the cokernel of the map $\varphi_N \colon X_N \longrightarrow P_N, x_N \longmapsto (x_N, 0)$. Thus $p_S = \mathrm{proj}\,(p_{S_N})_{N \in \mathbb{N}}$ is the cokernel of $\varphi := \mathrm{proj}\,(\varphi_N)_{N \in \mathbb{N}}$ in $\mathrm{PLS_w}$ by (2.2.14) ii) and g is a semi-stable cokernel.

$\square$

Theorem 2.2.17. *The class $\mathcal{E}_{max}^{\mathrm{PLS_w}}$, which consists of all kernel-cokernel pairs (f, g) in $\mathrm{PLS_w}$ such that g is surjective, is the maximal exact structure in $\mathrm{PLS_w}$. The same is true in PLH.*

Proof. By Sieg and Wegner's characterization (2.1.4) the maximal exact structure of a preabelian category consists of all kernel-cokernel pairs that consist of a semi-stable kernel and a semi-stable cokernel. Thus the determination of semi-stable kernels and cokernels (2.2.16) yields the assertion. $\square$

Definition and Remark 2.2.18. When analyzing if the vanishing of Ext groups is passed down to subcategories of the locally convex spaces, three-space properties are an important tool. As we investigate categories whose exact structure differs from the short topologically exact sequences, we recall the general concept of fully exact and exact subcategories:

Let $(C, \mathcal{E})$ be an exact category.

i) A full subcategory C' of an exact category $(C, \mathcal{E})$ is called a *fully exact subcategory* of $(C, \mathcal{E})$ if for any short exact sequence

$$X \xrightarrow{\;f\;} Y \xrightarrow{\;g\;} Z$$

25

in C, X and Z in C' implies Y in C'. Then being a three-space property translates into being a fully exact subcategory of LCS.

We do not know, whether the categories $\mathrm{LS_w}, \mathrm{PLS_w}, \mathrm{LS}$ or PLS are fully exact subcategories of LCS – i.e. if being an $\mathrm{LS}, \mathrm{LS_w}, \mathrm{PLS}$ or $\mathrm{PLS_w}$ space is a three-space property – but we do know the following:

a) LH and PLH are not fully exact subcategories of LCS, as they are full subcategories of the category of hilbertizable lcs by (4.2.6) i), which is not a fully exact subcategory of LCS. To see this, we consider the short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

of Banach spaces, which Enflo, Lindenstrauss and Pisier constructed in [ELP75], where X and Z are Hilbert spaces but Y is not. Then Y is not a hilbertizable lcs.

b) $\mathrm{LS_w}$ is a fully exact subcategory of $\mathrm{PLS_w}$, the same is true for LH in PLH: Let

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

be a short exact sequence of $\mathrm{PLS_w}$ spaces, where X and Z are $\mathrm{LS_w}$ spaces. By (2.2.15) we have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\
& & \Big\| & & \Big\downarrow{\scriptstyle Y^1_\infty} & & \Big\| & & \\
0 & \longrightarrow & X & \xrightarrow[f_1]{} & Y_1 & \xrightarrow[g_1]{} & Z & \longrightarrow & 0
\end{array}
$$

with rows that are exact in $\mathrm{PLS_w}$ and Y_1 is an $\mathrm{LS_w}$ space. Thus the Five Lemma for exact categories, cf. [Büh10, (3.2)], implies that Y^1_∞ is an isomorphism, hence Y is an $\mathrm{LS_w}$ space. The same proof shows the assertion for the pairing LH / PLH. Note, that ii) is the complete analogon to the result by Sieg about the pairing LS / PLS, cf. [Sie10, (5.1.3)].

ii) If we have an exact structure $\mathcal{E}'$ on C', then $(C', \mathcal{E}')$ is called an *exact subcategory* of $(C, \mathcal{E})$ if the restriction of $\mathcal{E}$ to C', referred to as $\mathcal{E} \cap C'$, coincides with $\mathcal{E}'$.

a) By (2.2.17) we know that $\mathrm{LS_w}, \mathrm{PLS_w}, \mathrm{LH}$ and PLH are not even exact subcategories of LCS (or $\overline{\mathrm{(LCS)}}$), as the class of short topologically exact sequences – although being an exact structure on all of the above categories by [DS12, (3.3) & (3.6)] – is a proper subclass of the maximal exact structure by (2.2.6), whereas Sieg proved in [Sie10, (3.1.1) & (3.2.1)] that both LS and PLS are exact subcategories of LCS.

b) We conclude this section by showing that neither PLH nor PLS_w is quasi-abelian or even semiabelian, cf. [Sie10, ((3.1.5))]:

Any non-surjective linear partial differential operator with constant coefficients $P(D) : \mathscr{D}'(\Omega) \longrightarrow \mathscr{D}'(\Omega)$ on the space of Schwartz distributions over an open subset $\Omega \subset \mathbb{R}^d$ that is surjective as an operator $P(D) : \mathscr{C}^\infty(\Omega) \longrightarrow \mathscr{C}^\infty(\Omega)$ yields a kernel $\mathscr{N}_P(\Omega)$ in $\mathscr{D}'(\Omega)$ with $\mathrm{proj}^1 \mathscr{N}_P(\Omega) \neq 0$ by [Wen03, (3.4.5)], hence the cokernel of the embedding of $\mathscr{N}_P(\Omega)$ into $\mathscr{D}'(\Omega)$, which is given by the Hausdorff completion of the quotient $\mathscr{D}'(\Omega)/_{\mathscr{N}_P(\Omega)}$ by (2.2.11) ii), is not surjective by [DV00b, (1.4)]. Classical results of Malgrange [Mal56, Chapitre 1, Théorème 4] and Hörmander [Hör62] reduce this problem to finding an open subset $\Omega \subset \mathbb{R}^d$ which is P-convex but not strongly P-convex. For $d \geq 3$ such pairs (P, Ω) are easy to find, see e.g. [Dom04, (3.5) a)]. Note that Kalmes showed in [Kal11] that this is not possible for $d = 2$, thus solving an old conjecture of Trève in the affirmative. Using the above example to construct a morphism f in PLH such that the induced morphism $\tilde{f}$ from the coimage of f into its image in the canonical factorization the same way as in [Sie10, (3.1.6)], we obtain that neither PLH nor PLS_w is even semi-abelian.

2.3 Yoneda Ext functors and the long exact sequence in PLS_w / PLH

Thanks to Sieg's, Wegner's and Bühler's work on exact structures [Büh10, Sie10, SW11], we can establish the whole homological toolbox, in particular the long exact sequence (2.3.2), for the Yoneda Ext groups, which leads us to the connection of Ext and proj and the splitting for $\mathscr{D}'(\Omega)$. We give an outline of Sieg's approach, cf. [Sie10, chap. 4 & 5]:

Definition and Remark 2.3.1. Let $(C, \mathcal{E})$ be an exact category.

i) A sequence

$$A: \quad 0 \longrightarrow X \xrightarrow{f_k} Y_{k-1} \longrightarrow \ldots \longrightarrow Y_1 \xrightarrow{f_1} Y_0 \xrightarrow{f_0} Z \longrightarrow 0$$

in C is called *exact* (in C, with left end X and right end Z), if every morphism f_l factorizes as $f_l = m_l \circ e_l$ for an admissible cokernel $e_l : Y_l \longrightarrow I_l$ and an admissible kernel $m_l : I_l \longrightarrow Y_{l-1}$, such that (m_l, e_{l-1}) is in $\mathcal{E}$ for $0 \leq l \leq k$. The integer k is called the *length* of the sequence.

ii) Let

$$A': \quad 0 \longrightarrow X' \longrightarrow Y'_{k-1} \longrightarrow \ldots \longrightarrow Y'_1 \longrightarrow Y'_0 \longrightarrow Z' \longrightarrow 0$$

be another exact sequence of length k in C. A morphism with fixed ends $\Phi: A \longrightarrow A'$

between the exact sequences A and A' of length k is a commutative diagram

$$A: 0 \longrightarrow X \longrightarrow Y_{k-1} \longrightarrow Y_{k-2} \longrightarrow \ldots \longrightarrow Y_1 \longrightarrow Y_0 \longrightarrow Z \longrightarrow 0$$

$$\Bigg\| \quad \Big\downarrow \Phi_{k-1} \quad \Big\downarrow \Phi_{k-2} \quad\quad \Big\downarrow \Phi_1 \quad \Big\downarrow \Phi_0 \quad \Bigg\|$$

$$A': 0 \longrightarrow X' \longrightarrow Y'_{k-1} \longrightarrow Y'_{k-2} \longrightarrow \ldots \longrightarrow Y'_1 \longrightarrow Y'_0 \longrightarrow Z' \longrightarrow 0.$$

iii) For Z and X in C we denote by $\mathrm{E}^k_C(Z, X)$ the exact sequences of length k with right end Z and left end X. On $\mathrm{E}^k_C(Z, X)$ we define the following equivalence relation:

Two elements A and A' of $\mathrm{E}^k_C(Z, X)$ are said to be equivalent, $A \sim A'$, if there is a sequence $A = A_0, A_1, \ldots, A_{l-1}, A_l = A'$ of elements of $\mathrm{E}^k_C(Z, X)$, such that for every $0 \le i \le l-1$ there is a morphism with fixed ends either from A_i to A_{i+1} or from A_{i+1} to A_i. We define

$$\mathrm{Ext}^k_C(Z, X) = \mathrm{E}^k_C(Z, X)/_\sim$$

and denote by $[A]_C$ the equivalence class of A in $\mathrm{Ext}^k_C(Z, X)$.

iv) Using the pullback and pushout constructions (2.1.7) it is possible to endow the equivalence classes $\mathrm{Ext}^k_C(Z, X)$ with an abelian group structure via the Baer sum and assign each morphism $\lambda: X \longrightarrow X'$ in C a homomorphism of abelian groups $\mathrm{Ext}^k_C(Z, \lambda):$ $\mathrm{Ext}^k_C(Z, X) \longrightarrow \mathrm{Ext}^k_C(Z, X')$, cf. [Sie10, (4.1.5)].

Then we have:

Theorem 2.3.2. *Let $(C, \mathcal{E})$ be an exact category and let E be an object of C. For each $k \ge 1$ the assignment*

$$\mathrm{Ext}^k_C(E, \_): C \longrightarrow (\mathrm{AB}), \, X \mapsto \mathrm{Ext}^k_C(E, X)$$

$$\lambda \mapsto \mathrm{Ext}^k_C(E, \lambda)$$

is a covariant, additive, abelian-group-valued functor, which induces for every short exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ a long exact sequence

$$
\begin{aligned}
0 \longrightarrow \quad & \mathrm{Hom}(E, X) && \longrightarrow \mathrm{Hom}(E, Y) && \longrightarrow \mathrm{Hom}(E, Z) && \xrightarrow{\delta_0} \\
\xrightarrow{\delta_0} \quad & \mathrm{Ext}^1_C(E, X) && \longrightarrow \mathrm{Ext}^1_C(E, Y) && \longrightarrow \mathrm{Ext}^1_C(E, Z) && \xrightarrow{\delta_1} \\
\xrightarrow{\delta_1} \quad & \mathrm{Ext}^2_C(E, X) && \longrightarrow \quad \ldots && \longrightarrow \mathrm{Ext}^{k-1}_C(E, Z) && \xrightarrow{\delta_{k-1}} \\
\xrightarrow{\delta_{k-1}} \quad & \mathrm{Ext}^k_C(E, X) && \longrightarrow \mathrm{Ext}^k_C(E, Y) && \longrightarrow \mathrm{Ext}^k_C(E, Z) && \xrightarrow{\delta_k} \\
\xrightarrow{\delta_k} \quad & \mathrm{Ext}^{k+1}_C(E, X) && \longrightarrow \quad \ldots && .
\end{aligned}
$$

Proof. See [Sie10, Theorem (4.1.1)]. $\qquad\square$

Remark 2.3.3. Using the concept of universal δ functors it is possible to show that if an exact category $(C, \mathcal{E})$ has enough injective objects, for all objects E in C the $\mathrm{Ext}_C^k (E, \_)$ groups are isomorphic to the derived functors of $\mathrm{Hom}_C (E, \_)$ in the sense of Palamodov, cf. [Pal71] and [Sie10, p. 36].

Proposition 2.3.4. *Let $(C, \mathcal{E})$ be an exact category.*

i) *For any short exact sequence*

$$0 \longrightarrow X \xrightarrow{f} G \xrightarrow{g} E \longrightarrow 0$$

in C the following are equivalent:

   a) *f has a left inverse.*

   b) *g has a right inverse.*

   c) *There is a commutative diagram*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\binom{1}{0}} & X \oplus E & \xrightarrow{(0,1)} & E & \longrightarrow & 0 \\
& & \| & & \downarrow{\beta} & & \| & & \\
0 & \longrightarrow & X & \xrightarrow{f} & G & \xrightarrow{g} & E & \longrightarrow & 0,
\end{array}
$$

   such that β is an isomorphism.

ii) *For two objects X and E in C the following are equivalent:*

   a) *$\mathrm{Ext}_C^1 (E, X) = 0$*

   b) *Every exact sequence $0 \longrightarrow X \xrightarrow{f} G \xrightarrow{g} E \longrightarrow 0$ in C is split exact.*

Proof. See [Sie10, (4.2.1) & (4.2.2)]. □

With exactly the same proofs as in [Sie10, (5.1.6), (5.1.7) & (5.1.8)] we obtain the results connecting $\mathrm{Ext}_{\mathrm{PLS_w}}^k$ and $\mathrm{Ext}_{\mathrm{LS_w}}^k$ respectively $\mathrm{Ext}_{\mathrm{PLH}}^k$ and $\mathrm{Ext}_{\mathrm{LH}}^k$:

Proposition 2.3.5.

i) *If X and E are $\mathrm{LS_w}$ spaces, then for all $k \in \mathbb{N}$ there is an isomorphism of abelian groups $\mathrm{Ext}_{\mathrm{LS_w}}^k (E, X) \cong \mathrm{Ext}_{\mathrm{PLS_w}}^k (E, X)$.*

ii) *Let X be an $\mathrm{LS_w}$ space, $E = \mathrm{proj}\left(E_N, E_M^N\right)$ be a $\mathrm{PLS_w}$ space and $k \in \mathbb{N}$. Then the vanishing of all $\mathrm{Ext}_{\mathrm{LS_w}}^k (E_N, X)$ $(N \in \mathbb{N})$ implies the vanishing of $\mathrm{Ext}_{\mathrm{PLS_w}}^k (E, X)$.*

iii) Let $(X_N)_{N \in \mathbb{N}}$ be a sequence of $\mathrm{LS_w}$ spaces and $E = \mathrm{proj}_{N \in \mathbb{N}} \left(E_N, E_M^N \right)$ be a $\mathrm{PLS_w}$ space. Then the vanishing of all $\mathrm{Ext}^k_{\mathrm{LS_w}} (E_N, X_M)$ $(M, N \in \mathbb{N})$ implies the vanishing of $\mathrm{Ext}^k_{\mathrm{PLS_w}} (E, \prod_{N \in \mathbb{N}} E_N)$.

The statements remain valid if we replace $\mathrm{PLS_w}$ by PLH and $\mathrm{LS_w}$ by LH.

By applying the long exact sequence (2.3.2) to the canonical resolution of a PLS space X, Sieg established the same connection between $\mathrm{Ext}^1_{\mathrm{PLS}} (E, X)$ and $\mathrm{proj}^1 \left(\mathrm{L}\, (E, X_N), X_M^{N\,*} \right)$ in [Sie10, (5.2)], which we are used to from the Fréchet setting (see e.g. [Wen03, (5.1.5)]). Here $X_M^{N\,*}$ denotes as usual $\mathrm{Hom}_{\mathrm{LCS}} \left(E, X_M^N \right)$. The same proofs yield the same results in the categories $\mathrm{PLS_w}$ and PLH. We just have to check, that under the correct assumption, i.e. $\mathrm{proj}^1 \left(X_N, X_M^N \right) = 0$, the canonical resolution of X is an exact sequence of $\mathrm{PLS_w}$ (respectively PLH) spaces. Hence we recall:

Given a $\mathrm{PLS_w}$ space $X = \mathrm{proj} \left(X_N, X_M^N \right)$ we define the map

$$\Psi_X : \prod_{N \in \mathbb{N}} X_N \longrightarrow \prod_{N \in \mathbb{N}} X_N, (x_N)_{N \in \mathbb{N}} \longmapsto \left(X_{N+1}^N x_{N+1} - x_N \right)_{N \in \mathbb{N}}.$$

As X is the kernel of Ψ_X in $\mathrm{PLS_w}$ and $\left(X_N, X_M^N \right)$ is strongly reduced, [Wen03, (3.3.1)] yields that Ψ_X is open onto its range. Since it has dense range, it is the cokernel of its kernel. Thus the canonical resolution of X

$$0 \longrightarrow X \hookrightarrow \prod_{N \in \mathbb{N}} X_N \xrightarrow{\Psi_X} \prod_{N \in \mathbb{N}} X_N$$

is an element of $\mathcal{E}^{\mathrm{PLS_w}}_{\max}$ if and only if Ψ_X is surjective. Since $\mathrm{proj}^1 \left(X_N, X_M^N \right) = \prod_{N \in \mathbb{N}} X_N / {\mathrm{im}\, (\Psi_X)}$ by [Wen03, (3.1.4)], this is equivalent to the vanishing of $\mathrm{proj}^1 \left(X_N, X_M^N \right)$. Thus we arrive at the analogons of [Sie10, (5.2.1) & (5.2.3)]:

Theorem 2.3.6. *Let $X = \mathrm{proj} \left(X_N, X_M^N \right)$ and $E = \mathrm{proj} \left(E_N, E_M^N \right)$ be two $\mathrm{PLS_w}$ spaces. Then we have:*

i) If $\mathrm{proj}^1 \left(X_N, X_M^N \right) = 0$ and $\mathrm{Ext}^1_{\mathrm{PLS_w}} (E, X) = 0$, then $\mathrm{proj}^1 \left(\mathrm{L}\, (E, X_N), X_M^{N\,*} \right) = 0$.

ii) If

  a) $\mathrm{proj}^1 \left(X_N, X_M^N \right) = 0$ and

  b) $\mathrm{Ext}^k_{\mathrm{LS_w}} (E_N, X_M) = 0$ for all $M, N \in \mathbb{N}$ and all $1 \leq k \leq k_0$ for a $k_0 \in \mathbb{N}$,

  then

  α) $\mathrm{Ext}^1_{\mathrm{PLS_w}} (E, X) = \mathrm{proj}^1 \left(\mathrm{L}\, (E, X_N), X_M^{N\,*} \right)$ and

  β) $\mathrm{Ext}^k_{\mathrm{PLS_w}} (E, X) = 0$ for all $2 \leq k \leq k_0$.

The results remain true if we replace $\mathrm{LS_w}$ by LH and $\mathrm{PLS_w}$ by PLH.

Domański and Vogt [DV00b] respectively Wengenroth [Wen01] have proven with functional analytic methods that the space of Schwartz distributions $\mathscr{D}'(\Omega)$ plays the same role in the splitting theory of PLS spaces as the space of rapidly decreasing sequences s does in the splitting theory for Fréchet spaces, i.e. if E is isomorphic to a closed subspace of $\mathscr{D}'(\Omega)$ and X is isomorphic to a complete Hausdorff quotient of $\mathscr{D}'(\Omega)$, then $\operatorname{Ext}^k_{\mathrm{PLS}}(E, X) = 0$ for all $k \geq 1$. Sieg has given a purely homological proof in [Sie10, (5.3.6)], that we can transfer with minor adaptions to the categories $\mathrm{PLS_w}$ and PLH. We give an outline of the argumentation:

The sequence space representation $\mathscr{D}'(\Omega) = (s')^{\mathbb{N}}$ – which is due to Valdivia [Val78] and (independently) Vogt [Vog83b] – and (2.3.6) ii) are used to show that $\operatorname{Ext}^k_{\mathrm{PLS_w}}(E, X) = 0$ for all $k \geq 2$. As (2.3.5) iii) yields that $\operatorname{Ext}^1_{\mathrm{PLS_w}}(E, \mathscr{D}'(\Omega)) = 0$, we obtain $\operatorname{Ext}^1_{\mathrm{PLS_w}}(E, X) = 0$ by applying the long exact sequence (2.3.2) to the (even) topologically exact sequence

$$0 \longrightarrow \mathscr{D}'(\Omega) \longrightarrow \mathscr{D}'(\Omega) \longrightarrow X \longrightarrow 0,$$

which exists by [DV00b, (2.1)].

To apply (2.3.6) ii) and (2.3.5) iii) we need to prove that $\operatorname{proj}^1\left(X_N, X_M^N\right) = 0$ and $\operatorname{Ext}^k_{\mathrm{LS_w}}(E_N, X_M) = 0 = \operatorname{Ext}^k_{\mathrm{LH}}(E_N, X_M)$ for all $M, N, k \geq 1$. The first statement is a consequence of [DV00b, (1.1)], since X is ultrabornological as a quotient of $\mathscr{D}'(\Omega)$. In (2.2.6) we have seen the duality of $\mathcal{E}^{\mathrm{LS_w}}_{\max}$ and $\mathcal{E}^{\mathrm{FS_w}}_{\max}$ respectively $\mathcal{E}^{\mathrm{LH}}_{\max}$ and $\mathcal{E}^{\mathrm{FH}}_{\max}$. Hence the same reasoning as in [Sie10, (5.3.2)] yields that for two $\mathrm{LS_w}$ spaces X and E and any $k \geq 1$ the vanishing of $\operatorname{Ext}^k_{\mathrm{LS_w}}(E, X)$ is equivalent to the vanishing of $\operatorname{Ext}^k_{\mathrm{FS_w}}(X', E')$ and the same holds in the dual pair LH / FH. Thus, to obtain the second statement, we need to prove that $\operatorname{Ext}^k_{\mathrm{FS_w}}(E, X)$ and $\operatorname{Ext}^k_{\mathrm{FH}}(E, X)$ vanish for all $k \geq 1$ for closed subspaces E of s and Hausdorff quotients X of s. The assertion in the category of Fréchet spaces is yielded e.g. by [Wen03, (5.2.9) & (5.1.5)]. As being a reflexive Banach space is a three-space property by [RD81, (4.3)], being an $\mathrm{FS_w}$ space is a three-space property when we apply (2.2.15) to localize short exact sequences of Fréchet spaces. Thus the same argument as in [Sie10, (5.3.4)], i.e. Yoneda Ext are inherited by fully exact subcategories, cf. (2.2.18) i), yields the assertion in $\mathrm{FS_w}$.

The category of Fréchet-Hilbert spaces is not a fully exact subcategory of the category of Fréchet spaces by (2.2.18) i) a). However the case $k = 1$ is contained in [MV97, (30.1)], [Vog77a, (1.3)] and [VW80, (1.8)] (or [MV97, (31.5), (31.6), & (30.1)]) and the case $k \geq 2$ is contained in [Vog11]. Thus we arrive at

Theorem 2.3.7. *If E and X are $\mathrm{PLS_w}$ spaces such that E is isomorphic to a closed subspace of $\mathscr{D}'(\Omega)$ and X is isomorphic to a complete Hausdorff quotient of $\mathscr{D}'(\Omega)$, then $\operatorname{Ext}^k_{\mathrm{PLS_w}}(E, X) = 0$ for all $k \geq 1$.*

The statement remains valid if we replace $\mathrm{PLS_w}$ by PLH.

3 Splitting theory

Splitting of short exact sequences has always been approached directly or indirectly by characterizations of the vanishing of proj[1] for projective spectra of operator spaces, cf. e.g. [Wen03, BD06, BD08]. Theorem (2.3.6)ii) allows for the same in our situation. Given PLH spaces $E = \text{proj}\left(E_N, E_M^N\right)$ and $X = \text{proj}\left(X_N, X_M^N\right)$, we will consider the projective spectrum $\mathscr{L} := \left(\text{L}_b\left(E, X_N\right), X_M^{N\,*}\right)$. If E and X are proper PLH spaces, the theory of webs has to be applied to obtain conditions for the vanishing of proj[1] $\mathscr{L}$. This has been investigated by Frerick, Kunkle and Wengenroth in [FKW03, Kun01]. Thus we will restrict ourselves to the cases where E is either a Fréchet-Hilbert or an LH space, since then we will be able to access the vanishing of proj[1] $\mathscr{L}$ by using duality relations between the vanishing of proj[1] of spectra of LB spaces and (weak) acyclicity of pre-dual LF spaces, which can in turn be described through variants of the condition (M) of Retakh, cf. [Ret70], due to Palamodov [Pal71], Vogt [Vog92] and Wengenroth [Wen95, Wen96, Wen03]. We will use an interpolation result due to Domański and Mastyło [DM07, (3.1)] in Vogt's variant [Vog11, (1.1)] to translate those into applicable conditions describing the interrelation of the defining seminorms by norm inequalities preceded by a long sequence of quantifiers. This type of condition has a rather long tradition that was initiated amongst others by Vogt in the mid 1970's in the Fréchet space setting with a strong emphasis on the functional analytic aspects [Vog77a, Vog77b, Vog87]. A first milestone was the classical splitting result, the famous (DN)-(Ω) splitting theorem due to Vogt and Wagner [VW80]. Under the premise of four so-called standard assumptions, one of the spaces E or X being either a Köthe sequence space or nuclear, Vogt was able to give sufficient and necessary conditions on the pair (E, X) for the splitting of each exact sequence

$$0 \longrightarrow X \overset{f}{\longrightarrow} G \overset{g}{\longrightarrow} E \longrightarrow 0 \qquad\qquad (\star)$$

in [Vog87], where the sufficient condition had been proposed by Apiola [Api83]. A complete characterization in the nuclear case was proven by Frerick [Fre96] and further results on sufficient conditions in the general Fréchet setting were given by Frerick and Wengenroth [FW96] and Wengenroth [Wen03] with a more pronounced emphasis on the homological aspects. Finally in 2007 Domański and Mastyło were able to give a complete characterization of splitting in the category of Fréchet-Hilbert spaces [DM07]. The elegant formulation of the condition (S) they used is due to Langenbruch [Lan04]. Their interpolation result for nuclear operators [DM07, (3.1)] was based on ideas from Ovchinnikov [Ovc98], a significantly shorter proof has been presented by Vogt [Vog11]. With this last characterization splitting theory for Fréchet-Hilbert spaces can be considered rather complete. In [BD06, BD08] Bonet and Domański presented characterizations of splitting of each exact sequence $(\star)$ of PLS spaces if E either is a Fréchet-Schwartz or an LS space again under assumptions about nuclearity and being a Köthe sequence space, that we will call successors of the four standard assumptions. Although their conditions $(G)\,/\,(G_\varepsilon)$ and $(H)\,/\,(H_\varepsilon)$ coincide with our conditions $(T)\,/\,(T_\varepsilon)$, the methods to prove characterizations are completely different, as theirs are based on nuclearity, hence follow the general path of [Wen03, section 5.2], whilst ours follow Domański, Mastyło and Vogt, cf. [DM07, Vog11].

We start by organizing the well-known results about (pre-)duality and the vanishing of proj^1 for PLH spaces together with the standard conditions in a first section:

3.1 Preliminary remarks

In general the well-known duality relation between the vanishing of proj^1 and weak acyclicity reads as follows, cf. [Pal71], [Vog92, (4.1)] or [Wen03, p.110]:

Proposition 3.1.1. *Let $\mathscr{H} = \left(H_N, H_N^M\right)$ be an inductive spectrum of locally convex spaces and let $\mathscr{H}' = \left(H_N', \left(H_N^M\right)^t\right)$ be the dual projective spectrum. Then $\mathscr{H}$ is weakly acyclic if and only if $\text{proj}^1 \mathscr{H}' = 0$.*

The following characterizations of acyclicity for LF spaces are due to Palamodov [Pal71] and Vogt [Vog92] concerning "*i*) $\Leftrightarrow$ *ii*)" and Wengenroth [Wen03, (6.4)] concerning "*ii*) $\Leftrightarrow$ *iii*)", which is just [Wen95, (3.4)] in Langenbruch's formulation of the quantifiers, cf. [Lan04].

Proposition 3.1.2. *Let $\mathscr{H} = \left(H_N, H_N^M\right)$ be an inductive spectrum of Fréchet spaces. Then the following are equivalent:*

i) $\mathscr{H}$ is acyclic.

ii) $\mathscr{H}$ has property (M) of Retakh, i.e. there is an increasing sequence $(U_N)_{N\in\mathbb{N}}$ of absolutely convex 0-neighbourhoods $U_N \subset H_N, N \in \mathbb{N}$, such that

$$\underset{N\in\mathbb{N}}{\forall} \ \underset{M\geq N}{\exists} \ \underset{K\geq M}{\forall} \ \underset{V\in\mathscr{U}_0(H_M)}{\forall} \ \underset{W\in\mathscr{U}_0(H_K)}{\exists} \ W \cap U_N \subset V.$$

iii) $\mathscr{H}$ has property $(Q)_0$, i.e.

$$\underset{N\in\mathbb{N}}{\forall} \ \underset{M\geq N}{\exists} \ \underset{K\geq M}{\forall} \ \underset{U\in\mathscr{U}_0(H_N)}{\exists} \ \underset{V\in\mathscr{U}_0(H_M)}{\forall} \ \underset{W\in\mathscr{U}_0(H_K)}{\exists} \ W \cap U \subset V.$$

Regarding PLH spaces, this leads to variants of the condition (P_3) for projective spectra of regular LB spaces. Before we introduce those, we specify the general assumptions we may impose on PLH spaces and the PLH spectra giving rise to them and fix some notation:

Remark 3.1.3. Let $X = \text{proj}_{N\in\mathbb{N}} \text{ind}_{n\in\mathbb{N}} X_{N,n}$ be a PLH space. By $B_{N,n}$ we refer to the closed unit balls of the Hilbert spaces $X_{N,n}, N, n \in \mathbb{N}$. By multiplying the norms of the steps with the continuity estimates of the respective inclusions, we may assume that $B_{N,n}$ is a subset of $B_{N,n+1}$ for all $N, n \in \mathbb{N}$. Furthermore, by applying a simple thinning procedure similar to the one used in the proof of (3.2.4) ii), we may assume that $X_{N+1}^N \left(B_{N+1,n}\right)$ is a subset of $B_{N,n}$ for all $n, N \in \mathbb{N}$. For any step $N \in \mathbb{N}$ and any linear functional $f \in X_N'$ we denote $\|f\|_{N,n}^* := \sup_{x\in B_{N,n}} |f(x)| \in [0, \infty]$.

Definition and Remark 3.1.4. A PLH space $X = \text{proj}_{N \in \mathbb{N}} \text{ind}_{n \in \mathbb{N}} X_{N,n}$ has property

i) $(P_3)_0$ if

$$\underset{N \in \mathbb{N}}{\forall} \underset{M \geq N}{\exists} \underset{K \geq M}{\forall} \underset{\substack{n \in \mathbb{N} \\ \varepsilon > 0}}{\exists} \underset{\substack{m \geq n \\ c > 0}}{\forall} \underset{k \geq m}{\exists} X_M^N (B_{M,m}) \subset \varepsilon B_{N,n} + c X_K^N (B_{K,k}),$$

which – by the bipolar theorem – is equivalent to its dual variant $(P_3^*)_0$:

$$\underset{N \in \mathbb{N}}{\forall} \underset{M \geq N}{\exists} \underset{K \geq M}{\forall} \underset{\substack{n \in \mathbb{N} \\ \varepsilon > 0}}{\exists} \underset{\substack{m \geq n \\ c > 0}}{\forall} \underset{k \geq m}{\exists} \underset{f \in X_N'}{\forall} \left\| \left(X_M^N \right)^t f \right\|_{M,m}^* \leq \varepsilon \|f\|_{N,n}^* + c \left\| \left(X_K^N \right)^t f \right\|_{K,k}^*.$$

ii) (P_3) if

$$\underset{N \in \mathbb{N}}{\forall} \underset{M \geq N}{\exists} \underset{K \geq M}{\forall} \underset{n \in \mathbb{N}}{\exists} \underset{\substack{m \geq n \\ c > 0}}{\forall} \underset{k \geq m}{\exists} X_M^N (B_{M,m}) \subset c \left(B_{N,n} + X_K^N (B_{K,k}) \right),$$

which – again by the bipolar theorem – is equivalent to its dual variant (P_3^*):

$$\underset{N \in \mathbb{N}}{\forall} \underset{M \geq N}{\exists} \underset{K \geq M}{\forall} \underset{n \in \mathbb{N}}{\exists} \underset{\substack{m \geq n \\ c > 0}}{\forall} \underset{k \geq m}{\exists} \underset{f \in X_N'}{\forall} \left\| \left(X_M^N \right)^t f \right\|_{M,m}^* \leq c \left(\|f\|_{N,n}^* + \left\| \left(X_K^N \right)^t f \right\|_{K,k}^* \right).$$

First of all the question arises, whether the above conditions are linear topological invariants, i.e. if they are invariant under passing to equivalent spectra. The positive answer to this is contained in (3.2.1) iii). Furthermore, by (3.1.2), the conditions $(P_3)_0$ and $(P_3^*)_0$ are equivalent to the dual LFH spectrum $\left(X_N', \left(X_{N+1}^N \right)^t \right)$ having condition (M) of Retakh or being acyclic. Concerning the conditions (P_3) and (P_3^*) and weak acyclicity the situation is different. We only know that weak acyclicity of $\left(X_N', \left(X_{N+1}^N \right)^t \right)$ implies the following condition (wQ) by [Vog92, (2.3) b)]:

$$\underset{\substack{N \in \mathbb{N} \\ U \in \mathcal{U}_0(X_N')}}{\forall} \underset{\substack{M \geq N \\ V \in \mathcal{U}_0(X_M')}}{\exists} \underset{K \geq M}{\forall} \underset{\substack{W \in \mathcal{U}_0(X_L') \\ c > 0}}{\exists} W \cap \left(X_K^N \right)^t (U) \subset c \left(X_K^M \right)^t (V),$$

which obviously implies (P_3^*), thus also (P_3). The other implication, i.e. the equivalence of (wQ) and (wQ^*) remains a conjecture, cf. [Wen95, (5.3)]. Thus we have the implications

$$(P_3)_0 \Leftrightarrow (P_3^*)_0 \Rightarrow \text{proj}^1 X = 0 \Rightarrow (P_3^*) \Leftrightarrow (P_3).$$

If X is a PLS space, all of the above are equivalent by [Wen03, (3.2.18)]. This is not true for

PLH spaces, as there is even a Fréchet-Hilbert space X that is not quasinormable by [MV97, (27.23)c)], hence it does not satisfy $(P_3)_0$ by [MV97, (26.14)], whereas $\operatorname{proj}^1 X$ always vanishes by [Wen03, (3.2.1)] as we may choose a reduced projective spectrum of Hilbert spaces giving rise to X.

In the case of $\mathscr{L}$, i.e. in the case of operator spectra, more detailed statements are possible. First of all the pre-duality relation from (3.1.1) is given by projective tensor products via the following relation, which is well-known:

Proposition 3.1.5. *i) Let Y and Z be two Fréchet spaces. Then the map*

$$\varphi \colon (Y \tilde{\otimes}_\pi Z)' \longrightarrow L(Y, Z'), f \mapsto T_f \colon \Big(Y \ni y \mapsto (Z \ni z \mapsto f(y \otimes z)) \Big)$$

is an algebraic isomorphism.

ii) Given four Fréchet spaces Y_0, Y_1, Z_0, Z_1 and linear and continuous operators $Y_0^1 \colon Y_0 \longrightarrow Y_1, Z_0^1 \colon Z_0 \longrightarrow Z_1$, the following diagram commutes:

$$
\begin{array}{ccc}
(Y_1 \tilde{\otimes}_\pi Z_1)' & \xrightarrow{\ \varphi_1\ } & L\Big(Y_1, Z_1'\Big) \\
{\scriptstyle (Y_0^1 \tilde{\otimes}_\pi Z_0^1)^t} \Big\downarrow & & \Big\downarrow {\scriptstyle Y_{0\,*}^1 \circ (Z_0^1)^{t\,*}} \\
(Y_0 \tilde{\otimes}_\pi Z_0)' & \xrightarrow[\ \varphi_0\ ]{} & L\Big(Y_0, Z_0'\Big),
\end{array}
$$

where φ_0, φ_1 are the maps defined in i) and $Y_{0\,*}^1$ denotes as usual $\operatorname{Hom}_{\mathrm{LCS}}\Big(Y_0^1, Z_0'\Big)$.

Proof. i) The map φ is linear, well-defined and injective, which is guaranteed by the basic properties of the π tensor product. Furthermore, φ is surjective as any separately continuous bilinear form on a product of Fréchet spaces is jointly continuous by [Köt79, §40, 2, (1) (a)].

ii) For $f \in (Y_1 \tilde{\otimes}_\pi Z_1)'$, $y_0 \in Y_0$ and $z_0 \in Z_0$ we calculate

$$\varphi_0 \left(\Big(Y_0^1 \tilde{\otimes}_\pi Z_0^1 \Big)^t (f) \right) (y_0)(z_0) = f\Big(Y_0^1 y_0 \otimes Z_0^1 z_0 \Big) = \Big(Y_{0\,*}^1 \circ \big(Z_0^1 \big)^{t\,*} \Big) \big(\varphi_1(f) \big)(y_0)(z_0).$$

$\square$

This tensor product representation allows us to transform (weak) acyclicity of the pre-dual spectra into evaluable conditions that we call (T_ε) respectively (T), cf. [Dom10, (5.2)], [BD06, (4.1)] and [BD08, (3.1)]. In the following section we will introduce those conditions and the notion of *deep reducedness*, which we will need to apply Vogt's version (3.2.6) of Domański's and Mastyło's interpolation result [DM07, (3.1)], which will be stated here as well, with all the necessary technical requirements.

3.2 Remarks on the splitting conditions and deep reducedness.

Definition and Remark 3.2.1. Let $\mathscr{X} = \left(X_N, X_M^N\right)$ and $\mathcal{E} = \left(E_T, E_S^T\right)$ be two PLH space spectra, where $X_N = \operatorname{ind}_{n\in\mathbb{N}} X_{N,n}$ and $E_T = \operatorname{ind}_{t\in\mathbb{N}} E_{T,t}$, $N, T \in \mathbb{N}$.

  i) The pair $(\mathcal{E}, \mathscr{X})$ satisfies the property

    a) (T_ε) if the following holds:

$$
\underset{\substack{N\in\mathbb{N}\\T\in\mathbb{N}}}{\forall}\ \underset{\substack{M\geq N\\S\geq T}}{\exists}\ \underset{\substack{K\geq M\\R\geq S}}{\forall}\ \underset{\substack{n\in\mathbb{N}\\t\in\mathbb{N}}}{\exists}\ \underset{\substack{m\geq n\\s\geq t\\\varepsilon>0}}{\forall}\ \underset{\substack{k\geq m\\r\geq s\\c>0}}{\exists}\ \underset{\substack{f\in X_N'\\g\in E_T'}}{\forall}
$$

$$
\left\|\left(X_M^N\right)^t f\right\|_{M,m}^* \left\|\left(E_S^T\right)^t g\right\|_{S,s}^* \leq \varepsilon \|f\|_{N,n}^* \|g\|_{T,t}^* + c \left\|\left(X_K^N\right)^t f\right\|_{K,k}^* \left\|\left(E_R^T\right)^t f\right\|_{R,r}^*.
$$

    b) (T) if the following holds:

$$
\underset{\substack{N\in\mathbb{N}\\T\in\mathbb{N}}}{\forall}\ \underset{\substack{M\geq N\\S\geq T}}{\exists}\ \underset{\substack{K\geq M\\R\geq S}}{\forall}\ \underset{\substack{n\in\mathbb{N}\\t\in\mathbb{N}}}{\exists}\ \underset{\substack{m\geq n\\s\geq t}}{\forall}\ \underset{\substack{k\geq m\\r\geq s\\c>0}}{\exists}\ \underset{\substack{f\in X_N'\\g\in E_T'}}{\forall}
$$

$$
\left\|\left(X_M^N\right)^t f\right\|_{M,m}^* \left\|\left(E_S^T\right)^t g\right\|_{S,s}^* \leq c \left(\|f\|_{N,n}^* \|g\|_{T,t}^* + \left\|\left(X_K^N\right)^t f\right\|_{K,k}^* \left\|\left(E_R^T\right)^t g\right\|_{R,r}^* \right).
$$

  ii)   a) (T_ε) is equivalent to (T_ε^d):

$$
\underset{N\in\mathbb{N}}{\forall}\ \underset{M\geq N}{\exists}\ \underset{K\geq M}{\forall}\ \underset{n\in\mathbb{N}}{\exists}\ \underset{\substack{m\geq n\\\varepsilon>0}}{\forall}\ \underset{\substack{k\geq m\\c>0}}{\exists}\ \underset{\substack{f\in X_N'\\g\in E_N'}}{\forall}
$$

$$
\left\|\left(X_M^N\right)^t f\right\|_{M,m}^* \left\|\left(E_M^N\right)^t g\right\|_{M,m}^* \leq \varepsilon \|f\|_{N,n}^* \|g\|_{N,n}^* + c \left\|\left(X_K^N\right)^t f\right\|_{K,k}^* \left\|\left(E_K^N\right)^t g\right\|_{K,k}^*.
$$

    b) (T) is equivalent to (T^d):

$$
\underset{N\in\mathbb{N}}{\forall}\ \underset{M\geq N}{\exists}\ \underset{K\geq M}{\forall}\ \underset{n\in\mathbb{N}}{\exists}\ \underset{m\geq n}{\forall}\ \underset{\substack{k\geq m\\c>0}}{\exists}\ \underset{\substack{f\in X_N'\\g\in E_N'}}{\forall}
$$

$$
\left\|\left(X_M^N\right)^t f\right\|_{M,m}^* \left\|\left(E_M^N\right)^t g\right\|_{M,m}^* \leq c \left(\|f\|_{N,n}^* \|g\|_{N,n}^* + \left\|\left(X_K^N\right)^t f\right\|_{K,k}^* \left\|\left(E_K^N\right)^t g\right\|_{K,k}^* \right).
$$

Proof. We only prove ii)a).

Necessity: Let $N \in \mathbb{N}$ be given. We chose $M \geq N$ and $S \geq T = N$ according to (T_ε). Given $K \geq M_0 := \max\{M, S\}$, we chose $n, t \in \mathbb{N}$ for $R = K$. Given $\varepsilon > 0$ and $s = m \geq n_0 := \max\{n, t\}$, we define $c_1 := \left\|\left(X_{M_0}^M\right)^t\right\|_{L\left(X_{M,m}', X_{M_0,m}'\right)}, c_2 := \left\|\left(E_{M_0}^S\right)^t\right\|_{L\left(E_{S,s}', E_{M_0,s}'\right)}$

and find $k, r \in \mathbb{N}$ and a $c > 0$ for $s = m$ and $\frac{\varepsilon}{c_1 c_2}$. Defining $c := c_1 c_2 c_3$, we obtain for any $f \in X'_N$ and $g \in E'_N$

$$
\begin{aligned}
\left\|\left(X_{M_0}^N\right)^t f\right\|_{M_0,m}^* \left\|\left(E_{M_0}^N\right)^t g\right\|_{M_0,m}^* &\leq c_1 c_2 \left\|\left(X_M^N\right)^t f\right\|_{M,m}^* \left\|\left(E_S^T\right)^t g\right\|_{S,s}^* \\
&\leq \varepsilon \|f\|_{N,n}^* \|g\|_{T,t}^* + c \left\|\left(X_K^N\right)^t f\right\|_{K,k}^* \left\|\left(E_R^T\right)^t g\right\|_{R,r}^* \\
&\leq \varepsilon \|f\|_{N,n_0}^* \|g\|_{N,n_0}^* + c \left\|\left(X_K^N\right)^t f\right\|_{K,k_0}^* \left\|\left(E_K^N\right)^t g\right\|_{K,k_0}^* .
\end{aligned}
$$

Sufficiency: Let $N, T \in \mathbb{N}$. We chose $S = M \geq N_0 := \max\{N, T\}$. Given $K, R \geq M$, we chose $n = t \in \mathbb{N}$ for $K_0 := \max\{K, R\}$. Given $m \geq n, s \geq t$ and $\varepsilon > 0$, we define $c_1 := \left\|\left(X_{N_0}^N\right)^t\right\|_{\mathrm{L}\left(X'_{N,n}, X'_{N_0,n}\right)}, c_2 := \left\|\left(E_{N_0}^T\right)^t\right\|_{\mathrm{L}\left(E'_{T,n}, E'_{N_0,n}\right)}, c_3 := \left\|\left(X_{K_0}^K\right)^t\right\|_{\mathrm{L}\left(X'_{K,k}, E'_{K_0,k}\right)}, c_4 := \left\|\left(E_{K_0}^R\right)^t\right\|_{\mathrm{L}\left(E'_{R,k}, E'_{K_0,k}\right)}$ and find $k = r \geq m_0 := \max\{m, s\}$ and $c_5 > 0$ for $\frac{\varepsilon}{c_1 c_2}$ and m_0. Defining $c := c_3 c_4 c_5$, we obtain for any $f \in X'_N$ and $g \in E'_T$

$$
\begin{aligned}
\left\|\left(X_M^N\right)^t f\right\|_{M,m}^* \left\|\left(E_S^T\right)^t g\right\|_{S,s}^* \leq &\ \left\|\left(X_M^{N_0}\right)^t \left(X_{N_0}^N\right)^t f\right\|_{M,m_0}^* \left\|\left(E_M^{N_0}\right)^t \left(E_{N_0}^T\right)^t g\right\|_{M,m_0}^* \\
\leq &\ \frac{\varepsilon}{c_1 c_2} \left\|\left(X_{N_0}^N\right)^t f\right\|_{N,n}^* \left\|\left(E_{N_0}^T\right)^t g\right\|_{N,t}^* + \\
&\ c_5 \left\|\left(X_{K_0}^{N_0}\right)^t \left(X_{N_0}^N\right)^t f\right\|_{K_0,k}^* \left\|\left(E_{K_0}^{N_0}\right)^t \left(E_{N_0}^T\right)^t g\right\|_{K_0,k}^* \\
\leq &\ \varepsilon \|f\|_{N,n}^* \|g\|_{T,t}^* + c \left\|\left(X_K^N\right)^t f\right\|_{K,k}^* \left\|\left(E_R^T\right)^t g\right\|_{R,r}^* .
\end{aligned}
$$

$\square$

iii) The conditions (T_ε) and (T) are passed on to equivalent spectra.

Proof. Again we only prove the assumption for (T_ε). Furthermore, since the condition is symmetric, it is sufficient to consider one equivalent PLH space spectrum. Thus let $\mathscr{Y} = \left(Y_N, Y_M^N\right)$, where $Y_N = \mathrm{ind}_{n \in \mathbb{N}} Y_{N,n}$, is a PLH space spectrum that is equivalent to $\mathscr{X}$. First we show the following

Claim: If $(L(N))_{N \in \mathbb{N}}$ is a strictly increasing sequence of natural numbers and $\tilde{\mathscr{X}} = \left(X_{L(N)}, X_{L(M)}^{L(N)}\right)$ is the induced PLH space spectrum, then $(\mathcal{E}, \mathscr{X})$ satisfies (T_ε) iff so does $\left(\mathcal{E}, \tilde{\mathscr{X}}\right)$:

Necessity: Given $N, T \in \mathbb{N}$, we chose $M \geq L(N), S \geq T$ for $L(N), T$. Given $K \geq M, R \geq S$, we chose n, t for $L(K), R$. Given $m \geq n, s \geq t, \varepsilon > 0$ we define $c_1 := \left\|\left(X_{L(M)}^M\right)^t\right\|_{\mathrm{L}\left(X'_{M,m}, X'_{L(M),m}\right)}$ and choose $k \geq m, r \geq s, c_2 > 0$ for $m, s, \frac{\varepsilon}{c_1}$. With $c := c_1 c_2$ we

obtain for all $f \in X'_{L(N)}, g \in E'_T$

$$\left\|\left(X^{L(N)}_{L(M)}\right)^t f\right\|^*_{L(M),m} \left\|\left(E^T_S\right)^t g\right\|^*_{S,s} \leq c_1 \left\|\left(X^{L(N)}_M\right)^t f\right\|^*_{M,m} \left\|\left(E^T_S\right)^t g\right\|^*_{S,s}$$

$$\leq \varepsilon \|f\|^*_{L(N),n} \|g\|^*_{T,t} + c \left\|\left(X^{L(N)}_{L(K)}\right)^t f\right\|^*_{L(K),k} \left\|\left(E^T_R\right)^t g\right\|^*_{R,r}. \diamond$$

Sufficiency: Let $N, T \in \mathbb{N}$. We choose $M \geq L(N), S \geq T$ for $L(N), T$.

Given $K \geq M, R \geq S$, we choose n, t for $L(K), R$. Given $m \geq n, s \geq t, \varepsilon > 0$, we define $c_1 := \left\|\left(X^N_{L(N)}\right)^t\right\|_{\mathrm{L}\left(X'_{N,n}, X'_{L(N),n}\right)}$ and choose $k \geq m, r \geq s, c_3 > 0$ for m, s and $\frac{\varepsilon}{c_1}$. Defining $c_2 := \left\|\left(X^K_{L(K)}\right)^t\right\|_{\mathrm{L}\left(X'_{K,k}, X'_{L(K),k}\right)}$ and $c := c_2 c_3$, we obtain for all $f \in X'_N, g \in E'_T$

$$\left\|\left(X^N_{L(M)}\right)^t f\right\|^*_{L(M),m} \left\|\left(E^T_S\right)^t g\right\|^*_{S,s} = \left\|\left(X^{L(N)}_{L(M)}\right)^t \left(X^N_{L(N)}\right)^t f\right\|^*_{L(M),m} \left\|\left(E^T_S\right)^t g\right\|^*_{S,s}$$

$$\leq \frac{\varepsilon}{c_1} \left\|\left(X^N_{L(N)}\right)^t f\right\|^*_{L(N),n} \|g\|^*_{T,t} + c_3 \left\|\left(X^N_{L(K)}\right)^t f\right\|^*_{L(K),k} \left\|\left(E^T_R\right)^t g\right\|^*_{R,r}$$

$$\leq \varepsilon \|f\|^*_{N,n} \|g\|^*_{T,t} + c \left\|\left(X^N_K\right)^t f\right\|^*_{K,k} \left\|\left(E^T_R\right)^t g\right\|^*_{R,r}. \diamond$$

Now suppose that $(\mathcal{E}, \mathcal{X})$ has (T_ε). With what we have just shown, we may assume that the equivalence of $\mathcal{X}$ and $\mathcal{Y}$ instantiates in operators $\alpha_N \in \mathrm{L}(Y_N, X_N), \beta_N \in \mathrm{L}(X_{N+1}, Y_N)$ with $\alpha_N \circ \beta_N = X^N_{N+1}$ and $\beta_N \circ \alpha_{N+1} = Y^N_{N+1}$, $N \in \mathbb{N}$.

Let $N, T \in \mathbb{N}$. We choose $M \geq N + 1, S \geq T$ for $N + 1, T$. Given $K \geq M, R \geq S$, we choose n, t for $K + 1, R$. By the factorization theorem there is an integer $n_1 \geq n$ such that $c_1 := \left\|\beta^t_N\right\|_{\mathrm{L}\left(Y'_{N,n_1}, X'_{N+1,n}\right)} < \infty$. Given $m \geq n, s \geq t, \varepsilon > 0$, again the factorization theorem provides integers $m_1 \geq m_0 \geq m$ such that $c_2 := \left\|\alpha^t_M\right\|_{\mathrm{L}\left(X'_{M,m_0}, Y'_{M,m_1}\right)} < \infty$. We choose $k \geq m_0, r \geq s, c_4 > 0$ for m_0, s and $\frac{\varepsilon}{c_1 c_2}$. Then there is an integer $k_1 \geq k$ such that $c_3 := \left\|\beta^t_K\right\|_{\mathrm{L}\left(Y'_{K,k_1}, X'_{K+1,k}\right)} < \infty$. Defining $c := c_2 c_3 c_4$, we obtain for all $f \in Y'_N, g \in E'_T$

$$\left\|\left(Y^N_M\right)^t f\right\|^*_{M,m_1} \left\|\left(E^T_S\right)^t g\right\|^*_{S,s} = \left\|\alpha^t_M \left(X^{N+1}_M\right)^t \beta^t_N f\right\|^*_{M,m_1} \left\|\left(E^T_S\right)^t g\right\|^*_{S,s}$$

$$\leq \frac{\varepsilon}{c_1} \left\|\beta^t_N f\right\|^*_{N,n} \|g\|^*_{T,t} + c_2 c_4 \left\|\left(X^{N+1}_{K+1}\right)^t \beta^t_N f\right\|^*_{K+1,k} \left\|\left(E^T_R\right)^t g\right\|^*_{R,r}$$

$$\leq \varepsilon \|f\|^*_{N,n_1} \|g\|^*_{T,t} + c \left\|\left(Y^N_K\right)^t f\right\|^*_{K,k_1} \left\|\left(E^T_R\right)^t g\right\|^*_{R,r}.$$

$\square$

iv) Based on iii) and (2.2.2) ii), i.e. the fact that all strongly reduced PLH space spectra that give rise to the same PLH space are equivalent, we may define all four properties for pairs (E, X) of PLH *spaces* (instead of *spectra*).

v) As X'_N and E'_T are dense in $X'_{N,k}$ and $E'_{T,r}$ for all N, T, k and $r \in \mathbb{N}$ respectively, we may replace in all four conditions the quantifier $\forall f \in X'_N, g \in E'_T$ by $\forall f \in X'_{N,k}, g \in E'_{T,r}$.

The following result connects (T_ε) and (T), compare [Dom10, (5.2)"(2)$\Rightarrow$(3)"].

Proposition 3.2.2. *Let* $E = \mathrm{proj}_{T\in\mathbb{N}} \, \mathrm{ind}_{t\in\mathbb{N}} \, E_{T,t}$ *and* $X = \mathrm{proj}_{N\in\mathbb{N}} \, \mathrm{ind}_{n\in\mathbb{N}} \, X_{N,n}$ *be PLH spaces both non-trivial, i.e. both not the zero space. Then* (T_ε) *holds for* (E, X) *if and only if* E *and* X *both satisfy* $(P_3^*)_0$ *and* (T) *holds for the pair* (E, X).

Proof. Necessity is trivial. Sufficiency: We assume $(P_3^*)_0$ in the forms

$$
\underset{\substack{N\in\mathbb{N} \\ }}{\forall} \; \underset{\substack{N_1\geq N \\ \varepsilon_X>0}}{\exists} \; \underset{\substack{K\geq N_1 \\ c_X>0}}{\forall} \; \underset{\substack{n_1\in\mathbb{N} \\ }}{\exists} \; \underset{\substack{m\geq n_1 \\ }}{\forall} \; \underset{\substack{k_1\geq m \\ }}{\exists} \; \underset{\substack{f\in X'_N \\ }}{\forall}
$$

$$
\left\| \left(X_{N_1}^N\right)^t f \right\|_{N_1,m}^* \leq \varepsilon_X \|f\|_{N,n_1}^* + c_X \left\| \left(X_K^N\right)^t f \right\|_{K,k_1}^*
$$

for X, and

$$
\underset{\substack{T\in\mathbb{N} \\ }}{\forall} \; \underset{\substack{T_1\geq T \\ \varepsilon_E>0}}{\exists} \; \underset{\substack{R\geq T_1 \\ c_E>0}}{\forall} \; \underset{\substack{t_1\in\mathbb{N} \\ }}{\exists} \; \underset{\substack{s\geq t_1 \\ }}{\forall} \; \underset{\substack{r_1\geq s \\ }}{\exists} \; \underset{\substack{g\in E'_T \\ }}{\forall}
$$

$$
\left\| \left(E_{T_1}^T\right)^t g \right\|_{T_1,s}^* \leq \varepsilon_E \|g\|_{T,t_1}^* + c_E \left\| \left(E_R^T\right)^t g \right\|_{R,r_1}^*
$$

for E, as well as (T) for (E, X) in the form

$$
\underset{\substack{N_1\in\mathbb{N} \\ T_1\in\mathbb{N}}}{\forall} \; \underset{\substack{M\geq N_1 \\ S\geq T_1}}{\exists} \; \underset{\substack{K\geq M \\ R\geq S}}{\forall} \; \underset{\substack{n_0\in\mathbb{N} \\ t_0\in\mathbb{N}}}{\exists} \; \underset{\substack{m\geq n_0 \\ s\geq t_0}}{\forall} \; \underset{\substack{k_0\geq m \\ r_0\geq s \\ c_0>0}}{\exists} \; \underset{\substack{f\in X'_{N_1} \\ g\in E'_{T_1}}}{\forall}
$$

$$
\left\| \left(X_M^{N_1}\right)^t f \right\|_{M,m}^* \left\| \left(E_S^{T_1}\right)^t g \right\|_{S,s}^* \leq c_0 \left(\|f\|_{N_1,n_0}^* \|g\|_{T_1,t_0}^* + \left\| \left(X_K^{N_1}\right)^t f \right\|_{K,k_0}^* \left\| \left(E_R^{T_1}\right)^t g \right\|_{R,r_0}^* \right).
$$

Now let $N, T \in \mathbb{N}$. We choose $N_1 \geq N, T_1 \geq T$ according to $(P_3^*)_0$ for E and X respectively. Then we choose $M \geq N_1, S \geq T_1$ according to (T) for N_1, T_1. Given $K \geq M, R \geq S$, we choose $n_1, t_1 \in \mathbb{N}$ according to $(P_3^*)_0$ for $K \geq N_1 \geq N$ and $R \geq T_1 \geq T$ respectively, as well as n_0, t_0 according to (T) for $K \geq M \geq N_1, R \geq S \geq T_1$. Given $m \geq n := \max\{n_0, n_1\}, s \geq t := \max\{t_0, t_1\}$, we choose $k_0 \geq m, r_0 \geq s, c_0 > 0$ according to (T) for $m \geq n_0, s \geq t_0$. Given $\varepsilon > 0$, we choose $r_1 \geq s, c_E \geq 1$ according to $(P_3^*)_0$ for E and $s \geq t_1, \varepsilon_E := \frac{\varepsilon}{2c_0}$, as well as $k_1 \geq m, c_X > 0$ according to $(P_3^*)_0$ for X and $m \geq n_1, \varepsilon_X := \frac{\varepsilon}{2c_0 c_E}$. Denoting $k := \max\{k_0, k_1\}, r := \max\{r_0, r_1\}$ and $c := c_0(c_X c_E + 1)$, we obtain for all $f \in X'_N, g \in E'_T$

$$
\left\| \left(X_M^N\right)^t f \right\|_{M,m}^* \left\| \left(E_S^T\right)^t g \right\|_{S,s}^* = \left\| \left(X_M^{N_1}\right)^t \left(X_{N_1}^N\right)^t f \right\|_{M,m}^* \left\| \left(E_S^{T_1}\right)^t \left(E_{T_1}^T\right)^t g \right\|_{S,s}^* \leq
$$

$$
c_0 \left\| \left(X_{N_1}^N\right)^t f \right\|_{N_1,n_0}^* \left\| \left(E_{T_1}^T\right)^t g \right\|_{T_1,t_0}^* + c_0 \left\| \left(X_K^N\right)^t f \right\|_{K,k_0}^* \left\| \left(E_R^T\right)^t g \right\|_{R,r_0}^* =: (\star).
$$

If $\left\|\left(E_{T_1}^T\right)^t g\right\|_{T_1,t_0}^* \leq \left\|\left(E_R^T\right)^t g\right\|_{R,r}^*$, we apply $(P_3^*)_0$ for X to the first term and obtain

$$
\begin{aligned}
(\star) \leq \quad & c_0 \varepsilon_X \|f\|_{N,n_1}^* \left\|\left(E_{T_1}^T\right)^t g\right\|_{T_1,t_0}^* + c_0 c_X \left\|\left(X_K^N\right)^t f\right\|_{K,k_1}^* \left\|\left(E_{T_1}^T\right)^t g\right\|_{T_1,t_0}^* + \\
& c_0 \left\|\left(X_K^N\right)^t f\right\|_{K,k_0}^* \left\|\left(E_R^T\right)^t g\right\|_{R,r_0}^* \\
\leq \quad & \varepsilon \|f\|_{N,n}^* \left\|\left(E_{T_1}^T\right)^t g\right\|_{T_1,t}^* + c_0 c_X \left\|\left(X_K^N\right)^t f\right\|_{K,k}^* \left\|\left(E_R^T\right)^t g\right\|_{R,r}^* + c_0 \left\|\left(X_K^N\right)^t f\right\|_{K,k}^* \left\|\left(E_R^T\right)^t g\right\|_{R,r}^* \\
\leq \quad & \varepsilon \|f\|_{N,n}^* \|g\|_{T,t}^* + c \left\|\left(X_K^N\right)^t f\right\|_{K,k}^* \left\|\left(E_R^T\right)^t g\right\|_{R,r}^* .
\end{aligned}
$$

If $\left\|\left(E_{T_1}^T\right)^t g\right\|_{T_1,t_0}^* > \left\|\left(E_R^T\right)^t g\right\|_{R,r}^*$, we apply $(P_3^*)_0$ for E to the first term and obtain

$$
\begin{aligned}
(\star) \leq \quad & c_0 \varepsilon_E \left\|X_{N_1}^N f\right\|_{N_1,n_0}^* \|g\|_{T,t_1}^* + c_0 c_E \left\|\left(X_{N_1}^N\right)^t f\right\|_{N_1,n_0}^* \left\|\left(E_R^T\right)^t g\right\|_{R,r_1}^* + \\
& c_0 \left\|\left(X_K^N\right)^t f\right\|_{K,k_0}^* \left\|\left(E_R^T\right)^t g\right\|_{R,r_0}^* =: (\star\star) .
\end{aligned}
$$

Applying $(P_3^*)_0$ for X to the second term, we arrive at

$$
\begin{aligned}
(\star\star) \leq & c_0 \varepsilon_E \left\|\left(X_{N_1}^N\right)^t f\right\|_{N_1,n_0}^* \|g\|_{T,t_1}^* + c_0 c_E \varepsilon_X \|f\|_{N,n_1}^* \left\|\left(E_R^T\right)^t g\right\|_{R,r_1}^* + \\
& c_0 c_E c_X \left\|\left(X_K^N\right)^t f\right\|_{K,k_1}^* \left\|\left(E_R^T\right)^t g\right\|_{R,r_1}^* + c_0 \left\|\left(X_K^N\right)^t f\right\|_{K,k_0}^* \left\|\left(E_R^T\right)^t g\right\|_{R,r_0}^* \\
\leq & c_0 \varepsilon_E \|f\|_{N,n}^* \|g\|_{T,t}^* + c_0 c_E \varepsilon_X \|f\|_{N,n}^* \left\|\left(E_{T_1}^T\right)^t g\right\|_{T_1,t_0}^* + \\
& c_0 c_E c_X \left\|\left(X_K^N\right)^t f\right\|_{K,k}^* \left\|\left(E_R^T\right)^t g\right\|_{R,r}^* + c_0 \left\|\left(X_K^N\right)^t f\right\|_{K,k}^* \left\|\left(E_R^T\right)^t g\right\|_{R,r}^* \\
\leq & \varepsilon \|f\|_{N,n}^* \|g\|_{T,t}^* + c \left\|\left(X_K^N\right)^t f\right\|_{K,k}^* \left\|\left(E_R^T\right)^t g\right\|_{R,r}^* .
\end{aligned}
$$

$\square$

In order to establish the aspired splitting results, we need interpolation results for couples of duals of the Hilbert space steps $X_{N,n}$ giving rise to X. As we do not have those at hand for generalized couples, we need the injectivity of the transposed of the restrictions of the connecting morphisms $X_M^N|_{X_{M,n}}$ and the inclusions $X_{N,n} \hookrightarrow X_{N,n+1}$, $M \geq N, n \in \mathbb{N}$. Hence we introduce the following property, see [Dom10, (5.1)]:

Definition and Remark 3.2.3. i) A PLH space X is called *deeply reduced*, if there is a PLH space spectrum $\left(X_N, X_M^N\right)$ for X, such that the following hold:

$$
\underset{N \in \mathbb{N}}{\forall} \; \underset{M \geq N}{\exists} \; \underset{K \geq M}{\forall} \; \underset{n \in \mathbb{N}}{\exists} \; \underset{m \geq n}{\forall} \; \underset{k \geq m}{\exists} \quad
\begin{array}{l}
(1) \; X_M^N \left(X_{M,m}\right) \subset \overline{X_K^N \left(X_{K,k}\right)}^{X_{N,k}} , \\[2mm]
(2) \; X_M^N \left(X_{M,m}\right) \subset \overline{X_{N,n}}^{X_{N,k}} .
\end{array}
$$

If only (1) holds, X is called *deeply reduced in columns*, and if only (2) holds, it is called *deeply reduced in rows*.

ii) Deep reducedness in any of its forms is passed on to equivalent PLH space spectra:

Let $X = \text{proj}\left(X_N, X_M^N\right)$ be a deeply reduced (in columns, in rows) PLH space and $Y = \text{proj}\left(Y_T, Y_T^S\right)$ a PLH space such that the spectra $\left(X_N, X_M^N\right)$ and $\left(Y_T, Y_T^S\right)$ are equivalent. Let $T \in \mathbb{N}$. By equivalence of spectra, see (2.2.7), we may chose $N \in \mathbb{N}$ and a linear and continuous operator $\gamma\colon X_N \longrightarrow Y_T$. Then we chose $M \geq N$ according to the deep reducedness of X and $S \geq T$ and $\beta \in \mathrm{L}\left(Y_S, X_M\right)$ with $\gamma \circ X_M^N \circ \beta = Y_S^T$. Given $R \geq S$, we choose again by equivalence an integer $K \geq M$ and an operator $\alpha \in \mathrm{L}\left(X_K, Y_R\right)$ such that $X_K^M = \beta \circ Y_R^S \circ \alpha$. By deep reducedness we choose $n \in \mathbb{N}$ and with the factorization theorem we choose an integer $t \in \mathbb{N}$ with $\gamma\left(X_{N,n}\right) \subset Y_{T,t}$. Given $s \geq t$, we choose in the same way an integer $m \geq n$ such that $\beta\left(Y_{S,s}\right) \subset X_{M,m}$ and choose an integer $k \geq m$ by deep reducedness. For k we choose $r_1 \geq s$ with $\alpha\left(Y_{X_{K,k}}\right) \subset Y_{R,r_1}$ and $r_2 \geq s$ with $\gamma\left(X_{N,k}\right) \subset Y_{T,r_2}$. We define $r := \max\{r_1, r_2\}$ and obtain:

$$Y_S^T\left(Y_{S,s}\right) \subset \gamma\left(X_M^N\left(X_{M,m}\right)\right) \subset \overline{\gamma\left(X_K^N\left(X_{K,k}\right)\right)}^{Y_{t,r_2}} \cap \overline{\gamma\left(X_{N,n}\right)}^{Y_{T,r_2}} \subset \overline{Y_R^T\left(Y_{R,r}\right)}^{Y_{T,r}} \cap \overline{\gamma\left(Y_{T,t}\right)}^{Y_{T,r}}.$$

To see the other forms of deep reducedness, we just leave out the respective factor.

Now we verify that deep reducedness indeed yields the desired injectivity of the transposed, compare [Dom10, (5.1)]:

Proposition 3.2.4. *Let* $X = \text{proj}\left(X_N, X_M^N\right) = \text{proj}_{N\in\mathbb{N}} \text{ind}_{n\in\mathbb{N}} X_{N,n}$ *be a* PLH *space.*

i) *If X is deeply reduced in rows, then there is an equivalent* PLH *space spectrum* $\left(Y_N, Y_M^N\right)$ *such that*

$$\underset{N\in\mathbb{N}}{\forall}\ \underset{n\in\mathbb{N}}{\exists}\ \underset{m\geq n}{\forall}\ \overline{Y_{N,m}}^{Y_{N,m+1}} = Y_{N,m+1}.$$

ii) *If X is even deeply reduced, we can choose* $\left(Y_N, Y_M^N\right)$ *with i) and*

$$\underset{N\in\mathbb{N}}{\forall}\ \underset{n\in\mathbb{N}}{\exists}\ \underset{m\geq n}{\forall}\ \overline{Y_{N+1}^N\left(Y_{N+1,m}\right)}^{Y_{N,m}} = Y_{N,m}.$$

Proof.   i) As X is deeply reduced in rows, we have

$$\underset{\substack{N\in\mathbb{N}\\ }}{\forall}\ \underset{\substack{M(N)\geq N\\ n(N)\in\mathbb{N}}}{\exists}\ \underset{m\geq n(N)}{\forall}\ \underset{k\geq m}{\exists}\ X_{M(M)}^N\left(X_{M(N),m}\right) \subset \overline{X_{N,n}}^{X_{N,k}}.$$

For integers $N, n \in \mathbb{N}$ we define:

$$Y_{N,n} := \begin{cases} X_{N,n} & , \text{if } n < n(N), \\ \overline{X_{N,n(N)}}^{X_{N,n}} & , \text{if } n \geq n(N), \end{cases}$$

endowed with the relative topology. Then $Y_N := \text{ind}_{n \in \mathbb{N}} Y_{N,n}$ is an LH space and defining $Y_M^N := X_M^N|_{Y_M}, M \geq N$, as the restrictions of the spectral maps, the continuity of which yields the closed graph theorem, we obtain a projective spectrum (Y_N, Y_M^N) of LH spaces that is equivalent to (X_N, X_M^N), since Y_N is continuously embedded into X_N and $X_{M(N)}^N$ is linear and continuous from $X_{M(N)}$ to Y_N for each $N \in \mathbb{N}$ by assumption and again the closed graph theorem. Since Y_N is a subspace of $X_N, N \in \mathbb{N}$, the spectrum (Y_N, Y_M^N) is strongly reduced, hence i) holds.

ii) By i) we may assume that

$$\underset{N \in \mathbb{N}}{\forall} \underset{n(N) \in \mathbb{N}}{\exists} \underset{m \geq n(N)}{\forall} \underset{k(N,m) \geq m}{\exists} X_{N+1}^N (X_{N+1,m}) \subset \overline{X_{N+2}^N (X_{N+2,k(N,m)})}^{X_{N,k(N,m)}}$$

and $X_{N,m}$ is dense in $X_{N,m+1}$, where $(k(N,m))_{N,m \in \mathbb{N}}$ may be chosen strictly increasing in each component.

For $N, m \in \mathbb{N}$ we define:

$$Z_{N,m} := \begin{cases} \overline{X_{N+2}^N (X_{N+2,k(N,n(N))})}^{X_{N,k(N,n(N))}} \subset X_{N,k(N,n(N))} & , m < n(N), \\ \overline{X_{N+2}^N (X_{N+2,k(N,m)})}^{X_{N,k(N,m)}} \subset X_{N,k(N,m)} & , m \geq n(N). \end{cases}$$

Then $Z_{N,m}$ is continuously and densely embedded in $Z_{N,m+1}$ for all $N, m \in \mathbb{N}$, since $X_{N+2}^N (X_{N+2,k(N,n(N))})$ is dense in all of those sets. Hence $Z_N := \text{ind}_{m \in \mathbb{N}} Z_{N,m}$ is an LH space and defining $Z_{N+1}^N := X_{N+1}^N|_{Z_{N+1}} \in L(Z_{N+1}, Z_N)$, we obtain a PLH space spectrum that is equivalent to (X_N, X_M^N), since Z_N is continuously embedded into X_N and X_{N+1}^N is continuous from X_{N+1} to Z_N for each $N \in \mathbb{N}$ by assumption and the closed graph theorem. The spectrum (Z_M, Z_M^N) does not yet yield ii), since $Z_{N+1}^N (Z_{N+1,m})$ is only a subset of $\overline{X_{N+3}^N (X_{N+3,k(N+1,m)})}^{X_{N,k(N+1,m)}}$, which might not be contained in $Z_{N,m}$, since $k(N+1,m)$ might be strictly greater that $k(N,m)$. But as $X_{N+2}^N (X_{N+2,m})$ is a dense subset of $\overline{Z_{N+1}^N (Z_{N+1,m})}^{X_{N,k(N+1,m)}} = \overline{X_{N+3}^N (X_{N+3,k(N+1,m)})}^{X_{N,k(N+1,m)}}$ as well as it a dense subset of each $Z_{N,v}$ for all $v \in \mathbb{N}$ with $k(N,v) > k(N+1,m)$ by assumption, a thinning construction also implying (3.1.3) leads to the following:

We define $Y_{1,m} := Z_{1,m}, m \in \mathbb{N}$ and choose a strictly increasing sequence $(l_1(m))_{m \in \mathbb{N}}$ of natural numbers such that $k(2,m) \leq k(1, l_1(m))$ which yields that $Z_2^1 Z_{2,m}$ is a dense

subset of $Y_{1,l_1(m)}$ for all $m \in \mathbb{N}$. Hence we define

$$Y_{2,\nu} := \begin{cases} \{0\} & , \text{ for all } 1 \leq \nu < l_1(1) \text{ and} \\ Z_{2,m} & , \text{ for all } m, \nu \in \mathbb{N} \text{ with } l_1(m) \leq \nu < l_1(m+1). \end{cases}$$

Then $Y_{2,\nu}$ is a dense subspace of $Y_{2,\nu+1}$ for all $\nu \geq l_1(n(2))$ and the spectra $(Y_{2,\nu})_{\nu \in \mathbb{N}}$ and $(Z_{2,m})_{m \in \mathbb{N}}$ are obviously equivalent, hence $Y_2 := \text{ind}_{\nu \in \mathbb{N}} Y_{2,\nu} = Z_2$ and defining $Y_2^1 := Z_2^1$, we obtain for all $m \geq n(1)$ and $l_1(m) \leq \nu < l_1(m+1)$

$$Y_2^1(Y_{2,\nu}) = Z_2^1(Z_{2,m}) \subset Z_{1,l_1(m)} \subset Y_{1,\nu},$$

where all inclusions are dense. Now we start a recursion:

We choose a strictly increasing sequence $(l_2(m))_{m \in \mathbb{N}}$ of natural numbers such that $k(3,m) \leq k(2, l_2(m))$ for all $m \in \mathbb{N}$. Then we choose another strictly increasing sequence $(s_r)_{r \in \mathbb{N}}$ of natural numbers such that $l_1(s_r) \geq l_2(r)$ for all $r \in \mathbb{N}$ and define

$$Y_{3,\nu} := \begin{cases} \{0\} & , \text{ for all } 1 \leq \nu < l_1(n_1) \text{ and} \\ Z_{3,r} & , \text{ for all } r, \nu \in \mathbb{N} \text{ with } l_1(s_r) \leq \nu < l_1(s_{r+1}). \end{cases}$$

Then $Y_{3,\nu}$ is a dense subset of $Y_{3,\nu+1}$ for all $\nu \geq l_1(s_{n(3)})$ and the spectra $(Y_{3,\nu})_{\nu \in \mathbb{N}}$ and $(Z_{3,m})_{m \in \mathbb{N}}$ are obviously equivalent, hence $Y_3 := \text{ind}_{\nu \in \mathbb{N}} Y_{3,\nu} = Z_3$ and defining $Y_3^2 := Z_3^2$, we obtain for all $r \in \mathbb{N}$ with $l_2(r) \geq n(2)$

$$Y_3^2(Y_{3,\nu}) = Z_3^2(Z_{3,r}) \subset Z_{2,l_2(r)} \subset Z_{2,l_1(s_r)} = Y_{2,\nu}.$$

Proceeding recursively we arrive at the PLH space spectrum $\left(Y_N, Y_M^N\right)$, which is equivalent to $\left(X_N, X_M^N\right)$ and yields ii).

$\square$

Our last statement about deep reducedness in this section involves examples taken from [Dom10, p. 15].

Remark 3.2.5. The space of real analytic functions is deeply reduced, as well as the $\mathcal{E}_\omega(\Omega)$ for quasianalytic weights. Moreover, all Köthe type PLH spaces are deeply reduced, hence $\mathscr{D}'_{(\omega)}(\Omega)$ and $\mathcal{E}_\omega(\Omega)$ are deeply reduced for non-quasianalytic weights ω. All these spaces satisfy an even stronger condition, the *dual interpolation estimate* (*DIE*) in one variant. This will be investigated in chapter 5.

Fréchet-Hilbert spaces are always deeply reduced and an LH space is deeply reduced if and only if its strong dual is countably normed, cf. (3.4.1) i).

Finally we state Vogt's version [Vog11, (1.1)] of Domański's and Mastyło's interpolation result [DM07, (3.1)] for the readers convenience:

Theorem 3.2.6. *Let $\alpha, \beta > 0$ and let $(F_0, F_1), (G_0, G_1)$ be two pairs of pre-Hilbert spaces and F, G two pre-Hilbert spaces with $F \subset F_0 \cap F_1, G \subset G_0 \cap G_1$ such that*

$$\|x\|_F \|y\|_G \leq \alpha \|x\|_{F_0} \|y\|_{G_0} + \beta \|x\|_{F_1} \|y\|_{G_1} \text{ for all } x \in F, y \in G.$$

Then

$$\|u\|_{F \otimes_\pi G} \leq \alpha \|u\|_{F_0 \otimes_\pi G_0} + \beta \|u\|_{F_1 \otimes_\pi G_1} \text{ for all } u \in F \otimes_\pi G.$$

Now we can start the investigation of the vanishing of $\mathrm{proj}^1 \mathscr{L}$ in the three announced cases:

3.3 E **is a Hilbert space**

If E is a Hilbert space, the methodology to prove necessity differs significantly from the other cases. Thus we treat it separately: If E is a finite dimensional Hilbert space, then $\mathrm{proj}^1 \mathscr{L}$ vanishes if and only if $\mathrm{proj}^1 X$ vanishes. Thus (3.1.4) implies on the one hand that (T_ε) is sufficient and that (T) is necessary for the vanishing of $\mathrm{proj}^1 \mathscr{L}$ but on the other hand that the other implications are either not true or out of reach. If E is of infinite dimension, we prove a complete characterization. To prove necessity, we generalize [Wen03, (3.3.15)], which characterizes under which circumstances X has $(P_3^*)_0$. In the Fréchet-Hilbert case Domański and Mastyło use a similar statement without giving a proof, cf. [DM07, (4.1)], and Vogt uses a dual statement (also without proof) in the proof of necessity [Vog11, (4.2), "3.$\Rightarrow$ 1."]. To prove sufficiency we use a condition due to Frerick and Wengenroth, cf. [FW96, (2.3)] in the more elegant form of [Wen03, (3.2.14)].

Proposition 3.3.1. *For a* PLH *space* $X = \mathrm{proj}\, \mathscr{X} = \mathrm{proj}\left(X_N, X_M^N\right)$ *with* $\Psi_{\mathscr{X}} : \prod_{N \in \mathbb{N}} X_N \longrightarrow \prod_{N \in \mathbb{N}} X_N, (x_N)_{N \in \mathbb{N}} \mapsto \left(x_N - X_{N+1}^N (x_{N+1})\right)_{N \in \mathbb{N}}$ *the following are equivalent:*

i) X has $(P_3^*)_0$.

ii) $\Psi_{\mathscr{X}}$ lifts bounded sets that span ℓ_2.

Proof. Since $(P_3)_0$ and $(P_3^*)_0$ are equivalent, [Wen03, (3.3.15)] yields that i) is equivalent to $\Psi_{\mathscr{X}}$ lifting bounded sets, which obviously implies ii). Hence it remains to show that if X does not fulfill $(P_3^*)_0$, then there is a bounded subset $B \subset \prod_{N \in \mathbb{N}} X_N$ with $([B], p_B) \cong \ell_2$ that is not lifted by $\Psi_{\mathscr{X}}$. To obtain such a set B, we construct a quotient spectrum $\mathscr{Y} = \left(Y_N, Y_M^N\right)$ of $\mathscr{X}$ consisting of separable quotients Y_N of X_N not fulfilling $(P_3^*)_0$. We consider the LFH spectrum $\mathscr{X}' = \left(X_N', \left(X_{N+1}^N\right)^t\right)$, which does not fulfill condition (M) of Retakh. Not having (M) is equivalent to either not fulfilling condition (wQ) or not being boundedly stable by [Wen95, (3.8)]. ii) implies the surjectivity of $\Psi_{\mathscr{X}}$ which characterizes the vanishing of $\mathrm{proj}^1 \mathscr{X}$ by [Wen03, (3.1.4)], which is equivalent to $\mathscr{X}$ being weakly acyclic by (3.1.1). As weak acyclicity implies condition (wQ) by [Vog92, (2.3) b)], $\mathscr{X}$ is not boundedly stable. Hence there is a step N_0 and a bounded subset $A_0 \subset X_{N_0}$ such that for every $M \geq N_0$ there is

a $K(M) \geq M$ such that the topology of X'_M is strictly finer on A_0 than that of $X'_{K(M)}$, i.e. for every $M \geq N_0$ there is a sequence $\left(z_l^{(M)}\right)_{l \in \mathbb{N}}$ in A_0 which is convergent in $X'_{K(M)}$ but not in X'_M. We define

$$A := \left\{z_l^{(M)} : l \in \mathbb{N}, M \geq N_0\right\} \text{ and } Y'_N := \begin{cases} \overline{\left(X_N^{N_0}\right)^t([A])}^{X'_N} \subset X'_N & , N \geq N_0 \\ \{0\} & , N < N_0. \end{cases}$$

Defining $\left(Y_{N+1}^N\right)^t$ as the restriction of $\left(X_{N+1}^N\right)^t$ to Y'_{N+1}, we obtain an LFH spectrum $\mathscr{Y}' = \left(Y'_N, \left(Y_{N+1}^N\right)^t\right)$ of separable subspaces of X'_N. As X'_N is the projective limit of the spectrum induced by the sequence of Hilbert spaces $\left(X'_{N,n}\right)_{n \in \mathbb{N}}$ and the transposed of the inclusions $X_{N,n} \hookrightarrow X_{N,n+1}, N, n \in \mathbb{N}$, the spaces Y'_N ($N \in \mathbb{N}$) are projective limits of sequences $\left(Y'_{N,n}\right)_{n \in \mathbb{N}}$ of separable subspaces of $X'_{N,n}, n \in \mathbb{N}$. Thus the strong dual Y_N of Y'_N is a separable quotient of X_N as it is the inductive limit of the sequence $(Y_{N,n})_{n \in \mathbb{N}}$ of (strong) duals of $Y'_{N,n}, n \in \mathbb{N}$. Hence $\mathscr{Y} := \left(Y_N, Y_M^N\right)$, where Y_M^N is the transposed of $\left(Y_M^N\right)^t$, is a quotient spectrum of $\mathscr{X}$, which is dual to $\mathscr{Y}'$. Thus the same argument as before, just backwards, yields, that $(P_3^*)_0$ does not hold for $\mathscr{Y}$, as $\mathscr{Y}'$ is not boundedly stable by construction. Then [Wen03, (3.3.15)] guarantees a bounded subset $C \subset \prod_{N \in \mathbb{N}} Y_N$ that is not lifted by $\Psi_{\mathscr{Y}}$. Without loss of generality we may assume that there is a strictly increasing sequence $(n(N))_{N \in \mathbb{N}}$ of natural numbers and a sequence of positive scalars $(\omega_n)_{n \in \mathbb{N}}$ such that C is the closed unit ball of the ℓ_2-sum of the sequence $\left(Y_{N,n(N)}, \omega_n \|\cdot\|_{N,n(N)}^Y\right)_{N \in \mathbb{N}}$, cf. (4.2.5). Here $\|\cdot\|_{N,n}^Y$ denotes the quotient norm for $N, n \in \mathbb{N}$. Since it is easy to verify that the ℓ_2-sum of separable Banach spaces is separable, C spans ℓ_2. Now we lift C to the desired bounded subset $B \subset \prod_{N \in \mathbb{N}} X_N$, which spans ℓ_2 and does not lift with respect to $\Psi_{\mathscr{X}}$.

Denoting by $q_N : X_N \longrightarrow Y_N$ ($N \in \mathbb{N}$) the quotient map and by E_Y, E_X the ℓ_2-sums of $\left(Y_{N,n(N)}, \omega_n \|\cdot\|_{N,n(N)}^Y\right)_{N \in \mathbb{N}}$ and $\left(X_{N,n(N)}, \omega_n \|\cdot\|_{N,n(N)}^X\right)_{N \in \mathbb{N}}$ respectively, we have induced quotient maps $(q_N)_{N \in \mathbb{N}} : \prod_{N \in \mathbb{N}} X_N \longrightarrow \prod_{N \in \mathbb{N}} Y_N, q : X \longrightarrow Y := \text{proj} \mathscr{Y}$ and $Q : E_X \longrightarrow E_Y$ as well as a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & X & \overset{i_X}{\longrightarrow} & \prod_{N \in \mathbb{N}} X_N & \overset{\Psi_{\mathscr{X}}}{\longrightarrow} & \prod_{N \in \mathbb{N}} X_N & \overset{j_X}{\longleftarrow} & E_X \\
& & \downarrow q & & \downarrow (q_N)_{N \in \mathbb{N}} & & \downarrow (q_N)_{N \in \mathbb{N}} & & \downarrow Q \\
0 & \longrightarrow & X & \overset{i_Y}{\longrightarrow} & \prod_{N \in \mathbb{N}} Y_N & \overset{\Psi_{\mathscr{Y}}}{\longrightarrow} & \prod_{N \in \mathbb{N}} Y_N & \overset{j_Y}{\longleftarrow} & E_Y,
\end{array}
$$

where i_X, i_Y, j_X and j_Y are the canonical inclusions, i.e. component-by-component, and $\Psi_{\mathscr{Y}}$ is the canonical map. As E_X and E_Y are Hilbert spaces, there is a linear and continuous right inverse $R : E_Y \longrightarrow E_X$ of Q. We define $B := j_X(R(C))$. Then B is a bounded subset of $\prod_{N \in \mathbb{N}} X_N$ that spans ℓ_2. Assuming that B can be lifted by $\Psi_{\mathscr{X}}$, we arrive at a bounded subset $A \subset \prod_{N \in \mathbb{N}} X_N$ with $\Psi_{\mathscr{Y}}\left((q_N)_{N \in \mathbb{N}}(A)\right) = (q_N)_{N \in \mathbb{N}}(\Psi_{\mathscr{X}}(A)) = (q_N)_{N \in \mathbb{N}}(j_X(R(C))) = B$, a contradiction, since $(q_N)_{N \in \mathbb{N}}(A)$ is a bounded subset of $\prod_{N \in \mathbb{N}} Y_N$. $\qquad \square$

Theorem 3.3.2. *Let E be Hilbert space of infinite dimension and $X = \operatorname{proj}_{N \in \mathbb{N}} \operatorname{ind}_{n \in \mathbb{N}} X_{N,n}$ be a PLH space. Denoting $\mathscr{L} := \left(\mathrm{L}(E, X_N), X_M^{N\,*} \right)$, the following holds:*

$$\operatorname{proj}^1 \mathscr{L} = 0 \text{ if and only if } (E, X) \text{ satisfies } (T_\varepsilon) \left(\text{if and only if } X \text{ satisfies } (P_3^*)_0 \right).$$

Proof.

Necessity: Assuming that (T_ε) or equivalently $(P_3^*)_0$ does not hold, (3.3.1) yields a bounded subset $B \subset \prod_{N \in \mathbb{N}} X_N$ with $([B], p_B) \cong \ell_2$ that does not lift with respect to $\Psi_{\mathscr{X}}$. As the dimension of E is not finite, their is a quotient map $T \colon E \longrightarrow \ell_2 \cong ([B], p_B) \hookrightarrow \prod_{N \in \mathbb{N}} X_N$. As $T(B_E) = B$, T cannot lift with respect to $\Psi_{\mathscr{X}}$. Hence $\operatorname{proj}^1 \mathscr{L} \neq 0$ by [Wen03, (3.1.4)], since $\Psi_{\mathscr{L}} \colon \prod_{N \in \mathbb{N}} \mathrm{L}(E, X_N) \longrightarrow \prod_{N \in \mathbb{N}} \mathrm{L}(E, X_N)$ is not surjective.

Sufficiency: As usual, we endow the steps $\mathrm{L}(E, X_N)$ of $\mathscr{L}$ with the associated ultrabornological topology to the topology of convergence on all bounded subsets of E, which coincides with the inductive limit topology $\mathrm{L}_b(E, X_N) = \operatorname{ind}_{n \in \mathbb{N}} \mathrm{L}(E, X_{N,n}) =: L_N, N \in \mathbb{N}$, cf. [Wen03, p.86]. Hence the system $\{ W(B_E, B_{N,n}) : n \in \mathbb{N} \}$, where $W(B_E, B_{N,n}) := \{ T \in \mathrm{L}(E, X_{N,n}) : T(B_E) \subset B_{N,n} \}$, is a fundamental system of Banach discs in $\mathrm{L}_b(E, X_N)$ by the factorization theorem for all $N \in \mathbb{N}$. Thus $(P_3^*)_0$ in the form $(P_3)_0$ implies that the following condition holds for $\mathscr{L}$:

$$\underset{N \in \mathbb{N}}{\forall} \; \underset{M \geq N}{\exists} \; \underset{K \geq M}{\forall} \; \underset{B \in \mathscr{BD}(L_N)}{\exists} \; \underset{R \in \mathscr{BD}(L_M)}{\forall} \; \underset{S \in \mathscr{BD}(L_K)}{\exists}$$
$$X_M^{N\,*}(R) \subset B + X_K^{N\,*}(S),$$

which is sufficient for the vanishing of $\operatorname{proj}^1 \mathscr{L}$ by [Wen03, (3.2.14)]. $\qquad\square$

3.4 E is a Fréchet-Hilbert space

We prove that the vanishing of $\operatorname{proj}^1 \mathscr{L}$ is characterized by condition (T_ε) for the pair (E', X) if E is a proper Fréchet-Hilbert space. The analogon for a PLS space X and an FS space E in the successors of the four standard cases can be found in [BD06, (4.1)]. The basic idea of the proof is similar to that of the Fréchet-Hilbert case in [Vog11]: Necessity needs a variant of [Vog87, (3.3)] to guarantee $(P_3^*)_0$ for X, cf. (3.4.4), and as we need the interpolation theorem (3.2.6) to prove sufficiency, we need X to be deeply reduced and E to be countably normed, which leads to the following well-known definitions and a proof by cases, cf. [Vog87, Wen03, BD06]:

Definition and Remark 3.4.1.   i) A Fréchet-Hilbert space E is called *countably normed* if there is a strongly reduced projective spectrum $\left(E_\nu, E_\mu^\nu \right)$ of Hilbert spaces giving rise to E such that

$$\underset{\nu \in \mathbb{N}}{\exists} \; \underset{\mu \geq \nu}{\forall} \; \underset{\kappa \geq \mu}{\exists} \; \underset{x \in E_\kappa}{\forall} \; E_\kappa^\nu(x) = 0 \Rightarrow E_\kappa^\mu(x) = 0. \; (\star)$$

It is a well-known fact that this is equivalent to the existence of a strongly reduced representing projective spectrum $\left(F_\nu, F_\mu^\nu \right)$ with injective spectral maps. Thus, we only give

an outline of the proof: Necessity is obvious. Sufficiency: Let ν be chosen according to ($\star$) and define $F_1 := E_\nu$. Then we choose $\kappa \geq \mu := \nu + 1$ according to ($\star$) and define $F_2 := E_\kappa^\mu E_\kappa$, endowed with the quotient topology and $F_2^1 := E_\mu^\nu|_{F_2}$, which is injective by construction. Proceeding recursively yields the assertion.

ii) A PLH space X is called *strict* if there is a representing PLH space spectrum $\left(X_N, X_M^N\right)$ such that

$$\bigvee_{N \in \mathbb{N}} \ \exists_{M \geq N} \ \bigvee_{K \geq M} X_M^N X_M \subset X_K^N X_K. \ (\star\star)$$

In complete analogy to (3.4.1) i) this is equivalent to the existence of a representing spectrum with surjective linking maps: Necessity is obvious. Sufficiency: For every $N \in \mathbb{N}$ we choose $M(N) \geq N$ according to ($\star\star$) and define $Y_N := X_{M(N)}^N X_{M(N)}$ endowed with the quotient topology. Furthermore, we define $Y_{N+1}^N := X_{M(N+1)}^{M(N)}|_{Y_{N+1}}$, obtaining a PLH spectrum $\left(Y_N, Y_M^N\right)$ yielding the assertion. Of course this characterization holds for arbitrary projective spectra of locally convex spaces. Furthermore, this implies that strict projective spectra of locally convex spaces (or vector spaces) $\mathscr{X}$ satisfy $\mathrm{proj}^1 \, \mathscr{X} = 0$, since the surjectivity of the linking maps implies the surjectivity of the canonical map $\Psi_{\mathscr{X}}$.

Proposition 3.4.2. *Let X be a* PLH *space and let E be a Fréchet-Hilbert space such that (E', X) satisfies condition (T_ε). If E is not countably normed, then X is strict.*

Proof. Given $N \in \mathbb{N}$, we choose $M \geq N$ according to (T_ε), as well as $n, v \in \mathbb{N}$ for every $K \geq M$. As E is not countably normed, we may choose $\mu \geq v$ according to the negation of (3.4.1) i). Given $m \geq n$, we choose $k \geq m, \kappa \geq \mu$ and $c > 0$ for $m \geq n, \mu \geq v$ and $\varepsilon := 1$ according to (T_ε). Again with (3.4.1) i) we choose an $x \in E_\kappa$ with $E_\kappa^\mu(x) \neq 0$ and $E_\kappa^v(x) = 0$. As (T_ε) holds for all $x \in E_\kappa$ as well by (3.2.1) v), the following holds for all $f \in X_N'$:

$$\left\|\left(X_M^N\right)^t f\right\|_{M,m}^* = \left\|\left(X_M^N\right)^t f\right\|_{M,m}^* \frac{\left\|E_\kappa^\mu x\right\|_\mu}{\left\|E_\kappa^\mu x\right\|_\mu}$$

$$\leq \|f\|_{N,n}^* \frac{\left\|E_\kappa^v x\right\|_v}{\left\|E_\kappa^\mu x\right\|_\mu} + c \left\|\left(X_K^N\right)^t f\right\|_{K,k}^* \frac{\|x\|_\kappa}{\left\|E_\kappa^\mu x\right\|_\mu}$$

$$= c \left\|\left(X_K^N\right)^t f\right\|_{K,k}^* \frac{\|x\|_\kappa}{\left\|E_\kappa^\mu x\right\|_\mu}.$$

Hence the bipolar theorem yields that

$$X_M^N X_M = \bigcup_{m \in \mathbb{N}} X_M^N B_{M,m} \subset \bigcup_{\substack{k \in \mathbb{N} \\ c > 0}} c X_K^N B_{K,k} \subset X_K^N X_K,$$

e.g. X is strict. $\qquad\square$

Proposition 3.4.3. *Let* $X = \text{proj}\left(X_N, X_M^N\right)$ *be a* PLH *space,* E *a Fréchet-Hilbert space and* $\mathscr{L} := \left(\text{L}\left(E, X_N\right), X_M^{N\,*}\right)$. *If X is strict, then so is* $\mathscr{L}$.

Proof. As usual, we endow the steps $\text{L}\left(E, X_N\right)$ of $\mathscr{L}$ with the associated bornological topology to the topology of convergence on all bounded subsets of E, which coincides with the inductive limit topology $\text{L}_b\left(E, X_N\right) = \text{ind}_{\nu,n\in\mathbb{N}} \text{L}\left(E_\nu, X_{N,n}\right) =: L_N, N \in \mathbb{N}$, cf. [Wen03, p.86]. Hence the system $\{W\left(B_\nu, B_{N,n}\right) : n, \nu \in \mathbb{N}\}$, where $W\left(B_\nu, B_{N,n}\right) := \{T \in \text{L}\left(E_\nu, X_{N,n}\right) : T\left(B_\nu\right) \subset B_{N,n}\}$, is a fundamental system of Banach discs in L_N by the factorization theorem for all $N \in \mathbb{N}$. Given $N \in \mathbb{N}$ we choose $M \geq N$ such that $X_M^N X_M \subset X_K^N X_K$ for all $K \geq M$. Then the factorization theorem yields for any $m \in \mathbb{N}$ an integer $k \geq m$ and a constant $c > 0$ such that $X_M^N B_{M,m} \subset c X_K^N B_{K,k}$, hence $X_M^{N\,*}\left(W\left(B_\nu, B_{M,m}\right)\right) \subset X_K^{N\,*}\left(W\left(B_\nu, c B_{K,k}\right)\right)$ for all $\nu \in \mathbb{N}$. Thus $X_M^{N\,*}\left(L_M\right) \subset X_K^{N\,*}\left(L_K\right)$, i.e. $\mathscr{L}$ is strict. $\qquad\square$

The following result is a variant of a construction due to Vogt, cf. proof of [Vog87, (3.3)], which will allow us to forgo condition $(P_3^*)_0$ for X in the splitting result (3.4.5).

Proposition 3.4.4. *Let* $E = \text{proj}\left(E_\nu, E_\mu^\nu\right)$ *be a non-normable Fréchet-Hilbert space and* $X = \text{proj}_{N\in\mathbb{N}} \text{ind}_{n\in\mathbb{N}} X_{N,n}$ *a* PLH *space such that* (E', X) *satisfies condition* (T). *Then X satisfies the condition* $(P_3^*)_0$.

Proof. To begin with, we prove the following:

$$\underset{\nu\in\mathbb{N}}{\forall}\,\underset{\mu\geq\nu}{\exists}\,\underset{\delta>0}{\forall}\,\underset{x\in E}{\exists}\,0 < \left\|E_\infty^\nu x\right\|_\nu \leq \delta \left\|E_\infty^\mu x\right\|_\mu . \;(\star)$$

For every $\nu \in \mathbb{N}$ there is a $\mu \geq \nu$ such that the seminorms $\left\|E_\infty^\nu\left(\cdot\right)\right\|_\nu$ and $\left\|E_\infty^\mu\left(\cdot\right)\right\|_\mu$ are not equivalent on E, since E is not normable. Furthermore, for every $\delta > 0$ there is an $x \in E$ such that $\delta \left\|E_\infty^\mu\left(x\right)\right\|_\mu > \left\|E_\infty^\nu\left(x\right)\right\|_\nu$, since E_μ^ν is continuous. We may assume additionally $\left\|E_\infty^\nu\left(x\right)\right\|_\nu > 0$, since presuming that $\left\|E_\infty^\nu\left(x\right)\right\|_\nu > 0$ implies $\delta \left\|E_\infty^\mu\left(x\right)\right\|_\mu \leq \left\|E_\infty^\nu\left(x\right)\right\|_\nu$ for all $x \in E$, we arrive at the openness of E_μ^ν, since the restricted projection E_∞^μ has dense range, a contradiction to the assumption. $\diamond$

Since E is a Fréchet-Hilbert space, we may write condition (T) for (E', X) in the following form:

$$\underset{N\in\mathbb{N}}{\forall}\,\underset{M\geq N}{\exists}\,\underset{K\geq M}{\forall}\,\underset{\substack{n\in\mathbb{N}\\\nu\in\mathbb{N}}}{\exists}\,\underset{\substack{m\geq n\\\mu\geq\nu}}{\forall}\,\underset{\substack{k\geq m\\\kappa\geq\mu\\c_0>0}}{\exists}\,\underset{\substack{f\in X_N'\\x\in E}}{\forall}$$

$$\left\|\left(X_M^N\right)^t f\right\|_{M,m}^* \left\|E_\infty^\mu\left(x\right)\right\|_\mu \leq c_0 \left(\|f\|_{N,n}^* \left\|E_\infty^\nu\left(x\right)\right\|_\nu + \left\|\left(X_K^N\right)^t f\right\|_{K,k}^* \left\|E_\infty^\kappa\left(x\right)\right\|_\kappa\right).$$

Now let $N \in \mathbb{N}$. We choose $M \geq N$ according to (T), as well as $n, \nu \in \mathbb{N}$ for given $K \geq M$. Then we choose $\mu \geq \nu$ according to $(\star)$. Given $m \geq n, \varepsilon > 0$, we choose $k \geq m, \kappa \geq \mu, c_0 > 0$

according to (T) and $x \in E$ according to $(\star)$ for $\delta := \frac{\varepsilon}{c_0}$. Thus, defining $c := \frac{c_0 \left\| E_\infty^\kappa x \right\|_\kappa}{\left\| E_\infty^\mu x \right\|_\mu}$, we obtain for all $f \in X'_N$

$$
\begin{aligned}
\left\| \left(X_M^N \right)^t f \right\|_{M,m}^* &= \left\| \left(X_M^N \right)^t f \right\|_{M,m}^* \frac{\left\| E_\infty^\mu x \right\|_\mu}{\left\| E_\infty^\mu x \right\|_\mu} \\
&\leq c_0 \left\| f \right\|_{N,n}^* \frac{\left\| E_\infty^\nu x \right\|_\nu}{\left\| E_\infty^\mu x \right\|_\mu} + c_0 \left\| \left(X_K^N \right)^t f \right\|_{K,k}^* \frac{\left\| E_\infty^\kappa x \right\|_\kappa}{\left\| E_\infty^\mu x \right\|_\mu} \\
&\leq \varepsilon \left\| f \right\|_{N,n}^* + c \left\| \left(X_K^N \right)^t f \right\|_{K,k}^*,
\end{aligned}
$$

which is $(P_3^*)_0$ for X. $\qquad\square$

The following theorem gives an answer to a variation of [BD06, problem (9.9)]:

Theorem 3.4.5. *Let $X = \mathrm{proj}_{N \in \mathbb{N}} \, \mathrm{ind}_{n \in \mathbb{N}} X_{N,n}$ be a deeply reduced PLH space and $E = \mathrm{proj}_{\nu \in \mathbb{N}} E_\nu$ a non-normable Fréchet-Hilbert space. Then the following are equivalent:*

   i) $\mathrm{proj}^1 \mathscr{L} = 0$.

   ii) (E', X) *satisfies* (T_ε).

   iii) (E', X) *satisfies* (T).

Proof. According to (3.1.5) the LF spectrum $\mathscr{H} := \left(H_N, H_N^{N+1} \right) := \left(E \tilde{\otimes}_\pi X'_N, \mathrm{id}_E \tilde{\otimes}_\pi \left(X_{N+1}^N \right)^t \right)$ is pre-dual to $\mathscr{L}$. Let us check quickly that the operators $\mathrm{id}_E \otimes_\pi \left(X_{N+1}^N \right)^t$ are injective: First of all, as X_{N+1}^N has dense range by (2.2.10) ii), its transposed $\left(X_{N+1}^N \right)^t$ is injective. Furthermore, since the spaces E, X'_N and X'_{N+1} are Fréchet-Hilbert spaces, i.e. complete with approximation property by [Köt79, §43, 1. (3)], the respective projective tensor products coincide with the respective spaces of bilinear mappings by [Köt79, §43, 2. (12)]. Thus the unique extension of $\mathrm{id}_E \otimes_\pi \left(X_{N+1}^N \right)^t$ to the completion is injective. Hence with (3.1.1) the vanishing of $\mathrm{proj}^1 \mathscr{L}$ is equivalent to the weak acyclicity of $\mathscr{H}$.

Necessity of (T): As in (3.1.4), $\mathscr{H}$ has property (wQ), which implies by exchange of quantifiers the following condition:

$$
\mathop{\forall}_{N \in \mathbb{N}} \mathop{\exists}_{M \geq N} \mathop{\forall}_{K \geq M} \mathop{\exists}_{\substack{U \in \mathscr{U}_0(H_N)}} \mathop{\forall}_{V \in \mathscr{U}_0(H_M)} \mathop{\exists}_{\substack{W \in \mathscr{U}_0(H_K) \\ c > 0}} H_N^K(U) \cap W \subset c H_M^K(V).
$$

Since $H_L = \mathrm{proj}_{l \in \mathbb{N}} E_l \tilde{\otimes}_\pi X_{L,l}$ for every $L \in \mathbb{N}$ by [Köt79, §41, 6. (3)] and $\pi \left(y \otimes g ; E_l, X'_{L,l} \right) =$

$\|y\|_l \|g\|^*_{L,l}$ for all $L, l \in \mathbb{N}$ and $y \in E_l, g \in X'_{L,l}$ by [DF93, 3.2 (3)], this implies

$$\underset{N \in \mathbb{N}}{\forall} \; \underset{M \geq N}{\exists} \; \underset{K \geq M}{\forall} \; \underset{n \in \mathbb{N}}{\exists} \; \underset{m \geq n}{\forall} \; \underset{\substack{k \geq m \\ c > 0}}{\exists} \; \underset{\substack{f \in X'_N \\ x \in E}}{\forall}$$

If $\pi\left(E^n_\infty(x) \otimes f; E_n, X'_{N,n}\right) \leq 1$ and $\pi\left(E^k_\infty(x) \otimes \left(X^N_K\right)^t (f); E_k, X'_{K,k}\right) \leq 1,$

then $\pi\left(E^m_\infty(x) \otimes \left(X^N_M\right)^t (f); E_m, X'_{M,m}\right) \leq c,$

which is condition (T^d) for (E', X), hence (T) holds for (E', X) by (3.2.1) ii) b).

(T) implies (T_ε): Since E is not normable, X satisfies condition $(P^*_3)_0$ by (3.4.4). If E' is an LH space the second case in the proof of (3.2.2) is not possible, hence we have (T_ε) for (E', X) with (3.2.2).

Sufficiency: If E is not countably normed, then X is strict by (3.4.2), hence so is $\mathscr{L}$ with (3.4.3), which implies $\mathrm{proj}^1 \mathscr{L} = 0$ by (3.4.1) ii). Thus we may assume by (3.4.1) that E is the projective limit of a (strongly reduced) spectrum $\left(E_\nu, E^\nu_\mu\right)$ of Hilbert spaces with injective spectral maps. First of all, in complete analogy to the proof of necessity, just backwards, condition (T_ε) in the form (3.2.1) v), i.e.

$$\underset{\substack{N \in \mathbb{N}}}{\forall} \; \underset{M \geq N}{\exists} \; \underset{K \geq M}{\forall} \; \underset{\substack{n \in \mathbb{N} \\ \nu \in \mathbb{N}}}{\exists} \; \underset{\substack{m \geq n \\ \mu \geq \nu \\ \varepsilon > 0}}{\forall} \; \underset{\substack{k \geq m \\ \kappa \geq \mu \\ c > 0}}{\exists} \; \underset{\substack{f \in X'_{N,k} \\ x \in E_\kappa}}{\forall}$$

$$\left\|\left(X^N_M\right)^t f\right\|^*_{M,m} \left\|E^\mu_\kappa x\right\|_\mu \leq \varepsilon \|f\|^*_{N,n} \left\|E^\nu_\kappa x\right\|_\nu + c \left\|\left(X^N_K\right)^t f\right\|^*_{K,k} \|x\|_\kappa,$$

corresponds exactly to the norm inequality, that is the equivalent of condition $(Q)_0$ for $\mathscr{H}$:

$$\underset{N \in \mathbb{N}}{\forall} \; \underset{M \geq N}{\exists} \; \underset{K \geq M}{\forall} \; \underset{U \in \mathscr{U}_0(H_N)}{\exists} \; \underset{V \in \mathscr{U}_0(H_M)}{\forall} \; \underset{W \in \mathscr{U}_0(H_K)}{\exists} \; H^K_N(U) \cap W \subset H^K_M(V),$$

though only for elementary tensors $x \otimes f \in E_\kappa \tilde{\otimes} X'_{N,k}$. To deduce from that the norm inequality for all elements of the tensor product $E_\kappa \tilde{\otimes}_\pi X'_{N,k}$, we need Vogt's interpolation result (3.2.6). This is where we need E to be countably normed and X to be deeply reduced: We define $F_0 := X'_{N,n}, F_1 := X'_{K,k}$ and $F := \left(X'_{N,k}, \left\|\left(X^N_M\right)^t (\cdot)\right\|^*_{M,m}\right)$, as well as $G_0 := E_\nu, G_1 := E_\kappa$ and $G := \left(E_\kappa, \left\|E^\nu_\mu(\cdot)\right\|_\mu\right)$ and $\alpha := \varepsilon, \beta := c$. Since X is deeply reduced in rows, $F = X'_{N,k}$ is a subspace of $F_0 = X'_{N,n}$ and since X is deeply reduced in columns, F is a subspace of $F_1 = X'_{K,k}$. Furthermore, the deep reducedness of X implies that F is a pre-Hilbert space and F_0 as well as F_1 are injectively and continuously embedded into $X'_{K,n}$, hence we have a Hilbert couple (F_0, F_1) with $F \subset F_0 \cap F_1$. At length, since E is countably normed, (G_0, G_1) is a Hilbert

couple (which is even ordered) and $G \subset G_0 \cap G_1$. Hence (3.2.6) yields that the desired norm inequality holds for all $u \in E_\kappa \otimes_\pi X'_{N,k}$. Since the inclusions $E_\kappa \tilde{\otimes}_\pi X'_{N,k} \hookrightarrow Z$ are continuous for $Z \in \left\{ E_\mu \tilde{\otimes}_\pi X'_{M,m}, E_\nu \tilde{\otimes}_\pi X'_{N,n}, E_\kappa \tilde{\otimes}_\pi X'_{K,k} \right\}$, the desired inequality holds for all $u \in E_\kappa \tilde{\otimes}_\pi X'_{N,k}$, which yields $(Q)_0$ for $\mathscr{H}$, which hence is acyclic by (3.1.2). Since acyclicity implies weak acyclicity, (3.1.1) gives the conclusion. $\qquad\square$

Together with (2.3.6) ii) the previous theorem (3.4.5) and (3.3.2) yield the following splitting result that generalizes Domański's and Mastyło's splitting result for Fréchet-Hilbert spaces [DM07, (4.2)]:

Corollary 3.4.6. *Let* $E = \mathrm{proj}\left(E_\nu, E_\mu^\nu \right)$ *be a Fréchet-Hilbert space and* $X = \mathrm{proj}\left(X_N, X_M^N \right)$ *be a deeply reduced* PLH *space with* $\mathrm{proj}^1 X = 0$. *If* $\mathrm{Ext}^1_{\mathrm{LH}}\left(E_\nu, X_N \right) = 0$ *for all* $\nu, N \in \mathbb{N}$, *then* $\mathrm{Ext}^1_{\mathrm{PLH}}\left(E, X \right) = 0$ *if and only if the pair* (E', X) *has property* (T_ε). *Furthermore,* $\mathrm{Ext}^k_{\mathrm{PLH}}\left(E, X \right) = 0$ *for all* $k \geq 2$.

Our proof of sufficiency of (T_ε) for splitting in (3.4.5) depends heavily on X being deeply reduced, as otherwise we can not interpolate. Although general necessity statements are not possible, we can show the following:

Remark 3.4.7. If E is a countably normed Fréchet-Hilbert space and X is a PLH space such that (E', X) satisfies (T_ε), then either X is deeply reduced in columns or E is a Hilbert space. Note that in the latter case $\mathrm{proj}^1 \mathscr{L} = 0$ by (3.3.2):

Let $K \geq M \geq N, k \geq m \geq n, \kappa \geq \mu \geq \nu, \varepsilon > 0$ and $c > 0$ be given by (T_ε) and its quantifiers. If X is not deeply reduced in columns, there is an $f \in X'_{N,k}$ such that $\left\| \left(X_M^N \right)^t f \right\|^*_{M,m}$ does not vanish but $\left\| \left(X_K^N \right)^t f \right\|^*_{K,k}$ does. Then (T_ε) yields, that for all $\mu \geq \nu$ and for all $x \in E$ we have $\|x\|_\mu \leq c \dfrac{\|f\|^*_{N,n}}{\left\| \left(X_M^N \right)^t f \right\|^*_{M,m}} \|x\|_\nu$, hence E is a Hilbert space.

3.5 E **is an** LH **space**

If E is a proper LH space, the situation is more complex than in the Fréchet-Hilbert case: First of all, as we want to use methods for proj^1 of spectra of separated LB spaces, we have to consider the spectrum $\mathscr{K} := \left(\mathrm{L}\left(E_N, X_N \right), E_{N}^{M} \circ X_M^{N\,*} \right)$ instead of $\mathscr{L}$. We start by establishing the connection between $\mathrm{proj}^1 \mathscr{L}$ and $\mathrm{proj}^1 \mathscr{K}$ with the help of Grothendieck's spectral sequences. A similar result in the PLS context has been proven by Bonet and Domański in [BD08, (3.4)] with functional analytic methods and without the assumption of local splitting.

Proposition 3.5.1. *Let* $X = \mathrm{proj}\left(X_N, X_M^N \right)$ *be a* PLH *space and* $E = \mathrm{ind}\left(E_\nu, E_\nu^\mu \right)$ *an* LH *space. Considering the induced projective spectra of operator spaces* $\mathscr{L} := \left(\mathrm{L}\left(E, X_N \right), X_M^{N\,*} \right)$, $\mathscr{L}_N := \left(\mathrm{L}\left(E_\nu, X_N \right), E_{\nu\,*}^\mu \right), N \in \mathbb{N}$ *and* $\mathscr{K} := \left(\mathrm{L}\left(E_N, X_N \right), E_{N\,*}^{M} \circ X_M^{N\,*} \right)$, *we obtain*

   i) If $\mathrm{proj}^1 \mathscr{K} = 0$, *then* $\mathrm{proj}^1 \mathscr{L} = 0$.

ii) If $\mathrm{proj}^1 \mathscr{L}_N = 0$ *for all* $N \in \mathbb{N}$*, then* $\mathrm{proj}^1 \mathscr{K} \cong \mathrm{proj}^1 \mathscr{L}$.

Proof. Since $\mathscr{K}$ is the diagonal spectrum of the double tower $\tilde{\mathscr{K}} := \left(\mathrm{L}\left(E_\nu, X_N\right), E_{\nu\,*}^\mu \circ X_M^{N\,*} \right)$, hence equivalent to $\tilde{\mathscr{K}}$, since for $M \geq N, \mu \geq \nu, L := \max\{M, \mu\}, K := \min\{N, \nu\}$ the diagram

$$
\begin{array}{ccc}
\mathrm{L}\left(E_L, X_L\right) & \xrightarrow{\;E_{K\,*}^L \circ X_L^{K\,*}\;} & \mathrm{L}\left(E_K, X_K\right) \\[4pt]
{\scriptstyle E_{\mu\,*}^L \circ X_L^{M\,*}} \Big\downarrow & \circlearrowleft & \Big\uparrow {\scriptstyle E_{\nu\,*}^K \circ X_N^{K\,*}} \\[4pt]
\mathrm{L}\left(E_\mu, X_M\right) & \xrightarrow[\;E_{\nu\,*}^\mu \circ X_M^{N\,*}\;]{} & \mathrm{L}\left(E_\nu, X_N\right)
\end{array}
$$

commutes, the vector spaces $\mathrm{proj}^k \mathscr{K}$ and $\mathrm{proj}^k \tilde{\mathscr{K}}$ are isomorphic for each $k \geq 0$ by [Wen03, (4.1.7)]. The application of Grothendieck's spectral sequence, "proj^1 of a double tower", see e.g. [Wei94, (5.8.7)], yields an exact sequence of vector spaces

$$
0 \longrightarrow \mathrm{proj}^1 \mathrm{proj}\,\mathscr{L}_N \longrightarrow \mathrm{proj}^1 \tilde{\mathscr{K}} \longrightarrow \mathrm{proj}\,\mathrm{proj}^1 \mathscr{L}_n \longrightarrow 0,
$$

which yields the assertions since $\mathrm{proj}\,\mathscr{L}_N = \mathscr{L}$. $\qquad\square$

Another problem is, that, in general, we can not expect condition (T_ε) to characterize the vanishing of $\mathrm{proj}^1 \mathscr{K}$. As the following, rather trivial, example shows, even very nice spaces E can exhibit very bad behavior regarding characterizing conditions:

Example 3.5.2. Let $E = \varphi = \bigoplus_{\nu \in \mathbb{N}} \mathbb{K}^\nu$ and $X = \mathrm{proj}\left(X_N, X_M^N\right)$ a Fréchet-Hilbert space with $\mathrm{proj}^1\left(X_N, X_M^N\right) = 0$ that is not quasinormable, cf. [MV97, (27.33)]. Since, as a direct sum of finite dimensional Hilbert spaces, E is a projective object in the category of LH spaces (even in LCS), we have $\mathrm{Ext}^1_{\mathrm{LH}}\left(E, X_N\right) = 0, N \in \mathbb{N}$, hence $0 = \mathrm{Ext}^1_{\mathrm{PLH}}\left(E, X\right) = \mathrm{proj}^1 \mathscr{L} = \mathrm{proj}^1 \mathscr{K}$ by (2.3.6) ii) and (3.5.1). However, the pair (E', X) can not have property (T_ε), since X does not have property $(P_3^*)_0$ by [MV97, (26.14)].

Of course, if E contains a complemented copy of ℓ_2, then the vanishing of $\mathrm{proj}^1 \mathscr{L}$ implies $(P_3^*)_0$ for X:

Remark 3.5.3. If $X = \mathrm{proj}\,\mathscr{X} = \mathrm{proj}\left(X_N, X_M^N\right)$ is a PLH space and $E = \mathrm{ind}_{\nu \in \mathbb{N}} E_\nu$ is an LH space that contains a complemented copy of ℓ_2 such that $\mathrm{proj}^1 \mathscr{L} = 0$, then X has $(P_3^*)_0$:

We use (3.3.1) to prove the claim: Without loss of generality we may assume $E_1 = \ell_2$ with continuous projection $P \colon E \longrightarrow E_1$. Let B be a bounded subset of $\prod_{N \in \mathbb{N}} X_N$ that spans $\ell_2 \cong E_1$. Then P is a linear and continuous operator from E to $([B], p_B) \hookrightarrow \prod_{N \in \mathbb{N}} X_N$, hence there is a linear and continuous operator $R \colon E \longrightarrow \prod_{N \in \mathbb{N}} X_N$ with $\Psi_{\mathscr{X}} \circ R = P$, as $\mathrm{proj}^1 \mathscr{L} = 0$. Thus $R\left(B_{\ell_2}\right)$ is a lifting of B with respect to $\Psi_{\mathscr{X}}$ and (3.3.1) yields the conclusion.

In general, we can neither renounce the acyclicity of E nor condition $(P_3^*)_0$ of X for the characterization. Even characterizing condition $(P_3^*)_0$ for special spaces E, as the dual S' of the space s of rapidly decreasing sequences, that does not contain a complemented copy of ℓ_2,

but is rather nice in any other sense, as nuclearity, separability etc., seems to be out of reach. However, we are able to prove the following analogon of the characterization of the vanishing of $\mathrm{proj}^1\,\mathscr{K}$ in the PLS setting [BD08, (3.1)]:

Theorem 3.5.4. *Let* $E = (E_\nu, E_\nu^\mu)$ *an* LH *space and* $X = \mathrm{proj}\left(X_N, X_M^N\right)$ *a deeply reduced* PLH *space. If* E *is acyclic and* X *has* $(P_3^*)_0$, *then* $\mathrm{proj}^1\,\mathscr{K}$ *vanishes if and only if the condition* (T_ε) *holds for the pair* (E', X) *if and only if the condition* (T) *holds for the pair* (E', X). *Here* $\mathscr{K} = \left(\mathrm{L}\left(E_N, X_N\right), E_{N\ *}^M \circ X_M^{N\ *}\right)$, *cf. (3.5.1).*

Proof. The proof is very similar to that of (3.4.5):

According to (3.1.5), $\mathscr{H} := \left(H_N, H_N^{N+1}\right) := \left(E_N \tilde{\otimes}_\pi X_N', E_N^{N+1} \tilde{\otimes}_\pi \left(X_{N+1}^N\right)^t\right)$ is an LF spectrum with injective linking maps that is pre-dual to $\mathscr{K}$. Hence with (3.1.1) the vanishing of $\mathrm{proj}^1\,\mathscr{K}$ is equivalent to the weak acyclicity of $\mathscr{H}$.

<u>Necessity of (T)</u>: In complete analogy to the proof of necessity in (3.4.5) we have that (E', X) has property (T):

Since $H_N = \mathrm{proj}_{n\in\mathbb{N}} E_N \tilde{\otimes}_\pi X_{N,n}'$, by [Köt79, §41, 6. (3)], exchanging quantifiers in condition (wQ) implies the following condition

$$\underset{N\in\mathbb{N}}{\forall}\ \underset{M\geq N}{\exists}\ \underset{K\geq M}{\forall}\ \underset{n\in\mathbb{N}}{\exists}\ \underset{m\geq n}{\forall}\ \underset{\substack{k\geq m\\c>0}}{\exists}\ \underset{\substack{f\in X_N'\\x\in E_N}}{\forall}$$

$$\text{If } \pi\left(x\otimes f; E_N, X_{N,n}'\right) \leq 1 \text{ and } \pi\left(E_N^K(x)\otimes\left(X_K^N\right)^t(f); E_K, X_{K,k}'\right) \leq 1,$$

$$\text{then } \pi\left(E_K^M(x)\otimes\left(X_M^N\right)^t(f); E_M, X_{M,m}'\right) \leq c,$$

which is condition (T^d) for (E', X), which is equivalent to (T) by (3.2.1) ii) b). As E is acyclic, E' satisfies $(P_3^*)_0$ by (3.1.4).

<u>(T) implies (T_ε)</u>: is (3.2.2).

<u>Sufficiency of (T_ε)</u>: Exactly as in the proof of sufficiency of (3.4.5) we see that condition (T_ε) in the form (3.2.1) v) corresponds exactly to the norm inequality, that correlates to condition $(Q)_0$ for $\mathscr{H}$, though only for elementary tensors $x\otimes f \in E_\nu \tilde{\otimes} X_{N,k}'$. The same application of the interpolation result (3.2.6) yields that this inequality holds for all elements of the tensor product $E_\nu \otimes_\pi X_{N,k}'$ and again the continuity of the respective inclusions yields the inequality for all $u \in E_\nu \tilde{\otimes}_\pi X_{N,k}'$. Hence $(Q)_0$ holds for $\mathscr{H}$, which implies (M) by (3.1.2), hence $\mathscr{H}$ is acyclic. Thus $\mathrm{proj}^1\,\mathscr{K} = 0$ by (3.1.1). $\qquad\square$

Together with (2.3.6) ii) the previous theorem (3.5.4) and (3.5.1) yield the following splitting result:

Corollary 3.5.5. *Let* $E = \mathrm{ind}\left(E_\nu, E_\nu^\mu\right)$ *be an* LH *space and* $X = \mathrm{proj}\left(X_N, X_M^N\right)$ *a deeply reduced* PLH *space with* $\mathrm{proj}^1 X = 0$ *and* $(P_3^*)_0$ *such that* $\mathrm{Ext}_{\mathrm{LH}}^1(E, X_N) = 0$ *for all* $N \in \mathbb{N}$. *Then the following characterizations hold:*

   i) If X has at least one step X_{N_0} with infinite dimension, then $\mathrm{Ext}^1_{\mathrm{PLH}}(E, X) = 0$ if and only if the pair (E', X) has property (T_ε). Furthermore, $\mathrm{Ext}^k_{\mathrm{PLH}}(E, X) = 0$ for all $k \geq 2$.

   ii) If all X_N are of finite dimension and E is acyclic, then $\mathrm{Ext}^1_{\mathrm{PLH}}(E, X) = 0$ if and only if the pair (E', X) has property (T_ε) and in addition $\mathrm{Ext}^k_{\mathrm{PLH}}(E, X) = 0$ for all $k \geq 2$.

Proof. First of all the vanishing of the local Ext groups $\mathrm{Ext}^1_{\mathrm{LH}}(E, X_N)$, $N \in \mathbb{N}$, is equivalent to the vanishing of the dual Ext groups $\mathrm{Ext}^1_{\mathrm{FH}}\left(X'_N, E'\right)$, $N \in \mathbb{N}$, by (2.2.6), which in turn is equivalent to the vanishing of $\mathrm{proj}^1 \mathscr{L}'_N$ by (2.3.6), where $\mathscr{L}'_N = \left(\mathrm{L}\left(X'_N, E'_\nu\right), (E^\mu_\nu)^{t\,*}\right)$. Using a simple duality argument, this is equivalent to the vanishing of $\mathrm{proj}^1 \mathscr{L}_N$, where $\mathscr{L}_N = (\mathrm{L}(E_\nu, X_N), E^\mu_{\nu\,*})$. Thus the vanishing of the local Ext groups is exactly the requirement we need for (3.5.1), hence (3.5.4) yields the assertion, as the vanishing of $\mathrm{Ext}^1_{\mathrm{FH}}\left(X'_{N_0}, E'\right)$, $N \in \mathbb{N}$, yields the acyclicity of E with (3.4.6) for (3.5.5) i). $\qquad\square$

   As in section 3.4, the proof of sufficiency of (T_ε) for splitting in (3.5.4) depends heavily on X being deeply reduced, as otherwise we can not interpolate. Although general necessity statements are not possible, the following holds in complete analogy to (3.4.7):

Remark 3.5.6. If E is an LH space and X is a PLH space such that (E', X) satisfies (T_ε), then either X is deeply reduced in rows or E is a strict LH space.

4 The Hilbert tensor product

Locally convex tensor products were introduced and studied extensively by A. Grothendieck [Gro55] in the fifties of the past century. One of the many reasons for the prominence of this area is the tensor product representation of many spaces of vector valued functions as the space of vector valued distributions, cf. [Sch58], and the spaces of holomorphic, real analytic or smooth functions, cf. [Dom11], which connects the parameter dependence problem for linear partial differential operators with constant coefficients to the problem under which conditions the tensorized operator remains surjective when extended to the completed tensor product. In this context the vanishing of proj[1] for tensorized spectra of PLS and PLN (or FN) spaces has been studied by Domański and Bonet and a complete characterization with a large variety of applications is given in [BD06, BD07, BD08, Dom10, Dom11]. Due to its rather complex structure, the space of real analytic functions received particular interest i.a. by Bonet, Domański, Frerick and Vogt, cf. [Vog75, Vog83a, BD98, DV00a, BD01, BDV02, DFV03, Vog04, Dom10].

In order to implement such a theory for PLH spaces, we can not limit ourselves to the ε or π tensor product as even $\ell_2 \tilde{\otimes}_\alpha \ell_2$ is not reflexive for $\alpha \in \{\varepsilon, \pi\}$, cf. [MV97, (16.27)]. Hence, contrary to the PLS situation, we can not expect the local tensor products $E_N \tilde{\otimes}_\alpha X_N, \alpha \in \{\varepsilon, \pi\}$ for two PLH spaces $E = \mathrm{proj}\left(E_T, E_S^T\right)$ and $X = \mathrm{proj}\left(X_N, X_M^N\right)$ to be ultrabornological, reflexive or even an LH space. For positive results in this direction the premises are very restrictive, cf. [Hol80]. Hence the results for the vanishing of proj[1] for projective spectra of separated LB spaces are not applicable in the context of neither the ε nor the π tensor product. Thus, our first aim is to establish a suitable tensor topology $\mathcal{T}_\sigma$ on the tensor product of two hilbertizable lcs E and F – i.e. lcs with fundamental systems of Hilbert seminorms (see [MV97, p. 344]) – creating a hilbertizable lcs $E \otimes_\sigma F$, which is in a certain sense compatible with the PLH space structure, see (4.2.14), as the ε and π topologies are compatible with the PLS space structure, cf. [Pis10, p. 158].

In contrast to the standard method to define a tensor topology for locally convex spaces, cf. [DF93, chap. 35], we will define in section 4.1 a Hilbert tensor seminorm directly on the tensor products of the local seminormed spaces and prove that it is finitely generated and both injective and projective. In section 4.2 we define the corresponding tensor topology $\mathcal{T}_\sigma$ and prove that it is compatible with the LH space structure to establish the announced compatibility with the PLH space structure, which will in fact involve the positive solution of Grothendieck's *problème des topologies* for the tensor topology $\mathcal{T}_\sigma$. Moreover, we will prove a surjectivity criterion for tensorized operators and the tensor topology $\mathcal{T}_\sigma$. Based on the results of section 4.2 we will investigate the vanishing of proj[1] for tensorized spectra of PLH spaces in section 4.3. To begin with we establish the traditional connection between surjectivity of tensorized operators and the vanishing of proj[1] for tensorized spectra. Then we prove that the vanishing of proj[1] is passed on from PLH spaces to the σ tensor product with Hilbert spaces. Finally we characterize the vanishing of proj[1] for the tensorized spectrum of both a nuclear Fréchet space and a PLN space E and a PLH space X under mild assumptions with our condition (T_ε), thus proving Bonet and Domański's result [BD06, BD08, Dom10] in the category of PLH spaces and generalizing Varol's result [Var02, (4.2.2)]. Since we only have interpolation in the sense of (3.2.6) for nuclear operators we can not forgo nuclearity.

4.1 Semi-unitary spaces

Given two Banach spaces E and F, there is a variety of reasonable tensor norms on the tensor product $E \otimes F$ (see e.g. [DF93]), the ε norm being the smallest, the π norm the largest of those norms. This variety creates the problem of finding the "right" tensor norm to deal with for specific applications. As mentioned in the introduction of this chapter, we are interested in a semi-scalar product on the tensor product of *semi-unitary* spaces, i.e. spaces carrying semi-scalar products.

Given two semi-unitary spaces E and F, the problem of finding the "right" semi-scalar product on $E \otimes F$ is much simpler, as a semi-scalar product on $E \otimes F$, which is induced by the semi-scalar products on E and F, is uniquely determined. Concerning a scalar product on the tensor product of two Hilbert spaces, this is a well-known fact (see e.g. [Mur90, Theorem (6.3.1)]); we just need some minor adaptions to prove the analogon for semi-unitary spaces, compare [Bou87, TVS V.25 §3 No. 1 & No. 2]:

Proposition 4.1.1. *Given semi-unitary spaces E and F, there is a unique semi-scalar product $\langle \cdot, \cdot \rangle_\sigma$ on $E \otimes F$, such that*

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle_\sigma = \langle x_1, x_2 \rangle_E \langle y_1, y_2 \rangle_F \qquad (\star)$$

holds for all $x_1, x_2 \in E$ and $y_1, y_2 \in F$.

Proof. The proof of the existence and uniqueness of such a sesqui-linear form $\langle \cdot, \cdot \rangle_\sigma$ is the same as in [Mur90, Theorem (6.3.1)]:

Given two elements $x \in E$ and $y \in F$, we denote by τ_x and τ_y the induced conjugate linear forms $\tau_x(x') := \langle x, x' \rangle_E$ $(x' \in E)$ and $\tau_y(y') := \langle y, y' \rangle_F$ $(y' \in F)$ on E and F respectively. Then there is a unique linear map M from $E \otimes F$ to the space of conjugate linear forms on $E \otimes F$ that realizes the tensor product of the induced conjugate linear forms on elementary tensors, i.e. $M(x \otimes y) = \tau_x \otimes \tau_y$ for all $x \in E$ and $y \in F$. Then it is easy to check that

$$\langle \cdot, \cdot \rangle_\sigma \colon (E \otimes F)^2 \longrightarrow \mathbb{K}, (z, z') \mapsto M(z)(z')$$

is a sesqui-linear form on $E \otimes F$ satisfying (4.1.1) $(\star)$. The uniqueness is obvious.

It remains to show that $\langle \cdot, \cdot \rangle_\sigma$ is positive semi-definite. Here we have to adjust Murphy's proof: Given an element $z = \sum_{i=1}^n x_i \otimes y_i \in E \otimes F$, we have to show that $\langle z, z \rangle_\sigma = \sum_{i,j=1}^n \langle x_i, x_j \rangle_E \langle y_i, y_j \rangle_F \geq 0$. Considering the subspaces $E_z := \text{span}\{x_1, \dots, x_n\}$ and $F_z := \text{span}\{y_1, \dots, y_n\}$ of E respectively F, it is sufficient to calculate the above sum in the finite dimensional semi-unitary spaces E_z and F_z. Now we choose a basis $(e_1, \dots, e_m)$ of F_z such that $\|e_k\|_F^2 = \|e_k\|_F$ $(1 \leq k \leq m)$ and $\langle e_k, e_l \rangle_F = 0$ if $k \neq l$. Then there are $x'_1, \dots, x'_m \in E_z$, such that $z = \sum_{k=1}^n x'_k \otimes e_k$ and we obtain:

$$\langle z, z \rangle_\sigma = \sum_{k=1}^m \langle x'_k, x'_k \rangle_E \langle e_k, e_k \rangle_F = \sum \left\| x'_k \right\|_E^2 \|e_k\|_F^2 \geq 0.$$

$\square$

We use (4.1.1) to define the tensor seminorm σ for semi-unitary spaces:

Definition and Remark 4.1.2.   i) For two semi-unitary spaces E and F we define the *tensor seminorm* σ as $\sigma(z; E, F) := \sqrt{\langle z, z \rangle_\sigma}$ for $z \in E \otimes F$. If no confusion concerning the spaces E and F is possible, we will just write $\sigma(z)$. Furthermore, we denote by $E \otimes_\sigma F$ the σ tensor product of E and F equipped with this seminorm and by $E \tilde{\otimes}_\sigma F$, as usual, its Hausdorff completion, also called the completed σ tensor product.

ii) The proof of (4.1.1) implies that $\langle \cdot, \cdot \rangle_\sigma$ is an inner product if E and F are pre-Hilbert spaces and that the Hausdorff space associated to $E \otimes_\sigma F$ is the σ tensor product of the Hausdorff spaces associated to E respectively F if E and F are semi-unitary spaces.

iii) By construction the transposition map $x \otimes y \mapsto y \otimes x$ is an isometric isomorphism from $E \otimes_\sigma F$ to $F \otimes_\sigma E$; i.e. σ is *symmetric* for semi-unitary spaces E and F.

iv) In the theory of C^*-algebras the so-called *Hilbert tensor product* is introduced via a universal property involving weak Hilbert-Schmidt mappings (see e.g. [KR83, (2.6)]). As we won't need the tensor product of more than two spaces, we are content with the above formulation.

In the following, we will establish the natural realization of $E \tilde{\otimes}_\sigma F$ as the space of Hilbert-Schmidt operators $S_2(E', F)$ from E' to F if E and F are Hilbert spaces, and prove the *metric mapping property* of σ. For the general theory of Hilbert-Schmidt operators, we refer to [MV97, chap. III, §16].

Remark 4.1.3.   i) Let E and F be Hilbert spaces.

a) The space $\mathcal{F}(E', F)$ of weak-*-continuous finite rank operators from E' to F together with $\underline{\otimes}: E \times F \longrightarrow \mathcal{F}(E', F), (x, y) \mapsto x \underline{\otimes} y: E' \longrightarrow F, u \mapsto u(x) y$ is a realization of $E \otimes F$ by [Jar81, (15.3), 6.]. Thus, if we can show that (4.1.1) ($\star$) holds for the Hilbert-Schmidt scalar product $\langle z^{(1)}, z^{(2)} \rangle_{\nu_2}$ of two rank 1 operators $\underline{z}^{(1)} = x^{(1)} \underline{\otimes} y^{(1)}$ and $\underline{z}^{(2)} = x^{(2)} \underline{\otimes} y^{(2)}$, we have proven that the space $S_2(E', F)$ of Hilbert-Schmidt operators together with $\underline{\otimes}$ is a realization of $E \tilde{\otimes}_\sigma F$. The scalar product in question is given by the trace of the nuclear operator $\left(\underline{z}^{(2)} \right)^* \circ \underline{z}^{(1)} \in \mathcal{N}(E', E')$:

$$\langle \underline{z}^{(1)}, \underline{z}^{(2)} \rangle_{\nu_2} = \mathrm{tr}_{E'} \left(\left(\underline{z}^{(2)} \right)^* \circ \underline{z}^{(1)} \right) = \sum_{\iota \in I} \left\langle \left(\left(\underline{z}^{(2)} \right)^* \circ \underline{z}^{(1)} \right) u_\iota, u_\iota \right\rangle_{E'}$$

for any orthonormal basis $(u_\iota)_{\iota \in I}$ of E'. Using the antilinear Riesz identification of E and E' and by choosing a suitable basis $(u_\iota)_{\iota \in I}$ of E', we arrive at (4.1.1) ($\star$). Whenever we want to treat elements z of $E \tilde{\otimes}_\sigma F$ as operators, we will write $\underline{z} \in E \tilde{\otimes}_\sigma F$. This yields a useful representation of the elements of $E \tilde{\otimes}_\sigma F$:

b) Given an element z of the completed σ tensor product of two Hilbert spaces E and F, there are orthonormal systems $(u_k)_{k \in \mathbb{N}}$ and $(y_k)_{k \in \mathbb{N}}$ in E' and F respectively, and a decreasing ℓ_2-series $(s_k)_{k \in \mathbb{N}}$ with positive entries such that

$$\underline{z}(u) = \sum_{k=1}^{\infty} s_k \langle u, u_k \rangle_{E'} y_k \quad \text{for all } u \in E'.$$

Again the antilinear Riesz identification of E and E' yields an orthonormal system $(x_k)_{k \in \mathbb{N}}$ in E corresponding to $(u_k)_{k \in \mathbb{N}}$ such that

$$z = \sum_{k=1}^{\infty} s_k x_k \otimes y_k$$

in $E \tilde{\otimes}_\sigma F$. The norm of z is then given by $\sigma(z; E, F) = \|(s_k)_{k \in \mathbb{N}}\|_{\ell_2} = \sqrt{\sum_{k \in \mathbb{N}} s_k^2}$.

The properties of the Hilbert tensor product of maps can be proven in the same way as for the ε product:

ii) Given maps $S \in L(E, G)$ and $T \in L(F, H)$ between semi-unitary spaces E, F, G and H, we have the usual linear map $S \otimes T$ from $E \otimes F$ to $G \otimes H$. Using the notation $S \otimes_\sigma T$ for the map $S \otimes T : E \otimes_\sigma F \longrightarrow G \otimes_\sigma H$, we will show the *metric mapping property* of σ, i.e. that $\|S \otimes_\sigma T\| \leq \|S\| \|T\|$ holds. As linear and continuous maps between seminormed spaces factor uniquely and isometrically over the associated normed spaces, we can restrict ourselves to pre-Hilbert spaces E, F, G and H. We consider the continuous extensions to the (Hausdorff) completions $\tilde{S} \in L(\tilde{E}, \tilde{G})$ and $\tilde{T} \in L(\tilde{F}, \tilde{H})$. Using the realization of the completed σ tensor product from above, we obtain:

$$\left(\tilde{S} \otimes_\sigma \tilde{T}\right) z = \tilde{T} \circ \underline{z} \circ \tilde{S}^t \text{ for all } z \in \tilde{E} \otimes_\sigma \tilde{F}, \qquad (\star)$$

and thus $\|S \otimes_\sigma T\| = \|\tilde{S} \otimes_\sigma \tilde{T}\| \leq \|\tilde{S}\| \|\tilde{T}\| = \|S\| \|T\|$. Denoting as usual the continuous extension of $S \otimes_\sigma T$ to $E \tilde{\otimes}_\sigma F$ by $S \tilde{\otimes}_\sigma T$, we obtain that $S \tilde{\otimes}_\sigma T$ is injective whenever S and T are injective and E and F are Hilbert spaces as $(\star)$ extends to the completion.

Having established σ as a Hilbert seminorm on the tensor product of two semi-unitary spaces that is a reasonable tensor norm for pre-Hilbert spaces, we will prove in the rest of this section that the σ tensor product respects isometric inclusions and metric surjections for semi-unitary spaces, and the completed σ tensor product does the same for Hilbert spaces. The preservation of isometric inclusions won't be problematic, whereas concerning the metric surjections, we need the following lemma:

Lemma 4.1.4. *Let $q \colon F \longrightarrow G$ be a metric surjection with closed kernel between semi-unitary spaces. Furthermore, let G be of finite dimension. Then for every $\varepsilon > 0$ there is a linear and continuous right inverse R_ε of q with $\|R_\varepsilon\| \leq 1 + \varepsilon$.*

Proof. Let $1 > \varepsilon > 0$. There is a finite dimensional subspace L_0 of F such that $q(L_0) = G$. As $q\left(\overline{\{0\}}^F\right) = \{0\}$, we may assume that L_0, equipped with the relative topology of F, is Hausdorff. As L_0 and G are Hausdorff spaces of finite dimension, the restriction $q|_{L_0} : L_0 \longrightarrow G$ is open, which implies that $q(U_{L_0})$ is open as well, where U_L is the relative open unit ball for any subspace L of F. As $q(U_F) \supset U_G \supset (1 - \varepsilon) B_G$ and $q\left(\overline{\{0\}}^F\right) = \{0\}$, the system

$$\left\{q(U_L) : L \text{ is a finite dimensional subspace of } F \text{ with } L_0 \subset L \text{ and } \overline{\{0\}}^F \cap L = \{0\}\right\}$$

is an open cover of the compact set $(1 - \varepsilon) B_G$. Thus we obtain a finite dimensional separated subspace L of F such that $q(U_L) \supset (1 - \varepsilon) B_G$. Endowing G with the norm topology of the gauge functional p of the set $q(U_L)$, we obtain a Hilbert space (G, p) that is isometrically isomorphic to a closed subspace H of the Hilbert space L. With the orthogonal projection from L to H we have a linear and continuous right inverse R_L of $q|_L : L \longrightarrow G$ with norm 1 with respect to G equipped with p. We compute

$$\|R_L\|_{\mathrm{L}(G,L)} = \frac{1}{1 - \varepsilon} \sup_{y \in (1-\varepsilon)B_G} \|R_L(y)\|_L \leq \frac{1}{1 - \varepsilon} \sup_{y \in q(U_L)} \|R_L(y)\|_L \leq \frac{1}{1 - \varepsilon} \sup_{y \in B_p} \|R_L(y)\|_L = \frac{1}{1 - \varepsilon}.$$

Defining R_ε as the composition of the inclusion of L into F with R_L completes the proof. $\square$

The following example shows that the previous lemma (4.1.4) is sharp:

Example 4.1.5. There is a pre-Hilbert space F and a metric surjection $q \colon F \longrightarrow \mathbb{K}$ such that there is no linear and continuous right inverse R of q with $\|R\| = 1$. In particular, there is no projection from F to the kernel of q with norm 1.

Proof. Let F be the space of continuous (real or complex valued) functions on the interval $[-1, 1]$ endowed with the L_2-norm and $q \colon F \longrightarrow \mathbb{K}, f \mapsto \int_{[0,1]} f d\lambda$. Then F is a pre-Hilbert space and q is continuous, as $q(f) = \langle f, \chi_{[0,1]}\rangle_{\mathrm{L}_2}$ for all $f \in F$. q is even a metric surjection, as $q(U_F) = U_{\mathbb{K}}$ holds. Assuming there is a linear and continuous right inverse R of q with $\|R\| = 1$, we obtain a function $f_0 \in F$ with $R(\mu) = \mu f_0$ for all $\mu \in \mathbb{K}$. Furthermore, we have

$$1 = (q \circ R)(1) = \int_{[0,1]} f_0 d\lambda \text{ and } 1 = \|R(1)\|_{\mathrm{L}_2} = \left(\int_{[-1,1]} |f_0|^2 d\lambda\right)^{1/2}.$$

Thus the Cauchy-Schwarz inequality holds for f_0 and $\chi_{[0,1]}$ with equality, which implies that the equivalence classes of f_0 and $\chi_{[0,1]}$ are linearly dependent in $\mathrm{L}_2([-1, 1])$. This is not possible as there is no continuous function on the interval $[-1, 1]$ that differs from $\chi_{[0,1]}$ only on a zero set. $\square$

Now we are ready to prove:

Proposition 4.1.6. *i) The σ tensor product respects metric inclusions and metric surjections for semi-unitary spaces.*

ii) The completed σ tensor product respects metric inclusions and metric surjections for Hilbert spaces.

Proof. i) Let E be a semi-unitary space and

$$0 \longrightarrow H \xrightarrow{\ i\ } F \xrightarrow{\ q\ } G \longrightarrow 0$$

a short algebraically exact sequence of semi-unitary spaces, where i is a metric inclusion and q a metric surjection.

As the σ tensor product is symmetric, cf. (4.1.2) iii), it is sufficient to prove, that $\mathrm{id}_E \otimes_\sigma i$ is a metric inclusion and $\mathrm{id}_E \otimes_\sigma q$ is a metric surjection.

We start with $\mathrm{id}_E \otimes_\sigma i$:
If $z = \sum_{k=1}^n x_k \otimes y_k \in E \otimes_\sigma H$ is given, we can compute

$$\sigma(z; E, H)^2 = \sum_{k=1}^n \sum_{l=1}^n \langle x_k, x_l \rangle_E \langle y_k, y_l \rangle_H = \sum_{k=1}^n \sum_{l=1}^n \langle x_k, x_l \rangle_E \langle i(y_k), i(y_l) \rangle_F$$
$$= \sigma((\mathrm{id}_E \otimes_\sigma i)(z); E, F)^2.$$

Now we will first use (4.1.4) to show that $\mathrm{id}_E \otimes_\sigma q$ is a metric surjection whenever the kernel H of q is closed in F. Let H be closed. We need to prove that $\mathrm{id}_E \otimes_\sigma q\,(U_{E \otimes_\sigma F})$ and $U_{E \otimes_\sigma G}$ have the same gauge functional. The metric mapping property of σ, cf. (4.1.3) ii), yields the inclusion

$$\mathrm{id}_E \otimes_\sigma q\,(U_{E \otimes_\sigma F}) \subset U_{E \otimes_\sigma G},$$

from which we obtain a corresponding inequality of the gauge functionals. To prove the other inequality let E_1 and G_1 be finite dimensional subspaces of E and G respectively. We define $F_1 := q^{-1}(G_1), q_1 := q|_{F_1}, H_1 := \ker q_1 = H \cap F_1$ and $i_1 := i|_{G_1}$. Then

$$0 \longrightarrow H_1 \xrightarrow{\ i_1\ } F_1 \xrightarrow{\ q_1\ } G_1 \longrightarrow 0$$

is a topologically exact sequence of semi-unitary spaces, i_1 is a metric inclusion and q_1 is a metric surjection. Furthermore, since H is closed in F, G is Hausdorff, hence G_1 is a finite dimensional Hilbert space and H_1 is closed in F_1. Thus (4.1.4) yields for every $\varepsilon > 0$ a linear and continuous right inverse R_ε of q_1 with $\|R_\varepsilon\| \leq 1 + \varepsilon$, and we obtain:

$$U_{E_1 \otimes_\sigma G_1} = ((\mathrm{id}_{E_1} \otimes_\sigma q_1) \circ (\mathrm{id}_{E_1} \otimes_\sigma R_\varepsilon))(U_{E_1 \otimes_\sigma G_1}) \subset (1 + \varepsilon)\,(\mathrm{id}_{E_1} \otimes_\sigma q_1)(U_{E_1 \otimes_\sigma F_1}).$$

Since $\varepsilon > 0$ is arbitrary, we have

$$U_{E_1 \otimes_\sigma G_1} \subset \bigcap_{\varepsilon > 0} (1 + \varepsilon) \left(\mathrm{id}_{E_1} \otimes_\sigma q_1 \right) \left(U_{E_1 \otimes_\sigma F_1} \right) \subset \overline{\left(\mathrm{id}_E \otimes_\sigma q \right) \left(U_{E \otimes_\sigma F} \right)}^{E \otimes_\sigma G}.$$

As we have proven in the first part of (4.1.6) i) that the σ tensor product respects metric inclusions, we obtain:

$$U_{E \otimes_\sigma G} = \bigcup \{ U_{E_1 \otimes_\sigma G_1} : E_1 \subset E, G_1 \subset G \text{ finite dimensional subspaces} \}$$

$$\subset \overline{\left(\mathrm{id}_E \otimes_\sigma q \right) \left(U_{E \otimes_\sigma F} \right)}^{E \otimes_\sigma G},$$

which yields the assertion for closed subspaces.

If H is not closed, we consider the exact sequence

$$0 \longrightarrow \overline{H} \xrightarrow{\bar{i}} F \xrightarrow{\bar{q}} F/\overline{H} \longrightarrow 0,$$

where $\bar{i}$ is the metric inclusion of the closure of H into F and $\bar{q}$ is the corresponding quotient map. Then $F/\overline{H}$ is a version of the Hausdorff space associated to G, hence the map $\varphi \colon G \longrightarrow F/\overline{H}, q(x) \mapsto \bar{q}(x)$ is a metric surjection with linear and continuous right inverse with norm 1. Considering the following commutative diagram

$$
\begin{array}{ccc}
E \otimes_\sigma F & \xrightarrow{\mathrm{id}_E \otimes_\sigma q} & E \otimes_\sigma G \\
{\scriptstyle \mathrm{id}_E \otimes_\sigma \bar{q}} \downarrow & \swarrow {\scriptstyle \mathrm{id}_E \otimes_\sigma \varphi} & \\
E \otimes_\sigma F/\overline{H} & &
\end{array}
,$$

we have for $z \in E \otimes_\sigma F$:

$$\sigma \left((\mathrm{id}_E \otimes_\sigma q)(z) ; E, G \right) = \sigma \left(\left((\mathrm{id}_E \otimes_\sigma \varphi) \circ (\mathrm{id}_E \otimes_\sigma q) \right)(z) ; E, F/\overline{H} \right)$$

$$= \sigma \left((\mathrm{id}_E \otimes_\sigma \bar{q})(z) ; E, F/\overline{H} \right)$$

$$= \inf \left\{ \sigma(\bar{z}; E, F) : z - \bar{z} \in \ker(\mathrm{id}_E \otimes_\sigma \bar{q}) \right\}$$

$$= \inf \left\{ \sigma(\bar{z}; E, F) : z - \bar{z} \in \ker(\mathrm{id}_E \otimes_\sigma q) \right\},$$

since $\mathrm{id}_E \otimes_\sigma \varphi$ has a linear and continuous right inverse with norm 1, $\mathrm{id}_E \otimes_\sigma \bar{q}$ is a metric surjection, as we have proven in the first part, and $E \otimes_\sigma H$ is dense in $E \otimes_\sigma \overline{H}$. Thus $\mathrm{id}_E \otimes_\sigma q$ is a quotient map and the proof of i) is complete.

ii) is shown analogously to i):

The preservation of metric inclusions is proven with the representation of the elements of $E \tilde{\otimes}_\sigma H$ in (4.1.3) i) b). The proof of the preservation of metric surjections of the

completed σ tensor product becomes much easier since we can work with the linear and continuous right inverse mappings with norm 1 of the quotient maps instead of using (4.1.4) as we are dealing with closed subspaces only.

$\square$

Remark 4.1.7.   i) σ is finitely generated and co-finitely generated, i.e.:

$$E \otimes_{\vec{\sigma}} F = E \otimes_{\sigma} F = E \otimes_{\overleftarrow{\sigma}} F \text{ for all Hilbert spaces } E \text{ and } F,$$

where

$$\vec{\sigma}(z; E, F) :=$$
$$\inf \{ \sigma(z; G, H) : G \subset E, H \subset F \text{ finite dimensional subspaces with } z \in G \otimes H \}$$

 and

$$\overleftarrow{\sigma}(z; E, F) := \sup \left\{ \sigma \left(z; E/_G, F/_H \right) : G \subset E, H \subset F \text{co-finite dimensional subspaces} \right\}$$
for all $z \in E \otimes F$.

  ii) Harksen has shown analogous results for finitely generated tensor norms α on the category of normed spaces in [Har82, (1.16) & (1.19)]. His proof involves the theory of the dual tensor norm α' and the (to α) associated co-finitely generated tensor norm $\tilde{\alpha}$. Using the concept of *tensorized seminorms* (see [Har82, p. 357]) and some auxiliary lemmata ([Har82, (2.4) & (2.5)]) about quotient seminorms and metric surjections between seminormed spaces, he is able to prove the analogon of (4.1.6) i) for metric surjections between seminormed spaces in [Har82, (2.6)]. A sorrow analysis of his proofs shows that they are valid for the σ tensor product as well. However, we have chosen to give direct proofs.

 iii) As Varol has shown in [Var02, Var07], it is well possible to approach problems concerning tensor norms in a homological manner. Indeed, we could endow the categories of semi-unitary spaces, pre-Hilbert spaces and Hilbert spaces with natural exact structures and investigate the preservation properties under consideration in (4.1.6) in context of some kind of *metric exactness* of the tensor product functors $\mathrm{id}_E \otimes_{\sigma} \cdot$ and $\mathrm{id}_E \tilde{\otimes}_{\sigma} \cdot$ on the respective categories. Since chapter 4.1 is more a means to an end for a part of this thesis to establish the tensor topology $\mathcal{T}_{\sigma}$ on the category of PLH spaces, we have chosen to provide the needed statements in a conventional functional analytic way.

4.2 Hilbertizable locally convex spaces

Given two hilbertizable locally convex spaces E and F with corresponding fundamental systems of Hilbert seminorms $\mathcal{P}$ and $\mathcal{R}$ respectively, we endow $E \otimes F$ with the locally convex topology $\mathcal{T}_{\sigma}$ induced by the directed system of Hilbert seminorms $(p \otimes_{\sigma} r)_{(p,r) \in \mathcal{P} \times \mathcal{R}}$, where $p \otimes_{\sigma} r(\cdot) := \sigma(\cdot; (E, p), (F, r))$, obtaining the hilbertizable locally convex space $E \otimes_{\sigma} F$. It is

easy to see that this construction is independent of the choice of the fundamental systems of seminorms. As usual, we denote the Hausdorff completion of $E \otimes_\sigma F$ with $E \tilde{\otimes}_\sigma F$.

A very similar construction of the extension of a finitely generated tensor norm α for normed spaces to a tensor topology $\mathcal{T}_\alpha$ on the tensor product of two locally convex spaces was introduced by Harksen in his dissertation in 1979 ([Har79], see also [Har82] and [DF93, chap. 35]). As he was dealing with tensor norms for normed spaces only, he needs the concept of *tensorized seminorms* (see [Har82, p. 357]), which involves dealing with the kernels of the seminorms. We can avoid this onerosity as we managed to define σ directly for the tensor product of semi-unitary spaces in (4.1.2).

Given linear and continuous maps $S \in L(E, G)$ and $T \in L(F, H)$ between hilbertizable locally convex spaces E and G respectively F and H, we have the linearisation $S \otimes_\sigma T : E \otimes_\sigma F \longrightarrow G \otimes_\sigma H$ of $S \times T$, which is easily seen to be continuous, and denote its extension to the completion by $S \tilde{\otimes}_\sigma T$. We expand the notation of (4.1.2) ii) and will speak of the (completed) σ tensor product on the category of hilbertizable lcs. Now we show that $S \tilde{\otimes}_\sigma T$ is injective whenever E, F, G, and H are complete and S and T are injective:

Proposition 4.2.1. *Let E, F, G and H be complete hilbertizable lcs, $S \in L(E, G)$ and $T \in L(F, H)$ injective operators. Then $S \tilde{\otimes}_\sigma T$ is injective.*

Proof. Let $\left(p_\iota^Z \right)_{\iota \in I_Z}$ be a corresponding fundamental system of seminorms and $(Z_\iota, Z_\kappa^\iota)_{\iota \le \kappa \in I_Z}$ be the corresponding projective spectra of local Hilbert spaces for $Z \in \{E, F, G, H\}$. As the ε-product respects reduced projective limits [Köt79, §44. 5. (5)] and the projective limit functor is left exact by [Wen03, (3.1.3)], we have the following commutative diagram:

$$
\begin{array}{ccccccc}
\operatorname*{proj}_{(\iota_E, \iota_F)} E_{\iota_E} \tilde{\otimes}_\sigma F_{\iota_F} & == & E \tilde{\otimes}_\sigma F & \xrightarrow{S \tilde{\otimes}_\sigma T} & G \tilde{\otimes}_\sigma H & == & \operatorname*{proj}_{(\iota_G, \iota_H)} G_{\iota_G} \tilde{\otimes}_\sigma H_{\iota_H} \\
{\scriptstyle \operatorname{proj}(i_{\iota_E, \iota_F})} \Big\downarrow & & & & & & \Big\downarrow {\scriptstyle \operatorname{proj}(i_{\iota_G, \iota_H})} \\
\operatorname*{proj}_{(\iota_E, \iota_F)} E_{\iota_E} \varepsilon F_{\iota_F} & == & E \varepsilon F & \xrightarrow{S \varepsilon T} & G \varepsilon H & == & \operatorname*{proj}_{(\iota_G, \iota_H)} G_{\iota_G} \varepsilon H_{\iota_H} \,,
\end{array}
$$

where i_{ι_E, ι_F} and i_{ι_G, ι_H} are the canonical (injective) inclusions of the Hilbert tensor product into the ε-product for $\iota_Z \in I_Z, Z \in \{E, F, G, H\}$. Then the injectivity of $S \varepsilon T$ implies the injectivity of $S \tilde{\otimes}_\sigma T$. $\qquad\square$

Before we turn our attention to the completed σ tensor product of two PLH spaces, we expand (4.1.6) to the category of hilbertizable lcs:

Corollary 4.2.2.

    i)   a) *The σ tensor product respects topological subspaces of hilbertizable lcs.*

           b) *The completed σ tensor product respects closed topological subspaces in the category of complete hilbertizable lcs.*

    ii) *The σ tensor product respects open mappings for hilbertizable lcs.*

Proof. The statements for the not completed case, (4.2.2) i) a) and ii), are direct consequences of the seminorm-wise statements that are yielded by (4.1.6) i). Considering the injectivity of the linearisation of the inclusions in (4.2.2) i) b), we have to apply (4.2.1), the proof of which together with (4.1.6) ii) yields that the inclusion is not only continuous but also topological.

□

Remark 4.2.3. The question whether the extension to the completion of the tensor product of two quotient maps between complete hilbertizable lcs is again surjective remains unattended in (4.2.2). The reason is rather simple: We can not prove general statements as (4.1.6). Be that as it may, in the following we will determine some classes of spaces that allow a positive answer, as the category of Fréchet-Hilbert spaces and the class of LH spaces. Furthermore, we will start investigating the even larger class of PLH spaces in section (4.3). To be able to apply the σ tensor product to the mentioned spaces we prove the necessary stability properties of the category hLCS of hilbertizable lcs.

Proposition 4.2.4.   i) *The category* hLCS *is stable with respect to the formation of*

   a) *subspaces,*

   b) *quotients,*

   c) *arbitrary products and*

   d) *countable direct sums.*

 ii) *For any set I the space $\bigoplus_{\iota \in I} \mathbb{K}$ is hilbertizable if and only if the set I is countable. Thus* hLCS *is not stable with respect to the formation of arbitrary direct sums.*

   Proof.   i) a) and b) are true, since the restriction of a Hilbert seminorm to a subspace and every quotient seminorm of a Hilbert seminorm are again Hilbert seminorms.

   A fundamental system of seminorms for the product of a family $(E_\iota)_{\iota \in I}$ of hilbertizable lcs E_ι with fundamental systems of Hilbert seminorms $\mathrm{cs}\,(E_\iota)$ $(\iota \in I)$ is given e.g. by the family

$$\{\max\{(p_\iota \circ \pi_\infty^\iota)(\cdot), \iota \in E\} : E \subset I, |E| < \infty, p_\iota \in \mathrm{cs}\,(E_\iota) \text{ for all } \iota \in E.\},$$

   where π_∞^ι denotes as usual the canonical projection on the ι-th component for every $\iota \in I$, see e.g. [MV97, remark on p. 276]. Each of those maximum seminorms is equivalent to the usual Hilbert sum of the family of Hilbert seminorms $((p_\iota \circ \pi_\infty^\iota)(\cdot))_{\iota \in E}$, since the set E is finite. This yields c).

   d) requires a detailed proof:

   Let $(E_n)_{n \in \mathbb{N}}$ be a sequence of hilbertizable lcs E_n with fundamental system of Hilbert seminorms $\mathrm{cs}\,(E_n)$ for each $n \in \mathbb{N}$. Then by [MV97, definition on p. 276]

a fundamental system of seminorms for $\bigoplus_{n\in\mathbb{N}} E_n$ is given by the family

$$\left\{ \bigoplus_{n\in\mathbb{N}} E_n \ni (x_n)_{n\in\mathbb{N}} \longmapsto \sum_{n\in\mathbb{N}} p_n(x_n) : p_n \in \mathrm{cs}(E_n) \text{ for all } n \in \mathbb{N}. \right\}.$$

Furthermore, it is easy to see that for every sequence $\lambda = (\lambda_n)_{n\in\mathbb{N}} \in [0,\infty)^{\mathbb{N}}$ and every sequence of Hilbert seminorms $p = (p_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathrm{cs}(E_n)$ the map

$$\|(x_n)_{n\in\mathbb{N}}\|_{p,\lambda} := \left(\sum_{n\in\mathbb{N}} \lambda_n p_n(x_n)^2 \right)^{1/2}$$

is a continuous Hilbert seminorm on $\bigoplus_{n\in\mathbb{N}} E_n$. Using the usual trick with the geometrical series, we prove that

$$\left\{ \|\cdot\|_{p,\lambda} : p \in \prod_{n\in\mathbb{N}} \mathrm{cs}(E_n), \lambda \in [0,\infty)^{\mathbb{N}} \right\}$$

is a fundamental system of seminorms for $\bigoplus_{n\in\mathbb{N}} E_n$: Let $p(\cdot) = \sum_{n\in\mathbb{N}} (p_n \circ \pi_\infty^n)(\cdot)$, where π_∞^n is the canonical projection from the direct sum to the n-th component, be a continuous seminorm on $\bigoplus_{n\in\mathbb{N}} E_n$. Defining $\lambda_n = (2^n)^2$ for every $n \in \mathbb{N}$, we obtain that $p(\cdot) \le \|\cdot\|_{p,\lambda}$, since for every $(x_n)_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} E_n, \|x\|_{p,\lambda} \le 1$ implies $p_n(x_n)^2 \le \frac{1}{(2^n)^2}$, hence $p(x) = \sum_{n\in\mathbb{N}} p_n(x_n) \le 1$.

ii) Part i) d) yields that $\bigoplus_{\iota\in I} \mathbb{K}$ is hilbertizable if I is countable. We show that if the set I is uncountable, there is not even a fundamental system of seminorms on $\bigoplus_{\iota\in I} \mathbb{K}$ that imply reflexivity: Assuming the opposite, we obtain a seminorm $\|\cdot\|$ on $\bigoplus_{\iota\in I} \mathbb{K}$ such that $\left(\bigoplus_{\iota\in I} \mathbb{K}, \|\cdot\| \right)$ is reflexive and $\sum_{\iota\in I} |x_\iota| \le \|x\|$ for all $x = (x_\iota)_{\iota\in I} \in \bigoplus_{\iota\in I} \mathbb{K}$. Since $I = \bigcup_{N\in\mathbb{N}} \{\kappa \in I : \|e_\kappa\| \le N\}$ is uncountable – where e_κ denotes as usual the κ-th unit vector $(\delta_{\kappa,\iota})_{\iota\in I}$ in $\bigoplus_{\iota\in I} \mathbb{K}$ for every $\kappa \in I$ – we obtain a positive number $R \ge 1$ and an infinite subset J of I such that $\|e_\kappa\| \le R$ for all $\kappa \in J$. Regarding $\bigoplus_{\iota\in J} \mathbb{K}$ as a subspace of $\bigoplus_{\iota\in I} \mathbb{K}$ via the natural embedding, we obtain for every $x = (x_\iota)_{\iota\in J} \in \bigoplus_{\iota\in J} \mathbb{K}$ that

$$\|x\|_{\ell_1(J)} = \sum_{\iota\in J} |x_\iota| \le \|x\| = \left\| \sum_{\iota\in J} x_\iota e_\iota \right\| \le \sum_{\iota\in J} |x_\iota| \|e_\iota\| \le R \|x\|_{\ell_1(J)}$$

holds. Hence $\ell_1(J)$ is reflexive, which is false even for countable infinite sets J, see e.g. [MV97, (7.10)].

$\square$

Dual to (4.2.4) i) we have

Remark 4.2.5. i) If E is a lcs with a fundamental system of bounded sets (fsb) which consists of Hilbert discs and F is a closed subspace of E, then F has an fsb consisting of Hilbert discs, too, since the gauge functional of the intersection of a Hilbert disc B (in E) with a closed subspace F of E is the restriction of the gauge functional of B to the intersection of the linear span of B with F, which is a closed subspace of $([B], p_B)$, since F is closed in E.

ii) If $(E_n)_{n\in\mathbb{N}}$ is a sequence of lcs that have fsb's consisting of Hilbert discs, then $\prod_{n\in\mathbb{N}} E_n$ has an fsb consisting of Hilbert discs: Given a sequence $(\lambda_n)_{n\in\mathbb{N}}$ of strictly positive numbers and any sequence $B = (B_n)_{n\in\mathbb{N}}$ of Hilbert discs B_n in E_n $(n \in \mathbb{N})$, we equip the space

$$E_{\lambda,B} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} E_n \colon \sum_{n\in\mathbb{N}} \lambda_n p_{B_n}(x_n)^2 < \infty \right\}$$

with the norm $\|x\|_{\lambda,B} := \left(\sum_{n\in\mathbb{N}} \lambda_n p_{B_n}(x_n)^2 \right)^{1/2}$, $x \in E_{\lambda,B}$. Then it is easy to check that $E_{\lambda,B}$ is a Hilbert space and its closed unit ball $B_{\lambda,B}$ is a bounded subset of $\prod_{n\in\mathbb{N}} E_n$ as it is contained in $\prod_{n\in\mathbb{N}} \frac{1}{\sqrt{\lambda_n}} B_n$. We prove that the set

$$\left\{ B_{\lambda,B} \colon \lambda \in (0,\infty)^{\mathbb{N}}, B \in \prod_{n\in\mathbb{N}} \mathscr{H}\mathscr{D}(E_n) \right\},$$

where $\mathscr{H}\mathscr{D}(E_n)$ denotes the set of all Hilbert discs in E_n $(n \in \mathbb{N})$, is an fsb of $\prod_{n\in\mathbb{N}} E_n$: If C is a bounded subset of $\prod_{n\in\mathbb{N}} E_n$, then for each $n \in \mathbb{N}$ there is a positive scalar μ_n and a Hilbert disc $B_n \in \mathscr{H}\mathscr{D}(E_n)$, such that $\overline{E_\infty^n(C)}^{E_n}$ is contained in $\mu_n B_n$. Now we define $\lambda_n := \frac{1}{2^n \mu_n}$ and $B := (B_n)_{n\in\mathbb{N}}$, obtaining a Hilbert space $E_{\lambda,B}$ whose closed unit ball $B_{\lambda,B}$ contains C by construction.

iii) For any set I the space $\prod_{\iota\in I} \mathbb{K}$ has an fsb consisting of Hilbert discs if and only if I is countable. The "if" part is just an application of ii). If I is not countable, then it contains an uncountable subset J with cardinality smaller then the smallest inaccessible cardinal and we may assume $I = J$. As $\prod_{\iota\in J} \mathbb{K}$ is ultrabornological by [Köt79, §35, 7., (8)], given any fsb $\mathscr{B}$ of Banach discs, it arises as the inductive limit of all the Banach spaces spanned by the discs $B \in \mathscr{B}$. Thus the strong dual of $\prod_{\iota\in J} \mathbb{K}$ – the direct sum $\bigoplus_{\iota\in J} \mathbb{K}$ – arises as projective limit of all the dual Banach spaces $[B]'$, $B \in \mathscr{B}$. Hence if $\prod_{\iota\in J} \mathbb{K}$ has an fsb consisting of Hilbert discs, then $\bigoplus_{\iota\in J} \mathbb{K}$ is hilbertizable, a contradiction to (4.2.4) ii).

An easy consequence of (4.2.4) i) and (4.2.5) is the following corollary that allows us to apply $\mathcal{T}_\sigma$ to PLH spaces and yields that PLH spaces allow for fsb's of Hilbert discs:

Corollary 4.2.6. *i)* PLH *spaces are hilbertizable.*

ii) PLH spaces have fundamental systems of bounded sets consisting of Hilbert discs.

Proof. i) A PLH space E is the projective limit of a strongly reduced projective spectrum $\left(E_N, E_M^N\right)$ of LH spaces E_N, i.e. $E_N = \text{ind}_{n\in\mathbb{N}} E_{N,n}$ is the inductive limit of an embedding spectrum $(E_{N,n})_{n\in\mathbb{N}}$ of Hilbert spaces, cf. (2.2.8) ii), (2.2.1) ii) and (2.2.7). Denoting for every $N \in \mathbb{N}$ with $\Sigma_N\colon \bigoplus_{n\in\mathbb{N}} X_{N,n} \longrightarrow \text{ind}_{n\in\mathbb{N}} E_{N,n}$ the summation map, we obtain by [MV97, (24.8)] that E_N is topologically isomorphic to the quotient $\bigoplus_{n\in\mathbb{N}} E_{N,n}\big/_{\ker \Sigma_N}$, hence it is hilbertizable by (4.2.4) i) d) and i) b). As E is topologically isomorphic to a (closed) subspace of $\prod_{N\in\mathbb{N}} E_N$, it is hilbertizable by (4.2.4) i) c) and i) a).

ii) Let $E = \text{proj}_{N\in\mathbb{N}}\left(E_N, E_M^N\right)$ be a PLH space. As in every LH space $E_N = \text{ind}_{n\in\mathbb{N}} E_{N,n}$ the sequence $(B_{N,n})_{n\in\mathbb{N}}$ of the closed unit balls of the steps $E_{N,n}$ $(n \in \mathbb{N})$ are an fsb by [MV97, (25.19) 2.], (4.2.5) i) and ii) yield the assertion.

$\square$

We start our investigation with the class of LH spaces by showing that the completed σ tensor product of two LH spaces is the LH space created by the completed σ tensor product of the corresponding embedding spectra, which can be assumed to be an embedding spectrum, since the completed σ tensor product preserves injectivity of linear and continuous operators between Hilbert spaces by (4.1.6) ii). We will divide the proof in three steps. At first we will attend the (uncompleted) σ tensor product of direct sums, which will be a matter of mere calculation and has been done by Harksen in [Har82, (2.10)]. Then we will use that the not completed σ tensor product preserves quotient maps between hilbertizable lcs (4.2.2) ii) to transfer the isomorphism obtained for direct sums to the (not completed) σ tensor product of LH spaces. Again this has been done by Harksen in [Har82, (2.11) a)] for finitely generated tensor norms for normed spaces. To obtain the desired statement for the completed σ tensor product of LH spaces, we will need the following Lemma, which is a consequence of a result due to Palamodov (see [Pal71, (4.2)], also [Wen03, (2.2.2)]).
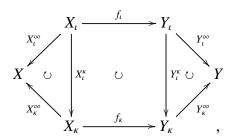
Lemma 4.2.7. *Let $(X_\iota, X_\iota^\kappa)_{\iota\leq\kappa\in I}$ and $(Y_\iota, Y_\iota^\kappa)_{\iota\leq\kappa\in I}$ be two inductive spectra with inductive limits X and Y respectively, and let $(f_\iota)_{\iota\in I}$ be a morphism between the spectra – i.e. for each $\iota \in I$ a linear and continuous map $f_\iota\colon X_\iota \longrightarrow Y_\iota$ such that for each $\iota \leq \kappa \in I$ we have $f_\kappa \circ X_\iota^\kappa = Y_\iota^\kappa \circ f_\iota$ – such that each f_ι is a topological isomorphism onto its dense range. Then the induced linear map $f\colon X \longrightarrow Y$ is a topological isomorphism onto its dense range.*

Proof. Let $\left(X, (X_\iota^\infty)_{\iota\in I}\right)$ and $\left(Y, (Y_\iota^\infty)_{\iota\in I}\right)$ be the inductive limits of the given spectra, i.e. X is a lcs with linear and continuous operators $X_\iota^\infty\colon X_\iota \longrightarrow X$ such that for each lcs Z and each family of maps $(T_\iota)_{\iota\in I} \in \prod_{\iota\in I} L(X_\iota, Z)$ such that $T_\kappa \circ X_\iota^\kappa = T_\iota$ for each $\iota \leq \kappa \in I$ we have a unique linear and continuous map $T\colon X \longrightarrow Z$ with $T \circ X_\iota^\infty = T_\iota$ for all $\iota \in I$. With this universal property we obtain a unique linear and continuous map $f\colon X \longrightarrow Y$ such that $f \circ X_\iota^\infty = Y_\iota^\infty \circ f_\iota$ for all $\iota \in I$. Since $X = \text{span}\{X_\iota^\infty(X_\iota) : \iota \in I\}$ (see e.g. [Köt69, §19, 1.]) f is injective and has dense range.

To see the isomorphy of f, we use a consequence of the Bipolar theorem due to Palamodov

[Pal71, (4.2)] (also see [Wen03, (2.2.2)]), which yields that f is a homomorphism if the map $f^* \colon \mathrm{L}\left(Y, \ell_M^\infty\right) \longrightarrow \mathrm{L}\left(X, \ell_M^\infty\right), f^*(T) := T \circ f$ for all $T \in \mathrm{L}\left(Y, \ell_M^\infty\right)$ is surjective for all sets M.

Let now M be an arbitrary set and T be a linear and continuous map from Y to ℓ_M^∞. Since for any $\iota \in I$ the composition $T \circ X_\iota^\infty$ is linear and continuous, there is a unique $S_\iota \in \mathrm{L}\left(Y_\iota, \ell_M^\infty\right)$ with $S_\iota \circ f_\iota = T_\iota \circ X_\iota^\infty$. Considering the following commutative diagram

we have for $\iota \leq \kappa \in I$:

$$S_\kappa \circ Y_\iota^\kappa \circ f_\iota = S_\kappa \circ f_\kappa \circ X_\iota^\kappa = T \circ X_\kappa^\infty \circ X_\iota^\kappa = T \circ X_\iota^\infty.$$

Thus the universal property of Y yields a unique linear and continuous map $S \colon Y \longrightarrow \ell_M^\infty$ such that $S \circ Y_\iota^\infty = S_\iota$ for all $\iota \in I$. Finally we compute $S \circ f \circ X_\iota^\infty = S \circ Y_\iota^\infty \circ f_\iota = S_\iota \circ f_\iota = T \circ X_\iota^\infty$, obtaining the desired property $S \circ f = T$. □

Now we prove the announced statements for the not completed σ tensor product:

Proposition 4.2.8. *Let E be a hilbertizable lcs that has the countable neighbourhood property (c.n.p.), i.e. for any sequence $(p_n)_{n\in N}$ of continuous seminorms on E there is a sequence λ of positive numbers and a continuous seminorm p on E such that $p_n \leq \lambda_n p$ for all $n \in \mathbb{N}$. Then the following hold:*

i) *The map $\Phi \colon \bigoplus_{n\in\mathbb{N}} (E \otimes_\sigma F_n) \longrightarrow E \otimes_\sigma \bigoplus_{n\in\mathbb{N}} F_n$ that is induced by the natural inclusions $E \otimes_\sigma F_n \hookrightarrow E \otimes_\sigma \bigoplus_{n\in\mathbb{N}} F_n, n \in \mathbb{N}$, is a topological isomorphism for any sequence $(F_n)_{n\in\mathbb{N}}$ of Hilbert spaces (see [Har82, (2.10)] for the same result for the inverse map).*

ii) *The map $\Psi \colon \mathrm{ind}_{n\in\mathbb{N}} (E \otimes_\sigma F_n) \longrightarrow E \otimes_\sigma \mathrm{ind}_{n\in\mathbb{N}} F_n$ that is induced by the natural inclusions $E \otimes_\sigma F_n \hookrightarrow E \otimes_\sigma \mathrm{ind}_{n\in\mathbb{N}} F_n, n \in \mathbb{N}$, is a topological isomorphism for any LH space $F = \mathrm{ind}_{n\in\mathbb{N}} F_n$ (see [Har82, (2.11) a)] for tensor topologies $\mathcal{T}_\alpha$ induced by right-projective tensor norms α).*

Proof.   i) The universal property of the locally convex direct sum yields that the map

$$\Phi \colon \bigoplus_{n\in\mathbb{N}} (E \otimes_\sigma F_n) \longrightarrow E \otimes_\sigma \bigoplus_{n\in\mathbb{N}} F_n, \sum_{n=1}^{n_0} \sum_{j=1}^{k_n} x_j^{(n)} \otimes y_j^{(n)} \longmapsto \left(\sum_{n=1}^{n_0} \sum_{j=1}^{k_n} x_j^{(n)} \otimes \left(y_j^{(n)} \cdot \delta_{n,l} \right) \right)_{l\in\mathbb{N}}$$

is continuous. Furthermore, Φ is an algebraic isomorphism as we have the inverse map

$$\Phi^{-1}\colon E \otimes_\sigma \bigoplus_{n \in \mathbb{N}} F_n \longrightarrow \bigoplus_{n \in \mathbb{N}} (E \otimes_\sigma F_n), \sum_{j=1}^{k} x_j \otimes \left(y_{j,n}\right)_{n \in \mathbb{N}} \mapsto \left(\sum_{j=1}^{k} x_j \otimes y_{j,n}\right)_{n \in \mathbb{N}}.$$

Before we prove the continuity of Φ^{-1}, we observe that Φ maps the image of the natural inclusion of $E \otimes_\sigma F_k$ into $\bigoplus_{n \in \mathbb{N}} (E \otimes_\sigma F_n)$ onto the image of the natural inclusion of $E \otimes_\sigma F_k$ into $E \otimes_\sigma \bigoplus_{n \in \mathbb{N}} F_n$ for any $k \in \mathbb{N}$. Thus we obtain the following estimate for continuous seminorms p and q_n $(n \in \mathbb{N})$ on E respectively F_n $(n \in \mathbb{N})$:

$$(p \otimes_\sigma q_k)(z_k) \le \left(p \otimes_\sigma \left(\sum_{n \in \mathbb{N}} q_n\right)\right)(\Phi(z)) \text{ for all } z \in E \otimes_\sigma \bigoplus_{n \in \mathbb{N}} F_n \text{ and all } k \in \mathbb{N}, \; (\star)$$

where $\sum_{n \in \mathbb{N}} q_n$ is the point wise summation of the seminorms $(q_n)_{n \in \mathbb{N}}$, since (F_k, q_k) is topologically complemented in $\left(\bigoplus_{n \in \mathbb{N}} F_n, \sum_{n \in \mathbb{N}} q_n\right)$ with projection that has norm 1 for all $k \in \mathbb{N}$.

Let now $\sum_{n \in \mathbb{N}} p_n \otimes_\sigma r_n$ be a defining seminorm on $\bigoplus_{n \in \mathbb{N}} E \otimes_\sigma F_n$ and let λ and p be chosen for $(p_n)_{n \in \mathbb{N}}$ according to the c.n.p. of E, i.e. $p_n \le \lambda_n p$ for all $n \in \mathbb{N}$. Defining $q_n := 2^n \lambda_n r_n$ $(n \in \mathbb{N})$, we obtain with $(\star)$ the following estimate for $z \in \bigoplus_{n \in \mathbb{N}} (E \otimes_\sigma F_n)$:

$$\left(\sum_{n \in \mathbb{N}} p_n \otimes_\sigma r_n\right)(z) \le \sum_{n \in \mathbb{N}} ((\lambda_n p) \otimes_\sigma r_n)(z_n) = \sum_{n \in \mathbb{N}} 2^{-n} (p \otimes_\sigma (\lambda_n 2^n r_n))(z_n)$$

$$\le \sum_{n \in \mathbb{N}} 2^{-n} \left(p \otimes_\sigma \left(\sum_{k \in \mathbb{N}} q_k\right)\right)(\Phi(z)) = \left(p \otimes_\sigma \left(\sum_{k \in \mathbb{N}} q_k\right)\right)(\Phi(z)).$$

Since $p \otimes_\sigma \left(\sum_{k \in \mathbb{N}} q_k\right)$ is a defining seminorm on $E \otimes_\sigma \bigoplus_{k \in \mathbb{N}} F_k$, we can complete the proof by applying the above estimate to $\Phi^{-1}(z), z \in E \otimes_\sigma \bigoplus_{n \in \mathbb{N}} F_n$.

ii) The universal property of the locally convex inductive limit implies that Ψ is continuous and it is easy to verify that it is an algebraic isomorphism. To show that Ψ is open, we denote by Σ and Σ_F the continuous and open summation maps from $\bigoplus_{n \in \mathbb{N}} E \otimes_\sigma F_n$ respectively $\bigoplus_{n \in \mathbb{N}} F_n$ onto the inductive limits $\text{ind}_{n \in \mathbb{N}} (E \otimes_\sigma F_n)$ respectively $\text{ind}_{n \in \mathbb{N}} F_n$ and we obtain the commutative diagram

$$
\begin{array}{ccc}
\bigoplus_{n \in \mathbb{N}} (E \otimes_\sigma F_n) & \xrightarrow{\;\Phi\;} & E \otimes_\sigma \bigoplus_{n \in \mathbb{N}} F_n \\
{\scriptstyle \Sigma}\Big\downarrow & \circlearrowleft & \Big\downarrow{\scriptstyle id_E \otimes_\sigma \Sigma_F} \\
\text{ind}_{n \in \mathbb{N}} (E \otimes_\sigma F_n) & \xrightarrow{\;\Psi\;} & E \otimes_\sigma \text{ind}_{n \in \mathbb{N}} F_n.
\end{array}
$$

Let $U = \Sigma(V)$ be a typical 0-neighbourhood of $\text{ind}_{n \in \mathbb{N}} (E \otimes_\sigma F_n)$, where V is a 0-

neighborhood in $\bigoplus_{n\in\mathbb{N}} (E \otimes_\sigma F_n)$. Then the first part, (4.2.8) i), yields that $\Phi(V)$ is open in $E \otimes_\sigma \bigoplus_{n\in\mathbb{N}} F_n$. As Σ_F is open, (4.2.2) ii) implies that $\mathrm{id}_E \otimes_\sigma \Sigma_F$ is open as well. Thus $\Psi(U) = (\mathrm{id}_E \otimes_\sigma \Sigma_F)(\Phi(V))$ is open and we have proven the assertion.

$\square$

Theorem 4.2.9. *Given two* LH *spaces* $E = \mathrm{ind}_{n\in\mathbb{N}} E_n$ *and* $F = \mathrm{ind}_{n\in\mathbb{N}} F_n$, *we have*

$$\mathop{\mathrm{ind}}_{n\in\mathbb{N}} (E_n \tilde{\otimes}_\sigma F_n) = E \tilde{\otimes}_\sigma F$$

topologically.

Proof. As $\mathcal{T}_\sigma$ is symmetric, a twofold application of (4.2.8) ii) yields that for LH spaces $E = \mathrm{ind}_{n\in\mathbb{N}} E_n$ and $F = \mathrm{ind}_{n\in\mathbb{N}} F_n$ the map $\Psi_{E,F}$ from $\mathrm{ind}_{n\in\mathbb{N}} (E_n \otimes_\sigma F_n)$ to $E \otimes_\sigma F$ induced by the inclusions $E_n \otimes_\sigma F_n \hookrightarrow E \otimes_\sigma F$ $(n \in \mathbb{N})$ – consult as usual (4.2.2) i) b) for injectivity – is a topological isomorphism since E and F_k $(k \in \mathbb{N})$ have the c.n.p. as LB spaces by [PCB87, (8.3.5)]. By (4.2.7) $\mathrm{ind}_{n\in\mathbb{N}} (E_n \otimes_\sigma F_n)$ is a dense topological subspace of $\mathrm{ind}_{n\in\mathbb{N}} (E_n \tilde{\otimes}_\sigma F_n)$, which is an LH space, as by (4.1.3) ii) the induced maps $E_n \tilde{\otimes}_\sigma F_n \hookrightarrow E_{n+1} \tilde{\otimes}_\sigma F_{n+1}$ are injective. Thus $\mathrm{ind}_{n\in\mathbb{N}} (E_n \tilde{\otimes}_\sigma F_n)$ is complete, hence $\Psi_{E,F}^{-1}$ can be extended to a topological isomorphism from $E \tilde{\otimes}_\sigma F$ to $\mathrm{ind}_{n\in\mathbb{N}} (E_n \tilde{\otimes}_\sigma F_n)$.

$\square$

Now we will prove a series of corollaries, the first of which is the positive solution of Grothendieck's *problème des topologies* for Fréchet-Hilbert spaces and the tensor topology $\mathcal{T}_\sigma$. To prove this, we need some auxiliary remarks about duality:

Remark 4.2.10. i) If E and F are semi-unitary spaces, then the strong dual of the σ tensor product of E and F is isometrically isomorphic to the completed σ tensor product of the strong duals of E and F extending the duality

$$\langle x \otimes y, x' \otimes y' \rangle = \langle x, x' \rangle \langle y, y' \rangle, x \in E, x' \in E', y \in F, y' \in \mathrm{F}',$$

since we have:

$$(E \otimes_\sigma F)' \cong (E \tilde{\otimes}_\sigma F)' \underset{(\star)}{\cong} E \tilde{\otimes}_\sigma F \cong \tilde{E} \tilde{\otimes}_\sigma \tilde{F} \underset{(\star)}{\cong} \tilde{E}' \tilde{\otimes}_\sigma \tilde{F}' \cong E' \tilde{\otimes}_\sigma F',$$

where the stars stand for the (map induced by the) respective antilinear isometric Riesz identification of a Hilbert space with its strong dual.

 ii) From (4.2.9) we deduce easily that (4.2.10) i) also holds for two Fréchet-Hilbert spaces $E = \mathrm{proj}_{n\in\mathbb{N}} E_n$ and $F = \mathrm{proj}_{n\in\mathbb{N}} F_n$, as we have

$$(E \tilde{\otimes}_\sigma F)' = (\mathrm{proj}_{n\in\mathbb{N}} E_n \tilde{\otimes}_\sigma F_n)' \cong \mathrm{ind}_{n\in\mathbb{N}} (E_n \tilde{\otimes}_\sigma F_n)' \cong \mathrm{ind}_{n\in\mathbb{N}} (E_n' \tilde{\otimes}_\sigma F_n')$$

$$\cong (\mathrm{ind}_{n\in\mathbb{N}} E_n') \tilde{\otimes}_\sigma (\mathrm{ind}_{n\in\mathbb{N}} F_n') \cong E' \tilde{\otimes}_\sigma F'.$$

Now we turn our attention to Grothendieck's *problème des topologies*: For bounded subsets $B \subset E$ and $C \subset F$ of Fréchet spaces E and F the closure of $\Gamma(B \otimes C)$ in $E \tilde{\otimes}_\pi F$ is a bounded subset of $E \tilde{\otimes}_\pi F$. In his thesis [Gro55] Grothendieck asked whether every bounded subset of $E \tilde{\otimes}_\pi F$ is contained in such a set. In general, this problem, which is called Grothendieck's *problème des topologies*, has a negative solution as Taskinen showed in [Tas86], whereas it has a positive solution for Fréchet-Hilbert spaces as Grothendieck stated in his thesis and Kürsten proved in [Kür91]. We prove the following formulation of the problem for Fréchet-Hilbert spaces and the tensor topology $\mathcal{T}_\sigma$:

Corollary 4.2.11. *Given Fréchet-Hilbert spaces $E = \mathrm{proj}_{n \in \mathbb{N}} E_n$, and $F = \mathrm{proj}_{n \in \mathbb{N}} F_n$ and a Hilbert space Z with continuous inclusion $i \colon Z \hookrightarrow E \tilde{\otimes}_\sigma F$, there are Hilbert spaces X and Y with continuous inclusions $X \hookrightarrow E$ and $Y \hookrightarrow F$ such that we have continuous inclusions*

$$Z \hookrightarrow X \tilde{\otimes}_\sigma Y \, (\hookrightarrow E \tilde{\otimes}_\sigma F) \, .$$

Proof. Let E, F, Z and i be given as above. Then the transposed of i is a linear and continuous map from $(E \tilde{\otimes}_\sigma F)'$ to the Hilbert space Z', hence i^t factorizes over a local Hilbert space of $(E \tilde{\otimes}_\sigma F)'$. As $(E \tilde{\otimes}_\sigma F)'$ equals $E' \tilde{\otimes}_\sigma F'$ topologically by (4.2.10) ii), there are two local Hilbert spaces X of E' and Y of F' with the canonical linear and continuous maps with dense range $\omega_X \colon E' \longrightarrow X$ and $\omega_Y \colon F' \longrightarrow Y$ and a linear and continuous map j from $X \tilde{\otimes}_\sigma Y$ to Z' such that $i^t = j \circ (\omega_X \tilde{\otimes}_\sigma \omega_Y)$, i.e. the following diagram commutes:

$$(E \tilde{\otimes}_\sigma F)' = E' \tilde{\otimes}_\sigma F' \xrightarrow{\ \omega_X \tilde{\otimes}_\sigma \omega_Y\ } X \tilde{\otimes}_\sigma Y \xrightarrow{\ j\ } Z'$$
$$\underbrace{\phantom{E' \tilde{\otimes}_\sigma F' \xrightarrow{\ \omega_X \tilde{\otimes}_\sigma \omega_Y\ } X \tilde{\otimes}_\sigma Y}}_{i^t}$$

As $E \tilde{\otimes}_\sigma F$ equals $(E' \tilde{\otimes}_\sigma F')'$ and $(X \tilde{\otimes}_\sigma Y)'$ equals $X' \tilde{\otimes}_\sigma Y'$ topologically (again by (4.2.10) i) and ii)), we obtain the continuous inclusions

$$Z = Z'' \xhookrightarrow{\ j^t\ } X' \tilde{\otimes}_\sigma Y' \xhookrightarrow{\ \omega_X^t \tilde{\otimes}_\sigma \omega_Y^t\ } E \tilde{\otimes}_\sigma F,$$

where j^t is injective, because j has dense range, which is true since i^t has dense range, and $\omega_X^t \tilde{\otimes}_\sigma \omega_Y^t$ is injective with (4.2.1) as ω_X^t and ω_Y^t are injective. Thus the claim is valid. $\quad\square$

(4.2.9) and (4.2.2) yield preliminary positive results for the surjectivity problem:

Remark 4.2.12. Given complete hilbertizable lcs E, Y and Z and a quotient map $q \colon Y \longrightarrow Z$, the induced operator $\mathrm{id}_E \tilde{\otimes}_\sigma q$ is a quotient map if

    i) E, Y, Z are LH spaces as $\mathcal{T}_\sigma$ respects LH spaces by (4.2.9) and quotient maps between Hilbert spaces by (4.1.6) ii), because

$$E \tilde{\otimes}_\sigma Z = \mathrm{ind}_{n \in \mathbb{N}} E_n \tilde{\otimes}_\sigma Z_n = \mathrm{ind}_{n \in \mathbb{N}} E_n \tilde{\otimes}_\sigma q_n (Y_n) = \mathrm{ind}_{n \in \mathbb{N}} E_n \tilde{\otimes}_\sigma \mathrm{ind}_{n \in \mathbb{N}} q_n (Y_n) = E \tilde{\otimes}_\sigma q (Y) \, ,$$

where $E = \text{ind}_{n\in\mathbb{N}} E_n, Y_n = \text{ind}_{n\in\mathbb{N}} Y_n, Z = \text{ind}_{n\in\mathbb{N}} Z_n$ and $q_n \colon Y_n \longrightarrow Z_n$ is the (surjective) restriction of q to Y_n (with values in Z_n, cf. (2.2.3) for detailed explanation).

ii) E, Y, Z are Fréchet-Hilbert spaces as $\text{id}_E \otimes_\sigma q$ is open with (4.2.2), hence $\text{id}_E \tilde{\otimes}_\sigma q$ is almost open, thus open by the Schauder Lemma ([MV97, (3.9)]).

As mentioned in (4.1.7) iii), the σ tensor product can also be approached from a homological point of view:

Remark 4.2.13. Given a hilbertizable locally convex space E, we have the tensor product functor $\text{id}_E \tilde{\otimes}_\sigma \cdot$, that acts on the category of hilbertizable lcs and takes values in the same category. We can easily prove the following:

i) If $E = \text{ind}_{n\in\mathbb{N}} E_n$ is an LH space, then $\text{id}_E \tilde{\otimes}_\sigma \cdot$ is an exact functor from LH to LH as for any short exact sequence $0 \longrightarrow \text{ind}_{n\in\mathbb{N}} X_n \longrightarrow \text{ind}_{n\in\mathbb{N}} Y_n \longrightarrow \text{ind}_{n\in\mathbb{N}} Z_n \longrightarrow 0$ of LH spaces (cf. (2.2.5) for the canonical exact structure on LH) the tensorized sequence arises by (4.2.9) as inductive limit of the tensorized sequences of the steps

$$0 \longrightarrow E_n \tilde{\otimes}_\sigma X_n \longrightarrow E_n \tilde{\otimes}_\sigma Y_n \longrightarrow E_n \tilde{\otimes}_\sigma Z_n \longrightarrow 0 \ (n \in \mathbb{N})$$

that are topologically exact as the completed σ tensor product respects closed subspaces and quotients of Hilbert spaces by (4.1.6) i) a) and ii). Thus the tensorized sequence is again exact in LH by (2.2.5) and the construction of kernels and cokernels in LH (cf. (2.2.3)).

ii) If $E = \text{proj}_{n\in\mathbb{N}} E_n$ is a Fréchet-Hilbert space, then $\text{id}_E \tilde{\otimes}_\sigma \cdot$ is an exact functor from the category of Fréchet-Hilbert spaces FH into itself, as for any short exact sequence $0 \longrightarrow \text{proj}_{n\in\mathbb{N}} X_n \longrightarrow \text{proj}_{n\in\mathbb{N}} Y_n \longrightarrow \text{proj}_{n\in\mathbb{N}} Z_n \longrightarrow 0$ of FH spaces the tensorized sequence arises by definition as projective limit of the local tensorized sequences

$$0 \longrightarrow E_n \tilde{\otimes}_\sigma X_n \longrightarrow E_n \tilde{\otimes}_\sigma Y_n \longrightarrow E_n \tilde{\otimes}_\sigma Z_n \longrightarrow 0 \ (n \in \mathbb{N}) \,,$$

that are topologically exact as the completed σ tensor product respects closed subspaces and quotients of Hilbert spaces by (4.1.6) i) a) and ii). Thus the tensorized sequence is again exact in FH as the completed σ tensor product respects closed subspaces of complete hilbertizable lcs and Hausdorff quotients of Fréchet-Hilbert spaces by (4.2.2) i) a) and (4.2.12) ii).

Since it is well-known that FH is quasi-abelian and has many injective objects – Hilbert spaces are easily seen to be injective objects in FH – cf. [DM07, Vog11], we can construct the right derivative of the tensor product functor $\text{id}_E \tilde{\otimes}_\sigma \cdot$ on FH in the sense of Palamodov (see e.g. [Wen03, section 2.1]) for any Fréchet-Hilbert space E, if we consider it as a functor taking values in the category VS of vector spaces. As usual, we denote the derivative with $\text{Tor}_\sigma^1(E, \cdot)$. Then the exactness of $\text{id}_E \tilde{\otimes}_\sigma \cdot$ yields, that $\text{Tor}_\sigma^1(E, \cdot)$ vanishes, see [Var02, (2.1.14)].

From the general theory about the ε and π tensor product [Köt79, §41 and §44] we know that those tensor topologies are compatible with the PLS space structure in the following sense:

For $\alpha \in \{\varepsilon, \pi\}$ the completed α tensor product of two PLS spaces is the PLS space arising from the α tensor product of the spectra giving rise to the original PLS spaces. Now we will deduce from (4.2.9) that the same is true for PLH spaces and the tensor topology $\mathcal{T}_\sigma$:

Corollary 4.2.14. *Given two* PLH *spaces*

$$E = \underset{N\in\mathbb{N}}{\operatorname{proj}}\left(\underset{n\in\mathbb{N}}{\operatorname{ind}}\,(E_{N,n})\,, E_M^N\right) = \underset{N\in\mathbb{N}}{\operatorname{proj}}\left(E_N, E_M^N\right)$$

and

$$F = \underset{N\in\mathbb{N}}{\operatorname{proj}}\left(\underset{n\in\mathbb{N}}{\operatorname{ind}}\,(F_{N,n})\,, F_M^N\right) = \underset{N\in\mathbb{N}}{\operatorname{proj}}\left(F_N, F_M^N\right)$$

the following isomorphies hold topologically:

$$E\tilde{\otimes}_\sigma F \underset{(1)}{\cong} \underset{N\in\mathbb{N}}{\operatorname{proj}}\left(E_N\tilde{\otimes}_\sigma F_N, E_M^N\tilde{\otimes}_\sigma F_M^N\right) \underset{(2)}{\cong} \underset{N\in\mathbb{N}}{\operatorname{proj}}\left(\underset{n\in\mathbb{N}}{\operatorname{ind}}\,(E_{N,n}\tilde{\otimes}_\sigma F_{N,n})\,, E_M^N\tilde{\otimes}_\sigma F_M^N\right).$$

Furthermore, the spectra on the right-hand side are strongly reduced, i.e. $E\tilde{\otimes}_\sigma F$ is a PLH *space.*

Proof. By construction $\mathcal{T}_\sigma$ respects projective limits of hilbertizable lcs, i.e. the completed σ tensor product of two hilbertizable (possibly uncountable) projective limits is the projective limit created by the completed σ tensor product of the corresponding projective spectra (see e.g. [Har82, (2.8)]). Thus (1) holds. (2) is given by (4.2.9). Since the tensor norm σ respects dense subspaces, so does $\mathcal{T}_\sigma$. Thus the projective spectrum arising from two strongly reduced projective spectra by applying $\mathcal{T}_\sigma$ is strongly reduced. $\qquad\square$

Considering the surjectivity problem of (4.2.3) for PLH spaces we need to impose further assumptions to obtain positive results even for Hilbert spaces E. In the PLS space situation the property *lifting of bounded subsets* of the quotient map under consideration seems to be the key property. In our situation the lifting of local c_0- or local ℓ_2-sequences seems to be more fitting, as quotient maps between Fréchet-Hilbert spaces always have this property because Fréchet spaces always have the strict Mackey condition (see [PCB87, (5.1.30)]), whereas those quotient maps do not need to lift bounded sets, cf. (2.2.6) for a counterexample. We start with a proper definition:

Definition and Remark 4.2.15. Let $q\colon Y \longrightarrow Z$ be a linear and continuous operator between hilbertizable lcs Y and Z.

   i) The operator *q lifts local c_0-* (*respectively local ℓ_2-sequences*) if for all Hilbert spaces G with continuous inclusion $G \hookrightarrow Z$ and every $(z_k)_{k\in\mathbb{N}} \in c_0\,(G)$ (respectively $\ell_2\,(G)$) there is a Hilbert space H with continuous inclusion $H \hookrightarrow Y$ and a sequence $(y_k)_{k\in\mathbb{N}} \in c_0\,(H)$ (respectively $\ell_2\,(H)$) such that $qy_k = z_k$ for all $k \in \mathbb{N}$.

ii) If q lifts local c_0-sequences then q lifts local ℓ_2-sequences: Let G be a Hilbert space with continuous inclusion $G \hookrightarrow Z$ and $(z_k)_{k \in \mathbb{N}} \in \ell_2 (G)$. Then there is a monotonically decreasing zero sequence $(\gamma_k)_{k \in \mathbb{N}}$ of strictly positive numbers such that $\sum_{k \in \mathbb{N}} \frac{\|z_k\|_G^2}{\gamma_k^2} < \infty$. As we may assume without loss of generality that each z_k is not zero, we may define $\widetilde{z_k} := \frac{\gamma_k}{\|z_k\|_G} z_k$ and obtain that $(\widetilde{z_k})_{k \in \mathbb{N}} \in c_0 (G)$. Hence the hypothesis yields a Hilbert space H with continuous inclusion $H \hookrightarrow Y$ and a sequence $(\widetilde{y_k})_{k \in \mathbb{N}} \in c_0 (H)$ such that $q\widetilde{y_k} = \widetilde{z_k}$ for all $k \in \mathbb{N}$. We define $y_k := \frac{\|z_k\|_G}{\gamma_k} \widetilde{y_k}$ for all $k \in \mathbb{N}$ and obtain on the one hand $q y_k = z_k$ for all $k \in \mathbb{N}$ and on the other hand $\sum_{k \in \mathbb{N}} \|y_k\|_H^2 = \sum_{k \in \mathbb{N}} \frac{\|z_k\|_G^2}{\gamma_k^2} \|\widetilde{y_k}\|_H^2$, which is finite by choice of $(\gamma_k)_{k \in \mathbb{N}}$ and since $(\widetilde{y_k})_{k \in \mathbb{N}} \in c_0 (H) \subset \ell_\infty (H)$. Thus $(y_k)_{k \in \mathbb{N}} \in \ell_2 (H)$ and the proof is complete.

Now we can prove the announced result:

Proposition 4.2.16. *Let* $q \colon Y \longrightarrow Z$ *be an open operator between* PLH *spaces* Y *and* Z *that lifts local* ℓ_2*-sequences. Then* $\mathrm{id}_E \widetilde{\otimes}_\sigma q \colon E \widetilde{\otimes}_\sigma Y \longrightarrow E \widetilde{\otimes}_\sigma Z$ *is surjective for every Hilbert space* E.

Proof. Let $(p_\iota)_{\iota \in I}$ be a fundamental system of seminorms of Z and $(Y_\iota, Y_\kappa^\iota)_{\iota \leq \kappa \in I}$ be the corresponding projective spectra of Hilbert spaces, i.e. $Z = \mathrm{proj}_{N \in \mathbb{N}} \, \mathrm{ind}_{n \in \mathbb{N}} \, Z_{N,n} = \mathrm{proj}_{\iota \in I} (Y_\iota, Y_\kappa^\iota)$. Now let z be an element of $E \widetilde{\otimes}_\sigma Z$. This space is isomorphic to $\mathrm{proj}_{\iota \in I} E \widetilde{\otimes}_\sigma Y_\iota = \mathrm{proj}_{N \in \mathbb{N}} E \widetilde{\otimes}_\sigma Z_N$ $= \mathrm{proj}_{N \in \mathbb{N}} \, \mathrm{ind}_{n \in \mathbb{N}} E \widetilde{\otimes}_\sigma Z_{N,n}$ by definition of $\mathcal{T}_\sigma$ respectively by (4.2.14), and is a subspace of $E \varepsilon Z$ by (4.2.1). Using as usual $\underline{z}$ as notation to emphasize z being a linear operator from E' to Z, we have for all $N \in \mathbb{N}$: $z_N := Z_\infty^N z = Z_\infty^N \circ \underline{z} = \underline{z}_N \in \mathrm{ind}_{n \in \mathbb{N}} S_2 (E', Z_{N,n})$ by (4.2.9). Thus for all $N \in \mathbb{N}$ there is an $n(N) \in \mathbb{N}$ such that $\underline{z}_N \in S_2 (E', Z_{N,n(N)})$. Hence there is an orthonormal system (ONS) $(e_k)_{k \in \mathbb{N}}$ in E' such that $\underline{z}$ vanishes on the orthogonal complement of $(e_k)_{k \in \mathbb{N}}$ in E' since countable unions of countable sets are countable. We define $(g_k)_{k \in \mathbb{N}}$ as the family of the images of $(e_k)_{k \in \mathbb{N}}$ under $\underline{z}$. Then the characterization of Hilbert-Schmidt operators via square summability of images of orthonormal bases [MV97, (16.8)] yields that

$$ c_N := \sqrt{\sum_{k \in \mathbb{N}} \left\| Z_\infty^N (g_k) \right\|_{N,n(N)}^2} = \sqrt{\sum_{k \in \mathbb{N}} \left\| \underline{z}_N (e_k) \right\|_{N,n(N)}^2} $$

is finite for each $N \in \mathbb{N}$.

Thus we have an $\ell_2 (Z_{N,n(N)})$-sequence $\left(Z_\infty^N (g_k) \right)_{k \in \mathbb{N}}$ that lifts separately to $\ell_2 (Y_{N,n(N)})$ for each $N \in \mathbb{N}$, since the local quotient maps $q_N \colon Y_N \longrightarrow Z_N$ giving rise to q, cf. (2.2.15), even lift bounded sets as LH spaces are regular. To obtain a simultaneous lifting, we need to use the concept of *vector valued sequence spaces*: Given a lcs X with fundamental system of seminorms $\mathrm{cs} (X)$, we define

$$ \ell_2 (X) := \left\{ (x_k)_{k \in \mathbb{N}} \in X^{\mathbb{N}} \colon (p (x_k))_{k \in \mathbb{N}} \in \ell_2 \text{ for all } p \in \mathrm{cs} (X) \right\} $$

and for any absolutely convex subset C of X:

$$\ell_2(C) := \left\{ (x_k)_{k\in\mathbb{N}} \in \ell_2(X) \cap [C]^\mathbb{N} : (p_C(x_k))_{k\in\mathbb{N}} \in B_{\ell_2} \right\}.$$

In this terminology we have that for each $N \in \mathbb{N}$ the sequence $\left(Z_\infty^N(g_k) \right)_{k\in\mathbb{N}}$ is an element of $\ell_2(c_N B_{N,n(N)})$, where $B_{N,n(N)}$ denotes the closed unit ball of $Z_{N,n(N)}$. As the gauge functional of the preimage of a set equals the composition of the gauge functional of the set with the map, i.e.

$$p_{(Z_\infty^N)^{-1}(c_N B_{N,n(N)})}(z) = p_{c_N B_{N,n(N)}}\left(Z_\infty^N(z) \right)$$

for all $z \in Z$ and $N \in \mathbb{N}$, and since $Y_\infty^\iota \circ \underline{z} \in S_2(E', Y_\iota)$ for all $\iota \in I$, we have

$$(g_k)_{k\in\mathbb{N}} \in \bigcap_{N\in\mathbb{N}} \ell_2 \left(\left(Z_\infty^N \right)^{-1}(c_N B_{N,n(N)}) \right).$$

The crucial step for the simultaneous lifting of the sequence $(g_k)_{k\in\mathbb{N}}$ to Y is the application of [Fre94, (5.3)], which provides

$$\bigcap_{N\in\mathbb{N}} \ell_2 \left(\left(Z_\infty^N \right)^{-1}(c_N B_{N,n(N)}) \right) \subset \|\alpha\|_{\ell_1} \, \ell_2 \left(\bigcap_{N\in\mathbb{N}} \frac{1}{\alpha_N} \left(Z_\infty^N \right)^{-1}(c_N B_{N,n(N)}) \right)$$

for any ℓ_1-sequence $\alpha = (\alpha_N)_{N\in\mathbb{N}}$. Choosing α in B_{ℓ_1}, e.g. the geometric series, the right-hand side equals $\ell_2 \left(\prod_{N\in\mathbb{N}} \frac{c_N}{\alpha_N} B_{N,n(N)} \cap Z \right)$ and since the bounded subset $\prod_{N\in\mathbb{N}} \frac{c_N}{\alpha_N} B_{N,n(N)} \cap Z$ of Z is contained in a Hilbert disc $B \subset Z$ by (4.2.6) ii), we obtain a Hilbert space $G = [B]$ with continuous inclusion $G \hookrightarrow Z$ and that $(g_k)_{k\in\mathbb{N}}$ is an $\ell_2(G)$-sequence. Now the hypothesis yields a Hilbert space H with continuous inclusion $H \hookrightarrow Y$ and a $\ell_2(H)$-sequence $(f_k)_{k\in\mathbb{N}}$ such that $qf_k = g_k$ for all $k \in \mathbb{N}$. Thus we may define $\underline{y}$ to be the unique linear and continuous operator from E' to H that fulfills $\underline{y}(e_k) = f_k$ for all $k \in \mathbb{N}$ and vanishes on the orthogonal complement of the family $\{e_k : k \in \mathbb{N}\}$ in E'. Then by [MV97, (16.8)] $\underline{y}$ is a Hilbert-Schmidt operator from E' to G and there is a corresponding element y in $E\tilde{\otimes}_\sigma Y$ since $Y_\infty^\iota \circ \underline{y} \in S_2(E', Y_\iota)$ by [MV97, (16.7) 3.] that satisfies $q(y) = z$ since $q \circ \underline{y} = \underline{z}$ by construction. $\qquad\square$

4.3 The vanishing of proj^1 for tensorized PLH space spectra

We start our investigation by establishing the connection to the problem of surjectivity of tensorized operators. The analogous results for PLS spaces and the ε product can be found in [Dom10, (4.5)].

Proposition 4.3.1. *Let*

$$0 \longrightarrow X \xrightarrow{\ i\ } Y \xrightarrow{\ q\ } Z \longrightarrow 0 \qquad (\star)$$

be a short exact sequence of PLH *spaces (cf. (2.2.17)) and* $E = \mathrm{proj}_{N\in\mathbb{N}}\left(E_N, E_M^N\right)$ *be a* PLH *space. If* $\mathrm{proj}^1\left(E_N\tilde{\otimes}_\sigma X_N, id_{E_N}\tilde{\otimes}_\sigma X_M^N\right) = 0$, *then the tensorized map* $\mathrm{id}_E\tilde{\otimes}_\sigma q\colon E\tilde{\otimes}_\sigma Y \longrightarrow E\tilde{\otimes}_\sigma Z$ *is surjective. If* $\mathrm{proj}^1\left(E_N\tilde{\otimes}_\sigma Y_N, id_{E_N}\tilde{\otimes}_\sigma Y_M^N\right) = 0$, *the condition is also necessary.*

Proof. By (2.2.15) the sequence $(\star)$ arises as the projective limit of a sequence of short exact sequences $0 \longrightarrow X_N \xrightarrow{i_N} Y_N \xrightarrow{q_N} Z_N \longrightarrow 0$ $(N \in \mathbb{N})$ of LH spaces. As the tensorized sequences

$$0 \longrightarrow E_N\tilde{\otimes}_\sigma X_N \xrightarrow{\mathrm{id}_{E_N}\tilde{\otimes}_\sigma i_N} E_N\tilde{\otimes}_\sigma Y_N \xrightarrow{\mathrm{id}_{E_N}\tilde{\otimes}_\sigma q_N} E_N\tilde{\otimes}_\sigma Z_N \longrightarrow 0 \ (N \in \mathbb{N})$$

are exact sequences of LH spaces by (4.2.13) ii), and give rise to the tensorized sequence

$$0 \longrightarrow E\tilde{\otimes}_\sigma X \xrightarrow{\mathrm{id}_E\tilde{\otimes}_\sigma i} E\tilde{\otimes}_\sigma Y \xrightarrow{\mathrm{id}_E\tilde{\otimes}_\sigma q} E\tilde{\otimes}_\sigma Z,$$

where $\mathrm{id}_E\tilde{\otimes}_\sigma i$ is a kernel of $\mathrm{id}_E\tilde{\otimes}_\sigma q$ in PLH by (4.2.14) and (4.2.2) i) b), the long exact sequence of the projective limit functor (e.g. [Wen03, (3.1.5)]) yields the existence of an algebraically exact sequence

$$0 \longrightarrow E\tilde{\otimes}_\sigma X \xrightarrow{\mathrm{id}_E\tilde{\otimes}_\sigma i} E\tilde{\otimes}_\sigma Y \xrightarrow{\mathrm{id}_E\tilde{\otimes}_\sigma q} E\tilde{\otimes}_\sigma Z \longrightarrow \mathop{\mathrm{proj}^1}_{N\in\mathbb{N}}\left(E_N\tilde{\otimes}_\sigma X_N\right) \longrightarrow$$

$$\longrightarrow \mathop{\mathrm{proj}^1}_{N\in\mathbb{N}}\left(E_N\tilde{\otimes}_\sigma Y_N\right) \longrightarrow \mathop{\mathrm{proj}^1}_{N\in\mathbb{N}}\left(E_N\tilde{\otimes}_\sigma Z_N\right) \longrightarrow 0,$$

which yields the assertion. $\qquad\square$

We prepare the proof of the most simple case, where X is a PLH space with $\mathrm{proj}^1 X = 0$ and E is a Hilbert space. To this end we need the following result, a condition characterizing the vanishing of proj^1 for spectra of regular LB spaces that we can tensorize with a Hilbert space. The proof of the following proposition is contained in [Wen03, (3.2.17) & (3.2.18)]. As we change the assumptions on the sequence of Banach discs $(B_N)_{N\in\mathbb{N}}$, we state the proof.

Proposition 4.3.2. *Let* $X = \mathrm{proj}_{N\in\mathbb{N}}\left(X_N, X_M^N\right)$ *be a projective limit of regular* LB *spaces* $X_N = \mathrm{ind}_{n\in\mathbb{N}} X_{N,n}$ *and let* $(B_N)_{N\in\mathbb{N}} \in \prod_{N\in\mathbb{N}} \mathscr{BD}\left(X_N\right)$. *Then the following are equivalent:*

i) $\mathop{\forall}_{N\in\mathbb{N}} \mathop{\exists}_{M\geq N} \mathop{\forall}_{K\geq M} X_M^N X_M \subset X_K^N X_K + B_N.$

ii) $\left(X_N, X_M^N\right)$ *is reduced and* $\mathop{\forall}_{N\in\mathbb{N}} \mathop{\exists}_{\substack{M\geq N \\ A\in\mathscr{BD}(X_M)}} \mathop{\forall}_{K\geq M} \mathop{\exists}_{\substack{D\in\mathscr{BD}(X_K) \\ c>0}} X_M^N A \subset X_K^N D + cB_N.$

iii) $\left(X_N, X_M^N\right)$ *is reduced and* $\mathop{\forall}_{N\in\mathbb{N}} \mathop{\exists}_{M\geq N} \mathop{\forall}_{\substack{K\geq M \\ m\in\mathbb{N}}} \mathop{\exists}_{k\geq m} X_M^N X_{M,m} \subset X_K^N X_{K,k} + [B_N].$

Furthermore, the existence of a sequence $(B_N)_{N\in\mathbb{N}} \in \prod_{N\in\mathbb{N}} \mathscr{BD}\left(X_N\right)$ *such that* $X_{N+1}^N B_{N+1} \subset B_N$ *holds for all* $N \in \mathbb{N}$ *together with one of the properties i) – iii) is equivalent to the vanishing of* $\mathrm{proj}^1\left(X_N, X_M^N\right)$.

Proof. "*i*) ⇒ *ii*)" i) implies reducedness and the second condition is a consequence of the factorization theorem: Let $K \geq M \geq N$ be chosen according to i) and $R \in \mathscr{BD}(X_M)$. Then X_M^N is continuous as an operator from $[R]$ to X_N and by assumption its image is contained in $\bigcup_{l \in \mathbb{N}} \left(X_K^N X_{K,l} + [B_N] \right)$. As $\left(X_K^N X_{K,l} + [B_N] \right)$ carries the quotient topology for all $K \geq N, l \in \mathbb{N}$, hence is a Banach space since $X_K^N X_{K,l}$ carries the quotient topology as well, we may apply the factorization theorem and obtain an integer k such that $X_M^N([R]) \subset X_K^N X_{K,k} + [B_N]$, where the inclusion is continuous with the closed graph theorem. This yields ii). Also see [Wen03, (3.2.17)].

"*ii*) ⇒ *i*)" Let $M \geq N$ be chosen according to ii) and $K \geq M$ such that $X_K^M X_K \subset \overline{X_L^M X_L}^{X_M}$ for all $L \geq K$. For every $L \geq K$ and for every $m \in \mathbb{N}$ there is a $c_m > 0$ such that $X_L^N(c_m B_{M,m}) \subset X_L^N X_L + B_N$. Hence $X_M^N U \subset X_L^N X_L + B_N$, where $U = \Gamma\left(\bigcup_{m \in \mathbb{N}} c_m B_{M,m} \right)$. Thus we arrive at $X_K^N X_K = X_M^N X_K^M X_K \subset X_M^N \left(X_L^M X_L + U \right) \subset X_L^N X_L + B_N$ which is i). Also see [Wen03, (3.2.18), 3. ⇒ 2.].

"*ii*) ⇔ *iii*)" is then obvious.

The additional statement is precisely the theorem of Retakh and Palamodov, cf. [Wen03, (3.2.9)].   □

Now we can prove the following characterization, which is the extension of [BD98, Lemma 33] from PLN to PLH.

Theorem 4.3.3. *Let* $X = \mathrm{proj}_{N \in \mathbb{N}} \mathscr{X}$, *where* $\mathscr{X} = \left(X_N, X_M^N \right)$, *be a* PLH *space. Then* $\mathrm{proj}^1 \mathscr{X} = 0$ *if and only if* $\mathrm{proj}^1 \left(E \tilde{\otimes}_\sigma X_N, \mathrm{id}_E \tilde{\otimes}_\sigma X_M^N \right) = 0$ *for any Hilbert space E.*

Proof. Let $\mathrm{proj}^1 \mathscr{X} = 0$. We use the characterization of the vanishing of $\mathrm{proj}^1 \mathscr{X}$ with the additional statement of (4.3.2) to obtain a sequence $(B_N)_{N \in \mathbb{N}} \in \prod_{N \in \mathbb{N}} \mathscr{BD}(X_N)$ with $X_{N+1}^N B_{N+1} \subset B_N$ for all $N \in \mathbb{N}$ such that (4.3.2) iii) holds and choose $K \geq M \geq N$ and $k \geq m$ accordingly.

To be able to tensorize this condition with the Hilbert space E, we need to ensure that the spaces $[B_N]$ can be assumed to be Hilbert:

Claim 1: For each $N \in \mathbb{N}$ the set B_N can be chosen as Hilbert disc.

Proof of Claim 1: We recall the necessary parts of the proof of [Wen03, (3.2.9)]:
In each step X_N the family $(nB_{N,n})_{n \in \mathbb{N}}$ of multiples of the unit balls $B_{N,n}$ of $X_{N,n}$ is a fundamental system of bounded sets covering X_N. Hence the Baire argument [Wen03, (3.2.6)] yields the existence of a subsequence $(n(J))_{J \in \mathbb{N}}$ of the natural numbers such that each set B_N can be chosen as

$$\bigcap_{J=1}^{N} \left(X_N^J \right)^{-1} \left(n(J) B_{J,n(J)} \right).$$

Each of those sets is a Hilbert disc, since [Wen03, (3.2.10)] remains true if we replace "Banach disc" by "Hilbert disc", which we prove with the next Claim. ◇

Claim 2: Let $f\colon X \longrightarrow Y$ be a continuous linear map between separated lcs and let $A \subset X$ and $B \subset Y$ be Hilbert discs. Then $f(A) + B$ and $A \cap f^{-1}(B)$ are again Hilbert discs.

Proof of Claim 2: The sequence

$$0 \longrightarrow \left[A \cap f^{-1}(B)\right] \overset{i}{\longrightarrow} [A] \times [B] \overset{q}{\longrightarrow} [f(A) + B] \longrightarrow 0,$$

where $i(x) = (x, -f(x))$ and $q(x, y) = f(x) + y$, is topologically exact and all spaces are separated. Hence the claim is valid since closed subspaces and Hausdorff quotients of Hilbert spaces are again Hilbert spaces. ◇

Thus we arrive at a sequence of Hilbert spaces $(E \tilde{\otimes}_\sigma [B_N])_{N \in \mathbb{N}}$ the unit balls of which are decreasing, i.e.:

$$\left(\mathrm{id}_E \,\tilde{\otimes}_\sigma X_{N+1}^N\right)(B_{E \tilde{\otimes}_\sigma [B_{N+1}]}) \subset B_{E \tilde{\otimes}_\sigma [B_N]} \ (N \in \mathbb{N}),$$

since σ has the metric mapping property by (4.1.3) ii). To compute the tensorized condition properly, we need a last ingredient:

Claim 3: If F, G and H are Hilbert spaces such that G and H embed into the same vector space V, then $\psi(x \otimes (y + z)) := x \otimes y + x \otimes z$ induces an isometric isomorphism Ψ from $F \tilde{\otimes}_\sigma (G + H)$ to $F \tilde{\otimes}_\sigma G + F \tilde{\otimes}_\sigma H$.

Proof of Claim 3: We consider the commutative diagram

$$
\begin{array}{ccc}
F \tilde{\otimes}_\sigma (G \times H) & \overset{\Phi}{\longrightarrow} & F \tilde{\otimes}_\sigma G \times F \tilde{\otimes}_\sigma H \\
\downarrow & & \downarrow \\
F \tilde{\otimes}_\sigma (G + H) & \overset{\Psi}{\longrightarrow} & F \tilde{\otimes}_\sigma G + F \tilde{\otimes}_\sigma H,
\end{array}
$$

where Φ is the natural isomorphism (given by (4.2.8) i)), and the morphisms in the columns are the tensorized quotient map on the left and the natural quotient map on the right. Thus the claim is valid since σ respects metric surjections by (4.1.6) ii). ◇

Now we can complete the proof: As the inclusion

$$X_M^N X_{M,m} \hookrightarrow X_K^N X_{K,k} + [B_N]$$

is continuous and σ respects inclusions, metric surjections and sums of Hilbert spaces by (4.1.6) ii) and claim 3, we obtain the continuous tensorized inclusion, hence condition (4.3.2) iii) for the tensorized spectrum $\left(E \tilde{\otimes}_\sigma X_N, \mathrm{id}_E \,\tilde{\otimes}_\sigma X_M^N\right)$. As this spectrum is even strongly reduced (cf. (4.2.14)) the additional statement of (4.3.2) yields $\mathrm{proj}^1 E \tilde{\otimes}_\sigma X = 0$. The other implication can easily be seen through the characterization of the vanishing of $\mathrm{proj}^1 \mathscr{X}$ by the surjectivity of the map $\Psi_{\mathscr{X}}$, cf. [Wen03, (3.1.4)]. □

To forgo condition $(P_3^*)_0$ for X in the characterization of the vanishing of proj^1 the tensorized spectrum in case of a PLN space E, we need to extend (3.4.4):

Proposition 4.3.4. *Let* $X = \mathrm{proj}_{N\in\mathbb{N}}\,\mathrm{ind}_{n\in\mathbb{N}}\,X_{N,n}$ *and* $E = \mathrm{proj}_{T\in\mathbb{N}}\,\mathrm{ind}_{t\in\mathbb{N}}\,E_{T,t}$ *two PLH spaces such that E is deeply reduced and (E, X) satisfies (T). Then either X satisfies $(P_3^*)_0$ or E is a Fréchet-Hilbert space.*

Proof. First we assume that the following holds:

$$\underset{T\in\mathbb{N}}{\exists}\,\,\underset{\substack{S\geq T\\ t\in\mathbb{N}}}{\forall}\,\,\underset{s\geq t}{\exists}\,\,\underset{\delta>0}{\forall}\,\,\underset{g\in E'_T}{\exists}\,\,0 < \|g\|^*_{T,t} < \delta\left\|\left(E_S^T\right)^t g\right\|^*_{S,s}. \qquad (\star)$$
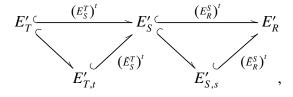
We show that then X satisfies $(P_3^*)_0$: Let $N \in \mathbb{N}$. We choose $T \in \mathbb{N}$ according to $(\star)$ and $M \geq N, S \geq T$ according to (T). Given $K \geq M$, we choose $n, t \in \mathbb{N}$ for K and $S + 1$ according to (T) and $s \geq t$ with $(\star)$. For any $m \geq n$ we choose $k \geq m, r \geq s$ and $c_1 > 0$ according to (T). Given $\varepsilon > 0$ we find by $(\star)$ a $g \in E'_T$ for $\delta := \frac{\varepsilon}{c_1}$ and define $c := c_1\dfrac{\left\|\left(E_{S+1}^T\right)^t g\right\|^*_{S+1,r}}{\left\|\left(E_S^T\right)^t g\right\|^*_{S,s}}$. Then (T) grants for all $f \in X'_N$:

$$
\begin{aligned}
\left\|\left(X_M^N\right)^t f\right\|^*_{M,m} &\leq c_1 \|f\|^*_{N,n}\,\frac{\|g\|^*_{T,t}}{\left\|\left(E_S^T\right)^t g\right\|^*_{S,s}} + c_1 \left\|\left(X_K^N\right)^t f\right\|^*_{K,k}\,\frac{\left\|\left(E_{S+1}^S\right)^t g\right\|^*_{S+1,r}}{\left\|\left(E_S^T\right)^t g\right\|^*_{S,s}}\\
&\leq \varepsilon \|f\|^*_{N,n} + c\left\|\left(X_K^N\right)^t f\right\|^*_{K,k},
\end{aligned}
$$

which is $(P_3^*)_0$ for X. Now if $(\star)$ does not hold, then we have

$$\underset{T\in\mathbb{N}}{\forall}\,\,\underset{\substack{S\geq T\\ t\in\mathbb{N}}}{\exists}\,\,\underset{s\geq t}{\forall}\,\,\underset{\delta>0}{\exists}\,\,\underset{g\in E'_T}{\forall}\,\,\text{Either } \|g\|^*_{T,t} = 0 \text{ or } \left\|\left(E_S^T\right)^t g\right\|^*_{S,s} \leq \tfrac{1}{\delta}\|g\|^*_{T,t}.\,(\star\star)$$

Since E is deeply reduced, we may assume by (3.2.4) that E'_T is a dense subspace of $E'_{T,t}$ for all $T, t \in \mathbb{N}$. Hence $(\star\star)$ yields that the restriction $\left(\tilde{E}_S^T\right)^t$ of $\left(E_S^T\right)^t$ to $E'_{T,t}$ is continuous from $E'_{T,t}$ to E'_S. If we choose $S \geq T, t \in \mathbb{N}$ for any $T \in \mathbb{N}$ according to $(\star\star)$ and $R \geq S, s \in \mathbb{N}$ for S by $(\star\star)$ we have a commutative diagram

hence E' is an LH space. Thus it has a countable fundamental system of bounded sets, i.e E is metrizable. Since E is complete and hilbertizable by (4.2.6) i), it is a Fréchet-Hilbert

space. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Now we are ready to characterize the vanishing of proj^1 for tensorized PLH space spectra ate least if one of them is a PLN space. For two PLH spaces $X = \mathrm{proj}_{N\in\mathbb{N}}\,\mathrm{ind}_{n\in\mathbb{N}}\,X_{N,n}$ and $E = \mathrm{proj}_{T\in\mathbb{N}}\,\mathrm{ind}_{t\in\mathbb{N}}\,E_{T,t}$ we define the projective spectra $\mathscr{P}_\alpha := \left(E_N\tilde{\otimes}_\alpha X_N, E_M^N\tilde{\otimes}_\alpha X_M^N\right)$ and the inductive spectra of Fréchet spaces $\mathscr{H}_\alpha := \left(E'_N\tilde{\otimes}_\alpha X'_N, \left(E_{N+1}^N\right)^t\tilde{\otimes}_\alpha \left(X_{N+1}^N\right)^t\right)$ for $\alpha \in \{\pi,\sigma,\varepsilon\}$. Note that the operators $\left(E_{N+1}^N\right)^t\tilde{\otimes}_\alpha\left(X_{N+1}^N\right)^t$ are indeed injective for $N \in \mathbb{N}$ and $\alpha \in \{\pi,\sigma,\varepsilon\}$: For $\alpha = \varepsilon$ this is [Köt79, §44. 4. (5)], for $\alpha = \sigma$ this is (4.2.1) and for $\alpha = \pi$ this is shown with the same argument as in the proof of (3.4.5). The next result generalizes [Var02, 4.2/4.3] as well as [Dom10, (5.2)] partially.

Theorem 4.3.5. *Let* $E = \mathrm{proj}_{T\in\mathbb{N}}\,\mathrm{ind}_{t\in\mathbb{N}}\,E_{T,t}$ *be a* PLN *space with* $\mathrm{proj}^1 E = 0$ *and* $X = \mathrm{proj}_{N\in\mathbb{N}}\,\mathrm{ind}_{n\in\mathbb{N}}\,X_{N,n}$ *be a* PLH *space, both deeply reduced. Then* $\mathscr{P}_\pi = \mathscr{P}_\sigma = \mathscr{P}_\varepsilon =: \mathscr{P}$ *and* $\mathrm{proj}^1 \mathscr{P} = 0$ *if and only if the pair* (E,X) *satisfies* (T_ε) *if and only if the pair* (E,X) *satisfies* (T).

Proof. Since E_N is nuclear as the strong dual of a nuclear Fréchet space by [Pie72, (4.3.3) Theorem], the three projective spectra $\mathscr{P}_\alpha, \alpha \in \{\pi,\sigma,\varepsilon\}$ coincide by [Pie72, (7.3.3) Theorem]. In complete analogy the three inductive spectra $\mathscr{H}_\alpha, \alpha \in \{\pi,\sigma,\varepsilon\}$ coincide as well. Since $\mathscr{H}_\sigma$ is pre-dual to $\mathscr{P}_\sigma$ by (4.2.10) ii), (3.1.1) yields that $\mathrm{proj}^1 \mathscr{P}$ vanishes if and only if $\mathscr{H}_\pi$ is weakly acyclic. If E is an LN space, the acyclicity of $\mathscr{H}_\pi$ is characterized by (E,X) satisfying (T_ε) respectively (T) by (3.4.5). Hence we may assume that E is not an LN space.
Necessity of (T): If $\mathscr{H}_\pi$ is weakly acyclic, we can prove just as in (3.4.5) via condition (wQ) that (E,X) satisfies (T).
(T) implies (T_ε): Since E is not an LN space, X satisfies $(P_3^*)_0$ by (4.3.4). As $\mathrm{proj}^1 E = 0$ by assumption, E satisfies $(P_3^*)_0$ by [Wen03, (3.2.18)], hence (E,X) satisfies (T_ε) by (3.2.2).
Sufficiency of (T_ε): Let $K \geq M \geq N, k \geq m \geq n, c > 0, \varepsilon > 0$ be given by $\mathcal{T}_\varepsilon$ in the formulation (T_ε^d), cf. (3.2.1) ii) a). Just as in (3.4.5) we use the nuclear interpolation of (3.2.6) on the spaces $F_0 := X'_{N,n}, F_1 := X'_{K,k}$ and $F := \left(X'_{N,k}, \left\|\left(X_M^N\right)^t(\cdot)\right\|^*_{M,m}\right)$, as well as $G_0 := E'_{N,n}, G_1 := E'_{K,k}$ and $G := \left(E'_{N,k}, \left\|\left(E_M^N\right)^t(\cdot)\right\|^*_{M,m}\right)$ and the continuity of the inclusions $E'_{N,k}\tilde{\otimes}_\pi X'_{N,k} \hookrightarrow Z, Z \in \{F\tilde{\otimes}_\pi G, F_i\tilde{\otimes}_\pi G_i, i = 0, 1\}$ to obtain from (T_ε^d) condition $(Q)_0$ for $\mathscr{H}_\pi$ which implies acyclicity by (3.1.2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

As LN spaces are strongly nuclear by [Jar81, 21.8.6] and this property is inherited by arbitrary products and subspaces by [Jar81, (21.1)], PLN spaces are even strongly nuclear. Thus only strongly nuclear Fréchet spaces can be endowed with a PLN structure, since there are nuclear Fréchet-spaces that are not strongly nuclear, e.g. the space s of rapidly decreasing sequences by [Jar81, (21.8.3) a)]. Thus we have to distinguish the case E being a nuclear Fréchet space, cf. [Pis10, p. 155].

Theorem 4.3.6. *Let* $E = \mathrm{proj}\left(E_N, E_M^N\right)$ *be an* FN *space and* $X = \mathrm{proj}_{N\in\mathbb{N}}\,\mathrm{ind}_{n\in\mathbb{N}}\,X_{N,n}$ *a deeply reduced* PLH *space that satisfies* $(P_3^*)_0$. *Then the three projective spectra* $\mathscr{P}_\pi, \mathscr{P}_\sigma$ *and* $\mathscr{P}_\varepsilon$ *are*

equivalent and $\mathrm{proj}^1 \mathscr{P}_\varepsilon$ *vanishes if and only if the pair* (E, X) *satisfies condition* (T_ε) *if and only if the pair* (E, X) *satisfies condition* (T).

Proof. As E is nuclear, we may assume the linking maps E_{N+1}^N to be nuclear for all $N \in \mathbb{N}$. As nuclear mappings are integral by [Pie72, (4.1.5) Proposition], for every local Hilbert space G of X_N the maps $E_{N+1}^N \otimes \mathrm{id}_G$ are continuous from $E_{N+1} \otimes_\varepsilon G$ to $E_N \otimes_\pi G$ by [DF93, (35.5)], hence the extension of $E_{N+1}^N \otimes \mathrm{id}_{X_N}$ to the completion is continuous from $E_{N+1} \tilde{\otimes}_\varepsilon X_N$ to $E_N \tilde{\otimes}_\pi X_N$, $N \in \mathbb{N}$. Thus all three spectra are equivalent. As for each $N \in \mathbb{N}$ the transposed of E_{N+1}^N is nuclear (again by [Pie72, (4.3.3) Theorem]), the same argument as above yields the equivalence of all three inductive spectra $\mathscr{H}_\pi$, $\mathscr{H}_\sigma$ and $\mathscr{H}_\varepsilon$. Now, since $E_N \tilde{\otimes}_\sigma X_N$ is reflexive by (4.2.10) ii) for all $N \in \mathbb{N}$, the LFH spectrum $\mathscr{H}_\sigma$ is pre-dual to the PLH spectrum $\mathscr{P}_\sigma$. Hence (3.1.1) yields that $\mathrm{proj}^1 \mathscr{P}_\sigma$ vanishes if and only if $\mathscr{H}_\sigma$ is weakly acyclic. Since $\mathscr{H}_\pi$ and $\mathscr{H}_\sigma$ are equivalent, $\mathscr{H}_\pi$ is weakly acyclic if and only if $\mathscr{H}_\sigma$ is weakly acyclic by [Vog92, (1.3)]. Thus we have shown that $\mathrm{proj}^1 \mathscr{P}_\varepsilon$ vanishes if and only if $\mathscr{H}_\pi$ is weakly acyclic. In (3.5.4) we have proven that if E' is acyclic, which is the same as E being quasinormable by [MV97, (26.14)] and (3.1.2), then the weak acyclicity of $\mathscr{H}_\pi$ is equivalent to (E'', X) satisfying (T_ε) respectively (T). As E is nuclear, hence Schwartz and reflexive by (2.2.2) iv), this yields the assertion. □

Combining these results with the splitting results (3.4.6) and (3.5.5), we arrive at the following characterizations:

Corollary 4.3.7. *Let E be an* LN *space and* $X = \mathrm{proj}\left(X_N, X_M^N\right)$ *be a deeply reduced* PLH *space with* $\mathrm{proj}^1 X = 0$ *and* $(P_3^*)_0$ *such that* $\mathrm{Ext}_{\mathrm{LH}}^1(E, X_N) = 0$ *for all* $N \in \mathbb{N}$. *Then* $\mathrm{Ext}_{\mathrm{PLH}}^1(E, X) = 0$ *if and only if* $\mathrm{proj}^1 E' \tilde{\otimes}_\varepsilon X = 0$. *The same is true for an* LH *space E and a* PLN *space X.*

Corollary 4.3.8. *Let $E = \mathrm{proj}\left(E_\nu, E_\mu^\nu\right)$ be a nuclear Fréchet space and $X = \mathrm{proj}\left(X_N, X_M^N\right)$ be a deeply reduced* PLH *space with* $\mathrm{proj}^1 X = 0$. *If* $\mathrm{Ext}_{\mathrm{LH}}^1(E_\nu, X_N) = 0$ *for all* $\nu, N \in \mathbb{N}$ *then* $\mathrm{Ext}_{\mathrm{PLH}}^1(E, X) = 0$ *if and only if* $\mathrm{proj}^1 E' \tilde{\otimes}_\varepsilon X = 0$. *The same is true for a Fréchet-Hilbert space E and a* PLN *space X.*

Combining the proofs of (3.4.7) and (3.5.6) we arrive at the following statement about necessity of deep reducedness, cf. [Dom10, (5.4)]:

Proposition 4.3.9. *If a pair (E, X) of* PLH *spaces satifies (T_ε) then either X is strict or E is deeply reduced in rows and either E is deeply reduced in columns or X is a Fréchet-Hilbert space.*

5 Applications

The most prominent application of splitting theory are of course linear partial differential operators $P(D)$ as the classical examples of the Laplace operator $\Delta u = \sum_{j=1}^{n} \partial^2 u / \partial x_j^2$ (which is *elliptic*), the heat operator $\partial u / \partial t - \Delta u$ (which is *parabolic*), and the wave operator $\partial^2 u / \partial t^2 - \Delta u$ (which is *hyperbolic*). The motivation for splitting theory then reads as follows:

Considering real world applications even under the assumption of surjectivity of $P(D)$ a continuative question arises: data in nature can not be conceived as rigidly fixed since the mere process of measuring them involves small errors. Therefore a mathematical problem can not be considered as realistically corresponding to physical phenomena unless a small variation of the given data leads to a small change in the solution.

Hence the existence of continuous solution operators is of importance, which corresponds to the splitting of the corresponding exact sequence

$$0 \longrightarrow \ker(P(D)) \hookrightarrow G \xrightarrow{P(D)} E \longrightarrow 0.$$

Here the domain G of $P(D)$ – the setting in which we consider $P(D)$ – plays an important role, motivating not only the analysis of various spaces of smooth, holomorphic, real analytic or ultradifferentiable functions but also various spaces of distributions and ultradistributions, cf. (5.1.3). Most of these spaces are either nuclear Fréchet or PLN spaces, hence are covered by the current approach of splitting theory and the parameter dependence problem for PLS spaces under the successors of the four standard assumptions. In this final chapter we want to consider applications of the structure theory established for spaces that are not PLS. To achieve this, we divide in the first section the condition (T_ε) for the pair (E, X) into two conditions of (DN)-(Ω) type for PLH spaces on E and X respectively and prove sufficiency in complete analogy to [Dom10]. This includes the analogon of the famous (DN)-(Ω) splitting result for Fréchet-Hilbert spaces, e.g. [MV97, (30.1)] in PLH. Note that these variants of the *dual interpolation estimate* (DIE) are due to Bonet and Domański in [BD07] in the PLS framework. In the remaining sections we consider three examples: The Fréchet-Hilbert spaces $B_{2,\kappa}^{loc}(\Omega)$ of all distributions that locally behave like $B_{2,\kappa}$ but are unrestricted at infinity or the boundary, the space $\mathscr{D}_{L_2}$ of all $\mathscr{C}^\infty$ functions all of whose derivatives are L_2 functions, its dual $\mathscr{D}'_{L_2}$, and the Köthe PLH spaces. A splitting theory for the $B_{2,\kappa}^{loc}(\Omega)$ spaces has been established by Hermanns [Her05] and real analytic parameter dependence has been characterized by Vogt [Vog04]. We extend Vogt's results and show that for elliptic partial differential operators $P(D)$ on $B_{2,\kappa}^{loc}(\Omega)$ admitting a continuous linear right inverse is a lot more than having a parameter dependence for a PLN space E with (DIE) for big θ. Concerning $\mathscr{D}_{L_2}$ and its strong dual, we prove that partial differential operators with constant coefficients on $\mathscr{D}'(\Omega)$ for open and convex $\Omega \subset \mathbb{R}^d$ as well as surjective convolution operators T_μ on $\mathscr{D}'_{(\omega)}(\mathbb{R}^d)$ have the corresponding parameter dependence. We conclude by giving a complete characterization of all the problems under consideration for Köthe PLH spaces in terms of the matrices without any nuclearity assumptions.

5.1 (*DN*)-(Ω) **type conditions for** PLH **spaces**

However complicated the characterizing condition (T_ε) established in the preceding sections may seem, it proves to be evaluable in practice. To this end, Bonet and Domański introduced the PLS variants $(P\Omega), (P\overline{\overline{\Omega}}), (PA)$ and $(P\underline{A})$ of the Fréchet space conditions (DN) (or rather its dual variant (A), cf. [Vog75]) and (Ω) in [BD06, BD08] and proved corresponding splitting results for PLS spaces under the successors of the four standard assumptions. In [BD07] they introduced a new condition, the *dual interpolation estimate*, henceforth (DIE), in three variants – for small, big and all θ –, unifying the other four conditions mentioned above. In the following, we will transfer the PLS space results from [BD06, BD07, BD08, Pis10] regarding (DIE) to the category of PLH spaces, thus proving the analogon of the famous (DN)-(Ω) splitting result for Fréchet-Hilbert spaces, e.g. [MV97, (30.1)] also for PLH spaces. We start by defining the conditions, cf. [BD07, p. 433]:

Definition 5.1.1. A PLH space $X = \operatorname{proj}_{N\in\mathbb{N}} \operatorname{ind}_{n\in\mathbb{N}} X_{N,n}$ satisfies the *dual interpolation estimate* (DIE) for small θ if the following holds:

$$
\underset{N\in\mathbb{N}}{\forall}\ \underset{M\geq N}{\exists}\ \underset{K\geq M}{\forall}\ \underset{n\in\mathbb{N}}{\exists}\ \underset{m\geq n}{\forall}\ \underset{\substack{0<\theta_0<1 \\ c>0}}{\exists}\ \underset{0<\theta\leq\theta_0}{\forall}\ \underset{k\geq m}{\exists}\ \underset{f\in X'_N}{\forall}
$$

$$
\left\| \left(X_M^N\right)^t f \right\|_{M,m}^* \leq c \left(\left\| \left(X_K^N\right)^t f \right\|_{K,k}^* \right)^{1-\theta} \left(\|f\|_{N,n}^* \right)^{\theta}.
$$

Replacing the quantifier $\underset{0<\theta\leq\theta_0}{\forall}$ by $\underset{\theta_0\leq\theta<1}{\forall}$, we arrive at ($DIE$) for big θ and if we have the estimate even for all $0 < \theta < 1$, then X has (DIE).

We collect the connections to the Fréchet space conditions (DN) and (Ω) as well as to the $(PA), (P\Omega)$ variants in the following remark. As it is meant for experts, we will not explicitly state the definition of the other conditions. The proof of the former connection is mere writing down the conditions, the proofs of the latter ones uses the characterization (5.1.5), cf. [BD06, (5.1)], [BD07, (1.1)] and [Pis10, 1. & 2.].

Remark 5.1.2.   i) A Fréchet-Hilbert space E has (DIE) for big θ iff it has (Ω) and it has (DIE) (or equivalently (DIE) for small θ) iff it has $(\overline{\overline{\Omega}})$.

  ii) An LH space E has (DIE) (or equivalently (DIE) for big θ) iff its strong dual E' has (DN) and it has (DIE) for small θ iff its strong dual E' has $(\underline{DN})$.

 iii) A PLH space X has (DIE) iff it has (PA) and $(P\overline{\overline{\Omega}})$.

 iv) A PLH space X has (DIE) for small θ iff it has $(P\underline{A})$ and $(P\overline{\overline{\Omega}})$.

 v) A PLH space X has (DIE) for big θ iff it has (PA) and $(P\Omega)$.

Before we prove the announced characterization, we compile examples of well-known spaces having or not having variants of (*DIE*), cf. [BD07, (1.1), (2.1), (2.2)] and [Pis10, 1.]:

Remark 5.1.3.   i) The space $\mathscr{D}'(\Omega)$ of Schwartz distributions has (*DIE*).

ii) The space $\mathscr{S}'$ of tempered distributions has (*DIE*).

iii) The space $\mathcal{E}'(\Omega)$ of distributions with compact support has no form of (*DIE*).

iv) The space $\mathscr{D}'_{(\omega)}(\Omega)$ of ultradistributions of Beurling type has (*DIE*).

v) If Ω is an open and convex subset of $\mathbb{R}^d$, then kernels of linear PDOs with constant coefficients on $\mathscr{D}'(\Omega)$ have (*DIE*) for big θ.

vi) The space $\mathcal{E}_{\{\omega\}}(\Omega)$ of quasianalytic functions of Roumieu type has (*DIE*) for small θ if Ω is open and convex and if ω satisfies:

$$\underset{c_1}{\exists} \; \underset{W \geq 1}{\forall} \; \underset{c_2}{\exists} \underset{z}{\forall} \omega\,(W\,|z| + W) \leq W c_1 \omega\,(z) + c_2.$$

vii) In the non-quasianalytic case $\mathcal{E}_{\{\omega\}}(\Omega)$ has (*DIE*) for small θ.

viii) The space $\mathscr{A}(\Omega)$ of real analytic functions has (*DIE*) for small θ.

The mentioned characterization (5.1.5) of (*DIE*) is based on a simple inequality for real numbers, the proof of which in turn involves the calculation of the minimum of a real-valued function, cf. [MV97, (29.13)]. Even though this might seem rather trivial, we want to execute the proof as it constitutes the fundament of the rest of the section:

Lemma 5.1.4. *For every $\gamma > 0$ there is $c > 0$ such that with $\theta := \frac{\gamma}{\gamma+1}$ the following holds for all a, b and $r > 0$:*

$$ca^{1-\theta}b^{\theta} \leq r^{\gamma}a + \frac{1}{r}b.$$

Proof. Let a, b and $\gamma > 0$ and $\varphi \colon (0, \infty) \longrightarrow \mathbb{R}, r \mapsto r^{\gamma}a + \frac{1}{r}b$. We calculate the first two derivatives of φ to $\varphi'(r) = \gamma r^{\gamma-1}a - \frac{1}{r^2}b$ and $\varphi''(r) = \gamma(\gamma-1)r^{\gamma-2}a + \frac{2}{r^3}b$ for all $r > 0$. Furthermore, $\varphi'(r)$ vanishes exactly in $r_0 = \left(\frac{b}{\gamma a}\right)^{\frac{1}{\gamma+1}}$ and

$$\varphi''(r_0) = \gamma(\gamma-1)\left(\frac{b}{\gamma a}\right)^{\frac{\gamma+1-3}{\gamma+1}} a + 2\left(\frac{b}{\gamma a}\right)^{\frac{-3}{\gamma+1}} b > 0,$$

hence φ reaches its minimum in r_0 with value

$$
\begin{aligned}
\varphi\left(r_0\right) &= \gamma^{\frac{-\gamma}{\gamma+1}} b^{\frac{\gamma}{\gamma+1}} a^{1-\frac{\gamma}{\gamma+1}} + \gamma^{\frac{1}{\gamma+1}} b^{1-\frac{1}{\gamma+1}} a^{\frac{1}{\gamma+1}} \\
&= \left(\gamma^{\frac{-\gamma}{\gamma+1}} + \gamma^{\frac{1}{\gamma+1}}\right) a^{1-\frac{\gamma}{\gamma+1}} b^{\frac{\gamma}{\gamma+1}}.
\end{aligned}
$$

$\square$

Now characterizing (DIE) remains an easy task, cf. [Pis10, 3.]:

Proposition 5.1.5. *For a* PLH *space* $X = \mathrm{proj}_{N\in\mathbb{N}} \, \mathrm{ind}_{n\in\mathbb{N}} \, X_{N,n}$ *the following are equivalent:*

i) X *has* (DIE) *for small* θ,

ii)

$$
\underset{N\in\mathbb{N}}{\forall} \; \underset{M\geq N}{\exists} \; \underset{K\geq M}{\forall} \; \underset{n\in\mathbb{N}}{\exists} \; \underset{m\geq n}{\forall} \; \underset{\gamma_0>0}{\exists} \; \underset{\substack{0<\gamma\leq\gamma_0 \\ c>0}}{\forall} \; \underset{k\geq m}{\exists} \; \underset{r>0}{\forall} \; \underset{f\in X_N'}{\forall}
$$

$$
\left\|\left(X_M^N\right)^t f\right\|_{M,m}^* \leq c\left(r^\gamma \left\|\left(X_K^N\right)^t f\right\|_{K,k}^* + \tfrac{1}{r}\|f\|_{N,n}^*\right),
$$

iii)

$$
\underset{N\in\mathbb{N}}{\forall} \; \underset{M\geq N}{\exists} \; \underset{K\geq M}{\forall} \; \underset{n\in\mathbb{N}}{\exists} \; \underset{m\geq n}{\forall} \; \underset{\gamma_0>0}{\exists} \; \underset{\substack{0<\gamma\leq\gamma_0 \\ c>0}}{\forall} \; \underset{k\geq m}{\exists} \; \underset{r>0}{\forall}
$$

$$
X_M^N B_{M,m} \subset c\left(r^\gamma X_K^N B_{K,k} + \tfrac{1}{r} B_{N,n}\right).
$$

Replacing the quantifier $\underset{0<\gamma\leq\gamma_0}{\forall}$ *by* $\underset{\gamma\geq\gamma_0}{\forall}$ *, we obtain* (DIE) *for big* θ *and if the statements hold for all* $\gamma > 0$ *we have* (DIE).

Proof. Applying Lemma (5.1.4) to $a := \left\|\left(X_K^N\right)^t f\right\|_{K,k}^*$ and $b := \|f\|_{N,n}^*$, using that the function $\gamma \mapsto \frac{\gamma}{\gamma+1}$ is monotonously increasing on the interval $(0, \infty)$ and an application of the Bipolar theorem yield the assertions. $\square$

These characterizations are very useful to compare (DIE) with our conditions of type (P_3):

Corollary 5.1.6. *For* PLH *spaces* $X = \mathrm{proj}_{N\in\mathbb{N}} \, \mathrm{ind}_{n\in\mathbb{N}} \, X_{N,n}$ *any form of* (DIE) *implies* $(P_3^*)_0$.

Also concluding our splitting condition (T_ε) from (DIE) in any form is an easy task by applying (5.1.4) to $a := \left\|\left(X_K^N\right)^t f\right\|_{K,k}^* \left\|\left(E_R^T\right)^t g\right\|_{R,r}^*$ and $b := \|g\|_{T,t}^*$, cf. [Dom10, (5.6)]:

Corollary 5.1.7. *Let* $E = \mathrm{proj}_{T\in\mathbb{N}} \, \mathrm{ind}_{t\in\mathbb{N}} \, E_{T,t}$ *and* $X = \mathrm{proj}_{N\in\mathbb{N}} \, \mathrm{ind}_{n\in\mathbb{N}} \, X_{N,n}$ *be two* PLH *spaces that satisfy the same form of* (DIE). *Then the pair* (E, X) *satisfies* (T_ε).

Deep reducedness is also a consequence of any form of (*DIE*), cf. [Pis10, 8.]:

Corollary 5.1.8. PLH *spaces* $X = \text{proj}_{N \in \mathbb{N}} \text{ind}_{n \in \mathbb{N}} X_{N,n}$ *satisfying any form of* (*DIE*) *are deeply reduced.*

Proof. Let $K \geq M \geq N, k \geq m \geq n$ and $\gamma_0 > 0, c > 0$ be chosen according to (5.1.5) iii). Then we have

$$X_M^N B_{M,m} \subset cr^{\gamma_0} X_K^N B_{K,k} + \frac{c}{r} B_{N,n} \text{ for all } r > 0,$$

which yields on the one hand

$$X_M^N B_{M,m} \subset \bigcap_{r>0} cr^{\gamma_0} B_{N,k} + X_{N,n} \subset \overline{X_{N,n}}^{X_{N,k}},$$

i.e., X is deeply reduced in rows, and on the other hand

$$X_M^N B_{M,m} \subset \bigcap_{r>0} X_K^N X_{K,k} + \frac{c}{r} B_{N,k} \subset \overline{X_K^N X_{K,k}}^{X_{N,k}},$$

i.e., X is deeply reduced in columns. □

The previous corollaries together with the splitting results (3.4.6) and (3.5.5) yield the generalization of the famous (*DN*)-(Ω) splitting result to the category of PLH spaces, cf. [BD06, (5.5)] and [BD08, (4.1)] for the PLS analogons:

Theorem 5.1.9. *Let X be a* PLH *space and E either a Fréchet-Hilbert space or an* LH *space such that E' and X satisfy the same form of* (*DIE*). *Then* $\text{Ext}_{\text{PLH}}^1 (E, X) = 0$ *if all the local* Ext *groups* $\text{Ext}_{\text{LH}}^1 (E_T, X_N)$ *vanish if E is* FH, *respectively* $\text{Ext}_{\text{LH}}^1 (E, X_N) = 0$ *if E is* LH, *for all $N, T \in \mathbb{N}$.*

In the theory for partial differential equations results about the regularity of solutions are important. One way to classify regularity is the behavior of the Fourier transform $\hat{u}$ of a distribution or function at infinity. When we ask for which weight function k it is true that $k\hat{u}$ is p-integrable, we arrive at the spaces $B_{p,k}$, cf. [Hör05, (10.1)], which will be the basic modules for our hilbertizable but non-nuclear function respectively distribution spaces $\mathscr{D}_{L_2}$ and $B_{2,k}^{\text{loc}} (\Omega)$:

A positive function k on $\mathbb{R}^d$ is called a *temperate weight function* if there are $C > 0$ and $N \in \mathbb{N}$ such that

$$k (\xi + \eta) \leq \left(1 + C|\xi|^N\right) k (\eta) \text{ for all } \xi, \eta \in \mathbb{R}^d$$

and by $\mathscr{K}$ we denote the set of all those functions. For $1 \leq p \leq \infty$ we define $B_{p,k}$ to be the (Banach) space of temperate distributions $u \in \mathscr{S}'$ such that $\hat{u} \in L_1^{\text{loc}} \left(\mathbb{R}^d\right)$ and $k\hat{u} \in L_p \left(\mathbb{R}^d\right)$,

equipped with the norm

$$
\|u\|_{p,k} := \begin{cases} (2\pi)^{-\frac{d}{p}} \|k\hat{u}\|_{L_p}, & 1 \le p < \infty, \\ \|k\hat{u}\|_{L_\infty}, & p = \infty. \end{cases}
$$

5.2 Parameter dependence of $P(D)$ in $\mathrm{B}^{\mathrm{loc}}_{2,k}(\Omega)$ spaces

For $1 \le p \le \infty$ and an open subset $\Omega \subset \mathbb{R}^d$ we denote by $\mathrm{B}^{\mathrm{loc}}_{p,k}(\Omega)$ the space of all distributions $u \in \mathscr{D}'(\Omega)$ that locally behave like $\mathrm{B}_{p,k}$ but are unrestricted at infinity or the boundary, i.e. $u \cdot \varphi \in \mathrm{B}_{p,k}$ for all $\varphi \in \mathscr{D}(\Omega)$, obtaining by [Hör05, (10.1.26)] a Fréchet space with continuous inclusions

$$
\mathscr{C}^\infty(\Omega) \hookrightarrow \mathrm{B}^{\mathrm{loc}}_{p,k}(\Omega) \hookrightarrow \mathscr{D}'(\Omega).
$$

Given a polynomial P in d variables with degree $m \in \mathbb{N}$, we define

$$
\tilde{P}(\xi) := \left(\sum_{|\alpha| \le m} |\partial^\alpha P(\xi)|^2 \right)^{\frac{1}{2}}, \xi \in \mathbb{R}^d
$$

and obtain by [Hör05, (10.6.7)] that for $1 \le p < \infty$ and a P-convex open subset $\Omega \subset \mathbb{R}^d$ we have a short exact sequence of Fréchet spaces

$$
0 \longrightarrow \mathscr{N}_{p,\tilde{P}k}(\Omega) \longrightarrow \mathrm{B}^{\mathrm{loc}}_{p,\tilde{P}k}(\Omega) \xrightarrow{P(D)} \mathrm{B}^{\mathrm{loc}}_{p,k}(\Omega) \longrightarrow 0 \qquad (\star)
$$

where $\mathscr{N}_{p,\tilde{P}k}(\Omega)$ is the kernel of $P(D)$ on $\mathrm{B}^{\mathrm{loc}}_{p,\tilde{P}k}(\Omega)$. In his thesis [Her05, Theorem A & B] Hermanns extended the sequence space representation $\mathrm{B}^{\mathrm{loc}}_{1,k}(\Omega) \cong \prod_{k\in\mathbb{N}} \ell^1$ due to Vogt [Vog83b] to $1 \le p < \infty$ and proved for $p = 1,2$ that each exact sequence $(\star)$ splits iff $\mathscr{N}_{p,\tilde{P}k}(\Omega)$ is a quojection.

In [Vog04, (3.3)] Vogt proved the following result about real analytic parameter dependence: Given an open subset $T \subset \mathbb{R}^d$ or a coherent subvariety $T \subset \mathbb{R}^d$ with $\dim(T) \ge 1$, the induced operator

$$
P(D) \tilde{\otimes}_\varepsilon \mathrm{id}_{\mathscr{A}(T)} : \ \mathrm{B}^{\mathrm{loc}}_{p,\tilde{P}k}(\Omega) \tilde{\otimes}_\varepsilon \mathscr{A}(T) \longrightarrow \mathrm{B}^{\mathrm{loc}}_{p,k}(\Omega) \tilde{\otimes}_\varepsilon \mathscr{A}(T)
$$

is surjective if and only if $\mathscr{N}_{p,\tilde{P}k}(\Omega)$ satisfies $\overline{\overline{(\Omega)}}$, which is almost never fulfilled if P is elliptic and Ω is convex except in the case $d = 1$, cf. [Vog06, Theorem 3].

We are able to contribute the following

Corollary 5.2.1. *Let E be a* PLN *space. Then in the above situation*

$$
P(D) \tilde{\otimes}_\varepsilon \mathrm{id}_E : B^{loc}_{2,\tilde{P}k}(\Omega) \tilde{\otimes}_\varepsilon E \longrightarrow B^{loc}_{2,k}(\Omega) \tilde{\otimes}_\varepsilon E
$$

is surjective whenever either

i) *E has (DIE) for big θ and* $\mathscr{N}_{2,\tilde{P}k}(\Omega)$ *satisfies* (Ω)*, or*

ii) *E has (DIE) for small θ and* $\mathscr{N}_{2,\tilde{P}k}(\Omega)$ *satisfies* $(\overline{\overline{\Omega}})$.

Proof. The pair $\left(\mathscr{N}_{2,\tilde{P}k}(\Omega), E\right)$ satisfies (T_ε) in both cases by (5.1.7) and both spaces are deeply reduced by (5.1.8). Hence (4.3.5) yields that $\text{proj}^1 \mathscr{N}_{2,\tilde{P}k}(\Omega)\tilde{\otimes}_\varepsilon E = 0$, which characterizes the surjectivity of $P(D)\tilde{\otimes}_\varepsilon \text{id}_E$ by (4.3.1) since with the same argument as above $\text{proj}^1 B^{\text{loc}}_{2,\tilde{P}k}(\Omega)\tilde{\otimes}_\varepsilon E$ vanishes. $\qquad\qquad\square$

In particular, the case (5.2.1) i) contains the Schwartz distributions $\mathscr{D}'(\Omega)$, the ultradistributions of Beurling type $\mathscr{D}'_{(\omega)}(\Omega)$, the tempered distributions $\mathscr{S}'$, the germs of holomorphic functions over a one-point set $\mathscr{H}(\{0\})$ (cf. [BD06, p. 331] and $\mathscr{D}'(K)$ for a compact subset $K \subset \mathbb{R}^n$ with non-void interior, cf. [MV97, (31.10)]. The second case (5.2.1) ii) contains the space of ultradifferentiable functions of Roumieu type $\mathcal{E}_{\{\omega\}}(\Omega)$ for a quasianalytic weight with condition (5.1.3) vi) or any non-quasianalytic weight.

The case P being an elliptic polynomial shows that having parameter dependence for a space E with (DIE) for big θ, i.e. the situation in which (5.2.1) i) holds is much less than admitting a continuous linear right inverse:

Example 5.2.2. Let $p = 2, d > 1$ and P be an elliptic polynomial. First of all by [Hör05, (10.8.2)] every open subset $\Omega \subset \mathbb{R}^n$ is P-convex, hence we have an exact sequence $(\star)$. By [Hör03, (4.4.2)] the topological kernels of $P(D)$ in $\mathscr{C}^\infty(\Omega)$ and $\mathscr{D}'(\Omega)$ coincide. Hence $\mathscr{N}_{2,\tilde{P}k}(\Omega)$ does never satisfy $(\overline{\overline{\Omega}})$ but always satisfies (Ω) by [Vog83a, (2.5)]. Thus the sequence $(\star)$ never splits as being a quojection trivially implies satisfying $(\overline{\overline{\Omega}})$ whereas

$$P(D)\tilde{\otimes}_\varepsilon \text{id}_E \colon B^{\text{loc}}_{2,\tilde{P}k}(\Omega)\tilde{\otimes}_\varepsilon E \longrightarrow B^{\text{loc}}_{2,k}(\Omega)\tilde{\otimes}_\varepsilon E$$

is surjective for all PLN spaces E with (DIE) for big θ. That the kernel of $P(D)$ satisfies (Ω) was also shown by Frerick and Kalmes, who proved in [FK10, Corollary 3] that in the above situation the augmented operator $P^+(D)$ is surjective on $\mathscr{D}'(\Omega \times \mathbb{R})$. This is equivalent to $(P\Omega)$ for the kernel of $P(D)$ by [BD06, (8.3)], which yields the assertion considering that $(P\Omega)$ coincides with (Ω) for Fréchet spaces. Note that this is not true for hypoelliptic operators and $n \geq 3$ as Kalmes proved in [Kal12].

5.3 $\mathscr{D}_{L_2}$ **parameter dependence**

Choosing the weight function $k_s(\xi) = \left(1 + |\xi|^2\right)^{s/2}, \xi \in \mathbb{R}^d, s \in \mathbb{R}$ and $p = 2$, we arrive at the well-known Sobolev spaces $H_{(s)} = B_{2,k_s}$, cf. [Hör05, (10.1.2) & (7.9)]. The Sobolev Embedding Theorem [MV97, (14.16)] yields that the space $\mathscr{D}_{L_2} := \bigcap_{s>0} H_{(s)}$ is the space of all $\mathscr{C}^\infty$-functions f on $\mathbb{R}^d$ with $D^\alpha f \in L_2$ for all $\alpha \in \mathbb{N}_0^d$, endowed with the seminorms induced by the embeddings into $H_{(k)}, k \in \mathbb{N}$. Thus $\mathscr{D}_{L_2}$ is a Fréchet-Hilbert space containing the space

$\mathscr{C}_c^\infty$ of $\mathscr{C}^\infty$-functions with compact support as a dense subspace. It satisfies (DN) as well as (Ω) but is not nuclear since it is not even Montel by [MV97, (31.15)].

Results as in (5.2) are not of great use since if the kernel of a surjective operator T in $\mathscr{D}_{L_2}$ satisfies (Ω), then T admits a linear and continuous right inverse by [MV97, 30.1], which in turn implies the surjectivity of the tensorized operator $T\tilde{\otimes}_\varepsilon \mathrm{id}_E$ for every lcs E on its own.

However, it seems reasonable to consider $\mathscr{D}_{L_2}$ and its dual $\mathscr{D}'_{L_2} = \mathrm{ind}_{k\in\mathbb{N}} H_{(-k)}$ as a form of parameter dependence in analogy to $\mathscr{C}^\infty(\Omega)$, $\mathscr{H}(\Omega)$ or $\mathscr{H}(K)$ parameter dependence, which leads to the following corollary:

Corollary 5.3.1.   *i) Let $P(D) : \mathscr{D}'(\Omega) \longrightarrow \mathscr{D}'(\Omega)$ be a linear partial differential operator with constant coefficients on the space of Schwartz distributions over an open and convex subset $\Omega \subset \mathbb{R}^d$. Then the tensorized operators $P(D)\tilde{\otimes}_\varepsilon \mathrm{id}_{\mathscr{D}_{L_2}}$ and $P(D)\tilde{\otimes}_\varepsilon \mathrm{id}_{\mathscr{D}'_{L_2}}$ are surjective.*

ii) Let $T_\mu \colon \mathscr{D}'_{(\omega)}(\mathbb{R}^d) \longrightarrow \mathscr{D}'_{(\omega)}(\mathbb{R}^d)$ be a surjective convolution operator on the space of ultradistributions of Beurling type for an ultradistribution μ of compact support. Then the tensorized operators $T_\mu\tilde{\otimes}_\varepsilon \mathrm{id}_{\mathscr{D}_{L_2}}$ and $T_\mu\tilde{\otimes}_\varepsilon \mathrm{id}_{\mathscr{D}'_{L_2}}$ are surjective.

Proof. As $\mathscr{D}_{L_2}$ and its strong dual satisfy (DIE) for big θ by (5.1.2), the completed ε tensor product with PLN spaces that satisfy (DIE) for big θ as well yields $\mathrm{proj}^1 = 0$ by (5.1.7) and (4.3.5). Thus (4.3.1) yields the claim if the kernel of both $P(D)$ and T_μ satisfy (DIE) for big θ. For $P(D)$ this is (5.1.3) v) and considering T_μ we consult [BD06, (8.5)] to obtain $(P\Omega)$ and [BD08, (5.7)] to obtain (PA) which yield (DIE) for big θ by (5.1.2) v). □

5.4 Köthe PLH **spaces**

Following [DV00a, p.189], we define Köthe PLB spaces, also see [Dom04, (1.7)]:

Definition and Remark 5.4.1.   i) A scalar matrix $A = \left(a_{j;N,n}\right)_{j,N,n\in\mathbb{N}}$ is called a *Köthe* PLB-*matrix* if the following holds:

    a) for all $j \in \mathbb{N}$ there is an $N \in \mathbb{N}$ such that $a_{j;N,n} > 0$ for all $n \in \mathbb{N}$,

    b) for all $j, N, n \in \mathbb{N}$ we have $a_{j,N+1,n} \geq a_{j;N,n}$ and

    c) for all $j, N, n \in \mathbb{N}$ we have $a_{j;N,n+1} \leq a_{j,N,n}$.

ii) A *Köthe* PLB *space* $E_p(A)$ for a Köthe PLB-matrix A and $1 \leq p \leq \infty$ is defined to be

$$E_p(A) := \left\{ x = (x_j)_{j\in\mathbb{N}} \in \mathbb{K}^\mathbb{N} : \underset{N\in\mathbb{N}}{\forall}\ \underset{n\in\mathbb{N}}{\exists}\ \|x\|_{N,n}^{(p)} := \left\| \left(x_j a_{j;N,n}\right)_{j\in I_N} \right\|_{\ell_p} < \infty \right\}$$

endowed with its natural PLB topology $E_p(A) = \mathrm{proj}_{N\in\mathbb{N}} E_p^N(A)$, where

$$E_p^N(A) = \mathrm{ind}_{n\in\mathbb{N}} E_p^{N,n}(A),$$

$$E_p^{N,n}(A) = \left\{ x = \left(x_j \right)_{j \in I_N} \in \mathbb{K}^{I_N} : \; \|x\|_{N,n}^{(p)} < \infty \right\} \text{ and } I_N := \left\{ j \in \mathbb{N} : a_{j;N,n} > 0 \; \text{ for all } n \in \mathbb{N} \right\}.$$

iii) Given a a Köthe PLB-matrix A and $1 \le p \le \infty$, $E_p(A)$ is a deeply reduced PLB space as the family $(e_k)_{k \in I_N}$ of all unit vectors $e_k := \left(\delta_{k,j} \right)_{j \in I_N}$ is a basis in every $E_p^{N,n}(A)$, $N, n \in \mathbb{N}$. The sequence $(I_N)_{N \in \mathbb{N}}$ satisfies $I_N \subset I_{N+1}$, $N \in \mathbb{N}$ because of i) b) and covers $\mathbb{N}$ by i) a). Thus the linking maps $E_M^N(A) : E_p^M(A) \longrightarrow E_p^N(A)$ are given by the restrictions of the canonical projections from $\mathbb{K}^{I_M}$ to $\mathbb{K}^{I_N}$ for all $M \ge N$. If no confusion concerning the Köthe matrix A is possible, we omit it. $E_p(A)$ is a PLS space if $\lim\limits_{n \to \infty} \frac{a_{j;N,n+1}}{a_{j;N,n}} = 0$ by [MV97, (27.8)] and a PLN space if $\sum\limits_{j \in \mathbb{N}} \frac{a_{j;N,n+1}}{a_{j;N,n}} < \infty$.

This type of spaces is worth investigating not only because their structure is easy to handle, but also as they appear commonly in applications:

Remark 5.4.2.   i) By [DV00a, (2.1)] for every PLN space E with an equicontinuous basis there is a Köthe PLB matrix A (not depending on p) such that $E \cong E_p(A)$.

ii) Many spaces as for example the non-quasianalytic Roumieu classes $\mathcal{E}_{\{\omega\}}$ and spaces of Beurling ultradistributions $\mathscr{D}'_{(\omega)}$ are isomorphic to Köthe PLB spaces, cf. [Vog83b, Val78].

iii) The kernels of surjective convolution operators on $\mathscr{D}'_{(\omega)}(\mathbb{R})$, $\mathcal{E}_{\{\omega\}}(\mathbb{R})$ or $\mathcal{E}_{\{\omega\}}(]-1, 1[)$ are even PLS power series spaces, i.e. Köthe PLB spaces with special matrices A, cf. [BD08, p. 578]. Splitting theory for the latter ones has been investigated in [Kun01] by Kunkle.

We give some simple characterizations of our regularity properties of type (P_3) and (DIE). Please note the duality to Vogt's discourse [Vog92, section 5] about LF sequence spaces and the properties (Q) and (wQ), which are just (P_3^*) and $(P_3^*)_0$ in the old formulation of the quantifiers.

Proposition 5.4.3. *Let $1 \le p < \infty$. A Köthe* PLB *space $E_p(A)$ satisfies*

i) (P_3^*) *if and only if* $\underset{N \in \mathbb{N}}{\forall} \; \underset{M \ge N}{\exists} \; \underset{K \ge M}{\forall} \; \underset{n \in \mathbb{N}}{\exists} \; \underset{m \ge n}{\forall} \; \underset{\substack{k \ge m \\ c > 0}}{\exists} \; \underset{j \in I_N}{\forall} \; \dfrac{1}{a_{j;M,m}} \le \dfrac{c}{a_{j;N,n}} + \dfrac{c}{a_{j;K,k}}.$

ii) $(P_3^*)_0$ *if and only if* $\underset{N \in \mathbb{N}}{\forall} \; \underset{M \ge N}{\exists} \; \underset{K \ge M}{\forall} \; \underset{n \in \mathbb{N}}{\exists} \; \underset{\substack{m \ge n \\ \varepsilon > 0}}{\forall} \; \underset{\substack{k \ge m \\ c > 0}}{\exists} \; \underset{j \in I_N}{\forall} \; \dfrac{1}{a_{j;M,m}} \le \dfrac{\varepsilon}{a_{j;N,n}} + \dfrac{c}{a_{j;K,k}}.$

iii) (DIE) *if and only if*

$$\underset{N \in \mathbb{N}}{\forall} \; \underset{M \ge N}{\exists} \; \underset{K \ge M}{\forall} \; \underset{n \in \mathbb{N}}{\exists} \; \underset{m \ge n}{\forall} \; \underset{0 < \theta < 1}{\forall} \; \underset{\substack{k \ge m \\ c > 0}}{\exists} \; \underset{j \in I_N}{\forall} \; \dfrac{1}{a_{j;M,m}} \le c \left(\dfrac{1}{a_{j;K,k}} \right)^{1-\theta} \left(\dfrac{1}{a_{j;N,n}} \right)^{\theta}.$$

As usual, replacing the quantifier $\forall_{0 < \theta < 1}$ by $\exists_{0 < \theta_o < 1} \forall_{0 < \theta \le \theta_o}$, we arrive at (DIE) for small θ and replacing it by $\exists_{0 < \theta_o < 1} \forall_{\theta_o \le \theta < 1}$, we obtain (DIE) for big θ.

Proof. The proof of the assertions is basically straightforward calculation. We give the main ideas. To see necessity, we apply the conditions to the functionals induced by the unit vectors e_j, $j \in I_N$. For sufficiency we calculate the terms appearing in the inequalities of the conditions. Let $S \geq T$, $s \in \mathbb{N}$ and $f \in E_p^T(A)'$. We denote $f_j := f(e_j)$, $j \in I_T$, use the Hölder conjugate q of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$ with the common agreement $\frac{1}{\infty} := 0$, and obtain with the duality of weighted sequence spaces, cf. [MV97, (27.13)]:

$$
\begin{aligned}
\left\| \left(E_S^T \right)^t f \right\|_{S,s}^{(p)*} &= \sup \left\{ \left| f \left(E_S^T(x) \right) \right| : x \in \mathbb{K}^{I_S}, \|x\|_{S,s}^{(p)} \leq 1 \right\} \\
&= \sup \left\{ \left| \sum_{j \in I_T} \left(\frac{1}{a_{j;S,s}} f_j \right) \left(a_{j;S,s} x_j \right) \right| : \left\| \left(a_{j;S,s} x_j \right)_{j \in I_T} \right\|_{\ell_p} \leq 1 \right\} \\
&= \left\| \left(\frac{1}{a_{j;S,s}} f_j \right)_{j \in I_T} \right\|_{\ell_q}.
\end{aligned}
$$

Applying the inequalities involving the entries of the Köthe PLB matrix component-wise yields sufficiency since ℓ_r are normal Banach sequence spaces for all $1 \leq r \leq \infty$, cf. [Fre94, (2.2)]. □

Enhancing the proof of sufficiency in the last proposition (5.4.3) slightly, we can compute a convenient characterization of the conditions (T_ε) and (T) for Köthe PLB spaces:

Proposition 5.4.4. *Let $1 \leq p < \infty$. Two Köthe* PLB *space $E_p(A)$ and $E_p(B)$ satisfy*

i) (T_ε) *if and only if*

$$
\mathop{\forall}_{N \in \mathbb{N}} \mathop{\exists}_{M \geq N} \mathop{\forall}_{K \geq M} \mathop{\exists}_{n \in \mathbb{N}} \mathop{\forall}_{\substack{m \geq n \\ \varepsilon > 0}} \mathop{\exists}_{\substack{k \geq m \\ c > 0}} \mathop{\forall}_{\substack{i \in I_N \\ j \in J_N}} \frac{1}{a_{i;M,m} b_{j;M,m}} \leq \frac{\varepsilon}{a_{i;N,n} b_{j;N,n}} + \frac{c}{a_{i;K,k} b_{j;K,k}},
$$

ii) (T) *if and only if*

$$
\mathop{\forall}_{N \in \mathbb{N}} \mathop{\exists}_{M \geq N} \mathop{\forall}_{K \geq M} \mathop{\exists}_{n \in \mathbb{N}} \mathop{\forall}_{m \geq n} \mathop{\exists}_{\substack{k \geq m \\ c > 0}} \mathop{\forall}_{\substack{i \in I_N \\ j \in J_N}} \frac{1}{a_{i;M,m} b_{j;M,m}} \leq \frac{c}{a_{i;N,n} b_{j;N,n}} + \frac{c}{a_{i;K,k} b_{j;K,k}},
$$

where $I_N := \{ i \in \mathbb{N} : a_{i;N,n} > 0$ for all $n \in \mathbb{N} \}$, $J_N := \left\{ j \in \mathbb{N} : b_{j;N,n} > 0$ for all $n \in \mathbb{N} \right\}$, $N \in \mathbb{N}$.

Proof. As in (5.4.3) the proof is basically straight forward calculation. We give the main idea. For necessity, we again apply the conditions to the functionals induced by the unit vectors e_i, $i \in I_N$ and e_j, $j \in J_N$. For sufficiency, we calculate the terms appearing in the inequalities of the conditions. Let $S \geq T$, $s \in \mathbb{N}$, $f \in E_p^T(A)'$, $g \in E_p^T(B)'$ and q be the Hölder conjugate of p. Furthermore, let $I_T = \{ i(\nu) : \nu \in \mathbb{N} \}$ and $J_T = \{ j(\mu) : \mu \in \mathbb{N} \}$ be countings of I_T respectively

J_T, which we may assume to be infinite. Then we have

$$
\left\| \left(E_S^T (A) \right)^t f \right\|_{S,s}^{(p)*} \left\| \left(E_S^T (B) \right)^t g \right\|_{S,s}^{(p)*} = \left\| \left(\frac{1}{a_{i;S,s}} f_i \right)_{i \in I_T} \right\|_{\ell_q} \left\| \left(\frac{1}{b_{j;S,s}} g_j \right)_{j \in J_T} \right\|_{\ell_q}
$$

$$
= \left(\left(\sum_{\nu=1}^{\infty} \left| \frac{f_{i(\nu)}}{a_{i(\nu);S,s}} \right|^q \right) \left(\sum_{\mu=1}^{\infty} \left| \frac{g_{j(\mu)}}{b_{j(\mu);S,s}} \right|^q \right) \right)^{1/q}
$$

$$
= \left(\sum_{\nu=1}^{\infty} \sum_{\mu=1}^{\nu} \left| \frac{f_{i(\mu)} g_{i(\nu)-j(\mu)}}{a_{i(\mu);S,s} b_{i(\nu)-j(\mu);S,s}} \right|^q \right)^{1/q},
$$

which allows us to apply the inequalities involving the entries of the Köthe PLB matrix component-wise yielding sufficiency. $\qquad\square$

Choosing $p = 2$, we obtain the Köthe PLH spaces, which allow for a complete characterization of the vanishing of proj^1 of σ tensorized spectra, since the unit vectors e_i, $i \in \mathbb{N}$, allow for common orthogonal systems which enable us to interpolate Hilbert-Schmidt operators. Note that under mild assumptions, cf. e.g. (3.4.4) and (4.3.4), only one of the two matrices has to satisfy $(P_3^*)_0$.

Theorem 5.4.5. *Let A and B be two Köthe* PLB *matrices that satisfy* $(P_3^*)_0$.
Then $\mathrm{proj}^1 E_2 (A) \tilde{\otimes}_\sigma E_2 (B) = 0$ *if and only if*

$$
\underset{N \in \mathbb{N}}{\forall} \; \underset{M \geq N}{\exists} \; \underset{K \geq M}{\forall} \; \underset{n \in \mathbb{N}}{\exists} \; \underset{\substack{m \geq n \\ \varepsilon > 0}}{\forall} \; \underset{\substack{k \geq m \\ c > 0}}{\exists} \; \underset{\substack{i \in I_N \\ j \in J_N}}{\forall} \quad \frac{1}{a_{i;M,m} b_{j;M,m}} \leq \frac{\varepsilon}{a_{i;N,n} b_{j;N,n}} + \frac{c}{a_{i;K,k} b_{j;K,k}}
$$

if and only if

$$
\underset{N \in \mathbb{N}}{\forall} \; \underset{M \geq N}{\exists} \; \underset{K \geq M}{\forall} \; \underset{n \in \mathbb{N}}{\exists} \; \underset{m \geq n}{\forall} \; \underset{\substack{k \geq m \\ c > 0}}{\exists} \; \underset{\substack{i \in I_N \\ j \in J_N}}{\forall} \quad \frac{1}{a_{i;M,m} b_{j;M,m}} \leq \frac{c}{a_{i;N,n} b_{j;N,n}} + \frac{c}{a_{i;K,k} b_{j;K,k}}.
$$

Proof. By (5.4.4) the two conditions are equivalent to (T_ε) respectively (T) for the pair $(E_2 (A), E_2 (B))$ which are equivalent by (3.2.2) since both spaces satisfy $(P_3^*)_0$. For the rest of the proof we use the same notation as in (4.3.5) and (4.3.6), i.e.

$$
\mathscr{P}_\sigma := \left(E_2^N (A) \tilde{\otimes}_\sigma E_2^N (B), E_M^N (A) \tilde{\otimes}_\sigma E_M^N (B) \right) \text{ and}
$$

$$
\mathscr{H}_\sigma := \left(\left(E_2^N (A) \right)' \tilde{\otimes}_\sigma \left(E_2^N (B) \right)', \left(E_{N+1}^N (A) \right)^t \tilde{\otimes}_\sigma \left(E_{N+1}^N (B) \right)^t \right).
$$

The second condition is necessary for the vanishing of $\mathrm{proj}^1 \mathscr{P}_\sigma$ in analogy to e.g. (4.3.5). Indeed, the vanishing of $\mathrm{proj}^1 \mathscr{P}_\sigma$ is equivalent to $\mathscr{H}_\sigma$ being weakly acyclic by (3.1.1) as the steps of $\mathscr{P}_\sigma$ are reflexive by (4.2.10) ii). Hence $\mathscr{H}_\sigma$ satisfies (wQ) by [Vog92, (2.3) b)] which yields (T) for the pair $(E_2 (A), E_2 (B))$ by exchanging the quantifiers and applying to

elementary tensors. Again similar to the proof of (4.3.5) the first condition implies even the acyclicity of $\mathcal{H}_\sigma$: (T_ε) implies $(Q)_0$ for elementary tensors. To obtain the norm inequality for all tensors of the completed σ tensor product, we only need interpolation in the sense of (3.2.6) for Hilbert-Schmidt operators, which is trivial if we have common orthogonal bases. Indeed, let $\alpha, \beta > 0$ and $(F_0, F_1), (G_0, G_1)$ be two pairs of pre-Hilbert spaces and F, G two pre-Hilbert spaces with $F \subset F_0 \cap F_1, G \subset G_0 \cap G_1$ such that F and G are of countably infinite, but without loss of generality infinite, dimension and bases $(x_i)_{i \in \mathbb{N}}$ in F and $\left(y_j\right)_{j \in \mathbb{N}}$ in G that are orthogonal with regard to each of the three respective scalar products. Furthermore, let

$$\|x\|_F \|y\|_G \leq \alpha \|x\|_{F_0} \|y\|_{G_0} + \beta \|x\|_{F_1} \|y\|_{G_1} \text{ for all } x \in F, y \in G.$$

Since all norms are equivalent on $\mathbb{R}^2$, this yields in particular

$$\|x\|_F^2 \|y\|_G^2 \leq 2\alpha^2 \|x\|_{F_0}^2 \|y\|_{G_0}^2 + 2\beta^2 \|x\|_{F_1}^2 \|y\|_{G_1}^2 \text{ for all } x \in F, y \in G. \qquad (\star)$$

Now let $z = \sum_{\nu=1}^\infty \lambda_\nu x_\nu \otimes y_\nu \in F \otimes G$. By Parseval's equation and $(\star)$ we obtain

$$
\begin{aligned}
\sigma(z; F, G)^2 &= \sum_{\nu=1}^\infty |\lambda_\nu|^2 \|x_\nu\|_F^2 \|y_\nu\|_G^2 \\
&\leq 2\alpha^2 \sum_{\nu=1}^\infty |\lambda_\nu|^2 \|x_\nu\|_{F_0}^2 \|y_\nu\|_{G_0}^2 + 2\beta^2 \sum_{\nu=1}^\infty |\lambda_\nu|^2 \|x_\nu\|_{F_1}^2 \|y_\nu\|_{G_1}^2 \\
&= 2\alpha^2 \sigma(z; F_0, G_0)^2 + 2\alpha^2 \sigma(z; F_1, G_1)^2,
\end{aligned}
$$

i.e. we have interpolation with the loss of $\sqrt{2}$. By (5.4) ii) we may apply this result to our interpolation problem for Köthe PLH spaces, which completes the proof.

$\square$

References

[Api83] H. Apiola, *Characterization of subspaces and quotients of nuclear $L_f(\alpha, \infty)$-spaces*, Compositio Math. **50** (1983), no. 1, 65–81.

[BD98] J. Bonet and P. Domański, *Real analytic curves in Fréchet spaces and their duals*, Monatsh. Math. **126** (1998), no. 1, 13–36.

[BD01] ———, *Parameter dependence of solutions of partial differential equations in spaces of real analytic functions*, Proc. Amer. Math. Soc. **129** (2001), no. 2, 495–503 (electronic).

[BD06] ———, *Parameter dependence of solutions of differential equations on spaces of distributions and the splitting of short exact sequences*, J. Funct. Anal. **230** (2006), no. 2, 329–381.

[BD07] ———, *The structure of spaces of quasianalytic functions of Roumieu type*, Arch. Math. (Basel) **89** (2007), no. 5, 430–441.

[BD08] ———, *The splitting of exact sequences of PLS-spaces and smooth dependence of solutions of linear partial differential equations*, Adv. Math. **217** (2008), no. 2, 561–585.

[BDV02] J. Bonet, P. Domański, and D. Vogt, *Interpolation of vector-valued real analytic functions*, J. London Math. Soc. (2) **66** (2002), no. 2, 407–420.

[Bou87] N. Bourbaki, *Topological vector spaces. Chapters 1–5*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1987, Translated from the French by H. G. Eggleston and S. Madan. MR 910295 (88g:46002)

[Bou07] ———, *Éléments de mathématique. Algèbre. Chapitre 10. Algèbre homologique*, Springer-Verlag, Berlin, 2007, Reprint of the 1980 original [Masson, Paris; MR0610795].

[Bro62] F. E. Browder, *Analyticity and partial differential equations. I*, Amer. J. Math. **84** (1962), 666–710.

[Büh10] T. Bühler, *Exact categories*, Expo. Math. **28** (2010), no. 1, 1–69.

[DF93] A. Defant and K. Floret, *Tensor norms and operator ideals*, North-Holland Mathematics Studies, vol. 176, North-Holland Publishing Co., Amsterdam, 1993.

[DFJP74] W. J. Davis, T. Figiel, W. B. Johnson, and A. Pełczyński, *Factoring weakly compact operators*, J. Functional Analysis **17** (1974), 311–327.

[DFV03] P. Domański, L. Frerick, and D. Vogt, *Fréchet quotients of spaces of real-analytic functions*, Studia Math. **159** (2003), no. 2, 229–245.

References

[DM07] P. Domański and M. Mastyło, *Characterization of splitting for Fréchet-Hilbert spaces via interpolation*, Math. Ann. **339** (2007), no. 2, 317–340.

[Dom04] P. Domański, *Classical PLS-spaces: spaces of distributions, real analytic functions and their relatives*, Orlicz centenary volume, Banach Center Publ., vol. 64, Polish Acad. Sci., Warsaw, 2004, pp. 51–70.

[Dom10] ———, *Real analytic parameter dependence of solutions of differential equations*, Rev. Mat. Iberoam. **26** (2010), no. 1, 175–238.

[Dom11] ———, *Real analytic parameter dependence of solutions of differential equations over Roumieu classes*, Funct. Approx. Comment. Math. **44** (2011), no. part 1, 79–109.

[DS12] B. Dierolf and D. Sieg, *Exact subcategories of the category of locally convex spaces*, Math. Nachr. **285** (2012), no. 17-18, 2106–2119.

[DV00a] P. Domański and D. Vogt, *The space of real-analytic functions has no basis*, Studia Math. **142** (2000), no. 2, 187–200.

[DV00b] ———, *A splitting theory for the space of distributions*, Studia Math. **140** (2000), no. 1, 57–77.

[ELP75] P. Enflo, J. Lindenstrauss, and G. Pisier, *On the "three space problem"*, Math. Scand. **36** (1975), no. 2, 199–210.

[FK10] L. Frerick and T. Kalmes, *Some results on surjectivity of augmented semi-elliptic differential operators*, Math. Ann. **347** (2010), no. 1, 81–94.

[FKW03] L. Frerick, D. Kunkle, and J. Wengenroth, *The projective limit functor for spectra of webbed spaces*, Studia Math. **158** (2003), no. 2, 117–129.

[Fre94] L. Frerick, *On vector valued sequence spaces*, Dissertation, Trier, 1994.

[Fre96] ———, *A splitting theorem for nuclear Fréchet spaces*, Functional analysis (Trier, 1994), de Gruyter, Berlin, 1996, pp. 163–167.

[FW68] K. Floret and J. Wloka, *Einführung in die Theorie der lokalkonvexen Räume*, Lecture Notes in Mathematics, No. 56, Springer-Verlag, Berlin, 1968.

[FW96] L. Frerick and J. Wengenroth, *A sufficient condition for vanishing of the derived projective limit functor*, Arch. Math. (Basel) **67** (1996), no. 4, 296–301.

[Gro55] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **1955** (1955), no. 16, 140.

[Har79] J. Harksen, *Tensornormtopologien*, Dissertation, Kiel, 1979.

References

[Har82]   ——, *Charakterisierung lokalkonvexer Räume mit Hilfe von Tensornormtopologien*, Math. Nachr. **106** (1982), 347–374.

[Her05]   V. Hermanns, *Zur Existenz von Rechtsinversen linearer partieller Differentialoperatoren mit konstanten Koeffizienten auf $B_{p,\kappa}^{loc}(\Omega)$-Räumen*, Dissertation, Wuppertal, 2005.

[Hol80]   R. Hollstein, *Inductive limits and ε-tensor products*, J. Reine Angew. Math. **319** (1980), 38–62.

[Hör62]   L. Hörmander, *On the range of convolution operators*, Ann. of Math. (2) **76** (1962), 148–170.

[Hör03]   Lars Hörmander, *The analysis of linear partial differential operators. I*, Classics in Mathematics, Springer-Verlag, Berlin, 2003, Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].

[Hör05]   L. Hörmander, *The analysis of linear partial differential operators. II*, Classics in Mathematics, Springer-Verlag, Berlin, 2005, Differential operators with constant coefficients, Reprint of the 1983 original.

[Jar81]   H. Jarchow, *Locally convex spaces*, B. G. Teubner, Stuttgart, 1981, Mathematische Leitfäden. [Mathematical Textbooks].

[Kal11]   T. Kalmes, *Every P-convex subset of $\mathbb{R}^2$ is already strongly P-convex*, Math. Z. **269** (2011), no. 3-4, 721–731.

[Kal12]   ——, *The augmented operator of a surjective partial differential operator with constant coefficients need not be surjective*, Bull. Lond. Math. Soc. **44** (2012), no. 3, 610–614.

[Kar06]   X. Karidopoulou, *Ein Splittingsatz für gradierte L-zahme Räume*, Dissertation Bergische Universität Wuppertal, 2006.

[Köt69]   G. Köthe, *Topological vector spaces. I*, Translated from the German by D. J. H. Garling. Die Grundlehren der mathematischen Wissenschaften, Band 159, Springer-Verlag New York Inc., New York, 1969.

[Köt79]   ——, *Topological vector spaces. II*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 237, Springer-Verlag, New York, 1979.

[KR83]   R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras. Vol. I*, Pure and Applied Mathematics, vol. 100, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1983, Elementary theory.

References

[Kun01] D. Kunkle, *Splitting of Power series spaces of (PLS)-type*, Dissertation, Wuppertal, 2001.

[Kür91] K.-D. Kürsten, *Bounded sets in projective tensor products of Hilbertizable locally convex spaces*, Studia Math. **99** (1991), no. 3, 185–198.

[Lan04] M. Langenbruch, *Characterization of surjective partial differential operators on spaces of real analytic functions*, Studia Math. **162** (2004), no. 1, 53–96.

[Mal56] B. Malgrange, *Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution*, Ann. Inst. Fourier, Grenoble **6** (1955–1956), 271–355.

[Man90] F. Mantlik, *Linear equations depending differentiably on a parameter*, Integral Equations Operator Theory **13** (1990), no. 2, 231–250.

[Man91] _____, *Partial differential operators depending analytically on a parameter*, Ann. Inst. Fourier (Grenoble) **41** (1991), no. 3, 577–599.

[Man92] _____, *Fundamental solutions for hypoelliptic differential operators depending analytically on a parameter*, Trans. Amer. Math. Soc. **334** (1992), no. 1, 245–257.

[Mur90] G. J. Murphy, *C^*-algebras and operator theory*, Academic Press Inc., Boston, MA, 1990.

[MV97] R. Meise and D. Vogt, *Introduction to functional analysis*, Oxford Graduate Texts in Mathematics, vol. 2, The Clarendon Press Oxford University Press, New York, 1997, Translated from the German by M. S. Ramanujan and revised by the authors.

[Ovc98] V. I. Ovchinnikov, *Coherently nuclear operators in couples of Hilbert spaces*, Mat. Zametki **63** (1998), no. 6, 866–872.

[Pal71] V. P. Palamodov, *Homological methods in the theory of locally convex spaces*, Uspekhi Mat. Nauk **26** (1) (1971) 3–66 (in Russian), English transl., Russian Math. Surveys **26** (1) (1971), 1–64.

[Pal72] _____, *The projective limit functor in the category of topological linear spaces*, Mat. Sb. **75** (1968) 567–603 (in Russian), English transl., Math. USSR Sbornik **17** (1972), 189–315.

[PCB87] P. Pérez Carreras and J. Bonet, *Barrelled locally convex spaces*, North-Holland Mathematics Studies, vol. 131, North-Holland Publishing Co., Amsterdam, 1987, Notas de Matemática [Mathematical Notes], 113.

[Pie72] A. Pietsch, *Nuclear locally convex spaces*, Springer-Verlag, New York, 1972, Translated from the second German edition by William H. Ruckle, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 66.

References

[Pis10] K. Piszczek, *On a property of PLS-spaces inherited by their tensor products*, Bull. Belg. Math. Soc. Simon Stevin **17** (2010), no. 1, 155–170.

[RD81] W. Roelcke and S. Dierolf, *On the three-space-problem for topological vector spaces*, Collect. Math. **32** (1981), no. 1, 13–35.

[Ret70] V. S. Retakh, *The subspaces of a countable inductive limit*, Dokl. Akad. Nauk SSSR **194** (1970), 1277–1279.

[RW77] F. Richman and E. A. Walker, *Ext in pre-Abelian categories*, Pacific J. Math. **71** (1977), no. 2, 521–535.

[Sch58] L. Schwartz, *Théorie des distributions à valeurs vectorielles. I,II*, Ann. Inst. Fourier, Grenoble **7,8** (1957,1958), 1–141,1–209.

[Sie10] D. Sieg, *A homological approach to the splitting theory of PLS-spaces*, Dissertation, Trier, 2010.

[SW11] D. Sieg and S.-A. Wegner, *Maximal exact structures on additive categories*, Math. Nachr. **284** (2011), no. 16, 2093–2100.

[Tas86] J. Taskinen, *Counterexamples to "problème des topologies" of Grothendieck*, Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes (1986), no. 63, 25.

[Trè62a] F. Trèves, *Fundamental solutions of linear partial differential equations with constant coefficients depending on parameters*, Amer. J. Math. **84** (1962), 561–577.

[Trè62b] _____, *Un théorème sur les équations aux dérivées partielles à coefficients constants dépendant de paramètres*, Bull. Soc. Math. France **90** (1962), 473–486.

[Val78] M. Valdivia, *Representations of the spaces $\mathscr{D}(\Omega)$ and $\mathscr{D}'(\Omega)$*, Rev. Real Acad. Cienc. Exact. Fís. Natur. Madrid **72** (1978), no. 3, 385–414.

[Var02] O. Varol, *Kriterien für $Tor_\alpha^1(E,F)= 0$ für (DF)- und Frécheträume*, Dissertation, Wuppertal, 2002.

[Var07] _____, *On the derived tensor product functors for (DF)- and Fréchet spaces*, Studia Math. **180** (2007), no. 1, 41–71.

[Vog75] D. Vogt, *Vektorwertige Distributionen als Randverteilungen holomorpher Funktionen*, Manuscripta Math. **17** (1975), no. 3, 267–290.

[Vog77a] _____, *Charakterisierung der Unterräume von s*, Math. Z. **155** (1977), no. 2, 109–117.

[Vog77b] _____, *Subspaces and quotient spaces of (s)*, Functional analysis: surveys and recent results (Proc. Conf., Paderborn, 1976), North-Holland, Amsterdam, 1977, pp. 167–187. North–Holland Math. Studies, Vol. 27; Notas de Mat., No. 63.

References

[Vog83a] _____, *On the solvability of P(D)f = g for vector valued functions*, RIMS Koky-oroku **508** (1983), 168–182.

[Vog83b] _____, *Sequence space representations of spaces of test functions and distributions*, Functional analysis, holomorphy, and approximation theory (Rio de Janeiro, 1979), Lecture Notes in Pure and Appl. Math., vol. 83, Dekker, New York, 1983, pp. 405–443.

[Vog87] _____, *On the functors* $\mathrm{Ext}^1(E, F)$ *for Fréchet spaces*, Studia Math. **85** (1987), no. 2, 163–197.

[Vog89] _____, *Topics on projective spectra of (LB)-spaces*, Advances in the theory of Fréchet spaces (Istanbul, 1988), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 287, Kluwer Acad. Publ., Dordrecht, 1989, pp. 11–27.

[Vog92] _____, *Regularity properties of (LF)-spaces*, Progress in functional analysis (Peñíscola, 1990), North-Holland Math. Stud., vol. 170, North-Holland, Amsterdam, 1992, pp. 57–84.

[Vog04] _____, *Fréchet valued real analytic functions*, Bull. Soc. Roy. Sci. Liège **73** (2004), no. 2-3, 155–170.

[Vog06] _____, *Invariants and spaces of zero solutions of linear partial differential operators*, Arch. Math. (Basel) **87** (2006), no. 2, 163–171.

[Vog11] _____, *On the splitting relation for Fréchet-Hilbert spaces*, Funct. Approx. Comment. Math. **44** (2011), no. part 2, 215–225.

[VW80] D. Vogt and M. J. Wagner, *Charakterisierung der Quotientenräume von s und eine Vermutung von Martineau*, Studia Math. **67** (1980), no. 3, 225–240.

[Wei94] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

[Wen95] J. Wengenroth, *Retractive (LF)-spaces*, Dissertation, Trier, 1995.

[Wen96] _____, *Acyclic inductive spectra of Fréchet spaces*, Studia Math. **120** (1996), no. 3, 247–258.

[Wen01] _____, *A splitting theorem for subspaces and quotients of $\mathscr{D}'$*, Bull. Polish Acad. Sci. Math. **49** (2001), no. 4, 349–354.

[Wen03] _____, *Derived functors in functional analysis*, Lecture Notes in Mathematics, vol. 1810, Springer-Verlag, Berlin, 2003.