Universität Trier

Universality of Composition Operators with Applications to Complex Dynamics

Dissertation

zur Erlangung des akademischen Grades eines
Doktors der Naturwissenschaften

Dem Fachbereich IV der Universität Trier
vorgelegt von

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September 2015

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Preface

In recent years, the study of dynamical systems has developed into a central research area in mathematics. Actually, in combination with keywords such as "chaos" or "butterfly effect", parts of this theory have been incorporated in other scientific fields, e.g. in physics, biology, meteorology and economics.

In general, a discrete dynamical system is given by a set X and a map $f : X \to X$. The set X can be interpreted as the state space of the system and the function f describes the temporal development of the system. If the system is in state $x \in X$ at time zero, its state at time $n \in \mathbb{N}$ is denoted by $f^n(x)$, where

$$f^n := f \circ \ldots \circ f \quad (n \text{ times})$$

stands for the n-th iterate of the map f. Typically, one is interested in the long-time behaviour of the dynamical system, i.e. in the behaviour of the sequence $(f^n(x))_{n\in\mathbb{N}}$ for an arbitrary initial state $x \in X$ as the time n increases. On the one hand, it is possible that there exist certain states $x \in X$ such that the system behaves stably, which means that $f^n(x)$ approaches a state of equilibrium for $n \to \infty$. On the other hand, it might be the case that the system runs unstably for some initial states $x \in X$ so that the sequence $((f^n(x))_{n\in\mathbb{N}}$ somehow shows chaotic behaviour.

As this work is attributed to the mathematical discipline of complex analysis, we will usually consider discrete dynamical systems which are given by an open subset D of the complex plane $\mathbb{C}$ and a holomorphic function $f : D \to D$. The simplest non-trivial examples of such systems are given by quadratic polynomials

$$P_c : \mathbb{C} \to \mathbb{C}, \;\; P_c(z) := z^2 + c,$$

for constants $c \in \mathbb{C}$. Even here, the dynamics can get quite complicated – for example when considering the celebrated Mandelbrot set, which is the set of all points $c \in \mathbb{C}$ such that the sequence $(P_c^n(0))_{n\in\mathbb{N}}$ is bounded.

In case of a non-linear entire function f, the complex plane always decomposes into two disjoint parts, the *Fatou set* F_f of f and the *Julia set* J_f of f (which are named after the French mathematicians Pierre Fatou (1878 - 1929) and Gaston Julia (1893 - 1978)). These two sets are defined in such a way that the sequence of iterates (f^n) behaves quite "wildly" or "chaotically" on J_f whereas, on the other hand, the behaviour of (f^n) on F_f is rather "nice" and well-understood (cf. Section 1.2). However, this nice behaviour of the iterates on the Fatou set can "change dramatically" if we compose the iterates from the left with just one other suitable holomorphic function, i.e. if we consider sequences of the form

$$\left(g \circ f^n\big|_D\right)_{n\in\mathbb{N}},$$

where D is an open subset of F_f with $f(D) \subset D$ and g is holomorphic on D. The general aim of this work is to study the long-time behaviour of such modified sequences. In particular, we want to prove the existence of holomorphic functions g on D having the property that the behaviour of the sequence of compositions $(g \circ f^n)$ on the set D (which is contained in the Fatou set of f) becomes quite similarly chaotic as the behaviour of the sequence (f^n) on the Julia set of f.

With this approach, we immerse ourselves into the theory of universal families and hypercyclic operators, which itself has developed into an own branch of research. In general, for topological spaces X, Y and a family $\{T_\iota : \iota \in I\}$ of continuous functions $T_\iota : X \to Y$, an element $x \in X$ is called *universal* for the family $\{T_\iota : \iota \in I\}$ if the set $\{T_\iota(x) : \iota \in I\}$ is dense in Y. In case that X is a topological vector space and $T : X \to X$ is a continuous linear operator, a vector $x \in X$ is called *hypercyclic* for T if it is universal for the family $\{T^n : n \in \mathbb{N}\}$, i.e. if its orbit $\{T^n(x) : n \in \mathbb{N}\}$ is dense in X. Thus, roughly speaking, universality and hypercyclicity can be described via the following two aspects: There exists a *single object* which allows us, via simple analytical operations, to approximate *every* element of a whole class of objects (cf. [27], p. 346). The first example of a universal object in complex analysis goes back to G. D. Birkhoff, who proved the existence of an entire function f which has the property that for each entire function g there exists a sequence (a_n) in $\mathbb{C}$ such that we have locally uniform convergence $f(\bullet + a_n) \to g$ on $\mathbb{C}$.

In the above situation, i.e. for a non-linear entire function f and an open subset D of F_f with $f(D) \subset D$, we endow the space $H(D)$ of holomorphic functions on D with the topology of locally uniform convergence and we consider the map

$$C_f : H(D) \to H(D), \;\; C_f(g) := g \circ f\big|_D,$$

which is called the *composition operator* with *symbol* f. The transform C_f is a continuous linear operator on the Fréchet space $H(D)$. In order to show that the above-mentioned "nice" behaviour of the sequence of iterates (f^n) on the set $D \subset F_f$ can "change dramatically" if we compose the iterates from the left with another suitable holomorphic function, our aim consists in finding functions $g \in H(D)$ which are hypercyclic for C_f. Indeed, for each hypercyclic function g for C_f, the set of compositions $\{g \circ f^n|_D : n \in \mathbb{N}\}$ is dense in $H(D)$ so that the sequence of compositions $(g \circ f^n|_D)$ is kind of "maximally divergent" – meaning that each function in $H(D)$ can be approximated locally uniformly on D via subsequences of $(g \circ f^n|_D)$. This kind of behaviour stands in sharp contrast to the fact that the sequence of iterates (f^n) itself converges, behaves like a rotation or shows some "wandering behaviour" on each component of F_f (cf. the classification theorem of Fatou components; see Section 1.2, Theorem 1.2.7). In order to find functions $g \in H(D)$ which are hypercyclic for C_f, we will use certain strategies which are mainly based on the work of L. Bernal-González and A. Montes-Rodríguez [13] as well as on the work of K.-G. Grosse-Erdmann and R. Mortini [28].

To put it in a nutshell, this work combines the theory of *non-linear* complex dynamics in the complex plane with the theory of dynamics of continuous *linear* operators on spaces of holomorphic functions. As far as the author knows, this approach has not been investigated before. We will see that the common belief that "chaos" is always linked with non-linearity is not true (cf. the textbook "Linear Chaos" [29] by Grosse-Erdmann) – even if the fictional character Sheldon Cooper, who plays the role of a brilliant physicist in the famous US sitcom "The Big Bang Theory" states that *"Chaos theory suggests that even in a deterministic system, if the equations describing its behaviour are NON-LINEAR, a tiny change in the initial conditions can lead to a cataclysmic and unpredictable result"* (cf. Series 7, Episode 16 of "The Big Bang Theory" by Chuck Lorre and Bill Prady).

I would like to thank my mentor and supervisor apl. Professor Dr. Jürgen Müller for his great support during the last years, his inspiring ideas, his valuable advice, his helpful comments and for the opportunity to learn from his extraordinary expertise. Moreover, I want to express my gratitude to Professor Dr. Luis Bernal-González for kindly and spontaneously accepting to be the second reviewer of this thesis. I would further like to thank my colleagues at the University of Trier, as well as one colleague at the Polytechnic University of Valencia, for the pleasant working atmosphere and many interesting academic and non-academic conversations in the last years. In addition, I wish to acknowledge the financial support of the "Stipendienstiftung

Rheinland-Pfalz". Last, but definitely not least, many thanks to my parents and my girlfriend for their invaluable and encouraging support. Without them, it would not have been possible to write this thesis.

Contents

Chapter 1

Introduction and Motivation

The purpose of this first chapter is to prepare and introduce the topic of this work. On the one hand, we want to explain and motivate the research questions and the aims which we will pursue in the following and, on the other hand, we want to create certain bases by providing relevant notations and mathematical tools.

In order to do this, the theories of complex dynamics and hypercyclic composition operators will be introduced and we will state some well-known results in these areas. Subsequently, we will have a look at rational and polynomial approximation of holomorphic functions and, finally, we will consider bounded connected components of subsets of the complex plane and how to "fill them up" relative to given supersets.

1.1 Basic Notations and Preliminaries

In this section, we want to introduce some basic notations which we will use throughout this work. Moreover, several topological preliminaries will be provided.

As usual, we endow the complex plane $\mathbb{C}$ with the topology $\mathcal{T}_{\mathbb{C}}$ induced by the Euclidean metric so that $(\mathbb{C}, \mathcal{T}_{\mathbb{C}})$ becomes a locally compact Hausdorff space. By means of Alexandrov's one-point compactification, we obtain the extended complex plane $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ and its topology $\mathcal{T}_\infty = \mathcal{T}_{\mathbb{C}} \cup \{\mathbb{C}_\infty \backslash K : K \subset \mathbb{C} \text{ compact}\}$. The space $(\mathbb{C}_\infty, \mathcal{T}_\infty)$ is compact and metrizable. A metric on $\mathbb{C}_\infty$ inducing the topology $\mathcal{T}_\infty$ is given for example by the chordal metric χ. Subsets of $\mathbb{C}$ are open in $(\mathbb{C}, \mathcal{T}_{\mathbb{C}})$ if and only if they are open in $(\mathbb{C}_\infty, \mathcal{T}_\infty)$ and bounded subsets of $\mathbb{C}$ are closed in $(\mathbb{C}, \mathcal{T}_{\mathbb{C}})$ if and only if they are closed in $(\mathbb{C}_\infty, \mathcal{T}_\infty)$. For a set $M \subset \mathbb{C}$, we denote by $M^\circ, \overline{M}$ and ∂M its interior, closure and boundary with respect to $(\mathbb{C}, \mathcal{T}_{\mathbb{C}})$. The boundary of a set $A \subset \mathbb{C}_\infty$ with respect to $(\mathbb{C}_\infty, \mathcal{T}_\infty)$ is denoted by $\partial_\infty A$.

For a set X, we denote by $|X|$ the cardinality of X, and for a function $f : X \to X$, we denote by

$$f^n := f \circ \ldots \circ f \quad (n \text{ times}), \ n \in \mathbb{N}_0,$$

the n-th iterate of f (more precisely, this means $f^0 := \mathrm{id}_X$ and $f^n := f \circ f^{n-1}$ for $n \in \mathbb{N}$). If S is a subset of X, we write

$$f^{-n}(S) := \{x \in X : f^n(x) \in S\}$$

for the n-th preimage of S under f. We call S *(forward) invariant* under f if $f(S) \subset S$ and we call S *backward invariant* under f if $f^{-1}(S) \subset S$. Moreover, if S is invariant and backward invariant under f, it is called *completely invariant* under f. This is exactly the case if S and $X \backslash S$ are invariant under f or, equivalently, if $f^{-1}(S) = S$. The *forward orbit* and the *backward orbit* of S under f are given by

$$O_f^+(S) := \bigcup_{n \in \mathbb{N}} f^n(S) \quad \text{and} \quad O_f^-(S) := \bigcup_{n \in \mathbb{N}} f^{-n}(S),$$

respectively. For $x \in X$, we write $O_f^+(x) := O_f^+(\{x\})$ and $O_f^-(x) := O_f^-(\{x\})$. Finally, for a further set Y, we denote by Y^X the set of all functions from X to Y.

Now, let X be a topological space. Subsets S of X will always be endowed with the relative topology on S, and we denote by $S^{\circ X}$ and $\overline{S}^X$ the interior and the closure of S with respect to X. The space X is called *σ-compact* if it is a countable union of compact subsets of X. A sequence (K_n) of compact subsets of X is called a *compact exhaustion* of X if it fulfils $X = \bigcup_{n \in \mathbb{N}} K_n$ and $K_n \subset K_{n+1}^{\circ X}$ for all $n \in \mathbb{N}$. By a *component* of X we mean a maximal connected subset of X, i.e. a connected subset of X which is not contained in any other connected subset of X. A subset of X is called *comeager* if its complement is of first Baire category in X. We say that a property is fulfilled by *comeager many* $x \in X$ if it is fulfilled on a comeager subset of X. Countable intersections and supersets of comeager sets are again comeager. If X is a Baire space, comeager sets are exactly those sets which contain a dense G_δ-set. For a further topological space Y and functions $f_n, f : X \to Y$, $n \in \mathbb{N}$, we write $f_n \to f$ if the sequence (f_n) converges to f pointwise on X. Finally, we denote by $C(X, Y)$ the set of all continuous functions from X to Y.

Now, let (X, d) be a metric space. For $R, S \subset X$, $x_0 \in X$ and $\varepsilon > 0$, we denote

$$\begin{aligned} \mathrm{dist}(R, S) &:= \inf\{d(x, y) : x \in R, y \in S\}, \\ \mathrm{dist}(x_0, R) &:= \mathrm{dist}\left(\{x_0\}, R\right), \end{aligned}$$

$$\begin{aligned} U_\varepsilon(R) &:= \{x \in X : \operatorname{dist}(x, R) < \varepsilon\}, \\ U_\varepsilon[R] &:= \{x \in X : \operatorname{dist}(x, R) \leq \varepsilon\}. \end{aligned}$$

For a topological space Z, a sequence (f_n) of functions $f_n : Z \to X$ and a set $X_0 \subset X$, we write

$$f_n \to X_0 \text{ (locally) uniformly}$$

if the sequence $\big(\operatorname{dist}(f_n(\cdot), X_0)\big)_{n\in\mathbb{N}}$ converges to 0 (locally) uniformly on Z.

The following definition is of great importance in the theory of complex dynamics:

Definition 1.1.1. Let X be a topological space and let Y be a metric space. A family $\mathcal{F} \subset C(X, Y)$ is called *normal* if each sequence in $\mathcal{F}$ has a subsequence which converges uniformly on all compact subsets of X to a function $f \in C(X, Y)$.

Remark 1.1.2. If X is locally compact and has a compact exhaustion, normality is a local property, i.e. a family $\mathcal{F} \subset C(X, Y)$ is normal if and only if for each point $x \in X$ there exists a neighbourhood W of x such that $\{f\big|_W : f \in \mathcal{F}\}$ is a normal family in $C(W, Y)$ (cf. [46], Theorem 2.1.2, in case that X is a domain in $\mathbb{C}$).

We write $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$, $\mathbb{R}^- := \{x \in \mathbb{R} : x < 0\}$ and $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. For $M, L \subset \mathbb{C}$ and $z_0 \in \mathbb{C}$ as well as $r > 0$, we define

$$\begin{aligned} U_r(z_0) &:= \{z \in \mathbb{C} : |z - z_0| < r\}, \\ U_r[z_0] &:= \{z \in \mathbb{C} : |z - z_0| \leq r\}, \\ K_r[z_0] &:= \{z \in \mathbb{C} : |z - z_0| = r\}, \\ z_0 M &:= \{z_0 z : z \in M\}, \\ M + L &:= \{z + w : z \in M, w \in L\}, \\ z_0 + M &:= \{z_0\} + M. \end{aligned}$$

Moreover, for sets $M \subset \mathbb{C}$, open sets $U \subset \mathbb{C}$ and compact sets $K \subset \mathbb{C}$, we introduce the function spaces

$$\begin{aligned} C(M) &:= \{f : M \to \mathbb{C} : f \text{ continuous}\}, \\ H(U) &:= \{f : U \to \mathbb{C} : f \text{ holomorphic}\}, \\ A(K) &:= \{f : K \to \mathbb{C} : f \text{ continuous on } K \text{ and holomorphic on } K^\circ\}. \end{aligned}$$

Now, we fix a subset M of the complex plane and we denote by $\mathcal{T}_M$ the relative topology of $\mathcal{T}_\mathbb{C}$ on M. If the space $(M, \mathcal{T}_M)$ is locally compact and σ-compact, there exists a compact exhaustion (K_n) of $(M, \mathcal{T}_M)$ (cf. [30], Satz 4.4-3). Hence, (K_n) is a sequence of compact subsets of $\mathbb{C}$ with $M = \bigcup_{n\in\mathbb{N}} K_n$ and $K_n \subset K_{n+1}^{\circ M}$ for all $n \in \mathbb{N}$. For $f \in C(M)$ and $n \in \mathbb{N}$, we define

$$||f||_n := \max_{z\in K_n} |f(z)|.$$

The increasing family of seminorms $(||\cdot||_n)_{n\in\mathbb{N}}$ turns $C(M)$ into a Fréchet space. A metric on $C(M)$ having the same open sets, convergent sequences and Cauchy sequences as this Fréchet space is given for example by

$$d_{C(M)} : C(M) \times C(M) \to \mathbb{R}, \quad d_{C(M)}(f, g) := \sup_{n\in\mathbb{N}} \min\left(1/n, \|f - g\|_n\right).$$

For functions $f_n, f \in C(M)$, $n \in \mathbb{N}$, the sequence (f_n) converges to f in the Fréchet space $C(M)$ if and only if (f_n) converges to f uniformly on all compact subsets of M. (As M is locally compact, this is equivalent to locally uniform convergence $f_n \to f$ on M.) The topology induced by the family of seminorms $(||\cdot||_n)_{n\in\mathbb{N}}$ is called the *topology of locally uniform convergence*. This topology does not depend on the choice of the sets K_n. In the following, the space $C(M)$ will always be endowed with this topology.

Remark 1.1.3.

i) Clearly, each compact set $K \subset \mathbb{C}$ is locally compact and σ-compact. Choosing in the above situation $K_n = K$ for all $n \in \mathbb{N}$ and defining

$$||f||_K := \max_{z\in K} |f(z)|, \;\; f \in C(K),$$

we obtain the Banach space $(C(K), ||\cdot||_K)$. As a closed subspace of this space, $(A(K), ||\cdot||_K)$ is a Banach space, too. It is well-known that $C(K)$ is separable (see e.g. [18], Lemma 1.49). As subsets of separable metric spaces are separable, $A(K)$ is also separable. For $f \in A(K)$ and $\varepsilon > 0$, we set

$$U_{\varepsilon,K}(f) := \left\{g \in A(K) : \|f - g\|_K < \varepsilon\right\}.$$

ii) Each open set $U \subset \mathbb{C}$ is locally compact. Moreover, defining

$$K_n(U) := U_n[0] \cap \{z \in U : \text{dist}(z, \mathbb{C}\backslash U) \geq 1/n\}, \;\; n \in \mathbb{N},$$

we see that $(K_n(U))$ is a sequence of compact subsets of U for which we have $U = \bigcup_{n\in\mathbb{N}} K_n(U)$ and $K_n(U) \subset K_{n+1}(U)^\circ$ for all $n \in \mathbb{N}$. In particular, U

is also σ-compact. The sequence $(K_n(U))$ is called the *standard (compact) exhaustion* of U. Choosing in the above situation $K_n = K_n(U)$, $n \in \mathbb{N}$, we obtain the Fréchet space

$$\left(C(U), \left(||\cdot||_{K_n(U)}\right)_{n\in\mathbb{N}}\right).$$

As $H(U)$, endowed with the same family of seminorms, is a closed subspace of this space, it is a Fréchet space, too. In particular, $C(U)$ and $H(U)$ are Baire spaces. Denoting by $d_{H(U)}$ the restriction of $d_{C(U)}$ to $H(U) \times H(U)$, one can show by an application of Runge's theorem on rational approximation (see Theorem 1.4.1 i) below) that the metric space $(H(U), d_{H(U)})$ is separable (cf. [29], p. 111 and Exercise 4.3.1). Hence, the Fréchet space $H(U)$ is second-countable. For $f \in H(U), K \subset U$ compact and $\varepsilon > 0$, we set

$$V_{\varepsilon,K,U}(f) := \{g \in H(U) : \|f - g\|_K < \varepsilon\}.$$

A subset $\mathcal{V}$ of $H(U)$ is open if and only if for each $f \in \mathcal{V}$ there exist a compact subset K of U and an $\varepsilon > 0$ with $V_{\varepsilon,K,U}(f) \subset \mathcal{V}$.

A complex-valued function h, which is defined on a neighbourhood of ∞, is called *holomorphic* at ∞ if the function $z \mapsto h(1/z)$ is holomorphic at 0. In this situation, we put

$$h'(\infty) := \frac{d}{dz}\, h(1/z)\Big|_{z=0}.$$

For open sets $U, V \subset \mathbb{C}_\infty$, we call a function $\psi : U \to V$ *conformal* if it is holomorphic and bijective. In this case, the inverse function $\psi^{-1} : V \to U$ is also conformal and U and V are called *conformally equivalent*. For holomorphic functions $f : U \to U$ and $g : V \to V$, we say that a function $\varphi : U \to V$ *(conformally) conjugates* f to g if φ is conformal and fulfils $\varphi \circ f = g \circ \varphi$. Then φ is called a *conjugation map* between f and g and we say that f is *conjugated* to g *via* φ. Inductively, we obtain

$$\varphi \circ f^n = g^n \circ \varphi$$

for all $n \in \mathbb{N}$. Conjugacy defines an equivalence relation between holomorphic self-maps of open subsets of the extended complex plane. If, in the above situation, φ is only holomorphic and has dense range, we say that φ *quasiconjugates* f to g.

1.2 Complex Dynamics

In the theory of complex dynamics, the long-time behaviour of certain discrete topological dynamical systems is investigated. The systems which are studied are given by the iteration of a transcendental entire function or a rational function. We will see that the local behaviour of the considered function near fixed points plays an important role in the study of the long-time behaviour of the corresponding dynamical system.

Definition 1.2.1. Let $z_0 \in \mathbb{C}_\infty$ be a *fixed point* of a function g (that is $g(z_0) = z_0$) which is defined on a neighbourhood of z_0. Then the *multiplier* of g at z_0, which is denoted by λ, is defined as follows:

i) If $z_0 \in \mathbb{C}$ and g is holomorphic at z_0, we put $\lambda := g'(z_0)$.

ii) If $z_0 = \infty$ and $1/g$ is holomorphic at ∞, we put $\lambda := (1/g)'(\infty)$.

Moreover, the fixed point z_0 is called:

$$\begin{aligned} &\textit{superattracting} && \text{if} && \lambda = 0, \\ &\textit{attracting} && \text{if} && 0 < |\lambda| < 1, \\ &\textit{rationally indifferent} && \text{if} && |\lambda| = 1 \text{ and } \lambda \text{ is a root of unity}, \\ &\textit{neutral} && \text{if} && z_0 \text{ is rationally indifferent with } \lambda = 1, \\ &\textit{irrationally indifferent} && \text{if} && |\lambda| = 1 \text{ but } \lambda \text{ is not a root of unity}, \\ &\textit{repelling} && \text{if} && |\lambda| > 1. \end{aligned}$$

We now fix a function f which shall be either a transcendental entire function $f : \mathbb{C} \to \mathbb{C}$ or a rational function $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ of degree $d \geq 2$. If z_0 is a (super-)attracting fixed point of f, we define $A_f(z_0)$ to be the set of all points $z \in \mathbb{C}$ (the set of all points $z \in \mathbb{C}_\infty$, respectively) which fulfil $f^n(z) \to z_0$. This set is called the *basin of attraction* of z_0 under f.

Definition 1.2.2. The *Fatou set* F_f of f is defined as the set of all points $z \in \mathbb{C}$ (the set of all points $z \in \mathbb{C}_\infty$, respectively) for which there exists a neighbourhood W of z such that $\left\{ f^n\big|_W : n \in \mathbb{N} \right\}$ is a normal family in $C(W, \mathbb{C}_\infty)$. The complement of F_f is called the *Julia set* of f and it is denoted by J_f.

Remark 1.2.3. By definition, F_f is an open set. Thus, J_f is closed. According to Remark 1.1.2 and Remark 1.1.3 ii), $\left\{ f^n\big|_{F_f} : n \in \mathbb{N} \right\}$ is a normal family in $C(F_f, \mathbb{C}_\infty)$.

It is well-known that the Julia set of f is completely invariant under f (in case of transcendental entire f, see e.g. [47] p. 299 and Theorem 1.7; in case of rational f, see e.g. [15], Theorem III.1.3). Hence, the same is also true for the Fatou set of f. Moreover, J_f is a perfect set, i.e. J_f has no isolated points (see [47], Theorem 1.4 and [15], Theorem III.1.8), and we have $F_{f^n} = F_f$ as well as $J_{f^n} = J_f$ for all $n \in \mathbb{N}$ (see [47], p. 299 and [15], Theorem III.1.4). Furthermore, (super-)attracting fixed points of f are contained in the Fatou set of f, whereas repelling and rationally indifferent fixed points of f belong to the Julia set of f. Irrationally indifferent fixed points of f can be contained in F_f or in J_f (see e.g. [10], p. 157). In case of a (super-)attracting fixed point z_0 of f, the basin of attraction $A_f(z_0)$ is an open subset of F_f (see e.g. [15], Theorem III.2.1).

A reason for splitting the complex plane (the extended complex plane, respectively) into the two disjoint subsets F_f and J_f is the following: It is well-known that the sequence of iterates (f^n) behaves quite "chaotically" on J_f whereas, on the other hand, the behaviour of (f^n) on F_f is quite "nice" and well-understood.

The "chaotic" behaviour of (f^n) on J_f can be described as follows: For each point $z \in J_f$ and each arbitrarily small open neighbourhood V of z, we have

$$\left|\mathbb{C}_\infty \setminus \bigcup_{n\in\mathbb{N}} f^n(V)\right| \leq 2.$$

Thus, every value in the extended complex plane, with at most two exceptions, is assumed on V under the iteration of f. This result is a direct application of Montel's theorem (see [38], p. 284f.):

Theorem 1.2.4. (Montel's Theorem, 1916) *Let $\mathcal{F}$ be a family of meromorphic functions on a domain $G \subset \mathbb{C}$. If there are three fixed values in $\mathbb{C}_\infty$ which are omitted by every $g \in \mathcal{F}$, then $\mathcal{F}$ is a normal family in $C(G, \mathbb{C}_\infty)$.*

A proof of this result can be found for example in [15], Theorem I.3.2. In order to point out that the sequence of iterates (f^n) behaves "nicely" on F_f, we now consider a component G of F_f. For each $n \in \mathbb{N}$, the continuity of f^n and the invariance of F_f under f imply that $f^n(G)$ is a connected subset of F_f. Hence, there exists a component G_n of F_f with $f^n(G) \subset G_n$. If f is rational, we always have $f^n(G) = G_n$ (see e.g. [49], Theorem 1 on p. 39). But, in general, this does not hold if f is transcendental entire: For example, it is well-known that for $0 < \lambda < e^{-1}$ and $g : \mathbb{C} \to \mathbb{C}$, $g(z) := \lambda e^z$, the Fatou set F_g is connected with $0 \in F_g$ (see e.g. [11], p. 1857), but of course we have $0 \notin g(F_g)$. Nevertheless, we obtain for transcendental entire f that $|G_n \backslash f^n(G)| \leq n$ holds for all $n \in \mathbb{N}$ (see e.g. the Theorem in [11] on p. 1857, which states $|G_1 \backslash f(G)| \leq 1$). Using the notation $G_0 := G$, we call G *preperiodic* if there exist integers $p > m \geq 0$ with $G_p = G_m$. If G is preperiodic and if in particular $m = 0$, we have $f^p(G) \subset G$ and G is called *periodic* with *period* p. Finally, if G is periodic with period 1, we have $f(G) \subset G$, i.e. the component G of F_f is invariant (under f).

Definition 1.2.5.

i) Let G be an invariant component of F_f. Then G is called an *(invariant)*

- *Böttcher domain* if G contains a superattracting fixed point z_0 of f and $f^n\big|_G \to z_0$,
- *Schröder domain* if G contains an attracting fixed point z_0 of f and $f^n\big|_G \to z_0$,
- *Leau domain* if $\partial_\infty G$ contains a neutral fixed point z_0 of f and $f^n\big|_G \to z_0$,
- *Siegel disc* if f is conjugated on G to an irrational rotation on $\mathbb{D}$ (i.e. there exist a conformal map $\varphi : G \to \mathbb{D}$ and an $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ with $\varphi \circ f = e^{2\pi i\alpha} \cdot \varphi$),
- *Arnol'd-Herman ring* if f is conjugated on G to an irrational rotation on an annulus $A_r := \{z \in \mathbb{C} : 1 < |z| < r\}$, $r > 1$,
- *Baker domain* if f is transcendental entire and $f^n\big|_G \to \infty$.

ii) Let G be a periodic component of F_f with period p. Then G is an invariant component of F_{f^p} for the function f^p and G is called a *(periodic) Böttcher domain*, *Schröder domain*, *Leau domain*, *Siegel disc*, *Arnol'd-Herman ring* or *Baker domain*, respectively, if it is invariant of the corresponding type for f^p.

iii) Let G be a preperiodic component of F_f. Then there exist $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$ such that G_m is a periodic component of F_f with period k and we call G a *(preperiodic) Böttcher domain*, *Schröder domain*, *Leau domain*, *Siegel disc*, *Arnol'd-Herman ring* or *Baker domain*, respectively, if G_m is periodic of the corresponding type.

iv) A component of F_f which is not preperiodic is called a *wandering domain*.

Remark 1.2.6.

i) In case of a Böttcher domain, a Schröder domain, a Leau domain or a Baker domain, the required pointwise convergence of (f^n) on G is already locally uniformly on G. Indeed, for each open subset U of F_f on which the sequence of iterates (f^n) converges pointwise to some function g, we have locally uniform convergence $f^n \to g$ on U. This follows from the spherical version of the Vitali-Porter theorem, which states the following: Given a domain $\Omega \subset \mathbb{C}$, a normal family $\mathcal{F} \subset C(\Omega, \mathbb{C}_\infty)$ and a sequence (g_n) of meromorphic functions in $\mathcal{F}$ having the property that (g_n) converges pointwise on a subset of Ω that has an accumulation point in Ω, then (g_n) already converges locally uniformly on Ω (see e.g. [46], Theorem 3.2.3 and Theorem 3.2.1).

ii) By the maximum modulus principle, Arnol'd-Herman rings do not exist if f is a transcendental entire function or a polynomial (see e.g. [31], p. 189).

iii) In 1985, D. Sullivan was able to prove his famous "No Wandering Domains Theorem", which states that rational functions do not have any wandering domains (see [51], Theorem 1). Sullivan's proof as well as shortened versions of this proof (see e.g. [15], Theorem IV.1.3, [37], Theorem F.1 or [49], p. 47ff.) all use the theory of quasiconformal mappings.

iv) As each set $G_n \backslash f^n(G)$ is finite, one can show that G is a wandering domain of f if and only if $f^p(G) \cap f^m(G) = \emptyset$ holds for all integers $p > m \geq 0$.

The following theorem gives a complete description of the behaviour of the sequence of iterates (f^n) on components of the Fatou set of f. It is mainly due to P. Fatou in 1919 and H. Cremer in 1932 (cf. [10], p. 163).

Theorem 1.2.7. (Classification Theorem of Fatou Components)

i) If f is a transcendental entire function, each component of F_f is either a Böttcher domain, a Schröder domain, a Leau domain, a Siegel disc, a Baker domain or a wandering domain.

ii) If f is a rational function of degree $d \geq 2$, each component of F_f is either a Böttcher domain, a Schröder domain, a Leau domain, a Siegel disc or an Arnol'd-Herman ring.

A proof of the classification theorem can be found for example in [39], Theorem 2.1.10. In case of transcendental entire f, see also [47], Theorem 2.1, and in case of rational f, see also [15], Theorem IV.2.1, [37], Theorem 16.1 and [49], p. 54.

The important thing about the classification theorem is that there exist only a few possibilities of how the iterates of f can behave on the Fatou set of f. For this reason, we stated above that the behaviour of (f^n) on F_f is "nice" and well-understood.

Now, the main motivation of this work is the following: Composing the iterates from the left with one other suitable holomorphic function, we will see that this nice behaviour of the sequence of iterates on the Fatou set can "change dramatically" such that it becomes quite similarly chaotic as on the Julia set! The general aim of this work is to study the long-time behaviour of such modified sequences on the Fatou set. Hence, in the following, we will consider sequences of the form $(g \circ f^n)_{n\in\mathbb{N}}$ for suitable holomorphic functions g. This leads us to the definition of composition operators.

1.3 Composition Operators and Universality

We consider an open set $D \subset \mathbb{C}$, which shall be fixed throughout this whole section.

Definition 1.3.1. Let $f : D \to D$ be holomorphic. The *composition operator* with *symbol* f is defined as

$$C_f : H(D) \to H(D), \;\; C_f(g) := g \circ f.$$

Iterating C_f, we obtain $(C_f)^n = C_{f^n}$ for all $n \in \mathbb{N}$. Clearly, C_f is a linear transform, and it is easy to verify that C_f is continuous on $H(D)$. Indeed, for a sequence (g_n) in $H(D)$ which converges to 0 in the Fréchet space $H(D)$, it follows for each compact subset K of D that

$$\|C_f(g_n)\|_K = \|g_n \circ f\|_K = \|g_n\|_{f(K)} \to 0.$$

Hence, we have locally uniform convergence $C_f(g_n) \to 0$ on D. We now introduce the usual definitions of "universality" of a sequence of composition operators and of "hypercyclicity" of a composition operator.

Definition 1.3.2.

i) Let (f_n) be a sequence of holomorphic self-maps of D. A function $g \in H(D)$ is called *universal* for (C_{f_n}) if the set

$$\{C_{f_n}(g) : n \in \mathbb{N}\}$$

is dense in $H(D)$. The sequence of composition operators (C_{f_n}) is called *universal* if there exists a universal function for (C_{f_n}).

ii) Let f be a holomorphic self-map of D. A function $g \in H(D)$ is called *hypercyclic* for C_f if it is universal for (C_{f^n}), i.e. if the set

$$\{(C_f)^n(g) : n \in \mathbb{N}\}$$

is dense in $H(D)$. The composition operator C_f is called *hypercyclic* if there exists a hypercyclic function for C_f.

The first work on universality of composition operators on general open subsets of the complex plane is due to W. Luh in 1993. In case that all components of D are simply connected, he proved the existence of a sequence (f_n) of conformal self-maps of D and the existence of a function $g \in H(D)$ having the property that for each compact subset K of D with connected complement the set $\left\{C_{f_n}(g)\big|_K : n \in \mathbb{N}\right\}$ is dense in $A(K)$ (see [34], Theorem on p. 161).

In 1995, L. Bernal-González and A. Montes-Rodríguez were the first to characterize universality of a sequence of composition operators (C_{f_n}) for a given sequence (f_n) of conformal self-maps of D (cf. [13]). In order to state their main result, we introduce the following definition, which was coined by them:

Definition 1.3.3. A sequence (f_n) of holomorphic self-maps of a domain $\Omega \subset \mathbb{C}$ is called a *run-away sequence* if for each compact subset K of Ω there exists an $N \in \mathbb{N}$ such that $f_N(K) \cap K = \emptyset$.

Now, the main result of Bernal and Montes reads as follows (see [13], Theorem 3.6):

Theorem 1.3.4. (Bernal, Montes, 1995) *Let $\Omega \subset \mathbb{C}$ be a domain and let (f_n) be a sequence of conformal self-maps of Ω.*

i) If Ω is not conformally equivalent to $\mathbb{C}\backslash\{0\}$, the sequence of composition operators (C_{f_n}) is universal if and only if (f_n) is a run-away sequence. In case that one of these conditions holds, the set of universal functions for (C_{f_n}) is comeager in $H(\Omega)$.

ii) There exists a function $g \in H(\Omega)$ having the property that for each compact subset K of Ω with connected complement the set $\left\{C_{f_n}(g)\big|_K : n \in \mathbb{N}\right\}$ is dense in $A(K)$ if and only if (f_n) is a run-away sequence. In case of existence, the set of such functions is comeager in $H(\Omega)$.

Moreover, Bernal and Montes gave an example that the statement in i) does not hold for $\Omega = \mathbb{C}\backslash\{0\}$ (see [13], Remark on p. 55).

Fourteen years later, in 2009, K.-G. Grosse-Erdmann and R. Mortini were able to characterize universality of composition operators without the assumption of conformality of the corresponding symbols; they only required (f_n) to be a sequence of holomorphic self-maps of Ω (cf. [28]). Their main result concerning the case of iteration of one holomorphic function is the following (see [28], Theorem 3.21(a)):

Theorem 1.3.5. (Grosse-Erdmann, Mortini, 2009) *Let $\Omega \subset \mathbb{C}$ be a simply connected domain and let f be a holomorphic self-map of Ω. Then C_f is hypercyclic if and only if (f^n) is a run-away sequence and f is injective.*

It is easy to see that injectivity of the symbol is necessary in order to obtain hypercyclicity of the corresponding composition operator. Indeed, if f is not injective in the situation of Theorem 1.3.5, we can choose points $z_1, z_2 \in \Omega$ with $z_1 \neq z_2$ and $f(z_1) = f(z_2)$. Considering a function $h \in H(\Omega)$ with $h(z_1) \neq h(z_2)$ (e.g. $h(z) := z - z_1$ for $z \in \Omega$) and assuming that there exists a hypercyclic function $g \in H(\Omega)$ for C_f, we could find a sequence (n_j) in $\mathbb{N}$ such that $(g \circ f^{n_j})$ converges to h locally uniformly on Ω. This would imply

$$h(z_1) = \lim_{j\to\infty} g\left(f^{n_j}(z_1)\right) = \lim_{j\to\infty} g\left(f^{n_j}(z_2)\right) = h(z_2),$$

a contradiction. Again, if one of the two conditions in Theorem 1.3.5 holds, the set of hypercyclic functions for C_f is comeager in $H(\Omega)$. The reason for obtaining comeager sets of universal and hypercyclic functions in Theorems 1.3.4 and 1.3.5 is that these theorems can be proved by an application of the following criterion, which goes back to Grosse-Erdmann in 1987 (cf. [26], Satz 1.1.7):

Theorem 1.3.6. (Universality Criterion) *Let X be a Baire space, Y a second-countable space and $T_\iota : X \to Y$ continuous, $\iota \in I$. Then, for*

$$S := \big\{x \in X : \{T_\iota(x) : \iota \in I\} \text{ dense in } Y\big\},$$

the following assertions are equivalent:

i) S is dense in X.

ii) S is comeager in X.

iii) For every pair of non-empty open subsets U of X and V of Y there exists an index $\iota \in I$ with $T_\iota(U) \cap V \neq \emptyset$.

If one of these conditions holds, S is a dense G_δ-set in X.

In the above form, the universality criterion and the main idea of its poof can be found in [27], Theorem 1. If the family $\{T_\iota : \iota \in I\}$ fulfils the third condition, it is called *topologically transitive*.

Now, we return to the general situation of an open set $D \subset \mathbb{C}$. For a holomorphic self-map f of D, we introduced in Definition 1.3.2 ii) the usual definition of hypercyclicity of the composition operator with symbol f – namely by calling a function $g \in H(D)$ hypercyclic for C_f if the set $\{g \circ f^n : n \in \mathbb{N}\}$ is dense in $H(D)$. This is equivalent to the fact that for each $h \in H(D)$ there exists a strictly increasing sequence (n_k) in $\mathbb{N}$ such that $(g \circ f^{n_k})$ converges to h locally uniformly on D, i.e. each function in $H(D)$ can be approximated locally uniformly on D via subsequences of $(g \circ f^n)$. For many situations which will occur in this work, it is useful to modify this definition of hypercyclicity with regard to the following two aspects:

- On the one hand, we want to obtain hypercyclic functions for C_f which are defined (and holomorphic) on larger open sets than D.
- On the other hand, for hypercyclic functions g for C_f and suitable sets $M \subset D$, we want to approximate all elements of certain subclasses of $C(M)$ on M via subsequences of $\left(g \circ f^n\big|_M\right)$, i.e. we want to approximate classes of functions which are larger than $H(D)$ on sets which are smaller than D.

In order to comply with these two modifications, we now fix a domain $\Omega \subset \mathbb{C}$ with $\Omega \supset D$ and we introduce the following definition:

Definition 1.3.7. Let $f : D \to D$ be holomorphic and let $M \subset D$ be locally compact and σ-compact. Moreover, let $\mathcal{F} \subset C(M)$ be a family of functions in the Fréchet space $C(M)$. A function $g \in H(\Omega)$ is called Ω-$\mathcal{F}$*-universal* for C_f if

$$\mathcal{F} \subset \overline{\left\{g \circ f^n\big|_M : n \in \mathbb{N}\right\}}^{C(M)}.$$

The composition operator C_f is called Ω-$\mathcal{F}$*-universal* if there exists an Ω-$\mathcal{F}$-universal function for C_f. If it is clear from the context which domain Ω is considered, Ω-$\mathcal{F}$-universality will briefly be referred to as $\mathcal{F}$-universality.

Remark 1.3.8.

i) Defining for sets $X, Y \subset \mathbb{C}$ with $Y \subset X$ the restriction map

$$r_{X,Y} : \mathbb{C}^X \to \mathbb{C}^Y, \;\; r_{X,Y}(F) := F\big|_Y,$$

the condition in Definition 1.3.7 can be written as

$$\mathcal{F} \subset \overline{\left\{ \big(r_{D,M} \circ (C_f)^n \circ r_{\Omega,D} \big)(g) : n \in \mathbb{N} \right\}}^{C(M)}.$$

ii) In case of $\left\{ g \circ f^n\big|_M : n \in \mathbb{N} \right\} \subset \mathcal{F}$, the condition in Definition 1.3.7 is equivalent to the fact that the set $\left\{ g \circ f^n\big|_M : n \in \mathbb{N} \right\}$ is dense in $\mathcal{F}$.

iii) For $\Omega = D$, a function $g \in H(D)$ is hypercyclic for C_f if and only if it is $H(D)$-universal for C_f.

We are mainly interested in the following question: For open subsets U of D (compact subsets K of D), we want to find conditions on f and U (on K) such that the composition operator C_f is $H(U)$-universal ($A(K)$-universal). In Chapter 2, we will formulate and prove several statements of this kind.

Subsequently, we will apply these general results to Fatou sets of transcendental entire or rational functions f. More precisely, this means that we will prove $H(U)$-universality and $A(K)$-universality of C_f for suitable open subsets U of F_f and suitable compact subsets K of F_f. This will be the above-mentioned “dramatic change of behaviour” of the sequence of iterates (f^n) composed with some universal function g for C_f because, in this situation, the set of compositions $\{g \circ f^n : n \in \mathbb{N}\}$ will be dense in some space of holomorphic functions, which means that the sequence of compositions $(g \circ f^n)$ will be kind of “maximally divergent”. But this stands in sharp contrast to the fact that the sequence of iterates (f^n) itself converges, behaves like a rotation or is wandering on all components of F_f, as stated in the classification theorem of Fatou components (see Theorem 1.2.7).

1.4 Approximation Theorems

Studying Bernal's and Montes's proof of Theorem 1.3.4 and Grosse-Erdmann's and Mortini's proof of Theorem 1.3.5, one can recognize the following rough outline of how to prove universality of a sequence of composition operators: Using the required injectivity of the symbols and the run-away behaviour of the sequence of symbols, a suitable function is constructed in such a way that it can be approximated uniformly by rational functions having poles only outside a given compact set. This approximation, which is the crucial step of the above-mentioned proofs, is obtained by an application of Runge's theorem on rational approximation. In this work, we will not only need Runge's theorem but also several other approximation theorems in the complex plane, which shall be collected in this section.

In order to formulate these theorems in a short way, we introduce the following notations for compact sets $K \subset \mathbb{C}$ and sets $B \subset \mathbb{C}_\infty \backslash K$:

- $H(K)$ shall be the set of all complex-valued functions on K which can be extended holomorphically to an open neighbourhood of K,
- $R_B(K)$ shall be the closure (with respect to $A(K)$) of the set of all restrictions of rational functions to K having poles only in B,
- $R(K)$ shall be the closure of the set of all restrictions of rational functions to K having poles outside of K, i.e. $R(K) = R_{\mathbb{C}_\infty \backslash K}(K)$,
- $P(K)$ shall be the closure of the set of all restrictions of polynomials to K, i.e. $P(K) = R_{\{\infty\}}(K)$.

The starting point is Runge's theorem on rational and polynomial approximation, which goes back to C. Runge in 1885 (cf. [45]):

Theorem 1.4.1. (Runge's Theorem) *Let $K \subset \mathbb{C}$ be compact.*

i) For any set $B \subset \mathbb{C}_\infty \backslash K$ which contains at least one point from each component of $\mathbb{C}_\infty \backslash K$, we have

$$H(K) \subset R_B(K).$$

ii) If, in particular, $\mathbb{C} \backslash K$ (or equivalently $\mathbb{C}_\infty \backslash K$) is connected, we can choose $B = \{\infty\}$ in i), i.e. then we have

$$H(K) \subset P(K).$$

A proof of this theorem can be found for example in [44], Theorem 13.6.

For a compact subset K of the complex plane, Runge's theorem provides conditions under which complex-valued functions on K, that can be extended holomorphically to an open neighbourhood of K, can be approximated uniformly on K by rational functions or polynomials. The following two theorems give an answer to the question of polynomial and rational approximation of functions which are continuous on K but holomorphic only on the interior of K, i.e. of functions in $A(K)$. These theorems are due to S. N. Mergelyan in 1951 and 1952 (see [35], Theorem on p. 288 and [36], p. 317):

Theorem 1.4.2. (Mergelyan's Theorem on Polynomial Approximation) *Let $K \subset \mathbb{C}$ be compact and suppose that $\mathbb{C}\backslash K$ is connected. Then we have*

$$A(K) = P(K).$$

Theorem 1.4.3. (Mergelyan's Theorem on Rational Approximation) *Let $K \subset \mathbb{C}$ be compact and suppose that there exists some $\delta > 0$ such that all components of $\mathbb{C}\backslash K$ have diameter greater than δ. Then we have*

$$A(K) = R(K).$$

Proofs of Mergelyan's theorems can be found for example in [24], Satz 1 on p. 92 and Satz 4 on p. 110.

Finally, we will need the following generalization of the well-known Weierstrass approximation theorem, which is due to M. H. Stone in 1937 (cf. [50]):

Theorem 1.4.4. (Complex Stone-Weierstrass Theorem) *Let X be a compact topological space and let $\mathcal{A}$ be a subalgebra of $C(X, \mathbb{C})$ such that*

- *$\mathcal{A}$ contains all constant functions,*
- *$\mathcal{A}$ separates points, i.e. for all points $x, y \in X$ with $x \neq y$ there exists an $f \in \mathcal{A}$ with $f(x) \neq f(y)$,*
- *$\mathcal{A}$ is closed under complex conjugation, i.e. $f \in \mathcal{A}$ implies $\overline{f} \in \mathcal{A}$.*

Then $\mathcal{A}$ is dense in $(C(X, \mathbb{C}), \|\cdot\|_X)$, where $\|f\|_X := \max_{x \in X} |f(x)|$ for $f \in C(X, \mathbb{C})$.

A proof of this statement can be found for example in [53], Satz VIII.4.7.

1.5 Holes and Holomorphic Hulls

As we have seen in Section 1.3, topological properties of sets play an important role for obtaining universality of a given sequence of composition operators, i.e. the above-mentioned theorems of Luh, Bernal and Montes as well as Grosse-Erdmann and Mortini hold for sets which have connected complement. Moreover, the statements of Runge's and Mergelyan's approximation theorems in the previous section also depend on topological properties of the considered sets. In the following, the number of bounded components of subsets of the complex plane will be crucial. The following definition and notations will be useful in this context:

Definition 1.5.1. Let $M \subset \mathbb{C}$ be a set.

i) A component H of $\mathbb{C}_\infty \backslash M$ with $\infty \notin H$ is called a *hole* of M.

ii) We denote

$$\begin{aligned}\mathcal{K}(M) &:= \{K \subset M : K \text{ compact}, K \neq \emptyset\}, \\ \mathcal{K}_0(M) &:= \{K \in \mathcal{K}(M) : K \text{ has no holes}\}, \\ \mathcal{U}(M) &:= \{U \subset M : U \text{ open in } (\mathbb{C}, \mathcal{T}_\mathbb{C}),\ U \neq \emptyset\}, \\ \mathcal{U}_0(M) &:= \{U \in \mathcal{U}(M) : U \text{ has no holes}\}.\end{aligned}$$

We remark that a subset M of the complex plane has no holes if and only if $\mathbb{C}_\infty \backslash M$ is connected. In this work, we will often use the fact that for an open set $U \subset \mathbb{C}$ and an injective holomorphic function $\varphi : U \to \mathbb{C}$ the image $\varphi(U)$ has the same number of holes as U. An analogous statement holds if $K \subset \mathbb{C}$ is compact and φ is injective and holomorphic on an open neighbourhood of K, i.e. then K and $\varphi(K)$ have the same number of holes (see [43], p. 276 and cf. [28], p. 360). Whenever we use one of these statements, we call this the *invariance of the number of holes.*

Sometimes, we want to "fill up" holes of compact sets relative to a given open superset. In order to specify what this means, the following definition is important:

Definition 1.5.2. Let $U \subset \mathbb{C}$ be open and let $K \in \mathcal{K}(U)$.

i) The set
$$\widehat{K}_U := \left\{z \in U : |f(z)| \leq ||f||_K \text{ for all } f \in H(U)\right\}$$
is called the *holomorphically convex hull* of K with respect to U.

ii) K is called *U-convex* if $K = \widehat{K}_U$.

iii) If, in particular, $U = \mathbb{C}$, we write $\widehat{K} := \widehat{K}_\mathbb{C}$ and we call this set the *polynomially convex hull* of K.

Remark 1.5.3.

i) $\widehat{K}_U$ is compact (see e.g. [43], p. 263).

ii) By definition, the equality $\|f\|_K = \|f\|_{\widehat{K}_U}$ holds for all $f \in H(U)$. Hence, we obtain $\widehat{(\widehat{K}_U)}_U = \widehat{K}_U$ and thus that $\widehat{K}_U$ is U-convex.

iii) It can be shown that $\widehat{K}_U$ is the union of K with all holes of K which are contained in U (cf. [43], p. 264 or [16], p. 202).

iv) Using part iii), we see that K is U-convex if and only if each hole of K contains a point in $\mathbb{C}\backslash U$. Moreover, this is equivalent to the fact that each hole of K contains a hole of U. Thus, $\widehat{K}_U$ is the smallest compact superset of K in U having the property that each of its holes contains a hole of U (cf. [43], p. 264).

v) For the standard compact exhaustion (K_n) of U, it is well-known that each set K_n is U-convex (see e.g. [14], Theorem 1.31 (3)). Hence, if U has no holes, each set K_n has no holes as well.

vi) By an application of Runge's theorem, one can show that

$$\widehat{K} = \big\{z \in \mathbb{C} : |P(z)| \leq ||P||_K \text{ for all polynomials } P\big\}$$

(see also [43], p. 266).

vii) Denoting by C_K the component of $\mathbb{C}_\infty\backslash K$ which contains ∞, part iii) yields $\widehat{K} = \mathbb{C}_\infty\backslash C_K$. As C_K is connected, we obtain that $\widehat{K}$ has no holes.

viii) If $U \in \mathcal{U}_0(\mathbb{C})$, we have $\widehat{K} \subset U$ and hence $\widehat{K} \in \mathcal{K}_0(U)$. Indeed, as U has no holes, $\mathbb{C}_\infty\backslash U$ is a connected subset of $\mathbb{C}_\infty\backslash K$ so that there exists a component V of $\mathbb{C}_\infty\backslash K$ with $\mathbb{C}_\infty\backslash U \subset V$. This implies $\infty \in V$ and hence $V = C_K$. Thus, we obtain $\widehat{K} = \mathbb{C}_\infty\backslash C_K \subset U$.

Chapter 2

General Universality Statements for Composition Operators with Holomorphic Symbol

Throughout this chapter, we fix an open set $D \subset \mathbb{C}$, a domain $\Omega \subset \mathbb{C}$ with $\Omega \supset D$ and a holomorphic function $f : D \to D$.

As already mentioned in Section 1.3, our aim will be the following: For open subsets U of D and compact subsets K of D, we want to find conditions on f and U (and on K, respectively) such that the composition operator C_f is $H(U)$-universal ($A(K)$-universal).

2.1 $H(U)$-Universality

In view of the above-mentioned hypercyclicity result of Grosse-Erdmann and Mortini (see Theorem 1.3.5), the following definition is reasonable:

Definition 2.1.1. We denote by $\mathcal{U}_0(D, \Omega, f)$ the set of all sets $U \in \mathcal{U}_0(D)$ for which the following two conditions hold:

i) The sequence of iterates (f^n) fulfils $f^n\big|_U \to \partial_\infty \Omega$ locally uniformly.

ii) The restrictions $f^n\big|_U$ are injective for all $n \in \mathbb{N}$.

Here, the first condition corresponds to the run-away property of the sequence of iterates (f^n) as defined by Bernal and Montes (see Definition 1.3.3) and as stated in Theorem 1.3.5. At first view, the second condition seems to be stronger than the condition of injectivity of the symbol of the composition operator in Grosse-Erdmann's and Mortini's theorem. However, in the situation of Theorem 1.3.5, the symbol is a self-map of the considered domain so that its injectivity implies the injectivity of all of its iterates, which is crucial for the proof of Theorem 1.3.5.

Now, we can state and prove the following universality result, which will be the basis for several further universality statements. The proof will run similarly to the proof of Theorem 3.2, (c)$\Longrightarrow$(a), in [28].

Theorem 2.1.2. *Let $U \in \mathcal{U}_0(D,\Omega,f)$. Then the set of all functions in $H(\Omega)$ which are $H(U)$-universal for C_f is a comeager set in $H(\Omega)$.*

Proof: For $n \in \mathbb{N}$, we consider the composition operators

$$C_{f^n,U} : H(\Omega) \to H(U), \;\; C_{f^n,U}(g) := g \circ f^n\big|_U.$$

According to the universality criterion (see Theorem 1.3.6), it suffices to show that the sequence $(C_{f^n,U})$ is topologically transitive. In order to do so, let $\emptyset \neq \mathcal{V} \subset H(\Omega)$ and $\emptyset \neq \mathcal{W} \subset H(U)$ be open. For $g \in \mathcal{V}$, there exist $\varepsilon_1 > 0$ and $K_1 \in \mathcal{K}(\Omega)$ such that $V_{\varepsilon_1,K_1,\Omega}(g) \subset \mathcal{V}$ (cf. Remark 1.1.3 ii)). Defining $K := \widehat{(K_1)}_\Omega$, Remark 1.5.3 ii) implies that K is Ω-convex with

$$V_{\varepsilon_1,K,\Omega}(g) = V_{\varepsilon_1,K_1,\Omega,}(g) \subset \mathcal{V}.$$

For $h \in \mathcal{W}$, there exist $\varepsilon_2 > 0$ and $L_1 \in \mathcal{K}(U)$ with $V_{\varepsilon_2,L_1,U}(h) \subset \mathcal{W}$. Considering the polynomially convex hull $L := \widehat{L_1}$, Remark 1.5.3 viii) yields $L \in \mathcal{K}_0(U)$, and we have

$$V_{\varepsilon_2,L,U}(h) = V_{\varepsilon_2,L_1,U}(h) \subset \mathcal{W}.$$

Putting $\varepsilon := \min(\varepsilon_1, \varepsilon_2) > 0$, we obtain

$$V_{\varepsilon,K,\Omega}(g) \subset V_{\varepsilon_1,K,\Omega}(g) \subset \mathcal{V} \quad \text{and} \quad V_{\varepsilon,L,U}(h) \subset V_{\varepsilon_2,L,U}(h) \subset \mathcal{W}.$$

Now, let $\delta := \operatorname{dist}(K, \partial_\infty\Omega) > 0$. Because of $U \in \mathcal{U}_0(D,\Omega,f)$, we obtain the uniform convergence $f^n\big|_L \to \partial_\infty\Omega$. Thus, there exists an $N \in \mathbb{N}$ with $\operatorname{dist}(f^N(z), \partial_\infty\Omega) < \delta$ for all $z \in L$, which implies

$$K \cap f^N(L) = \emptyset.$$

According to $U \in \mathcal{U}_0(D,\Omega,f)$, the restriction $f^N\big|_U$ is injective. Hence, as L has no holes, the invariance of the number of holes implies that $f^N(L)$ has no holes as well.

Therefore, the disjoint union of the compact set K with the compact set $f^N(L)$ does not produce a new hole so that the Ω-convexity of K implies that $K \cup f^N(L)$ is Ω-convex again (cf. Remark 1.5.3 iv)). Thus, we can choose from each hole of $K \cup f^N(L)$ a point which lies in $\mathbb{C}\backslash\Omega$. Let A be the union of these points and let $B := A \cup \{\infty\}$. Then we have $B \subset \mathbb{C}_\infty\backslash\Omega \subset \mathbb{C}_\infty\backslash(K \cup f^N(L))$ and $B \cap C \neq \emptyset$ for all components C of $\mathbb{C}_\infty\backslash(K \cup f^N(L))$. We now consider the function

$$\varphi : K \cup f^N(L) \to \mathbb{C}, \;\; \varphi(z) := \begin{cases} g(z), & \text{if } z \in K \\ h\big((f^N\big|_U)^{-1}(z)\big), & \text{if } z \in f^N(L) \end{cases} .$$

As the disjoint sets K and $f^N(L)$ are compact with $K \subset \Omega$ and $f^N(L) \subset f^N(U)$ and as we have $g \in H(\Omega)$ as well as $h \in H(U)$, we see that φ can be extended holomorphically to an open neighbourhood of $K \cup f^N(L)$. Hence, Runge's theorem yields a rational function R having poles only in $B \subset \mathbb{C}_\infty\backslash\Omega$ such that

$$\|\varphi - R\|_{K\cup f^N(L)} < \varepsilon.$$

The restriction $R\big|_\Omega$ is holomorphic on Ω, and due to $\varphi = g$ on K we obtain

$$\|g - R\|_K = \|\varphi - R\|_K \leq \|\varphi - R\|_{K\cup f^N(L)} < \varepsilon$$

and thus $R\big|_\Omega \in V_{\varepsilon,K,\Omega}(g) \subset \mathcal{V}$. Because of $\varphi = h \circ (f^N\big|_U)^{-1}$ on $f^N(L)$, we further obtain

$$\begin{aligned} \left\|h - C_{f^N,U}\left(R\big|_\Omega\right)\right\|_L &= \left\|h - R \circ f^N\right\|_L = \left\|h \circ (f^N\big|_U)^{-1} - R\right\|_{f^N(L)} \\ &= \|\varphi - R\|_{f^N(L)} \leq \|\varphi - R\|_{K\cup f^N(L)} < \varepsilon \end{aligned}$$

and therefore $C_{f^N,U}\left(R\big|_\Omega\right) \in V_{\varepsilon,L,U}(h) \subset \mathcal{W}$. Altogether, it follows

$$C_{f^N,U}\left(R\big|_\Omega\right) \in C_{f^N,U}(\mathcal{V}) \cap \mathcal{W} \neq \emptyset,$$

so that the topological transitivity of the sequence $(C_{f^n,U})$ is shown. □

Remark 2.1.3.

i) The difference between the above theorem and the result of Grosse-Erdmann and Mortini (Theorem 1.3.5) is the following: Applying Theorem 2.1.2, we obtain $H(U)$-universal functions for C_f which are actually defined and holomorphic on the "large" domain Ω – and not only on the set U itself which is contained in the open set D on which the symbol f is defined.

ii) Going through the proof of Theorem 2.1.2, one can check that both conditions in Definition 2.1.1 can be weakened in the following way: For $U \in \mathcal{U}_0(D, \Omega, f)$, it suffices that there exists a subsequence (f^{n_k}) of (f^n) with $f^{n_k}\big|_U \to \partial_\infty \Omega$ locally uniformly and that the restrictions $f^n\big|_U$ are eventually injective. However, in all situations which we will consider in this work, we will have locally uniform convergence $f^n\big|_U \to \partial_\infty \Omega$ and all restrictions $f^n\big|_U$ will be injective. Hence, there is no advantage in using this weakened definition, which is inconvenient to handle. Therefore, we remain with the two conditions for sets $U \in \mathcal{U}_0(D, \Omega, f)$ as stated in Definition 2.1.1.

Now, it is our aim to extend Theorem 2.1.2 in the following way: For $D_0 \subset D$ open, we want to find conditions such that the composition operator C_f is $H(U)$-universal for all sets $U \in \mathcal{U}_0(D_0)$. This intention originates from the above-mentioned second result of Bernal and Montes (see Theorem 1.3.4 ii)). In view of Theorem 2.1.2, it is natural to require the existence of a sequence in $\mathcal{U}_0(D, \Omega, f)$, which fulfils the following property in D_0:

Definition 2.1.4. Let $V \subset \mathbb{C}$ be open. A sequence (M_n) of subsets of V is called $\mathcal{K}_0(V)$*-exhausting* if for each set $L \in \mathcal{K}_0(V)$ there exists an $N \in \mathbb{N}$ with $L \subset M_N^\circ$.

Remark 2.1.5. Each open set $V \subset \mathbb{C}$ has a $\mathcal{K}_0(V)$-exhausting sequence in $\mathcal{U}_0(V)$. Indeed, Bernal and Montes have shown that each domain $G \subset \mathbb{C}$ has a $\mathcal{K}_0(G)$-exhausting sequence in $\mathcal{K}_0(G)$ (see [13], Lemma 2.9). Considering their proof of this statement, one can verify that it also holds for arbitrary open subsets of the complex plane. Hence, there exists a $\mathcal{K}_0(V)$-exhausting sequence (K_n) in $\mathcal{K}_0(V)$. As we can find for each $n \in \mathbb{N}$ a set $U_n \in \mathcal{U}_0(V)$ with $U_n \supset K_n$ (see e.g. [26], Satz 2.2.4 and the first part of its proof), we obtain that (U_n) is a $\mathcal{K}_0(V)$-exhausting sequence in $\mathcal{U}_0(V)$.

Now, we can state and prove the following universality result:

Corollary 2.1.6. *Let $D_0 \subset D$ be open and let there exist a $\mathcal{K}_0(D_0)$-exhausting sequence in $\mathcal{U}_0(D, \Omega, f)$. Then the set of all functions in $H(\Omega)$ which are $H(U)$-universal for C_f for all $U \in \mathcal{U}_0(D_0)$ is a comeager set in $H(\Omega)$.*

Proof: Let (U_m) be a $\mathcal{K}_0(D_0)$-exhausting sequence in $\mathcal{U}_0(D, \Omega, f)$. Moreover, for $m \in \mathbb{N}$, let $\mathcal{G}_m$ be the set of all functions in $H(\Omega)$ which are $H(U_m)$-universal for

C_f. Due to Theorem 2.1.2, each set $\mathcal{G}_m$ is comeager in $H(\Omega)$ so that the same is also true for the countable intersection

$$\mathcal{G} := \bigcap_{m\in\mathbb{N}} \mathcal{G}_m.$$

Let $\mathcal{H}$ be the set of all functions in $H(\Omega)$ which are $H(U)$-universal for C_f for all $U \in \mathcal{U}_0(D_0)$. We now show the inclusion $\mathcal{G} \subset \mathcal{H}$, which will complete the proof. For this purpose, let $g \in \mathcal{G}$ and $U \in \mathcal{U}_0(D_0)$. As we have to show the denseness of the set $\{g \circ f^n|_U : n \in \mathbb{N}\}$ in $H(U)$, let moreover $h \in H(U), K \in \mathcal{K}(U)$ and $\varepsilon > 0$. Because of $U \in \mathcal{U}_0(D_0) \subset \mathcal{U}_0(\mathbb{C})$, Remark 1.5.3 viii) yields $\widehat{K} \in \mathcal{K}_0(U) \subset \mathcal{K}_0(D_0)$. Runge's theorem implies the existence of a polynomial P with $\|P - h\|_{\widehat{K}} < \varepsilon/2$. As (U_m) is $\mathcal{K}_0(D_0)$-exhausting, there exists an $M \in \mathbb{N}$ with $\widehat{K} \subset U_M$. Due to $g \in \mathcal{G} \subset \mathcal{G}_M$ and $P|_{U_M} \in H(U_M)$, we obtain an $N \in \mathbb{N}$ with

$$\left\| g \circ f^N|_{U_M} - P \right\|_{\widehat{K}} < \varepsilon/2.$$

Altogether, we have

$$\left\| g \circ f^N|_U - h \right\|_K \leq \left\| g \circ f^N|_{U_M} - P \right\|_{\widehat{K}} + \|P - h\|_{\widehat{K}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, we have shown that $\{g \circ f^n|_U : n \in \mathbb{N}\}$ is dense in $H(U)$, which means that g is $H(U)$-universal for C_f. Therefore, we obtain $g \in \mathcal{H}$ so that $\mathcal{G} \subset \mathcal{H}$ is shown. □

In particular, observing Remark 2.1.5, the assumptions of Corollary 2.1.6 are fulfilled for $D_0 = D$ if f is injective and if $f^n|_D \to \partial_\infty\Omega$ locally uniformly. Thus, we obtain:

Corollary 2.1.7. *Let f be injective and let $f^n \to \partial_\infty\Omega$ locally uniformly on D. Then the set of all functions in $H(\Omega)$ which are $H(U)$-universal for C_f for all $U \in \mathcal{U}_0(D)$ is a comeager set in $H(\Omega)$.*

In the special case that D has no holes, we obtain the following result:

Corollary 2.1.8. *Let $D \in \mathcal{U}_0(\mathbb{C})$, let f be injective and let $f^n \to \partial_\infty\Omega$ locally uniformly on D. Then the set of all functions in $H(\Omega)$ which are $H(D)$-universal for C_f is a comeager set in $H(\Omega)$.*

2.2 $A(K)$-Universality

Now, we want to find conditions such that the composition operator C_f is $A(K)$-universal for compact subsets K of D. We start with the following lemma, which relates $H(U)$-universality of C_f for all $U \in \mathcal{U}_0(D)$ with $A(K)$-universality of C_f for all $K \in \mathcal{K}_0(D)$. For proving that the first implies the latter, we will need Mergelyan's theorem on polynomial approximation.

Lemma 2.2.1. *Let $D_0 \subset D$ be open and let $g \in H(\Omega)$. Then the following are equivalent:*

i) g is $H(U)$-universal for C_f for all $U \in \mathcal{U}_0(D_0)$.

ii) g is $A(K)$-universal for C_f for all $K \in \mathcal{K}_0(D_0)$.

Proof: In order to show that i) implies ii), let g be $H(U)$-universal for C_f for all $U \in \mathcal{U}_0(D_0)$ and let $K \in \mathcal{K}_0(D_0)$. As we have to show the denseness of the set $\left\{g \circ f^n\big|_K : n \in \mathbb{N}\right\}$ in $A(K)$, let moreover $h \in A(K)$ and $\varepsilon > 0$. According to Mergelyan's theorem on polynomial approximation, there exists a polynomial P with $\|P - h\|_K < \varepsilon/2$. Moreover, we can find some $U_0 \in \mathcal{U}_0(D_0)$ with $K \subset U_0$ (see [26], Satz 2.2.4). As g is $H(U_0)$-universal for C_f, there exists an $N \in \mathbb{N}$ with $\|g \circ f^N\big|_{U_0} - P\|_K < \varepsilon/2$. This yields

$$\left\|g \circ f^N\big|_K - h\right\|_K \leq \left\|g \circ f^N\big|_K - P\right\|_K + \|P - h\|_K < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, we have shown that $\left\{g \circ f^n\big|_K : n \in \mathbb{N}\right\}$ is dense in $A(K)$, which means that g is $A(K)$-universal for C_f.

For proving that ii) implies i), let g be $A(K)$-universal for C_f for all $K \in \mathcal{K}_0(D_0)$ and let $U \in \mathcal{U}_0(D_0)$. As we have to show the denseness of the set $\left\{g \circ f^n\big|_U : n \in \mathbb{N}\right\}$ in $H(U)$, let moreover $h \in H(U), L \in \mathcal{K}(U)$ and $\varepsilon > 0$. Remark 1.5.3 viii) implies $K_0 := \widehat{L} \in \mathcal{K}_0(U) \subset \mathcal{K}_0(D_0)$. As g is $A(K_0)$-universal for C_f, there exists an $N \in \mathbb{N}$ with

$$\varepsilon > \left\|g \circ f^N\big|_{K_0} - h\big|_{K_0}\right\|_{K_0} \geq \left\|g \circ f^N\big|_U - h\right\|_L.$$

Thus, we have shown that $\left\{g \circ f^n\big|_U : n \in \mathbb{N}\right\}$ is dense in $H(U)$, which means that g is $H(U)$-universal for C_f. □

Later on, we will be interested in the question under what conditions on a single set $K \in \mathcal{K}_0(D)$ we can obtain $A(K)$-universality of C_f. Similarly to the case of open sets above, now the following definition is reasonable:

Definition 2.2.2. We denote by $\mathcal{K}_0(D,\Omega,f)$ the set of all sets $K \in \mathcal{K}_0(D)$ for which the following two conditions hold:

i) The sequence of iterates (f^n) fulfils $f^n\big|_K \to \partial_\infty\Omega$ uniformly.

ii) For all $n \in \mathbb{N}$ there exists an open neighbourhood $U_n \subset D$ of K such that the restriction $f^n\big|_{U_n}$ is injective.

Before stating a universality result for sets in $\mathcal{K}_0(D,\Omega,f)$, we observe the following relation between the existence of $\mathcal{K}_0(D)$-exhausting sequences in $\mathcal{U}_0(D,\Omega,f)$ and in $\mathcal{K}_0(D,\Omega,f)$, respectively.

Lemma 2.2.3. *Let $D_0 \subset D$ be open. Then the following are equivalent:*

i) There exists a $\mathcal{K}_0(D_0)$-exhausting sequence in $\mathcal{U}_0(D,\Omega,f)$.

ii) There exists a $\mathcal{K}_0(D_0)$-exhausting sequence in $\mathcal{K}_0(D,\Omega,f)$.

Proof: Let us first assume that (U_n) is a $\mathcal{K}_0(D_0)$-exhausting sequence in $\mathcal{U}_0(D,\Omega,f)$. As already mentioned above, there always exists a $\mathcal{K}_0(D_0)$-exhausting sequence (K_n) in $\mathcal{K}_0(D_0) \subset \mathcal{K}_0(D)$ (see Remark 2.1.5). Moreover, as (U_n) is $\mathcal{K}_0(D_0)$-exhausting, there exists for each $n \in \mathbb{N}$ an $m(n) \in \mathbb{N}$ with $K_n \subset U_{m(n)}$. Now, let $N \in \mathbb{N}$ be fixed. Due to $U_{m(N)} \in \mathcal{U}_0(D,\Omega,f)$, we have locally uniform convergence $f^n\big|_{U_{m(N)}} \to \partial_\infty\Omega$ and injectivity of all restrictions $f^n\big|_{U_{m(N)}}$, which implies $K_N \in \mathcal{K}_0(D,\Omega,f)$. Thus, (K_n) is a $\mathcal{K}_0(D_0)$-exhausting sequence in $\mathcal{K}_0(D,\Omega,f)$.
For proving the reverse implication, let (K_n) be a $\mathcal{K}_0(D_0)$-exhausting sequence in $\mathcal{K}_0(D,\Omega,f)$ and let us fix $n_0 \in \mathbb{N}$. As the compact set $K_{n_0} \subset D_0$ has no holes, there exists a bounded set $U_{n_0} \in \mathcal{U}_0(D_0) \subset \mathcal{U}_0(D)$ such that $K_{n_0} \subset U_{n_0} \subset \overline{U_{n_0}} \subset D_0$ and such that $\mathbb{C}_\infty \backslash \overline{U_{n_0}}$ is connected (see [26], Satz 2.2.4). Thus, we have $\overline{U_{n_0}} \in \mathcal{K}_0(D_0)$. As (K_n) is $\mathcal{K}_0(D_0)$-exhausting, there exists an $N \in \mathbb{N}$ with $\overline{U_{n_0}} \subset K_N$. Due to K_N in $\mathcal{K}_0(D,\Omega,f)$, we have uniform convergence $f^n\big|_{K_N} \to \partial_\infty\Omega$ and for all $n \in \mathbb{N}$ there exists an open neighbourhood $V_n \subset D$ of K_N such that the restriction $f^n\big|_{V_n}$ is injective. Because of $U_{n_0} \subset \overline{U_{n_0}} \subset K_N$, this implies in particular $U_{n_0} \in \mathcal{U}_0(D,\Omega,f)$. Hence, we obtain for all $n \in \mathbb{N}$ a set $U_n \in \mathcal{U}_0(D,\Omega,f)$ with $K_n \subset U_n$ so that (U_n) is a $\mathcal{K}_0(D_0)$-exhausting sequence in $\mathcal{U}_0(D,\Omega,f)$. □

Applying the previous two lemmas, we can rewrite Corollary 2.1.6 equivalently to:

Corollary 2.2.4. *Let $D_0 \subset D$ be open and let there exist a $\mathcal{K}_0(D_0)$-exhausting sequence in $\mathcal{K}_0(D,\Omega,f)$. Then the set of all functions in $H(\Omega)$ which are $A(K)$-universal for C_f for all $K \in \mathcal{K}_0(D_0)$ is a comeager set in $H(\Omega)$.*

Now, we state and prove – similarly to Theorem 2.1.2 – the following universality result for a given compact subset of D. The proof will run analogously to the proof of Theorem 2.1.2 until defining the auxiliary function φ. But now, in addition to Runge's theorem, we will also need Mergelyan's theorem on rational approximation.

Theorem 2.2.5. *Let $K \in \mathcal{K}_0(D, \Omega, f)$. Then the set of all functions in $H(\Omega)$ which are $A(K)$-universal for C_f is a comeager set in $H(\Omega)$.*

Proof: For $n \in \mathbb{N}$, we consider the composition operators

$$C_{f^n,K} : H(\Omega) \to A(K), \;\; C_{f^n,K}(g) := g \circ f^n\big|_K.$$

According to the universality criterion, it suffices to show that the sequence $(C_{f^n,K})$ is topologically transitive. Let $\emptyset \neq \mathcal{V} \subset H(\Omega)$ and $\emptyset \neq \mathcal{W} \subset A(K)$ be open. As in the proof of Theorem 2.1.2, there exist functions $g \in \mathcal{V}$ and $h \in \mathcal{W}$, an Ω-convex subset L of Ω and an $\varepsilon > 0$ with

$$V_{\varepsilon,L,\Omega}(g) \subset \mathcal{V} \quad \text{and} \quad U_{\varepsilon,K}(h) \subset \mathcal{W}.$$

Due to $K \in \mathcal{K}_0(D, \Omega, f)$, we can now find an $N \in \mathbb{N}$ and an open neighbourhood $U \subset D$ of K such that

$$L \cap f^N(K) = \emptyset$$

and such that $f^N\big|_U$ is injective. We now consider the function

$$\varphi : L \cup f^N(K) \to \mathbb{C}, \;\; \varphi(z) := \begin{cases} g(z), & \text{if } z \in L \\ h\big((f^N\big|_U)^{-1}(z)\big), & \text{if } z \in f^N(K) \end{cases}.$$

Lemma A.1 yields $(f^N(K))^\circ = f^N(K^\circ)$ so that we obtain $\varphi \in A(L \cup f^N(K))$. Moreover, the invariance of the number of holes implies that $f^N(K)$ has no holes. Hence, as in the proof of Theorem 2.1.2, the union $L \cup f^n(K)$ is Ω-convex. As Ω-convex sets can only have a finite number of holes (see e.g. [28], Lemma 3.10), Mergelyan's theorem on rational approximation implies the existence of a rational function S having poles only in $\mathbb{C}_\infty \backslash (L \cup f^N(K))$ such that $\|\varphi - S\|_{L \cup f^N(K)} < \varepsilon/2$. Because of $S\big|_{L \cup f^N(K)} \in H(L \cup f^N(K))$ and analogously to the proof of Theorem 2.1.2, Runge's theorem and the Ω-convexity of $L \cup f^N(K)$ now yield a rational function R having poles only in $\mathbb{C}_\infty \backslash \Omega$ such that $\|S - R\|_{L \cup f^N(K)} < \varepsilon/2$. The restriction $R\big|_\Omega$ is holomorphic on Ω and we have

$$\|\varphi - R\|_{L \cup f^N(K)} \leq \|\varphi - S\|_{L \cup f^N(K)} + \|S - R\|_{L \cup f^N(K)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Similarly to the proof of Theorem 2.1.2, we finally obtain

$$C_{f^N,K}\left(R\big|_\Omega\right) \in C_{f^N,K}(\mathcal{V}) \cap \mathcal{W} \neq \emptyset,$$

so that the topological transitivity of the sequence $(C_{f^n,K})$ is shown. □

Remark 2.2.6.

i) The following consideration shows that Theorem 2.2.5 implies Theorem 2.1.2. Indeed, let $U \in \mathcal{U}_0(D,\Omega,f)$ and let $\mathcal{G}$ be the set of all functions in $H(\Omega)$ which are $H(U)$-universal for C_f. Moreover, let (K_n) be the standard compact exhaustion of U. As U has no holes, Remark 1.5.3 v) yields $K_n \in \mathcal{K}_0(D)$ for all $n \in \mathbb{N}$. Hence, $U \in \mathcal{U}_0(D,\Omega,f)$ implies $K_n \in \mathcal{K}_0(D,\Omega,f)$ for all $n \in \mathbb{N}$. Thus, we obtain from Theorem 2.2.5 that

$$\mathcal{R} := \bigcap_{n\in\mathbb{N}} \left\{g \in H(\Omega) : g\ A(K_n)\text{-universal for } C_f\right\}$$

is comeager in $H(\Omega)$. Now, let $g \in \mathcal{R}$ and let $h \in H(U)$, $K \in \mathcal{K}(U)$ and $\varepsilon > 0$. Then there exists an $n_0 \in \mathbb{N}$ with $K \subset K_{n_0}$ so that $h\big|_{K_{n_0}} \in A(K_{n_0})$ implies the existence of an $N \in \mathbb{N}$ with

$$\varepsilon > \left\| g \circ f^N\big|_{K_{n_0}} - h\big|_{K_{n_0}} \right\|_{K_{n_0}} \geq \left\| g \circ f^N\big|_U - h \right\|_K .$$

Hence, the set $\left\{g \circ f^n\big|_U : n \in \mathbb{N}\right\}$ is dense in $H(U)$, which means that we have $g \in \mathcal{G}$. Thus, we obtain that $\mathcal{G} \supset \mathcal{R}$ is comeager in $H(\Omega)$.

ii) In spite of the consideration in part i), we have proved Theorem 2.1.2 at the beginning of Section 2.1 independently of Theorem 2.2.5. The reason for choosing this approach is the fact that we were able to prove Theorem 2.1.2 by an application of Runge's theorem – in contrast to the fact the we had to apply Mergelyan's theorem, which is much stronger than Runge's theorem, in order to prove Theorem 2.2.5.

iii) Theorem 2.1.2 and Mergelyan's theorem do not imply Theorem 2.2.5. The reason for this is the following: Although we can find for $K \in \mathcal{K}_0(D,\Omega,f)$ a set $U \in \mathcal{U}_0(D)$ with $K \subset U$ (see [26], Satz 2.2.4), we cannot guarantee that we have $U \in \mathcal{U}_0(D,\Omega,f)$ in this situation. Therefore, we cannot apply Theorem 2.1.2, and for this reason we had to modify the proof of Theorem 2.1.2 in order to prove Theorem 2.2.5.

2.3 $C(E)$-Universality

In the proof of Theorem 2.2.5, we used twice that f^N is not only injective on K but actually on an open neighbourhood U of K. On the one hand, we used this fact for obtaining the equality $(f^N(K))^\circ = f^N(K^\circ)$ by Lemma A.1, which implied $\varphi \in A(L \cup f^N(K))$. On the other hand, we used the injectivity of f^N on U in order to apply the invariance of the number of holes so that we could ensure that $f^N(K)$ had no holes, which implied the Ω-convexity of $L \cup f^N(K)$.

If we consider a finite set $E \subset D$ instead of a set $K \in \mathcal{K}_0(D, \Omega, f)$ in the situation of Theorem 2.2.5, we can drop the assumption that the iterates f^n have to be injective on open neighbourhoods of E. In this case, it suffices to require that all iterates of f are injective on E in order to prove an analogous statement. Indeed, this is true because then the finite set $f^N(E)$ has neither any interior points nor any holes so that the condition $\varphi \in A(L \cup f^N(E))$ as well as the Ω-convexity of $L \cup f^N(E)$ are guaranteed. Therefore, the following definition is reasonable:

Definition 2.3.1. We denote by $\mathcal{E}(D, \Omega, f)$ the set of all finite subsets E of D for which the following two conditions hold:

i) The sequence of iterates (f^n) fulfils $f^n\big|_E \to \partial_\infty \Omega$.

ii) The restrictions $f^n\big|_E$ are injective for all $n \in \mathbb{N}$.

Observing the above consideration and the equality $A(E) = C(E)$ $(= \mathbb{C}^E)$ for finite sets $E \subset \mathbb{C}$, the same approach as in the proof of Theorem 2.2.5 now yields the following result:

Corollary 2.3.2. *Let $E \in \mathcal{E}(D, \Omega, f)$. Then the set of all functions in $H(\Omega)$ which are $C(E)$-universal for C_f is a comeager set in $H(\Omega)$.*

Chapter 3

Applications: Local Theory

In this chapter, we want to state some first applications of the main results of the previous chapter to the theory of complex dynamics. These applications will all be of local nature, meaning that we will prove universality of composition operators with locally defined symbols which are holomorphic near a fixed point. In view of the classification theorem of Fatou components, we will consider attracting, neutral and superattracting fixed points of the symbols – nevertheless, the results in this chapter do not depend on results of the Fatou-Julia theory, i.e. here we consider symbols which only need to be defined locally near a fixed point.

In order to proceed as requested, we now fix a complex-valued function f which shall be holomorphic on an open neighbourhood of a point $z_0 \in \mathbb{C}$ with $f(z_0) = z_0$.

3.1 Attracting Fixed Points

In this section, we consider the case that z_0 is an attracting fixed point of f, i.e. for $\lambda := f'(z_0)$ we have $0 < |\lambda| < 1$. The main consideration that we need now is the concept of conformal conjugation. Due to G. Kœnigs' linearization theorem, which goes back to the year 1884, we know that there exist open neighbourhoods U of z_0 and V of 0 as well as a conformal map $\varphi : U \to V$ which conjugates the map $f\big|_U : U \to U$ to the linear function $F : V \to V$, $F(w) := \lambda w$, i.e.

$$\varphi \circ f^n = F^n \circ \varphi = \lambda^n \cdot \varphi$$

holds on U for all $n \in \mathbb{N}$ (see e.g. [15], Theorem II.2.1). In particular, we obtain $\varphi(z_0) = \varphi(f(z_0)) = \lambda\varphi(z_0)$ and hence $\varphi(z_0) = 0$. For $K \in \mathcal{K}(U)$ and $w \in \varphi(K)$, we have $F^n(w) = \lambda^n w \to 0$, and this convergence is uniform on the compact set $\varphi(K)$.

Due to Lemma A.2, this implies the uniform convergence $\varphi^{-1} \circ F^n \to \varphi^{-1}(0) = z_0$ on $\varphi(K)$. Therefore, we obtain

$$\|f^n - z_0\|_K = \|\varphi^{-1} \circ F^n \circ \varphi - z_0\|_K = \|\varphi^{-1} \circ F^n - z_0\|_{\varphi(K)} \to 0,$$

which means that we have locally uniform convergence $f^n \to z_0$ on U. Now, we can formulate and prove the following universality result:

Theorem 3.1.1. *In the above situation, the set of all functions in* $H(\mathbb{C}\backslash\{z_0\})$ *which are* $H(W)$*-universal for* C_f *for all* $W \in \mathcal{U}_0(U\backslash\{z_0\})$ *is a comeager set in* $H(\mathbb{C}\backslash\{z_0\})$*.*

Proof: We consider the domain $\Omega := \mathbb{C}\backslash\{z_0\}$ and the open set $D := U\backslash\{z_0\} \subset \Omega$. As F, φ and φ^{-1} are injective, the same is also true for $f\big|_U = \varphi^{-1} \circ F \circ \varphi$. Hence, the restriction $f\big|_D : D \to D$ is injective. Due to the locally uniform convergence $f^n\big|_D \to z_0 \in \partial_\infty\Omega$, the assertion now follows from Corollary 2.1.7. □

Remark 3.1.2. In the situation of Theorem 3.1.1, we denote by $\mathcal{G}$ the set of all functions in $H(\mathbb{C}\backslash\{z_0\})$ which are $H(W)$-universal for C_f for all $W \in \mathcal{U}_0(U\backslash\{z_0\})$. Then the following observations are true:

i) Due to Lemma 2.2.1, the set of all functions in $H(\mathbb{C}\backslash\{z_0\})$ which are $A(K)$-universal for C_f for all $K \in \mathcal{K}_0(U\backslash\{z_0\})$ is equal to $\mathcal{G}$ and hence also comeager in $H(\mathbb{C}\backslash\{z_0\})$.

ii) Each function $g \in \mathcal{G}$ must have an essential singularity at z_0. Indeed, assuming that there exists some $g \in \mathcal{G}$ which has a removable singularity at z_0, there would exist a constant $C > 0$ and some $\delta > 0$ with $U_\delta(z_0) \subset U$ and $\|g\|_{U_\delta(z_0)\backslash\{z_0\}} \leq C$. Then, for an arbitrary set $K \in \mathcal{K}_0(U\backslash\{z_0\})$, the locally uniform convergence $f^n\big|_U \to z_0$ would yield an $N \in \mathbb{N}$ such that we have $f^n(K) \subset U_\delta(z_0)\backslash\{z_0\}$ for all $n \geq N$. Thus, we would obtain $|g(f^n(z))| \leq C$ for all $z \in K$ and for all $n \geq N$. But this contradicts the fact that the set $\left\{g \circ f^n\big|_K : n \geq N\right\}$ is dense in $A(K)$ due to part i) (observe that the space $A(K)$ has no isolated points). The assumption that g has a pole at z_0 analogously leads to a contradiction because in this case there would exist some $\rho > 0$ with $U_\rho(z_0) \subset U$ and $\|g\|_{U_\rho(z_0)\backslash\{z_0\}} \geq 1$.

iii) As the open set $U\backslash\{z_0\}$ has a hole (namely $\{z_0\}$), one can show that there does not exist a function in $H(U\backslash\{z_0\})$ which is $H(U\backslash\{z_0\})$-universal for C_f (cf. the Remark in [13] on p. 55 and observe that $f\big|_{U\backslash\{z_0\}}$ is conjugated to the linear function $w \mapsto \lambda w$ on $V\backslash\{0\}$).

3.2 Neutral Fixed Points

In this section, we consider the case that z_0 is a neutral fixed point of f, i.e. we have $f'(z_0) = 1$. Moreover, we require that f is not the identity map. Then, for $m := \min\{n \in \mathbb{N} : f^{(n+1)}(z_0) \neq 0\}$ and $a := f^{(m+1)}(z_0)/(m+1)!$, we have $m \in \mathbb{N}$ and $a \in \mathbb{C}\backslash\{0\}$, and f is given near z_0 by

$$\begin{aligned} f(z) &= z_0 + (z - z_0) + a(z - z_0)^{m+1} + \cdots \\ &= z + a(z - z_0)^{m+1} + \cdots . \end{aligned}$$

Example 3.2.1. As a special case, we consider for $m \in \mathbb{N}$ and $a \in \mathbb{C}\backslash\{0\}$ the polynomial

$$p : \mathbb{C} \to \mathbb{C}, \;\; p(z) := z + a(z - z_0)^{m+1}$$

(compare [1], p. 10). First, let $v \in \partial\mathbb{D}$ with $av^m \in \mathbb{R}^+$. Then, for $c > 0$, we have

$$p(z_0 + vc) = z_0 + vc + a(vc)^{m+1} = z_0 + v(1 + c^m av^m)c = z_0 + vc_1,$$

where $c_1 := (1 + c^m av^m)c > c$. Hence, the ray $z_0 + v\mathbb{R}^+$ is invariant under p and it is repelled from the neutral fixed point z_0. On the other hand, now let $w \in \partial\mathbb{D}$ with $aw^m \in \mathbb{R}^-$. Then, for $0 < c < |a|^{-1/m}$, we analogously obtain

$$p(z_0 + wc) = z_0 + wc_2,$$

where $c_2 := (1 + c^m aw^m)c$ now fulfils $0 < c_2 < c$. Therefore, the line segment $z_0 + w\{x \in \mathbb{R} : 0 < x < |a|^{-1/m}\}$ is invariant under p and it is attracted by z_0.

Returning to the general case described above, the previous example leads us to the following definition:

Definition 3.2.2. A unit vector $v \in \partial\mathbb{D}$ is called an *attracting direction* for f at z_0 if $av^m \in \mathbb{R}^-$, and it is called a *repelling direction* for f at z_0 if $av^m \in \mathbb{R}^+$.

Now, let $\theta \in \mathbb{R}$ such that $v := e^{i\theta}$ is an attracting direction for f at z_0. Then we have $av^m \in \mathbb{R}^-$ and $|av^m| = |a|$. This yields $av^m = -|a|$ and hence $v^m = -|a|/a$. Putting $\alpha := \arg a$, we obtain

$$e^{i\theta m} = v^m = -\frac{|a|}{a} = -\frac{|a|}{|a|e^{i\alpha}} = -e^{-i\alpha} = e^{i(\pi-\alpha)}.$$

This is exactly the case if there exists an integer k with $\theta m = \pi - \alpha + 2k\pi$. Hence, there are exactly m equally spaced attracting directions $e^{i\theta_1}, \dots, e^{i\theta_m}$ for f at z_0 which are given by

$$\theta_k := \frac{(2k+1)\pi - \alpha}{m}, \quad k = 1, \dots, m.$$

A similar calculation shows that there exist also m equally spaced repelling directions $e^{i\sigma_1}, \dots, e^{i\sigma_m}$ for f at z_0 which are given by

$$\sigma_k := \frac{2k\pi - \alpha}{m}, \quad k = 1, \dots, m$$

(cf. [1], p. 10 or [15], p. 40). Because of $f'(z_0) = 1 \neq 0$, the inverse function f^{-1} exists locally near z_0. Observing the equality $(f^{-1})'(z_0) = 1/f'(f^{-1}(z_0)) = 1/f'(z_0) = 1$, Faà di Bruno's formula for calculating higher derivatives of compositions of two functions implies $(f^{-1})^{(k)}(z_0) = 0$ for all $k \in \{2, \dots, m\}$ and

$$(f^{-1})^{(m+1)}(z_0) = -f^{(m+1)}(z_0).$$

Therefore, the repelling (attracting) directions for f at z_0 are exactly the attracting (repelling) directions for f^{-1} at z_0 (cf. [1], p. 10).

In order to describe the dynamics of f locally near the neutral fixed point z_0, we now fix an open neighbourhood U_f of z_0 such that $f\big|_{U_f}$ is injective.

Definition 3.2.3. Let $v \in \partial\mathbb{D}$ be an attracting direction for f at z_0.

i) Let $z \in U_f \cap f(U_f)$ be a point such that all iterates $f^n(z)$ are defined and not equal to z_0. We say that the sequence of iterates $(f^n(z))$ *tends to* z_0 *in direction* v if $f^n(z) \to z_0$ and $f^n(z)/|f^n(z)| \to v$.

ii) A set $P \in \mathcal{U}_0(U_f \cap f(U_f))$ is called an *attracting petal* for f in direction v at z_0 if it is invariant under f and if the following two conditions are fulfilled:

- $f^n\big|_P \to z_0$ uniformly.
- For each $z \in U_f \cap f(U_f)$, the sequence $(f^n(z))$ tends to z_0 in direction v if and only if $O_f^+(z) \cap P \neq \emptyset$.

iii) Let $w \in \partial\mathbb{D}$ be a repelling direction for f at z_0. A set $Q \in \mathcal{U}_0(U_f \cap f(U_f))$ is called a *repelling petal* for f in direction w at z_0 if Q is an attracting petal for f^{-1} in direction w at z_0.

By definition, z_0 is a boundary point of each attracting or repelling petal for f. Moreover, if f is a transcendental entire function or a rational function of degree $d \geq 2$, then each attracting petal for f is contained in F_f.

Now, we will apply the Leau-Fatou flower theorem, which goes back to the years 1897 (Leau) and 1919–1920 (Fatou), in order to describe the local behaviour of f near the neutral fixed point z_0 (see e.g. [1], Theorem 3.6, [37], Theorem 10.5 or [49], p. 75). This theorem states that for each attracting direction $e^{i\theta_k}$ (for each repelling direction $e^{i\sigma_k}$) for f at z_0 there exists an attracting petal P_k (a repelling petal Q_k) for f in direction $e^{i\theta_k}$ (in direction $e^{i\sigma_k}$) at z_0 such that the union of these $2m$ petals, together with z_0, forms an open neighbourhood of z_0. Moreover, the petals can be chosen in such a way that they are arranged cyclically around z_0 so that two petals intersect if and only if the angle between their corresponding attracting (repelling) directions is π/m. Furthermore (see e.g. [49], p. 74), for each $k \in \{1, \ldots, m\}$, the attracting petal P_k can be chosen such that it is bounded by a piecewise analytic Jordan curve which is contained in the sector

$$\{z \in \mathbb{C} : \sigma_k < \arg(z - z_0) < \sigma_{k+1}\}$$

(where $\sigma_{m+1} := \sigma_1$) and which is symmetric about the ray $\{z \in \mathbb{C} : \arg(z - z_0) = \theta_k\}$ and has the two rays $\{z \in \mathbb{C} : \arg(z - z_0) = \sigma_k\}$ and $\{z \in \mathbb{C} : \arg(z - z_0) = \sigma_{k+1}\}$ as tangents at z_0. As repelling petals for f are attracting petals for f^{-1}, this symmetry also holds for all repelling petals Q_k, where now the roles of the angles θ_k and σ_k are interchanged. The following figure, which is a slightly modified version of Figure 19 in [37] on p. 106, displays the Leau-Fatou flower in the case $m = 3$ and $\alpha = 0$:

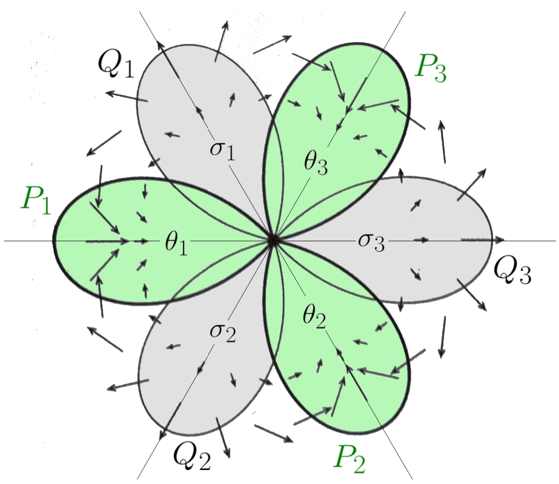

Figure 3.1

In case of $m = 1$, each attracting petal P_1 of the only existing attracting direction $e^{i\theta_1}$ for f at z_0 looks like a cardioid (cf. the following figure, which is a slightly modified version of Figure 1 in [15] on p. 37 with $z_0 = 0$):

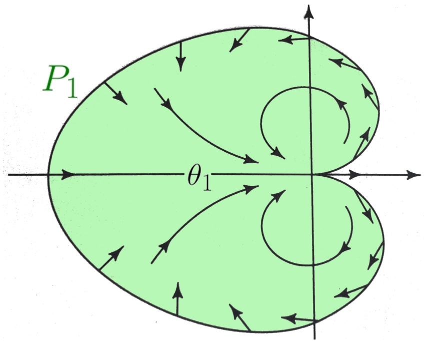

Figure 3.2

Finally, for each $k \in \{1, \dots, m\}$, the Leau-Fatou flower theorem yields a constant $c_k > 0$, a domain $V_k \in \mathcal{U}_0(\mathbb{C})$ containing the right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > c_k\}$ and a conformal map $\varphi_k : P_k \to V_k$ which conjugates $f\big|_{P_k} : P_k \to P_k$ to the translation $F_k : V_k \to V_k$, $F_k(w) := w + 1$, i.e.

$$\varphi_k \circ f^n = F_k^n \circ \varphi_k = \varphi_k + n$$

holds on P_k for all $n \in \mathbb{N}$ (see e.g. [1], Theorem 3.6 or [37], Theorem 10.7). Defining $P := \bigcup_{k=1}^m P_k$ to be the union of all attracting petals, we can now formulate and prove the following universality result:

Theorem 3.2.4. *In the above situation, the set $\mathcal{G}$ of all functions in $H(\mathbb{C}\backslash\{z_0\})$ which are $H(P)$-universal for C_f is a comeager set in $H(\mathbb{C}\backslash\{z_0\})$.*

Proof: We consider the domain $\Omega := \mathbb{C}\backslash\{z_0\}$ and the open set $D := P \in \mathcal{U}_0(\Omega)$. First, we show that $f\big|_D : D \to D$ is injective. To this end, let $z, w \in D$ with $f(z) = f(w)$. Then there exist $k, l \in \{1, \dots, m\}$ with $z \in P_k$ and $w \in P_l$, and the invariance of P_k and P_l under f implies $f(z) \in P_k \cap P_l$. As all attracting petals are pairwise disjoint, we obtain $k = l$ so that the injectivity of $f\big|_{P_k} = \varphi_k^{-1} \circ (\varphi_k + 1)$ yields $z = w$. Due to the uniform convergence $f^n\big|_D \to z_0 \in \partial_\infty \Omega$, the assertion now follows from Corollary 2.1.8. □

Remark 3.2.5. The set $\widetilde{\mathcal{G}}$ of all functions in $H(\mathbb{C}\backslash\{z_0\})$ which are $H(P_k)$-universal for C_f for all $k \in \{1, \dots, m\}$ is a superset of $\mathcal{G}$ and hence also comeager in $H(\mathbb{C}\backslash\{z_0\})$. Indeed, for $g \in \mathcal{G}$, the set $\{g \circ f^n\big|_P : n \in \mathbb{N}\}$ is dense in $H(P)$. Now, let $k \in \{1, \dots, m\}$ and consider $h \in H(P_k), K \in \mathcal{K}(P_k)$ and $\varepsilon > 0$. As P_k has no holes, Remark 1.5.3 viii) yields $\widehat{K} \in \mathcal{K}_0(P_k)$. Hence, Runge's theorem implies the existence of a polynomial q with $\|q - h\|_{\widehat{K}} < \varepsilon/2$. Because of $q\big|_P \in H(P)$ and $K \in \mathcal{K}(P)$, we obtain an $N \in \mathbb{N}$ such that $\|g \circ f^N\big|_P - q\|_K < \varepsilon/2$ and hence

$$\left\|g \circ f^N\big|_{P_k} - h\right\|_K \leq \left\|g \circ f^N\big|_P - q\right\|_K + \|q - h\|_{\widehat{K}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, each set $\left\{g \circ f^n\big|_{P_k} : n \in \mathbb{N}\right\}$ is dense in $H(P_k)$, which means $g \in \widetilde{\mathcal{G}}$.

In all situations in this work, we will guarantee the existence of universal functions for composition operators by an application of Theorem 2.1.2, Theorem 2.2.5 or of one of their corollaries. The proofs of Theorems 2.1.2 and 2.2.5 are based on the universality criterion which is formulated for families of continuous functions on Baire spaces (see Theorem 1.3.6). In fact, the existence of "universal objects" in the literature is usually proved by an application of the Baire category theorem or via a countable inductive process (cf. [27], p. 370 or see the two different proofs of the "Hypercyclicity Criterion" in [6] on p. 5). Hence, in general, there is only little information on "individual" universal objects – with at least one exception. In 1975, S. Voronin has shown that the Riemann zeta function $\zeta \in H(\mathbb{C}\backslash\{1\})$ fulfils a special kind of "translation universality" on the right half of the critical strip $\{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$. Defining for open sets $U \subset \mathbb{C}$ and compact sets $K \subset \mathbb{C}$ the sets

$$\begin{aligned} H_{\neq 0}(U) &:= \{h \in H(U) : h(z) \neq 0 \text{ for all } z \in U\}, \\ A_{\neq 0}(K) &:= \{h \in A(K) : h(z) \neq 0 \text{ for all } z \in K\}, \end{aligned}$$

Voronin proved that for each $0 < r < 1/4$, each function $h \in A_{\neq 0}(U_r[0])$ and each $\varepsilon > 0$, there exists some $t_0 \in \mathbb{R}$ such that

$$\left\|\zeta\left(\bullet + \frac{3}{4} + it_0\right) - h\right\|_{U_r[0]} < \varepsilon$$

(see the Theorem in [52] on p. 443). In the sequel, Voronin's result has been improved, and one version, which is due to A. Laurinčikas in 1995, reads as follows: Denoting by $S := \{z \in \mathbb{C} : 1/2 < \operatorname{Re} z < 1\}$ the right half of the critical strip, Laurinčikas proved that for all $K \in \mathcal{K}_0(S)$, $h \in A_{\neq 0}(K)$ and $\varepsilon > 0$ we have

$$\liminf_{\tau \to \infty} \frac{1}{\tau} \lambda_1\Big(\Big\{t \in [0, \tau] : \|\zeta(\bullet + it) - h\|_K < \varepsilon\Big\}\Big) > 0,$$

where λ_1 stands for the one-dimensional Lebesgue measure (see [33], Theorem B).

Moreover, in 1980, A. Reich has proved that the values of the real number t_0 in Voronin's theorem may be chosen from a fixed arithmetic progression $(n\Delta)_{n\in\mathbb{N}}$ for an arbitrary $\Delta > 0$ (see [42], Satz 3.1). In particular, we obtain that for all $K \in \mathcal{K}_0(S)$, $h \in A_{\neq 0}(K)$ and $\varepsilon > 0$, there exists an integer $N \in \mathbb{N}$ with $\|\zeta(\bullet + iN) - h\|_K < \varepsilon$. Thus, we have

$$A_{\neq 0}(K) \subset \overline{\left\{\zeta(\bullet + in)\big|_K : n \in \mathbb{N}\right\}}^{A(K)}$$

for all $K \in \mathcal{K}_0(S)$. Defining the translation $T : S \to S$, $T(w) := w + i$, this means that ζ is $\mathbb{C}\backslash\{1\}$-$A_{\neq 0}(K)$-universal for the composition operator C_T for all $K \in \mathcal{K}_0(S)$.

We now return to the general situation of Theorem 3.2.4 considering an attracting petal P_k for f. Due to the above-stated translation universality of ζ and according to the fact that $f\big|_{P_k}$ is conjugated to the translation $w \mapsto w + 1$, it is now our aim to construct a function which is P_k-$H_{\neq 0}(P_{k,0})$-universal for C_f for a suitable open subset $P_{k,0}$ of P_k with $z_0 \in \partial P_{k,0}$. In order to do so, we define the horizontal strip

$$W := \{z \in \mathbb{C} : -1 < \operatorname{Im} z < -1/2\} = -iS$$

and the function

$$Z : \mathbb{C}\backslash\{-i\} \to \mathbb{C}, \ \ Z(s) := \zeta(is),$$

which is holomorphic on $\mathbb{C}\backslash\{-i\}$. Now, let $K \in \mathcal{K}_0(W)$, $h \in A_{\neq 0}(K)$ and $\varepsilon > 0$. Then the rotated set iK fulfils $iK \in \mathcal{K}_0(iW) = \mathcal{K}_0(S)$ and the map

$$\widetilde{h} : iK \to \mathbb{C}, \ \ \widetilde{h}(z) := h(-iz),$$

is contained in $A_{\neq 0}(iK)$. Considering the translation $F : W \to W$, $F(w) := w + 1$, the $\mathbb{C}\backslash\{1\}$-$A_{\neq 0}(iK)$-universality of ζ for C_T yields an $N \in \mathbb{N}$ such that we have

$$\begin{aligned} \varepsilon &> \left\|\zeta(\bullet + iN) - \widetilde{h}\right\|_{iK} = \left\|\zeta(i\,\bullet + iN) - \widetilde{h}(i\,\bullet)\right\|_K \\ &= \|Z(\bullet + N) - h\|_K = \left\|Z \circ F^N - h\right\|_K . \end{aligned}$$

Hence, Z is $\mathbb{C}\backslash\{-i\}$-$A_{\neq 0}(K)$-universal for C_F for all $K \in \mathcal{K}_0(W)$. Analogously to the proof of Lemma 2.2.1, this is equivalent to the fact that Z is $\mathbb{C}\backslash\{-i\}$-$H_{\neq 0}(U)$-universal for C_F for all $U \in \mathcal{U}_0(W)$ (compare [42], Corollar 4.1).

We remark that the existence of a set $U \in \mathcal{U}_0(W)$ having the property that Z is $H(U)$-universal for C_F would contradict the Riemann hypothesis. Indeed, assuming this to be true, we consider a function $h \in H(U)$ which has a zero in U but is not constant zero on U. According to the $H(U)$-universality of Z for C_F, there would exist a strictly increasing sequence (n_j) in $\mathbb{N}$ with $Z \circ F^{n_j} \to h$ locally uniformly on U. Thus, Hurwitz's theorem would imply that the functions $Z \circ F^{n_j}$ have zeros in U for large values of j. In particular, there would be a point $w_0 \in U \subset W$ and an integer $j_0 \in \mathbb{N}$ such that $0 = Z(F^{n_{j_0}}(w_0)) = Z(w_0 + n_{j_0}) = \zeta(iw_0 + in_{j_0})$. Due to $iw_0 \in iW = S$, this would imply that $iw_0 + in_{j_0} \in S$ is a zero of ζ, contradicting the Riemann hypothesis (compare [42], p. 450).

Now, we consider the open set $P_{k,0} := \varphi_k^{-1}(W \cap V_k)$, which is the image of the intersection of the horizontal strip W and the simply connected domain V_k containing the right half-plane $\{z \in \mathbb{C} : \operatorname{Re} z > c_k\}$ under the inverse of the map φ_k (which conjugates $f\big|_{P_k}$ to the translation $V_k \ni w \mapsto w + 1 = F_k(w)$). As $V_k \cap W$ has no holes, we obtain that $P_{k,0} \in \mathcal{U}_0(P_k)$. Moreover, as $\big(\varphi_k^{-1}(c_k + n + 1 - 3i/4)\big)$ is a sequence in $P_{k,0}$ and due to the conjugation $\varphi_k \circ f = F_k \circ \varphi_k$ on P_k and the convergence $f^n\big|_{P_k} \to z_0$, we have

$$\begin{aligned} & \varphi_k^{-1}\left(c_k + n + 1 - \frac{3}{4}i\right) = \varphi_k^{-1}\left(F_k^n\left(c_k + 1 - \frac{3}{4}i\right)\right) \\ = \; & f^n\left(\varphi_k^{-1}\left(c_k + 1 - \frac{3}{4}i\right)\right) \to z_0. \end{aligned}$$

Therefore, we obtain $z_0 \in \overline{P_{k,0}}$ and hence $z_0 \in \partial P_{k,0}$. Without loss of generality, let us now assume that $-i \notin V_k$ so that we have $Z \circ \varphi_k \in H(P_k)$. Now, we can state the following universality result concerning the existence of an individual universal function for C_f:

Theorem 3.2.6. *In the above situation, $Z \circ \varphi_k$ is P_k-$H_{\neq 0}(P_{k,0})$-universal for C_f.*

Proof: As we have $W \cap V_k \in \mathcal{U}_0(W)$, the above consideration yields that Z is $H_{\neq 0}(W \cap V_k)$-universal for C_F. Now, let $h \in H_{\neq 0}(P_{k,0})$, $K \in \mathcal{K}(P_{k,0})$ and $\varepsilon > 0$. Then we have $h \circ \varphi_k^{-1}\big|_{W \cap V_k} \in H_{\neq 0}(W \cap V_k)$ and $\varphi_k(K) \in \mathcal{K}(W \cap V_k)$. Hence, there exists an $N \in \mathbb{N}$ such that

$$\begin{aligned} \varepsilon \; > \; & \left\| Z \circ F^N\big|_{W \cap V_k} - h \circ \varphi_k^{-1}\big|_{W \cap V_k} \right\|_{\varphi_k(K)} = \left\| Z \circ F_k^N - h \circ \varphi_k^{-1} \right\|_{\varphi_k(K)} \\ = \; & \left\| Z \circ \varphi_k \circ f^N \circ \varphi_k^{-1} - h \circ \varphi_k^{-1} \right\|_{\varphi_k(K)} = \left\| (Z \circ \varphi_k) \circ f^N\big|_{P_{k,0}} - h \right\|_K . \end{aligned}$$

Thus, we have shown that $H_{\neq 0}(P_{k,0}) \subset \overline{\left\{(Z \circ \varphi_k) \circ f^n\big|_{P_{k,0}} : n \in \mathbb{N}\right\}}^{H(P_{k,0})}$. □

If we had some information about the conjugation map φ_k, Theorem 3.2.6 would imply that $Z \circ \varphi_k$ is an "explicit" universal function for the composition operator C_f. As the existence of φ_k can be proved constructively (see e.g. [1], proof of Theorem 3.6 or [37], proofs of Lemma 10.1 and Theorem 10.7), this is indeed the case. In the following concluding example, we will have a look at the construction of the conjugation map in case of a very simple function having a neutral fixed point at the origin.

Example 3.2.7. We consider the polynomial $p : \mathbb{C} \to \mathbb{C}$, $p(z) := z - z^2$, which has a neutral fixed point at $z_0 = 0$ (cf. Example 3.2.1 with $m = 1$ and $a = -1$). In the following, we will adapt Abate's proof of the Leau-Fatou flower theorem (cf. [1], Theorem 3.6) in order to construct a map which conjugates p on an attracting petal of the only existing attracting direction $v = 1$ for p at the origin to the translation $w \mapsto w + 1$ (see also [7], Example 6.5.3, for a construction of an attracting petal). For $\delta > 0$, we define the open disc $P_\delta := \{z \in \mathbb{C} : |z - \delta| < \delta\}$, and we define the conformal map

$$\psi : \mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}, \;\; \psi(z) := \frac{1}{z}.$$

Then we have $\psi^{-1} = \psi$ and $\psi(P_\delta) = \{w \in \mathbb{C} : \operatorname{Re} w > 1/(2\delta)\} =: H_\delta$. Putting

$$F : \mathbb{C}\backslash\overline{\mathbb{D}} \to \mathbb{C}\backslash\{0\}, \;\; F := \psi \circ p \circ \psi^{-1},$$

we obtain $F \in H(\mathbb{C}\backslash\overline{\mathbb{D}})$, and for all $w \in \mathbb{C}\backslash\overline{\mathbb{D}}$ it follows that

$$\begin{aligned} F(w) &= \psi(p(\psi^{-1}(w))) = \psi\left(p\left(1/w\right)\right) = \frac{1}{p\left(1/w\right)} = \frac{1}{1/w - 1/w^2} \\ &= \frac{w^2}{w-1} = \frac{w^2 - 1 + 1}{w-1} = \frac{(w+1)(w-1)+1}{w-1} = w + 1 + \frac{1}{w-1}. \end{aligned}$$

Following Abate's proof in [1], we can choose constants $R, C > 0$ such that we have $|F(w) - w - 1| \leq C/|w|$ for all $|w| > R$ (this is fulfilled here e.g. for $R := C := 2$). Moreover, there exist $\varepsilon \in (0, 1)$ and $\delta > 0$ with $4\delta < 1/2$ and $16\delta < \varepsilon$ (this is fulfilled here e.g. for $\varepsilon := 1/2$ and $\delta := 1/48$). Putting $M_\varepsilon := (1 + \varepsilon)/(2\delta)$, the set

$$\begin{aligned} H &:= \{w \in \mathbb{C} : |\operatorname{Im} w| > -\varepsilon \operatorname{Re} w + M_\varepsilon\} \cup H_\delta \\ &= \{w \in \mathbb{C} : |\operatorname{Im} w| > -0.5 \cdot \operatorname{Re} w + 36\} \cup \{w \in \mathbb{C} : \operatorname{Re} w > 24\} \end{aligned}$$

is invariant under F and $P := \psi^{-1}(H)$ is an attracting petal for p in direction 1 at the origin (see [1], proof of Theorem 3.6). The following figure shows a MATLAB plot of the sets H and P:

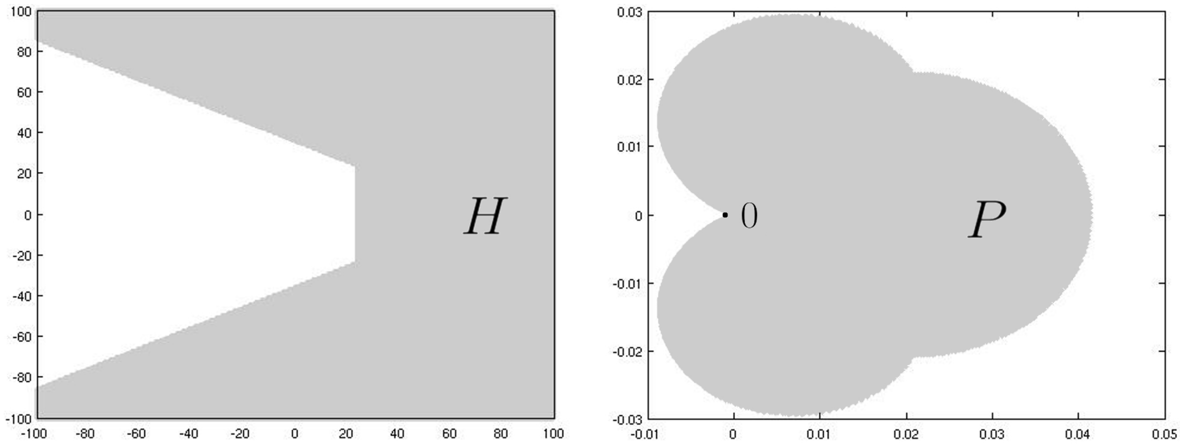

Figure 3.3

We now fix some point $w_0 \in H$, e.g. $w_0 := 25$, and for $k \in \mathbb{N}$, we define

$$\widetilde{\varphi_k} : H \to \mathbb{C}, \;\; \widetilde{\varphi_k}(w) := F^k(w) - F^k(w_0).$$

According to Abate's proof in [1], there exists an injective function $\widetilde{\varphi} : H \to \mathbb{C}$ such that $\widetilde{\varphi_k} \to \widetilde{\varphi}$ holds locally uniformly on H and such that $\widetilde{\varphi}(H)$ contains a right half-plane. Moreover, $\widetilde{\varphi}$ fulfils the equation $\widetilde{\varphi} \circ F = \widetilde{\varphi} + 1$ on H. Therefore, the map

$$\varphi : P \to \widetilde{\varphi}(H), \;\; \varphi := \widetilde{\varphi} \circ \psi\big|_P$$

conformally conjugates p on P to the translation $w \mapsto w + 1$. For each $z \in P$, the value of z under φ is given by

$$\begin{aligned}\varphi(z) &= \widetilde{\varphi}(\psi(z)) = \widetilde{\varphi}(1/z) = \lim_{k\to\infty} \widetilde{\varphi_k}(1/z) = \lim_{k\to\infty} F^k(1/z) - F^k(w_0) \\ &= \lim_{k\to\infty} \left(\left(w \mapsto w + 1 + \frac{1}{w-1} \right)^k \bigg|_{w=1/z} - \left(w \mapsto w + 1 + \frac{1}{w-1} \right)^k \bigg|_{w=w_0} \right).\end{aligned}$$

3.3 Superattracting Fixed Points

In this section, we consider the case that z_0 is a superattracting fixed point of f, i.e. we have $f'(z_0) = 0$. Moreover, we require that f is not constant. Then, for $p := \min\{n \in \mathbb{N} : f^{(n)}(z_0) \neq 0\}$, we have $p \in \mathbb{N}$ with $p \geq 2$, and f is given near z_0 by

$$f(z) = z_0 + \frac{f^{(p)}(z_0)}{p!}(z - z_0)^p + \cdots .$$

Again, the concept of conformal conjugation will be the main consideration now. Due to Böttcher's theorem, which goes back to the year 1904, we know that there exist open neighbourhoods U of z_0 and V of 0 as well as a conformal map $\varphi : U \to V$ which conjugates $f\big|_U : U \to U$ to the p-th monomial Q on V, i.e.

$$\varphi(f^n(z)) = Q^n(\varphi(z)) = (\varphi(z))^{p^n}$$

holds for all $z \in U$ and for all $n \in \mathbb{N}$ (see e.g. [15], Theorem II.4.1). Without loss of generality, we can assume $V \subset \mathbb{D}$ so that the equality $\varphi(z_0) = \varphi(f(z_0)) = \varphi(z_0)^p$ implies $\varphi(z_0) = 0$. Thus, analogously to the considerations before Theorem 3.1.1, we obtain locally uniform convergence $f^n \to z_0$ on U.

Similarly to the previous two sections, it is now our aim to formulate and prove a universality result for the composition operator C_f which holds locally near z_0. The problem compared to the situations of attracting or neutral fixed points is the following: As we have $f(z) = \varphi^{-1}\big((\varphi(z))^p\big)$ for all $z \in U$, we see that the symbol f now is not injective on U. Hence, we cannot apply Corollaries 2.1.7 or 2.1.8 for proving $H(W)$-universality of C_f for suitable sets $W \in \mathcal{U}_0(U\backslash\{z_0\})$. However, for $n \in \mathbb{N}$, the p^n-th monomial is injective on each angular sector of the complex plane with arc length smaller than $2\pi/p^n$. In view of the proof of Theorem 2.1.2, we need injectivity of all iterates of f in order to obtain universality of C_f (cf. also the considerations on p. 54 below). Because of $2\pi/p^n \to 0$ and due to the above conjugation, this means that for sets $M \subset U\backslash\{z_0\}$ and families $\mathcal{F} \subset C(M)$, we only have a chance of obtaining $\mathcal{F}$-universality of C_f if M has empty interior (cf. Remark 3.3.6 iii) below). It will be our aim to prove that C_f is $A(K)$-universal for suitable compact subsets K of $U\backslash\{z_0\}$ with $K^\circ = \emptyset$. As compensation for only considering such "small" subsets, we want to obtain "many" of them.

In order to specify what this means, we consider for $\delta > 0$ with $U_\delta[0] \subset V$ the set

$$B_\delta := \varphi^{-1}(U_\delta[0]),$$

which is a compact neighbourhood of z_0 that is contained in U. According to Lemma A.1, we have $\varphi(B_\delta^\circ) = \varphi(B_\delta)^\circ = U_\delta(0)$ so that

$$\varphi\big|_{B_\delta^\circ\backslash\{z_0\}} : B_\delta^\circ\backslash\{z_0\} \to U_\delta(0)\backslash\{0\}$$

is a conformal map. In particular, we obtain $f(B_\delta^\circ\backslash\{z_0\}) \subset B_\delta^\circ\backslash\{z_0\}$. Endowing the set $\mathcal{K}(B_\delta)$ with the Hausdorff distance $d_{\mathcal{K}(B_\delta)}$, we obtain the complete metric space

$\big(\mathcal{K}(B_\delta), d_{\mathcal{K}(B_\delta)}\big)$ (cf. Definition B.1 and Remark B.2 in Appendix B). In the following, we want to show that C_f is $A(K)$-universal for comeager many $K \in \mathcal{K}(B_\delta)$.

The starting point is the following universality result, which holds for finite sets:

Corollary 3.3.1. *Let $E \subset B_\delta^\circ \backslash \{z_0\}$ be a finite set such that all iterates f^n are injective on E. Then the set of all functions in $H(\mathbb{C}\backslash\{z_0\})$ which are $C(E)$-universal for C_f is a comeager set in $H(\mathbb{C}\backslash\{z_0\})$.*

Proof: We consider the domain $\Omega := \mathbb{C}\backslash\{z_0\}$ and the open set $D := B_\delta^\circ\backslash\{z_0\} \subset \Omega$. Due to the injectivity of all iterates $f^n\big|_E$ and the convergence $f^n\big|_E \to z_0 \in \partial_\infty\Omega$, we obtain $E \in \mathcal{E}(D, \Omega, f)$ so that the assertion now follows from Corollary 2.3.2. □

In order to extend the universality statement of Corollary 3.3.1 to comeager many compact subsets of $B_\delta^\circ\backslash\{z_0\}$, we denote $A_\delta := (B_\delta^\circ\backslash\{z_0\}) \cap (\mathbb{Q}+i\mathbb{Q})$ and we introduce the countable set

$$\mathcal{E}_\delta := \left\{E \subset A_\delta : E \neq \emptyset,\ E \text{ finite},\ f^n\big|_E \text{ injective for all } n \in \mathbb{N}\right\},$$

for which the following holds:

Lemma 3.3.2. *Let $\emptyset \neq E \subset B_\delta$ be finite and let $\varepsilon > 0$. Then there exists a set $\widetilde{E} \in \mathcal{E}_\delta$ with $d_{\mathcal{K}(B_\delta)}(E, \widetilde{E}) < \varepsilon$.*

Proof: We will prove the lemma by induction over $m = |E|$.

i) First, let $E = \{z_1\}$ for a point $z_1 \in B_\delta$. As A_δ is dense in B_δ, there exists some $\widetilde{z_1} \in A_\delta$ with $|z_1 - \widetilde{z_1}| < \varepsilon$ so that the set $\widetilde{E} := \{\widetilde{z_1}\}$ is as desired.

ii) Now, let the assertion be true for all m-element subsets of B_δ and consider an $(m+1)$-element set $E \subset B_\delta$ of the form $E = \{z_1, \ldots, z_m, z_{m+1}\}$. Due to the injectivity of φ, we may assume without loss of generality that $\varphi(z_{m+1}) \neq 0$. Denoting $F := \{z_1, \ldots, z_m\}$, the induction hypothesis yields a set $\widetilde{F} \in \mathcal{E}_\delta$ with $d_{\mathcal{K}(B_\delta)}(F, \widetilde{F}) < \varepsilon$. We write $\widetilde{F} = \{\widetilde{z_1}, \ldots, \widetilde{z_M}\}$ for an integer $M \in \mathbb{N}$ and points $\widetilde{z_1}, \ldots, \widetilde{z_M} \in A_\delta$.

iii) We show that all iterates of the p-th monomial Q are injective on the set $\varphi(\widetilde{F}) = \{\varphi(\widetilde{z_1}), \ldots, \varphi(\widetilde{z_M})\}$. In order to do so, let $n \in \mathbb{N}$ and $k, l \in \{1, \ldots, M\}$ with $Q^n(\varphi(\widetilde{z_k})) = Q^n(\varphi(\widetilde{z_l}))$. Then we have $\varphi(f^n(\widetilde{z_k})) = \varphi(f^n(\widetilde{z_l}))$ and the injectivity of φ implies $f^n(\widetilde{z_k}) = f^n(\widetilde{z_l})$. Because of $\widetilde{F} \in \mathcal{E}_\delta$, it follows that $\widetilde{z_k} = \widetilde{z_l}$ so that we obtain $\varphi(\widetilde{z_k}) = \varphi(\widetilde{z_l})$.

iv) As φ^{-1} is holomorphic on V and as $U_\delta[0]$ is a convex compact subset of V, the map $\varphi^{-1}\big|_{U_\delta[0]} : U_\delta[0] \to B_\delta$ is Lipschitz. Now, let c be a Lipschitz constant of this function. As the set $\varphi(\widetilde{F}) \subset U_\delta(0)\backslash\{0\}$ is finite, there exists some $0 < \rho_0 < \min(1, \varepsilon/c)$ such that we have

$$(*) \qquad |\varphi(\widetilde{z_k})| \neq (1-\rho)|\varphi(z_{m+1})|$$

for all $0 < \rho < \rho_0$ and for each $k \in \{1, \ldots, M\}$. We consider the point $v_0 := (1 - \rho_0/2) \cdot \varphi(z_{m+1})$ and the radius $r := (\rho_0/2) \cdot |\varphi(z_{m+1})| > 0$ (cf. the following figure in case of $M = 5$).

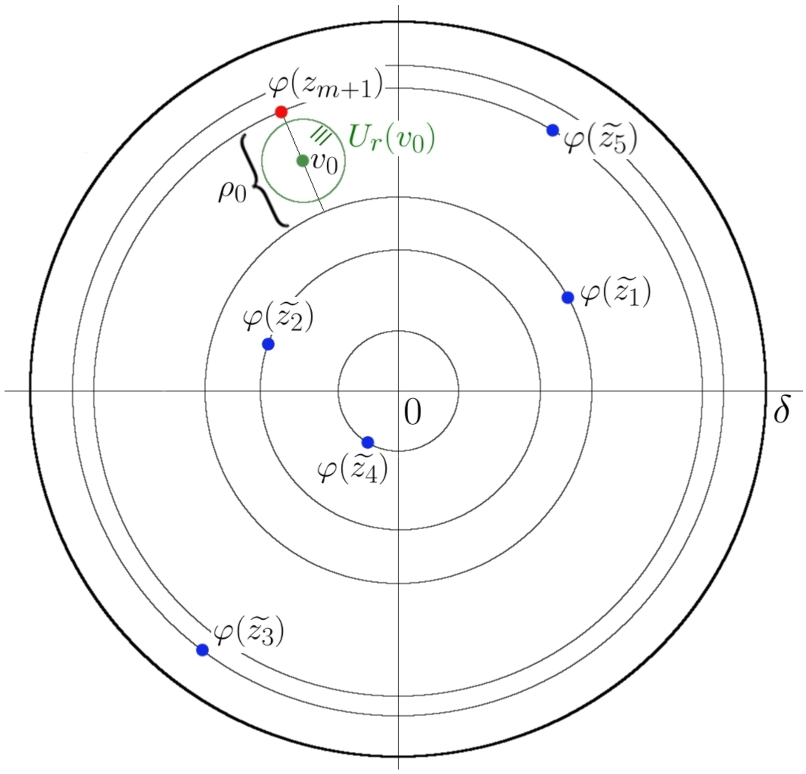

Figure 3.4

As $\mathbb{Q}+i\mathbb{Q}$ is dense in $\mathbb{C}$, there exists a point $\widetilde{v_0} \in U_r(v_0)$ with $\varphi^{-1}(\widetilde{v_0}) \in \mathbb{Q}+i\mathbb{Q}$. We obtain

$$|\widetilde{v_0}| \leq |\widetilde{v_0} - v_0| + |v_0| < \frac{\rho_0}{2}|\varphi(z_{m+1})| + \left(1 - \frac{\rho_0}{2}\right)|\varphi(z_{m+1})| = |\varphi(z_{m+1})|$$

and

$$\begin{aligned} |\widetilde{v_0}| &\geq |v_0| - |\widetilde{v_0} - v_0| > \left(1 - \frac{\rho_0}{2}\right)|\varphi(z_{m+1})| - \frac{\rho_0}{2}|\varphi(z_{m+1})| \\ &= (1-\rho_0)|\varphi(z_{m+1})|. \end{aligned}$$

In particular, we have $\widetilde{v_0} \in U_\delta(0)\backslash\{0\}$ and hence

$$\widetilde{z_{M+1}} := \varphi^{-1}(\widetilde{v_0}) \in B_\delta^\circ\backslash\{z_0\} \cap (\mathbb{Q} + i\mathbb{Q}) = A_\delta.$$

According to $(*)$, we obtain $|\varphi(\widetilde{z_k})| \neq |\widetilde{v_0}| = |\varphi(\widetilde{z_{M+1}})|$ for all $k \in \{1, \dots, M\}$ and thus

$$Q^n(\varphi(\widetilde{z_k})) = \varphi(\widetilde{z_k})^{p^n} \neq \varphi(\widetilde{z_{M+1}})^{p^n} = Q^n(\varphi(\widetilde{z_{M+1}})).$$

Therefore, part iii) implies that all iterates of Q are injective on the set $\{\varphi(\widetilde{z_1}), \dots, \varphi(\widetilde{z_M}), \varphi(\widetilde{z_{M+1}})\}$.

v) We define

$$\widetilde{E} := \widetilde{F} \cup \{\widetilde{z_{M+1}}\} = \{\widetilde{z_1}, \dots, \widetilde{z_M}, \widetilde{z_{M+1}}\}.$$

Then $\widetilde{E}$ is a non-empty finite subset of A_δ. Moreover, all iterates of f are injective on $\widetilde{E}$. Indeed, for $k, l \in \{1, \dots, M+1\}$ with $f^n(\widetilde{z_k}) = f^n(\widetilde{z_l})$ we obtain $\varphi(f^n(\widetilde{z_k})) = \varphi(f^n(\widetilde{z_l}))$ and hence $Q^n(\varphi(\widetilde{z_k})) = Q^n(\varphi(\widetilde{z_l}))$. Part iv) yields $\varphi(\widetilde{z_k}) = \varphi(\widetilde{z_l})$ so that the injectivity of φ finally implies $\widetilde{z_k} = \widetilde{z_l}$. Hence, we have $\widetilde{E} \in \mathcal{E}_\delta$.

vi) We now show that $d_{\mathcal{K}(B_\delta)}(E, \widetilde{E}) < \varepsilon$. Because of $\widetilde{F} \subset \widetilde{E}$, we obtain for all $j \in \{1, \dots, m\}$ that

$$\operatorname{dist}(z_j, \widetilde{E}) \leq \operatorname{dist}(z_j, \widetilde{F}) \leq \max_{z \in F} \operatorname{dist}(z, \widetilde{F}) \leq d_{\mathcal{K}(B_\delta)}(F, \widetilde{F}) < \varepsilon.$$

According to part iv), the Lipschitz continuity of $\varphi^{-1}\big|_{U_\delta[0]}$ implies

$$\begin{aligned}
\operatorname{dist}(z_{m+1}, \widetilde{E}) &\leq |z_{m+1} - \widetilde{z_{M+1}}| \\
&= |\varphi^{-1}(\varphi(z_{m+1})) - \varphi^{-1}(\widetilde{v_0})| \\
&\leq c \cdot |\varphi(z_{m+1}) - \widetilde{v_0}| \\
&\leq c \cdot \big(|\varphi(z_{m+1}) - v_0| + |v_0 - \widetilde{v_0}|\big) \\
&\leq c \cdot \left(\frac{\rho_0}{2}|\varphi(z_{m+1})| + \frac{\rho_0}{2}|\varphi(z_{m+1})|\right) \\
&= c \cdot \rho_0 |\varphi(z_{m+1})| \leq c \cdot \rho_0 < c \cdot \frac{\varepsilon}{c} = \varepsilon,
\end{aligned}$$

so that we obtain $\max_{z \in E} \operatorname{dist}(z, \widetilde{E}) < \varepsilon$. Because of $F \subset E$, we have for all $k \in \{1, \dots, M\}$ that

$$\operatorname{dist}(\widetilde{z_k}, E) \leq \operatorname{dist}(\widetilde{z_k}, F) \leq \max_{\widetilde{z} \in \widetilde{F}} \operatorname{dist}(\widetilde{z}, F) \leq d_{\mathcal{K}(B_\delta)}(F, \widetilde{F}) < \varepsilon.$$

As $\operatorname{dist}(\widetilde{z_{M+1}}, E) \leq |\widetilde{z_{M+1}} - z_{m+1}| < \varepsilon$, if follows that $\max_{\widetilde{z}\in\widetilde{E}} \operatorname{dist}(\widetilde{z}, E) < \varepsilon$. Altogether, we finally obtain

$$d_{\mathcal{K}(B_\delta)}(E, \widetilde{E}) = \max\left(\max_{z\in E} \operatorname{dist}(z, \widetilde{E}),\ \max_{\widetilde{z}\in\widetilde{E}} \operatorname{dist}(\widetilde{z}, E)\right) < \varepsilon,$$

which completes the proof. □

Corollary 3.3.3. *The set $\mathcal{E}_\delta$ is dense in $\big(\mathcal{K}(B_\delta), d_{\mathcal{K}(B_\delta)}\big)$.*

Proof: Let $K \in \mathcal{K}(B_\delta)$ and $\varepsilon > 0$. As $\bigcup_{z\in K} U_{\varepsilon/2}(z)$ is an open cover of K, there exist an $N \in \mathbb{N}$ and points $z_1, \ldots, z_N \in K$ with $K \subset \bigcup_{k=1}^{N} U_{\varepsilon/2}(z_k)$. Putting $F := \{z_1, \ldots, z_N\}$, we obtain $F \subset K$ and hence $\max_{z\in F} \operatorname{dist}(z, K) = 0$. For $w_0 \in K$, there exists some $k \in \{1, \ldots N\}$ with $w_0 \in U_{\varepsilon/2}(z_k)$ so that we obtain

$$\operatorname{dist}(w_0, F) \leq |w_0 - z_k| + \operatorname{dist}(z_k, F) = |w_0 - z_k| < \varepsilon/2.$$

This yields $\max_{w\in K} \operatorname{dist}(w, F) < \varepsilon/2$ and thus $d_{\mathcal{K}(B_\delta)}(F, K) < \varepsilon/2$. According to Lemma 3.3.2, there exists a set $E \in \mathcal{E}_\delta$ with $d_{\mathcal{K}(B_\delta)}(F, E) < \varepsilon/2$, and we obtain

$$d_{\mathcal{K}(B_\delta)}(E, K) \leq d_{\mathcal{K}(B_\delta)}(E, F) + d_{\mathcal{K}(B_\delta)}(F, K) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

which completes the proof. □

Before stating the main universality result of this section, we will briefly verify that $\mathcal{K}(B_\delta^\circ \backslash \{z_0\})$ is a comeager subset of $\mathcal{K}(B_\delta)$. Indeed, the following holds:

Lemma 3.3.4. *The set $\mathcal{K}(B_\delta^\circ \backslash \{z_0\})$ is open and dense in $\big(\mathcal{K}(B_\delta), d_{\mathcal{K}(B_\delta)}\big)$.*

Proof: Let $K \in \mathcal{K}(B_\delta^\circ \backslash \{z_0\})$. Putting $\delta := \operatorname{dist}(\{z_0\} \cup \partial B_\delta, K) > 0$, we obtain

$$\big\{L \in \mathcal{K}(B_\delta) : d_{\mathcal{K}(B_\delta)}(K, L) < \delta\big\} \subset \mathcal{K}(B_\delta^\circ \backslash \{z_0\}).$$

Indeed, let $L \in \mathcal{K}(B_\delta)$ with $d_{\mathcal{K}(B_\delta)}(K, L) < \delta$. Assuming that there exists a point $\widetilde{z} \in L \cap (\{z_0\} \cup \partial B_\delta)$, we would obtain

$$d_{\mathcal{K}(B_\delta)}(K, L) \geq \max_{z\in L} \operatorname{dist}(z, K) \geq \operatorname{dist}(\widetilde{z}, K) \geq \operatorname{dist}(\{z_0\} \cup \partial B_\delta, K) = \delta,$$

a contradiction. Thus, we have $L \cap (\{z_0\} \cup \partial B_\delta) = \emptyset$ and hence $L \in \mathcal{K}(B_\delta^\circ \backslash \{z_0\})$. Due to $\mathcal{K}(B_\delta^\circ \backslash \{z_0\}) \supset \mathcal{E}_\delta$, Corollary 3.3.3 implies the denseness of $\mathcal{K}(B_\delta^\circ \backslash \{z_0\})$ in $\big(\mathcal{K}(B_\delta), d_{\mathcal{K}(B_\delta)}\big)$. □

Now, we can extend the statement of Corollary 3.3.1 in the following way:

Theorem 3.3.5. *For each $\delta > 0$ with $U_\delta[0] \subset V$, comeager many functions in $H(\mathbb{C}\backslash\{z_0\})$ are $C(K)$-universal for C_f for comeager many $K \in \mathcal{K}(B_\delta)$.*

Proof: The proof will run similarly to the proof of Lemma 2 in [40].

i) For $g \in H(\mathbb{C}\backslash\{z_0\})$, we define

$$\mathcal{K}_g := \Big\{K \in \mathcal{K}(B_\delta^\circ\backslash\{z_0\}) : \big\{g \circ f^n\big|_K : n \in \mathbb{N}\big\} \text{ dense in } C(K)\Big\}.$$

Denoting by $\mathcal{P}$ the set of all complex-valued polynomials in two real variables with Gaussian rational coefficients, i.e.

$$\mathcal{P} := \left\{\mathbb{C} \ni z \mapsto \sum_{\substack{(\nu,\mu)\in\mathbb{N}_0^2 \\ \nu+\mu\leq n}} a_{\nu,\mu}(\operatorname{Re} z)^\nu(\operatorname{Im} z)^\mu \,:\, n \in \mathbb{N}_0,\, a_{\nu,\mu} \in \mathbb{Q} + i\mathbb{Q}\right\},$$

the complex Stone-Weierstrass theorem implies that the set $\big\{p\big|_L : p \in \mathcal{P}\big\}$ is dense in $C(L)$ for each $L \in \mathcal{K}(\mathbb{C})$ (see Theorem 1.4.4). Hence, we obtain

$$\mathcal{K}_g = \bigcap_{\substack{j\in\mathbb{N}\\ p\in\mathcal{P}}} \bigcup_{n\in\mathbb{N}} \left\{K \in \mathcal{K}(B_\delta^\circ\backslash\{z_0\}) : \big\|g \circ f^n\big|_K - p\big\|_K < \frac{1}{j}\right\}.$$

ii) Let $g \in H(\mathbb{C}\backslash\{z_0\})$, $p \in \mathcal{P}$ and $n \in \mathbb{N}$ be fixed. We show that the function

$$\psi : \mathcal{K}(B_\delta^\circ\backslash\{z_0\}) \to [0,\infty), \;\; \psi(K) := \big\|g \circ f^n\big|_K - p\big\|_K$$

is continuous. In order to do so, we denote $q := g \circ f^n\big|_{B_\delta^\circ\backslash\{z_0\}} - p$. Now, let $K \in \mathcal{K}(B_\delta^\circ\backslash\{z_0\})$ and $\varepsilon > 0$. We choose some $\delta_1 > 0$ with $U_{\delta_1}[K] \subset B_\delta^\circ\backslash\{z_0\}$. As q is uniformly continuous on the compact set $U_{\delta_1}[K]$, there exists a $\delta_2 > 0$ such that we have

$$(*) \qquad |q(z_1) - q(z_2)| < \varepsilon$$

for all points $z_1, z_2 \in U_{\delta_1}[K]$ with $|z_1 - z_2| < \delta_2$. We put $\delta := \min(\delta_1, \delta_2)$ and we consider a set $L \in \mathcal{K}(B_\delta^\circ\backslash\{z_0\})$ with $d_{\mathcal{K}(B_\delta)}(L, K) < \delta$. Then, for $z \in L$, there exists a point $w \in K$ with $|z - w| < \delta \leq \delta_2$, and we have $\operatorname{dist}(z, K) \leq |z - w| < \delta$ which yields $z \in U_{\delta_1}[K]$. Therefore, $(*)$ implies

$$|q(z)| \leq |q(z) - q(w)| + |q(w)| < \varepsilon + \|q\|_K.$$

As this inequality holds for all $z \in L$, we obtain $\|q\|_L \leq \varepsilon + \|q\|_K$ and hence $\|q\|_L - \|q\|_K \leq \varepsilon$. Analogously, it follows that $\|q\|_K - \|q\|_L \leq \varepsilon$, which finally yields

$$|\psi(L) - \psi(K)| = \big|\|q\|_L - \|q\|_K\big| \leq \varepsilon.$$

Thus, the continuity of ψ is shown so that for each $j \in \mathbb{N}$ the set

$$\Big\{K \in \mathcal{K}(B_\delta^\circ \backslash \{z_0\}) : \big\|g \circ f^n\big|_K - p\big\|_K < 1/j\Big\}$$

is open in $\mathcal{K}(B_\delta^\circ \backslash \{z_0\})$. Due to Lemma 3.3.4, this implies that these sets are also open in $\mathcal{K}(B_\delta)$. Therefore, part i) yields that $\mathcal{K}_g$ is a G_δ-set in $\mathcal{K}(B_\delta)$.

iii) For each $E \in \mathcal{E}_\delta$, Corollary 3.3.1 implies that the set

$$\mathcal{G}_E := \big\{g \in H(\mathbb{C}\backslash\{z_0\}) : g\ C(E)\text{-universal for } C_f\big\}$$

is comeager in $H(\mathbb{C}\backslash\{z_0\})$. As $\mathcal{E}_\delta$ is countable, we obtain that $\mathcal{G} := \bigcap_{E \in \mathcal{E}_\delta} \mathcal{G}_E$ is also comeager in $H(\mathbb{C}\backslash\{z_0\})$. Moreover, as each function $g \in \mathcal{G}$ is $C(E)$-universal for C_f for all sets $E \in \mathcal{E}_\delta$, we have $\mathcal{E}_\delta \subset \mathcal{K}_g$ for all $g \in \mathcal{G}$. As $\mathcal{E}_\delta$ is dense in $\mathcal{K}(B_\delta)$ due to Corollary 3.3.3, it follows that $\mathcal{K}_g$ is dense in $\mathcal{K}(B_\delta)$ for all $g \in \mathcal{G}$. Therefore, part ii) yields that for comeager many $g \in H(\mathbb{C}\backslash\{z_0\})$ the set $\mathcal{K}_g$ is a dense G_δ-set in $\mathcal{K}(B_\delta)$ and hence, in particular, comeager in $\mathcal{K}(B_\delta)$. Thus, comeager many functions $g \in H(\mathbb{C}\backslash\{z_0\})$ have the property that for comeager many $K \in \mathcal{K}(B_\delta)$ the set $\big\{g \circ f^n\big|_K : n \in \mathbb{N}\big\}$ is dense in $C(K)$. □

Remark 3.3.6.

i) It is well-known that comeager many sets in $\mathcal{K}(B_\delta)$ are Cantor sets, i.e. perfect and totally disconnected (cf. [8], Remark 2 on p. 236). In particular, we have $K^\circ = \emptyset$ and hence $A(K) = C(K)$ for comeager many $K \in \mathcal{K}(B_\delta)$. For this reason, we may replace $C(K)$-universality by $A(K)$-universality in the statement of Theorem 3.3.5. However, for $K \in \mathcal{K}(B_\delta^\circ \backslash \{z_0\})$ with $K^\circ \neq \emptyset$, we will see in part iii) that there does not exist a function in $H(U\backslash\{z_0\})$ which is $A(K)$-universal for C_f (in particular, then C_f also cannot be $C(K)$-universal). Therefore, stating $C(K)$-universality instead of $A(K)$-universality in Theorem 3.3.5 is more to the point.

ii) Let $L \in \mathcal{K}(U_\delta(0)\backslash\{0\})$ such that L contains an arc around the origin, i.e. there exists a radius $0 < r < \delta$, a point $w_0 \in K_r[0]$ and an $\varepsilon > 0$ with

$$L \supset K_r[0] \cap U_\varepsilon(w_0) =: A.$$

Denoting by $M := \bigcup_{n\in\mathbb{N}} \{z \in \mathbb{C} : z^{p^n} = 1\}$ the set of all p^n-th roots of unity, the set $w_0 M$ is dense in $K_r[0]$. As $A\backslash\{w_0\}$ is non-empty and open in $K_r[0]$, there exists an integer $N \in \mathbb{N}$ and a p^N-th root of unity $w \in M$ with $w_1 := w_0 w \in A\backslash\{w_0\}$. It follows that

$$Q^N(w_1) = w_1^{p^N} = w_0^{p^N} w^{p^N} = w_0^{p^N} = Q^N(w_0)$$

so that we obtain $Q^n(w_1) = Q^n(w_0)$ for all $n \geq N$. Analogously to the considerations after Theorem 1.3.5, this implies that there does not exist a function in $H(V\backslash\{0\})$ which is $A(L)$-universal for C_Q.

iii) Now, let $K \in \mathcal{K}(B_\delta^\circ\backslash\{z_0\})$ such that the image $\varphi(K) \in \mathcal{K}(U_\delta(0)\backslash\{0\})$ contains an arc around the origin. (As Lemma A.1 yields $\varphi(K)^\circ = \varphi(K^\circ)$, this is in particular the case if $K^\circ \neq \emptyset$.) Assuming that there exists a function $g \in H(U\backslash\{z_0\})$ which is $A(K)$-universal for C_f, we consider the function

$$h := g \circ \varphi^{-1}\big|_{V\backslash\{0\}} \in H(V\backslash\{0\}).$$

For $F \in A(\varphi(K))$ and $\varepsilon > 0$, we would obtain $F \circ \varphi\big|_K \in A(K)$ so that the denseness of the set $\{g \circ f^n\big|_K : n \in \mathbb{N}\}$ in $A(K)$ would yield an $N \in \mathbb{N}$ with

$$\begin{aligned} \varepsilon &> \left\| g \circ f^N\big|_K - F \circ \varphi\big|_K \right\|_K = \left\| g \circ f^N \circ \varphi^{-1} - F \right\|_{\varphi(K)} \\ &= \left\| g \circ \varphi^{-1} \circ Q^N - F \right\|_{\varphi(K)} = \left\| h \circ Q^N\big|_{\varphi(K)} - F \right\|_{\varphi(K)}. \end{aligned}$$

Hence, the set $\{h \circ Q^n\big|_{\varphi(K)} : n \in \mathbb{N}\}$ would be dense in $A(\varphi(K))$, i.e. h would be $A(\varphi(K))$-universal for C_Q. But this contradicts part ii). Therefore, there does not exist a function in $H(U\backslash\{z_0\})$ which is $A(K)$-universal for C_f.

iv) In particular, part iii) shows that the universality statement of Theorem 3.3.5 is much weaker than the universality result which holds near attracting fixed points of f (see Theorem 3.1.1), where we actually obtained $H(W)$-universality of C_f for suitable open sets W.

In the following, we want to find sufficient conditions on sets $K \in \mathcal{K}(B_\delta^\circ\backslash\{z_0\})$ such that the composition operator C_f is $C(K)$-universal. We start with the following simple example concerning the p-th monomial Q.

Example 3.3.7. For $0 < a < b < \delta$, we consider the compact interval $L := [a, b]$.

Then, for each $n \in \mathbb{N}$, the angular sector

$$U_n := \left\{ re^{i\phi} : 0 < r < \delta,\ \phi \in \left[0, \frac{\pi}{p^n}\right) \cup \left(2\pi - \frac{\pi}{p^n}, 2\pi\right]\right\}$$

is an open neighbourhood of L such that the restriction $Q^n\big|_{U_n}$ is injective. Hence, we have $L \in \mathcal{K}_0\big(U_\delta(0)\backslash\{0\}, \mathbb{C}\backslash\{0\}, Q\big)$ so that Theorem 2.2.5 implies that comeager many functions in $H(\mathbb{C}\backslash\{0\})$ are $C(L)$-universal for C_Q.

Each interval $[a, b]$ from the previous example is the image of the unit interval $I := [0, 1]$ under the path

$$\psi_{a,b} : I \to U_\delta(0)\backslash\{0\},\ \ \psi_{a,b}(t) := a + t(b - a),$$

which has the property that its absolute value $|\psi_{a,b}|$ is injective. This motivates us to formulate and prove the following statement:

Lemma 3.3.8. *Let $K \in \mathcal{K}_0(B_\delta^\circ\backslash\{z_0\})$ and let there exist a path $\psi : I \to U_\delta(0)\backslash\{0\}$ with $\psi(I) = \varphi(K)$ and such that $|\psi|$ is injective. Then the set of all functions in $H(\mathbb{C}\backslash\{z_0\})$ which are $C(K)$-universal for C_f is a comeager set in $H(\mathbb{C}\backslash\{z_0\})$.*

Proof:

i) Without loss of generality, we may assume that $|\psi|$ is strictly increasing. Let $0 < r_0 < \delta$ and $\phi_0 \in \mathbb{R}$ with $\psi(1) = r_0 \cdot e^{i\phi_0}$ and let $R \in (r_0, \delta)$. We consider the affine linear paths

$$\psi_1 : I \to U_\delta(0)\backslash\{0\},\ \ \psi_1(t) := \frac{\psi(0)}{2} + t\left(\psi(0) - \frac{\psi(0)}{2}\right)$$

and

$$\psi_2 : I \to U_\delta(0)\backslash\{0\},\ \ \psi_2(t) := \psi(1) + t\left(Re^{i\phi_0} - \psi(1)\right),$$

and we define $\gamma := \psi_1 \bullet \psi \bullet \psi_2$ to be the path composition of ψ_1, ψ and ψ_2 (cf. Definition B.6 ii) in Appendix B). Then $|\gamma|$ is strictly increasing.

ii) Let $n \in \mathbb{N}$ be fixed. For $k \in \{0, \dots, p^n - 1\}$, we denote by $\zeta_k^{(n)} := e^{2\pi ik/p^n}$ the p^n-th roots of unity. Moreover, we define the radius $r_n := |\gamma(0)| \cdot |\zeta_1^{(n)} - 1|/2$ and the arcs

$$K_{t,n} := K_{|\gamma(t)|}[0] \cap U_{r_n}(\gamma(t)),\ \ t \in I.$$

Then the disjoint union $U_n := \bigcup_{t \in I^\circ} K_{t,n} \subset U_\delta(0)\backslash\{0\}$ contains the image $\psi(I) = \varphi(K)$ (cf. Figure 3.5 on the next page).

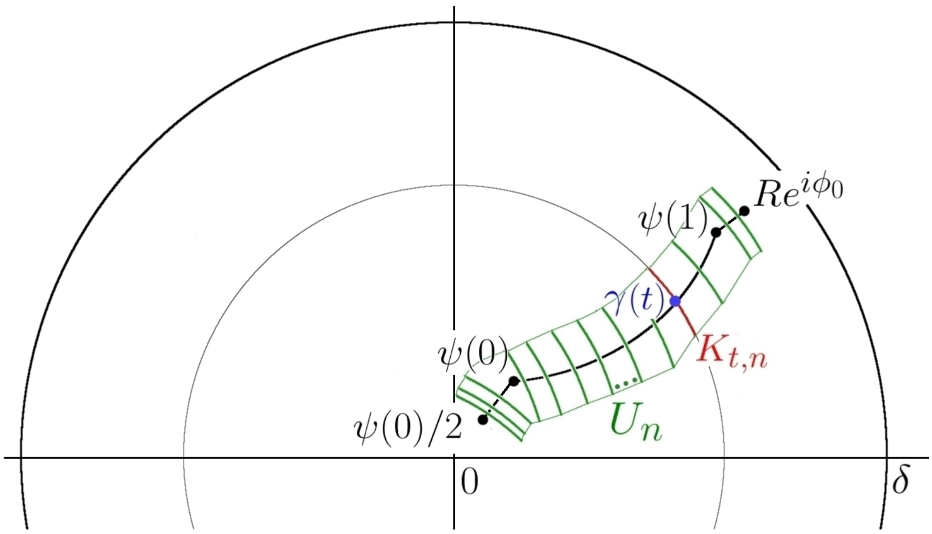

Figure 3.5

iii) We show that U_n is open. Assuming that this is not true, there would exist a point $w_0 \in U_n$ and a sequence (w_m) in $\mathbb{C}\backslash U_n$ with $w_m \to w_0$. By definition of the set U_n, then there exists a unique $t_0 \in I^\circ$ with $w_0 \in K_{t_0,n}$, i.e. we have $|w_0| = |\gamma(t_0)|$ and $|w_0 - \gamma(t_0)| < r_n$. Putting

$$\delta := \min\Big(|\gamma(t_0)| - |\gamma(0)|, |\gamma(1)| - |\gamma(t_0)|\Big) > 0,$$

we choose an integer $M_1 \in \mathbb{N}$ such that $|w_m - w_0| < \delta$ holds for all $m \geq M_1$. Hence, for each $m \geq M_1$, we obtain

$$|w_m| \leq |w_m - w_0| + |w_0| < \delta + |\gamma(t_0)| \leq |\gamma(1)| - |\gamma(t_0)| + |\gamma(t_0)| = |\gamma(1)|$$

as well as

$$|w_m| \geq |w_0| - |w_m - w_0| > |\gamma(t_0)| - \delta \geq |\gamma(t_0)| + |\gamma(0)| - |\gamma(t_0)| = |\gamma(0)|.$$

Therefore, the intermediate value theorem and the injectivity of $|\gamma|$ imply that for each $m \geq M_1$ there exists a unique $t_m \in I^\circ$ with $|\gamma(t_m)| = |w_m|$. Thus, we obtain

$$|\gamma(t_m)| = |w_m| \to |w_0| = |\gamma(t_0)|.$$

But this already implies the convergence $t_m \to t_0$. $\big($Indeed, if this does not hold, there would exist an $\varepsilon_0 > 0$ and a subsequence (t_{m_k}) of (t_m) such that $t_{m_k} \notin (t_0 - \varepsilon_0, t_0 + \varepsilon_0)$ for all $k \in \mathbb{N}$. Hence, the intermediate value theorem and the strict monotony of $|\gamma|$ would yield $|\gamma(t_{m_k})| \notin (|\gamma(t_0 - \varepsilon_0)|, |\gamma(t_0 + \varepsilon_0)|)$

for all $k \in \mathbb{N}$, which contradicts the convergence $|\gamma(t_{m_k})| \to |\gamma(t_0)|$.) Now, the continuity of γ implies $\gamma(t_m) \to \gamma(t_0)$.

Because of $|w_0 - \gamma(t_0)| < r_n$, we can choose an $\varepsilon > 0$ with $|w_0 - \gamma(t_0)| + \varepsilon < r_n$. According to $w_m \to w_0$ and $\gamma(t_m) \to \gamma(t_0)$, there exists an integer $M_2 \in \mathbb{N}$ such that we have $|w_m - w_0| < \varepsilon/2$ and $|\gamma(t_m) - \gamma(t_0)| < \varepsilon/2$ for all $m \geq M_2$ and hence

$$\begin{aligned} |w_m - \gamma(t_m)| &\leq |w_m - w_0| + |w_0 - \gamma(t_0)| + |\gamma(t_0) - \gamma(t_m)| \\ &< \varepsilon + |w_0 - \gamma(t_0)| < r_n, \quad m \geq M_2. \end{aligned}$$

Putting $M := \max(M_1, M_2)$, we obtain $w_m \in K_{t_m,n} \subset U_n$ for all $m \geq M$. But this contradicts the fact that (w_m) is a sequence in $\mathbb{C} \backslash U_n$. Hence, U_n must be open.

iv) We show that Q^n is injective on U_n. In order to do so, let $w_1, w_2 \in U_n$ with $Q^n(w_1) = Q^n(w_2)$. Then we have $w_1^{p^n} = w_2^{p^n}$ so that we obtain

$$w_1 \in \left\{ \zeta_0^{(n)} w_2,\, \zeta_1^{(n)} w_2,\, \ldots,\, \zeta_{p^n-1}^{(n)} w_2 \right\}$$

and in particular $|w_1| = |w_2|$. Now, let $t_1, t_2 \in I^\circ$ with $w_1 \in K_{t_1,n}$ and $w_2 \in K_{t_2,n}$. This yields $|\gamma(t_1)| = |w_1| = |w_2| = |\gamma(t_2)|$ so that the injectivity of $|\gamma|$ implies $t_1 = t_2$. Therefore, it follows that

$$\begin{aligned} |w_1 - w_2| &\leq |w_1 - \gamma(t_1)| + |\gamma(t_1) - w_2| < r_n + r_n = 2r_n \\ &= |\gamma(0)| \cdot |\zeta_1^{(n)} - 1|. \end{aligned}$$

Assuming that we have $w_1 = \zeta_k^{(n)} w_2$ for some $k \in \{1, \ldots, p^n - 1\}$, we would obtain

$$\begin{aligned} |w_1 - w_2| &= |\zeta_k^{(n)} w_2 - w_2| = |w_2| \cdot |\zeta_k^{(n)} - 1| = |\gamma(t_2)| \cdot |\zeta_k^{(n)} - 1| \\ &> |\gamma(0)| \cdot |\zeta_k^{(n)} - 1| \geq |\gamma(0)| \cdot |\zeta_1^{(n)} - 1|, \end{aligned}$$

a contradiction. Hence, we have $w_1 = \zeta_0^{(n)} w_2 = w_2$.

v) For each $n \in \mathbb{N}$, parts ii), iii) and iv) and the conformality of the conjugation map φ imply that $\varphi^{-1}(U_n) \subset B_\delta^\circ \backslash \{z_0\}$ is an open neighbourhood of K such that the iterate

$$f^n\big|_{\varphi^{-1}(U_n)} = \varphi^{-1} \circ Q^n \circ \varphi\big|_{\varphi^{-1}(U_n)}$$

is injective. Thus, we obtain $K \in \mathcal{K}_0\big(B_\delta^\circ\backslash\{z_0\}, \mathbb{C}\backslash\{z_0\}, f\big)$ so that the assertion now follows from Theorem 2.2.5. □

We want to conclude this section with the following consideration. For $\delta > 0$ with $U_\delta[0] \subset V$, Theorem 3.3.5 states that comeager many functions in $H(\mathbb{C}\backslash\{z_0\})$ are $C(K)$-universal for C_f for comeager many $K \in \mathcal{K}(B_\delta)$. Here, the second "comeager"-expression depends on the first one, meaning that the comeager subset of $\mathcal{K}(B_\delta)$ depends on the choice of a function from the comeager subset of $H(\mathbb{C}\backslash\{z_0\})$ (cf. the proof of Theorem 3.3.5). In the following, we want to show that this dependence can be interchanged.

For two sets X and Y and $x_0 \in X$, $y_0 \in Y$ as well as $A \subset X \times Y$, we denote

$$A(x_0, \cdot) := \{y \in Y : (x_0, y) \in A\} \quad \text{and} \quad A(\cdot, y_0) := \{x \in X : (x, y_0) \in A\}.$$

Then the following statement holds:

Lemma 3.3.9. *Let X and Y be Baire spaces and let Y be second-countable. Moreover, let $A \subset X \times Y$ such that $A(x, \cdot)$ is a G_δ-set in Y for comeager many $x \in X$ and such that $A(\cdot, y)$ is comeager in X for comeager many $y \in Y$. Then $A(x, \cdot)$ is comeager in Y for comeager many $x \in X$.*

Proof: According to the assumption, there exists a comeager subset S of Y such that $A(\cdot, y)$ is comeager in X for all $y \in S$. As Y is a Baire space, we obtain that S is dense in Y. The second-countability of Y implies that S is also second-countable and hence, in particular, separable. Thus, there exists a countable subset R of S which is dense in S. Then R is also dense in Y and the set $X_0 := \bigcap_{y \in R} A(\cdot, y)$ is comeager in X. For $x_0 \in X_0$, we have $(x_0, y) \in A$ and hence $y \in A(x_0, \cdot)$ for all $y \in R$. Thus, we obtain $A(x_0, \cdot) \supset R$ so that the denseness of R in Y implies that $A(x_0, \cdot)$ is dense in Y. Hence, $A(x, \cdot)$ is dense in Y for comeager many $x \in X$. The assertion now follows from the assumption that $A(x, \cdot)$ is also a G_δ-set in Y for comeager many $x \in X$. □

Remark 3.3.10. There exists a similar version of Lemma 3.3.9 which follows from the Kuratowski-Ulam theorem (see e.g. [32], p. 144). This version states that for a Baire space X, a second-countable Baire space Y and a set $A \subset X \times Y$ the following is true: If A is a G_δ-set in $X \times Y$ such that $A(\cdot, y)$ is comeager in X for all $y \in Y$, then $A(x, \cdot)$ is comeager in Y for comeager many $x \in X$. As the property of A being a G_δ-set in $X \times Y$ implies that $A(x, \cdot)$ is a G_δ in Y for all $x \in X$, Lemma 3.3.9 yields the same result as the above-mentioned version – but under weaker assumptions. In order to prove the following theorem, we will see that we do indeed need the statement of Lemma 3.3.9.

Theorem 3.3.11. *For each $\delta > 0$ with $U_\delta[0] \subset V$, comeager many $K \in \mathcal{K}(B_\delta)$ have the property that comeager many functions in $H(\mathbb{C}\backslash\{z_0\})$ are $C(K)$-universal for C_f.*

Proof: We denote $X := \mathcal{K}(B_\delta)$, $Y := H(\mathbb{C}\backslash\{z_0\})$ and

$$A := \Big\{(K, g) \in \mathcal{K}(B_\delta^\circ\backslash\{z_0\}) \times H(\mathbb{C}\backslash\{z_0\}) : \big\{g \circ f^n\big|_K : n \in \mathbb{N}\big\} \text{ dense in } C(K)\Big\}.$$

Then X and Y are Baire spaces, Y is second-countable and A is a subset of $X \times Y$. Denoting by $\mathcal{P}$ the set of all complex-valued polynomials in two real variables with Gaussian rational coefficients, the complex Stone-Weierstrass theorem implies, analogously to the proof of Theorem 3.3.5, that we have

$$\begin{aligned} A(K,\cdot) &= \Big\{g \in H(\mathbb{C}\backslash\{z_0\}) : \big\{g \circ f^n\big|_K : n \in \mathbb{N}\big\} \text{ dense in } C(K)\Big\} \\ &= \bigcap_{\substack{j\in\mathbb{N} \\ p\in\mathcal{P}}} \bigcup_{n\in\mathbb{N}} \Big\{g \in H(\mathbb{C}\backslash\{z_0\}) : \|g \circ f^n\big|_K - p\|_K < \frac{1}{j}\Big\} \end{aligned}$$

for all $K \in \mathcal{K}(B_\delta^\circ\backslash\{z_0\})$. For each $p \in \mathcal{P}, n \in \mathbb{N}$ and $K \in \mathcal{K}(B_\delta^\circ\backslash\{z_0\})$, the continuity of the composition operator $C_{f^n,K} : H(\mathbb{C}\backslash\{z_0\}) \to A(K)$, $C_{f^n,K}(g) := g \circ f^n\big|_K$, yields that the function

$$H(\mathbb{C}\backslash\{z_0\}) \ni g \mapsto \big\|g \circ f^n\big|_K - p\big\|_K$$

is continuous. Hence, each set $\big\{g \in H(\mathbb{C}\backslash\{z_0\}) : \|g \circ f^n\big|_K - p\|_K < 1/j\big\}$ is open in $H(\mathbb{C}\backslash\{z_0\})$ so that we obtain that $A(K,\cdot)$ is a G_δ-set in Y for all $K \in \mathcal{K}(B_\delta^\circ\backslash\{z_0\})$. As $\mathcal{K}(B_\delta^\circ\backslash\{z_0\})$ is a comeager subset of $\mathcal{K}(B_\delta)$ due to Lemma 3.3.4, it follows that $A(K,\cdot)$ is a G_δ-set in Y for comeager many $K \in X$. According to Theorem 3.3.5, we have that

$$A(\cdot, g) = \Big\{K \in \mathcal{K}(B_\delta^\circ\backslash\{z_0\}) : \big\{g \circ f^n\big|_K : n \in \mathbb{N}\big\} \text{ dense in } C(K)\Big\}$$

is comeager in X for comeager many $g \in Y$ (see also the proof of Theorem 3.3.5, where we have $A(\cdot, g) = \mathcal{K}_g$). Therefore, Lemma 3.3.9 implies that for comeager many $K \in X$ the set $A(K,\cdot)$ is comeager in Y. $\square$

Chapter 4

Applications: Global Theory

For a complex-valued function f which is holomorphic on an open neighbourhood of a (super-)attracting or a neutral fixed point $z_0 \in \mathbb{C}$ of f, Chapter 3 provides several universality statements for the composition operator with symbol f. If, in addition, the symbol f is not only locally defined but even a transcendental entire function or a rational function of degree $d \geq 2$, all these universality phenomenons occur on "small" parts of the Fatou set of f. (Note that (super-)attracting fixed points of f as well as attracting petals for f at neutral fixed points are always contained in the open set F_f.) More precisely, this means that we were able to find open sets $U \subset F_f$ or compact sets $K \subset F_f$ located near z_0 with the property that C_f is $H(U)$- or $A(K)$-universal, respectively. In case of $H(U)$-universality of C_f, there exist (comeager many) functions $g \in H(\mathbb{C}\backslash\{z_0\})$ for which we have

$$\overline{\left\{g \circ f^n\big|_U : n \in \mathbb{N}\right\}}^{H(U)} = H(U).$$

In this chapter, we will study the long-time behaviour of sequences of compositions of the form $(g \circ f^n)_{n\in\mathbb{N}}$ on "large" parts of the Fatou set of a transcendental entire or a rational symbol f, e.g. on whole components of F_f. Our aim is to determine the "richness" of sets of the form

$$\overline{\left\{g \circ f^n\big|_G : n \in \mathbb{N}\right\}}^{H(G)}$$

for "many" functions $g \in H(\Omega)$, where now G should be a "large" open invariant subset of F_f and Ω is a domain in $\mathbb{C}$ which contains G. (As this setting implies that $\infty \notin G \supset f(G)$, we see that the iterates f^n are holomorphic on G.) In contrast to the local situations which we have considered in the previous chapter, we now cannot expect that there exists a function $g \in H(\Omega)$ such that the closure of the set of compositions $\left\{g \circ f^n\big|_G : n \in \mathbb{N}\right\}$ equals the whole space $H(G)$, i.e. we cannot expect that C_f is $H(G)$-universal. The reason for this is that whenever f is not injective on G – which is likely to be the case for "large" subsets G of F_f – then there exists the following "natural restriction" for each function

$$h \in \overline{\left\{g \circ f^n\big|_G : n \in \mathbb{N}\right\}}^{H(G)} \setminus \left\{g \circ f^n\big|_G : n \in \mathbb{N}\right\}$$

(cf. the considerations after Theorem 1.3.5): Choosing a strictly increasing sequence (n_k) in $\mathbb{N}$ such that $(g \circ f^{n_k})$ converges to h locally uniformly on G, we obtain for any $N \in \mathbb{N}$ and any points $z, w \in G$ with $z \neq w$ and $f^N(z) = f^N(w)$ that

$$h(z) = \lim_{k\to\infty} g(f^{n_k}(z)) = \lim_{k\to\infty} g(f^{n_k}(w)) = h(w).$$

This means that we can approximate only those functions in $H(G)$ via subsequences of $(g \circ f^n)$ which have the property of assuming the same value at all points in G which eventually coincide under the iteration of f. As not all functions in $H(G)$ have this property, it follows that

$$\overline{\left\{g \circ f^n\big|_G : n \in \mathbb{N}\right\}}^{H(G)} \neq H(G).$$

Therefore, the composition operator C_f is not $H(G)$-universal. In this situation, the challenge now consists in determining the closure of the set of compositions $\left\{g \circ f^n\big|_G : n \in \mathbb{N}\right\}$ in $H(G)$. In order to do so, it is useful to consider the set

$$\omega(G, g, f)$$

which shall denote the set of all functions $h : G \to \mathbb{C}$ for which there exists a strictly increasing sequence (n_k) in $\mathbb{N}$ such that $(g \circ f^{n_k})$ converges to h locally uniformly on G. Using this terminology, we have

$$\overline{\left\{g \circ f^n\big|_G : n \in \mathbb{N}\right\}}^{H(G)} = \omega(G, g, f) \cup \left\{g \circ f^n\big|_G : n \in \mathbb{N}\right\}.$$

In the following first two sections, which are the main part of this chapter, we will characterize the sets $\omega(G, g, f)$ for comeager many functions $g \in H(\mathbb{C}\setminus\{z_0\})$ in case that G is a "large" open subset of a union of Schröder domains or a suitable superset of a union of Leau domains belonging to an attracting or a neutral fixed point z_0 of f, respectively. Subsequently, we will remark that the set $\omega\big(G\setminus O_f^-(z_0), g, f\big)$ only contains constant functions if G is a Böttcher domain of f which contains a superattracting fixed point z_0 of f and g is holomorphic on $\mathbb{C}\backslash\{z_0\}$, and that there exist constant functions which are not contained in the closure of $\left\{g \circ f^n\big|_G : n \in \mathbb{N}\right\}$ in $H(G)$ if G is an invariant rotation domain (i.e. a Siegel disc or an Arnol'd-Herman ring) of f and g is holomorphic on G. Finally, we will conclude this chapter by studying the case that G is a Baker domain or a wandering domain of f. Requiring that f is injective on G (and providing several examples of transcendental entire functions which are injective on Baker domains or on wandering domains), we will again obtain $H(G)$-universality of C_f in both situations.

4.1 Schröder Domains

Let f be a transcendental entire function or a rational function of degree $d \geq 2$ which has an attracting fixed point at $z_0 \in \mathbb{C}$. We denote by

$$A(z_0) := A_f(z_0) = \{z : f^n(z) \to z_0\}$$

the basin of attraction of z_0 under f, which is a union of components of F_f (see e.g. [15], Theorem III.2.1). In particular, the Schröder domain of f containing z_0 (i.e. the component of F_f which contains z_0) is a subset of $A(z_0)$ (cf. Definition 1.2.5 and Theorem 1.2.7). Moreover, we consider the backward orbit of z_0 under f, which is given by

$$O^-(z_0) := O_f^-(z_0) = \bigcup_{n\in\mathbb{N}} \{z : f^n(z) = z_0\}.$$

By definition, the set $A(z_0)\backslash O^-(z_0)$ is completely invariant under f. In particular, this yields that f is not injective on $A(z_0)\backslash O^-(z_0)$. (If f is a rational function, the component of $A(z_0)$ which contains z_0 even contains a critical point of f; see e.g. [15], Theorem III.2.2.) Throughout this whole section, we assume that $\infty \notin A(z_0)$ in case of a rational symbol f, so that the invariance of $A(z_0)$ under f implies that f is holomorphic on $A(z_0)$.

As already described in the introduction of this chapter, it is our aim to characterize sets of the form $\omega(G, g, f)$ for suitable "large" open subsets G of $A(z_0)\backslash O^-(z_0)$ and "many" functions $g \in H(\mathbb{C}\backslash\{z_0\})$. Due to the above considerations, each function $h \in \omega(G, g, f)$ must have the property of assuming the same value at all points $z, w \in G$ for which we have $f^N(z) = f^N(w)$ for some $N \in \mathbb{N}$. Thus, in order to determine the set $\omega(G, g, f)$, we have to look for holomorphic functions on G which fulfil this property. Clearly, this holds for all constant functions – but the crucial question we are facing here is whether there exist any non-constant functions for which this is true. As we do not have any information concerning this fact at first view, it is natural to remind ourselves that f is locally conjugated near the attracting fixed point z_0 to the linear function $w \mapsto \lambda w$ with $\lambda := f'(z_0)$ (where $0 < |\lambda| < 1$). This means that there exist open neighbourhoods U of z_0 and V of 0 with $f(U) \subset U$ and $\lambda V \subset V$ as well as a conformal map $\varphi : U \to V$ such that the equation

$$\varphi \circ f^n = \lambda^n \cdot \varphi$$

holds on U for all integers $n \in \mathbb{N}$ (cf. Section 3.1). According to the considerations before Theorem 3.1.1, we have $U \subset A(z_0)$. It is well-known that the conjugation map φ can be extended holomorphically to the entire basin of attraction $A(z_0)$ (see e.g. [15], p. 32). To see this, we set

$$\Phi : A(z_0) \to \mathbb{C}, \;\; \Phi(z) := \frac{\varphi(f^N(z))}{\lambda^N},$$

where $N \in \mathbb{N}_0$ is chosen so large that $f^N(z) \in U$ (because of $z \in A(z_0)$ and as U is an open neighbourhood of z_0 this is always possible). Then Φ is a well-defined holomorphic function on $A(z_0)$. Indeed, for $z \in A(z_0)$ and $N, M \in \mathbb{N}_0$ with $f^N(z), f^M(z) \in U$ and $N < M$, there exists some $k \in \mathbb{N}$ with $N + k = M$, so that the above conjugation implies

$$\frac{\varphi(f^M(z))}{\lambda^M} = \frac{\varphi\big(f^k(f^N(z))\big)}{\lambda^k} \Big/ \lambda^N = \frac{\lambda^k \cdot \varphi(f^N(z))}{\lambda^k} \Big/ \lambda^N = \frac{\varphi(f^N(z))}{\lambda^N}.$$

By definition, we have $\Phi\big|_U = \varphi$ and the function Φ fulfils the same functional equation as φ, but now on the whole set $A(z_0)$ (cf. [15], p. 32). In fact, for $z \in A(z_0)$, we choose some $N \in \mathbb{N}_0$ with $f^N(f(z)) \in U$ and we obtain

$$\Phi(f(z)) = \frac{\varphi\big(f^N(f(z))\big)}{\lambda^N} = \lambda \cdot \frac{\varphi(f^{N+1}(z))}{\lambda^{N+1}} = \lambda \cdot \Phi(z).$$

Inductively, it follows that

$$\Phi \circ f^n = \lambda^n \cdot \Phi$$

holds on $A(z_0)$ for all $n \in \mathbb{N}$. Hence, for $z, w \in A(z_0)$ and $N \in \mathbb{N}$ with $f^N(z) = f^N(w)$, we have

$$\Phi(z) = \frac{\Phi(f^N(z))}{\lambda^N} = \frac{\Phi(f^N(w))}{\lambda^N} = \Phi(w),$$

i.e. the non-constant holomorphic function Φ assumes the same value at all points in $A(z_0)$ which eventually coincide under the iteration of f.

Remark 4.1.1. In case that f is a rational function, it is well-known that we have $\Phi(A(z_0)) = \mathbb{C}$ (see e.g. [15], p. 32 or [49], p. 68f.). By slightly adapting the proof of this statement which can be found in the latter reference, one can check that $\Phi(A(z_0))$ is always a dense subset of the complex plane in case that f is a transcendental entire function. Hence, in the above situation, the function Φ quasi-conjugates $f\big|_{A(z_0)}$ to the linear function $w \mapsto \lambda w$ on $\mathbb{C}$.

Now, knowing that Φ fulfils the property of assuming the same value at all points in $A(z_0)$ which eventually coincide under the iteration of f, we obtain that this is already true for a whole class of functions – namely for all compositions $\psi \circ \Phi$, where ψ is defined on $\Phi(A(z_0))$. Therefore, we introduce the following notation:

Definition 4.1.2. For $G \subset A(z_0)$ open, we set

$$H_\Phi(G) := \left\{\psi \circ \Phi\big|_G : \psi \in H(\Phi(G))\right\}.$$

Then $H_\Phi(G)$ is a subspace of $H(G)$ and for each function $h \in H_\Phi(G)$ we have $h(z) = h(w)$ for all points $z, w \in G$ with $f^N(z) = f^N(w)$ for some $N \in \mathbb{N}$. Hence, for $G \subset A(z_0)\backslash O^-(z_0)$ open and $g \in H(\mathbb{C}\backslash\{z_0\})$, each function in $H_\Phi(G)$ is a possible candidate for being contained in the set $\omega(G, g, f)$. In the following, it will be our aim to prove the existence of a "large" open subset G_0 of $A(z_0)\backslash O^-(z_0)$ such that we actually have $\omega(G_0, g, f) = H_\Phi(G_0)$ and hence, in particular,

$$\overline{\left\{g \circ f^n\big|_{G_0} : n \in \mathbb{N}\right\}}^{H(G_0)} = H_\Phi(G_0) \cup \left\{g \circ f^n\big|_{G_0} : n \in \mathbb{N}\right\}$$

for "many" functions $g \in H(\mathbb{C}\backslash\{z_0\})$. In order to show that such a "large" set G_0 exists, we introduce the following definition:

Definition 4.1.3. A function $g \in H(A(z_0)\backslash\{z_0\})$ is called *locally universal* for C_f on U if the set $\left\{g \circ f^n\big|_W : n \in \mathbb{N}\right\}$ is dense in $H(W)$ for all $W \in \mathcal{U}_0(U\backslash\{z_0\})$.

Using this terminology, Theorem 3.1.1 states that the set

$$\mathcal{G} := \left\{g \in H(\mathbb{C}\backslash\{z_0\}) : g\big|_{A(z_0)\backslash\{z_0\}} \text{ locally universal for } C_f \text{ on } U\right\}$$

is a comeager set in $H(\mathbb{C}\backslash\{z_0\})$. If f is a transcendental entire function, Picard's little theorem implies either $f^n(\mathbb{C}) = \mathbb{C}$ for all $n \in \mathbb{N}$ or the existence of a point $a_f \in \mathbb{C}$ with $f^n(\mathbb{C}) = \mathbb{C}\backslash\{a_f\}$ for all $n \in \mathbb{N}$. As z_0 is a fixed point of f, we have $a_f \neq z_0$ in the latter case. Without loss of generality, let us now assume that we have $a_f \notin U$ (by shrinking, if necessary, the open neighbourhood U of z_0).

In view of Lemma A.3, it is natural to consider preimages of sets in $\mathcal{U}_0(U\backslash\{z_0\})$ under the iteration of f:

Lemma 4.1.4. *Let g be locally universal for C_f on U, $N \in \mathbb{N}_0$ and $W \in \mathcal{U}_0(U\backslash\{z_0\})$. Then we have*

$$\omega\big(f^{-N}(W), g, f\big) \supset H_\Phi\big(f^{-N}(W)\big).$$

Proof:

i) We first observe that we have $f^N(f^{-N}(W)) = W$. In fact, the inclusion $f^N(f^{-N}(W)) \subset W$ always holds, and the reverse inclusion is also true whenever the set W is contained in the range of f^N. As rational functions are surjective and as the possibly existing Picard exceptional value of f is not contained in $U \supset W$ in case that f is a transcendental entire function (cf. the above considerations), this is indeed the case. Hence, as the function Φ quasi-conjugates $f\big|_{A(z_0)}$ to $w \mapsto \lambda w$, we have

$$\Phi\big(f^{-N}(W)\big) = \frac{1}{\lambda^N} \cdot \Phi\big(f^N(f^{-N}(W))\big) = \frac{1}{\lambda^N} \cdot \Phi(W).$$

ii) Now, let $h \in H_\Phi\big(f^{-N}(W)\big)$. Then there exists some $\psi \in H\big(\Phi(f^{-N}(W))\big)$ with

$$h = \psi \circ \Phi\big|_{f^{-N}(W)} = \psi \circ \frac{1}{\lambda^N} \cdot \Phi \circ f^N\big|_{f^{-N}(W)} = \widetilde{h} \circ f^N\big|_{f^{-N}(W)},$$

where

$$\widetilde{h} := \psi \circ \frac{1}{\lambda^N} \cdot \Phi\big|_W \in H(W).$$

Due to $W \in \mathcal{U}_0(U\backslash\{z_0\})$ and the fact that g is locally universal for C_f on U, the set $\big\{g \circ f^n\big|_W : n \in \mathbb{N}\big\}$ is dense in $H(W)$. For $K \subset f^{-N}(W)$ compact, the set $f^N(K)$ is a compact subset of W so that there exists an $m_1 \in \mathbb{N}$ with

$$1 > \Big\|g \circ f^{m_1}\big|_W - \widetilde{h}\Big\|_{f^N(K)} = \|g \circ f^{m_1} \circ f^N - \widetilde{h} \circ f^N\|_K = \|g \circ f^{m_1+N} - h\|_K.$$

As the space $H(W)$ has no isolated points, each set $\big\{g \circ f^n\big|_W : n \geq m\big\}$ is dense in $H(W)$. Thus, we inductively find a strictly increasing sequence (m_j) in $\mathbb{N}$ with $\|g \circ f^{m_j+N} - h\|_K < 1/j$ for all $j \in \mathbb{N}$. Hence, we have uniform convergence $g \circ f^{m_j+N} \to h$ on K. Due to Lemma A.4, this already implies the existence of a strictly increasing sequence (n_k) in $\mathbb{N}$ with $g \circ f^{n_k} \to h$ locally uniformly on $f^{-N}(W)$, i.e. we have $h \in \omega\big(f^{-N}(W), g, f\big)$. □

Remark 4.1.5. For $g \in \mathcal{G}$, $W \in \mathcal{U}_0(U\backslash\{z_0\})$ and $N = 0$, Lemma 4.1.4 states $\omega(W, g, f) \supset H_\Phi(W)$. As each function $h \in H(W)$ can be written as

$$h = h \circ \varphi^{-1}\big|_{\varphi(W)} \circ \varphi\big|_W = \left(h \circ \varphi^{-1}\big|_{\Phi(W)}\right) \circ \Phi\big|_W \in H_\Phi(W),$$

we actually have $H_\Phi(W) = H(W)$ and hence $\omega(W, g, f) \supset H(W)$, which is equivalent to the fact that the set $\left\{g \circ f^n\big|_W : n \in \mathbb{N}\right\}$ is dense in $H(W)$. Therefore, the statement of Lemma 4.1.4 in case of $g \in \mathcal{G}$ and $N = 0$ is just the statement of Theorem 3.1.1.

Corollary 4.1.6. *Let g be locally universal for C_f on U and let $W \in \mathcal{U}_0(U\backslash\{z_0\})$ with $f(W) \subset W$. Then*

$$D := \bigcup_{n\in\mathbb{N}_0} f^{-n}(W)$$

is an open subset of $A(z_0)\backslash O^-(z_0)$ and we have $\omega(D, g, f) \supset H_\Phi(D)$.

Proof:

i) Because of $W \subset U \subset A(z_0)$, the backward invariance of $A(z_0)$ under f yields $D \subset A(z_0)$. Assuming that there exists a point $z \in D \cap O^-(z_0)$, there would exist integers $n \in \mathbb{N}_0$, $k \in \mathbb{N}$ with $f^n(z) \in W$ and $f^k(z) = z_0$. In case of $k \leq n$, we would obtain $z_0 = f^n(z) \in W \subset U\backslash\{z_0\}$, a contradiction. For $k > n$, it would follow that $z_0 = f^k(z) = f^{k-n}(f^n(z)) \in f^{k-n}(W) \subset W \subset U\backslash\{z_0\}$, the same contradiction. Thus, we have $D \subset A(z_0)\backslash O^-(z_0)$.

ii) Let $h \in H_\Phi(D)$. Then there exists some $\psi \in H(\Phi(D))$ with $h = \psi \circ \Phi\big|_D$. For $K \subset D$ compact, there exists a finite set $E \subset \mathbb{N}_0$ with $K \subset \bigcup_{n\in E} f^{-n}(W)$. Putting $N := \max E$, the invariance of W under f yields

$$f^N(K) \subset \bigcup_{n\in E} f^N(f^{-n}(W)) \subset \bigcup_{n\in E} f^{N-n}(W) \subset W.$$

Hence, K is a compact subset of $f^{-N}(W)$. According to Lemma 4.1.4, we have

$$h\big|_{f^{-N}(W)} = \psi\big|_{\Phi(f^{-N}(W))} \circ \Phi\big|_{f^{-N}(W)} \in H_\Phi\big(f^{-N}(W)\big) \subset \omega\big(f^{-N}(W), g, f\big).$$

Thus, there exists a strictly increasing sequence (m_j) in $\mathbb{N}$ such that $(g \circ f^{m_j})$ converges to h locally uniformly on $f^{-N}(W)$. In particular, we obtain uniform convergence $g \circ f^{m_j} \to h$ on K. Due to Lemma A.4, this implies the existence of a strictly increasing sequence (n_k) in $\mathbb{N}$ with $g \circ f^{n_k} \to h$ locally uniformly on D, i.e. we have $h \in \omega(D, g, f)$. □

In the following lemma, we will prove the existence of a set which fulfils the assumptions of Corollary 4.1.6, i.e. we will construct an open subset U_0 of $U\backslash\{z_0\}$ which has no holes and which is invariant under f. Moreover, we will show that the set $\bigcup_{n\in\mathbb{N}_0} f^{-n}(U_0)$ is an open dense subset of the basin of attraction $A(z_0)$.

Lemma 4.1.7.

i) Let $W \subset U$ be dense in U. Then the set $\bigcup_{n\in\mathbb{N}_0} f^{-n}(W)$ is dense in $A(z_0)$.

ii) There exists a set $U_0 \in \mathcal{U}_0(U\backslash\{z_0\})$ with $f(U_0) \subset U_0$ and such that U_0 is dense in U.

Proof:

i) Let $z \in A(z_0)$ and $\varepsilon > 0$. As $A(z_0)$ is an open subset of F_f, Remark 1.2.6 i) yields that we have locally uniform convergence $f^n \to z_0$ on $A(z_0)$. Thus, choosing $0 < r \leq \varepsilon$ with $U_r[z] \subset A(z_0)$, we have uniform convergence $f^n \to z_0$ on the set $U_r[z]$. Therefore, there exists an $N \in \mathbb{N}_0$ with $f^N(U_r[z]) \subset U$. As $f^N(U_r(z))$ is a non-empty open subset of U, the denseness of W in U implies the existence of a point $w \in W \cap f^N(U_r(z))$ so that there exists a point $\widetilde{z} \in U_r(z)$ with $f^N(\widetilde{z}) = w \in W$. Thus, we have $|\widetilde{z} - z| < r \leq \varepsilon$ and $\widetilde{z} \in f^{-N}(W) \subset \bigcup_{n\in\mathbb{N}_0} f^{-n}(W)$.

ii) Considering the conjugation map $\varphi : U \to V$, we may assume without loss of generality that we have $V = U_\delta(0)$ for some $\delta > 0$. We first show that there exists a set $V_0 \in \mathcal{U}_0(V\backslash\{0\})$ with $\lambda V_0 \subset V_0$ and such that V_0 is dense in V. If $\lambda \in \mathbb{R}$, we can simply choose $V_0 := V\backslash(-\delta, \delta)$ and all required properties are fulfilled. In case of $\lambda \notin \mathbb{R}$, we consider the connected set

$$S_0 := \big\{\delta\lambda^t : t \in (0, \infty)\big\} \cup \{0\},$$

which is the union of the trace of a logarithmic spiral and its “endpoint” 0 (cf. [20], p. 39f.). Putting $V_0 := V\backslash S_0$, we obtain that V_0 is an open subset of $V\backslash\{0\}$ which is dense in V and has no holes (cf. Figure 4.1 on the next page). In order to show that the inclusion $\lambda V_0 \subset V_0$ holds, now let $z \in V_0$. Then we have $z \in U_\delta(0)$ and $z \notin S_0$, and due to $0 < |\lambda| < 1$ it follows that $\lambda z \in V\backslash\{0\}$. Assuming that there exists some $t \in (0, \infty)$ with $\lambda z = \delta\lambda^t$, we would obtain $z = \delta\lambda^{t-1}$. On the one hand, this is impossible for $t \leq 1$ because in this case it would follow that $|z| = \delta|\lambda|^{t-1} \geq \delta$, a contradiction. On the other hand, $z = \delta\lambda^{t-1}$ is also not possible for $t > 1$ because we have $z \notin S_0$. Therefore, we

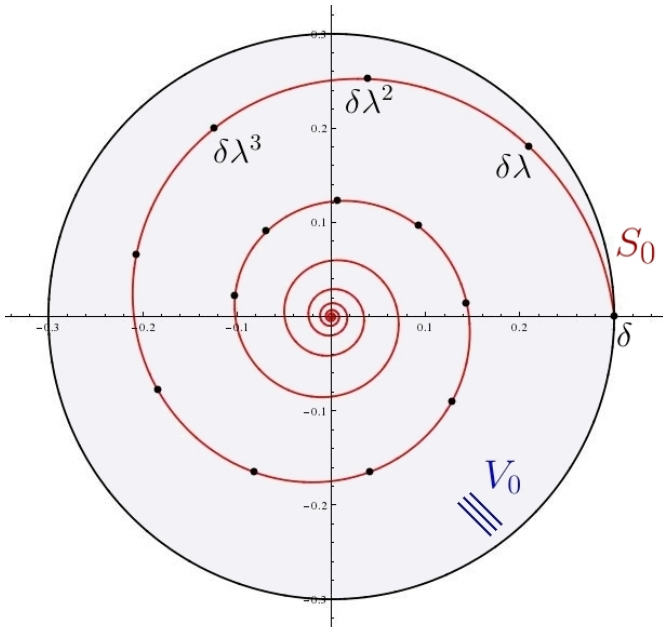

Figure 4.1: Mathematica plot of the sets S_0 and V_0 in case of $\delta = 0.3$ and $\lambda = 0.7 + 0.6i$

obtain that $\lambda z \in V \backslash S_0 = V_0$. Hence, there always exists a dense subset V_0 of V with $V_0 \in \mathcal{U}_0(V\backslash\{0\})$ and $\lambda V_0 \subset V_0$. We now consider the set

$$U_0 := \varphi^{-1}(V_0),$$

which is an open subset of $\varphi^{-1}(V\backslash\{0\}) = U\backslash\{z_0\}$. The denseness of V_0 in V and the continuity of $\varphi^{-1} : V \to U$ imply that $U_0 = \varphi^{-1}(V_0)$ is dense in $\varphi^{-1}(V) = U$. Moreover, as V_0 does not have any holes, the invariance of the number of holes implies that U_0 has no holes as well. Finally, as φ conjugates $f\big|_U$ to $w \mapsto \lambda w$, we obtain from $\lambda V_0 \subset V_0$ that

$$f(U_0) = f\big(\varphi^{-1}(V_0)\big) = \varphi^{-1}(\lambda V_0) \subset \varphi^{-1}(V_0) = U_0,$$

which completes the proof. □

Remark 4.1.8. According to Lemma 4.1.7 ii), there exists a dense subset U_0 of U with $U_0 \in \mathcal{U}_0(U\backslash\{z_0\})$ and $f(U_0) \subset U_0$. Lemma 4.1.7 i) and Corollary 4.1.6 yield that the set

$$G_0 := \bigcup_{n\in\mathbb{N}_0} f^{-n}(U_0)$$

is an open dense subset of $A(z_0)\backslash O^-(z_0)$ for which we have $\omega(G_0, g, f) \supset H_\Phi(G_0)$ for each locally universal function g for C_f on U. The following figure displays a

MATLAB plot of the set G_0 in case that f equals the finite Blaschke product

$$B : \mathbb{C}_\infty \to \mathbb{C}_\infty, \;\; B(z) := z \cdot \frac{z-\alpha}{1-\overline{\alpha}z},$$

where $\alpha = 0.4 + 0.6i$ and where U_0 is constructed as in the proof of Lemma 4.1.7 ii).

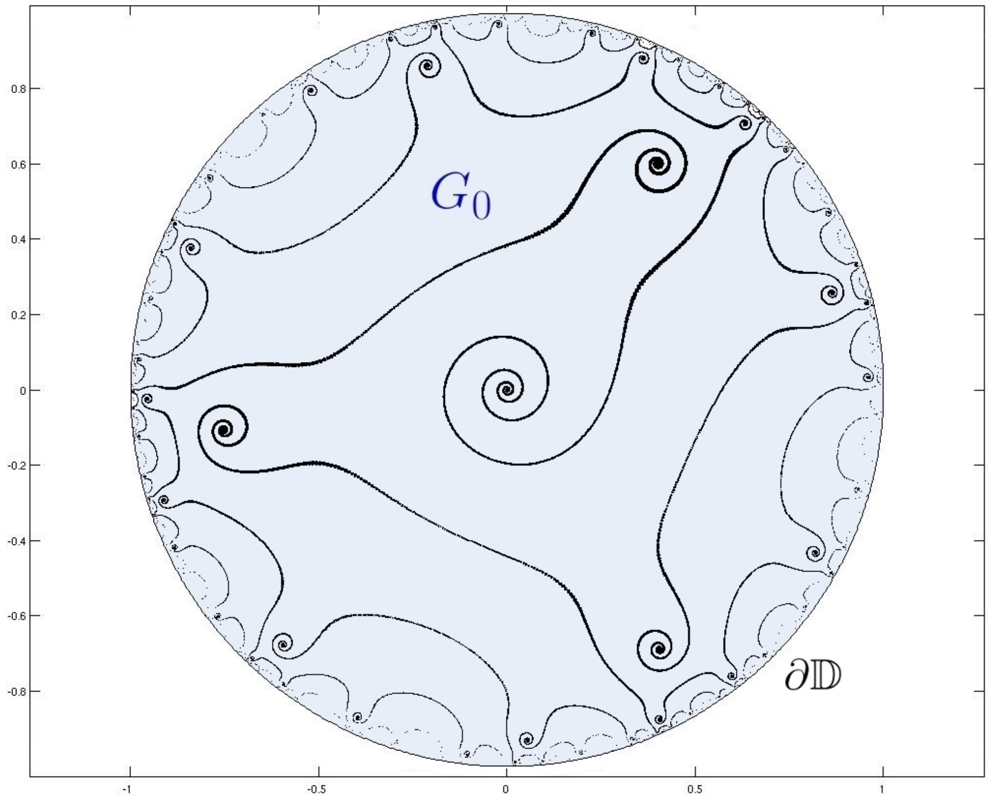

Figure 4.2

In this situation, we consider the attracting fixed point at the origin (with multiplier $B'(0) = -\alpha$) and its basin of attraction $A(0) = \mathbb{D}$ (cf. Remark 5.1.3 iv)). In fact, as G_0 is dense in $\mathbb{D}$, the set $\mathbb{D}\backslash G_0$ does not have any interior points, i.e. the bulges of the black curves in Figure 4.2 are just numerical errors. Moreover, the exceptional set $\mathbb{D}\backslash G_0$ looks like a union of pairwise disjoint traces of injective rectifiable paths.

It is well-known that the Hausdorff dimension of the trace of an injective rectifiable path defined on a non-singleton compact interval equals one (see e.g. [17], Theorem 6.2.7 and cf. Appendix B, p. 117). Considering the set U_0 which has been constructed in the proof of Lemma 4.1.7 ii) and putting $G_0 := \bigcup_{n\in\mathbb{N}_0} f^{-n}(U_0)$, we will now show that the Hausdorff dimension of the exceptional set $A(z_0)\backslash G_0$ is always one. Basic notations, definitions and properties concerning the concept of Hausdorff dimension are provided in Appendix B.

Lemma 4.1.9. *The set $A(z_0)\backslash G_0$ has Hausdorff dimension one.*

Proof:

i) Analogously to the proof of Lemma 4.1.7 ii), let $S_0 := \{\delta\lambda^t : t \in (0,\infty)\} \cup \{0\}$ be the union of the trace of the logarithmic spiral through the points $\delta\lambda^n$, $n \in \mathbb{N}$, and the origin (where $\delta > 0$ with $V = U_\delta(0)$). Then, for $V_0 := V\backslash S_0$, we have $U_0 = \varphi^{-1}(V_0)$. Moreover, putting $T_0 := \varphi^{-1}(S_0)$, we obtain

$$U = \varphi^{-1}(V) = \varphi^{-1}(V_0 \cup S_0) = \varphi^{-1}(V_0) \cup \varphi^{-1}(S_0) = U_0 \cup T_0$$

with $U_0 \cap T_0 = \emptyset$ and

$$f(T_0) = f(\varphi^{-1}(S_0)) = \varphi^{-1}(\lambda S_0) \subset \varphi^{-1}(S_0) = T_0.$$

ii) We show that the identity

$$A(z_0)\backslash G_0 = \bigcup_{n\in\mathbb{N}_0} f^{-n}(T_0)$$

holds. Indeed, for $z \in A(z_0)\backslash G_0$, we have convergence $f^n(z) \to z_0$ as well as $f^m(z) \notin U_0$ for all $m \in \mathbb{N}_0$. As U is an open neighbourhood of z_0, there exists an $N \in \mathbb{N}_0$ with $f^N(z) \in U$ so that we obtain $z \in f^{-N}(U\backslash U_0) = f^{-N}(T_0)$. In order to prove the reverse inclusion, we first observe that the inclusion $T_0 \subset U \subset A(z_0)$ holds. Hence, the backward invariance of $A(z_0)$ under f yields $\bigcup_{n\in\mathbb{N}_0} f^{-n}(T_0) \subset A(z_0)$. Moreover, for $w \in \bigcup_{n\in\mathbb{N}_0} f^{-n}(T_0)$, there exists an $N \in \mathbb{N}_0$ with $f^N(w) \in T_0$. Assuming that we have $w \in G_0$, there would exist an $\widetilde{N} \in \mathbb{N}_0$ with $f^{\widetilde{N}}(w) \in U_0$. But as the sets T_0 and U_0 are disjoint and invariant under f (see part i) and the end of the proof of Lemma 4.1.7 ii)), this is a contradiction. Thus, it follows that $w \notin G_0$.

iii) Defining the path $\sigma : (0,\infty) \to V$, $\sigma(t) := \delta\lambda^t$, we obtain that σ^* is an analytic arc with $S_0 = \sigma^* \cup \{0\}$. Hence, putting $\gamma := \varphi^{-1} \circ \sigma$, it follows that γ^* is also an analytic arc and that we have

$$T_0 = \varphi^{-1}(S_0) = \varphi^{-1}(\sigma^*) \cup \varphi^{-1}(\{0\}) = \gamma^* \cup \{z_0\}.$$

Thus, for $n \in \mathbb{N}_0$, Lemma B.7 and Lemma B.5 ii),iii) imply

$$\dim_H f^{-n}(T_0) = \max\left(\dim_H f^{-n}(\gamma^*),\, \dim_H f^{-n}\left(\{z_0\}\right)\right) = \max(1,0) = 1$$

(because of $\gamma^* \subset T_0 \subset A(z_0)$ and $\infty \notin A(z_0)$, the backward invariance of $A(z_0)$ under f yields $f^n(\infty) \notin \gamma^*$ in case that f is a rational function). Therefore, part ii) and Lemma B.5 ii) finally yield

$$\dim_H A(z_0) \backslash G_0 = \sup_{n \in \mathbb{N}_0} \dim_H f^{-n}(T_0) = 1,$$

which completes the proof. □

Remark 4.1.10. According to Corollary 4.1.6 and Lemma 4.1.7, we know that the inclusion

$$\omega(G_0, g, f) \supset H_\Phi(G_0)$$

holds for each locally universal function g for C_f on U, where the set G_0 is an open dense subset of $A(z_0) \backslash O^-(z_0)$. In general, open dense subsets of open sets in the complex plane can have arbitrarily small positive Lebesgue measure (indeed, for $D \subset \mathbb{C}$ open and $\varepsilon > 0$, consider the open set $\bigcup_{j \in \mathbb{N}} U_{r_j}(a_j)$, where $\{a_j : j \in \mathbb{N}\}$ is a dense subset of D and the radii r_j are defined by $r_j := \min(\sqrt{\varepsilon / 3^j \pi}, \operatorname{dist}(a_j, \partial D))$). However, in the above situation, we have $\dim_H A(z_0) \backslash G_0 = 1 < 2$ due to Lemma 4.1.9. Therefore, it follows from Lemma B.5 viii) that we have $\lambda_2(A(z_0) \backslash G_0) = 0$ and hence

$$\lambda_2(G_0) = \lambda_2(A(z_0)).$$

Moreover, as the set $A(z_0) \backslash G_0$ is not totally disconnected, Lemma B.5 vi) yields that 1 is the smallest possible value of the Hausdorff dimension of $A(z_0) \backslash G_0$. Thus, the open subset G_0 of $A(z_0) \backslash O^-(z_0)$, for which the inclusion $\omega(G_0, g, f) \supset H_\Phi(G_0)$ holds for each locally universal function g for C_f on U, is not only large in topological sense but also in measure-theoretic sense as well as in sense of Hausdorff dimension.

In the following, we will show that the reverse inclusion $\omega(D, g, f) \subset H_\Phi(D)$ indeed holds for each open subset D of $A(z_0) \backslash O^-(z_0)$ and all functions $g \in H(A(z_0) \backslash \{z_0\})$. As above, the special structure of the function Φ will play an important role here. For $z, w \in A(z_0)$, we write $z \sim w$ if there exists some $N \in \mathbb{N}_0$ with $f^N(z) = f^N(w)$. Then $\sim$ defines an equivalence relation on $A(z_0)$ for which the following is true:

Lemma 4.1.11. *Let $z, w \in A(z_0)$. Then we have $z \sim w$ if and only if $\Phi(z) = \Phi(w)$.*

Proof: Due to the considerations before Remark 4.1.1, $z \sim w$ implies $\Phi(z) = \Phi(w)$. On the other hand, in case of $\Phi(z) = \Phi(w)$, we choose an integer $N \in \mathbb{N}_0$ with

$f^N(z), f^N(w) \in U$ so that we obtain

$$\frac{\varphi(f^N(z))}{\lambda^N} = \Phi(z) = \Phi(w) = \frac{\varphi(f^N(w))}{\lambda^N}$$

by definition. Hence, we have $\varphi(f^N(z)) = \varphi(f^N(w))$, and the injectivity of φ yields $f^N(z) = f^N(w)$, which means $z \sim w$. □

Now, we can prove the following statement:

Lemma 4.1.12. *Let $g \in H(A(z_0)\backslash\{z_0\})$ and let D be an open subset of $A(z_0)\backslash O^-(z_0)$. Then we have*

$$\omega(D, g, f) \subset H_\Phi(D).$$

Proof: We write $\sim$ for the restriction of the above equivalence relation to D and $[z]_\sim := \{w \in D : z \sim w\}$ for the equivalence class of $z \in D$. Moreover, let $D/_\sim := \{[z]_\sim : z \in D\}$ be the quotient induced by $\sim$ and let

$$p : D \to D/_\sim, \;\; p(z) := [z]_\sim,$$

be the associated quotient map. For $h \in \omega(D, g, f)$, there exists a strictly increasing sequence (n_k) in $\mathbb{N}$ such that $(g \circ f^{n_k})$ converges to h locally uniformly on D. Thus, h is holomorphic on D, and it follows for all points $z, w \in D$ with $z \sim w$ that

$$h(z) = \lim_{k\to\infty} g(f^{n_k}(z)) = \lim_{k\to\infty} g(f^{n_k}(w)) = h(w).$$

Therefore, the map

$$\widetilde{h} : D/_\sim \to \mathbb{C}, \;\; \widetilde{h}\left([z]_\sim\right) := h(z),$$

is well-defined and fulfils $h = \widetilde{h} \circ p$. Moreover, Lemma 4.1.11 implies that the map

$$\widetilde{\Phi} : D/_\sim \to \Phi(D), \;\; \widetilde{\Phi}\left([z]_\sim\right) := \Phi(z),$$

is well-defined, bijective and fulfils $\Phi\big|_D = \widetilde{\Phi} \circ p$. Altogether, we have

$$h = \widetilde{h} \circ p = \widetilde{h} \circ \widetilde{\Phi}^{-1} \circ \widetilde{\Phi} \circ p = \left(\widetilde{h} \circ \widetilde{\Phi}^{-1}\right) \circ \Phi\big|_D.$$

As h is holomorphic on D and as $\Phi\big|_D : D \to \Phi(D)$ is holomorphic and surjective, Lemma A.5 yields

$$\widetilde{h} \circ \widetilde{\Phi}^{-1} \in H(\Phi(D))$$

so that we obtain $h \in H_\Phi(D)$. □

Combining Corollary 4.1.6, Lemma 4.1.7, Lemma 4.1.9 and Lemma 4.1.12, we have proved the following concluding statement:

Theorem 4.1.13. *The set* G_0 *is an open dense subset of* $A(z_0)\backslash O^-(z_0)$ *such that we have* $\dim_H A(z_0)\backslash G_0 = 1$ *and*

$$\omega(G_0, g, f) = H_\Phi(G_0)$$

for each locally universal function g *for* C_f *on* U*.*

Remark 4.1.14. In particular, according to the consideration after Definition 4.1.3, we obtain that the identity $\omega(G_0, g, f) = H_\Phi(G_0)$ holds for comeager many functions $g \in H(\mathbb{C}\backslash\{z_0\})$.

As the restriction $f\big|_{G_0} : G_0 \to G_0$ is not injective, the composition operator $C_{f|_{G_0}}$ cannot be hypercyclic on $H(G_0)$, i.e. there does not exist a function $g \in H(G_0)$ such that the set $\left\{g \circ f^n\big|_{G_0} : n \in \mathbb{N}\right\}$ is dense in $H(G_0)$ (cf. the considerations after Theorem 1.3.5). Now, it is our aim to change the above setting in such a way that we obtain "hypercyclic functions" for a composition operator $C_{\widetilde{f}}$, where the symbol $\widetilde{f} : \widetilde{G}_0 \to \widetilde{G}_0$ shall be a "suitable modification" of the function $f\big|_{G_0}$ and $\widetilde{G}_0$ shall be a "suitable modification" of the set G_0. As above, we consider on $G := A(z_0)$ the equivalence relation

$$z \sim w \quad :\Leftrightarrow \quad \text{there exists some } N \in \mathbb{N}_0 \text{ with } f^N(z) = f^N(w).$$

We write $[z]_\sim := \{w \in G : z \sim w\}$ for the equivalence class of $z \in G$ and we consider the quotient

$$G/_\sim := \{[z]_\sim : z \in G\}.$$

For $z, w \in G$, we have $z \sim w$ if and only if $\Phi(z) = \Phi(w)$ (cf. Lemma 4.1.11). Hence, the map

$$\widetilde{\Phi} : G/_\sim \to \mathbb{C}, \ \ \widetilde{\Phi}\left([z]_\sim\right) := \Phi(z),$$

is well-defined and injective. In case that f is a rational function, we have $\Phi(G) = \mathbb{C}$ (cf. Remark 4.1.1). If f is a transcendental entire function, Picard's little theorem implies either $f(\mathbb{C}) = \mathbb{C}$ or the existence of a point $a \in \mathbb{C}$ with $f(\mathbb{C}) = \mathbb{C}\backslash\{a\}$. In this situation, one can check that we have $\Phi(G) = \mathbb{C}$ if $f(\mathbb{C}) = \mathbb{C}$ or $a \notin G$, and that we have $\Phi(G) \supset \mathbb{C}\backslash\{\Phi(a)/\lambda^n : n \in \mathbb{N}_0\}$ if $a \in G$. Thus, we obtain that the set $\widetilde{\Phi}(G/_\sim) = \Phi(G)$ is always a domain in $\mathbb{C}$. As $\widetilde{\Phi}$ maps $G/_\sim$ bijectively onto $\Phi(G)$, we can consider the triple

$$\Big(G/_{\sim}, G/_{\sim}, \widetilde{\Phi}\Big)$$

as a Riemann surface (cf. Definition C.1). For the rest of this section, let us now assume that the possibly existing Picard exceptional value of f is not contained in the set G_0. Then the complete invariance of G_0 under f yields $f^n(G_0) = G_0$ for all integers $n \in \mathbb{N}$, and it follows that we not only have $\lambda^n \Phi(G_0) = \Phi\big(f^n(G_0)\big) \subset \Phi(G_0)$ but also

$$\lambda^n \Phi(G_0) = \Phi\big(f^n(G_0)\big) = \Phi(G_0), \;\; n \in \mathbb{N}$$

$\big($this identity will be crucial in the proof of Lemma 4.1.16 ii) below$\big)$. For $z \in G_0$ and $\zeta \in G$ with $z \sim \zeta$, there exist integers $N_1, N_2 \in \mathbb{N}_0$ with $f^{N_1}(z) \in U_0$ and $f^{N_2}(z) = f^{N_2}(\zeta)$. Hence, for $N := \max(N_1, N_2)$, the invariance of U_0 under f yields $f^N(\zeta) = f^N(z) \in U_0$ so that we obtain $\zeta \in G_0$. Thus, we have

$$[z]_{\sim} = \{w \in G : z \sim w\} = \{w \in G_0 : z \sim w\}.$$

For this reason, the set

$$G_0/_{\sim} := \big\{[z]_{\sim} : z \in G_0\big\}$$

is a well-defined subset of $G/_{\sim}$. As $\widetilde{\Phi}(G_0/_{\sim}) = \Phi(G_0)$ is an open subset of $\mathbb{C}$, the set $G_0/_{\sim}$ is an open subset of the Riemann surface $G/_{\sim}$, and we can consider the set of all complex-valued analytic functions on $G_0/_{\sim}$, which is given by

$$\widetilde{H}(G_0/_{\sim}) = \Big\{\widetilde{g} : G_0/_{\sim} \to \mathbb{C} : \widetilde{g} \circ \widetilde{\Phi}^{-1} \text{ is holomorphic on } \Phi(G_0)\Big\}$$

(cf. Definition C.2). Due to $\widetilde{\Phi}\big|_{G_0/_{\sim}} \in \widetilde{H}(G_0/_{\sim})$ and according to Lemma 4.1.11, we see that there exist non-constant functions which are contained in the set $\widetilde{H}(G_0/_{\sim})$. We now consider the map

$$\widetilde{f} : G_0/_{\sim} \to G_0/_{\sim}, \;\; \widetilde{f}\left([z]_{\sim}\right) := [f(z)]_{\sim}.$$

By definition of the equivalence relation $\sim$ and due to the invariance of G_0 under f, it follows that $\widetilde{f}$ is a well-defined injective self-map of $G_0/_{\sim}$. Moreover, according to $f(G_0) = G_0$, the map $\widetilde{f}$ is also surjective. Iterating $\widetilde{f}$, we have

$$\widetilde{f}^{\,n}\left([z]_{\sim}\right) = [f^n(z)]_{\sim}$$

for all $[z]_{\sim} \in G_0/_{\sim}$ and for all $n \in \mathbb{N}$. For $w \in \Phi(G_0)$, there exists some $z \in G_0$ with

$w = \Phi(z) = \widetilde{\Phi}\big([z]_\sim\big)$ so that we obtain

$$\widetilde{\Phi}\left(\widetilde{f}\left(\widetilde{\Phi}^{-1}(w)\right)\right) = \widetilde{\Phi}\left(\widetilde{f}\left([z]_\sim\right)\right) = \widetilde{\Phi}\big([f(z)]_\sim\big) = \Phi(f(z)) = \lambda\Phi(z) = \lambda w.$$

Hence, we can say that the map $\widetilde{\Phi}$ "conjugates" $\widetilde{f}$ to the linear function $w \mapsto \lambda w$ on $\Phi(G_0)$, i.e. $\widetilde{\Phi}\big|_{G_0/\sim} : G_0/_\sim \to \Phi(G_0)$ is bijective and fulfils

$$\widetilde{\Phi} \circ \widetilde{f}^{\,n} = \lambda^n \cdot \widetilde{\Phi}$$

on $G_0/_\sim$ for all $n \in \mathbb{N}$. In particular, it follows for each function $\widetilde{g} \in \widetilde{H}(G_0/_\sim)$ that the map

$$\left(\widetilde{g} \circ \widetilde{f}\right) \circ \widetilde{\Phi}^{-1}\big|_{\Phi(G_0)} = \left(\widetilde{g} \circ \widetilde{\Phi}^{-1}\big|_{\Phi(G_0)}\right) \circ \left(\widetilde{\Phi}\big|_{G_0/\sim} \circ \widetilde{f} \circ \widetilde{\Phi}^{-1}\big|_{\Phi(G_0)}\right)$$

is holomorphic. Therefore, the composition operator

$$C_{\widetilde{f}} : \widetilde{H}(G_0/_\sim) \to \widetilde{H}(G_0/_\sim),\;\; C_{\widetilde{f}}(\widetilde{g}) := \widetilde{g} \circ \widetilde{f},$$

is well-defined. Endowing the space $\widetilde{H}(G_0/_\sim)$ with the metric $\widetilde{d} := d_{\widetilde{H}(G_0/\sim)}$ which is defined in Appendix C, Lemma C.3 yields that the metric space $\big(\widetilde{H}(G_0/_\sim), \widetilde{d}\big)$ is complete and separable.

Lemma 4.1.15. *The composition operator $C_{\widetilde{f}}$ is continuous.*

Proof: Let $\widetilde{g} \in \widetilde{H}(G_0/_\sim)$ and let $(\widetilde{g_n})$ be a sequence in $\widetilde{H}(G_0/_\sim)$ with $\widetilde{g_n} \to \widetilde{g}$ in $\big(\widetilde{H}(G_0/_\sim), \widetilde{d}\big)$. Then we have locally uniform convergence

$$\widetilde{g_n} \circ \widetilde{\Phi}^{-1}\big|_{\Phi(G_0)} \to \widetilde{g} \circ \widetilde{\Phi}^{-1}\big|_{\Phi(G_0)}$$

(cf. the considerations before Lemma C.3). For $K \subset \Phi(G_0)$ compact, the set λK is a compact subset of $\lambda\Phi(G_0) \subset \Phi(G_0)$. According to the above considerations, we have $\widetilde{\Phi}\big(\widetilde{f}(\widetilde{\Phi}^{-1}(K))\big) = \lambda K$ so that we obtain

$$\begin{aligned}
& \left\|C_{\widetilde{f}}(\widetilde{g_n}) \circ \widetilde{\Phi}^{-1} - C_{\widetilde{f}}(\widetilde{g}) \circ \widetilde{\Phi}^{-1}\right\|_K \\
= \;& \left\|\widetilde{g_n} \circ \widetilde{f} \circ \widetilde{\Phi}^{-1} - \widetilde{g} \circ \widetilde{f} \circ \widetilde{\Phi}^{-1}\right\|_K \\
= \;& \left\|\widetilde{g_n} \circ \widetilde{\Phi}^{-1} \circ (\widetilde{\Phi} \circ \widetilde{f} \circ \widetilde{\Phi}^{-1}) - \widetilde{g} \circ \widetilde{\Phi}^{-1} \circ (\widetilde{\Phi} \circ \widetilde{f} \circ \widetilde{\Phi}^{-1})\right\|_K \\
= \;& \left\|\widetilde{g_n} \circ \widetilde{\Phi}^{-1} - \widetilde{g} \circ \widetilde{\Phi}^{-1}\right\|_{\lambda K} \to 0.
\end{aligned}$$

Thus, we have locally uniform convergence $C_{\widetilde{f}}(\widetilde{g_n}) \circ \widetilde{\Phi}^{-1} \to C_{\widetilde{f}}(\widetilde{g}) \circ \widetilde{\Phi}^{-1}$ on $\Phi(G_0)$, which is equivalent to convergence $C_{\widetilde{f}}(\widetilde{g_n}) \to C_{\widetilde{f}}(\widetilde{g})$ in $\big(\widetilde{H}(G_0/_\sim), \widetilde{d}\,\big)$. □

The following lemma states basic properties of holes of compact subsets of $\Phi(G_0)$:

Lemma 4.1.16. *Let $K \subset \Phi(G_0)$ be compact. Then the following statements hold:*

i) Each hole of K does not contain the origin.

ii) If K is $\Phi(G_0)$-convex, then $\lambda^n K$ is $\Phi(G_0)$-convex for all $n \in \mathbb{N}$.

Proof:

i) By definition and because of $\Phi \circ f = \lambda\Phi$ on G, we have

$$\begin{aligned} f^{-n}(U_0) &= f^{-n}\big(\varphi^{-1}(V_0)\big) \subset f^{-n}\big(\Phi^{-1}(V_0)\big) = (\Phi \circ f^n)^{-1}(V_0) \\ &= (\lambda^n \Phi)^{-1}(V_0) = \Phi^{-1}\left(\frac{1}{\lambda^n}\, V_0\right), \quad n \in \mathbb{N}. \end{aligned}$$

Considering the logarithmic spirals $S_n := \{\delta\lambda^t : t > -n\} \cup \{0\}$, we obtain

$$\frac{1}{\lambda^n}\, V_0 = U_{\delta/|\lambda|^n}(0) \backslash S_n, \quad n \in \mathbb{N}_0.$$

Hence, it follows that

$$\begin{aligned} \Phi(G_0) &= \Phi\left(\bigcup_{n \in \mathbb{N}_0} f^{-n}(U_0)\right) \subset \bigcup_{n \in \mathbb{N}_0} \Phi\left(\Phi^{-1}\left(\frac{1}{\lambda^n}\, V_0\right)\right) \subset \bigcup_{n \in \mathbb{N}_0} \frac{1}{\lambda^n}\, V_0 \\ &= \mathbb{C} \backslash \left(\{\delta\lambda^t : t \in \mathbb{R}\} \cup \{0\}\right). \end{aligned}$$

Thus, K is contained in the “spiral domain” $\mathbb{C} \backslash (\{\delta\lambda^t : t \in \mathbb{R}\} \cup \{0\})$. Now, let H be a hole of K. According to Lemma A.7 i), H is an open subset of $\mathbb{C}$. Assuming that we have $0 \in H$, it would follow that the boundary of H intersects the spiral $S := \{\delta\lambda^t : t \in \mathbb{R}\} \cup \{0\}$. Therefore, Lemma A.7 ii) would yield

$$\emptyset \neq S \cap \partial H \subset S \cap K \subset S \cap \mathbb{C} \backslash S = \emptyset,$$

a contradiction. Hence, we have $0 \notin H$.

ii) Let K be $\Phi(G_0)$-convex, $N \in \mathbb{N}$ and let H be a hole of $\lambda^N K$. According to Lemma A.7 iii), it follows that $\lambda^{-N} H$ is a hole of K. Thus, the $\Phi(G_0)$-convexity of K implies the existence of a point $w \in \lambda^{-N} H$ with $w \in \mathbb{C} \backslash \Phi(G_0)$, and the considerations on p. 67 yield

$$\lambda^N w \in H \cap \big(\mathbb{C}\backslash\lambda^N\Phi(G_0)\big) = H \cap \big(\mathbb{C}\backslash\Phi(G_0)\big)$$

so that the $\Phi(G_0)$-convexity of $\lambda^N K$ is shown (cf. Remark 1.5.3 iv)). □

Similarly to Definition 1.3.2 ii), we now introduce the following definition:

Definition 4.1.17. A function $\widetilde{g} \in \widetilde{H}(G_0/_\sim)$ is called *hypercyclic* for $C_{\widetilde{f}}$ if the set

$$\Big\{(C_{\widetilde{f}})^n(\widetilde{g}) : n \in \mathbb{N}\Big\}$$

is dense in $\widetilde{H}(G_0/_\sim)$. The composition operator $C_{\widetilde{f}}$ is called *hypercyclic* if there exists a hypercyclic function for $C_{\widetilde{f}}$.

As the modified symbol $\widetilde{f}$ is injective on the set $G_0/_\sim$ (in contrast to the fact that f is not injective on the set G_0), we have a chance of obtaining hypercyclicity of $C_{\widetilde{f}}$ (cf. the considerations after Theorem 1.3.5). Indeed, the following statement holds:

Theorem 4.1.18. *The set of all functions in $\widetilde{H}(G_0/_\sim)$ which are hypercyclic for $C_{\widetilde{f}}$ is a comeager set in $\widetilde{H}(G_0/_\sim)$.*

Proof: According to Lemma 4.1.15 and the preceding considerations, the space $\widetilde{H}(G_0/_\sim)$ is complete and separable, and the composition operator $C_{\widetilde{f}}$ is continuous. Therefore, the universality criterion yields that it suffices to show that the sequence $\big((C_{\widetilde{f}})^n\big)$ is topologically transitive (cf. Theorem 1.3.6). In order to do so, let $\widetilde{\mathcal{U}}_1$ as well as $\widetilde{\mathcal{U}}_2$ be non-empty open subsets of $\widetilde{H}(G_0/_\sim)$. Lemma C.4 implies that the sets

$$\mathcal{U}_j := \Big\{\widetilde{g} \circ \widetilde{\Phi}^{-1}\big|_{\Phi(G_0)} : \widetilde{g} \in \widetilde{\mathcal{U}}_j\Big\}, \;\; j = 1, 2,$$

are open subsets of $H(\Phi(G_0))$. Hence, for $\widetilde{h}_1 \in \widetilde{\mathcal{U}}_1$ and $\widetilde{h}_2 \in \widetilde{\mathcal{U}}_2$, there exist $\varepsilon_1, \varepsilon_2 > 0$ and non-empty compact subsets L_1, L_2 of $\Phi(G_0)$ with

$$V_{\varepsilon_j, L_j, \Phi(G_0)}\left(\widetilde{h}_j \circ \widetilde{\Phi}^{-1}\big|_{\Phi(G_0)}\right) \subset \mathcal{U}_j, \;\; j = 1, 2$$

(cf. Remark 1.1.3 ii)). Putting $\varepsilon := \min(\varepsilon_1, \varepsilon_2)$ and $h_j := \widetilde{h}_j \circ \widetilde{\Phi}^{-1}\big|_{\Phi(G_0)} \in H(\Phi(G_0))$ as well as $K_j := \widehat{(L_j)}_{\Phi(G_0)}$ for $j = 1, 2$, Remark 1.5.3 ii) implies that the sets K_1 and K_2 are $\Phi(G_0)$-convex with

$$V_{\varepsilon, K_j, \Phi(G_0)}(h_j) \subset V_{\varepsilon_j, L_j, \Phi(G_0)}(h_j) \subset \mathcal{U}_j, \;\; j = 1, 2.$$

The proof of Lemma 4.1.16 i) shows that we have $0 \notin \Phi(G_0)$. Hence, there exists some $r > 0$ with $K_1 \subset \mathbb{C}\backslash U_r(0)$. As z_0 is an attracting fixed point of f, we have $0 < |\lambda| < 1$ so that we can find an $N \in \mathbb{N}$ with $\lambda^N K_2 \subset U_r(0)$. In particular, we obtain

$$K_1 \cap \lambda^N K_2 = \emptyset$$

and we have $\lambda^N K_2 \subset \lambda^N \Phi(G_0) \subset \Phi(G_0)$. Moreover, according to Lemma 4.1.16 ii), the set $\lambda^N K_2$ is $\Phi(G_0)$-convex. Denoting by C_{K_1} and $C_{\lambda^N K_2}$ the components of $\mathbb{C}_\infty\backslash K_1$ and $\mathbb{C}_\infty\backslash\lambda^N K_2$ which contain ∞, respectively, we directly obtain

$$K_1 \subset \mathbb{C}_\infty\backslash U_r(0) \subset C_{\lambda^N K_2}.$$

Furthermore, because of $\lambda^N K_2 \subset U_r(0) \subset \mathbb{C}_\infty\backslash K_1$, there exists a component H of $\mathbb{C}_\infty\backslash K_1$ with $\lambda^N K_2 \subset H$. Assuming that we have $H \neq C_{K_1}$, we would obtain $\infty \notin H$, i.e. H would be a hole of K_1. Thus, Lemma 4.1.16 i) would yield $0 \notin H$. As we have $\partial H \subset K_1 \subset \mathbb{C}\backslash U_r(0)$ due to Lemma A.7 ii), it would follow that H must be contained in $\mathbb{C}\backslash U_r(0)$, and we would obtain that $\lambda^N K_2 \subset H \subset \mathbb{C}\backslash U_r(0)$, a contradiction. Therefore, we have $\lambda^n K_2 \subset H = C_{K_1}$. Hence, K_1 is contained in the component of $\mathbb{C}_\infty\backslash\lambda^N K_2$ which contains ∞, and $\lambda^N K_2$ is contained in the component of $\mathbb{C}_\infty\backslash K_1$ which contains ∞. For this reason, each hole of the disjoint union of the compact set K_1 with the compact set $\lambda^N K_2$ is either a hole of K_1 or a hole of $\lambda^N K_2$ so that the $\Phi(G_0)$-convexity of K_1 and $\lambda^N K_2$ implies that $K_1 \cup \lambda^N K_2$ is also $\Phi(G_0)$-convex. Thus, we can choose from each hole of $K_1 \cup \lambda^N K_2$ a point which lies in $\mathbb{C}\backslash\Phi(G_0)$. Let A be the union of these points and let $B := A \cup \{\infty\}$. Then we have $B \subset \mathbb{C}_\infty\backslash\Phi(G_0) \subset \mathbb{C}_\infty\backslash(K_1 \cup \lambda^N K_2)$ and $B \cap C \neq \emptyset$ for all components C of $\mathbb{C}_\infty\backslash(K_1 \cup \lambda^N K_2)$. We now consider the function

$$\psi : K_1 \cup \lambda^N K_2 \to \mathbb{C},\ \ \psi(w) := \begin{cases} h_1(w), & \text{if } w \in K_1 \\ h_2\left(\widetilde{\Phi}\left(\left(\widetilde{f}^N\right)^{-1}\left(\widetilde{\Phi}^{-1}(w)\right)\right)\right), & \text{if } w \in \lambda^N K_2 \end{cases}.$$

For $w \in \Phi(G_0)$, the "conjugation" $\widetilde{\Phi} \circ \widetilde{f} = \lambda \cdot \widetilde{\Phi}$ on $G_0/_\sim$ yields

$$\widetilde{\Phi}\left(\left(\widetilde{f}^N\right)^{-1}\left(\widetilde{\Phi}^{-1}(w)\right)\right) = \left(\widetilde{\Phi} \circ \widetilde{f}^N \circ \widetilde{\Phi}^{-1}\right)^{-1}(w) = \lambda^{-N} w$$

(cf. the considerations on p. 68). Thus, as h_1 and h_2 are holomorphic on $\Phi(G_0)$ and as the disjoint sets K_1 and $\lambda^N K_2$ are compact subsets of $\Phi(G_0)$, we see that ψ can be extended holomorphically to an open neighbourhood of $K_1 \cup \lambda^N K_2$. Hence, Runge's theorem implies the existence of a rational function R having poles only in $B \subset \mathbb{C}_\infty \backslash \Phi(G_0)$ such that

$$\|\psi - R\|_{K_1 \cup \lambda^N K_2} < \varepsilon.$$

The restriction $R\big|_{\Phi(G_0)}$ is holomorphic on $\Phi(G_0)$, and due to $\psi = h_1$ on K_1 we obtain

$$\|h_1 - R\|_{K_1} = \|\psi - R\|_{K_1} \leq \|\psi - R\|_{K_1 \cup \lambda^N K_2} < \varepsilon.$$

Therefore, we have $R\big|_{\Phi(G_0)} \in V_{\varepsilon, K_1, \Phi(G_0)}(h_1) \subset \mathcal{U}_1$ and thus $R\big|_{\Phi(G_0)} \circ \widetilde{\Phi}\big|_{G_0/\sim} \in \widetilde{\mathcal{U}}_1$. Moreover, according to the above "conjugation", we further obtain

$$\begin{aligned}
&\left\| h_2 - R \circ \left(\widetilde{\Phi} \circ \widetilde{f}^{\,N} \circ \widetilde{\Phi}^{-1} \right) \right\|_{K_2} \\
=\ & \left\| h_2 \circ \left(\widetilde{\Phi} \circ \widetilde{f}^{\,N} \circ \widetilde{\Phi}^{-1} \right)^{-1} - R \right\|_{\widetilde{\Phi}(\widetilde{f}^N(\widetilde{\Phi}^{-1}(K_2)))} \\
=\ & \left\| h_2 \circ \widetilde{\Phi} \circ \left(\widetilde{f}^{\,N} \right)^{-1} \circ \widetilde{\Phi}^{-1} - R \right\|_{\lambda^N K_2} \\
=\ & \|\psi - R\|_{\lambda^N K_2} \leq \|\psi - R\|_{K_1 \cup \lambda^N K_2} < \varepsilon.
\end{aligned}$$

Hence, we have $R\big|_{\Phi(G_0)} \circ \widetilde{\Phi}\big|_{G_0/\sim} \circ \widetilde{f}^{\,N} \circ \widetilde{\Phi}^{-1}\big|_{\Phi(G_0)} \in V_{\varepsilon, K_2, \Phi(G_0)}(h_2) \subset \mathcal{U}_2$ and thus

$$(C_{\widetilde{f}})^N \left(R\big|_{\Phi(G_0)} \circ \widetilde{\Phi}\big|_{G_0/\sim} \right) = R\big|_{\Phi(G_0)} \circ \widetilde{\Phi}\big|_{G_0/\sim} \circ \widetilde{f}^{\,N} \in \widetilde{\mathcal{U}}_2.$$

Altogether, it follows

$$(C_{\widetilde{f}})^N \left(R\big|_{\Phi(G_0)} \circ \widetilde{\Phi}\big|_{G_0/\sim} \right) \in (C_{\widetilde{f}})^N (\widetilde{\mathcal{U}}_1) \cap \widetilde{\mathcal{U}}_2 \neq \emptyset,$$

so that the topological transitivity of the sequence $\left((C_{\widetilde{f}})^n\right)$ is shown. □

4.2 Leau Domains

Let f be a transcendental entire function or a rational function of degree $d \geq 2$ which has a neutral fixed point at $z_0 \in \mathbb{C}$. We put

$$m := \min\left\{n \in \mathbb{N} : f^{(n+1)}(z_0) \neq 0\right\} \in \mathbb{N}.$$

As we have seen in Section 3.2, the Leau-Fatou flower theorem implies the existence of m attracting petals $P_1, \ldots, P_m$ for f, which are pairwise disjoint domains for which we have $P_k \in \mathcal{U}_0(\mathbb{C})$, $f(P_k) \subset P_k$, $z_0 \in \partial P_k$ and $f^n\big|_{P_k} \to z_0$ uniformly. Moreover, for each $k \in \{1, \ldots, m\}$, there exists a simply connected domain $V_k \subset \mathbb{C}$ containing a right half-plane with $V_k + 1 \subset V_k$ as well as a conformal map $\varphi_k : P_k \to V_k$ which conjugates $f\big|_{P_k}$ to the translation $w \mapsto w + 1$ on V_k, i.e. the equation

$$\varphi_k \circ f^n = \varphi_k + n$$

holds on P_k for all integers $n \in \mathbb{N}$ (cf. Section 3.2, p. 33f.). For each $k \in \{1, \ldots, m\}$, we now consider the open set

$$G_k := \bigcup_{n \in \mathbb{N}_0} f^{-n}(P_k),$$

which is completely invariant under f. As the Leau domain D_k of f which contains P_k (i.e. D_k is the component of F_f containing P_k) can be written as $D_k = \bigcup_{n \in \mathbb{N}_0} D_{k,n}$, where $D_{k,n}$ is the component of $f^{-n}(P_k)$ which contains P_k (see e.g. [49], p. 76), we obtain that D_k is a subset of G_k. Throughout this whole section, we assume that we have $\infty \notin G_k$ in case of a rational symbol f, so that the invariance of G_k under f implies that f is holomorphic on G_k. Analogously to the situation of Section 4.1, it is well-known that the conjugation map φ_k can be extended holomorphically to G_k (see e.g. [37], Corollary 10.9). To see this, we set

$$\Phi_k : G_k \to \mathbb{C}, \;\; \Phi_k(z) := \varphi_k(f^N(z)) - N,$$

where $N \in \mathbb{N}_0$ is chosen such that $f^N(z) \in P_k$. Then Φ_k is a well-defined holomorphic function on G_k. Indeed, for $z \in G_k$ and $N, M \in \mathbb{N}_0$ with $f^N(z), f^M(z) \in P_k$ and $N < M$, there exists some $p \in \mathbb{N}$ with $N + p = M$, so that the above conjugation implies

$$\begin{aligned} \varphi_k(f^M(z)) - M &= \varphi_k(f^p(f^N(z))) - p - N = \varphi_k(f^N(z)) + p - p - N \\ &= \varphi_k(f^N(z)) - N. \end{aligned}$$

By definition, we have $\Phi_k\big|_{P_k} = \varphi_k$ and the function Φ_k fulfils the same functional equation as φ_k, but now on the whole set G_k (cf. [37], Corollary 10.9). In fact, for $z \in G_k$ we choose some $N \in \mathbb{N}_0$ with $f^N(f(z)) \in P_k$ and we obtain

$$\Phi_k(f(z)) = \varphi_k(f^N(f(z))) - N = \varphi_k(f^{N+1}(z)) - (N+1) + 1 = \Phi_k(z) + 1.$$

Inductively, it follows that

$$\Phi_k \circ f^n = \Phi_k + n$$

holds on G_k for all $n \in \mathbb{N}$. Hence, for $z, w \in G_k$ and $N \in \mathbb{N}$ with $f^N(z) = f^N(w)$, we have

$$\Phi_k(z) = \Phi_k(f^N(z)) - N = \Phi_k(f^N(w)) - N = \Phi_k(w),$$

i.e. the non-constant holomorphic function Φ_k assumes the same value at all points in G_k which eventually coincide under the iteration of f.

Remark 4.2.1. In case that f is a rational function, it is well-known that we have $\Phi_k(G_k) \supset \Phi_k(D_k) = \mathbb{C}$ (see e.g. [49], p. 76). By slightly adapting the proof of this statement, one can check that $\Phi_k(G_k)$ is always a dense subset of the complex plane in case that f is a transcendental entire function. Hence, in the above situation, the function Φ_k quasiconjugates $f\big|_{G_k}$ to the translation $w \mapsto w + 1$ on $\mathbb{C}$.

For the same reasons as in Section 4.1, we consider for each $k \in \{1, \dots, m\}$ the set

$$H_{\Phi_k}(G_k) := \big\{\psi \circ \Phi_k : \psi \in H(\Phi_k(G_k))\big\},$$

which is a subspace of $H(G_k)$, and we introduce the following definition:

Definition 4.2.2. A function $g \in H(G_k)$ is called *locally universal* for C_f on P_k if the set $\big\{g \circ f^n\big|_{P_k} : n \in \mathbb{N}\big\}$ is dense in $H(P_k)$.

For each $k \in \{1, \dots, m\}$, Theorem 3.2.4 and Remark 3.2.5 state that the set

$$\mathcal{G}_k := \Big\{g \in H(\mathbb{C}\backslash\{z_0\}) : g\big|_{G_k} \text{ locally universal for } C_f \text{ on } P_k\Big\}$$

is a comeager set in $H(\mathbb{C}\backslash\{z_0\})$. Analogously to the situation of Section 4.1, we now assume that the possibly existing Picard exceptional value of f is not contained in the attracting petal P_k for each $k \in \{1, \dots, m\}$ in case that f is a transcendental entire function. Similarly to Theorem 4.1.13, the following statement holds:

Theorem 4.2.3. *For each $k \in \{1, \dots, m\}$, we have*

$$\omega(G_k, g, f) = H_{\Phi_k}(G_k)$$

for each locally universal function g for C_f on P_k. In particular, the identity $\omega(G_k, g, f) = H_{\Phi_k}(G_k)$ holds for comeager many functions $g \in H(\mathbb{C}\backslash\{z_0\})$.

Proof: Let $k \in \{1, \dots, m\}$.

i) Analogously to the proof of Lemma 4.1.4 and observing the above quasiconjugation, one can show that the inclusion $\omega\big(f^{-N}(P_k), g, f\big) \supset H_{\Phi_k}\big(f^{-N}(P_k)\big)$

holds for each locally universal function g for C_f on P_k and for each $N \in \mathbb{N}_0$. Hence, analogously to the proof of Corollary 4.1.6, it follows that we have $\omega(G_k, g, f) \supset H_{\Phi_k}(G_k)$ for each locally universal function g for C_f on P_k.

ii) For two points $z, w \in G_k$, we write $z \sim_k w$ if there exists some $N \in \mathbb{N}_0$ with $f^N(z) = f^N(w)$. Then $\sim_k$ defines an equivalence relation on G_k and we have $z \sim_k w$ if and only if $\Phi_k(z) = \Phi_k(w)$. Indeed, in case of $z \sim_k w$, the considerations before Remark 4.2.1 imply $\Phi_k(z) = \Phi_k(w)$. On the other hand, if $\Phi_k(z) = \Phi_k(w)$, the invariance of P_k under f allows us to choose an integer $N \in \mathbb{N}$ with $f^N(z), f^N(w) \in P_k$ so that we obtain

$$\varphi_k(f^N(z)) - N = \Phi_k(z) = \Phi_k(w) = \varphi_k(f^N(w)) - N$$

by definition. Hence, the injectivity of φ_k yields $f^N(z) = f^N(w)$, which means $z \sim_k w$.

iii) Applying part ii), one can show analogously to the proof of Lemma 4.1.12 that we have $\omega(G_k, g, f) \subset H_{\Phi_k}(G_k)$ for all functions $g \in H(G_k)$. □

Remark 4.2.4. In contrast to the situation of Theorem 4.1.13, we do not have to restrict ourselves to a "large" open subset of G_k in order to obtain the identity as stated in Theorem 4.2.3. The reason for this is that each attracting petal P_k has no holes and is invariant under f. Therefore, the inclusion $\omega(G_k, g, f) \supset H_{\Phi_k}(G_k)$ in part i) of the proof of Theorem 4.2.3 can be proved in exactly the same way as in the proofs of Lemma 4.1.4 and Corollary 4.1.6.

Defining $P := \bigcup_{k=1}^m P_k$ to be the union of all attracting petals, we now consider the set

$$G := \bigcup_{k=1}^{m} G_k = \bigcup_{k=1}^{m} \bigcup_{n \in \mathbb{N}_0} f^{-n}(P_k) = \bigcup_{n \in \mathbb{N}_0} f^{-n}(P).$$

As all attracting petals are pairwise disjoint and invariant under f, we obtain that all sets G_k are pairwise disjoint. Moreover, the complete invariance of each set G_k under f implies that G is also completely invariant under f. For the rest of this section, let us assume that we have $m \geq 2$. The question arises to determine the sets $\omega(G, g, f)$ for suitable functions $g \in H(G)$. In order to do so, it seems natural to consider the function

$$\Phi : G \to \mathbb{C}, \;\; \Phi(z) := \Phi_k(z), \;\text{ if } z \in G_k.$$

As each function Φ_k quasiconjugates $f\big|_{G_k}$ to the translation $w \mapsto w+1$ on $\mathbb{C}$ and as each set G_k is invariant under f, we obtain that Φ quasiconjugates $f\big|_G$ to $w \mapsto w+1$ on $\mathbb{C}$, i.e. the equation

$$\Phi \circ f^n = \Phi + n$$

holds on G for all integers $n \in \mathbb{N}$. For $D \subset G$ open, we consider the set

$$H_\Phi(D) := \big\{\psi \circ \Phi\big|_D : \psi \in H(\Phi(D))\big\},$$

which is a subspace of $H(D)$, and we introduce the following definition:

Definition 4.2.5. A function $g \in H(G)$ is called *locally universal* for C_f on P if the set $\big\{g \circ f^n\big|_P : n \in \mathbb{N}\big\}$ is dense in $H(P)$.

Using this terminology, Theorem 3.2.4 states that the set

$$\mathcal{G} := \big\{g \in H(\mathbb{C}\backslash\{z_0\}) : g\big|_G \text{ locally universal for } C_f \text{ on } P\big\}$$

is a comeager set in $H(\mathbb{C}\backslash\{z_0\})$. Due to the above quasiconjugation, the following statement can be proved analogously to the first part of the proof of Theorem 4.2.3 (using exactly the same techniques as in the proofs of Lemma 4.1.4 and Corollary 4.1.6):

Lemma 4.2.6. *The inclusion $\omega(G, g, f) \supset H_\Phi(G)$ holds for each locally universal function g for C_f on P.*

However, in this situation, we cannot prove the reverse inclusion as we have done in parts ii) and iii) of the proof of Theorem 4.2.3. The reason for this is the following: Considering on G the equivalence relation

$$z \sim w \quad :\Leftrightarrow \quad \text{there exists some } N \in \mathbb{N}_0 \text{ with } f^N(z) = f^N(w),$$

we obtain that the equality $\Phi(z) = \Phi(w)$ for two points $z, w \in G$ now does not imply $z \sim w$ anymore (however, due to the above quasiconjugation, $z \sim w$ still yields $\Phi(z) = \Phi(w)$). Indeed, let $k, l \in \{1, \dots, m\}$ with $k \neq l$. Due to Remark 4.2.1, the intersection $\Phi_k(G_k) \cap \Phi_l(G_l)$ is dense in $\mathbb{C}$ and in particular non-empty. Thus, there exist points $z_k \in G_k$ and $z_l \in G_l$ with $\Phi_k(z_k) = \Phi_l(z_l)$, i.e. we have $\Phi(z_k) = \Phi(z_l)$. But as G_k and G_l are disjoint and invariant under f, we have $f^n(z_k) \neq f^n(z_l)$ for all $n \in \mathbb{N}$ and hence $z_k \not\sim z_l$. According to this observation, it might be the case that the set $\omega(G,g,f)$ is a proper superset of the subspace $H_\Phi(G)$. In the following, we will see that this is indeed the case.

For two points $z,w \in G$ with $\Phi(z) = \Phi(w)$, we have just seen that we cannot conclude $z \sim w$ anymore, and we have seen that this implication fails because it is possible that z and w are contained in different sets G_k and G_l. In order to avoid this trouble, it is reasonable to consider the function

$$\Phi_* : G \to \mathbb{C} \times \{1,\dots,m\}, \;\; \Phi_*(z) := (\Phi_k(z), k), \text{ if } z \in G_k.$$

As each function Φ_k quasiconjugates $f\big|_{G_k}$ to the translation $w \mapsto w+1$ and as each set G_k is invariant under f, we obtain that Φ_* "quasiconjugates" $f\big|_G$ to the map $(w,k) \mapsto (w,k) + (1,0)$ on $\mathbb{C} \times \{1,\dots,m\}$, i.e. the equation

$$\Phi_* \circ f^n = \Phi_* + (n,0)$$

holds on G for all integers $n \in \mathbb{N}$. Thus, for $z,w \in G$ with $z \sim w$, it follows that $\Phi_*(z) = \Phi_*(w)$. Moreover, in contrast to the above considerations about the function Φ, now the reverse implication also holds. Indeed, for $z,w \in G$ with $z \in G_k$, $w \in G_l$ and $\Phi_*(z) = \Phi_*(w)$, we have $(\Phi_k(z),k) = (\Phi_l(w),l)$. Hence, we obtain $k = l$ and thus $\Phi_k(z) = \Phi_k(w)$. Analogously to part ii) of the proof of Theorem 4.2.3, this yields $z \sim w$. Similarly to the definition before Definition 4.2.5, we now introduce the following notations:

Definition 4.2.7. For $D \subset G$ open, we set

i) $H_*(\Phi_*(D)) := \bigcap_{k=1}^{m} \left\{\psi : \Phi_*(D) \to \mathbb{C} : \psi\left(\bullet\big|_{\Phi_k(G_k\cap D)}, k\right) \in H\big(\Phi_k(G_k \cap D)\big)\right\},$

ii) $H_{\Phi_*}(D) := \left\{\psi \circ \Phi_*\big|_D : \psi \in H_*(\Phi_*(D))\right\}.$

The following lemma lists some important properties of the sets $H_*(\Phi_*(D))$:

Lemma 4.2.8. *Let $D \subset G$ be open. Then the following statements are true:*

i) $H_{\Phi_*}(D)$ is a subspace of $H(D)$.

ii) $H_\Phi(D) \subset H_{\Phi_*}(D)$.

iii) If D is dense in G, the inclusion in part ii) holds properly.

Proof:

i) For $h \in H_{\Phi_*}(D)$, there exists a function $\psi : \Phi_*(D) \to \mathbb{C}$ such that the map $z \mapsto \psi(z,k)$ is holomorphic on $\Phi_k(G_k \cap D)$ for all $k \in \{1,\ldots,m\}$ and such that we have $h = \psi \circ \Phi_*\big|_D$. As each point in D is contained in exactly one of the sets $G_k \cap D$, the assertion follows from the fact that the functions $G_k \cap D \ni z \mapsto \Phi_k(z)$ and

$$\Phi_k(G_k \cap D) \ \ni\ \Phi_k(z) \ \mapsto\ \psi(\Phi_k(z),k) = \psi(\Phi_*(z)) = h(z)$$

are holomorphic.

ii) For $h \in H_\Phi(D)$, there exists some $\psi \in H(\Phi(D))$ with $h = \psi \circ \Phi\big|_D$. We define

$$\psi_* : \Phi_*(D) \to \mathbb{C}, \ \ \psi_*(w,k) := \psi(w).$$

Then, for all $k \in \{1,\ldots,m\}$, it follows that the function

$$w \mapsto \psi_* \left(\bullet\big|_{\Phi_k(G_k \cap D)}, k \right)(w) = \psi(w)$$

is holomorphic on the set $\Phi_k(G_k \cap D)$ so that we obtain $\psi_* \in H_*(\Phi_*(D))$. For each point $z \in D$, there exists exactly one $k \in \{1,\ldots,m\}$ with $z \in G_k \cap D$, and we have

$$\psi_*(\Phi_*(z)) = \psi_*(\Phi_k(z),k) = \psi(\Phi_k(z)) = \psi(\Phi(z)).$$

This yields $h = \psi \circ \Phi\big|_D = \psi_* \circ \Phi_*\big|_D \in H_{\Phi_*}(D)$.

iii) Let D be dense in G. Then, for each $k \in \{1,\ldots,m\}$, the set $G_k \cap D$ is dense in G_k so that the continuity of Φ_k and the denseness of $\Phi_k(G_k)$ in $\mathbb{C}$ (see Remark 4.2.1) imply that $\Phi_k(G_k \cap D)$ is dense in $\mathbb{C}$. Fixing $j,l \in \{1,\ldots,m\}$ with $j \neq l$, we obtain that the intersection $\Phi_j(G_j \cap D) \cap \Phi_l(G_l \cap D)$ is dense in $\mathbb{C}$ and in particular non-empty. Thus, there exist points $z_j \in G_j \cap D$ and $z_l \in G_l \cap D$ with $\Phi_j(z_j) = \Phi_l(z_l)$. Defining

$$\psi_* : \Phi_*(D) \to \mathbb{C}, \ \ \psi_*(w,k) := w - k,$$

it follows for all $k \in \{1,\ldots,m\}$ that the function

$$w \mapsto \psi_* \left(\bullet\big|_{\Phi_k(G_k \cap D)}, k \right)(w) = w - k$$

is holomorphic on $\Phi_k(G_k \cap D)$ so that we obtain $\psi_* \in H_*(\Phi_*(D))$. Therefore,

$h := \psi_* \circ \Phi_*\big|_D$ is an element of the subspace $H_{\Phi_*}(D)$. Assuming that we have $h \in H_\Phi(D)$, there would exist some $\psi \in H(\Phi(D))$ with $h = \psi \circ \Phi\big|_D$. But this would imply

$$\begin{aligned}\Phi_j(z_j) - l &= \Phi_l(z_l) - l = \psi_*(\Phi_l(z_l), l) = \psi_*(\Phi_*(z_l)) = h(z_l) = \psi(\Phi(z_l))\\ &= \psi(\Phi_l(z_l)) = \psi(\Phi_j(z_j)) = \psi(\Phi(z_j)) = h(z_j) = \psi_*(\Phi_*(z_j))\\ &= \psi_*(\Phi_j(z_j), j) = \Phi_j(z_j) - j\end{aligned}$$

and hence $l = j$, a contradiction. Thus, h is not contained in $H_\Phi(D)$. □

In particular, Lemma 4.2.8 implies that $H_\Phi(G)$ is a proper subset of $H_{\Phi_*}(G)$. For this reason, and in view of Theorem 4.2.6 and the subsequent considerations, it is now our aim to show that we actually have

$$\omega(G, g, f) = H_{\Phi_*}(G)$$

for each locally universal function g for C_f on P. Analogously to Lemma 4.1.4 in case of an attracting fixed point, we start with the following statement:

Lemma 4.2.9. *For each locally universal function g for C_f on P and for each $N \in \mathbb{N}_0$, we have*

$$\omega\big(f^{-N}(P), g, f\big) \supset H_{\Phi_*}\big(f^{-N}(P)\big).$$

Proof:

i) Analogously to the first part of the proof of Lemma 4.1.4, it follows that we have $f^N(f^{-N}(P)) = P$ (observe the considerations before Theorem 4.2.3). Hence, as Φ_* "quasiconjugates" $f\big|_G$ to the map $(w, k) \mapsto (w, k) + (1, 0)$, we obtain

$$\Phi_*\big(f^{-N}(P)\big) = \Phi_*\big(f^N(f^{-N}(P))\big) - (N, 0) = \Phi_*(P) - (N, 0).$$

ii) Now, let $h \in H_{\Phi_*}\big(f^{-N}(P)\big)$. Then there exists a function $\psi : \Phi_*(f^{-N}(P)) \to \mathbb{C}$ such that, for all $k \in \{1, \ldots, m\}$, the map $z \mapsto \psi(z, k)$ is holomorphic on the set $\Phi_k(G_k \cap f^{-N}(P)) = \Phi_k(f^{-N}(P_k))$ and such that we have

$$h = \psi \circ \Phi_*\big|_{f^{-N}(P)} = \psi \circ \big(\Phi_* - (N, 0)\big) \circ f^N\big|_{f^{-N}(P)}.$$

Putting $\widetilde{h} := \psi \circ \big(\Phi_* - (N, 0)\big)\big|_P$, we have $h = \widetilde{h} \circ f^N\big|_{f^{-N}(P)}$ and we obtain that $\widetilde{h}$ is holomorphic on P. Indeed, as each point in P is contained in exactly one of the sets P_k and as we have

$$\Phi_k(f^{-N}(P_k)) = \Phi_k(f^N(f^{-N}(P_k))) - N = \Phi_k(P_k) - N,$$

the holomorphy of the functions $P_k \ni z \mapsto \Phi_k(z) - N$ and

$$\begin{aligned}\Phi_k(f^{-N}(P_k)) &\ni \Phi_k(z) - N \mapsto \\ \psi\big(\Phi_k(z) - N, k\big) &= \psi\big((\Phi_k(z), k) - (N, 0)\big) = \psi\big(\Phi_*(z) - (N, 0)\big) = \widetilde{h}(z)\end{aligned}$$

yields that $\widetilde{h}$ is holomorphic on P. As g is locally universal for C_f on P, the set $\{g \circ f^n\big|_P : n \in \mathbb{N}\}$ is dense in $H(P)$. For $K \subset f^{-N}(P)$ compact, the set $f^N(K)$ is a compact subset of P so that there exists an $m_1 \in \mathbb{N}$ with

$$1 > \left\| g \circ f^{m_1}\big|_P - \widetilde{h} \right\|_{f^N(K)} = \|g \circ f^{m_1} \circ f^N - \widetilde{h} \circ f^N\|_K = \|g \circ f^{m_1+N} - h\|_K.$$

Analogously to the proof of Lemma 4.1.4, we can find a strictly increasing sequence (m_j) in $\mathbb{N}$ such that we have uniform convergence $g \circ f^{m_j+N} \to h$ on K. According to Lemma A.4, this already implies the existence of a strictly increasing sequence (n_k) in $\mathbb{N}$ with $g \circ f^{n_k} \to h$ locally uniformly on $f^{-N}(P)$, i.e. we have $h \in \omega\big(f^{-N}(P), g, f\big)$. □

Corollary 4.2.10. *For each locally universal function g for C_f on P, we have*

$$\omega(G, g, f) \supset H_{\Phi_*}(G).$$

Proof: Let $h \in H_{\Phi_*}(G)$. Then there exists some $\psi \in H_*(\Phi_*(G))$ with $h = \psi \circ \Phi_*$. For $K \subset G = \bigcup_{n\in\mathbb{N}_0} f^{-n}(P)$ compact, there exists a finite set $E \subset \mathbb{N}_0$ such that we have $K \subset \bigcup_{n\in E} f^{-n}(P)$. Putting $N := \max E$, the invariance of P under f yields

$$f^N(K) \subset \bigcup_{n\in E} f^N(f^{-n}(P)) \subset \bigcup_{n\in E} f^{N-n}(P) \subset P.$$

Hence, K is a compact subset of $f^{-N}(P)$. According to Lemma 4.2.9, we have

$$h\big|_{f^{-N}(P)} = \psi\big|_{\Phi_*(f^{-N}(P))} \circ \Phi_*\big|_{f^{-N}(P)} \in H_{\Phi_*}(f^{-N}(P)) \subset \omega(f^{-N}(P), g, f).$$

Thus, there exists a strictly increasing sequence (n_k) in $\mathbb{N}$ with $g \circ f^{n_k} \to h$ uniformly on K so that the assertion now follows from Lemma A.4. □

Lemma 4.2.11. *Let $g \in H(G)$. Then we have*

$$\omega(G, g, f) \subset H_{\Phi_*}(G).$$

Proof: We consider the equivalence relation $\sim$ which we have introduced after Lemma 4.2.6, i.e. for $z, w \in G$, we have $z \sim w$ if and only if there exists some $N \in \mathbb{N}_0$ with $f^N(z) = f^N(w)$. Let $[z]_\sim := \{w \in G : z \sim w\}$ be the equivalence class of $z \in G$, let $G/_\sim := \{[z]_\sim : z \in G\}$ be the quotient induced by $\sim$ and let

$$p : G \to G/_\sim, \;\; p(z) := [z]_\sim,$$

be the associated quotient map. For $h \in \omega(G, g, f)$, there exists a strictly increasing sequence (n_k) in $\mathbb{N}$ such that $(g \circ f^{n_k})$ converges to h locally uniformly on G. Thus, h is holomorphic on G, and for all points $z, w \in G$ with $z \sim w$, it follows that

$$h(z) = \lim_{k\to\infty} g(f^{n_k}(z)) = \lim_{k\to\infty} g(f^{n_k}(w)) = h(w).$$

Therefore, the map

$$\widetilde{h} : G/_\sim \to \mathbb{C}, \;\; \widetilde{h}\left([z]_\sim\right) := h(z),$$

is well-defined and fulfils $h = \widetilde{h} \circ p$. Moreover, for $z, w \in G$, the considerations before Definition 4.2.7 yield that we have $z \sim w$ if and only if $\Phi_*(z) = \Phi_*(w)$. Hence, the map

$$\widetilde{\Phi}_* : G/_\sim \to \Phi_*(G), \;\; \widetilde{\Phi}_*\big([z]_\sim\big) := \Phi_*(z),$$

is well-defined, bijective and fulfils $\Phi_* = \widetilde{\Phi}_* \circ p$. Altogether, we have

$$h = \widetilde{h} \circ p = \widetilde{h} \circ \widetilde{\Phi}_*^{-1} \circ \widetilde{\Phi}_* \circ p = \left(\widetilde{h} \circ \widetilde{\Phi}_*^{-1}\right) \circ \Phi_*.$$

As h is holomorphic on G and as each function $\Phi_k : G_k \to \Phi_k(G_k)$ is holomorphic and surjective, Corollary A.6 yields

$$\left(\widetilde{h} \circ \widetilde{\Phi}_*^{-1}\right)\left(\bullet\big|_{\Phi_k(G_k)}, k\right) \in H(\Phi_k(G_k))$$

for all $k \in \{1, \ldots, m\}$. Therefore, we obtain $\widetilde{h} \circ \widetilde{\Phi}_*^{-1} \in H_*(\Phi_*(G))$ and hence $h \in H_{\Phi_*}(G)$. □

Combining Corollary 4.2.10 and Lemma 4.2.11, we have proved the following concluding statement:

Theorem 4.2.12. *We have*

$$\omega(G, g, f) = H_{\Phi_*}(G)$$

for each locally universal function g for C_f on P.

Remark 4.2.13. In particular, according to the consideration after Definition 4.2.5, we obtain that the identity $\omega(G, g, f) = H_{\Phi_*}(G)$ holds for comeager many functions $g \in H(\mathbb{C}\backslash\{z_0\})$.

4.3 Böttcher Domains and Rotation Domains

Let f be a transcendental entire function or a rational function of degree $d \geq 2$. In this short section, which has the character of a remark, we want to show that the set $\omega\big(G\backslash O_f^-(z_0), g, f\big)$ only contains constant functions if G is a Böttcher domain of f which contains a superattracting fixed point z_0 of f and g is holomorphic on $\mathbb{C}\backslash\{z_0\}$, and that there exist constant functions which are not contained in the closure of $\big\{g \circ f^n\big|_G : n \in \mathbb{N}\big\}$ in $H(G)$ if G is an invariant rotation domain of f and g is holomorphic on G. Indeed, in these two situations, the composition operator C_f cannot be $H\big(G\backslash O_f^-(z_0)\big)$-universal or $H(G)$-universal, respectively, because of lack of injectivity of the symbol f on $G\backslash O_f^-(z_0)$ in the first case and because of lack of "run-away behaviour" of the sequence of iterates (f^n) on G in the latter case (cf. Definition 2.1.1 and the subsequent considerations).

First, let us assume that f has a superattracting fixed point at $z_0 \in \mathbb{C}$ and that f is not constant. Then, for $p := \min\{n \in \mathbb{N} : f^{(n)}(z_0) \neq 0\}$, we have $p \in \mathbb{N}\backslash\{1\}$ and we know that f is locally conjugated near z_0 to the p-th monomial Q. Thus, there exist open neighbourhoods U of z_0 with $f(U) \subset U$ and $V \subset \mathbb{D}$ of 0 as well as a conformal map $\varphi : U \to V$ with $\varphi(z_0) = 0$ such that the equation

$$\varphi(f^n(z)) = Q^n(\varphi(z)) = (\varphi(z))^{p^n}$$

holds for all $z \in U$ and for all $n \in \mathbb{N}$ (cf. Section 3.3). Denoting by G the Böttcher domain of f which contains z_0 (i.e. G is the component of F_f containing z_0) and assuming that $\infty \notin G$ in case of a rational symbol f, the following statement holds:

Lemma 4.3.1. *For $D \subset G\backslash O_f^-(z_0)$ open and $g \in H(\mathbb{C}\backslash\{z_0\})$, we have*

$$\omega(D, g, f) \subset \Big\{h \in \mathbb{C}^D : h \text{ constant on each component } \widetilde{D} \text{ of } D \text{ with } \widetilde{D} \cap U \neq \emptyset\Big\}.$$

Proof:

i) First, we consider a domain $W \subset V\backslash\{0\}$ and a function $g \in H(V\backslash\{0\})$, and we show that we have
$$\omega(W, g, Q) \subset \{\text{const.}\}$$
in this situation. Indeed, for an arbitrary point $w_0 \in W$, we choose some $\varepsilon > 0$ with $U_\varepsilon[w_0] \subset W$ and we put
$$A := K_{|w_0|}[0] \cap U_\varepsilon(w_0).$$
As the set $M := \bigcup_{n\in\mathbb{N}}\{z \in \mathbb{C} : z^{p^n} = 1\}$ of all p^n-th roots of unity is dense in $\partial\mathbb{D}$, the set w_0M is dense in $K_{|w_0|}[0]$ so that the openness of $U_\varepsilon(w_0)$ in $\mathbb{C}$ implies that the set
$$\widetilde{A} := w_0M \cap U_\varepsilon(w_0)$$
is dense in A. In particular, $\widetilde{A}$ has an accumulation point in $U_\varepsilon[w_0] \subset W$. For $w_1, w_2 \in \widetilde{A}$, there exist points $z_1, z_2 \in M$ with $w_1 = w_0z_1$ and $w_2 = w_0z_2$. Then we can find integers $N_1, N_2 \in \mathbb{N}$ with $z_1^{p^{N_1}} = 1$ and $z_2^{p^{N_2}} = 1$, and it follows for all $n \geq \max(N_1, N_2) =: N$ that $z_1^{p^n} = 1 = z_2^{p^n}$. Thus, we obtain for all $n \geq N$ that
$$Q^n(w_1) = (w_0z_1)^{p^n} = (w_0z_2)^{p^n} = Q^n(w_2).$$
Now, for $h \in \omega(W, g, Q)$, there exists a strictly increasing sequence (n_k) in $\mathbb{N}$ with $g \circ Q^{n_k} \to h$ locally uniformly on W. Hence, we have
$$h(w_1) = \lim_{k\to\infty} g(Q^{n_k}(w_1)) = \lim_{k\to\infty} g(Q^{n_k}(w_2)) = h(w_2).$$
Therefore, h is constant on $\widetilde{A}$ so that the identity theorem yields that h is constant on W.

ii) Now, let $D \subset G\backslash O_f^-(z_0)$ be open and let $g \in H(\mathbb{C}\backslash\{z_0\})$. For $h \in \omega(D, g, f)$, there exists a strictly increasing sequence (n_k) in $\mathbb{N}$ such that we have locally uniform convergence $g \circ f^{n_k} \to h$ on D. Let $\widetilde{D}$ be a component of D with $\widetilde{D} \cap U \neq \emptyset$. Then the non-empty open set $\varphi(\widetilde{D} \cap U)$ is contained in $V\backslash\{0\}$ and the function $g \circ \varphi^{-1}$ is holomorphic on $V\backslash\{0\}$. As we have locally uniform convergence
$$(g \circ \varphi^{-1}) \circ Q^{n_k} = g \circ \varphi^{-1} \circ Q^{n_k} \circ \varphi \circ \varphi^{-1} = g \circ f^{n_k} \circ \varphi^{-1} \to h \circ \varphi^{-1}$$

on $\varphi(\widetilde{D} \cap U)$, part i) yields for an arbitrary component W of $\varphi(\widetilde{D} \cap U)$ that

$$h \circ \varphi^{-1}\big|_W \in \omega\Big(W, g \circ \varphi^{-1}\big|_{V\setminus\{0\}}, Q\Big) \subset \{\text{const.}\}.$$

Thus, h is constant on the open subset $\varphi^{-1}(W)$ of $\widetilde{D}$ so that the identity theorem implies that h is constant on $\widetilde{D}$. □

Remark 4.3.2. Considering the basin of attraction $A_f(z_0)$ of z_0 under f, a short calculation shows that we have $A_f(z_0)\setminus O_f^-(z_0) = \bigcup_{n\in\mathbb{N}} f^{-n}\big(U\setminus\{z_0\}\big)$. Therefore, $A_f(z_0)\setminus O_f^-(z_0)$ is open so that the set $G\setminus O_f^-(z_0) = (A_f(z_0)\setminus O_f^-(z_0)) \cap G$ is also open. Moreover, as the backward orbit $O_f^-(z_0)$ is countable, the set $G\setminus O_f^-(z_0)$ is (path-)connected. Hence, Lemma 4.3.1 yields

$$\omega\big(G\setminus O_f^-(z_0), g, f\big) \subset \{\text{const.}\}$$

for all functions $g \in H(\mathbb{C}\setminus\{z_0\})$.

For the rest of this section, let us now assume that G is an invariant rotation domain (i.e. a Siegel disc or an Arnol'd-Herman ring) of f. In case that G is an invariant Siegel disc of f, there exists a conformal map $\varphi : G \to \mathbb{D}$ and some $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ such that the equation

$$\varphi \circ f^n = F_\alpha^n \circ \varphi$$

holds on G for all integers $n \in \mathbb{N}$, where $F_\alpha : \mathbb{D} \to \mathbb{D}$, $F_\alpha(w) := e^{2\pi i\alpha}w$, denotes the rotation on $\mathbb{D}$ by the angle $2\pi\alpha$ (cf. Definition 1.2.5). Putting $z_0 := \varphi^{-1}(0)$, we obtain

$$f(z_0) = f(\varphi^{-1}(0)) = \varphi^{-1}(F_\alpha(0)) = \varphi^{-1}(0) = z_0$$

as well as $\varphi'(f(z_0)) \cdot f'(z_0) = F_\alpha'(\varphi(z_0)) \cdot \varphi'(z_0)$ and thus $f'(z_0) = e^{2\pi i\alpha}$. Hence, $z_0 \in G$ is an irrationally indifferent fixed point of f (which is called the *center* of the Siegel disc G). Now, the following statement holds:

Lemma 4.3.3. *Let $U \subset G$ be open with $z_0 \in U$. Then, for each function $g \in H(G)$, we have*

$$\{\text{const.}\} \not\subset \overline{\big\{g \circ f^n\big|_U : n \in \mathbb{N}\big\}}^{H(U)}.$$

In particular, the composition operator C_f is not hypercyclic.

Proof: The following proof will run similarly to the proof of Theorem 3.5 in [13]. Considering the open set $V := \varphi(U) \subset \mathbb{D}$, we have $0 = \varphi(z_0) \in \varphi(U) = V$. Hence, there exists some $0 < r < 1$ with $K_r[0] \subset V$. Putting $K := \varphi^{-1}(K_r[0])$, we obtain that K is a compact subset of U for which we have

$$f^n(K) = \varphi^{-1}\big(F_\alpha^n(\varphi(K))\big) = \varphi^{-1}\big(F_\alpha^n(K_r[0])\big) = \varphi^{-1}\big(K_r[0]\big) = K$$

for all $n \in \mathbb{N}$. Thus, there exists a sequence (z_n) in K with $f^n(z_n) \in K$ for all $n \in \mathbb{N}$. Now, for $g \in H(G)$, we consider the constant function

$$h : U \to \mathbb{C}, \;\; h(z) :\equiv \|g\|_K + 1.$$

Then we have for all $n \in \mathbb{N}$ that

$$\begin{aligned} \|g \circ f^n - h\|_K \;&\geq\; |g(f^n(z_n)) - h(z_n)| \;\geq\; |h(z_n)| - |g(f^n(z_n))| \\ &=\; \|g\|_K + 1 - |g(f^n(z_n))| \;\geq\; \|g\|_K + 1 - \|g\|_K = 1, \end{aligned}$$

so that we obtain $h \notin \overline{\{g \circ f^n\big|_U : n \in \mathbb{N}\}}^{H(U)}$. □

Remark 4.3.4.

i) In case that G is an invariant Arnol'd-Herman ring of f, there exists some $R > 1$, a conformal map $\varphi : G \to \{z \in \mathbb{C} : 1 < |z| < R\} =: A_R$ and some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ such that we have $\varphi(f^n(z)) = e^{2\pi i\alpha\cdot n} \cdot \varphi(z)$ for all $z \in G$ and for all $n \in \mathbb{N}$. Analogously to the proof of Lemma 4.3.3, one can prove the following statement: For each open subset V of the annulus A_R, for which there exists some $1 < r < R$ with $K_r[0] \subset V$, we have

$$\{\text{const.}\} \not\subset \overline{\{g \circ f^n\big|_{\varphi^{-1}(V)} : n \in \mathbb{N}\}}^{H(\varphi^{-1}(V))}$$

for all functions $g \in H(G)$. In particular, the composition operator C_f is not hypercyclic.

ii) In the situations of Lemma 4.3.3 and part i) of this remark, we have shown that there always exist constant functions which are not contained in $\omega(G, g, f)$ – in contrast to the situations of Sections 4.1 and 4.2, where the considered sets $\omega(G, g, f)$ equal some subspace of $H(G)$ which contains all constant functions.

4.4 Baker Domains

Let f be a transcendental entire function and let G be an invariant Baker domain of f, i.e. G is an invariant component of F_f with $f^n \to \infty$ locally uniformly on G (cf. Definition 1.2.5 and Remark 1.2.6 i)). By definition, G is unbounded so that a theorem of Baker, which states that the Fatou set of a transcendental entire function does not have any unbounded multiply connected components (see [2], Theorem 1), yields that G is simply connected. Moreover, we require that f is injective on G. Then G is called a *univalent* Baker domain of f.

Example 4.4.1.

i) We consider the transcendental entire function

$$f : \mathbb{C} \to \mathbb{C}, \quad f(z) := 2 - \log 2 + 2z - e^z.$$

Then f has an invariant Baker domain G which contains the left half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < -2\}$ and the boundary $\partial_\infty G$ is a Jordan curve in $\mathbb{C}_\infty$ (see [9], Theorem 1 and its proof as well as Theorem 2). As a theorem of Baker and Weinreich states that a transcendental entire function is injective on each unbounded invariant component of the Fatou set whose boundary is a Jordan curve in $\mathbb{C}_\infty$ (see [4], Theorem 4), we obtain that G is a univalent Baker domain of f (cf. [9], p. 526).

ii) For suitable chosen constants $0 < \alpha < 2\pi$ and $0 < \beta < 1$, the transcendental entire function

$$f : \mathbb{C} \to \mathbb{C}, \quad f(z) := z + \alpha + \beta \sin z$$

has an invariant univalent Baker domain which is symmetric with respect to the real axis (see [5], subsection 5.2).

iii) For each $n \in \mathbb{N}$, there exists a transcendental entire function f which has n univalent Baker domains $U_1, \dots, U_n$ such that we have $\overline{U_i} \cap \overline{U_j} = \{z_0\}$ for all $i, j \in \{1, \dots, n\}$ with $i \neq j$, where $z_0 \in J_f$ is a repelling fixed point of f (see [48], Theorem 1).

iv) In general, Baker domains do not have to be univalent (see e.g. [12], Theorem 1.1 and Theorem 1.2).

For invariant univalent Baker domains, the following universality result holds:

Theorem 4.4.2. *In the situation at the beginning of this section, the set of all entire functions which are $H(G)$-universal for C_f is a comeager set in $H(\mathbb{C})$.*

Proof: We consider the domain $\Omega := \mathbb{C}$ and the open set $D := G \subset \Omega$. According to the above considerations, D is a simply connected domain, i.e. we have $D \in \mathcal{U}_0(\mathbb{C})$. As the map $f\big|_D : D \to D$ is injective and as we have locally uniform convergence $f^n\big|_D \to \infty \in \partial_\infty\Omega$, the assertion now follows from Corollary 2.1.8. □

4.5 Wandering Domains

Let f be a transcendental entire function and let G be a wandering domain of f, i.e. G is a component of F_f with $f^n(G) \cap f^m(G) = \emptyset$ for all integers $n > m \geq 0$ (cf. Definition 1.2.5 and Remark 1.2.6 iv)). Moreover, we require that all iterates f^n are injective on G and that we have convergence $f^n \to \infty$ on G (which is locally uniform due to Remark 1.2.6 i)).

Example 4.5.1.

i) We consider the transcendental entire function

$$f : \mathbb{C} \to \mathbb{C}, \ \ f(z) := z - 1 + e^{-z} + 2\pi i.$$

Then, for each $k \in \mathbb{Z}$, there exists a simply connected wandering domain G_k of f containing the point $2\pi ik$ (see [3], Example 5.1). As we have $f'(2\pi ik) = 0$ for each $k \in \mathbb{Z}$, we see that f is not injective on any set G_k. Moreover, we have $f^n(2\pi ik) = 2\pi i(k + n) \to \infty$, which implies $f^n\big|_{G_k} \to \infty$ for each $k \in \mathbb{Z}$. (Indeed, for each domain $D \subset F_f$ for which there exists a point $w \in D$ with $f^n(w) \to \infty$, it follows that $f^n\big|_D \to \infty$. To see this, we choose for $z \in D$ a path γ in D which connects z and w. Then γ^* is a compact subset of D so that the normality of the family $\{f^n\big|_D : n \in \mathbb{N}\}$ yields that for each subsequence $(f^{n_k}(z))$ of $(f^n(z))$ there exists a subsequence $(f^{n_{k_l}})$ of (f^{n_k}) which converges uniformly on γ^*. Due to $f^{n_{k_l}}(w) \to \infty$, we obtain $f^{n_{k_l}}\big|_{\gamma^*} \to \infty$ by continuity and hence, in particular, $f^{n_{k_l}}(z) \to \infty$. But this already implies $f^n(z) \to \infty$.)

ii) In 1987, Eremenko and Lyubich proved the existence of a transcendental entire function f which has a simply connected wandering domain G such that all iterates f^n are injective on G and such that $f^n\big|_G \to \infty$ (see [21], Example 2).

iii) For almost all $\lambda \in \partial\mathbb{D}$, the transcendental entire function

$$f_\lambda : \mathbb{C} \to \mathbb{C}, \ \ f_\lambda(z) := z + e^z + 1 - \lambda + 2\pi i$$

has a simply connected wandering domain $G_{\lambda,k}$ for each $k \in \mathbb{Z}$ such that all iterates f_λ^n are injective on each set $G_{\lambda,k}$ (see [54], Theorem 3.4). Looking at the proof of this theorem (see [54], p. 35f.), one sees that a point $z_0 \in \mathbb{C}$ with $e^{z_0} + 1 - \lambda = 0$ is chosen and that each set $G_{\lambda,k}$ is constructed such that the point $z_k := z_0 + 2\pi ik$ is contained in $G_{\lambda,k}$. Because of

$$\begin{aligned} f_\lambda(z_k) &= f_\lambda(z_0 + 2\pi ik) = z_0 + 2\pi ik + e^{z_0+2\pi ik} + 1 - \lambda + 2\pi i \\ &= z_0 + 2\pi i(k+1) + e^{z_0} + 1 - \lambda = z_0 + 2\pi i(k+1) = z_{k+1}, \end{aligned}$$

we obtain $f_\lambda^n(z_k) = z_{k+n} \to \infty$. As in part i), this yields $f_\lambda^n\big|_{G_{\lambda,k}} \to \infty$.

Remark 4.5.2. It is an open question whether there exists a wandering domain G of a transcendental entire function f such that its forward orbit $O_f^+(G) = \bigcup_{k\in\mathbb{N}} f^k(G)$ is bounded (see e.g. [47], Question 8.2). In any case, for all currently known examples of transcendental entire functions f having a wandering domain G, the set $O_f^+(G)$ is unbounded. But in contrast to the previous examples, we do not necessarily have convergence $f^n\big|_G \to \infty$ in this situation. Indeed, Eremenko and Lyubich proved the existence of a transcendental entire function f which has a wandering domain G such that for each point $z \in G$ the sequence $(f^n(z))$ has infinitely many accumulation points (see [21], Example 1).

For wandering domains, on which all iterates are injective and on which the iterates converge to the point at ∞, the following universality result holds:

Theorem 4.5.3. *In the situation at the beginning of this section, the set of all entire functions which are $H(W)$-universal for C_f for all $W \in \mathcal{U}_0(O_f^+(G))$ is a comeager set in $H(\mathbb{C})$.*

Proof:

i) In general, for a family $(U_\iota)_{\iota\in I}$ of pairwise disjoint open subsets of $\mathbb{C}$ and a compact subset K of $\bigcup_{\iota\in I} U_\iota$, each set

$$K_\iota := K \cap U_\iota$$

is compact. Indeed, let $\iota_0 \in I$ and let $(V_\kappa)_{\kappa\in J}$ be a family of open subsets of $\mathbb{C}$ with $K_{\iota_0} \subset \bigcup_{\kappa\in J} V_\kappa$. Then we have

$$K = K_{\iota_0} \cup (K \backslash U_{\iota_0}) \subset \bigcup_{\kappa\in J} V_\kappa \ \cup \bigcup_{\iota\in I\backslash\{\iota_0\}} U_\iota,$$

so that the compactness of K implies the existence of finite sets $E \subset J$ and $F \subset I$ with $K \subset \bigcup_{\kappa \in E} V_\kappa \cup \bigcup_{\iota \in F \setminus \{\iota_0\}} U_\iota$. Hence, as the sets U_ι are pairwise disjoint, it follows that

$$K_{\iota_0} = K \cap U_{\iota_0} \subset \bigcup_{\kappa \in E} V_\kappa \cap U_{\iota_0} \cup \bigcup_{\iota \in F \setminus \{\iota_0\}} U_\iota \cap U_{\iota_0} \subset \bigcup_{\kappa \in E} V_\kappa.$$

ii) We show that we have locally uniform convergence $f^n\big|_{O_f^+(G)} \to \infty$. To see this, let $K \subset O_f^+(G)$ be compact. Then there exists a finite set $E \subset \mathbb{N}$ with $K \subset \bigcup_{k \in E} f^k(G)$. As G is a wandering domain of f, the open sets $f^k(G)$ are pairwise disjoint so that part i) yields that each set $K_k := K \cap f^k(G)$ is a compact subset of $f^k(G)$. Therefore, as we have locally uniform convergence $f^n\big|_{f^k(G)} \to \infty$ by assumption (observe Remark 1.2.6 i)), we obtain uniform convergence $f^n\big|_{K_k} \to \infty$ for each $k \in E$, which implies uniform convergence $f^n \to \infty$ on the set $\bigcup_{k \in E} K_k = K$.

iii) We now consider the domain $\Omega := \mathbb{C}$ and the open set $D := O_f^+(G) \subset \Omega$. Then the map $f\big|_D : D \to D$ is injective. Indeed, for $z, w \in D$ with $f(z) = f(w)$ there exist integers $k, l \in \mathbb{N}$ with $z \in f^k(G)$ and $w \in f^l(G)$. Thus, we can find points $\widetilde{z}, \widetilde{w} \in G$ with $z = f^k(\widetilde{z})$ and $w = f^l(\widetilde{w})$, and it follows that $f^{k+1}(\widetilde{z}) = f^{l+1}(\widetilde{w})$. As G is a wandering domain of f, we have $k = l$ and hence $f^{k+1}(\widetilde{z}) = f^{k+1}(\widetilde{w})$. By assumption, f^{k+1} is injective on G so that we obtain $\widetilde{z} = \widetilde{w}$, which finally yields

$$z = f^k(\widetilde{z}) = f^k(\widetilde{w}) = f^l(\widetilde{w}) = w.$$

As we have locally uniform convergence $f^n\big|_D \to \infty \in \partial_\infty \Omega$, the assertion now follows from Corollary 2.1.7. □

In case of a simply connected wandering domain, we obtain the following statement:

Corollary 4.5.4. *If, in the situation at the beginning of this section, G is in addition simply connected, then the set of all entire functions which are $H(O_f^+(G))$-universal for C_f is a comeager set in $H(\mathbb{C})$.*

Proof: As G has no holes and as each iterate f^k is injective on G, the invariance of the number of holes yields that each set $f^k(G)$ has no holes as well. Thus, the disjoint union $O_f^+(G) = \bigcup_{k \in \mathbb{N}} f^k(G)$ has no holes, i.e. we have $O_f^+(G) \in \mathcal{U}_0(O_f^+(G))$. Therefore, the assertion follows directly from Theorem 4.5.3. □

Chapter 5

A Geometric Approach of Extending the Local Theory

In Chapter 3, we have seen several universality results for composition operators which hold locally on neighbourhoods of fixed points of the corresponding symbol. The key tool for proving these statements was the fact that the symbol is conformally conjugated on the considered neighbourhoods to a function which is quite easy to handle. In this chapter, it is our aim to extend the universality result which we have formulated for attracting fixed points (Theorem 3.1.1) in case that the symbol is a rational function.

More precisely, this means the following: We consider a rational function f of degree $d_f \geq 2$ which has an attracting fixed point at $z_0 \in \mathbb{C}$. As we have seen in Section 3.1, there exist open neighbourhoods U of z_0 and V of 0 as well as a conformal map $\varphi : U \to V$ which conjugates the map $f\big|_U : U \to U$ to the linear function $w \mapsto f'(z_0) \cdot w$ on V. Moreover, there exists a maximal radius $r_{max} > 0$ such that the inverse function φ^{-1} can be extended to a conformal function on the open disc $V_{max} := U_{r_{max}}(0)$ (see e.g. [37], Lemma 8.5). Hence, there exists a "maximal" open neighbourhood U_{max} of z_0 and a conformal map $\varphi_{max} : U_{max} \to V_{max}$ which conjugates $f\big|_{U_{max}} : U_{max} \to U_{max}$ to the linear function $w \mapsto f'(z_0) \cdot w$ on V_{max} (cf. the proof of Lemma 8.5 in [37]). Analogously to the proof of Theorem 3.1.1, we obtain that comeager many functions in $H(\mathbb{C}\backslash\{z_0\})$ are $H(W)$-universal for C_f for all $W \in \mathcal{U}_0(U_{\max}\backslash\{z_0\})$. In the following, we want to prove the existence of non-empty open sets $W \subset \mathbb{C}$ with $W \cap U_{max} = \emptyset$ such that C_f is $H(W)$-universal.

5.1 Finite Blaschke Products

In order to prove $H(W)$-universality of C_f for suitable open sets W which lie outside U_{max}, we need some information about f on a larger neighbourhood of the attracting fixed point z_0. Again, the concept of conformal conjugation will be the main consideration here. Apparently, f should now be conjugated to a "simpler" function on a "large" open set containing z_0. This can be achieved as follows (cf. [22], p. 586): We consider a simply connected invariant component G of the Fatou set F_f with $G \subset \mathbb{C}$, $G \neq \mathbb{C}$. Then the Riemann mapping theorem implies the existence of a conformal map $\psi : G \to \mathbb{D}$. Hence, ψ conjugates $f\big|_G : G \to G$ to the function

$$g := \psi \circ f \circ \psi^{-1} : \mathbb{D} \to \mathbb{D}.$$

Since rational functions map components of their Fatou sets properly onto each other (see e.g. [49], Theorem 1 on p. 39), we obtain that g is a proper self-map of $\mathbb{D}$ (i.e. $g^{-1}(K)$ is compact for each compact set $K \subset \mathbb{D}$). It is well-known that each proper self-map of $\mathbb{D}$ is the restriction to $\mathbb{D}$ of a finite Blaschke product (see e.g. [49], Exercise 6 on p. 7, or [43], p. 185):

Definition 5.1.1. Let $\theta \in \mathbb{R}$, $d \in \mathbb{N}$ and $\alpha_1, \ldots \alpha_d \in \mathbb{D}$. The function $B : \mathbb{C}_\infty \to \mathbb{C}_\infty$, defined by

$$B(z) := e^{i\theta} \prod_{k=1}^{d} \frac{z - \alpha_k}{1 - \overline{\alpha_k} z},$$

is called a *finite Blaschke product* of degree d.

Hence, in the above situation, there exists a finite Blaschke product B with $g = B\big|_{\mathbb{D}}$.

Definition 5.1.2. Let $\alpha \in \mathbb{D}$. The Möbius transformation $\varphi_\alpha : \mathbb{C}_\infty \to \mathbb{C}_\infty$, defined by

$$\varphi_\alpha(z) := \frac{z - \alpha}{1 - \overline{\alpha} z},$$

is called a *Blaschke factor*.

Remark 5.1.3.

i) For $\alpha \in \mathbb{D}$, the restriction $\varphi_\alpha\big|_{\mathbb{D}}$ is a conformal self-map of $\mathbb{D}$ and we have $\varphi_\alpha(\partial\mathbb{D}) = \partial\mathbb{D}$ (see e.g. [44], Theorem 12.4). As φ_α is a rational function of degree one, we further obtain that $\varphi_\alpha(\mathbb{C}_\infty \backslash \overline{\mathbb{D}}) = \mathbb{C}_\infty \backslash \overline{\mathbb{D}}$. Moreover, each conformal self-map of $\mathbb{D}$ is of the form $e^{i\theta}\varphi_\beta$ for some $\beta \in \mathbb{D}$ and $\theta \in \mathbb{R}$ (see e.g. [44], Theorem 12.6).

ii) For $\alpha \in \mathbb{D}\backslash\{0\}$, a simple calculation shows that φ_α has no fixed points in $\mathbb{D}$. Therefore, the Denjoy-Wolff theorem implies the existence of a point $w_\alpha \in \partial\mathbb{D}$ with $\varphi_\alpha^n \to w_\alpha$ locally uniformly on $\mathbb{D}$ (see e.g. [49], p. 43). Hence, according to Corollary 2.1.8, the set of all functions in $H(\mathbb{C}\backslash\{w_\alpha\})$ which are $H(\mathbb{D})$-universal for $C_{\varphi_\alpha|_\mathbb{D}}$ is a comeager set in $H(\mathbb{C}\backslash\{w_\alpha\})$.

iii) Due to part i), we have $B(\mathbb{D}) = \mathbb{D}$, $B(\partial\mathbb{D}) = \partial\mathbb{D}$ and $B(\mathbb{C}_\infty\backslash\overline{\mathbb{D}}) = \mathbb{C}_\infty\backslash\overline{\mathbb{D}}$ for each finite Blaschke product B. Thus, Montel's theorem implies $\mathbb{D} \subset F_B$ and $\mathbb{C}_\infty\backslash\overline{\mathbb{D}} \subset F_B$ so that we obtain $J_B \subset \partial\mathbb{D}$ (cf. [37], Problem 7-b on p. 70).

iv) Now, let B be a finite Blaschke product which has at least one Blaschke factor φ_0, i.e. there exist $d \in \mathbb{N}$, $N \in \{1, \cdots, d\}$, $\alpha_1, \ldots \alpha_{d-N} \in \mathbb{D}\backslash\{0\}$ and $\theta \in \mathbb{R}$ with

$$B(z) = e^{i\theta} z^N \prod_{k=1}^{d-N} \varphi_{\alpha_k}(z), \quad z \in \mathbb{C}_\infty.$$

Then 0 and ∞ are the only fixed points of B and we have

$$B'(0) = \begin{cases} e^{i\theta} \prod_{k=1}^{d-1}(-\alpha_k), & \text{if } N = 1 \\ 0, & \text{if } N \geq 2 \end{cases}$$

as well as

$$B'(\infty) = \frac{d}{dz}\,\frac{1}{B(1/z)}\bigg|_{z=0} = \begin{cases} e^{-i\theta} \prod_{k=1}^{d-1}(-\overline{\alpha_k}), & \text{if } N = 1 \\ 0, & \text{if } N \geq 2 \end{cases}.$$

Hence, the origin and the point at infinity are attracting fixed points of B in case of $N = 1$ and superattracting fixed points of B if $N \geq 2$. According to part iii), we have $J_B \subset \partial\mathbb{D}$. Assuming that $\partial\mathbb{D} \cap F_B \neq \emptyset$, we would obtain that F_B is connected. But as 0 and ∞ both are (super-)attracting fixed points of B, this contradicts the classification theorem of Fatou components. Thus, we have $J_B = \partial\mathbb{D}$ and hence $F_B = \mathbb{D} \cup (\mathbb{C}_\infty\backslash\overline{\mathbb{D}})$ (cf. [37], Problem 7-b on p. 70). The classification theorem now implies locally uniform convergence $B^n \to 0$ on $\mathbb{D}$ and $B^n \to \infty$ on $\mathbb{C}_\infty\backslash\overline{\mathbb{D}}$.

In order to extend the universality statement of Theorem 3.1.1, we now consider the simplest subclass of finite Blaschke products having an attracting fixed point at the origin. Due to the above considerations, this subclass is given by the functions

$$B_\alpha : \mathbb{C}_\infty \to \mathbb{C}_\infty, \;\; B_\alpha(z) := z \cdot \varphi_\alpha(z) = z \cdot \frac{z-\alpha}{1-\overline{\alpha}z}, \;\; \alpha \in \mathbb{D}\backslash\{0\}.$$

We fix some $\alpha \in \mathbb{D}\backslash\{0\}$ and we write $B := B_\alpha$. The following observations will be useful:

Remark 5.1.4.

i) As $\varphi_\alpha\big|_{\mathbb{D}}$ is a self-map of $\mathbb{D}$, we have $|B(z)| < |z|$ for each $z \in \mathbb{D}\backslash\{0\}$.

ii) Calculating $B'(z) = 0$, we see that there exist exactly two critical points of B which are given by

$$z_1 := \frac{1 - \sqrt{1 - |\alpha|^2}}{\overline{\alpha}} \qquad \text{and} \qquad z_2 := \frac{1 + \sqrt{1 - |\alpha|^2}}{\overline{\alpha}}.$$

Another simple calculation yields

$$0 < |z_1| < |\alpha| < 1 < \frac{1}{|\overline{\alpha}|} < |z_2|$$

so that z_1 is the only critical point of B in $\mathbb{D}$. Because of $\arg \alpha = \arg(1/\overline{\alpha})$, the points 0, z_1, α, $1/\overline{\alpha}$ and z_2 lie on the same straight line through 0. Moreover, the two distances $|z_1 - 1/\overline{\alpha}|$ and $|z_2 - 1/\overline{\alpha}|$ are equal.

iii) As B is a rational function of degree two, each point in $\mathbb{C}_\infty$ has exactly two preimages (counting multiplicity) under B. For $z, w \in \mathbb{C}_\infty$, a short computation shows that we have $B(z) = B(w)$ if and only if $w = z$ or $w = -\varphi_\alpha(z)$.

iv) For $z \in \mathbb{C}_\infty$, we compute that we have $-\varphi_\alpha(z) = z$ if and only if $z = z_1$ or $z = z_2$. Thus, the critical points of B are exactly the fixed points of $-\varphi_\alpha$.

The last two parts of the previous remark indicate that there is a close relation between the maps $-\varphi_\alpha$ and $B = (-\mathrm{id}_{\mathbb{C}_\infty}) \cdot (-\varphi_\alpha)$. Indeed, the map $-\varphi_\alpha$ will be a key factor in understanding the dynamics of B because its geometry on $\mathbb{D}$ is well-known. In order to describe this geometry, we introduce the following notations:

Definition 5.1.5.

i) For a straight line L in the complex plane, we denote by R_L the map which reflects each point in $\mathbb{C}$ with respect to L.

ii) Let $w \in \mathbb{C}$, $r > 0$ and consider the closed disc $C := U_r[w]$. For $z \in \mathbb{C}\backslash\{w\}$, we define $I_C(z)$ to be the point which lies on the ray $\{w + t(z - w) : t \geq 0\}$ and which has distance $r^2/|z - w|$ to w. The map I_C is called the *inversion* on C.

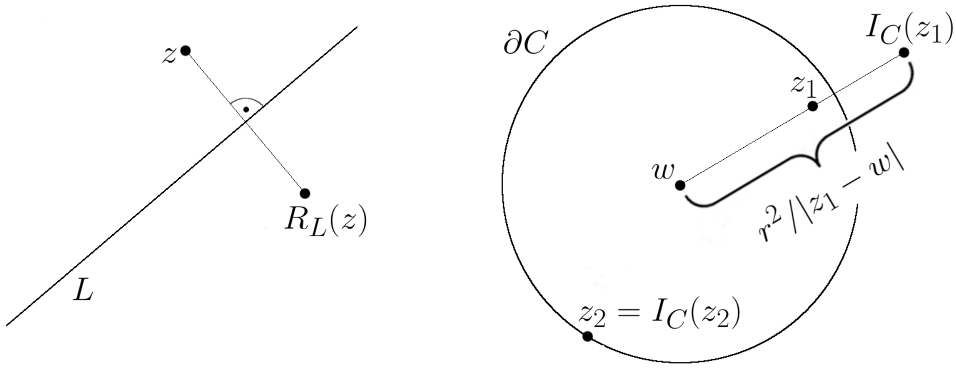

Figure 5.1

We obtain $I_C(C^\circ\backslash\{w\}) = \mathbb{C}\backslash C$ and $I_C(\mathbb{C}\backslash C) = C^\circ\backslash\{w\}$. Each point on ∂C is a fixed point of I_C. In case of $w = 0$ and $r = 1$, we have $I_{\overline{\mathbb{D}}}(z) = 1/\overline{z}$ for all $z \in \mathbb{C}\backslash\{0\}$. For a closed disc D which is orthogonal to C (i.e. the two tangents to D through the two points in $\partial C \cap \partial D$ pass through w), it follows that $I_C(D) = D$ (see [41], p. 149).

Considering the finite Blaschke product $B = B_\alpha$, now let L be the straight line which passes through 0 and α. Due to Remark 5.1.4 ii), we have z_1, z_2, $1/\overline{\alpha} \in L$ and $0 < |z_1| < |\alpha| < 1 < 1/|\overline{\alpha}| < |z_2|$ as well as $|z_1 - 1/\overline{\alpha}| = |z_2 - 1/\overline{\alpha}|$. Moreover, the two critical points z_1 and z_2 of B are exactly the two fixed points of $-\varphi_\alpha$ (see Remark 5.1.4 iv)). Therefore, there exists a closed disc C with center $1/\overline{\alpha}$ which is orthogonal to $\overline{\mathbb{D}}$ with $I_C(\alpha) = 0$ and $\partial C \cap L = \{z_1, z_2\}$ and such that $-\varphi_\alpha$ acts on $\mathbb{D}$ as the composition of the reflection R_L and the inversion I_C in any order, i.e. we have

$$(*) \qquad -\varphi_\alpha\big|_{\mathbb{D}} = (R_L \circ I_C)\big|_{\mathbb{D}} = (I_C \circ R_L)\big|_{\mathbb{D}}$$

(see [41], p. 207).

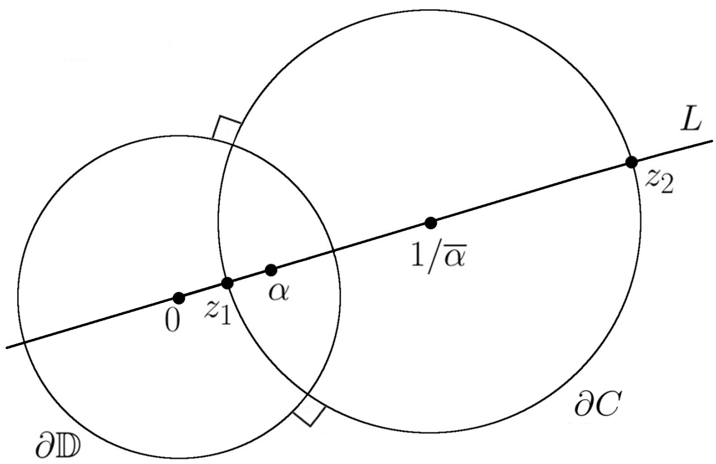

Figure 5.2

We define the disjoint subsets S_1, S_2, S_3 and S_4 of $\mathbb{D}$ as illustrated in the following figure:

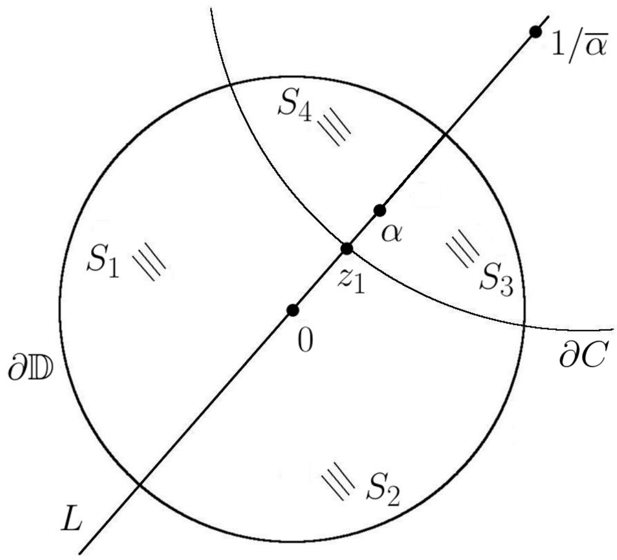

Figure 5.3

Defining $x_\alpha := \arg\alpha$, these are given by

$$\begin{aligned} S_1 &:= \left\{ z \in \mathbb{D} : x_\alpha \le \arg z \le x_\alpha + \pi,\ |z - 1/\overline{\alpha}| > |z_1 - 1/\overline{\alpha}| \right\}, \\ S_2 &:= \left\{ z \in \mathbb{D} : x_\alpha + \pi < \arg z < x_\alpha + 2\pi,\ |z - 1/\overline{\alpha}| > |z_1 - 1/\overline{\alpha}| \right\}, \\ S_3 &:= \left\{ z \in \mathbb{D} : x_\alpha + \pi \le \arg z \le x_\alpha + 2\pi,\ |z - 1/\overline{\alpha}| < |z_1 - 1/\overline{\alpha}| \right\}, \\ S_4 &:= \left\{ z \in \mathbb{D} : x_\alpha < \arg z < x_\alpha + \pi,\ |z - 1/\overline{\alpha}| < |z_1 - 1/\overline{\alpha}| \right\}. \end{aligned}$$

According to $(*)$, we see that $-\varphi_\alpha$ interchanges S_1 and S_3 as well as S_2 and S_4, i.e. we have $-\varphi_\alpha(S_1) = S_3$, $-\varphi_\alpha(S_3) = S_1$, $-\varphi_\alpha(S_2) = S_4$ and $-\varphi_\alpha(S_4) = S_2$.

Lemma 5.1.6. *The finite Blaschke product B is injective on the unions $S_k \cup S_l$ for all $k, l \in \{1,2,3,4\}$ with $\{k,l\} \notin \{\{1,3\},\{2,4\}\}$.*

Proof: Let $z, w \in S_1 \cup S_2$ with $B(z) = B(w)$. According to Remark 5.1.4 iii), it follows that $z = w$ or $-\varphi_\alpha(z) = w$. But the latter cannot be true because in this case we would obtain

$$w = -\varphi_\alpha(z) \in -\varphi_\alpha(S_1 \cup S_2) = -\varphi_\alpha(S_1) \cup -\varphi_\alpha(S_2) = S_3 \cup S_4,$$

a contradiction. Hence, B is injective on $S_1 \cup S_2$. Due to the geometry of $-\varphi_\alpha$, the injectivity of B on the other unions can be shown similarly. $\square$

In order to extend the universality statement of Theorem 3.1.1 in case of the finite Blaschke product B, we now have a look at the contour lines of B. For $0 < r < 1$ and $z \in \mathbb{D}$, we have $|B(z)| = r$ if and only if

$$(**) \qquad |z| \cdot |z - \alpha| = r \cdot |1 - \overline{\alpha} z|.$$

Ignoring the factor $|1 - \overline{\alpha} z|$ on the right-hand side, the points $z \in \mathbb{C}$ fulfilling the equation $|z| \cdot |z - \alpha| = r$ would form a *Cassini oval.* In general, for two points $w_1, w_2 \in \mathbb{C}$ and a constant $c > 0$, the Cassini oval $C(w_1, w_2, c)$ is defined as the set of all points in the complex plane having the property that the product of their distances to w_1 and w_2 has constant value c^2, i.e.

$$C(w_1, w_2, c) := \big\{ z \in \mathbb{C} : |z - w_1| \cdot |z - w_2| = c^2 \big\}.$$

The shapes of these sets according to the value of c are well-known. For small positive values of c, the set $C(w_1, w_2, c)$ consists of two disjoint Jordan curves which look like small circles around w_1 and w_2. Increasing c, these two components become more and more egg-shaped until they meet each other in the middle of the line segment between w_1 and w_2 for $c = |w_1 - w_2|/2$. The figure-eight-shaped set

$$L(w_1, w_2) := C(w_1, w_2, |w_1 - w_2|/2)$$

is called a *lemniscate*. For larger values $c > |w_1 - w_2|/2$, the Cassini ovals $C(w_1, w_2, c)$ consist of one component which first looks like a sand glass, then like an ellipse and finally like a large circle (cf. [41], p. 71f.).

In equation $(**)$, the factor $|1 - \overline{\alpha} z|$ on the right-hand side acts as an "error term" so that the sets

$$\widetilde{C}(0, \alpha, r) := \{ z \in \mathbb{D} : |z| \, |z - \alpha| = r \, |1 - \overline{\alpha} z| \}$$

look like "deformed" Cassini ovals. As z_1 is a critical point of B, the deformed lemniscate $\widetilde{L}(0, \alpha)$ is reached for $r = |B(z_1)|$, i.e. we have

$$\widetilde{L}(0, \alpha) := \{ z \in \mathbb{D} : |B(z)| = |B(z_1)| \}.$$

We define W_1 and W_2 as the components of the open set $\{ z \in \mathbb{D} : |B(z)| < |B(z_1)| \}$ which contain 0 and α, respectively. On the left-hand side of Figure 5.4 on the next page, a Mathematica plot of several deformed Cassini ovals is displayed in case of

$\alpha = 0.4 + 0.6i$. The right-hand side of Figure 5.4 shows a schematic plot of the deformed lemniscate $\widetilde{L}(0, \alpha)$ and the sets W_1 and W_2.

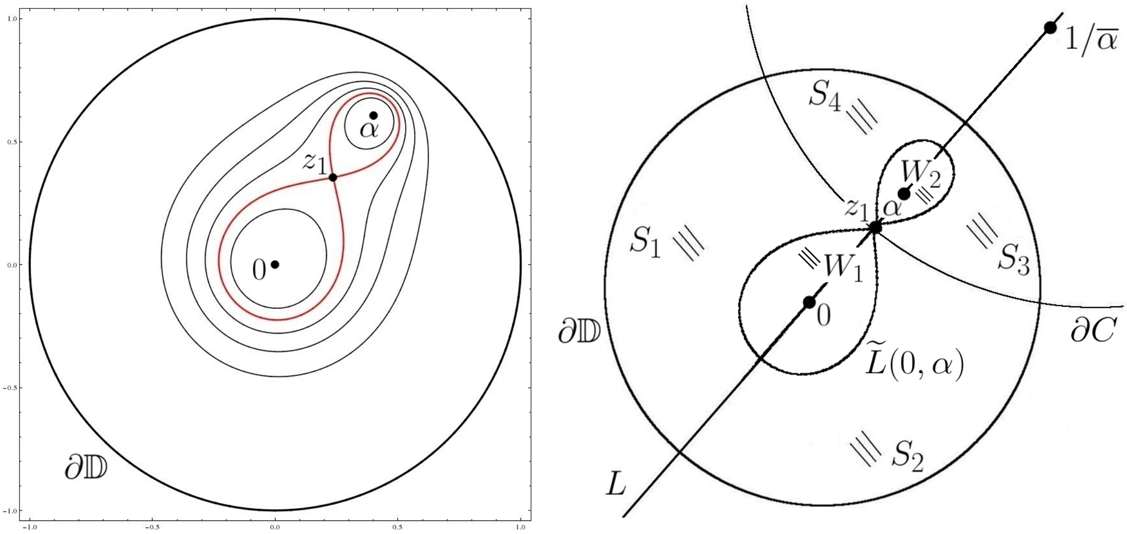

Figure 5.4

Lemma 5.1.7. *W_1 is invariant under B and we have $B(W_2) \subset W_1$.*

Proof: For $z \in W_1$, it follows that $|B(B(z))| \leq |B(z)| < |B(z_1)|$. Hence, we obtain $B(z) \in W_1 \cup W_2$ and thus $B(W_1) \subset W_1 \cup W_2$. We have $0 = B(0) \in B(W_1)$, and the continuity of B and the connectedness of W_1 imply that $B(W_1)$ is also connected. Due to $B(z) \in B(W_1)$, there exists a path in $B(W_1) \subset W_1 \cup W_2$ which connects 0 and $B(z)$. As this is not possible for $B(z) \in W_2$, we obtain $B(z) \in W_1$. Because of $0 = B(\alpha) \in B(W_2)$, the inclusion $B(W_2) \subset W_1$ can be proved in exactly the same way. □

Now, we can state and prove the following universality result:

Theorem 5.1.8. *The set of all functions in $H(\mathbb{C}\backslash\{0\})$ which are $H(U)$-universal for C_B for all $U \in \mathcal{U}_0(W_1\backslash\{0\}) \cup \mathcal{U}_0(W_2\backslash\{\alpha\})$ is a comeager set in $H(\mathbb{C}\backslash\{0\})$.*

Proof: We consider the domain $\Omega := \mathbb{C}\backslash\{0\}$, and we denote by $\mathcal{G}_1$ the set of all functions in $H(\Omega)$ which are $H(U)$-universal for C_B for all $U \in \mathcal{U}_0(W_1\backslash\{0\})$ and by $\mathcal{G}_2$ the set of all functions in $H(\Omega)$ which are $H(U)$-universal for C_B for all $U \in \mathcal{U}_0(W_2\backslash\{\alpha\})$.

i) Defining the open set $D := W_1\backslash\{0\}$, Lemma 5.1.6 and Lemma 5.1.7 imply that $B\big|_D : D \to D$ is injective. According to Remark 5.1.3 iv), we have locally uniform convergence $B^n\big|_D \to 0 \in \partial_\infty\Omega$ so that Corollary 2.1.7 yields that $\mathcal{G}_1$ is comeager in $H(\Omega)$.

ii) First, we show by induction that B^n is injective on W_2 for all $n \in \mathbb{N}$. For $n = 1$, this follows from Lemma 5.1.6. Now, let B^n be injective on W_2 and let $z, w \in W_2$ with $B^{n+1}(z) = B^{n+1}(w)$, i.e. $B(B^n(z)) = B(B^n(w))$. As we have $B^n(z), B^n(w) \in W_1$ by Lemma 5.1.7 and as B is injective on W_1 due to Lemma 5.1.6, we obtain $B^n(z) = B^n(w)$ and hence $z = w$. Considering the open sets $G := (W_1\backslash\{0\}) \cup (W_2\backslash\{\alpha\})$ and $G_0 := W_2\backslash\{\alpha\} \subset G$, Lemma 5.1.7 yields $B(G) \subset W_1\backslash\{0\} \subset G$. According to Remark 2.1.5, there exists a $\mathcal{K}_0(G_0)$-exhausting sequence in $\mathcal{U}_0(G_0)$. The injectivity of all iterates B^n on W_2 and the locally uniform convergence $B^n \to 0 \in \partial_\infty\Omega$ on $\mathbb{D}$ imply that this sequence is a sequence in $\mathcal{U}_0(G, \Omega, B)$. Thus, Corollary 2.1.6 yields that $\mathcal{G}_2$ is comeager in $H(\Omega)$. □

Remark 5.1.9. As already described in the introduction of this chapter, there exist "maximal" open neighbourhoods $U_{max} \subset \mathbb{D}$ and V_{max} of the origin with the property that there exists a conformal map $\varphi_{max} : U_{max} \to V_{max}$ which conjugates the map $B\big|_{U_{max}} : U_{max} \to U_{max}$ to the linear function $w \mapsto B'(0) \cdot w = -\alpha w$ on V_{max}. Hence, comeager many functions in $H(\mathbb{C}\backslash\{0\})$ are $H(U)$-universal for C_B for all $U \in \mathcal{U}_0(U_{max}\backslash\{0\})$. It is well-known that the inverse φ_{max}^{-1} can be extended homeomorphically to $\overline{V_{max}}$ (see e.g. [37], Lemma 8.5). Thus, there exists a function $\widetilde{\varphi}_{max} : \overline{U_{max}} \to \overline{V_{max}}$ with $\widetilde{\varphi}_{max} \in A\big(\overline{U_{max}}\big)$ such that

$$\widetilde{\varphi}_{max} \circ B = -\alpha \cdot \widetilde{\varphi}_{max}$$

holds on $\overline{U_{max}}$. Moreover, the image $\widetilde{\varphi}_{max}^{-1}(\partial V_{max})$ contains a critical point of B (see [37], Lemma 8.5) so that we have $z_1 \in \partial U_{max}$. Due to the above conjugation, we see that B is injective on $\overline{U_{max}}$. Because of $0 \in \overline{U_{max}}$, this implies that we have $\alpha \notin \overline{U_{max}}$. Hence, there exists an open neighbourhood U_α of α which is disjoint from U_{max}. Thus, we obtain the following result:

Corollary 5.1.10. *There exists a non-empty open set $W_\alpha \subset \mathbb{D}$ with $W_\alpha \cap U_{max} = \emptyset$ such that the set of all functions in $H(\mathbb{C}\backslash\{0\})$ which are $H(U)$-universal for C_B for all $U \in \mathcal{U}_0(W_\alpha)$ is a comeager set in $H(\mathbb{C}\backslash\{0\})$.*

Proof: Observing the above consideration and defining $W_\alpha := W_2\backslash\{\alpha\} \cap U_\alpha$, the assertion follows directly from Theorem 5.1.8. □

The three open subsets W_1, W_2 and U_{max} of $\mathbb{D}$ all have positive distance to $\partial\mathbb{D}$ (see e.g. [37], p. 78, for $\overline{U_{max}} \subset \mathbb{D}$). Hence, there exists some $\delta > 0$ such that, in the situations of Theorem 5.1.8 and Remark 5.1.9, all open sets U which admit $H(U)$-

universal functions for C_B have distance to $\partial\mathbb{D}$ greater than δ. In the following, we want to find compact sets $K \subset \mathbb{D}$ "arbitrarily close" to $\partial\mathbb{D}$ such that C_B is $A(K)$-universal – meaning that for each $\varepsilon > 0$, we want to find a set $K \in \mathcal{K}(\mathbb{D})$ with $\text{dist}(K, \partial\mathbb{D}) < \varepsilon$ such that C_B is $A(K)$-universal. As the straight line L and the boundary of the closed disc C define the geometry of the finite Blaschke product B, it is natural to look for compact line segments $K \subset L \cap \mathbb{D}$ or compact arcs $K \subset \partial C \cap \mathbb{D}$ having the property that C_B is $C(K)$-universal.

According to Lemma A.3, the backward orbit $O_B^-(z)$ has no accumulation point in $\mathbb{D}$ for each $z \in \mathbb{D}\backslash\{0\}$. Because of $O_B^-(0) = \{0, \alpha\} \cup O_B^-(\alpha)$, we obtain that $O_B^-(0)$ also has no accumulation point in $\mathbb{D}$. In particular, the set $\mathbb{D}\backslash O_B^-(0)$, which is invariant under B, is an open set. For simplicity, we now consider the case $\alpha \in \mathbb{R}$. Then we have $L = \mathbb{R}$ so that the reflection of a point $z \in \mathbb{C}$ with respect to L is given by $R_L(z) = \overline{z}$. Let K be a connected compact subset of $\partial C \cap \mathbb{D}$ such that $K \cap O_B^-(\{0, z_1\}) = \emptyset$ and $z_1 \notin K$. As each point of ∂C is a fixed point of I_C, it follows for all $z \in K$ that

$$B(z) = -z \cdot (-\varphi_\alpha(z)) = -z \cdot (R_L(I_C(z))) = -z \cdot (R_L(z)) = -z \cdot \overline{z} = -|z|^2.$$

Due to the connectedness of K and the continuity of B, this yields that $B(K)$ is a compact interval on the negative real axis. Because of $\alpha \in \mathbb{R}$, the real axis is invariant under B so that $B^n(K)$ is a compact interval on the real axis for all $n \in \mathbb{N}$. According to $K \cap O_B^-(z_1) = \emptyset$, we have $z_1 \notin B^n(K)$ for each $n \in \mathbb{N}$ and hence $B^n(K) \subset S_1 \cup S_2$ or $B^n(K) \subset S_3 \cup S_4$. Therefore, Lemma 5.1.6 implies that B is injective on $B^n(K)$ for each $n \in \mathbb{N}$.

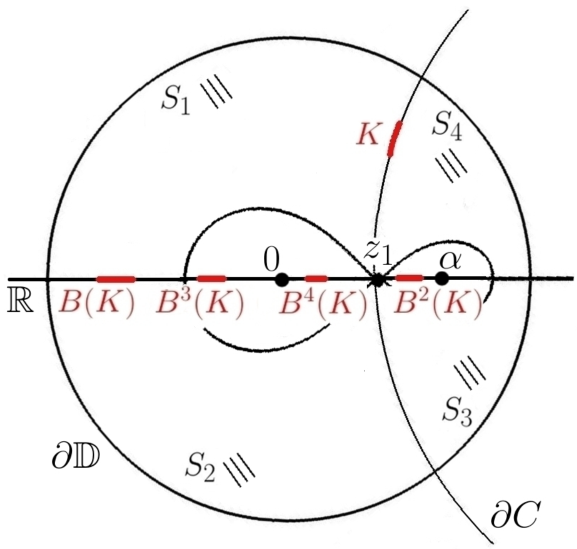

Figure 5.5

In order to apply Theorem 2.2.5 for showing that C_B is $C(K)$-universal, we have to verify that for each $n \in \mathbb{N}$ there exists an open neighbourhood of K on which B^n is injective. For $n \in \mathbb{N}$, Lemma A.8 yields the existence of an open connected neighbourhood $U_{n+1} \subset \mathbb{D}\backslash O_B^-(0)$ of K with $B^n(U_{n+1}) \cap (\partial C \cap \overline{\mathbb{D}}) = \emptyset$ so that we have $B^n(U_{n+1}) \subset S_1 \cup S_2$ or $B^n(U_{n+1}) \subset S_3 \cup S_4$. Without loss of generality, we may assume $U_{n+1} \subset U_n$ for all $n \in \mathbb{N}$. According to Lemma 5.1.6, we obtain that B is injective on $B^n(U_{n+1})$ for all $n \in \mathbb{N}$. Moreover, as K is a connected compact subset of $\partial C \cap \mathbb{D}$ with $z_1 \notin K$, Lemma 5.1.6 allows us to choose an open neighbourhood $U_1 \subset \mathbb{D}\backslash O_B^-(0)$ of K such that B is injective on U_1. Inductively, it follows that each iterate B^n is injective on U_n. Indeed, for $n = 1$, this is clear by construction. Assuming that B^n is injective on U_n, now let $z, w \in U_{n+1}$ with $B^{n+1}(z) = B^{n+1}(w)$, i.e. $B(B^n(z)) = B(B^n(w))$. The injectivity of B on $B^n(U_{n+1})$ implies $B^n(z) = B^n(w)$, and due to $z, w \in U_{n+1} \subset U_n$, it follows that $z = w$. Finally, the uniform convergence $B^n\big|_K \to 0$ yields

$$K \in \mathcal{K}_0\big(\mathbb{D}\backslash O_B^-(0), \mathbb{C}\backslash\{0\}, B\big).$$

Applying Theorem 2.2.5, we obtain the following universality result:

Theorem 5.1.11. *Let $\alpha \in \mathbb{R}$ and let K be a connected compact subset of $\partial C \cap \mathbb{D}$ or of $(-1,1)$ with $K \cap O_B^-(\{0, z_1\}) = \emptyset$ and $z_1 \notin K$. Then the set of all functions in $H(\mathbb{C}\backslash\{0\})$ which are $C(K)$-universal for C_B is a comeager set in $H(\mathbb{C}\backslash\{0\})$.*

Remark 5.1.12. As $O_B^-(\{0, z_1\})$ has no accumulation point in $\mathbb{D}$, there exist non-degenerate continua K arbitrarily close to $\partial\mathbb{D}$ which fulfil the assumptions of Theorem 5.1.11. However, these sets can be "small" near $\partial\mathbb{D}$. This is demonstrated in the following figure, which shows a Mathematica plot of the set $\bigcup_{n=1}^{5} B^{-n}(\{0\})$ in case of $\alpha = 1/2$:

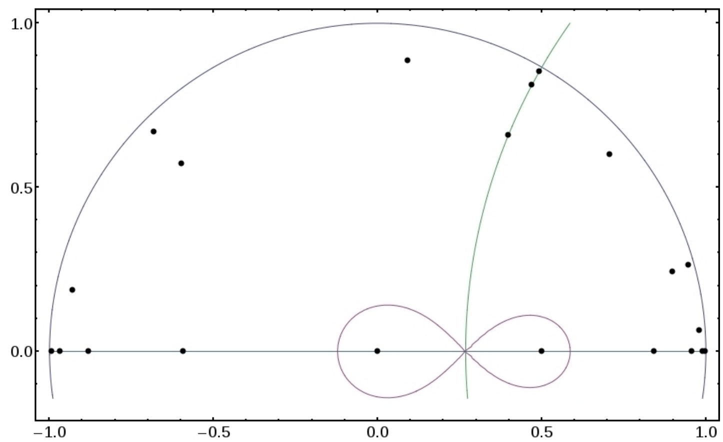

Figure 5.6

5.2 Rational Functions

According to the considerations at the beginning of Section 5.1, it is now our aim to transfer the universality statements of the previous section to the situation of arbitrary rational functions having an attracting fixed point.

In order to do so, we consider a rational function f of degree $d_f \geq 2$ and a simply connected component G of F_f with $G \subset \mathbb{C}$, $G \neq \mathbb{C}$, which contains an attracting fixed point z_0 of f. Then G is invariant under f so that there exists a conformal map $\psi : G \to \mathbb{D}$ which conjugates $f\big|_G : G \to G$ to the restriction $B\big|_{\mathbb{D}} : \mathbb{D} \to \mathbb{D}$ of a finite Blaschke product B of degree $d_B \in \mathbb{N}$ of the form

$$B(z) = e^{i\theta} \prod_{k=1}^{d_B} \frac{z - \alpha_k}{1 - \overline{\alpha_k} z}, \quad z \in \mathbb{C}_\infty,$$

where $\theta \in \mathbb{R}$ and $\alpha_1, \ldots, \alpha_d \in \mathbb{D}$ (cf. Section 5.1). This means that the equation

$$(*) \qquad \psi \circ f = B \circ \psi$$

holds on G. Without loss of generality, we may assume that $\psi(z_0) = 0$. Then we obtain $B(0) = B(\psi(z_0)) = \psi(f(z_0)) = \psi(z_0) = 0$ so that there must be some $k \in \{1, \ldots, d_B\}$ with $\alpha_k = 0$. Differentiating both sides of equation $(*)$, we obtain

$$(**) \qquad (\psi' \circ f) \cdot f' = (B' \circ \psi) \cdot \psi'$$

and hence $\psi'(z_0) \cdot f'(z_0) = B'(0) \cdot \psi'(z_0)$. As the derivative ψ' of the conformal map ψ does not vanish on G, it follows that $B'(0) = f'(z_0)$. Therefore, 0 is an attracting fixed point of B so that Remark 5.1.3 iv) implies that B is of the form

$$B(z) = e^{i\theta} z \prod_{k=1}^{d_B - 1} \frac{z - \alpha_k}{1 - \overline{\alpha_k} z}, \quad z \in \mathbb{C}_\infty,$$

where $\alpha_1, \ldots, \alpha_{d_B - 1} \in \mathbb{D} \backslash \{0\}$ (cf. [7], Exercise 6.3.2 on p. 109). In order to apply the results of Section 5.1, we now want to find situations in which we have $d_B = 2$. The following lemma provides a necessary and sufficient condition:

Lemma 5.2.1. *In the above situation, let G further be completely invariant under f. Then we have $d_B = 2$ if and only if $d_f = 2$.*

Proof: First, let $d_f = 2$. Because of $f'(z_0) \neq 0$, there exists a point $\widetilde{z} \in \mathbb{C} \backslash \{z_0\}$ with $f(\widetilde{z}) = f(z_0) \in G$. The backward invariance of G under f implies $\widetilde{z} \in f^{-1}(G) \subset G$, and due to $d_f = 2$, we have $f'(\widetilde{z}) \neq 0$. As ψ is injective, we obtain $\psi(z_0) \neq \psi(\widetilde{z})$, and according to the above conjugation, it follows that

$$B(\psi(z_0)) = \psi(f(z_0)) = \psi(f(\widetilde{z})) = B(\psi(\widetilde{z})).$$

Because of $f'(z_0) \neq 0$ and $f'(\widetilde{z}) \neq 0$, $(**)$ yields $B'(\psi(z_0)) \neq 0$ and $B'(\psi(\widetilde{z})) \neq 0$. Assuming that there exists a point $w \in \mathbb{D}\backslash\{\psi(z_0), \psi(\widetilde{z})\}$ with $B(w) = B(\psi(z_0))$, the conjugation of $f\big|_G$ to $B\big|_{\mathbb{D}}$ would imply that

$$f\big(\psi^{-1}(w)\big) = \psi^{-1}\big(B(w)\big) = \psi^{-1}\big(B(\psi(z_0))\big) = \psi^{-1}\big(\psi(f(z_0))\big) = f(z_0).$$

Due to $d_f = 2$ and $f(z_0) = f(\widetilde{z})$, we would obtain $\psi^{-1}(w) \in \{z_0, \widetilde{z}\}$ and hence $w \in \{\psi(z_0), \psi(\widetilde{z})\}$, a contradiction. Thus, we have

$$B^{-1}\big(\big\{B(\psi(z_0))\big\}\big) = \{\psi(z_0), \psi(\widetilde{z})\}.$$

According to $B'(\psi(z_0)) \neq 0$ and $B'(\psi(\widetilde{z})) \neq 0$, this yields $d_B = 2$. On the other hand, now let $d_f \geq 3$. As above, the backward invariance of G under B yields the existence of a point $w_0 \in G\backslash\{z_0\}$ with $f(w_0) = f(z_0)$. In case of $f'(w_0) \neq 0$, there must be another point $w_1 \in G\backslash\{z_0, w_0\}$ with $f(w_1) = f(z_0)$. Due to the injectivity of ψ, the three points $\psi(z_0), \psi(w_0)$ and $\psi(w_1)$ are distinct, and the above conjugation implies that they are all mapped onto $B(\psi(z_0))$ under B so that we have $d_B \geq 3$. If $f'(w_0) = 0$, $(**)$ yields $B'(\psi(w_0)) = 0$. As we have $\psi(z_0) \neq \psi(w_0)$ and $B(\psi(z_0)) = B(\psi(w_0))$, it follows again that $d_B \geq 3$. □

Corollary 5.2.2. *Let f be a rational function of degree two and let $G \subset \mathbb{C}$, $G \neq \mathbb{C}$, be a simply connected component of F_f which is completely invariant under f and which contains an attracting fixed point of f. Then there exists some $\alpha \in \mathbb{D}\backslash\{0\}$ such that $f\big|_G$ is conjugated to the restriction $B_\alpha\big|_{\mathbb{D}}$ of the finite Blaschke product*

$$B_\alpha : \mathbb{C}_\infty \to \mathbb{C}_\infty, \;\; B_\alpha(z) := z \cdot \frac{z - \alpha}{1 - \overline{\alpha} z}.$$

Proof: According to Lemma 5.2.1, there exist $\theta \in \mathbb{R}$ and $\beta \in \mathbb{D}\backslash\{0\}$ as well as a conformal map $\psi : G \to \mathbb{D}$ which conjugates $f\big|_G$ to $B_{\theta,\beta}$, where

$$B_{\theta,\beta} : \mathbb{D} \to \mathbb{D}, \;\; B_{\theta,\beta}(z) := e^{i\theta} z \, \frac{z - \beta}{1 - \overline{\beta} z}.$$

Putting $\alpha := e^{i\theta}\beta \in \mathbb{D}\backslash\{0\}$ and considering the rotation $\psi_\theta : \mathbb{D} \to \mathbb{D}$, $\psi_\theta(z) := e^{i\theta} z$, the following equation holds for all $z \in \mathbb{D}$:

$$\begin{aligned}\psi_\theta(B_{\theta,\beta}(z)) &= e^{i\theta} \cdot B_{\theta,\beta}(z) = e^{i\theta} \cdot e^{i\theta} z \cdot \frac{z-\beta}{1-\overline{\beta} z} = e^{i\theta} z \cdot \frac{e^{i\theta} z - e^{i\theta}\beta}{1-\overline{(e^{-i\theta}\alpha)}\, z} \\ &= e^{i\theta} z \cdot \frac{e^{i\theta} z - \alpha}{1-\overline{\alpha}\, e^{i\theta} z} = B_\alpha(e^{i\theta} z) = B_\alpha(\psi_\theta(z)).\end{aligned}$$

Thus, ψ_θ conjugates $B_{\theta,\beta}$ to $B_\alpha\big|_{\mathbb{D}}$ so that $\psi_\theta \circ \psi$ conjugates $f\big|_G$ to $B_\alpha\big|_{\mathbb{D}}$. □

Example 5.2.3. In the following situations, the assumptions of Corollary 5.2.2 are fulfilled:

i) Let P be a polynomial of degree two and let $z_0 \in \mathbb{C}$ be an attracting fixed point of P. Then the component $G(z_0)$ of F_P containing z_0 and the component $G(\infty)$ of F_P containing the superattracting fixed point ∞ of P both are completely invariant under P (see e.g. [15], p. 103). In this situation, $G(z_0)$ and $G(\infty)$ must be simply connected ([15], Theorem IV.1.1). Thus, $G(z_0)$ fulfils the assumptions of Corollary 5.2.2.

ii) Examples of polynomials P as required in part i) are given by the following families: For each $\lambda \in \mathbb{D}\backslash\{0\}$, the quadratic polynomial

$$P_\lambda : \mathbb{C}_\infty \to \mathbb{C}_\infty, \;\; P_\lambda(z) := z^2 + \lambda z,$$

has an attracting fixed point at 0 (cf. [49], p. 40), and for each $0 < c < 1/4$, the quadratic polynomial

$$Q_c : \mathbb{C}_\infty \to \mathbb{C}_\infty, \;\; Q_c(z) := z^2 + c,$$

has an attracting fixed point at $(1-\sqrt{1-4c})/2$ (cf. [49], Exercise 5 on p. 42).

iii) Moreover, there are examples of rational, non-polynomial functions in the situation of Corollary 5.2.2. Indeed, for each $0 < b < 1$, the rational function

$$R_b : \mathbb{C}_\infty \to \mathbb{C}_\infty, \;\; R_b(z) := \frac{z}{z^2 - bz + 1},$$

has an attracting fixed point at b and a neutral fixed point at 0. Furthermore, we have $\infty \in J_{R_b}$, and the component $G(b)$ of F_{R_b} containing b as well as the component of F_{R_b} containing an attracting petal at 0 are completely invariant under R_b (see [15], p. 83) and hence simply connected ([15], Theorem IV.1.1). Thus, $G(b)$ fulfils the assumptions of Corollary 5.2.2.

Now, we transfer the universality statements of Section 5.1 to the situation of a rational function f of degree two and a simply connected, completely invariant component $G \subset \mathbb{C}$, $G \neq \mathbb{C}$, of F_f which contains an attracting fixed point z_0 of f. According to Corollary 5.2.2, there exists an $\alpha \in \mathbb{D}\backslash\{0\}$, a finite Blaschke product B_α of the form $B_\alpha(z) := z(z-\alpha)/(1-\overline{\alpha}z)$, $z \in \mathbb{C}_\infty$, and a conformal map $\psi : G \to \mathbb{D}$ such that the equation

$$\psi \circ f = B_\alpha \circ \psi$$

holds on G. Moreover, as described in the introduction of this chapter, we denote by $U_{max} \subset G$ the "maximal" open neighbourhood of z_0 on which f is conjugated to the linear function $w \mapsto f'(z_0) \cdot w$. Then the following statement holds:

Theorem 5.2.4. *There exists a non-empty open set $V_\alpha \subset G$ with $V_\alpha \cap U_{max} = \emptyset$ such that the set of all functions in $H(\mathbb{C}\backslash\{z_0\})$ which are $H(U)$-universal for C_f for all $U \in \mathcal{U}_0(V_\alpha)$ is a comeager set in $H(\mathbb{C}\backslash\{z_0\})$.*

Proof:

i) For $k \in \{1, 2, 3, 4\}$, we consider the disjoint subsets S_k of $\mathbb{D}$ which we have defined before Lemma 5.1.6. Then the sets $\widetilde{S}_k := \psi^{-1}(S_k)$ are disjoint subsets of G and f is injective on each union $\widetilde{S}_k \cup \widetilde{S}_l$ for $\{k, l\} \notin \{\{1, 3\}, \{2, 4\}\}$. Indeed, for $\widetilde{z}, \widetilde{w} \in \widetilde{S}_k \cup \widetilde{S}_l$ with $f(\widetilde{z}) = f(\widetilde{w})$ and points $z \in S_k$, $w \in S_l$ with $\widetilde{z} = \psi^{-1}(z)$ and $\widetilde{w} = \psi^{-1}(w)$, the above conjugation yields

$$\psi^{-1}(B_\alpha(z)) = f(\psi^{-1}(z)) = f(\widetilde{z}) = f(\widetilde{w}) = f(\psi^{-1}(w)) = \psi^{-1}(B_\alpha(w)).$$

As ψ^{-1} is injective, we obtain $B_\alpha(z) = B_\alpha(w)$ so that Lemma 5.1.6 yields $z = w$ and hence $\widetilde{z} = \widetilde{w}$.

ii) Let z_1 be the critical point of B_α in $\mathbb{D}$ (see Remark 5.1.4 ii)) and let V_1 and V_2 be the components of the open set $\{z \in \mathbb{D} : |B_\alpha(z)| < |B_\alpha(z_1)|\}$ which contain 0 and α, respectively. Denoting $\widetilde{V}_1 := \psi^{-1}(V_1)$ and $\widetilde{V}_2 := \psi^{-1}(V_2)$, Lemma 5.1.7 and the above conjugation imply that $\widetilde{V}_1$ is invariant under f and that we have $f(\widetilde{V}_2) \subset \widetilde{V}_1$. Because of $V_1 \subset S_1 \cup S_2$ and $V_2 \subset S_3 \cup S_4$ (see Figure 5.4), we obtain $\widetilde{V}_1 \subset \widetilde{S}_1 \cup \widetilde{S}_2$ and $\widetilde{V}_2 \subset \widetilde{S}_3 \cup \widetilde{S}_4$ so that part i) yields that each iterate f^n is injective on $\widetilde{V}_2$.

iii) Let $\mathcal{G}$ be the set of all functions in $H(\mathbb{C}\backslash\{z_0\})$ which are $H(U)$-universal for C_f for all $U \in \mathcal{U}_0(\widetilde{V}_2\backslash\{\psi^{-1}(\alpha)\})$. We consider the domain $\Omega := \mathbb{C}\backslash\{z_0\}$ and the open sets $D := (\widetilde{V}_1\backslash\{z_0\}) \cup (\widetilde{V}_2\backslash\{\psi^{-1}(\alpha)\}) \subset \Omega$ and $D_0 := \widetilde{V}_2\backslash\{\psi^{-1}(\alpha)\} \subset D$.

Observing part ii), we obtain analogously to the proof of Theorem 5.1.8 that $\mathcal{G}$ is comeager in $H(\mathbb{C}\backslash\{z_0\})$.

iv) As in Remark 5.1.9, we see that f is injective on $\overline{U_{max}}$. Because of $z_0 \in \overline{U_{max}}$ and $f(z_0) = z_0$ as well as $f(\psi^{-1}(\alpha)) = \psi^{-1}(B_\alpha(\alpha)) = \psi^{-1}(0) = z_0$, we have $\psi^{-1}(\alpha) \notin \overline{U_{max}}$. Hence, there exists an open neighbourhood U_α of $\psi^{-1}(\alpha)$ which is disjoint from U_{max}. Defining $V_\alpha := \widetilde{V}_2\backslash\{\psi^{-1}(\alpha)\} \cap U_\alpha$, the assertion now follows from part iii). □

We conclude this chapter with the following universality result which holds for sets that may be chosen arbitrarily close to the boundary of the component of F_f which contains the attracting fixed point z_0 of f:

Theorem 5.2.5. *In the above situation, let $\alpha \in \mathbb{R}$. Then there exist non-degenerate continua $K \subset G$ arbitrarily close to $\partial_\infty G$ such that the set of all functions in $H(\mathbb{C}\backslash\{z_0\})$ which are $C(K)$-universal for C_f is a comeager set in $H(\mathbb{C}\backslash\{z_0\})$.*

Proof: Let z_1 be the critical point of B_α in $\mathbb{D}$ and let L be a connected compact subset of the interval $(-1, 1)$ with $L \cap O^-_{B_\alpha}(\{0, z_1\}) = \emptyset$ and $z_1 \notin L$. According to the considerations before Lemma 5.1.11, we have uniform convergence $B^n_\alpha\big|_L \to 0$ and for each $n \in \mathbb{N}$ there exists an open neighbourhood $V_n \subset \mathbb{D}\backslash O^-_{B_\alpha}(0)$ of L such that B^n_α is injective on V_n. We now consider the compact set $K := \psi^{-1}(L)$. The invariance of the number of holes implies that K has no holes, and due to the conjugation $\psi \circ f = B_\alpha \circ \psi$, we obtain $K \subset \psi^{-1}(\mathbb{D}\backslash O^-_{B_\alpha}(0)) = G\backslash O^-_f(z_0)$. Defining $U_n := \psi^{-1}(V_n) \subset G\backslash O^-_f(z_0)$ for each $n \in \mathbb{N}$, it follows that U_n is an open neighbourhood of K on which f^n is injective. Moreover, Lemma A.2 yields the uniform convergence

$$f^n \circ \psi^{-1}\big|_L = \psi^{-1} \circ B^n_\alpha\big|_L \to \psi^{-1}(0) = z_0$$

so that we obtain $f^n\big|_K \to z_0$ uniformly. Altogether, we have shown that

$$K \in \mathcal{K}_0\big(G\backslash O^-_f(z_0), \mathbb{C}\backslash\{z_0\}, f\big).$$

Hence, observing that $K^\circ = \psi^{-1}(L)^\circ = \psi^{-1}(L^\circ) = \psi^{-1}(\emptyset) = \emptyset$ (see Lemma A.1), Theorem 2.2.5 implies that the set of all functions in $H(\mathbb{C}\backslash\{z_0\})$ which are $C(K)$-universal for C_f is a comeager set in $H(\mathbb{C}\backslash\{z_0\})$. According to Remark 5.1.12, the set L can be chosen to be a non-degenerate continuum (which implies that K is also a non-degenerate continuum) and such that L is arbitrarily close to $\partial\mathbb{D}$. As the conformal map ψ^{-1} maps "boundary sequences of $\mathbb{D}$" onto "boundary sequences of G" $\big($i.e. for each sequence (z_n) in $\mathbb{D}$ which has no accumulation point in $\mathbb{D}$, the sequence $(\psi^{-1}(z_n))$ has no accumulation point in G; see e.g. [43], p. 184f.$\big)$, we obtain that L can be chosen such that K is arbitrarily close to $\partial_\infty G$. $\square$

Appendix A

Miscellanea

In this part of the appendix, we collect and prove several lemmas which we need in the course of this work.

Lemma A.1. *Let $U, V \subset \mathbb{C}$ be open, $\varphi : U \to V$ conformal and $B \subset U$. Then the following statements hold:*

i) $\varphi(B^\circ) = \varphi(B)^\circ$.

ii) If $\overline{B}$ is a compact subset of U, we have $\varphi(\overline{B}) = \overline{\varphi(B)}$ and $\varphi(\partial B) = \partial\varphi(B)$.

Proof:

i) As $\varphi(B^\circ)$ is an open subset of $\varphi(B)$ and as $\varphi(B)^\circ$ is the union of all open subsets of $\varphi(B)$, we obtain the inclusion $\varphi(B^\circ) \subset \varphi(B)^\circ$. Moreover, the inverse $\varphi^{-1} : V \to U$ is conformal and we have $\varphi(B) \subset V$. Thus, the same argument as above yields $\varphi^{-1}(\varphi(B)^\circ) \subset \varphi^{-1}(\varphi(B))^\circ = B^\circ$ so that we obtain the inclusion $\varphi(B)^\circ \subset \varphi(B^\circ)$.

ii) Let $\overline{B} \subset U$ be compact. The inclusion $\varphi(\overline{B}) \subset \overline{\varphi(B)}$ follows directly from the continuity of φ. For showing the reverse inclusion, we first observe that the compactness of $\overline{B}$ and the continuity of φ imply that $\varphi(\overline{B})$ is compact and thus in particular closed. As $\overline{\varphi(B)}$ is the intersection of all closed supersets of $\varphi(B)$, we obtain $\overline{\varphi(B)} \subset \varphi(\overline{B}) \subset \varphi(U) = V$. Hence, the compactness of $\varphi(\overline{B})$ implies that $\overline{\varphi(B)}$ is a compact subset of V, and the same argument as above yields

$$\varphi^{-1}\left(\overline{\varphi(B)}\right) \subset \overline{\varphi^{-1}(\varphi(B))} = \overline{B}.$$

Thus, we obtain $\overline{\varphi(B)} \subset \varphi(\overline{B})$ so that $\varphi(\overline{B}) = \overline{\varphi(B)}$ is shown. Finally, due to this equality and due to part i), the injectivity of φ implies

$$\varphi(\partial B) = \varphi\left(\overline{B} \backslash B^{\circ}\right) = \varphi(\overline{B}) \backslash \varphi(B^{\circ}) = \overline{\varphi(B)} \backslash \varphi(B)^{\circ} = \partial\varphi(B),$$

which completes the proof. □

Lemma A.2. *Let $K \subset \mathbb{C}$ be compact, $f \in C(K)$ and let $V \supset f(K)$ be open and $\varphi \in C(V)$. Moreover, let (f_n) be a sequence in V^K with $f_n \to f$ uniformly on K. Then $\varphi \circ f_n \to \varphi \circ f$ holds uniformly on K.*

Proof: Let $\varepsilon > 0$. We choose an $r > 0$ with $U_r[f(K)] \subset V$. Because of the uniform convergence $f_n \to f$ on K, there exists an $N_1 \in \mathbb{N}$ such that we have

$$r > |f_n(z) - f(z)| \geq \operatorname{dist}(f_n(z), f(K))$$

and hence $f_n(z) \in U_r[f(K)]$ for all $n \geq N_1$ and for all $z \in K$. As φ is uniformly continuous on the compact set $U_r[f(K)]$, there exists some $\delta > 0$ such that we have $|\varphi(z) - \varphi(w)| < \varepsilon$ for all points $z, w \in U_r[f(K)]$ with $|z - w| < \delta$. The uniform convergence $f_n \to f$ on K now implies the existence of an $N_2 \in \mathbb{N}$ with $|f_n(z) - f(z)| < \delta$ for all $n \geq N_2$ and for all $z \in K$. Putting $N := \max(N_1, N_2)$, we finally obtain $|\varphi(f_n(z)) - \varphi(f(z))| < \varepsilon$ for all $n \geq N$ and for all $z \in K$. □

Lemma A.3. *Let f be a transcendental entire function or a rational function of degree $d \geq 2$ which has a (super-)attracting fixed point at z_0. Moreover, let G be the basin of attraction of z_0 under f and let K be a compact subset of $G \backslash \{z_0\}$. Then, for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and each $z \in f^{-n}(K)$ we have dist$(z, \partial_\infty G) < \varepsilon$.*

Proof: We only consider the case that f is a rational function of degree $d \geq 2$, i.e. we have $z_0 \in \mathbb{C}_\infty$ and $G = \{z \in \mathbb{C}_\infty : f^n(z) \to z_0\}$. Let us assume that the assertion does not hold. Then there would exist an $\varepsilon_0 > 0$ and a strictly increasing sequence (n_k) in $\mathbb{N}$ such that for all $k \in \mathbb{N}$ there exists a point $z_k \in f^{-n_k}(K)$ with $\operatorname{dist}(z_k, \partial_\infty G) \geq \varepsilon_0$. As $(\mathbb{C}_\infty, \chi)$ is compact, there would exist a subsequence (z_{k_l}) of (z_k) and a point $w_0 \in \mathbb{C}_\infty$ with $z_{k_l} \to w_0$. The backward invariance of G under f implies that (z_{k_l}) is a sequence in G. Thus, if $w_0 \notin G$, we would obtain $w_0 \in \partial_\infty G$ so that there would exist a $j \in \mathbb{N}$ with

$$\varepsilon_0 > \chi(z_{k_j}, w_0) \geq \operatorname{dist}(z_{k_j}, \partial_\infty G).$$

But this contradicts the choice of the point z_{k_j}. On the other hand, if $w_0 \in G$, we would obtain that

$$L := \{z_{k_l} : l \in \mathbb{N}\} \cup \{w_0\}$$

is a compact subset of G. According to the compactness of K and due to $z_0 \notin K$, there exists some $\delta > 0$ with $\chi(z, z_0) \geq \delta$ for all $z \in K$. In particular, because of $z_k \in f^{-n_k}(K)$, we have $\chi(f^{n_k}(z_k), z_0) \geq \delta$ for each $k \in \mathbb{N}$. As G is an open subset of the Fatou set F_f, we obtain that the convergence $f^n\big|_L \to z_0$ is uniform (apply the statement of Remark 1.2.6 i) to each component of G). Hence, there would exist an $N \in \mathbb{N}$ such that we have $\chi(f^n(z), z_0) < \delta$ for all $n \geq N$ and for all $z \in L$. Thus, for $l \in \mathbb{N}$ with $n_{k_l} \geq N$, it would follow that $\chi(f^{n_{k_l}}(z_{k_l}), z_0) < \delta$, which is a contradiction. Therefore, the assertion holds in case that f is a rational function. If f is a transcendental entire function, the proof runs analogously. □

Lemma A.4. *Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ and let (f_n) be a sequence in $\mathbb{C}^U$. Then the following assertions are equivalent:*

- *i) For each compact subset K of U, there exists a strictly increasing sequence $(n_k) = (n_k(K))$ in $\mathbb{N}$ such that (f_{n_k}) converges to f uniformly on K.*
- *ii) There exists a strictly increasing sequence (n_k) in $\mathbb{N}$ such that (f_{n_k}) converges to f uniformly on all compact subsets of U.*

Proof: The implication ii) $\Rightarrow$ i) is trivial. In order to show that i) implies ii), let (K_n) be a compact exhaustion of U. Due to i), there exists an $n_1 \in \mathbb{N}$ with $\|f_{n_1} - f\|_{K_1} < 1$. Moreover, i) implies the existence of a strictly increasing sequence (m_j) in $\mathbb{N}$ such that $\|f_{m_j} - f\|_{K_2} \to 0$. Therefore, there exists an $n_2 > n_1$ with $\|f_{n_2} - f\|_{K_2} < 1/2$. Inductively, we find a strictly increasing sequence (n_k) in $\mathbb{N}$ with $\|f_{n_k} - f\|_{K_k} < 1/k$ for all $k \in \mathbb{N}$. Now, let $K \subset U$ be compact and let $\varepsilon > 0$. Choosing $m \in \mathbb{N}$ with $K \subset K_m$ and $1/m < \varepsilon$, we obtain for all $k \geq m$ that

$$\|f_{n_k} - f\|_K \leq \|f_{n_k} - f\|_{K_m} \leq \|f_{n_k} - f\|_{K_k} \leq \frac{1}{k} \leq \frac{1}{m} < \varepsilon.$$

Thus, (f_{n_k}) converges to f uniformly on K and ii) is shown. □

Lemma A.5. *Let $G, D \subset \mathbb{C}$ be open, $f : G \to D$ holomorphic and surjective and let $g : D \to \mathbb{C}$ be a function such that the composition $g \circ f : G \to \mathbb{C}$ is holomorphic. Then g is holomorphic on D.*

Proof: We first remark that the statement of Lemma A.5 can be found in the form of a question in [44], Exercise 14 on p. 228. However, in this reference, the statement is not proved. The following proof is a slightly adapted version of a proof of the above statement that can be found on the Internet platform "Mathematics Stack Exchange" (see *http://math.stackexchange.com/questions/206652/if-composition-of-one-function-and-the-other-holomorphic-function-is-holomorphic*; as of September 7, 2015).

i) We show that g is continuous on D. For $w_0 \in D$, the surjectivity of f yields a point $z_0 \in G$ with $w_0 = f(z_0)$. According to the local behaviour of the holomorphic function f, there exist an open neighbourhood $U \subset G$ of z_0, an integer $m \in \mathbb{N}$ and an injective function $\psi \in H(U)$ with

$$f(z) = w_0 + \psi^m(z)$$

for all $z \in U$ (see e.g. [44], Theorem 10.32; observe that the surjectivity of f and the openness of D imply that f is not constant). In particular, we have $\psi(z_0) = 0$ and the inverse function $\left(\psi\big|_U\right)^{-1} : \psi(U) \to U$ is holomorphic. Now, let (w_k) be a sequence in D which converges to w_0. As $f(U)$ is open and as we have $w_0 = f(z_0) \in f(U)$, there exists an $N \in \mathbb{N}$ with $w_k \in f(U)$ for all $k \geq N$. Thus, for each $k \geq N$, we can find a point $z_k \in U$ with $w_k = f(z_k)$. Due to

$$\psi^m(z_k) = f(z_k) - w_0 = w_k - w_0 \to 0,$$

we obtain the convergence $\psi(z_k) \to 0 = \psi(z_0)$ so that the continuity of $\left(\psi\big|_U\right)^{-1}$ implies that (z_k) converges to z_0. As $g \circ f$ is continuous, it follows that

$$g(w_k) = g(f(z_k)) \to g(f(z_0)) = g(w_0)$$

so that the continuity of g at w_0 is shown.

ii) We now show that g is holomorphic on D by using the continuity of g on D. Let $w_0 \in D$ and let $z_0 \in G$ with $f(z_0) = w_0$. First, we consider the case that $f'(z_0) \neq 0$. Then there exists an open neighbourhood $U \subset G$ of z_0 such that $f\big|_U : U \to f(U)$ is a conformal map (see e.g. [44], Theorem 10.30). Thus, the holomorphy of $g \circ f$ on U and the holomorphy of $\left(f\big|_U\right)^{-1}$ on $f(U)$ imply that

$$g\big|_{f(U)} = g \circ f\big|_U \circ \left(f\big|_U\right)^{-1} = (g \circ f) \circ \left(f\big|_U\right)^{-1}$$

is holomorphic on the set $f(U)$ which contains the point $f(z_0) = w_0$. Finally, we have to deal with the case that $f'(z_0) = 0$. As the set of zeros of f' does not have an accumulation point in G, we can find an open bounded neighbourhood V of z_0 with $\overline{V} \subset G$ and $f'(z) \neq 0$ for all $z \in V \backslash \{z_0\}$. Then $f(V)$ is an open neighbourhood of w_0, and for all $w \in f(V)\backslash\{w_0\}$ there exists some $z \in V$ with $w = f(z)$ and $f'(z) \neq 0$. Thus, the first case implies that g is holomorphic on $f(V)\backslash\{w_0\}$. As g is continuous according to part i), the compactness of $f(\overline{V})$ yields that g is bounded on $f(\overline{V}) \supset f(V)\backslash\{w_0\}$. Therefore, Riemann's theorem on removable singularities implies that g is holomorphic at w_0. □

Corollary A.6. *Let $G, D \subset \mathbb{C}$ be open and let I be a set such that there exist pairwise disjoint open sets G_k as well as open sets D_k, $k \in I$, with $G = \bigcup_{k\in I} G_k$ and $D = \bigcup_{k\in I} D_k$. Moreover, for each $k \in I$, let $f_k : G_k \to D_k$ be holomorphic and surjective, and let*

$$f : G \to D \times I, \;\; f(z) := (f_k(z), k)\,, \;\; \textit{if } z \in G_k.$$

Finally, let $g : f(G) \to \mathbb{C}$ be a function such that the composition $g \circ f : G \to \mathbb{C}$ is holomorphic. Then we have $g\big(\bullet\big|_{D_k}, k\big) \in H(D_k)$ for all $k \in I$.

Proof: Let $k \in I$ be fixed. Then G_k and D_k are open subsets of $\mathbb{C}$, $f_k : G_k \to D_k$ is holomorphic and surjective, and $g\big(\bullet\big|_{D_k}, k\big) : D_k \to \mathbb{C}$ is a function such that the composition

$$G_k \ni z \mapsto \left(g\left(\bullet\big|_{D_k}, k\right) \circ f_k\right)(z) = g(f_k(z), k) = g(f(z)) = (g \circ f)(z)$$

is holomorphic. Thus, Lemma A.5 implies that $g\big(\bullet\big|_{D_k}, k\big)$ is holomorphic on D_k. □

Lemma A.7. *Let $K \subset \mathbb{C}$ be compact and let H be a hole of K. Then the following statements hold:*

i) H is a domain in $\mathbb{C}$.

ii) We have $\partial H \subset K$.

iii) If U is an open neighbourhood of K which has no holes and $\varphi : U \to \mathbb{C}$ is holomorphic and injective, then $\varphi(H)$ is a hole of $\varphi(K)$.

Proof:

i) As K is closed and bounded in $\mathbb{C}$, it is also closed in $\mathbb{C}_\infty$ so that $\mathbb{C}_\infty \backslash K$ is an open subset of $\mathbb{C}_\infty$. Hence, as $\mathbb{C}_\infty$ is locally connected, each component W of $\mathbb{C}_\infty \backslash K$ is open in $\mathbb{C}_\infty$, and it follows that $W \cap \mathbb{C}$ is open in $\mathbb{C}$. By definition, H is a component of $\mathbb{C}_\infty \backslash K$ with $\infty \notin H$. As subsets of $\mathbb{C}$ are connected in $(\mathbb{C}, \mathcal{T}_\mathbb{C})$ if and only if they are connected in $(\mathbb{C}_\infty, \mathcal{T}_\infty)$, this yields that H is a domain in $\mathbb{C}$.

ii) Let C_K be the component of $\mathbb{C}_\infty \backslash K$ which contains ∞. Assuming that we have $\infty \in \partial H$, there would exist a sequence (z_n) in H which converges to ∞ so that the openness of C_K implies that we could find some $N \in \mathbb{N}$ with $z_N \in H \cap C_K = \emptyset$, a contradiction. Hence, we have $\infty \notin \partial H$. Now, let $z \in \partial H$. Assuming that $z \notin K$, there would exist a component V of $\mathbb{C}_\infty \backslash K$ with $z \in V$. According to the considerations in part i), the set $V_0 := V \cap \mathbb{C}$ is open in $\mathbb{C}$. Because of $z \neq \infty$, there would exist some $\varepsilon > 0$ with $U_\varepsilon(z) \subset V_0$, and because of $z \in \partial H$, we would obtain

$$\emptyset \neq U_\varepsilon(z) \cap H \subset V \cap H.$$

Thus, the two components H and V of $\mathbb{C}_\infty \backslash K$ would be equal, and we would obtain $z \in H$. But as H is open in $\mathbb{C}$ due to part i), this contradicts $z \in \partial H$.

iii) Let $U \in \mathcal{U}_0(\mathbb{C})$ with $K \subset U$ and let $\varphi : U \to \mathbb{C}$ be holomorphic and injective. As H is connected, it follows that the set $\varphi(H)$ is also connected. According to Remark 1.5.3 iii), viii), we have $H \subset \widehat{K} \subset U$, and the injectivity of φ implies

$$\varphi(H) \subset \varphi\big((\mathbb{C}_\infty \backslash K) \cap U\big) \subset \mathbb{C}_\infty \backslash \varphi(K).$$

As $\widehat{K}$ is compact (see Remark 1.5.3 i)), we obtain that $\overline{H}$ is a compact subset of U so that Lemma A.1 ii) and part ii) yield

$$\partial\varphi(H) = \varphi(\partial H) \subset \varphi(K).$$

Therefore, $\varphi(H)$ is a component of $\mathbb{C}_\infty \backslash \varphi(K)$. Because of $\infty \notin \varphi(H)$, it follows that $\varphi(H)$ is a hole of $\varphi(K)$. □

Lemma A.8. *Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ continuous and $K \subset U$ compact. Moreover, let $M \subset \mathbb{C}$ be closed with $f(K) \cap M = \emptyset$. Then there exists an open neighbourhood $V \subset U$ of K with $f(V) \cap M = \emptyset$.*

Proof: Putting $\varepsilon_1 := \mathrm{dist}(K, \partial U)$ and $\varepsilon_2 := \mathrm{dist}(f(K), M)$, the compactness of K and $f(K)$ yields $\varepsilon_1, \varepsilon_2 > 0$. For $\varepsilon := \min(\varepsilon_1/2, \varepsilon_2)$, we obtain that $U_\varepsilon[K]$ is a compact subset of U. As f is uniformly continuous on $U_\varepsilon[K]$, there exists some $\delta > 0$ such that we have $|f(z_1) - f(z_2)| < \varepsilon$ for all $z_1, z_2 \in U_\varepsilon[K]$ with $|z_1 - z_2| < \delta$. Putting $r := \min(\delta, \varepsilon)$, we consider the open neighbourhood $V := U_r(K) \subset U$ of K. For a point $w \in f(V)$, there exists some $\widetilde{z} \in V$ with $w = f(\widetilde{z})$. Choosing $z \in K$ with $|z - \widetilde{z}| < r \leq \delta$, it follows that $|f(z) - f(\widetilde{z})| < \varepsilon$. Assuming that we have $w \in M$, we would obtain

$$\mathrm{dist}(f(K), M) \leq |f(z) - w| = |f(z) - f(\widetilde{z})| < \varepsilon \leq \varepsilon_2,$$

a contradiction. Therefore, we have $w \notin M$ so that $f(V) \cap M = \emptyset$ is shown. □

Appendix B

Hausdorff Distance and Hausdorff Dimension

In this part of the appendix, we introduce the concepts of Hausdorff distance and Hausdorff dimension which are needed in Sections 3.3 and 4.1. Moreover, at the end of this appendix, we will formulate and prove a lemma concerning the Hausdorff dimension of preimages of analytic arcs under holomorphic functions.

Definition B.1. Let (X, d) be a metric space.

i) We denote by

$$\mathcal{K}(X) := \{K \subset X : K \text{ compact}, \ K \neq \emptyset\}$$

the set of all compact non-empty subsets of X.

ii) The function

$$d_{\mathcal{K}(X)} : \mathcal{K}(X)^2 \to \mathbb{R}, \ \ d_{\mathcal{K}(X)}(K, L) := \max\left(\max_{z \in K} \operatorname{dist}(z, L), \ \max_{w \in L} \operatorname{dist}(w, K)\right),$$

is called the *Hausdorff distance* on $\mathcal{K}(X)$.

Remark B.2.

i) For $K, L \in \mathcal{K}(X)$ and $\varepsilon > 0$, we have $d_{\mathcal{K}(X)}(K, L) < \varepsilon$ if and only if $K \subset U_\varepsilon(L)$ and $L \subset U_\varepsilon(K)$. Hence, we obtain that

$$d_{\mathcal{K}(X)}(K, L) = \inf\left\{r > 0 : K \subset U_r(L) \text{ and } L \subset U_r(K)\right\}.$$

ii) The Hausdorff distance $d_{\mathcal{K}(X)}$ defines a metric on the set $\mathcal{K}(X)$ (see e.g. [17], Theorem 2.4.1).

iii) If the metric space (X,d) is complete, then the metric space $\big(\mathcal{K}(X), d_{\mathcal{K}(X)}\big)$ is also complete (see e.g. [17], Theorem 2.4.4).

Definition B.3. Let (X,d) be a metric space and let $A \subset X$.

i) The *diameter* of A is defined by

$$\text{diam } A := \sup\{d(x,y) : x,y \in A\}.$$

ii) For $\delta > 0$, a countable family $\mathcal{B}$ of subsets of X is called a δ-*cover* of A if diam $B \leq \delta$ for all $B \in \mathcal{B}$ and $A \subset \bigcup_{B\in\mathcal{B}} B$.

iii) For $\delta > 0$ and $s \geq 0$, we put

$$\overline{\mathcal{H}^s_\delta}(A) := \inf\left\{\sum_{B\in\mathcal{B}} (\text{diam } B)^s : \mathcal{B} \text{ is a } \delta\text{-cover of } A\right\}.$$

Remark and Definition B.4.

i) Let $A \subset X$ and $s \geq 0$. As the set $\left\{\sum_{B\in\mathcal{B}} (\text{diam } B)^s : \mathcal{B} \text{ is a } \delta\text{-cover of } A\right\}$ decreases if δ decreases, its infimum increases if δ decreases. Hence, the limit

$$\overline{\mathcal{H}^s}(A) := \lim_{\delta\to 0} \overline{\mathcal{H}^s_\delta}(A) = \sup_{\delta>0} \overline{\mathcal{H}^s_\delta}(A)$$

exists in $[0,\infty]$. The function $\overline{\mathcal{H}^s} : \{A : A \subset X\} \to [0,\infty]$, $A \mapsto \overline{\mathcal{H}^s}(A)$, is a metric outer measure on X, i.e. we have

- $\overline{\mathcal{H}^s}(\emptyset) = 0$,
- $\overline{\mathcal{H}^s}(A) \leq \overline{\mathcal{H}^s}(B)$ for all sets $A \subset B \subset X$,
- $\overline{\mathcal{H}^s}\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_{n\in\mathbb{N}} \overline{\mathcal{H}^s}(A_n)$ for each sequence (A_n) of subsets of X,
- $\overline{\mathcal{H}^s}(A\cup B) = \overline{\mathcal{H}^s}(A) + \overline{\mathcal{H}^s}(B)$ for all sets $\emptyset \neq A, B \subset X$ with $\text{dist}(A,B) > 0$

(see e.g. [17], p. 147 and Theorem 5.4.2).

ii) We denote by $\mathcal{H}^s$ the restriction of $\overline{\mathcal{H}^s}$ to the set

$$\mathcal{A}_s := \left\{A \subset X : \overline{\mathcal{H}^s}(B) = \overline{\mathcal{H}^s}(B\cap A) + \overline{\mathcal{H}^s}(B\backslash A) \text{ for all sets } B \subset X\right\}.$$

It is well-known that $\mathcal{A}_s$ is a σ-algebra on X and that $\mathcal{H}^s$ is a measure on $\mathcal{A}_s$ (see e.g. [17], Theorem 5.2.3). $\mathcal{H}^s$ is called the *s-dimensional Hausdorff measure* on $\mathcal{A}_s$. The Borel σ-algebra on X is contained in $\mathcal{A}_s$ (see e.g. [19], Satz 9.3).

iii) Let $A \subset X$, $\delta > 0$ and $0 < s < t$. A short computation yields

$$\overline{\mathcal{H}^t_\delta}(A) \leq \delta^{t-s}\, \overline{\mathcal{H}^s_\delta}(A).$$

Hence, it follows that $\overline{\mathcal{H}^s}(A) < \infty$ implies $\overline{\mathcal{H}^t}(A) = 0$ (or, equivalently, that $\overline{\mathcal{H}^t}(A) > 0$ implies $\overline{\mathcal{H}^s}(A) = \infty$; see e.g. [17], Theorem 6.1.5).

iv) Let $A \subset X$. According to part iii), there exists a unique $s_0 \in [0, \infty]$ such that we have

$$\begin{aligned} \overline{\mathcal{H}^s}(A) = \infty \quad &\text{for all} \quad 0 \leq s < s_0 \qquad \text{and} \\ \overline{\mathcal{H}^s}(A) = 0 \quad &\text{for all} \quad s > s_0. \end{aligned}$$

This value s_0 is called the *Hausdorff dimension* of the set A. It is denoted by

$$s_0 = \dim_H A.$$

The following lemma lists some well-known properties of Hausdorff dimension:

Lemma B.5. *Let (X, d) be a metric space.*

i) For $A \subset B \subset X$, we have $\dim_H A \leq \dim_H B$.

ii) For $A_n \subset X$, $n \in \mathbb{N}$, we have $\dim_H \bigcup_{n\in\mathbb{N}} A_n = \sup_{n\in\mathbb{N}} \dim_H A_n$.

iii) For each countable set $A \subset X$, we have $\dim_H A = 0$.

In case of $X = \mathbb{R}^n$ for some $n \in \mathbb{N}$, the following statements also hold:

iv) For $A \subset \mathbb{R}^n$, we have $\dim_H A \leq n$.

v) For $A \subset \mathbb{R}^n$ with $A^\circ \neq \emptyset$, we have $\dim_H A = n$.

vi) For $A \subset \mathbb{R}^n$ with $\dim_H A < 1$, the set A is totally disconnected.

vii) Denoting by λ_n the n-dimensional Lebesgue measure, we have

$$\mathcal{H}^n(A) = \frac{\lambda_n(A)}{c_n}$$

for each Borel set A of $\mathbb{R}^n$, where c_n denotes the n-dimensional Lebesgue measure of an n-dimensional ball with diameter 1.

viii) For a Borel set A of $\mathbb{R}^n$ with $\dim_H A < n$, we have $\lambda_n(A) = 0$.

Proof: Short proofs of the statements in parts i) to vi) can be found in [23], p. 48ff. Part vii) is stated in [23] on p. 45, and the statement of part viii) follows by definition from part vii). □

Definition B.6.

i) Let $I \subset \mathbb{R}$ be an interval. We call a continuous function $\gamma : I \to \mathbb{C}$ a *path*. The image of I under γ is denoted by $\gamma^* := \gamma(I)$, and this set is called the *trace* of γ. If $I = [a, b]$ is a compact interval, the *length* of γ is defined by

$$L(\gamma) := \sup \left\{ \sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})| : n \in \mathbb{N},\ a = t_0 < t_1 < \ldots < t_n = b \right\},$$

and γ is called *rectifiable* if $L(\gamma) < \infty$.

ii) Let $\gamma : [0, 1] \to \mathbb{C}$ and $\psi : [0, 1] \to \mathbb{C}$ be paths with $\gamma(1) = \psi(0)$. Then the *path composition* of γ and ψ is defined by

$$\gamma \bullet \psi : [0,1] \to \mathbb{C},\ \ \gamma \bullet \psi(t) := \begin{cases} \gamma(2t), & \text{if } 0 \le t \le 1/2 \\ \psi(2t-1), & \text{if } 1/2 \le t \le 1 \end{cases},$$

which is again a path.

ii) We call a subset of the complex plane an *arc* if it is the trace of a path. An arc Γ is called *analytic* if every point of Γ has an open neighbourhood U for which there exists a conformal map $\varphi : \mathbb{D} \to U$ such that

$$\varphi(\mathbb{D} \cap \mathbb{R}) = U \cap \Gamma.$$

It is well-known that the one-dimensional Hausdorff measure of the trace of an injective path defined on a non-singleton compact interval equals the length of the path (see e.g. [17], Theorem 6.2.7). Therefore, if such a path is also rectifiable, it follows by definition that the Hausdorff dimension of its trace is one (cf. Remark and Definition B.4 iii), iv)). The following lemma states that this is also true for preimages of analytic arcs under holomorphic functions.

Lemma B.7. *Let* Γ *be an analytic arc and let* f *be a transcendental entire function or a rational function with* $f(\infty) \notin \Gamma$*. Then we have* $\dim_H f^{-1}(\Gamma) = 1$*.*

Proof:

i) Until the last part of the proof, we fix a point $w \in \Gamma$ and an integer $n \in \mathbb{N}$. As Γ is analytic, there exist an open neighbourhood U_w of w and a conformal map $\varphi_w : \mathbb{D} \to U_w$ with $\varphi_w(\mathbb{D} \cap \mathbb{R}) = U_w \cap \Gamma$. In particular, we have $w = \varphi_w(t_w)$ for a unique $t_w \in (-1, 1)$.

ii) We first consider the case that $U_n[0] \cap f^{-1}(\{w\})$ is non-empty. As $f^{-1}(\{w\})$ has no accumulation point in $\mathbb{C}$, the set $U_n[0] \cap f^{-1}(\{w\})$ is finite so that there exist pairwise distinct points $z_1, \ldots, z_d \in \mathbb{C}$, $d \in \mathbb{N}$, with

$$U_n[0] \cap f^{-1}(\{w\}) = \{z_1, \ldots, z_d\}.$$

Moreover, we can choose an $\varepsilon_{w,n} > 0$ with

$$U_{n+\varepsilon_{w,n}}[0] \cap f^{-1}(\{w\}) = U_n[0] \cap f^{-1}(\{w\})$$

and such that f has no poles in $K_{n+\varepsilon_{w,n}}[0]$. In particular, there exists some $\delta_0 > 0$ with $|f(z) - w| \geq \delta_0$ for all $z \in K_{n+\varepsilon_{w,n}}[0]$.

iii) For $k \in \{1, \ldots, d\}$, we choose constants $\rho_k, \delta_k > 0$ with $U_{\rho_k}(z_k) \subset U_{n+\varepsilon_{w,n}}(0)$ such that we have $f(z) \neq w$ for all $z \in U_{\rho_k}[z_k] \backslash \{z_k\}$ and $|f(z) - w| \geq \delta_k$ for all $z \in K_{\rho_k}[z_k]$. Without loss of generality, all sets $U_{\rho_k}(z_k)$ shall be pairwise disjoint. Moreover, as Γ is analytic, we can assume without loss of generality that there exists some $\theta_k \in \mathbb{R}$ such that the set $\Gamma \cap U_{\delta_k}(w) \backslash \{w\}$ is contained in the slit disc

$$W_k := U_{\delta_k}(w) \backslash \left\{ w + re^{i\theta_k} : 0 \leq r < \delta_k \right\}.$$

Denoting by m_k the order of the zero z_k of the function $f - w$, the local behaviour of holomorphic functions near critical points implies the existence of holomorphic injective functions $\sigma_{k,1}, \ldots \sigma_{k,m_k} : W_k \to U_{\rho_k}(z_k)$ such that the following is true: For each $\zeta \in W_k$, the points $\sigma_{k,1}(\zeta), \ldots, \sigma_{k,m_k}(\zeta)$ are pairwise distinct and we have

$$f^{-1}(\{\zeta\}) \cap U_{\rho_k}(z_k) = \{\sigma_{k,1}(\zeta), \ldots, \sigma_{k,m_k}(\zeta)\}$$

(see e.g. [25], p. 238).

iv) We put $\delta := \min(\delta_0, \delta_1, \ldots, \delta_d)$ and we define $V_{w,n} := U_w \cap U_\delta(w)$, which is an open neighbourhood of w. As φ_w is continuous, there exist points $x, y \in \mathbb{R}$ with $-1 < x < t_w < y < 1$ such that the set

$$M := \left(\varphi_w\big|_{[x,y]}\right)^*$$

is contained in $\Gamma \cap V_{w,n}$.

v) We show that the following identity holds:

$$f^{-1}(M) \cap U_{n+\varepsilon_{w,n}}(0) = \bigcup_{k=1}^{d} f^{-1}(M) \cap U_{\rho_k}(z_k).$$

As all sets $U_{\rho_k}(z_k)$ are contained in $U_{n+\varepsilon_{w,n}}(0)$, the set on the right-hand side is contained in the set on the left-hand side. Now, let us consider a point $z_0 \in f^{-1}(M) \cap U_{n+\varepsilon_{w,n}}(0)$. In case of $f(z_0) = w$, part ii) yields

$$z_0 \in U_{n+\varepsilon_{w,n}}(0) \cap f^{-1}(\{w\}) = U_n[0] \cap f^{-1}(\{w\}) = \{z_1, \ldots, z_d\}$$

so that z_0 is contained in the set on the right-hand side. Hence, it remains to consider the case $f(z_0) \neq w$. As we have

$$f(z_0) \in M \backslash \{w\} \subset \Gamma \cap U_{\delta_k}(w) \backslash \{w\} \subset W_k$$

for each $k \in \{1, \ldots, d\}$, part iii) yields the existence of m_k mutually distinct points $w_{k,1}, \ldots, w_{k,m_k} \in U_{\rho_k}(z_k)$ with $f(w_{k,1}) = \ldots = f(w_{k,m_k}) = f(z_0)$. As the sets $U_{\rho_k}(z_k)$ are pairwise disjoint, we obtain that all points $w_{k,l}$ are mutually distinct. Therefore, the function $f - f(z_0)$ has at least $\sum_{k=1}^{d} m_k =: N$ zeros in $U_{n+\varepsilon_{w,n}}(0)$. We now show that these are the only zeros of $f - f(z_0)$ in $U_{n+\varepsilon_{w,n}}(0)$. In order to do so, we consider the two meromorphic functions $f_1, f_2 : \mathbb{C} \to \mathbb{C}_\infty$, defined by

$$f_1(z) := f(z) - w, \quad f_2(z) :\equiv w - f(z_0).$$

According to $f(z_0) \in M \subset U_{\delta_0}(w)$, part ii) yields that f_1 has no zeros and no poles in $K_{n+\varepsilon_{w,n}}[0]$ and that we have

$$|f_2(z)| = |w - f(z_0)| < \delta_0 \leq |f(z) - w| = |f_1(z)|$$

for all $z \in K_{n+\varepsilon_{w,n}}[0]$. Therefore, Rouché's theorem implies that the functions

$f_1 = f - w$ and $f_1 + f_2 = f - f(z_0)$ have the same number of zeros, counting multiplicities, in $U_{n+\varepsilon_{w,n}}(0)$ (observe that f_1 and $f_1 + f_2$ have the same poles). Hence, as the function $f - w$ has exactly N zeros, counting multiplicities, in $U_{n+\varepsilon_{w,n}}(0)$ (see parts ii) and iii)), the same must be true for $f - f(z_0)$. Therefore, we obtain

$$z_0 \in \{w_{1,1}, \ldots, w_{1,m_1}, \ldots, w_{d,1}, \ldots w_{d,m_d}\} \subset \bigcup_{k=1}^{d} U_{\rho_k}(z_k).$$

vi) For $k \in \{1, \ldots, d\}$, we now prove the identity

$$f^{-1}(M) \cap U_{\rho_k}(z_k) = \{z_k\} \cup \bigcup_{l=1}^{m_k} \left(\sigma_{k,l} \circ \varphi_w\big|_{[x,y]\setminus\{t_w\}}\right)^* .$$

Indeed, due to part iv), we have $M = \left(\varphi_w\big|_{[x,y]}\right)^* \subset \Gamma \cap V_{w,n} \subset \Gamma \cap U_{\delta_k}(w)$ so that part iii) yields

$$\left(\varphi_w\big|_{[x,y]\setminus\{t_w\}}\right)^* = \left(\varphi_w\big|_{[x,y]}\right)^* \setminus\{w\} \subset \Gamma \cap U_{\delta_k}(w)\setminus\{w\} \subset W_k.$$

Thus, according to part iii), the set on the right-hand side is contained in the set on the left-hand side. For a point $z \in f^{-1}(M) \cap U_{\rho_k}(z_k)$, there exists some $t \in [x, y]$ with $f(z) = \varphi_w(t)$. In case of $t = t_w$, we obtain $f(z) = \varphi_w(t_w) = w$ so that the choice of ρ_k in part iii) and $z \in U_{\rho_k}(z_k)$ imply $z = z_k$. On the other hand, if $t \neq t_w$, the above consideration yields $\varphi_w(t) \in W_k$. Hence, due to $z \in f^{-1}(\{\varphi_w(t)\}) \cap U_{\rho_k}(z_k)$, there exists some $l \in \{1. \ldots, m_k\}$ with $z = \sigma_{k,l}(\varphi_w(t))$ (see part iii)).

vii) Defining the paths

$$\gamma_{k,l,p}^- := \sigma_{k,l} \circ \varphi_w\big|_{[x,\, t_w - 1/p]} \qquad \text{and} \qquad \gamma_{k,l,p}^+ := \sigma_{k,l} \circ \varphi_w\big|_{[t_w + 1/p,\, y]}$$

for $k \in \{1, \ldots, d\}$, $l \in \{1, \ldots, m_k\}$ and $p \in \mathbb{N}$, parts v) and vi) yield

$$f^{-1}(M) \cap U_{n+\varepsilon_{w,n}}(0) = \bigcup_{k=1}^{d} \left(\{z_k\} \cup \bigcup_{l=1}^{m_k} \bigcup_{p \in \mathbb{N}} \left(\gamma_{k,l,p}^-\right)^* \cup \left(\gamma_{k,l,p}^+\right)^* \right).$$

As the functions $\sigma_{k,l}$ and φ_w are injective and holomorphic, we obtain that the paths $\gamma_{k,l,p}^-$ and $\gamma_{k,l,p}^+$ are injective and infinitely differentiable. Therefore, observing that continuously differentiable paths are rectifiable, it follows from the considerations before Lemma B.7 that

$$\dim_H \left(\gamma_{k,l,p}^-\right)^* = \dim_H \left(\gamma_{k,l,p}^+\right)^* = 1.$$

Thus, Lemma B.5 ii), iii) yields

$$\dim_H \left(f^{-1}(M) \cap U_{n+\varepsilon_{w,n}}(0)\right) = 1.$$

Finally, choosing a constant $\delta_{w,n} > 0$ such that $M_{w,n} := \varphi_w(U_{\delta_{w,n}}(t_w)) \cap \Gamma$ is contained in M, Lemma B.5 i), vi) implies

$$\dim_H \left(f^{-1}(M_{w,n}) \cap U_{n+\varepsilon_{w,n}}(0)\right) = 1.$$

viii) Now, we consider the case that $U_n[0] \cap f^{-1}(\{w\})$ is empty. As $f^{-1}(\{w\})$ has no accumulation point in $\mathbb{C}$, there exists an $\varepsilon_{w,n} > 0$ with

$$U_{n+\varepsilon_{w,n}}[0] \cap f^{-1}(\{w\}) = \emptyset.$$

We choose $\delta_0, \delta_{w,n} > 0$ with $|f(z) - w| \geq \delta_0$ for all points $z \in U_{n+\varepsilon_{w,n}}[0]$ which are no poles of f and such that we have $\varphi_w(U_{\delta_{w,n}}(t_w)) \subset U_{\delta_0}(w)$. Putting $M_{w,n} := \varphi_w(U_{\delta_{w,n}}(t_w)) \cap \Gamma$, we obtain

$$f^{-1}(M_{w,n}) \cap U_{n+\varepsilon_{w,n}}(0) = \emptyset$$

and hence $\dim_H \left(f^{-1}(M_{w,n}) \cap U_{n+\varepsilon_{w,n}}(0)\right) = 0$.

ix) Let $I \subset \mathbb{R}$ be an interval and let $\gamma : I \to \mathbb{C}$ be a path such that $\Gamma = \gamma(I)$. Considering a sequence (I_m) of increasing compact intervals with $I = \bigcup_{m\in\mathbb{N}} I_m$, we obtain that $(\gamma(I_m))$ is a sequence of increasing compact sets such that $\Gamma = \bigcup_{m\in\mathbb{N}} \gamma(I_m)$. For fixed $n, m \in \mathbb{N}$, the family $\left(\varphi_w(U_{\delta_{w,n}}(t_w))\right)_{w\in\Gamma}$ forms an open cover of the compact set $\gamma(I_m)$ (observe that $w = \varphi_w(t_w)$ for all $w \in \Gamma$). Hence, there exists a finite set $E_{n.m} \subset \Gamma$ with

$$\gamma(I_m) \subset \bigcup_{w\in E_{n,m}} \varphi_w(U_{\delta_{w,n}}(t_w)) \cap \Gamma = \bigcup_{w\in E_{n,m}} M_{w,n}\,,$$

and we obtain

$$f^{-1}\left(\gamma(I_m)\right) \cap U_n[0] \subset \bigcup_{w\in E_{n,m}} f^{-1}\left(M_{w,n}\right) \cap U_{n+\varepsilon_{w,n}}(0).$$

Therefore, parts vii) and viii) as well as Lemma B.5 i), ii), vi) imply

$$\dim_H \left(f^{-1}\left(\gamma(I_m)\right) \cap U_n[0]\right) \leq 1$$

with equality for large n and m. Because of $f(\infty) \notin \Gamma$ in case of a rational function f, we have

$$f^{-1}(\Gamma) = \bigcup_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} f^{-1}\left(\gamma(I_m)\right) \cap U_n[0].$$

Therefore, we finally obtain

$$\dim_H f^{-1}(\Gamma) = \sup_{n,m\in\mathbb{N}} \dim_H \left(f^{-1}\left(\gamma(I_m)\right) \cap U_n[0]\right) = 1,$$

which completes the proof. □

Appendix C

Riemann Surfaces

In this part of the appendix, we briefly provide some basic definitions and notations concerning the concept of Riemann surfaces which we use in Section 4.1 (cf. [25], p. 418ff.). Moreover, we endow the space of complex-valued analytic functions on open subsets of suitable Riemann surfaces with a metric, and we show that the resulting metric space is complete and separable.

Definition C.1. A *Riemann surface* consists of a set X, a family $(X_\iota)_{\iota\in I}$ of subsets of X with $\bigcup_{\iota\in I} X_\iota = X$ and a family $(F_\iota)_{\iota\in I}$ of functions $F_\iota : X_\iota \to \mathbb{C}$ such that

- each set $F_\iota(X_\iota)$ is a domain in $\mathbb{C}$ and each function F_ι maps X_ι bijectively onto $F_\iota(X_\iota)$,
- each composition $F_\kappa \circ F_\iota^{-1}$ is a holomorphic function from $F_\iota(X_\iota \cap X_\kappa)$ to $F_\kappa(X_\iota \cap X_\kappa)$,
- X is "connected", i.e. for any two points $x, y \in X$, there exist finitely many sets $X_{\iota_1}, \dots, X_{\iota_n}$ with $x \in X_{\iota_1}$, $y \in X_{\iota_n}$ such that $X_{\iota_k} \cap X_{\iota_{k+1}} \neq \emptyset$ for all $k \in \{1, \dots, n-1\}$.
- X fulfils the "Hausdorff property", i.e. for any two points $x, y \in X$ with $x \neq y$ and any $\iota, \kappa \in I$ with $x \in X_\iota$ and $y \in X_\kappa$, there exist radii $r, s > 0$ with $U_r(F_\iota(x)) \subset F_\iota(X_\iota)$ and $U_s(F_\kappa(y)) \subset F_\kappa(X_\kappa)$ such that

$$F_\iota^{-1}\big(U_r(F_\iota(x))\big) \cap F_\kappa^{-1}\big(U_s(F_\kappa(y))\big) = \emptyset.$$

Definition C.2. Let $\big(X, (X_\iota)_{\iota\in I}, (F_\iota)_{\iota\in I}\big)$ be a Riemann surface.

i) A set $U \subset X$ is called *open* if for each $\iota \in I$ the set $F_\iota(U \cap X_\iota)$ is an open subset of $\mathbb{C}$.

ii) For an open subset U of X, a function $\widetilde{f} : U \to \mathbb{C}$ is called *analytic* if for each $\iota \in I$ the function $\widetilde{f} \circ F_\iota^{-1}$ is holomorphic on $F_\iota(U \cap X_\iota)$. We put

$$\widetilde{H}(U) := \left\{ \widetilde{f} : U \to \mathbb{C} : \widetilde{f} \text{ analytic} \right\}.$$

We now consider a Riemann surface of the form (X, X, F), i.e. X is a set and $F : X \to \mathbb{C}$ is a function which maps X bijectively onto the domain $F(X) \subset \mathbb{C}$. For an open subset U of X, the set $V := F(U)$ is an open subset of $\mathbb{C}$. Let (K_n) be a compact exhaustion of V. Then, according to Remark 1.1.3 and the preceding considerations, the map

$$d_{H(V)} : H(V) \times H(V) \to \mathbb{R}, \quad d_{H(V)}(f, g) := \sup_{n \in \mathbb{N}} \min \left(1/n, \|f - g\|_{K_n} \right),$$

defines a metric on $H(V)$ such that for $f_n, f \in H(V)$, $n \in \mathbb{N}$, the sequence (f_n) converges to f in the metric space $\big(H(V), d_{H(V)}\big)$ if and only if (f_n) converges to f locally uniformly on V. Therefore, the map

$$d_{\widetilde{H}(U)} : \widetilde{H}(U) \times \widetilde{H}(U) \to \mathbb{R}, \; d_{\widetilde{H}(U)}\big(\widetilde{f}, \widetilde{g}\big) := d_{H(V)}\Big(\widetilde{f} \circ F^{-1}\big|_V, \widetilde{g} \circ F^{-1}\big|_V\Big),$$

defines a metric on $\widetilde{H}(U)$ such that for $\widetilde{f}_n, \widetilde{f} \in \widetilde{H}(U)$, $n \in \mathbb{N}$, the sequence $\big(\widetilde{f}_n\big)$ converges to $\widetilde{f}$ in the metric space $\big(\widetilde{H}(U), d_{\widetilde{H}(U)}\big)$ if and only if $\big(\widetilde{f}_n \circ F^{-1}\big|_V\big)$ converges to $\widetilde{f} \circ F^{-1}\big|_V$ locally uniformly on V.

Lemma C.3. *In the above situation, the metric space* $\big(\widetilde{H}(U), d_{\widetilde{H}(U)}\big)$ *is complete and separable.*

Proof:

i) Let $\big(\widetilde{f}_n\big)$ be a Cauchy sequence in $\big(\widetilde{H}(U), d_{\widetilde{H}(U)}\big)$. Then $\big(\widetilde{f}_n \circ F^{-1}\big|_V\big)$ is a Cauchy sequence in $\big(H(V), d_{H(V)}\big)$, which is a complete metric space (cf. Remark 1.1.3 and the preceding considerations). Hence, there exists some $f \in H(V)$ such that we have locally uniform convergence

$$\widetilde{f}_n \circ F^{-1}\big|_V \to f = \big(f \circ F\big|_U\big) \circ F^{-1}\big|_V.$$

Due to $f \circ F\big|_U \in \widetilde{H}(U)$, the above considerations yield $\widetilde{f}_n \to f \circ F\big|_U$ in $\big(\widetilde{H}(U), d_{\widetilde{H}(U)}\big)$.

ii) As the metric space $\big(H(V), d_{H(V)}\big)$ is separable (cf. Remark 1.1.3 ii)), there exists a countable dense set $\mathcal{M} \subset H(V)$. Putting

$$\widetilde{\mathcal{M}} := \left\{ f \circ F\big|_U : f \in \mathcal{M} \right\},$$

we obtain that $\widetilde{\mathcal{M}}$ is a countable subset of $\widetilde{H}(U)$. Now, let $\widetilde{f} \in \widetilde{H}(U)$ and let $\varepsilon > 0$. Then we have $\widetilde{f} \circ F^{-1}\big|_V \in H(V)$ so that the denseness of $\mathcal{M}$ in $\big(H(V), d_{H(V)}\big)$ yields some $f \in \mathcal{M}$ with

$$\varepsilon > d_{H(V)} \left(f, \, \widetilde{f} \circ F^{-1}\big|_V \right) = d_{\widetilde{H}(U)} \left(f \circ F\big|_U, \widetilde{f} \right).$$

Because of $f \circ F\big|_U \in \widetilde{\mathcal{M}}$, we obtain that $\widetilde{\mathcal{M}}$ is dense in $\big(\widetilde{H}(U), d_{\widetilde{H}(U)}\big)$. □

Lemma C.4. *In the above situation, let $\widetilde{\mathcal{U}} \subset \widetilde{H}(U)$ and let*

$$\mathcal{U} := \left\{ \widetilde{f} \circ F^{-1}\big|_V : \widetilde{f} \in \widetilde{\mathcal{U}} \right\} \subset H(V).$$

Then $\widetilde{\mathcal{U}}$ is open in $\big(\widetilde{H}(U), d_{\widetilde{H}(U)}\big)$ if and only if $\mathcal{U}$ is open in $\big(H(V), d_{H(V)}\big)$.

Proof:

i) First, let $\widetilde{\mathcal{U}}$ be open in $\big(\widetilde{H}(U), d_{\widetilde{H}(U)}\big)$. For $f \in \mathcal{U}$, there exists some $\widetilde{f} \in \widetilde{\mathcal{U}}$ with $f = \widetilde{f} \circ F^{-1}\big|_V$, and we can find an $\varepsilon > 0$ with

$$U_\varepsilon(\widetilde{f}) := \left\{ \widetilde{g} \in \widetilde{H}(U) : d_{\widetilde{H}(U)}(\widetilde{f}, \widetilde{g}) < \varepsilon \right\} \subset \widetilde{\mathcal{U}}.$$

Now, let $g \in H(V)$ with $d_{H(V)}(f, g) < \varepsilon$. Then we have

$$\varepsilon > d_{H(V)} \left(\widetilde{f} \circ F^{-1}\big|_V, \, g \circ F\big|_U \circ F^{-1}\big|_V \right) = d_{\widetilde{H}(U)} \left(\widetilde{f}, \, g \circ F\big|_U \right),$$

which yields $g \circ F\big|_U \in U_\varepsilon(\widetilde{f}) \subset \widetilde{\mathcal{U}}$. Thus, we obtain $g = \big(g \circ F\big|_U\big) \circ F^{-1}\big|_V \in \mathcal{U}$, i.e. we have shown that

$$U_\varepsilon(f) := \left\{ g \in H(V) : d_{H(V)}(f, g) < \varepsilon \right\} \subset \mathcal{U}.$$

ii) Now, let $\mathcal{U}$ be open in $\big(H(V), d_{H(V)}\big)$. For $\widetilde{f} \in \widetilde{\mathcal{U}}$, we have $f := \widetilde{f} \circ F^{-1}\big|_V \in \mathcal{U}$, and there exists an $\varepsilon > 0$ with $U_\varepsilon(f) \subset \mathcal{U}$. Now, let $\widetilde{g}$ be an element of $U_\varepsilon(\widetilde{f})$. Then we obtain

$$\varepsilon > d_{\widetilde{H}(U)}(\widetilde{f}, \widetilde{g}) = d_{H(V)} \left(f, \, \widetilde{g} \circ F^{-1}\big|_V \right),$$

which yields $\widetilde{g} \circ F^{-1}\big|_V \in U_\varepsilon(f) \subset \mathcal{U}$. Hence, there exists some $\widetilde{h} \in \widetilde{\mathcal{U}}$ with

$$\widetilde{g} \circ F^{-1}\big|_V = \widetilde{h} \circ F^{-1}\big|_V,$$

and it follows that $\widetilde{g} = \widetilde{h} \in \widetilde{\mathcal{U}}$. Thus, we have shown that $U_\varepsilon(\widetilde{f}) \subset \widetilde{\mathcal{U}}$. □

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