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The optimal Berry-Esseen constant in the binomial case

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Abstract

The Berry-Esseen theorem belongs to the classical theorems of probability theory. The present work considers the case of $X, X_1, ..., X_n$ independent and identically distributed random variables with Bernoulli distribution with parameter $p \in (0,1)$. In this case we show that

$$(1) \qquad \Delta \;:=\; \sup_{x\in\mathbb{R}} \Big|\mathbb{P}\Big(\sum_{i=1}^{n} X_i \le x\Big) - \Phi\big(\tfrac{x-n\mathbb{E}X}{\sqrt{n\mathrm{Var}X}}\big)\Big| \;<\; c_{\mathrm{E}} \cdot \tfrac{\mathbb{E}|X-\mathbb{E}X|^3}{\sqrt{n}\cdot\sqrt{\mathrm{Var}X}^3}$$

with $c_{\mathrm{E}} := \frac{\sqrt{10}+3}{6\sqrt{2\pi}} = 0.4097...$ holds. Due to Esseen (1956) the constant c_{E} can not be replaced by a smaller one, that is why we talk about optimality in the title. The proof of the inequality (1) claims the biggest part of the present work. In the first part of the proof the points where the supremum Δ is attained are restricted by means of a comparison of densities and in the second one the distances at these points are estimated specifically. In the case of $p \in [\frac{1}{3}, \frac{2}{3}]$ we even show the sharper inequality

$$\Delta \;<\; \tfrac{1}{\sqrt{2\pi n\cdot\mathrm{Var}X}} \cdot \big(\tfrac{1}{2} + \tfrac{|\mathbb{E}(X-\mathbb{E}X)^3|}{6\cdot\mathrm{Var}X}\big),$$

which does not hold for p near to 0 or 1. Finally we consider the supremum distance of interval probabilities instead of distribution functions in (1), and for $n \ge 6$ and $p \in [\frac{1}{6}, \frac{5}{6}]$ we get

$$\sup_{I\subset\mathbb{R}\text{ interval}} \Big|\mathbb{P}\Big(\sum_{i=1}^{n} X_i \in I\Big) - \mathrm{N}_{n\mathbb{E}X,n\mathrm{Var}X}(I)\Big| \;<\; c_{\mathcal{I}} \cdot \tfrac{\mathbb{E}|X-\mathbb{E}X|^3}{\sqrt{n}\cdot\sqrt{\mathrm{Var}X}^3},$$

where $\mathrm{N}_{\mu,\sigma^2}$ denotes the normal distribution with mean μ and variance σ^2 and we have $c_{\mathcal{I}} := \frac{2}{\sqrt{2\pi}} = 0.7978... < 2 \cdot c_{\mathrm{E}}$, with again $c_{\mathcal{I}}$ unimprovable.

Zusammenfassung

Einer der klassischen Sätze der Wahrscheinlichkeitstheorie ist der Satz von Berry-Esseen. Die vorliegende Arbeit untersucht den Fall, dass $X, X_1, ..., X_n$ unabhängig und identisch verteilte Zufallsvariablen mit Bernoulliverteilung mit Parameter $p \in (0,1)$ sind. Für diesen Fall wird gezeigt, dass

$$\text{(1)} \qquad \Delta \;:=\; \sup_{x\in\mathbb{R}} \Big|\mathbb{P}\Big(\sum_{i=1}^{n} X_i \le x\Big) - \Phi\big(\tfrac{x-n\mathbb{E}X}{\sqrt{n\mathrm{Var}X}}\big)\Big| \;<\; c_{\mathrm{E}} \cdot \tfrac{\mathbb{E}|X-\mathbb{E}X|^3}{\sqrt{n}\cdot\sqrt{\mathrm{Var}X}^3}$$

mit $c_{\mathrm{E}} := \frac{\sqrt{10}+3}{6\sqrt{2\pi}} = 0.4097...$ gilt. Nach Esseen (1956) kann c_{E} nicht durch eine kleinere Konstante ersetzt werden, weswegen wir im Titel von Optimalität sprechen. Der Beweis der Ungleichung (1) nimmt den größten Teil dieser Arbeit ein. Im ersten Teil des Beweises werden die Stellen, an denen das Supremum Δ angenommen werden kann mithilfe eines Dichtevergleiches eingeschränkt und im zweiten Teil werden die Abstände an diesen Stellen dann konkret abgeschätzt. Im Fall $p \in [\frac{1}{3}, \frac{2}{3}]$ wird sogar die schärfere Ungleichung

$$\Delta \;<\; \tfrac{1}{\sqrt{2\pi n\cdot\mathrm{Var}X}} \cdot \big(\tfrac{1}{2} + \tfrac{|\mathbb{E}(X-\mathbb{E}X)^3|}{6\cdot\mathrm{Var}X}\big)$$

gezeigt, die aber für p nahe 0 oder 1 nicht mehr gilt. Abschließend wird noch der Supremumsabstand von Intervallwahrscheinlichkeiten statt der Verteilungsfunktion in (1) betrachtet und für $n \ge 6$ und $p \in [\frac{1}{6}, \frac{5}{6}]$ gezeigt, dass

$$\sup_{I\subset\mathbb{R}\text{ Intervall}} \Big|\mathbb{P}\Big(\sum_{i=1}^{n} X_i \in I\Big) - \mathrm{N}_{n\mathbb{E}X,n\mathrm{Var}X}(I)\Big| \;<\; c_{\mathcal{I}} \cdot \tfrac{\mathbb{E}|X-\mathbb{E}X|^3}{\sqrt{n}\cdot\sqrt{\mathrm{Var}X}^3},$$

wobei $\mathrm{N}_{\mu,\sigma^2}$ die Normalverteilung mit Erwartungswert μ und Varianz σ^2 bezeichne und es gilt $c_{\mathcal{I}} := \frac{2}{\sqrt{2\pi}} = 0.7978... < 2 \cdot c_{\mathrm{E}}$ und $c_{\mathcal{I}}$ kann ebenfalls nicht durch eine kleinere Konstante ersetzt werden.

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1. Introduction

The aim of this work is to determine the optimal Berry-Esseen constant in the binomial case. This is formulated in the following theorem:

Theorem 1. *Let $p \in (0,1)$ and $n \in \mathbb{N}$ and let $F_{n,p}$ denote the distribution function of the binomial distribution with parameters n and p. Then we have with $q := 1-p$*

$$\sup_{x\in\mathbb{R}} \left| F_{n,p}(x) - \Phi\left(\tfrac{x-np}{\sqrt{npq}}\right) \right| \quad < \quad \tfrac{\sqrt{10}+3}{6\sqrt{2\pi}} \cdot \tfrac{p^2+q^2}{\sqrt{npq}}.$$

In the case $p \in [\frac{1}{3}, \frac{2}{3}]$ we even have the sharper inequality

$$\sup_{x\in\mathbb{R}} \left| F_{n,p}(x) - \Phi\left(\tfrac{x-np}{\sqrt{npq}}\right) \right| \quad < \quad \tfrac{3+|p-q|}{6\sqrt{2\pi}\sqrt{npq}}.$$

To connect the theorem above to the Berry-Esseen theorem we firstly need some notations. In the present work we fequently consider the following four distribution. Let denote

- Ber_p the Bernoulli distribution with parameter $p \in [0,1]$
- $\mathrm{Bi}_{n,p}$ the binomial distribution with parameters $p \in [0,1]$ and $n \in \mathbb{N}_0$
- Poi_α the Poisson distribution with parameter $\alpha \in [0,\infty)$
- $\mathrm{N}_{0,1}$ the standard normal distribution with distribution function Φ.

Further for a $P \in \mathrm{Prob}_1(\mathbb{R})$:= set of all laws on $\mathbb{R}$ with a finite first moment, let $\mu(P) := \int x \mathrm{d}P(x)$, $\sigma^2(P) := \int (x-\mu(P))^2 \mathrm{d}P(x)$ and $\beta_3(P) := \int |x-\mu(P)|^3 \mathrm{d}P(x)$. Let

$$\mathcal{P}_3 := \{P \in \mathrm{Prob}_1(\mathbb{R}) \ : \ \sigma^2(P) > 0 \text{ and } \beta_3(P) < \infty\}.$$

and for $P \in \mathcal{P}_3$, let further denote $\varrho(P) := \frac{\beta_3(P)}{\sigma^3(P)}$ the standardized third absolute centred moment and $d_{\mathrm{K}}(P,Q) := \|F_P - F_Q\|_\infty$ the Kolmogorov distance of $P, Q \in \mathcal{P}_3$, where F_P denotes the distribution function of P, and

$$C_P := \sup_{n\in\mathbb{N}} \sqrt{n}\, d_{\mathrm{K}}(\widetilde{P^{*n}}, \mathrm{N}_{0,1}) \quad \text{for} \quad P \in \mathcal{P}_3$$

with $\widetilde{P}$ denoting the standardized probability measure of a $P \in \mathcal{P}_3$. From Theorem 1 it follows that

$$C_{\mathrm{Ber}_p} \quad = \quad \tfrac{3+|p-q|}{6\sqrt{2\pi}\sqrt{pq}} \quad \text{for} \quad p \in [\tfrac{1}{3}, \tfrac{2}{3}].$$

In the case $p = \frac{1}{2}$ Hipp/Mattner (2007) already showed that $C_{\mathrm{Ber}_{\frac{1}{2}}} = \frac{1}{\sqrt{2\pi}}$, which is a special case of the equation above.

Due to the Berry-Esseen theorem, proved independently by Berry (1941) and Esseen (1942), we have not only $C_P < \infty$ for every $P \in \mathcal{P}_3$, but more precisely

$$C_{\mathrm{BE}} \;:=\; \sup_{P\in\mathcal{P}_3} \tfrac{C_P}{\varrho(P)} \;<\; \infty. \tag{2}$$

Furthermore we are interested in $C_{\mathrm{BE},\mathcal{Q}} \;:=\; \sup_{P\in Q} \frac{C_P}{\varrho(P)}$ for $\mathcal{Q} \subset \mathcal{P}_3$ and we denote further $C_{\mathrm{BE},Q} \;:=\; C_{\mathrm{BE},\{Q\}}$ for $Q \in \mathcal{P}_3$. For $\mathcal{B} := \{\mathrm{Ber}_p : p \in (0,1)\}$, Theorem 1 yields

$$C_{\mathrm{BE},\mathcal{B}} \;=\; \tfrac{\sqrt{10}+3}{6\sqrt{2\pi}} \;=:\; c_{\mathrm{E}}.$$

Now we show that for $P \in \mathcal{P}_3$ already $C_{\mathrm{BE},P^{*n}} \leq C_{\mathrm{BE},P}$ holds for all $n \in \mathbb{N}$. To this end, let $X_1,...,X_n$ independent and identically distributed with $X_1 \sim P$; then for $S_n := X_1 + ... + X_n$ we have $S_n \sim P^{*n}$. Due to Cox/Kemperman (1983) we have $E|X+Y|^3 \geq E|X|^3 + E|Y|^3$ whenever X, Y are independent random variables with mean 0, and thus

$$\begin{aligned}
\varrho(P^{*n}) \;&=\; \mathbb{E}|\tfrac{S_n-\mu(P^{*n})}{\sigma(P^{*n})}|^3 \;=\; \mathbb{E}|\sum_{i=1}^n (X_i-\mu(P))|^3 \cdot \tfrac{1}{\sigma^3(P^{*n})} \\
&\geq\; \sum_{i=1}^n \mathbb{E}|(X_i-\mu(P))|^3 \cdot \tfrac{1}{\sigma^3(P^{*n})} \;=\; \tfrac{n\mathbb{E}|X_1-\mu(P)|^3}{(\sqrt{n}\,\sigma(P))^3} \;=\; \tfrac{\varrho(P)}{\sqrt{n}}.
\end{aligned}$$

Hence we finally have

$$\begin{aligned}
C_{\mathrm{BE},P^{*n}} \;&=\; \tfrac{C_{P^{*n}}}{\varrho(P^{*n})} \;\leq\; \tfrac{\sqrt{n}\cdot C_{P^{*n}}}{\varrho(P)} \;=\; \tfrac{\sup_{m\in\mathbb{N}} \sqrt{n\cdot m}\, d_{\mathrm{K}}(P^{*(n\cdot m)},\mathrm{N}_{0,1})}{\varrho(P)} \\
&\leq\; \tfrac{\sup_{m\in\mathbb{N}} \sqrt{m}\, d_{\mathrm{K}}(P^{*m},\mathrm{N}_{0,1})}{\varrho(P)} \;=\; C_{\mathrm{BE},P}.
\end{aligned}$$

Thus we have in particular $C_{\mathrm{BE},\mathrm{Bi}_{n,p}} \leq C_{\mathrm{BE},\mathrm{Ber}_p} = C_{\mathrm{BE},\mathrm{Bi}_{1,p}}$ for all $n \in \mathbb{N}$ and $p \in (0,1)$ and hence for $\mathcal{B}i := \{\mathrm{Bi}_{n,p} : n \in \mathbb{N}, p \in (0,1)\}$

$$C_{\mathrm{BE},\mathcal{B}} \;=\; C_{\mathrm{BE},\mathcal{B}i} \;=\; C_{\mathrm{BE},\mathcal{B}i\cup\{\mathrm{Poi}_\alpha:\alpha\in(0,\infty)\}} \;=\; c_{\mathrm{E}}$$

since the Poisson is a limiting case of the binomial distribution with $np \to \alpha$ $(n \to \infty)$ and we even have $C_{\mathrm{BE},\{\mathrm{Poi}_\alpha:\alpha\in(0,\infty)\}} < 0.3031$ due to Shevtsova (2013).

Next one may ask about upper bounds of the general constant C_{BE}. Esseen (1945) started with $C_{\mathrm{BE}} \leq 7.49$, up to now we have $C_{\mathrm{BE}} \leq 0.4690$ according to Shevtsova (2013). Again Esseen (1956) corrected the conjecture of Kolmogorov (1953) that $C_{\mathrm{BE}} = 1/\sqrt{2\pi}$ be the optimal constant, by showing that c_{E} is a lower bound for C_{BE}. He proved this result interestingly with the help of a binomial distribution; more precisely he showed for $p_{\mathrm{E}} := (4-\sqrt{10})/2$ that $C_{\mathrm{BE},\mathrm{Ber}_{p_{\mathrm{E}}}} \geq c_{\mathrm{E}}$ holds. Thus this result also yields $C_{\mathrm{BE},\mathcal{B}} \geq c_{\mathrm{E}}$. In the present work we now want to prove that also $C_{\mathrm{BE},\mathcal{B}} \leq c_{\mathrm{E}}$ holds.

Next we introduce for $P \in \mathcal{P}_3$ the constants

$$C_P^\infty := \limsup_{n\to\infty} \sqrt{n} \cdot d_{\mathrm{K}}(\widetilde{P^{*n}}, \mathrm{N}_{0,1}) \quad , \quad C_{\mathrm{BE}}^\infty := \sup_{P\in\mathcal{P}_3} \tfrac{C_P^\infty}{\varrho(P)}.$$

Esseen (1956) proved $C_{\mathrm{BE}}^{\infty} = c_{\mathrm{E}}$ and Chistyakov (2001) strengthen that result to

$$d_{\mathrm{K}}(\widetilde{P^{*n}}, \mathrm{N}_{0,1}) \quad \leq \quad c_{\mathrm{E}} \cdot \tfrac{\varrho(P)}{\sqrt{n}} + c \cdot (\tfrac{\varrho(P)}{\sqrt{n}})^{40/39} |\log \tfrac{\varrho(P)}{\sqrt{n}}|^{7/6}$$

with an absolute constant c. Later Shevtsova (2012) concretized Chistyakov's result by showing

$$d_{\mathrm{K}}(\widetilde{P^{*n}}, \mathrm{N}_{0,1}) \quad \leq \quad c_{\mathrm{E}} \cdot \tfrac{\varrho(P)}{\sqrt{n}} + 2.5786 \cdot \tfrac{\varrho^2(P)}{n}.$$

A very detailed overview of the history of the Berry-Esseen theorem one may find in Korolev/Shevtsova (2009).

Next we want to sketch the structure of the present work. The main part consists of the chapters 6-9, in which the already mentioned Theorem 1 is proved in two steps. In the first one it is shown that for all $x \in \mathbb{R}$ either an $x_0 \in \{\lfloor np \rfloor, \lceil np \rceil\}$ exists with

$$|F_{n,p}(x) - \Phi(\tfrac{x-np}{\sqrt{np(1-p)}})| \quad \leq \quad \max\Big\{\ |F_{n,p}(x_0) - \Phi(\tfrac{x_0-np}{\sqrt{np(1-p)}})|\ ,\ |\Phi(\tfrac{x_0-np}{\sqrt{np(1-p)}}) - F_{n,p}(x_0-)|\ \Big\}$$

and so the distance on the left hand side is smaller than the distance at an integer near the mean, or the distance is directly shown to be smaller than the desired right hand side.

Hence it remains to estimate the distance at the points $x_0 \in \{\lfloor np \rfloor, \lceil np \rceil\}$, this is done in the second part.

To present the proofs of these two parts more clearly, we prove the lemmas used in the proofs separately at the ends of the respective chapters.

In chapter 4 we present as Theorem 2 an analogue of Theorem 1 where, instead of intervals $(-\infty, x]$ occurring in the definition of $F_{n,p}(x)$, arbitrary intervals $I \subset \mathbb{R}$ are considered, that is

$$\sup_{I\subset\mathbb{R}\ \text{interval}} \left|\mathrm{Bi}_{n,p}(I) - \mathrm{N}_{np,np(1-p)}(I)\right| \quad < \quad \tfrac{2}{\sqrt{2\pi}} \cdot \tfrac{p^2+q^2}{\sqrt{np(1-p)}},$$

where N_{ν,τ^2} denotes the normal distribution with mean ν and variance τ^2. Theorem 2 is proved in chapter 10.

We present in chapter 3 several notations and also some elementary inequalities, which we often use in the later proofs. The aim of chapter 5 is to sketch the proof. This chapter does not claim to be a full proof, but it should convey the topic more descriptively with the help of describing text and several figures. Thus this describing chapter stands a bit in contrast to the proofs in the chapters 6-10.

In the next chapter we continue to explain the context of our present work, by presenting several other works, which are of a similar topic.

2. Related works

Berry-Esseen theorem. The present work can be understood in some way as a continuation of the works of Hipp/Mattner (2007) and Mattner/Schulz (2014), the structure of the proofs is very similar in all three works. In Hipp/Mattner it was shown that in the case of symmetric binomial distributions, that is $p = \frac{1}{2}$, the optimal Berry-Esseen constant is $1/\sqrt{2\pi}$, this means $C_{\mathrm{BE,Ber}_{1/2}} = 1/\sqrt{2\pi}$ with the notation of the introduction. This result was generalized to symmetric hypergeometric distributions by Mattner/Schulz, also with the constant $1/\sqrt{2\pi}$. In the present work it is extended to arbitrary binomial distributions, but with $1/\sqrt{2\pi}$ replaced by c_{E}. Furthermore in the later proof some inequalities of the work of Mattner/Schulz are used.

A work, related to Theorem 2 proved in chapter 10, where we consider interval probabilities instead of distribution functions, is the one of Dinev/Mattner (2012), where it was shown that the optimal asymptotic constant in the Berry-Esseen theorem for interval probabilities is $\sqrt{2/\pi}$.

Zolotarev (1997) devotes in his book several pages to the Berry-Esseen theorem and observes among other things on page 249 the behaviour of

$$D_n \;:=\; \sup_{p\in(0,1)} \left\{ \sup_{x\in\mathbb{R}} \left|F_{n,p}(x) - \Phi\left(\tfrac{x-np}{\sqrt{np(1-p)}}\right)\right| \cdot \tfrac{\sqrt{np(1-p)}}{p^2+(1-p)^2} \right\}, \quad n \geq 1.$$

He computed numerically D_n for $n \leq 21$ and received the following table:

n	1	3	5	7	9	11	13	15	17	19
D_n	0.3703	0.3981	0.4010	0.4015	0.4037	0.4063	0.4074	0.4078	0.4079	0.4077
n	2	4	6	8	10	12	14	16	18	20
D_n	0.3559	0.3951	0.4037	0.4060	0.4065	0.4064	0.4061	0.4078	0.4080	0.4083

Table 1. Approximate computation of the first 20 D_n. Source: Zolotarev (1997, p.250).

Based on these numbers, with the D_n almost increasing if one considers even and odd n separately, and with $\lim_{n\to\infty} D_n = c_{\mathrm{E}}$, which is true for example by Chistyakov (2001), already mentioned on page 3, Zolotarev conjectured that $\sup_{n\in\mathbb{N}} D_n = c_{\mathrm{E}}$.

Apparently the only progress so far towards proving Zolotarev's conjecture is due to Nagaev et al. in two recent papers. In Nagaev/Chebotarev (2011) it was shown that the optimal Berry-Esseen constant in the binomial case $C_{\mathrm{BE},\{\mathrm{Ber}_p:p\in(0,1)\}}$ must be smaller than 0.4215.

Somewhat later they improved that upper bound for $C_{\mathrm{BE},\{\mathrm{Ber}_p : p\in(0,1)\}}$ in the work of Nagaev/Chebotarev/Zolotukhin (2016) to $0.409953.... = c_{\mathrm{E}} + 0.00022...$, which is only very slightly larger than c_{E}. Nevertheless the present work can not necessarily be considered as a refinement of that work, because here completely other methods of proof are used. Nagaev/Chebotarev (2011) used smoothing methods and showed the claim in the case of $n < 200$ with direct numerical computation, while in the present proof only elementary calculations are used. In Nagaev/Chebotarev/Zolotukhin (2016) the claim was even shown in the case of $n < 5 \cdot 10^5$ only with the help of supercomputers.

Useful for the present work turned out to be the result of Bentkus/Kirsa (1989). There it was shown that for the constant in the general Berry-Esseen theorem in the case of $n = 1$ already 0.3704 is an upper bound, see proof of Theorem 1 in the present chapter 6. With the notations of the introduction, this means

$$\sup_{P\in\mathcal{P}_3} \frac{d_K(\widetilde{P},\mathrm{N}_{0,1})}{\varrho(P)} \quad = \quad 0.370352...$$

with the supremum attained at $P = \mathrm{Ber}_p$ with $p = p_{\mathrm{BK}} := 0.6095...$ or $p = 1 - p_{\mathrm{BK}}$. In the Bernoulli case that result yields $D_1 \leq 0.3704$ with the notations above. Thus in the present proof we do not have to consider the case $n = 1$ and we may assume always $n \geq 2$, if convenient.

Further binomial-normal inequalities. In this section we discuss some estimations of the binomial distribution by the normal distribution. Let $f_{n,p}$ denote the density of the binomial distribution with parameters n and p.

Especially interesting for the present work is the result of Zubkov/Serov (2012), who showed that for $H(x,p) := x\log(\frac{x}{p}) + (1-x)\log(\frac{1-x}{1-p})$ we have

$$(3) \quad \Phi\big(\mathrm{sgn}(\tfrac{k}{n}-p)\sqrt{2nH(\tfrac{k}{n},p)}\,\big) \;\leq\; F_{n,p}(k) \;\leq\; \Phi\big(\mathrm{sgn}(\tfrac{k+1}{n}-p)\sqrt{2nH(\tfrac{k+1}{n},p)}\,\big).$$

This estimation is very useful if $k - np \gg 0$ and is used in Lemma 7.4 for $k \geq np + \frac{3}{2}\sqrt{npq}$. Unfortunately if $k \approx np$, then (3) is not as good, for example it only yields $F_{n,\frac{k}{n}}(k) \geq \frac{1}{2}$, which can however be sharpened, see Lemma 11.9. The work of Zubkov/Serov is based on the ideas of Alfers/ Dinges (1984), whose result however did not have a clear representation and that's why they are not mentioned here, as well the work of Feller (1945), which considers already very early the normal approximation of the binomial distribution.

Further interesting inequalities in this topic originate from McKay (1989),where, besides two other asymptotic inequalities, it was shown that for $np \le k \le n$ and $z = (k-np)/\sigma$ we have

$$1 - F_{n,p}(k-1) = \sigma f_{n-1,p}(k-1) \cdot \tfrac{1-\Phi(z)}{\varphi(z)} \cdot \mathrm{e}^{E(k,n,p)/\sigma}$$

with $0 \le E(k,n,p) \le \min\{\sqrt{\pi/8}, 1/z\}$ holds. We consider again the case if $k \approx np$ by setting $p = \frac{k}{n}$ and hence $z = 0$ and $\sigma^2 = k(n-k)/n$. Then we have

$$\begin{aligned} F_{n,\frac{n-k}{n}}(n-k) &= 1 - F_{n,\frac{k}{n}}(k) = \sigma \cdot f_{n,\frac{k}{n}}(k) \cdot \tfrac{\sqrt{2\pi}}{2} \cdot \mathrm{e}^{E(k,n,k/n)/\sigma} \\ &\approx \tfrac{1}{2} + \tfrac{\sqrt{2\pi}\cdot E(k,n,k/n)}{2} \cdot f_{n,\frac{k}{n}}(k) \end{aligned}$$

since Lemma 11.4 yields $\sigma \cdot f_{n,\frac{k}{n}}(k) \cdot \sqrt{2\pi} \approx 1$ if σ large enough and thus we only get if σ large enough

$$\tfrac{1}{2} \ \le \ F_{n,\frac{n-k}{n}}(n-k) \ \le \ \tfrac{1}{2} + \tfrac{\pi}{4} \cdot f_{n,\frac{n-k}{n}}(n-k),$$

which yields like mentioned above not the desired accuracy, see Lemma 11.9.

Finally we want to mention the work of Slud (1977), which yielded many ideas for the proofs in chapter 7. In that work it was among other things shown that

$$f_{n,p}(k) \ \ge \ \Phi\big(\tfrac{k+1-np}{\sqrt{np(1-p)}}\big) - \Phi\big(\tfrac{k-np}{\sqrt{np(1-p)}}\big) \qquad \text{if } np \le k \le n(1-p) \text{ and } k \ge 2,$$

which corresponds to Lemma 7.3 for another range of k. Unfortunately the interval $[np, n(1-p)]$ there is not sufficient for our purposes, because if p near 0.5, that interval would be very small. That is why the proof for the desired range had to be done again in a way similar to the one in Slud. He also showed that

$$F_{n,p}(k-1) \ \le \ \Phi\big(\tfrac{k-np}{\sqrt{np(1-p)}}\big) \qquad \text{if } np \le k \le n(1-p) \text{ or } 0 \le p \le \tfrac{1}{4},\ np \le k \le n$$

holds, but neither this result nor its proof is used in the present work.

Median. In this section we have a look at the median of the binomial distribution. Let denote in the following

$$m_{n,p} := \min\{k \in \mathbb{N} : F_{n,p}(k) \ge \tfrac{1}{2}\}$$

the smallest median of a given binomial distribution with parameters n and p and let $p_{n,k}$ the unique $p \in [0,1]$ with $F_{n,p_{n,k}}(k) = \frac{1}{2}$ for $k < n$ (this uniquely exists since $\frac{\mathrm{d}}{\mathrm{d}p} F_{n,p}(k) = -n f_{n-1,p}(k) < 0$ and $F_{n,0}(k) = 1$ and $F_{n,1}(k) = 0$ if $k < n$).

Now the result of Göb (1994) yields

$$\begin{aligned} m_{n,p} &\in (np-q, np+q) \quad \text{if} \quad p<\tfrac{1}{2} \\ m_{n,p} &= \lfloor \tfrac{n}{2} \rfloor \quad \text{if} \quad p=\tfrac{1}{2} \\ m_{n,p} &\in (np-p, np+p) \quad \text{if} \quad p>\tfrac{1}{2} \end{aligned}$$

and it follows in particular that $m_{n,p} = \lfloor np \rfloor$ or $m_{n,p} = \lceil np \rceil$. Further we have that the distance between median and mean satisfies $|m_{n,p} - np| \leq \max(p,q) = \frac{1}{2} + \frac{|1-2p|}{2}$. Extending this result, Hamza (1995) showed that for every $n \in \mathbb{N}$ and for every $p \in [0,1]$ we have

$$|m_{n,p} - np| \;\leq\; \begin{cases} \max\{p,q\} & \text{if } 1-\log(2) \leq p \leq \log(2) \\ \log(2) & \text{else} \end{cases},$$

where the upper bound $\log(2)$ is optimal in the sense that

$$\sup_{n\in\mathbb{N}, p\in[0,1]} |m_{n,p} - np| = \log(2)$$

holds. This follows from $p_{n,0} = 1 - 1/(2^{1/n})$ and the convergence $np_{n,0} \to \log(2)$ for $n \to \infty$. Then we have as a conclusion that $m_{n,k} = \lfloor np \rfloor$, if $np - \lfloor np \rfloor \leq 1 - \log(2)$ and also $m_{n,k} = \lceil np \rceil$, if $np - \lfloor np \rfloor \leq \log(2)$ holds.

That result is refined in the present work if $\frac{n}{6} \leq k \leq \frac{5n}{6} - 1$. In this case Lemma 11.10 yields

$$p_{n,k} \;\in\; [\tfrac{k}{n} + \tfrac{1}{2n} + \tfrac{1-2(k+1)/n}{6n}, \tfrac{k}{n} + \tfrac{1}{2n} + \tfrac{1-2k/n}{6n}]$$

Thus it follows that $|m_{n,p} - np| \leq \frac{1}{2} + \frac{|1-2k/n|}{6n}$ in case of $\frac{1}{6} \leq \lfloor np \rfloor \leq \lceil np \rceil \leq \frac{5}{6}$. Since $\frac{|1-2k/n|}{6} \approx \frac{|1-2p|}{6}$ if n is not too small, we have at least in this range a better estimation for $m_{n,p}$ and $p_{n,k}$; in particular for $\frac{n}{6} \leq m := \lfloor np \rfloor \leq \frac{5n}{6} - 1$ we have

$$\begin{aligned} m_{n,p} &= \lfloor np \rfloor & &\text{if} \quad p \in [\tfrac{m}{n}, \tfrac{m}{n} + \tfrac{2}{3n} - \tfrac{m+1}{3n^2}] \\ m_{n,p} &\in \{\lfloor np \rfloor, \lceil np \rceil\} & &\text{if} \quad p \in (\tfrac{m}{n} + \tfrac{2}{3n} - \tfrac{m+1}{3n^2}, \tfrac{m}{n} + \tfrac{2}{3n} - \tfrac{m}{3n^2}) \\ m_{n,p} &= \lceil np \rceil & &\text{if} \quad p \in [\tfrac{m}{n} + \tfrac{2}{3n} - \tfrac{m}{3n^2}, \tfrac{m+1}{n}] \end{aligned}$$

yielding the smallest median exactly except for p in a small range.

The work of Jogdeo/Samuels (1968) considers a slightly other question. There for the $z_{n,k}$ defined by

$$F_{n,\frac{k}{n}}(k) = \tfrac{1}{2} + z_{n,k} \cdot f_{n,\frac{k}{n}}(k) \quad \text{if} \quad k \in \{0,...,n\}$$

it was shown that $z_{n,k} \in [\frac{1}{2}, \frac{2}{3}]$ if $k \leq n/2$. A partial refinement is now Lemma 11.9, which yields $z_{n,k} \in [\frac{4-2k/n}{6} - \frac{n-2k}{18k(n-k)}, \frac{4-2k/n}{6} - \frac{n-2k}{72k(n-k)}]$ if $\frac{n}{6} \leq k \leq \frac{n}{2}$.

Density maximum. It is well known that the binomial density $f_{n,p}$ is maximized at

$$\begin{cases} \lfloor (n+1)p \rfloor & \text{if } (n+1)p \notin \mathbb{N} \\ (n+1)p \text{ and } (n+1)p-1 & \text{if } (n+1)p \in \{1,...,n\} \\ n & \text{if } (n+1)p = n+1 \end{cases}$$

since $f_{n,p}(k) \leq f_{n,p}(k+1) \Leftrightarrow k \leq (n+1)p-1$ for $k \in \{0,...,n-1\}$. We now want to estimate the maximal value of the density. We have the very general result of Herzog (1947), where he showed

$$f_{n,p}(k) \quad < \quad \tfrac{1}{\sqrt{2\mathrm{e}np(1-p)}} \;, \qquad \text{for all } n \geq 1,\ 0 \leq k \leq n \text{ and } p \in (0,1).$$

It was also shown there that the constant $1/\sqrt{2\mathrm{e}}$ cannot be replaced by a smaller one, since $f_{n,\frac{1}{2n+2}}(0) \cdot \sqrt{n \cdot \frac{1}{2n+2} \cdot (1 - \frac{1}{2n+2})} = \sqrt{\frac{1}{2} \cdot \frac{n(2n+2)}{(n+1)(2n+1)}} \cdot (1 - \frac{1/2}{n+1})^{n+1} \to \sqrt{\frac{1}{2\mathrm{e}}}$ for $n \to \infty$. The present work yields a bit other estimation by replacing the denominator on the right hand side $\sqrt{np(1-p)}$ by $\sqrt{k(n-k)/n}$. First we have since

$$\tfrac{\mathrm{d}}{\mathrm{d}p} f_{n,p}(k) \quad = \quad -n \cdot f_{n-1,p}(k) \cdot \left(1 - \tfrac{k(1-p)}{(n-k)p}\right)$$

that $f_{n,p}(k)$ becomes maximal at the point $p = \frac{k}{n}$ and hence Lemma 11.4 yields

$$f_{n,p}(k) \quad \leq \quad f_{n,\frac{k}{n}}(k) \quad \leq \quad \tfrac{1}{\sqrt{2\pi k(n-k)/n}} \quad \text{for } n \in \mathbb{N},\ 0 < k < n,$$

where the optimality of $1/\sqrt{2\pi}$ also follows from Lemma 11.4. Thus in the case $p \approx \frac{k}{n}$ we have a better estimation, since $2\pi > 2\mathrm{e}$.

Further estimations. In this section we present more estimations of the distribution function of a binomial distribution. Very early Uspensky (1937) showed already for $X \sim \mathrm{Bi}(n,p)$ with $0 < p < 1$ that $\mathbb{P}(|\frac{X}{n} - p| \geq c) < 2\mathrm{e}^{-nc^2/2}$ for $c > 0$ holds. This estimation was improved by Okamoto (1958), who showed under the conditions above

$$\mathbb{P}(\tfrac{X}{n} - p \geq c) \;<\; \mathrm{e}^{-2nc^2} \quad \text{and} \quad \mathbb{P}(\tfrac{X}{n} - p \leq -c) \;<\; \mathrm{e}^{-2nc^2}.$$

If we set c very small, for example $c = 1/n$, we obtain $\mathbb{P}(X \geq np+1) < \mathrm{e}^{-2/n}$, which has no use and even if we set $c = \sigma/n := \sqrt{npq}/n$ we only obtain $\mathbb{P}(X \geq np + \sigma) < \mathrm{e}^{-2pq}$ and since $\mathrm{e}^{-2pq} \geq \mathrm{e}^{-1/2} > 1/2$ also nothing useful. Even the refinement of the estimation by Kambo/Kotz (1966), who showed

$$\mathbb{P}(\tfrac{X}{n} - p \geq c) \;<\; \mathrm{e}^{-2nc^2 - \frac{4}{3}nc^4} \quad \text{and} \quad \mathbb{P}(\tfrac{X}{n} - p \leq -c) \;<\; \mathrm{e}^{-2nc^2 - \frac{4}{3}nc^4}$$

under the same conditions as above, is in the case of $c \leq 1/\sqrt{n}$ of no use in the present work.

The result of Okamoto is a special case of the later result of Hoeffding (1963), where it was shown even for arbitrary independent random variables $X_1, ..., X_n$, with $0 \leq X_i \leq 1$ for $i = 1, ..., n$ that

$$\mathbb{P}(\overline{X} - \mathbb{E}\overline{X} \geq c) \;<\; \mathrm{e}^{-2nc^2},$$

where $\overline{X} := (X_1 + ... + X_n)/n$ denotes the sample mean.

We also do not profit for the present work from the result of Bahadur (1959) and works based on that one. Bahadur showed for $k, n \in \mathbb{N}$, $0 < p < 1$ the following representation

$$\sum_{i=k}^{n} f_{n,p}(i) \;=\; f_{n,p}(k) \cdot (1-p) \cdot H(n+1, 1; k+1; p),$$

where $H(n+1, 1; k+1; p) := 1 + \frac{(n+1)p}{k+1} + \frac{(n+1)(n+2)p^2}{(k+1)(k+2)} + ...$ denotes the hypergeometric series, and estimations based on this representation. But the terms in this series become small fast only if $p \ll k/n$ and hence on this way we can't get manageable estimations for $k \approx np$.

Poisson distribution. In this section we consider similar estimations, but this time with respect to the Poisson distribution. We start with Bohman (1963), who showed for the distribution function F_{P_α} of the Poisson distribution with parameter α that

$$F_{P_\alpha}(x) \;\leq\; \Phi\big(\tfrac{x+1-\alpha}{\sqrt{\alpha}}\big) \qquad \text{for all } x \in \mathbb{R}.$$

He denotes this as „dominating "of $\Phi\big(\frac{\bullet+1-\alpha}{\sqrt{\alpha}}\big)$ over F_{P_α}, thus due to this notion each Poisson distribution is dominated by a normal distribution. The more common mode of expression might be that $\Phi\big(\frac{\bullet+1-\alpha}{\sqrt{\alpha}}\big)$ is in the usual stochastic order, see for example Shaked/Shanthikumar (1994), less than F_{P_α}.

Of course, this inequality cannot hold if we replace F_{P_α} by the distribution function of a binomial distribution $F_{n,p}$, since $F_{n,p}(x) = 1$ if $x \geq n$, but like as already mentioned in the section "Further binomial-normal inequalities" above, we have $F(k) \leq \Phi\big(\frac{k+1-np}{\sqrt{np(1-p)}}\big)$ due to Slud (1977) at least if $np \leq k+1 \leq n(1-p)$ and if $p \leq \frac{1}{4}$ and also $np \leq k+1 \leq n$.

Similar to the works in the section Median above Adell/Jodrá (2005) considered the medians of the Poisson distribution. They showed that for α_n with $F_{P_{\alpha_n}}(n) = \frac{1}{2}$ and $c_n := \mathrm{e}^{-1} \cdot (1 + 1/n)^{n-1}$ we have for $n \in \mathbb{N}$

$$n + \tfrac{2}{3} \;<\; \alpha_n \;<\; \tfrac{2}{3} + \tfrac{8(1-c_{n+1})}{81 c_{n+1}}$$

and thus $-\frac{2}{3} - \frac{8(1-c_{n+1})}{81c_{n+1}} < m_\alpha(P_\alpha) - \alpha \;<\; \frac{1}{3}$, where $m_\alpha(P_\alpha)$ denotes the smallest median of the Poisson distribution.

Somewhat earlier Choi (1994) already showed $n+\frac{2}{3} < \alpha_n \leq \min\left\{n+\log 2, n+\frac{2}{3}+\frac{1}{2n+2}\right\}$, which is a worse upper bound and concluded that $-\log 2 \leq m_\alpha(P_\alpha) - \alpha < \frac{1}{3}$, which was already conjectured by Chen/Rubin (1986), but they proved it only with -1 instead of $-\log 2$ on the left hand side. For $\alpha \geq 0$ arbitrary, the bounds $-\log 2$ and $\frac{1}{3}$ are optimal due to Choi.

This topic is in close context to the question set up by Ramanujan (1907) about $\theta(n)$ defined by

$$\frac{\mathrm{e}^n}{2} \;=\; \sum_{k=0}^{n-1} \frac{n^k}{k!} + \theta(n) \cdot \frac{n^n}{n!}.$$

The conjecture of Ramanujan that $\frac{1}{3} < \theta(n) < \frac{1}{2}$ holds, was proved by Szegö (1928) and yields, with p_α denotes the density of F_{P_α},

$$F_{P_n}(n) - \frac{1}{2} \;=\; (1-\theta_n) \cdot p_n(n) \;\leq\; \frac{2}{3} \cdot p_n(n) \;\leq\; \frac{2}{3\sqrt{2\pi n}},$$

where in the last inequality $p_n(n) = n^n \mathrm{e}^{-n}/n! \leq 1/\sqrt{2\pi n}$ by the Stirling inequality was used. The above can, beside the error term in the Stirling inequality, be improved by the further conjecture of Ramanujan, proved by Flajolet et al. (1993), that even $\theta(n) = \frac{1}{3} + \frac{4}{135(n+k_n)}$ with $\frac{2}{21} \leq k_n \leq \frac{4}{45}$ holds. Further Adell/Jodrá (2007) refined the estimations of $\theta(n)$ up to an error less than $n^{-7}/100$, such that $F_{P_n}(n) - 1/2$ can be estimated with the help of the results mentioned above very exactly. This is analogous to Lemma 11.9 of the present work with Poisson instead of binomial distributions.

3. Notations and elementary inequalities

In the course of this work we often use the following notations:

- $\sigma_p^2 := np(1-p)$ for $p \in (0,1)$ and given fixed n
- $f_{n,p}(k) := \binom{n}{k} p^k (1-p)^{n-k}$ for $n \in \mathbb{N}_0$, $k \in \mathbb{Z}$, $p \in [0,1]$ and $\binom{n}{k} := 0$ if $k \notin \{0, ..., n\}$
- $F_{n,p}(x) := \sum_{k=0}^{\lfloor x \rfloor} f_{n,p}(k) = 1 - F_{n,q}(n - \lfloor x \rfloor - 1)$ for $x \in \mathbb{R}$, $n \in \mathbb{N}_0$ and $p \in [0,1]$
- $G_{n,p}(x) := \Phi\left(\frac{x-np}{\sqrt{np(1-p)}}\right) = 1 - G_{n,q}(n-x)$ for $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $p \in (0,1)$
- $g_{n,p}(x) := G_{n,p}(x) - G_{n,p}(x-1)$ for $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $p \in (0,1)$
- $c_{\mathrm{E}} := \frac{\sqrt{10}+3}{6\sqrt{2\pi}} = 0.409732...$, $\quad c_0 := \frac{2}{\sqrt{2\pi}\, c_{\mathrm{E}}} - 1 = \frac{12}{\sqrt{10}+3} - 1 = 0.947331...$
- $\varphi(t) := \frac{1}{\sqrt{2\pi}} \cdot \mathrm{e}^{-\frac{t^2}{2}}$ for $t \in \mathbb{R}$
- $\Phi(x) := \int_{-\infty}^{x} \varphi(t)\mathrm{d}t$ for $x \in \mathbb{R}$.

From now on, in this work let the given $n \in \mathbb{N} = \{1, 2, 3, ...\}$ and $p \in (0,1)$, unless noted otherwise, be fixed. With these two fixed parameters we put or have:

- $q := 1 - p$
- $m := \lfloor np \rfloor$
- $\ell := m + 1$
- $\lambda := \frac{\ell}{n} = \frac{m+1}{n}$
- $\delta := m + 1 - np \in (0,1]$
- $\sigma^2 := npq$
- $\sigma_\lambda^2 = n\lambda(1-\lambda) = \frac{\ell(n-\ell)}{n} = \frac{(m+1)(n-m-1)}{n}$
- $f(k) := f_{n,p}(k)$ for $k \in \{0, ..., n\}$
- $F(x) := F_{n,p}(x)$ for $x \in \mathbb{R}$
- $g(x) := g_{n,p}(x) = \Phi\left(\frac{x-np}{\sigma}\right) - \Phi\left(\frac{x-np-1}{\sigma}\right)$ for $x \in \mathbb{R}$
- $G(x) := G_{n,p}(x) = \Phi\left(\frac{x-np}{\sigma}\right)$ for $x \in \mathbb{R}$
- $\mathrm{e}^{S_n^k} := f_{n,\frac{k}{n}}(k) \cdot \sqrt{2\pi}\, \sigma_{k/n}$ for $k \in \{0, .., n\}$, in particular $\mathrm{e}^{S_n^\ell} = f_{n,\lambda}(\ell) \cdot \sqrt{2\pi}\, \sigma_\lambda$
- $\Delta(x) := |F(x) - G(x)|$ for $x \in \mathbb{R}$.

Furthermore we often use the following elementary inequalities:

(1) If $x \in [0,1)$ we have
$$\begin{aligned} \log(1+x) &\leq x - \tfrac{x^2}{2} + \tfrac{x^3}{3} \leq x, \\ \log(1+x) &\geq x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \tfrac{x^4}{4} \geq x - \tfrac{x^2}{2}, \\ \log(1-x) &\leq -\sum_{r=1}^{n} \tfrac{x^r}{r} \quad \text{for every } n \in \mathbb{N}_0. \end{aligned}$$

(2) If $x \in \mathbb{R}$ we have $\mathrm{e}^x \geq 1 + x$ and if $x \geq 0$ we also have
$$\mathrm{e}^{-x} \leq 1 - x + \tfrac{x^2}{2}.$$

(3) If $x \in [0,1]$ we have
$$\begin{aligned} 1 + \tfrac{x}{2} - \tfrac{x^2}{8} \leq \sqrt{1+x} &\leq 1 + \tfrac{x}{2} - \tfrac{x^2}{8} + \tfrac{x^3}{16} \leq 1 + \tfrac{x}{2}, \\ \sqrt{1-x} &\leq 1 - \tfrac{x}{2} - \tfrac{x^2}{8} \leq 1 - \tfrac{x}{2}. \end{aligned}$$

(4) For $0 \leq p_1 \leq p_2 \leq \frac{1}{2}$ we have
$$0 \leq p_1(1-p_1) \leq p_2(1-p_2) \leq \tfrac{1}{4}.$$

(5) If $a, b > 0$ we have $\min\limits_{x \in \mathbb{R}} ax^2 - bx = -\frac{b^2}{4a}$ and $\arg\min\limits_{x \in \mathbb{R}} ax^2 - bx = \frac{b}{2a}$.

(6) For $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and $p \in [0,1]$ we have
$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}p} f_{n,p}(k) &= -n \cdot (f_{n-1,p}(k) - f_{n-1,p}(k-1)), \\ \tfrac{\mathrm{d}}{\mathrm{d}p} F_{n,p}(k) &= -n f_{n-1,p}(k). \end{aligned}$$

Proof of (1)-(3): There are the corresponding McLaurin series in the respective ranges
$$\log(1+x) = \sum_{r=1}^{\infty} \tfrac{(-1)^{r+1} x^r}{r} \ ; \ \mathrm{e}^{-x} = \sum_{r=0}^{\infty} \tfrac{(-x)^r}{r!} \ ; \ \sqrt{1+x} = 1 + \tfrac{x}{2} + \sum_{r=2}^{\infty} \tfrac{(-1)^{r+1} x^r (2r-3)!!}{(2r)!!}.$$

By Pólya/Szegö (1978, Problem 140 and solution, pp. 33, 209) a function $h : [0,x] \to \mathbb{R}$ is enveloped by its McLaurin series at x if the modulus $|h^{(r)}|$ of its r-th derivative is decreasing on $[0,x]$ for each $r \in \mathbb{N}$. This is satisfied by the three functions above, since $|(\mathrm{e}^{-t})^{(r)}| = \mathrm{e}^{-t}$ and $|\log(1+t)^{(r)}| = \frac{1}{(r-1)!(1+t)^r}$ as well as $|\sqrt{1+t}^{(r)}| = \frac{c_r}{(1+x)^{r-1/2}}$ for some $c_r > 0$. Thus the series is alternately larger and smaller than $h(x)$ and hence we have (1)-(3), with the third inequality in (1) and the second inequality in (3) following directly from the convergence of the McLaurin series.

The statements (4), (5) and (6) result from simple differentiations.

4. Results

Theorem 1. *Let $p \in (0,1)$ and $n \in \mathbb{N}$ and let $F_{n,p}$ denote the distribution function of the binomial distribution with parameters n and p. Then we have*

$$\sup_{x\in\mathbb{R}} \left| F_{n,p}(x) - \Phi\left(\tfrac{x-np}{\sqrt{npq}}\right) \right| \quad < \quad \tfrac{\sqrt{10}+3}{6\sqrt{2\pi}} \cdot \tfrac{p^2+q^2}{\sqrt{npq}}. \tag{4}$$

In the case $p \in [\frac{1}{3}, \frac{2}{3}]$ we even have the sharper inequality

$$\sup_{x\in\mathbb{R}} \left| F_{n,p}(x) - \Phi\left(\tfrac{x-np}{\sqrt{npq}}\right) \right| \quad < \quad \tfrac{3+|p-q|}{6\sqrt{2\pi}\sqrt{npq}}. \tag{5}$$

Remark 4.1. The constant $c_{\mathrm{E}} = \frac{\sqrt{10}+3}{6\sqrt{2\pi}} = 0.4097...$ is optimal due to Esseen (1956), since there for $p = \frac{4-\sqrt{10}}{2}$ it is shown that

$$\lim_{n\to\infty} \sqrt{n} \cdot \sup_{x\in\mathbb{R}} \left| F_{n,p}(x) - \Phi\left(\tfrac{x-np}{\sigma}\right) \right| \quad = \quad c_{\mathrm{E}} \cdot \tfrac{p^2+q^2}{\sqrt{pq}}$$

as already explained in the introduction on page 2.

Remark 4.2. The right hand side of the sharper inequality in the case $p \in [\frac{1}{3}, \frac{2}{3}]$ in Theorem 1 is optimal for each p in the sense that

$$\lim_{n\to\infty} \sqrt{n} \cdot \sup_{x\in\mathbb{R}} \left| F_{n,p}(x) - \Phi\left(\tfrac{x-\mu}{\sigma}\right) \right| \quad = \quad \tfrac{3+|p-q|}{6\sqrt{2\pi}\sqrt{pq}}$$

for each $p \in (0,1)$, which holds by Esseen (1956). There it was proved more generally that if $X_1, ..., X_n, ...$ are independent and identically distributed random variables with distribution P and with $\nu := \mathbb{E}X_1$, $\tau^2 := \mathrm{Var}(X_1) \in (0,\infty)$, $\alpha_3 := \mathbb{E}(X_1 - \nu)^3 \in \mathbb{R}$ and $F_{S_n}(x) := \mathbb{P}(\sum_{i=1}^{n} X_i \leq x)$ for $x \in \mathbb{R}$, we have

$$\lim_{n\to\infty} \sqrt{n} \cdot \sup_{x\in\mathbb{R}} \left| F_{S_n}(x) - \Phi\left(\tfrac{x-n\nu}{\tau}\right) \right| = \tfrac{|\alpha_3|}{6\tau^3\sqrt{2\pi}}$$

if P is non lattice and, relevant in the binomial case,

$$\lim_{n\to\infty} \sqrt{n} \cdot \left| F_{S_n}(x) - \Phi\left(\tfrac{x-n\nu}{\tau}\right) \right| = \tfrac{1}{\sqrt{2\pi}} \cdot \left(\tfrac{h}{2\tau} + \tfrac{|\alpha_3|}{6\tau^3}\right)$$

if P is lattice, where h is the largest value satisfying $P(\{x_0 + h \cdot z \ : \ z \in \mathbb{Z}\}) = 1$ for some $x_0 \in \mathbb{R}$.

Remark 4.3. Unfortunately (5), the sharper inequality in Theorem 1, does not hold for all $p \in (0, \frac{1}{3})$, since we obtain by the Edgeworth expansion with $m_n = \lfloor np \rfloor$, $\varsigma_n := \sqrt{npq}$, $z_n := np - m_n \in [0,1)$ and $d_n := F_{n,p}(m_n) - \Phi(\frac{m_n - np}{\varsigma_n}) - \frac{4-2p}{6\sqrt{2\pi}\,\varsigma_n}$

$$\sqrt{2\pi}\,\varsigma_n^3 \cdot d_n + \tfrac{46-69p+21p^2+p^3}{540} - \tfrac{5-7p+p^2}{12} \cdot z_n + \tfrac{1-p}{2} \cdot z_n^2 \quad \to \quad 0 \quad (n \to \infty)$$

as is explained more precisely on page 25 in chapter 5. If now $p \in (0, \frac{1}{3})$ is irrational, then there is for every $z \in [0,1]$ a subsequence $(n_k)_{k\in\mathbb{N}}$ with $z_{n_k} \to z$ for $k \to \infty$. Thus we then have

$$\begin{aligned}
\lim_{k\to\infty} \sqrt{2\pi}\,\varsigma_{n_k}^3 \cdot d_{n_k} \quad &= \quad \lim_{k\to\infty} \left(\sqrt{2\pi}\,\varsigma_{n_k}^3 \cdot d_{n_k} + \tfrac{46-69p+21p^2+p^3}{540} - \tfrac{5-7p+p^2}{12} \cdot z_{n_k} + \tfrac{1-p}{2} \cdot z_{n_k}^2 \right) \\
&\quad - \lim_{k\to\infty} \left(\tfrac{46-69p+21p^2+p^3}{540} - \tfrac{5-7p+p^2}{12} \cdot z_{n_k} + \tfrac{1-p}{2} \cdot z_{n_k}^2 \right) \\
&= \quad -\tfrac{46-69p+21p^2+p^3}{540} + \tfrac{5-7p+p^2}{12} \cdot z - \tfrac{1-p}{2} \cdot z^2 \quad =: \quad e(z,p).
\end{aligned}$$

Now $e(z,p)$ becomes maximal with respect to $z \in \mathbb{R}$ for $z_p = \frac{5-7p+p^2}{12(1-p)} = \frac{5}{12} - \frac{p}{6} - \frac{p^2}{12(1-p)}$ and hence $z_p \in [0,1]$ since $p \in (0, \frac{1}{3})$ and thus

$$\max_{z\in[0,1]} e(z,p) \quad = \quad e(z_p, p) \quad = \quad \tfrac{7-130p+165p^2+50p^3-23p^4}{4320(1-p)}.$$

Since z_p only depends on p, and since $z \in [0,1]$ above was arbitrary, for given $p \in (0, \frac{1}{3})$ we have

$$\limsup_{n\to\infty} \sqrt{2\pi}\,\varsigma_n^3 \cdot d_n \quad = \quad e(z_p, p).$$

Now we observe by numerical calculation $e(z_p, p) > 0$ holds iff $p < p_0$ with $p_0 = 0.058\ldots$, from which $\sup_{n\in\mathbb{N}} d_n > 0$ follows. Thus (5) cannot hold for every $n \in \mathbb{N}$ if $p \in (0, 0.058)$ is irrational.

For those $p \in (0, \frac{1}{3})$ with $e(z_p, p) < 0$ we may conjecture by numerical computations that $d_n < 0$ not only holds asymptotically, but also for every $n \in \mathbb{N}$. By the symmetries

$$\Phi(\tfrac{n-k-np}{\sigma}) - F_{n,p}(n-k-1) \quad = \quad F_{n,q}(k) - \Phi(\tfrac{k-nq}{\sigma})$$

with $k = \lfloor nq \rfloor$, (5) neither holds universally if $p \in (1-p_0, 1)$, since then $q \in (0, p_0)$. Thus we conjecture that (5) holds for $p \in [p_0, 1-p_0] \approx [0.058, 0.942]$.

By the explanations above, one might suppose that the optimal Berry-Esseen constant in the Poisson case, which was denoted with $C_{\mathrm{BE},\{\mathrm{Poi}_\alpha:\alpha\in(0,\infty)\}}$ and also explained in the introduction not equals $\frac{2}{3\sqrt{2\pi}}$. In fact, if we set

$$d_{\mathrm{Poi}}(m,\alpha) \quad := \quad \tfrac{3\sqrt{2\pi\alpha}}{2} \cdot \left(F_{\mathrm{Poi}_\alpha}(m) - \Phi\left(\tfrac{m-\alpha}{\sqrt{\alpha}}\right) \right)$$

we get for example $d_{\mathrm{Poi}}(6, 6.4206) = 1.00018989\ldots > 1$, thus we have in particular $C_{\mathrm{BE},\{\mathrm{Poi}_\alpha:\alpha\in(0,\infty)\}} > \frac{2}{3\sqrt{2\pi}} = 0.2660\ldots$. By numerical computations we may conjecture that the absolute maximum of $d_{\mathrm{Poi}}(m,a)$ is actually attained at $m = 6$ and $\lambda \approx 6.4206$. Up to now we have $C_{\mathrm{BE},\{\mathrm{Poi}_\alpha:\alpha\in(0,\infty)\}} \leq 0.3031$ due to Shevtsova (2013), this is explained in chapter 5 in more detail.

Theorem 2. *Let* $n \in \mathbb{N}$, $n \geq 6$ *and* $p \in [\frac{1}{6}, \frac{5}{6}]$ *and* X *binomially distributed with parameters* n *and* p *and hence* $\mu := \mathbb{E}X = np$ *and* $\sigma^2 := \mathbb{E}(X - \mu)^2 = np(1-p)$. *Let further denote* $\mathrm{Bi}_{n,p}$ *the distribution of* X *and* $\mathrm{N}_{\mu,\sigma^2}$ *the normal distribution with mean* μ *and variance* σ^2. *Then we have*

$$\sup_{I \subset \mathbb{R} \text{ interval}} \left|\mathrm{Bi}_{n,p}(I) - \mathrm{N}_{\mu,\sigma^2}(I)\right| \;<\; \tfrac{2}{\sqrt{2\pi}} \cdot \tfrac{p^2+q^2}{\sigma}.$$

Remark 4.4. The constant $c_I := \frac{2}{\sqrt{2\pi}} = 0.7979...$ in Theorem 2 is optimal, since for example from the corollary in Dinev/Mattner (2012) it follows as a special case that if $p = \frac{1}{2}$, we have

$$\lim_{n\to\infty} \sqrt{n} \cdot \sup_{I \subset \mathbb{R} \text{ interval}} \left|\mathrm{Bi}_{n,p}(I) - \mathrm{N}_{\mu,\sigma^2}(I)\right| \;=\; c_I \cdot \tfrac{p^2+q^2}{\sqrt{pq}}.$$

The claim in Theorem 2 also holds if $p \in (0, 0.155]$ and $p \in [0.845, 1)$ due to Shevtsova (2013), because that result yields for these p in the second inequality

$$\sup_{I \subset \mathbb{R} \text{ interval}} \left|\mathrm{Bi}_{n,p}(I) - \mathrm{N}_{\mu,\sigma^2}(I)\right| \;\leq\; 2 \cdot \sup_{x \in \mathbb{R}} \left|F_{n,p}(x) - \Phi\left(\tfrac{x-\mu}{\sigma}\right)\right| \;\leq\; \tfrac{2}{\sqrt{2\pi}} \cdot \tfrac{p^2+q^2}{\sigma},$$

where the first inequality holds by the triangle inequality for all p and n. This is explained in more detail in chapter 5, see bottom of page 20.

5. An overview of the proof of Theorem 1

First we want to consider the result of Shevtsova (2013), cited already in chapters 1 and 4, in more detail, because we need that result later in the proof. Let again $X_1, \ldots, X_n$ be independent and identically distributed with $X_1 \sim \mathrm{Ber}(p)$ for a $p \in (0,1)$ and let further $S_n := X_1 + \cdots + X_n$. Then we firstly have

$$\begin{aligned}
\sup_{x\in\mathbb{R}} \left|\mathbb{P}(S_n \le x) - \Phi(\tfrac{x-np}{\sqrt{np(1-p)}})\right| &= \sup_{x\in\mathbb{R}} \left|\mathbb{P}(\tfrac{S_n-np}{\sqrt{np(1-p)}} \le \tfrac{x-np}{\sqrt{np(1-p)}}) - \Phi(\tfrac{x-np}{\sqrt{np(1-p)}})\right| \\
&= \sup_{x\in\mathbb{R}} \left|\mathbb{P}(\tfrac{S_n-np}{\sqrt{np(1-p)}} \le x) - \Phi(x)\right| \\
&= \sup_{x\in\mathbb{R}} \left|\mathbb{P}(\sum_{i=1}^{n} \tfrac{X_i-p}{\sqrt{np(1-p)}} \le x) - \Phi(x)\right| \\
&=: \Delta_{n,p}.
\end{aligned}$$

Now for $Y_i := \frac{X_i-p}{\sqrt{np(1-p)}}$ we have $\mathbb{E}Y_i = 0$, $\varsigma_i^2 := \mathrm{Var}(Y_i) = \frac{\mathbb{E}(X_i-p)^2}{np(1-p)} = \frac{1}{n}$ for $i = 1, ..., n$ and $l_n := \sum_{i=1}^{n} \mathbb{E}|Y_i|^3 = n \cdot \frac{(1-p)p^3+p(1-p)^3}{(np(1-p))^{3/2}} = \frac{p^2+(1-p)^2}{\sqrt{np(1-p)}}$ as well as $\tau_n := \sum_{i=1}^{n} \varsigma_i^3 = n \cdot \frac{1}{n^{3/2}} = \frac{1}{\sqrt{n}}$ and we have $\sum_{i=1}^{n} \varsigma_i^2 = 1$.

With $\varrho := \frac{l_n}{\tau_n} = \frac{p^2+(1-p)^2}{\sqrt{p(1-p)}}$ we have due to Shevtsova (2013), considering the identically distributed case there,

$$\begin{aligned}
\Delta_{n,p} &\le \min\{0.4690 l_n,\ 0.3322(l_n + 0.429\tau_n),\ 0.3031(l_n + 0.646\tau_n)\} \\
&= l_n \cdot \min\{0.4690,\ 0.3322(1 + \tfrac{0.429}{\varrho}),\ 0.3031(1 + \tfrac{0.646}{\varrho})\}.
\end{aligned}$$

Because the Poisson distribution Poi_α is a boundary case of the binomial distribution with $n \to \infty$ and $np \to \alpha$, hence in particular $p \to 0$ and $\varrho \to \infty$, it follows

$$\sup_{x\in\mathbb{R}} \left|F_{\mathrm{Poi}_\alpha}(x) - \Phi(\tfrac{x-\alpha}{\sqrt{\alpha}})\right| \le 0.3031 \cdot \tfrac{1}{\sqrt{\alpha}}$$

with F_{Poi_α} the distribution function of Poi_α. This also explains a part of Remark 4.3 in more detail.

The aim of this work is to prove $\Delta_{n,p} \le c_{\mathrm{E}} \cdot l_n$ for $c_{\mathrm{E}} = \frac{\sqrt{10}+3}{6\sqrt{2\pi}} = 0.4097...$, see Theorem 1. We now compare c_{E} with $\min\{0.4690,\ 0.3322(1 + \frac{0.429}{\varrho}),\ 0.3031(1 + \frac{0.646}{\varrho})\}$. Obviously 0.4690 is too large and for the other two terms in the minimum we have the equivalence

$$0.3322(1 + \tfrac{0.429}{\varrho}) \ \ge\ 0.3031(1 + \tfrac{0.646}{\varrho}) \ \Leftrightarrow\ \varrho \ge \tfrac{0.3031\cdot 0.646 - 0.3322\cdot 0.429}{0.3322-0.3031} \ =:\ \varrho_0 \approx 1.831\ldots.$$

Since for $\varrho \leq \varrho_0$ we have $0.3322(1+\frac{0.429}{\varrho}) \geq 0.3322(1+\frac{0.429}{\varrho_0}) = 0.41... > c_{\mathrm{E}}$, there only remains $0.3031(1+\frac{0.646}{\varrho})$ as an interesting upper bound. To see where this term is less than c_{E} we notice that

$$0.3031(1+\tfrac{0.646}{\varrho}) \quad \leq \quad c_{\mathrm{E}} \;\Leftrightarrow\; \varrho \;\geq\; \tfrac{0.3031\cdot 0.646}{c_{\mathrm{E}}-0.3031} \quad =: \; \varrho_1 \quad = \quad 1.8362\ldots .$$

Next we compute for which p now $\varrho = \frac{p^2+(1-p)^2}{\sqrt{p(1-p)}} \geq \varrho_1$ holds, see Figure 1. We first have the equivalence

$$\tfrac{p^2+(1-p)^2}{\sqrt{p(1-p)}} \;=\; \tfrac{1}{\sqrt{p(1-p}} - 2\sqrt{p(1-p)} \;\geq\; \varrho_1 \;\;\Leftrightarrow\;\; \sqrt{p(1-p)} \;\leq\; \sqrt{\tfrac{1}{2}+\tfrac{\varrho_1^2}{16}} - \tfrac{\varrho_1}{4}$$

On the right hand side we have equality iff $p = \frac{1}{2} \pm \sqrt{\frac{1}{4} - (\sqrt{\frac{1}{2}+\frac{\varrho_1^2}{16}} - \frac{\varrho_1}{4})^2} = \frac{1}{2} \pm 0.3202...$ and since $p(1-p)$ is increasing on $(0,\frac{1}{2}]$ and decreasing on $[\frac{1}{2},1)$, we have

$$\tfrac{p^2+(1-p)^2}{\sqrt{p(1-p)}} \;\geq\; \varrho_1 \qquad \text{if} \;\; p \in (0, 0.1797) \cup (0.8203, 1).$$

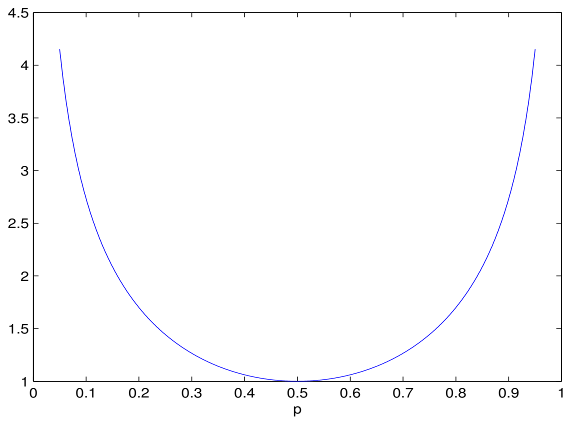

FIGURE 1. Graph of $\varrho(p) = \frac{p^2+(1-p)^2}{\sqrt{p(1-p)}}$.

Based on these considerations the proof rests in the cases $0 < p < \frac{1}{6}$ and $\frac{5}{6} < p < 1$ on Shevtsova (2013). Here $\frac{1}{6} = 0.1666...$ is a quite arbitrary number smaller than 0.1797 to have easier computations later.

In chapter 10 about the distance between interval probabilities we have the problem that we want to have $2/\sqrt{2\pi}$ on the right hand side, which is less than $2c_{\mathrm{E}}$. So that the result of Shevtsova can be applied, we have to consider in which range even $0.3031 \cdot (1+\frac{0.646}{\varrho}) \leq \frac{1}{\sqrt{2\pi}}$ holds and obtain analogously that it is the case if $\varrho \geq 2.05$ and that is the case if $p \in (0, 0.153)$ or $p \in (0.847, 1)$. Thus unfortunately we have a gap in Theorem 2 for $0.153 \leq p \leq 1/6$ and $5/6 \leq p \leq 0.847$, see also Remark 4.4.

Next we want to explain why on the one hand we sometimes use $\frac{3+|p-q|}{6\sqrt{2\pi}\sqrt{pq}}$ and on the other hand sometimes $c_{\mathrm{E}} \cdot \frac{p^2+q^2}{\sqrt{pq}}$ and how they are related. On the basis of the asymptotics

$$\lim_{n\to\infty} \sqrt{n} \cdot \sup_{x\in\mathbb{R}} \left| F_{n,p}(x) - \Phi\left(\tfrac{x-\mu}{\sigma}\right) \right| \quad = \quad \frac{3+|p-q|}{6\sqrt{2\pi}\sqrt{pq}}$$

due to Esseen (1956), one might conjecture $\frac{3+|p-q|}{6\sqrt{2\pi}\sqrt{pq}}$ as a natural upper bound, and in the case of $p \in [\frac{1}{3}, \frac{2}{3}]$ we show indeed inequality (5), that is $\sup_{x\in\mathbb{R}} \left|F_{n,p}(x) - \Phi\left(\frac{x-\mu}{\sigma}\right)\right| \le \frac{1}{\sqrt{2\pi n}} \cdot \frac{3+|p-q|}{6\sqrt{pq}}$, then holds. But for $p < 0.058$ this latter inequality does not hold in general any more, this is considered at the end of this chapter in more detail. If we want to compare inequality (5) in relation to the Berry-Esseen inequality (2) in the introduction on page 2, we need the standardized third absolute centred moment of Ber_p, that is $\frac{p^2+q^2}{\sqrt{pq}}$, instead of $\frac{3+|p-q|}{6\sqrt{pq}}$ on the right hand side. This leads to the following transformation

$$\frac{3+|p-q|}{6\sqrt{pq}} \quad = \quad \frac{p^2+q^2}{\sqrt{pq}} \cdot \frac{3+|p-q|}{6(p^2+q^2)} \quad =: \quad \frac{p^2+q^2}{\sqrt{pq}} \cdot \varepsilon(p),$$

and we now consider the function $\varepsilon(p)$.

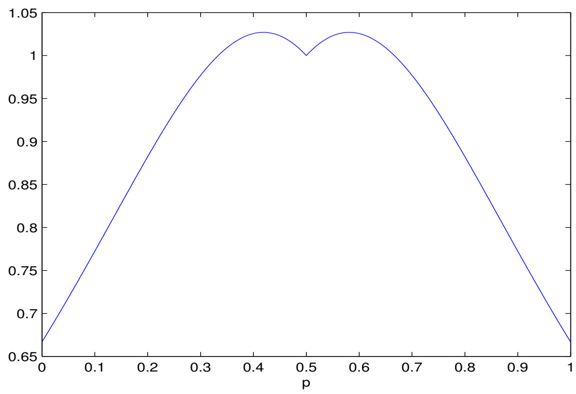

FIGURE 2. Graph of $\varepsilon(p) = \frac{3+|p-q|}{6(p^2+q^2)}$.

According to Esseen (1956), see also Nagaev/Chebotarev (2011), differentiation yields $\varepsilon(p) \le \frac{\sqrt{10}+3}{6} \approx 1.027$, see Figure 2, and thus we know the origin of the constant $c_{\mathrm{E}} := \frac{\sqrt{10}+3}{6\sqrt{2\pi}} \approx 0.4097$ and it is clear that

$$\frac{3+|p-q|}{6\sqrt{2\pi}} \quad \le \quad c_{\mathrm{E}} \cdot (p^2+q^2)$$

holds and hence inequality (5), if applicable, is sharper than the optimal Berry-Esseen inequality in the binomial case (4).

If we take a look at Figure 3, the graph of $c_{\mathrm{E}} \cdot (p^2+q^2) - \frac{3+|p-q|}{6\sqrt{2\pi}}$, since the symmetry we restrict on $p \in [0, 0.5]$, we see that between about 0.35 and 0.5 the difference is very small, this is why we can later show for $p \in [\frac{1}{3}, \frac{1}{2}]$ the sharper and more natural upper bound $\frac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}$ without significantly more effort.

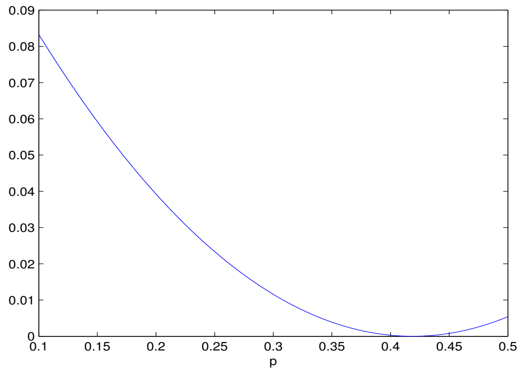

FIGURE 3. Graph of $p \mapsto c_{\mathrm{E}} \cdot (p^2 + q^2) - \frac{3+|p-q|}{6\sqrt{2\pi}}$.

Because the estimation used in the proof later becomes the more inaccurate the more p is away from 0.5, it was not possible for us to show in this way the stronger inequality in the case $p \in [\frac{1}{6}, \frac{1}{3}]$ without enormous calculations, thus we need the significant difference in Figure 3 for small p as a buffer and show only the weaker inequality (4).

We next want to explain the procedure of the present proof of Theorem 1. Since G is continuous and F the distribution function of a discrete distribution, the distance must become maximal at a jump discontinuity of F. In Figure 4 the functions $|F(k) - G(k)|$ and $|G(k) - F(k-1)|$ for $k \in \{0....n\}$ are shown in the case $p = \frac{1}{3}$ and $n = 30$.

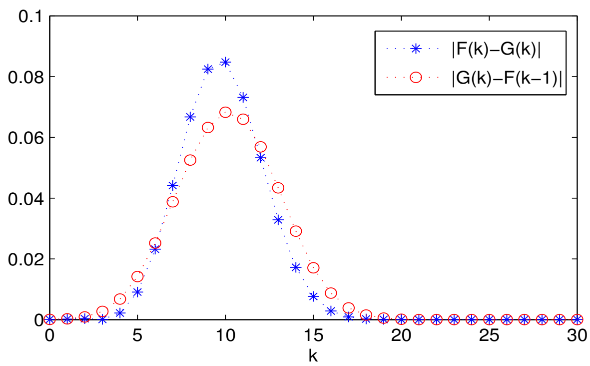

FIGURE 4. Graph of $k \mapsto |F(k) - G(k)|$ and $k \mapsto |G(k) - F(k-1)|$ in the case $n = 30, p = \frac{1}{3}$.

Here we can see that the distance becomes largest near the mean $np = 10$. For those k which are further away from the mean than for example $\frac{3}{2}\sigma \approx 3.9$ it becomes very small. Thus the first part of the proof of Theorem 1 consists of showing that the distance actually becomes largest at an integer around the mean, namely $\lfloor np \rfloor$ or $\lceil np \rceil$. The k further away from the mean np than $c \cdot \sigma$, mostly $c = \frac{3}{2}$, we estimate roughly, because there the distance is very small and hence we may exclude such k, which simplifies the computations a lot.

To show that the distance becomes largest near the mean for the restricted $k \in \{0, ..., n\}$, we compare the densities $f(k)$ respectively $f(k-1)$ and $g(k)$. Here one might see in Figure 5 that, firstly only for $k > \lceil np \rceil$, in the range of interest $f(k) \leq g(k)$ and $g(k) \leq f(k-1)$ hold. The case $k < \lfloor np \rfloor$ can be reduced without any problems to the case $k > \lceil np \rceil$. Further we obviously have $f(k) = 0 < g(k+1)$ if $k > n$, hence $g(k) \leq f(k-1)$ cannot hold for all $k > \lceil np \rceil$, but these boundary cases are already excluded above.

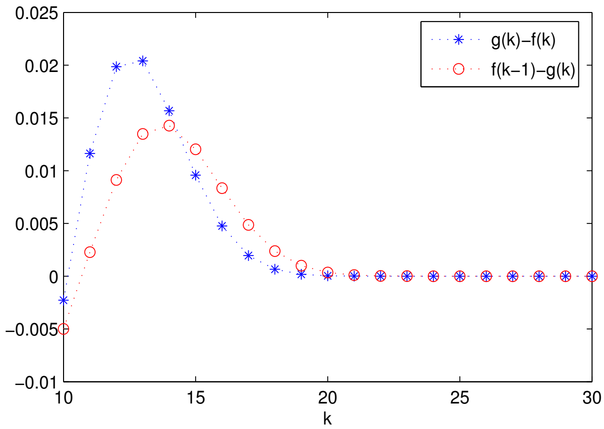

FIGURE 5. Graph of $k \mapsto g(k) - f(k)$ and $k \mapsto f(k-1) - g(k)$ in the case $n = 30, p = \frac{1}{3}$.

After reducing the problem to the integers around the mean in the first part, it remains to estimate the distances at these particular points, namely $\Delta(\lfloor np \rfloor), \Delta(\lfloor np \rfloor -), \Delta(\lceil np \rceil)$, $\Delta(\lceil np \rceil -)$ with $\Delta(x) = |F(x) - G(x)|$, in the second part. Here we may also assume without loss of generality $p \leq 0.5$. The four distances are illustrated in Figure 6 in case of $n = 30$, $p \in (\frac{1}{3}, \frac{1}{3} + \frac{1}{n})$ and $m = \lfloor np \rfloor < \lceil np \rceil = m + 1$.

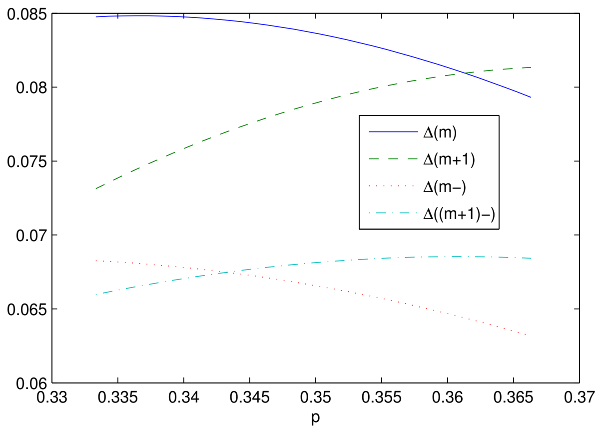

FIGURE 6. Graph of $p \mapsto \Delta(x)$ and $p \mapsto \Delta(x-)$ for $x \in \{m, m+1\}$ in the case $n = 30$.

As this figure might suggest the proof is most difficult for $\Delta(\lfloor np \rfloor)$ and thus claims the majority of the second part of the proof, while the other three possibilities need significantly less work.

Now we consider the function $\mathcal{D}(p) := \frac{4-2p}{6\sqrt{2\pi}\,\sigma} - |F_{n,p}(m) - G_{n,p}(m)|$ with $m = \lfloor np \rfloor$ in more detail and see in Figure 7, here for $n = 10$, that in each interval $[\frac{m}{n}, \frac{m+1}{n}]$ we have to prove that a convex and non-monotonic function with a minimum difficult to localize remains positive.

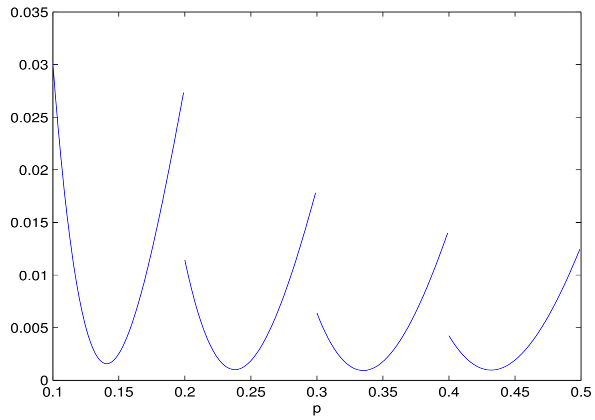

FIGURE 7. Graph of $p \mapsto \mathcal{D}(p)$ in the case $n = 10$.

We handle this problem for $p \in [\frac{1}{3}, \frac{1}{2}]$ in a number of steps. First we estimate downwards the distance at the base points $p = \frac{k+1}{n}$ for $k+1 \in \{\frac{n}{3}, ..., \lfloor \frac{n}{2} \rfloor\}$, at these points the distance is much easier to handle than at the other points, since here we have $np = k+1 \in \mathbb{N}$ and hence $G_{n,p}(k+1) = \frac{1}{2}$. Next we estimate the difference $\mathcal{D}(p) - \mathcal{D}(\frac{k+1}{n})$ for $p \in [\frac{k}{n}, \frac{k+1}{n}]$, in words, we observe how far the arcs lean in left direction downwards. From Figure 7 one might guess that the lowest point of an arc will be attained in the first half of the interval $[\frac{k}{n}, \frac{k+1}{n}]$. For this reason we firstly consider the left half of the interval and show that the arc leans less downwards than the lower bound associated to the right base point. The estimation for the right half of the interval can be reduced easily to the left half in the next step, which might be guessed from Figure 7.

In Figure 7 we can also see the problem in the case $p < \frac{1}{3}$. Not only the values of the difference at the base points but also the depth of the arc becomes significant larger and hence we need better estimations to accomplish out goal. Since this would apparently require enormously long computations, if successful at all, we use the buffer already mentioned above for those smaller p, which results from the worsening to the larger upper bound $c_{\mathrm{E}} \cdot \frac{p^2+q^2}{\sqrt{2\pi}\,\sigma}$ instead of $\frac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}$.

Finally we want to mention that the case $m = \frac{n-1}{2}$ is often considered separately as a special case, because then the base $\frac{m+1}{n}$ is no longer less than $\frac{1}{2}$ and we need to perform another kind of computation. Moreover n and $m+1$ must often have a minimum size in order that terms like $\frac{1}{n}$ or $\frac{1}{m+1}$ can be estimated reasonably well. This necessitates that finally we have to consider separately later a number of boundary cases excluded before.

Let us next consider the asymptotical behaviour of $F_{n,p}(x)$ for $n \to \infty$, which was already mentioned several times, in more detail, by using the Edgeworth expansion in the Bernoulli case. Let again $X_1, \dots, X_n$ independent identically distributed with $X_1 \sim \mathrm{Ber}_p$ for $p \in (0,1)$ and hence $\tau^2 := \mathrm{Var} X_1 = pq > 0$ and let $\gamma_i := \kappa_i/\tau^i$, where κ_i is the cumulant of order i, for $i \in \mathbb{N}$. Thus X_1 has a lattice distribution with step size $h = 1$. The first four terms of an Edgeworth expansion of $F_{n,p}(x) = \mathbb{P}(\sum_{i=0}^{n} X_i \le x)$ at integer points x are

$$\begin{aligned}
g_0(z) &= \Phi(z),\\
g_1(z) &= -\big(\tfrac{\gamma_3}{6}H_2(z) - \tfrac{h}{2\tau}\big)\cdot\varphi(z),\\
g_2(z) &= -\big(\tfrac{\gamma_3^2}{72}H_5(z) + \tfrac{\gamma_4}{24}H_3(z) - \tfrac{\gamma_3 h}{12\tau}H_3(z) + \tfrac{h^2}{12\tau^2}H_1(z)\big)\cdot\varphi(z),\\
g_3(z) &= -\big(\tfrac{\gamma_3^3 H_8(z)}{1296} + \tfrac{\gamma_3\gamma_4 H_6(z)}{144} + \tfrac{\gamma_5 H_4(z)}{120} - \tfrac{\gamma_3^2 h H_6(z)}{144\tau} - \tfrac{\gamma_4 h H_4(z)}{48\tau} + \tfrac{\gamma_3 h^2 H_4(z)}{72\tau^2}\big)\cdot\varphi(z),
\end{aligned}$$

where $z := \frac{x-np}{\sqrt{n}\cdot\tau}$ and $H_i(x)$ denotes the ith Hermite polynomial and $\varphi(t) = \frac{1}{\sqrt{2\pi}}\cdot \mathrm{e}^{-t^2/2}$ as supplied before the density of the standard normal distribution. This expansion is of interest to us, because in the present work we consider essentially the error term $|F_{n,p}(x) - g_0(\frac{x-np}{\sqrt{n}\tau})|$. It is well known that, extending the calculations for example of Gnedenko/Kolmogorov (1968, p.213) as in Esseen (1945, p.61), one can prove that we have

$$F_{n,p}(x) = \sum_{i=0}^{r} g_i\big(\tfrac{x-np}{\sqrt{n}\tau}\big)\cdot n^{-i/2} + o(n^{-r/2}) \qquad r = 0,1,2,3$$

for $n \to \infty$ uniformly in $x \in \mathbb{Z}$. If we set $m_n := \lfloor np \rfloor$ and $\delta_n := np - m_n \in [0,1)$ we obtain with $\gamma_3 = (1-2p)/\tau$, $\gamma_4 = (1-6p+6p^2)/\tau^2$ and $\gamma_5 = (1-14p+36p^2-24p^3)/\tau^3$

$$\begin{aligned}
F_{n,p}(m_n) &= \Phi\big(\tfrac{-\delta_n}{\tau\sqrt{n}}\big) - \tfrac{1}{\sqrt{2\pi n}}\cdot\big(\tfrac{1-2p}{6\tau}\big(\tfrac{\delta_n^2}{n\tau^2}-1\big) - \tfrac{1}{2\tau}\big)\cdot\big(1-\tfrac{\delta_n^2}{2n\tau^2}\big) + o(n^{-\frac{3}{2}})\\
&\quad - \tfrac{1}{\sqrt{2\pi n}}\cdot\big(-\tfrac{5(1-2p)^2}{24\tau^2}\cdot\tfrac{\delta_n}{n\tau} + \tfrac{1-6p+6p^2}{8\tau^2}\cdot\tfrac{\delta_n}{n\tau} - \tfrac{1-2p}{4\tau^2}\cdot\tfrac{\delta_n}{n\tau} - \tfrac{1}{12\tau^2}\cdot\tfrac{\delta_n}{n\tau}\big) - \tfrac{1}{\sqrt{2\pi}\,n^{3/2}}\cdot\\
&\quad \big(\tfrac{35(1-2p)^3}{432\tau^3} - \tfrac{5(1-2p)(1-6p+6p^2)}{48\tau^3} + \tfrac{1-14p+36p^2-24p^3}{40\tau^3} + \tfrac{5(1-2p)^2}{48\tau^3} - \tfrac{1-6p+6p^2}{16\tau^3} + \tfrac{1-2p}{24\tau^3}\big)\\
&= \Phi\big(\tfrac{-\delta_n}{\tau\sqrt{n}}\big) + \tfrac{1}{\sqrt{2\pi n}}\cdot\big(\tfrac{4-2p}{6\tau} - \tfrac{1-p}{2n\tau^3}\cdot\delta_n^2\big) + \tfrac{\delta_n}{\sqrt{2\pi n^3}\,\tau^3}\cdot\tfrac{5-7p+p^2}{12}\\
&\quad - \tfrac{\delta_n^2}{\sqrt{2\pi n^3}\,\tau^3}\cdot\tfrac{46-69p+21p^2+p^3}{540} + o(n^{-\frac{3}{2}})\\
&= \Phi\big(\tfrac{-\delta_n}{\tau\sqrt{n}}\big) + \tfrac{4-2p}{6\sqrt{2\pi n}\,\tau} + \tfrac{1}{\sqrt{2\pi n^3}\,\tau^3}\cdot h(\delta_n) + o(n^{-\frac{3}{2}})
\end{aligned}$$

for $n \to \infty$ where $h(t) := -\frac{1-p}{2}\cdot t^2 + \frac{5-7p+p^2}{12}\cdot t - \frac{46-69p+21p^2+p^3}{540}$ for $t \in \mathbb{R}$.

Unfortunately we cannot use this expansion in the proof of Theorem 1, because it only yields an asymptotic statement. But we can further consider now for which $p \in (0,1)$ the inequality $F_{n,p}(m_n) - \Phi(\frac{m_n-np}{\sqrt{n}\tau}) \le \frac{4-2p}{6\sqrt{2\pi}\sqrt{n}\tau}$ cannot hold for all $n \in \mathbb{N}$. We now assume that p is irrational such that there exists a $d \in [0,1)$ with $h(d) \ge 0$. Then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ with $\delta_{n_k} = n_k p - \lfloor n_k p \rfloor \to d$ and then we get that $F_{n,p}(m_n) - \Phi(\frac{m_n-np}{\sqrt{n}\tau}) > \frac{4-2p}{6\sqrt{2\pi}\sqrt{n}\tau}$ holds for some $n \in \mathbb{N}$ since $h(d) = \lim_{k\to\infty} h(\delta_{n_k}) > 0$. We now consider for which p such a $d \in [0,1)$ exists.

We obtain by differentiation that the concave function h becomes maximal at the point $d_0 := \frac{5-7p+p^2}{12(1-p)} = \frac{5}{12} - \frac{p}{6} - \frac{p^2}{12(1-p)} \in (-\infty, 1)$. Further we have $d_0 \geq 0$ iff $p \leq p_0 := \frac{7-\sqrt{29}}{2} = 0.8074...$ holds. Hence for $p \geq p_0$ we have $\max_{x\in[0,1)} h(x) = h(0)$ and we obtain that $\max_{x\in[0,1)} h(x) > 0$ if $p < 0.058$ and if $p > 0.96$, see also Figure 8.

This explains most of Remark 4.3.

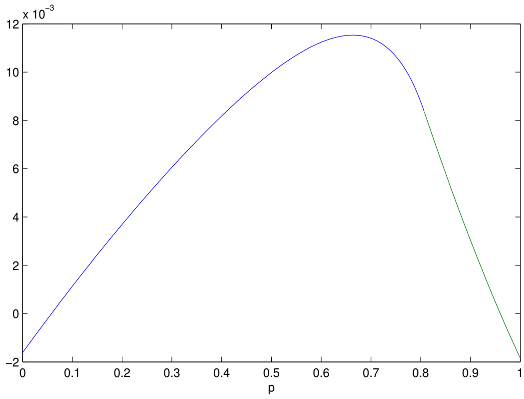

FIGURE 8. Graph of $p \mapsto \max_{x\in[0,1)} h(x)$.

At this point we want to consider a specific counterexample to inequality (5) on page 15 in case of $p \in (0, \frac{1}{3})$. Let now $n \in \mathbb{N}$ with $n \geq 7$ and hence $p(n) := \frac{6}{n} + \frac{5}{12n} \in (0,1)$ and we denote, $m_{n,p} := \lfloor np \rfloor$ and

$$\mathcal{D}_2(n,p) := F_{n,p}(m_{n,p}) - G_{n,p}(m_{n,p}) - \frac{4-2p}{6\sqrt{2\pi np(1-p)}}.$$

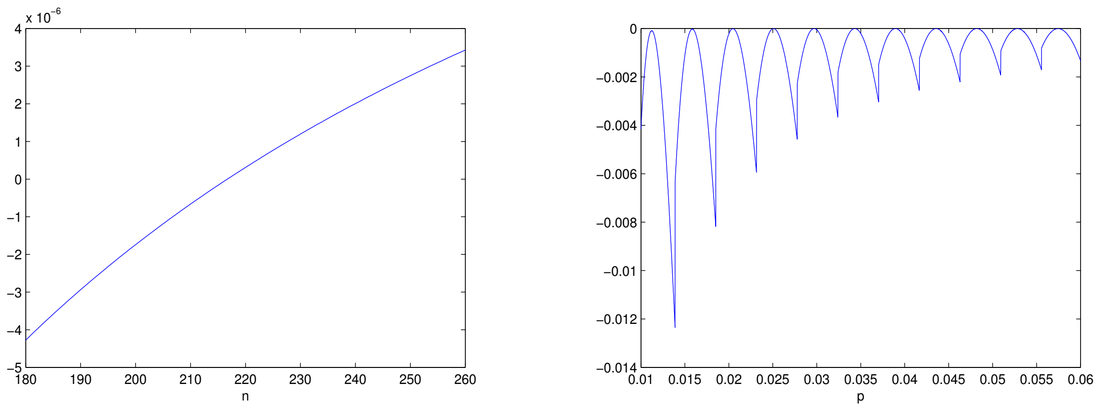

FIGURE 9. Graph of $n \mapsto \mathcal{D}_2(n, p(n))$ on the left and $p \mapsto \mathcal{D}_2(216, p)$ on the right hand side.

Then we find in Figure 9 on the left that from $n = 217$ upwards already $\mathcal{D}_2(n, p(n)) > 0$ holds, where here we have $p(217) = \frac{6+5/12}{217} \approx 0.0296$. Specifically, we have $\mathcal{D}_2(217, p(217)) \approx 2.97 \cdot 10^{-8} > 0$ and the right part of Figure 9 might suggest that $n = 217$ is the smallest n for which $\mathcal{D}_2(n, p) > 0$ can hold, since if $n = 216$ we apparently have $\mathcal{D}_2(n, p) < 0$ on the range, critical due to the Edgeworth expansion, $p \in [0, 0.058]$.

One may also put the Edgeworth expansion in relation to Lemma 11.9, where for $a, n \in \mathbb{N}$ with $\alpha := a/n \in [\frac{1}{6}, \frac{1}{2}]$ and $\sigma_\alpha^2 = n\alpha(1-\alpha)$ it was shown that

$$\frac{4-2\alpha}{6} - \frac{1-2\alpha}{18\sigma_\alpha^2} \;\leq\; \frac{F_{n,\alpha}(a)-1/2}{f_{n,\alpha}(a)} \;\leq\; \frac{4-2\alpha}{6} - \frac{1-2\alpha}{72\sigma_\alpha^2}.$$

Let now $\alpha \in [\frac{1}{6}, \frac{1}{2}]$ rational, then there exists a sequence $(n_k)_{k\in\mathbb{N}}$ with $a_{n_k} := n_k\alpha \in \mathbb{N}$ and hence $\frac{a_{n_k}}{n_k} = \alpha$ and $\delta_{n_k} := n_k\alpha - a_{n_k} = 0$ for all $k \in \mathbb{N}$. Let further $\tau_\alpha := \sqrt{\alpha(1-\alpha)}$. If we consider the expansion from page 25 above at the points a_{n_k} and with $p = \alpha$ there, we obtain

$$\begin{aligned}
F_{n_k,\alpha}(a_{n_k}) - \tfrac{1}{2} &= \frac{4-2\alpha}{6\sqrt{2\pi}\sqrt{n_k}\tau_\alpha} - \frac{46-69\alpha+21\alpha^2+\alpha^3}{540\sqrt{2\pi}\sqrt{n}^3\tau_\alpha^3} + o(n_k^{-3/2}) \\
&= f_{n_k,\alpha}(a_{n_k}) \cdot \Big(\frac{4-2\alpha}{6} \cdot \big(1 + \frac{1}{12n_k\tau_\alpha^2} - \frac{1}{12n_k}\big) - \frac{46-69\alpha+21\alpha^2+\alpha^3}{540n_k\tau_\alpha^2}\Big) + o(n_k^{-3/2}) \\
&= f_{n_k,\alpha}(a_{n_k}) \cdot \Big(\frac{4-2\alpha}{6} - \frac{(1-2\alpha)\cdot(4+2\alpha(1-\alpha))}{135n_k\tau_\alpha^2}\Big) + o(n_k^{-3/2})
\end{aligned}$$

for $n_k \to \infty$ since $\frac{1}{\sqrt{2\pi}\sqrt{n_k}\tau_\alpha} = f_{n_k,\alpha}(a_{n_k}) \cdot \mathrm{e}^{\frac{1}{12n_k\tau_\alpha^2} - \frac{1}{12n_k}} + o(n_k^{-3/2})$ for $n_k \to \infty$ due to Lemma 11.4 and of course we have

$$-\frac{1}{18} \;<\; -\frac{4+2\alpha(1-\alpha)}{135} \;<\; -\frac{1}{72}.$$

Thus one may conjecture that the estimations in Lemma 11.9 can be refined on both sides. Because there are for each $n \in \mathbb{N}$ only a finite number of computations in that proof (only the p of the form a/n for $a \in \mathbb{N}$ with $\frac{n}{6} \leq a \leq \frac{n}{2}$) have to be considered, with the help of computers the accuracy of the inequality might be increased arbitrarily.

We hope that the present chapter serves to explain the proofs following in the next chapters.

6. Proof of Theorem 1

Let us restate Theorem 1 for convenience:

Theorem 1. *Let $p \in (0,1)$, $n \in \mathbb{N}$ and let denote $F_{n,p}$ the distribution function of the binomial distribution with parameters n and p. Then we have*

$$\sup_{x\in\mathbb{R}} |F_{n,p}(x) - \Phi\left(\tfrac{x-np}{\sqrt{npq}}\right)| \;<\; \tfrac{\sqrt{10}+3}{6\sqrt{2\pi}} \cdot \tfrac{p^2+q^2}{\sqrt{npq}}. \tag{4}$$

In the case $p \in [\frac{1}{3}, \frac{2}{3}]$ we have the sharper inequality

$$\sup_{x\in\mathbb{R}} |F_{n,p}(x) - \Phi\left(\tfrac{x-np}{\sqrt{npq}}\right)| \;<\; \tfrac{3+|p-q|}{6\sqrt{2\pi}\sqrt{npq}}. \tag{5}$$

Proof. This proof uses results of Bentkus/Kirsa (1989) and Shevtsova (2013) and Propositions 6.1 and 6.2 stated here but proved below in chapters 7 and 8; these proofs in turn use lemmas proved using lemmas from chapter 11. Thus a logical, but somewhat unmotivated, order of reading in this work would be: First chapter 11, then Lemmas 7.2 - 7.4, then the boundary cases in chapter 9, then Proposition 6.1, then Lemmas 8.1 - 8.6, then Proposition 6.2 and finally the proof of Theorem 1.

Beforehand we have due to Bentkus/Kirsa (1989) for $n = 1$ even

$$\sup_{x\in\mathbb{R}} |F_{n,p}(x) - \Phi\left(\tfrac{x-np}{\sqrt{npq}}\right)| \;\le\; 0.3704 \cdot \tfrac{p^2+q^2}{\sigma} \;<\; \tfrac{p^2+q^2}{\sqrt{2\pi}\,\sigma}$$

and with $p^2 + q^2 \le \frac{3+|p-q|}{6}$ for $p \in [\frac{1}{3}, \frac{2}{3}]$ we also have the stronger inequality (5), thus we can assume below always $n \ge 2$, if convenient.

We divide the proof into the following two cases with respect to p:

1.) $p \in \left(0, \frac{1}{6}\right] \cup \left[\frac{5}{6}, 1\right)$

2.) $p \in \left[\frac{1}{6}, \frac{5}{6}\right]$

We start with 1.), so let $p \in (0, \frac{1}{6}] \cup [\frac{5}{6}, 1)$. Let us further put $\varrho(p) := \varrho(\mathrm{Ber}_p) = \frac{p^2+(1-p)^2}{\sqrt{p(1-p)}}$. Then we have $\varrho'(p) = \frac{(2p-2(1-p))}{\sqrt{p(1-p)}} - \frac{(1-2p)\cdot(p^2+(1-p)^2)}{2(p(1-p))^{3/2}} = \frac{2p-1}{(p(1-p))^{3/2}} \cdot (2p(1-p) + \frac{p^2+(1-p)^2}{2})$ and consequently for $p \in (0, \frac{1}{6}]$, we have $\varrho'(p) \le 0$ and thus $\varrho(p) \ge \varrho(\frac{1}{6}) = \frac{1/36+25/36}{\sqrt{5/36}} = \frac{13}{3\sqrt{5}}$, and for $p \in [\frac{5}{6}, 1)$, we have $\varrho'(p) \ge 0$ and thus as well $\varrho(p) \ge \varrho(\frac{5}{6}) = \varrho(\frac{1}{6})$.

For $\varrho(p) > 1.93$, which holds for $p \in (0, \frac{1}{6}] \cup [\frac{5}{6}, 1)$ by the above, we have due to Shevtsova (2013)

$$\sqrt{n} \cdot \sup_{x\in\mathbb{R}} |F_{n,p}(x) - \Phi\left(\tfrac{x-np}{\sigma}\right)| \;\leq\; 0.3031 \cdot (1 + \tfrac{0.646}{\varrho(p)}) \cdot \varrho(p) \;\leq\; 0.4045... \cdot \varrho(p),$$

and hence (4). Thus for these boundary cases of p the general estimation due to Shevtsova is sufficient, as was already discussed in section 5 in more detail.

Now we consider 2.), so let $p \in [\frac{1}{6}, \frac{5}{6}]$. Here we show firstly that the distance $\Delta(x) = |F_{n,p}(x) - \Phi\left(\frac{x-np}{\sqrt{npq}}\right)|$ becomes maximal around the mean np or satisfies a sharpening of the desired inequality directly:

Proposition 6.1. *Let $\frac{1}{6} \leq p \leq \frac{5}{6}$ and $n \in \mathbb{N}$. Then we have*

$$\sup_{x\in\mathbb{R}} \Delta(x) \;\leq\; \max\Big\{\Delta(\lfloor np\rfloor),\ \Delta(\lfloor np\rfloor-),\ \Delta(\lceil np\rceil),\ \Delta(\lceil np\rceil-),\ (\tfrac{2}{\sqrt{2\pi}\,c_{\mathrm{E}}} - 1)\cdot \tfrac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}\Big\}.$$

Here $(\frac{2}{\sqrt{2\pi}\,c_{\mathrm{E}}} - 1) = 0.9473... < 1$ and we prove here this sharper upper bound, because we use it in the proof of Theorem 2 in section 10 and furthermore to exclude the case $\Delta(x) = \frac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}$ in order to get with proposition 6.2 below strict inequalities in (4) and (5).

Secondly we estimate the remaining four critical distances around the mean:

Proposition 6.2. *Let $\frac{1}{6} \leq p \leq \frac{5}{6}$ and $n \in \mathbb{N}$. Then we have*

$$\max\Big\{\Delta(\lfloor np\rfloor),\ \Delta(\lfloor np\rfloor-),\ \Delta(\lceil np\rceil),\ \Delta(\lceil np\rceil-)\Big\} \;<\; \tfrac{p^2+q^2}{\sqrt{2\pi}\,\sigma}.$$

In the case $\frac{1}{3} \leq p \leq \frac{2}{3}$ we even have

$$\max\Big\{\Delta(\lfloor np\rfloor),\ \Delta(\lfloor np\rfloor-),\ \Delta(\lceil np\rceil),\ \Delta(\lceil np\rceil-)\Big\} \;<\; \tfrac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}.$$

□

In the following two chapters Proposition 6.1 and Proposition 6.2 are proved.

7. Proof of Proposition 6.1

Let us restate Proposition 6.1 for convenience:

Proposition 6.1. *Let* $\frac{1}{6} \leq p \leq \frac{5}{6}$ *and* $n \in \mathbb{N}$*. Then we have*

$$\sup_{x\in\mathbb{R}} \Delta(x) \quad \leq \quad \max\Big\{\Delta(\lfloor np\rfloor),\ \Delta(\lfloor np\rfloor-),\ \Delta(\lceil np\rceil),\ \Delta(\lceil np\rceil-),\ (\tfrac{2}{\sqrt{2\pi}\,c_{\mathrm{E}}}-1)\cdot\tfrac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}\Big\}.$$

We recall the abbreviation $\sigma := \sqrt{npq}$.

Remark 7.1. From Lemmas 7.2 and 7.3 in the proof of Proposition 6.1 below it further follows that for $n \geq 6$, $\frac{1}{6} \leq p \leq \frac{5}{6}$ and $k \geq \lceil np\rceil$ we have

$$F(k)-G(k) \quad \leq \quad F(\lceil np\rceil)-G(\lceil np\rceil) \quad , \quad \text{if } \begin{cases} k \leq np+\frac{3}{2}\sigma \ ,\ p \leq \frac{1}{2} \\ p \geq \frac{1}{2} \end{cases}$$

$$G(k)-F(k-1) \quad \leq \quad G(\lceil np\rceil)-F(\lceil np\rceil-1) \quad , \quad \text{if } \begin{cases} k \leq np+\frac{3}{2}\sigma+1 \ ,\ p \leq \frac{1}{2} \\ k \leq np+\sqrt{\frac{3}{2}}\,\sigma+1 \ ,\ \sigma \geq \frac{3}{2},\ p \geq \frac{1}{2} \\ k \leq np+\frac{8}{9}\sigma+1 \ ,\ \sigma < \frac{3}{2},\ p \geq \frac{1}{2}. \end{cases}$$

Proof of Proposition 6.1. We use Lemma 7.2 - Lemma 7.4, stated here but proved below on page 36 - 51 and for the boundary cases $n \leq 5$ Lemma 7.5 stated on page 34 and proved in chapter 9 on page 81 - 84 because in this way the structure of the present proof becomes clearer.

Let us denote the right hand side of the claimed inequality by A. Since we consider the difference between a continuous and a discrete distribution with support $\{0, ..., n\}$, it is enough to prove

$$\max\big\{\Delta(k), \Delta(k-)\big\} \quad \leq \quad A \qquad \text{for} \quad k \in \{0, ..., n\}. \tag{6}$$

We now prove the claim by comparing first the densities of the distributions not far from the mean. We always assume $k \in \mathbb{N}_0$ and $n \in \mathbb{N}$ in this chapter. For the further proof we need the following three Lemmas.

Lemma 7.2. *If* $p \leq \frac{1}{2}$*, then we have*

$$f(k) \quad \leq \quad g(k) \qquad \textit{if} \qquad np+1 \leq k \leq np+\tfrac{3}{2}\cdot\sqrt{npq}$$

provided that $npq \geq \frac{5}{6}$*, and hence in particular if* $\frac{1}{6} \leq p \leq \frac{1}{2}$ *and* $n \geq 6$.

If $\frac{1}{2} \leq p \leq \frac{5}{6}$ *and* $n \geq 6$*, we have*

$$f(k) \quad \leq \quad g(k) \qquad \textit{if} \qquad np+1 \leq k. \tag{7}$$

Lemma 7.3. *Let* $\frac{1}{6} \le p \le \frac{1}{2}$ *and* $n \ge 6$*. Then* $\sqrt{npq} \ge \sqrt{5/6}$ *and we have*

$$g(k+1) \;\le\; f(k) \qquad \text{if} \qquad np \le k \le np + \tfrac{3}{2} \cdot \sqrt{npq}.$$

For $\frac{1}{2} \le p \le \frac{5}{6}$ *and* $n \ge 6$*, we have in the case* $\sqrt{npq} \ge 2$

$$g(k+1) \;\le\; f(k) \qquad \text{if} \qquad np \le k \le np + \sqrt{\tfrac{3}{2} npq}$$

and in the case $\sqrt{5/6} \le \sqrt{npq} < 2$

$$g(k+1) \;\le\; f(k) \qquad \text{if} \qquad np \le k \le np + \sqrt{npq}.$$

Lemma 7.4. *Let* $n, k \in \mathbb{N}$ *and* $p \in [\frac{1}{6}, \frac{1}{2}]$*. Then if* $k > np + \frac{3}{2} \cdot \sqrt{npq}$ *we have*

$$|F(k) - G(k)| \;\le\; c_0 \cdot \tfrac{3+|p-q|}{6\sqrt{2\pi}\,\sigma} \quad , \quad |G(k+1) - F(k)| \;\le\; c_0 \cdot \tfrac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}$$

with $c_0 := \frac{2}{\sqrt{2\pi}\, c_{\mathrm{E}}} - 1 = 0.9473... < 1$.

The above Lemmas will be applied to

$$(8) \qquad F(k) - G(k) \;=\; F(\lceil np \rceil) - G(\lceil np \rceil) + \sum_{j=\lceil np \rceil + 1}^{k} (f(j) - g(j)) \quad \text{for } k \ge np$$

$$(9) \quad G(k) - F(k-1) \;=\; G(\lceil np \rceil) - F(\lceil np \rceil - 1) + \sum_{j=\lceil np \rceil}^{k-1} (g(j+1) - f(j)) \quad \text{for } k \ge np$$

$$(10) \qquad F(k) - G(k) \;=\; \sum_{j=k+1}^{\infty} (g(j) - f(j)) \quad \text{for } k \in \mathbb{N}_0$$

where in (10) we use $F(k) = 1 - \sum_{j=k+1}^{\infty} f(j)$ and $G(k) = 1 - \sum_{j=k+1}^{\infty} g(j)$.

We firstly prove inequality (6) in case of $n \ge 6$, $p \le \frac{1}{2}$ and $k \ge np$.

In the case of $np \le k \le np + \frac{3}{2}\sigma + 1$ Lemma 7.3 applied to (9) then yields

$$(11) \qquad G(k) - F(k-1) \;\le\; G(\lceil np \rceil) - F(\lceil np \rceil - 1)$$

and in case of $np + \frac{3}{2}\sigma + 1 < k$ Lemma 7.4 yields $G(k) - F(k-1) \le (\frac{2}{\sqrt{2\pi}\, c_{\mathrm{E}}} - 1) \cdot \frac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}$ and hence we have $G(k) - F(k-1) \le A$ for $k \ge np$.

Further in the case of $np \le k \le np + \frac{3}{2}\sigma$ Lemma 7.2 applied to (8) yields

$$F(k) - G(k) \;\; \le \;\; F(\lceil np \rceil) - G(\lceil np \rceil),$$

and in the case of $np + \frac{3}{2}\sigma < k$ Lemma 7.4 yields $F(k) - G(k) \le (\frac{2}{\sqrt{2\pi}\, c_{\mathrm{E}}} - 1) \cdot \frac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}$ and hence we also have $F(k) - G(k) \le A$ for $k \ge np$.

Thus with the relations $G(k)-F(k) \le G(k)-F(k-1)$ and $F(k-1)-G(k) \le F(k)-G(k)$, inequality (6) is proved in the present case.

Secondly we consider $p > \frac{1}{2}$ and again $n \ge 6$ and $k \ge np$. Due to (7) applied to (10) we get $F(k) - G(k) \ge 0$ and hence $|F(k) - G(k)| = F(k) - G(k) \le F(\lceil np \rceil) - G(\lceil np \rceil)$, where in the last inequality we used (7) applied to (8) .

A bit more difficult will be the part $|G(k) - F(k-1)|$ for $k \ge np$. If $F(k-1) > G(k)$ we have by the above $|G(k) - F(k-1)| = F(k-1) - G(k) \le F(k) - G(k) \le F(\lceil np \rceil) - G(\lceil np \rceil)$, thus we consider $G(k) - F(k-1)$ without absolute value.

1.) Let $\sigma < 2$. In this case Lemma 7.3 yields $G(k) - F(k-1) \le G(\lceil np \rceil) - F(\lceil np \rceil - 1)$ analogous to the computations in (11) only for $np \le k \le np + \sigma + 1$. For $k > np + \sigma + 1$ we now have with $F(k-1) \ge G(k-1)$ and $\frac{1}{2} \cdot (\frac{2}{\sqrt{2\pi}\, c_{\mathrm{E}}} - 1) = 0.4737...$

$$
\begin{aligned}
(12) \quad G(k) - F(k-1) \;\; &\le \;\; G(k) - G(k-1) \;\; = \;\; \Phi(\tfrac{k-np}{\sigma}) - \Phi(\tfrac{k-1-np}{\sigma}) \\
&\le \;\; \Phi(1 + \tfrac{1}{\sigma}) - \Phi(1) \;\; = \;\; \big(\Phi(1 + \tfrac{1}{\sigma}) - \Phi(1)\big) \cdot \sigma\sqrt{2\pi} \cdot \tfrac{1}{\sqrt{2\pi}\,\sigma} \\
&\le \;\; \big(\Phi(1 + \tfrac{1}{2}) - \Phi(1)\big) \cdot 2\sqrt{2\pi} \cdot \tfrac{1}{\sqrt{2\pi}\,\sigma} = 0.4605... \cdot \tfrac{1}{\sqrt{2\pi}\,\sigma} \\
&\le \;\; (\tfrac{2}{\sqrt{2\pi}\, c_{\mathrm{E}}} - 1) \cdot \tfrac{1}{2\sqrt{2\pi}\,\sigma} \;\; \le \;\; (\tfrac{2}{\sqrt{2\pi}\, c_{\mathrm{E}}} - 1) \cdot \tfrac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}.
\end{aligned}
$$

Here we used in the second and third inequality the monotonicity properties of the difference quotients of the concave function $\Phi|_{[0,\infty)}$.

2.) Let $\sigma > 2$. In this case Lemma 7.3 yields $G(k) - F(k-1) \le G(\lceil np \rceil) - F(\lceil np \rceil - 1)$ analogous to the computations in (11) for $np \le k \le np + \sqrt{3/2}\,\sigma + 1$. For $k > np + \sqrt{3/2}\,\sigma + 1$ we have

$$
\begin{aligned}
G(k) - F(k-1) \;\; &\le \;\; G(k) - G(k-1) \;\; \le \;\; \tfrac{1}{\sigma} \cdot \varphi(\tfrac{k-1-np}{\sigma}) \;\; \le \;\; \tfrac{1}{\sigma} \cdot \varphi(\sqrt{\tfrac{3}{2}}) \;\; = \;\; \tfrac{0.4724...}{\sqrt{2\pi}\,\sigma} \\
&\le \;\; (\tfrac{2}{\sqrt{2\pi}\, c_{\mathrm{E}}} - 1) \cdot \tfrac{1}{2\sqrt{2\pi}\,\sigma} \;\; \le \;\; (\tfrac{2}{\sqrt{2\pi}\, c_{\mathrm{E}}} - 1) \cdot \tfrac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}.
\end{aligned}
$$

Thus we get inequality (6) also in the case $p > \frac{1}{2}$, $n \ge 6$ and $k \ge np$.

Thirdly we consider the case $k \geq np$ and this time $n \leq 5$. Here we use the following Lemma, which is proved, as already mentioned, in chapter 9 on page 81 - 84.

Lemma 7.5. *Let* $n \in \mathbb{N},\ n \leq 5$. *If* $p \in [\frac{1}{2}, \frac{5}{6}]$, *then*

$$\begin{aligned} f(k) &\leq g(k) \quad \text{if} \quad np+1 \leq k \\ g(k+1) &\leq f(k) \quad \text{if} \quad np \leq k \leq np + \tfrac{8}{9} \cdot \sqrt{npq} \end{aligned}$$

and if $p \in [\frac{1}{6}, \frac{1}{2})$, *then*

$$\begin{aligned} f(k) &\leq g(k) \quad \text{if} \quad np+1 \leq k \leq np + npq \\ g(k+1) &\leq f(k) \quad \text{if} \quad np \leq k \leq np + npq \end{aligned}$$

and if $p \in [\frac{1}{6}, \frac{1}{2}]$*and* $k > np + npq$, *then with* $c_0 := \frac{2}{\sqrt{2\pi}\, c_{\mathrm{E}}} - 1$

$$|F(k) - G(k)| \leq c_0 \cdot \tfrac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}, \qquad |G(k+1) - F(k)| \leq c_0 \cdot \tfrac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}.$$

Due to this Lemma inequality (6) also holds for $n \leq 5$ and $k \geq np$ with almost the same argumentation like in the case $n \geq 6$ above, with the only difference that in case of $p \geq \frac{1}{2}$ we have $g(k+1) - f(k) \leq 0$ only for $k \leq np + \frac{8}{9} \cdot \sigma$. But here we have $\sigma \leq \sqrt{5/4}$ since $n \leq 5$ and hence we get for $k > np + \frac{8}{9}\sigma + 1$ analogous to the computation starting in (12) above

$$\begin{aligned} G(k) - F(k-1) &\leq \left(\Phi(\tfrac{8}{9} + \tfrac{1}{\sigma}) - \Phi(\tfrac{8}{9})\right) \cdot \sigma\sqrt{2\pi} \cdot \tfrac{1}{\sqrt{2\pi}\,\sigma} \\ &\leq \left(\Phi(\tfrac{8}{9} + \sqrt{4/5}) - \Phi(\tfrac{8}{9})\right) \cdot \sqrt{5/4}\sqrt{2\pi} \cdot \tfrac{1}{\sqrt{2\pi}\,\sigma} \\ &= 0.4197... \cdot \tfrac{1}{\sqrt{2\pi}\,\sigma} \leq (\tfrac{2}{\sqrt{2\pi}\,c_{\mathrm{E}}} - 1) \cdot \tfrac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}. \end{aligned}$$

Finally for $k \leq np$ inequality (6) now follows from

$$\begin{aligned} |F_{n,p}(k) - G_{n,p}(k)| &= |F_{n,q}(n-k-1) - G_{n,q}(n-k)| \\ |G_{n,p}(k) - F_{n,p}(k-1)| &= |F_{n,q}(n-k) - G_{n,q}(n-k)| \end{aligned}$$

with $n-k \geq nq$, which was already considered above and hence inequalitiy (6) is proved for all $k \in \{0, ..., n\}$. □

Remark 7.6. By numerical computations one might conjecture that the distance becomes maximal around the mean in each case and hence that in Proposition 6.1 the term $(\frac{2}{\sqrt{2\pi}\,c_{\mathrm{E}}}-1)\cdot\frac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}$ can be omitted and that it can be expanded to each $p\in[0,1]$. However statement (7) in Lemma 7.2 does not hold in case of $k=\lceil np\rceil+1$ for each $p\in(0,1)$ and $n\in\mathbb{N}$ and thus neither $F(k)-G(k)\leq F(\lceil np\rceil)-G(\lceil np\rceil)$ for all $k>\lceil np\rceil$. This follows from the Edgeworth expansion of $f(k)$, which we now consider below. Let p irrational, then there exists a sequence $(n_j)_{j\in\mathbb{N}}$, such that

$$z_{n_j} \;:=\; k_{n_j}-n_jp \;:=\; \lceil n_jp\rceil+1-n_jp \;\to\; 1 \;\text{ for }\; j\to\infty$$

since $\{\lceil np\rceil+1-np : n\in\mathbb{N}\}$ is dense in $[1,2]$. With $\varsigma_n:=\sqrt{npq}$ we get due to the Edgeworth expansion

$$\begin{aligned} f_{n_j,p}(k_{n_j})-g_{n_j,p}(k_{n_j}) \;&=\; \varphi(\tfrac{z_{n_j}}{\varsigma_{n_j}})\cdot\big(\tfrac{1}{\varsigma_{n_j}}-\tfrac{(1-2p)z_{n_j}}{2\varsigma_{n_j}^3}-\tfrac{5(1-2p)^2}{24\varsigma_{n_j}^3}+\tfrac{1-6p+6p^2}{8\varsigma_{n_j}^3}\big)\\ &\quad -\varphi(\tfrac{z_{n_j}}{\varsigma_{n_j}})\cdot\big(\tfrac{1}{\varsigma_{n_j}}+\tfrac{z_{n_j}}{2\varsigma_{n_j}^3}-\tfrac{1}{6\varsigma_{n_j}^3}\big)+o(n_j^{-3/2})\\ &=\; \varphi(\tfrac{z_{n_j}}{\varsigma_{n_j}})\cdot\tfrac{1}{\varsigma_{n_j}^3}\cdot\big(\tfrac{1+p-p^2}{12}-(1-p)z_{n_j}\big)+o(n_j^{-3/2}). \end{aligned}$$

Thus we have

$$\begin{aligned} \lim_{j\to\infty}\varsigma_{n_j}^3\cdot\big(f_{n_j,p}(k_{n_j})-g_{n_j,p}(k_{n_j})\big) \;>\; 0 \quad &\Leftrightarrow\quad \tfrac{1+p-p^2}{12}-(1-p) \;>\; 0\\ &\Leftrightarrow\quad p \;>\; 0.90983... \end{aligned}$$

and consequently for each $p\in(0.90983...,1)$ irrational, there exists arbitrary large $n\in\mathbb{N}$ with $f_{n,p}(\lceil np\rceil+1)>g_{n,p}(\lceil np\rceil+1)$ and hence (7) in case of $k=\lceil np\rceil+1$ can not hold for all $n\in\mathbb{N}$.

However we have for $m\in\{\lfloor np\rfloor,\lceil np\rceil\}$ by the Edgeworth expansion

$$F(m)-G(m)-\big(G(m+1)-F(m)\big) \;=\; \tfrac{1-2p}{3\sqrt{2\pi}\,\sigma}+o(\tfrac{1}{n}).$$

Thus it is to be conjectured that for $p\gg\frac{1}{2}$ now $G(m+1)-F(m)>F(m)-G(m)$ holds and that the distance also here becomes maximal around the mean, because with the notations above we have

$$\begin{aligned} f_{n_j,p}(k_{n_j}-1)-g_{n_j,p}(k_{n_j}) \;&=\; \varphi(\tfrac{z_{n_j}}{\varsigma_{n_j}})\cdot\big(\tfrac{1}{\varsigma_{n_j}}-\tfrac{(1-2p)z_{n_j}}{2\varsigma_{n_j}^3}-\tfrac{5(1-2p)^2}{24\varsigma_{n_j}^3}+\tfrac{1-6p+6p^2}{8\varsigma_{n_j}^3}\big)\\ &\quad -\varphi(\tfrac{z_{n_j}}{\varsigma_{n_j}})\cdot\big(\tfrac{1}{\varsigma_{n_j}}-\tfrac{z_{n_j}}{2\varsigma_{n_j}^3}-\tfrac{1}{6\varsigma_{n_j}^3}\big)+o(n^{-3/2})\\ &=\; \varphi(\tfrac{z_{n_j}}{\varsigma_{n_j}})\cdot\tfrac{1}{\varsigma_{n_j}^3}\cdot\big(\tfrac{1-p+p^2}{12}+pz_{n_j}\big)+o(n^{-3/2}) \end{aligned}$$

and because of $\frac{1-p+p^2}{12}+pz_{n_j}>0$, for every $j\in\mathbb{N}$ one may conjecture that $f(k)\geq g(k+1)$ for each $p\in[0,1]$ and each $\lceil np\rceil\leq k\leq\lceil np\rceil+z$, with $z\geq 1$ not too large, holds and thus $G(k)-F(k-1)\leq G(\lceil np\rceil)-F(\lceil np\rceil-1)$ for each $k\geq\lceil np\rceil$.

We now prove Lemmas 7.2 - 7.4 used above, using lemmas from chapter 11:

Lemma 7.2. *If* $p \leq \frac{1}{2}$*, then we have*

$$f(k) \;\leq\; g(k) \qquad \text{if} \qquad np+1 \leq k \leq np + \tfrac{3}{2} \cdot \sqrt{npq} \tag{13}$$

provided that $npq \geq \frac{5}{6}$*, and hence in particular if* $\frac{1}{6} \leq p \leq \frac{1}{2}$ *and* $n \geq 6$*.*

If $\frac{1}{2} \leq p \leq \frac{5}{6}$ *and* $n \geq 6$*, we have*

$$f(k) \;\leq\; g(k) \qquad \text{if} \qquad np+1 \leq k. \tag{14}$$

Proof. Let $np+1 \leq k \leq n$ and $z := k - np \in [1, nq]$ with the conditions above. We consider first all $p \in (0, \frac{5}{6}]$, before we distinguish between the two cases $p \in (0, \frac{1}{2}]$ and $p \in [\frac{1}{2}, \frac{5}{6}]$. Then we have due to Lemma 11.2 in the first inequality

$$\begin{aligned} g(k) &= \Phi(\tfrac{z}{\sigma}) - \Phi(\tfrac{z-1}{\sigma}) \\ &= \Phi(\tfrac{z-1/2}{\sigma} + \tfrac{1}{2\sigma}) - \Phi(\tfrac{z-1/2}{\sigma} - \tfrac{1}{2\sigma}) \\ &\geq \tfrac{1}{\sigma} \cdot \tfrac{1}{\sqrt{2\pi}} \cdot \mathrm{e}^{-\frac{(z-1/2)^2}{2\sigma^2}} \cdot \mathrm{e}^{\frac{(z-1/2)^2/\sigma^2 - 1}{24\sigma^2} - \frac{(z-1/2)^4/\sigma^4}{960\sigma^4}} \\ &= \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot \mathrm{e}^{-\frac{z^2}{2\sigma^2} + \frac{z}{2\sigma^2} - \frac{1}{6\sigma^2}} \cdot \mathrm{e}^{\frac{(z-1/2)^2}{24\sigma^4} \cdot (1 - \frac{(z-1/2)^2}{40\sigma^4})} \\ &\geq \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot \mathrm{e}^{-\frac{z^2}{2\sigma^2} + \frac{z}{2\sigma^2} - \frac{1}{6\sigma^2}} \end{aligned}$$

because of $\frac{(z-1/2)^2}{\sigma^4} \leq \frac{9}{4\sigma^2} \leq 40$, if $p \leq \frac{1}{2}$, $z \leq \frac{3}{2}\sigma$ and $\frac{(z-1/2)^2}{\sigma^4} \leq \frac{n^2q^2}{\sigma^4} = \frac{1}{p^2} \leq 40$, if $p \geq \frac{1}{2}$ in the second inequality. Thus it follows in both cases

$$\log(g(k)) \;\geq\; \log(\tfrac{1}{\sqrt{2\pi}\,\sigma}) - \tfrac{z^2}{2\sigma^2} + \tfrac{z}{2\sigma^2} - \tfrac{1}{6\sigma^2}. \tag{15}$$

We further have in case of $k < n$ with Lemma 11.4 on page 100 and $\sigma^2_{k/n} = \frac{k(n-k)}{n}$

$$\begin{aligned} \log(f(k) \cdot \sqrt{2\pi}\,\sigma) &= \log(f_{n,\frac{k}{n}}(k)) \cdot \sqrt{2\pi}\,\sigma) + \log((\tfrac{np}{k})^k \cdot (\tfrac{nq}{n-k})^{n-k}) \\ &\leq \log(\tfrac{\sigma}{\sigma_{k/n}}) - \tfrac{19}{320\sigma^2_{k/n}} + \log((\tfrac{np}{k})^k \cdot (\tfrac{nq}{n-k})^{n-k}) \\ &= (k + \tfrac{1}{2}) \cdot \log(\tfrac{np}{k}) + (n - k + \tfrac{1}{2}) \cdot \log(\tfrac{nq}{n-k}) - \tfrac{19}{320\sigma^2_{k/n}} \\ &= -(np + z + \tfrac{1}{2}) \cdot \log(1 + \tfrac{z}{np}) - (nq - z + \tfrac{1}{2}) \cdot \log(1 - \tfrac{z}{nq}) - \tfrac{19}{320\sigma^2_{k/n}}. \end{aligned}$$

For $0 \le x_1 < x_2$ we have furthermore $(x_2 + x_1)\log(1 + \frac{x_1}{x_2}) = x_1 + \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r(r+1)} \cdot \frac{x_1^{r+1}}{x_2^r} \ge x_1 + \frac{x_1^2}{2x_2} - \frac{x_1^3}{6x_2^2}$ and $(x_2 - x_1)\log(1 - \frac{x_1}{x_2}) = -x_1 + \sum_{r=1}^{\infty} \frac{1}{r(r+1)} \cdot \frac{x_1^{r+1}}{x_2^r} \ge -x_1 + \frac{x_1^2}{2x_2} + \frac{x_1^3}{6x_2^2}$.

Thus with $z < nq$ since $k < n$ we have

$$\begin{aligned}(np+z)\log(1+\tfrac{z}{np}) + (nq-z)\log(1-\tfrac{z}{nq}) &\ge \tfrac{z^2}{2np} - \tfrac{z^3}{6n^2p^2} + \tfrac{z^2}{2nq} + \tfrac{z^3}{6n^2q^2} \\ &= \tfrac{z^2}{2\sigma^2} - \tfrac{z^3\cdot(q-p)}{6\sigma^4}\end{aligned}$$

and hence

$$\log(f(k)\cdot\sqrt{2\pi}\,\sigma) \le -\tfrac{z^2}{2\sigma^2} + \tfrac{z^3\cdot(q-p)}{6\sigma^4} - \tfrac{1}{2}\cdot(\log(1+\tfrac{z}{np}) + \log(1-\tfrac{z}{nq})) - \tfrac{19}{320\sigma_{k/n}^2}.$$

Together with (15) we have in particular

$$\text{(16)} \quad \log(\tfrac{g(k)}{f(k)}) \ge \tfrac{z}{2\sigma^2} - \tfrac{1}{6\sigma^2} + \tfrac{19}{320\sigma_{k/n}^2} - \tfrac{z^3\cdot(q-p)}{6\sigma^4} + \tfrac{1}{2}\cdot(\log(1+\tfrac{z}{np}) + \log(1-\tfrac{z}{nq})).$$

Further we have due to $\frac{1}{1+y} \ge 1 - y$ for $1 + y > 0$ in case of $z < nq$

$$\text{(17)} \quad \tfrac{1}{\sigma_{k/n}^2} = \tfrac{1}{\sigma^2}\cdot\tfrac{\sigma^2\cdot n}{(np+z)(nq-z)} = \tfrac{1}{\sigma^2}\cdot\tfrac{1}{1+z(q-p)/\sigma^2 - z^2pq/\sigma^4} \ge \tfrac{1}{\sigma^2}\cdot(1 - \tfrac{z(q-p)}{\sigma^2} + \tfrac{z^2pq}{\sigma^4}).$$

Inequalities (15), (16), (17) are used for $p \le \frac{1}{2}$ in part (a) as well as for $p \ge \frac{1}{2}$ in part (b).

(a) Let $p \le 0.5$:.

We show (14) in the case $p \in (0, \frac{1}{2}]$ and $\sigma^2 \ge \frac{5}{6}$. Since here $z \in [1, \frac{3}{2}\sqrt{npq}]$ we have $\frac{z}{nq} \le \frac{3\sigma}{2nq} = \frac{3p}{2\sigma} \le \frac{3\sqrt{6}}{4\sqrt{5}}$ implying

$$\begin{aligned}\log(1+\tfrac{z}{np}) + \log(1-\tfrac{z}{nq}) &\ge \tfrac{z(q-p)}{\sigma^2} - \tfrac{z^2(p^2+q^2)}{2\sigma^4} - \tfrac{z^3p^3}{3\sigma^6} - \tfrac{z^4p^4}{4\sigma^8} - \tfrac{z^5}{5(nq)^5}\cdot\sum_{r=0}^{\infty}(\tfrac{3\sqrt{6}}{4\sqrt{5}})^r \\ &\ge \tfrac{z(q-p)}{\sigma^2} - \tfrac{z^2(p^2+q^2)}{2\sigma^4} - \tfrac{z^3p^3}{3\sigma^6} - \tfrac{z^4p^4}{4\sigma^8} - \tfrac{9z^5p^5}{8\sigma^{10}}.\end{aligned}$$

Here we also use in the first step that $j(x) := \log(1+x) - (x - \frac{x^2}{2}) \ge 0$ for $x \ge 0$, since $j'(x) = \frac{x^2}{1+x} \ge 0$ and hence $j(x) \ge j(0) = 0$, and in the last step we use $1/(1 - \frac{3\sqrt{6}}{4\sqrt{5}}) = 5.6048... \le 5.625 = \frac{45}{8}$. We use above the auxiliary function j, because we do not know if $\frac{z}{np} < 1$ holds, while since $\frac{z}{nq} \le \frac{3\sqrt{6}}{4\sqrt{5}} < 1$ we can use the power series of $\log(1 - \frac{z}{nq})$.

Since $\frac{z}{nq} < 1$ we have $k = np + z < n$ and thus we have together with (16) and (17)

$$\begin{aligned}\log(\tfrac{g(k)}{f(k)}) &\geq \tfrac{z}{2\sigma^2} - \tfrac{1}{6\sigma^2} + \tfrac{19}{320\sigma^2} \cdot (1 - \tfrac{z(q-p)}{\sigma^2} + \tfrac{z^2 pq}{\sigma^4}) - \tfrac{z^3(q-p)}{6\sigma^4} \\ &\quad + \tfrac{1}{2} \cdot (\tfrac{z(q-p)}{\sigma^2} - \tfrac{z^2(p^2+q^2)}{2\sigma^4} - \tfrac{z^3p^3}{3\sigma^6} - \tfrac{z^4p^4}{4\sigma^8} - \tfrac{9z^5p^5}{8\sigma^{10}}) \\ &=: h(z).\end{aligned}$$

Obviously $\frac{\mathrm{d}^2 h}{dz^2} < 0$ and hence h minimal at the boundary, where we have with $\sigma^2 \geq 5/6$

$$\begin{aligned}h(1) &= \tfrac{1}{2\sigma^2} - \tfrac{1}{6\sigma^2} + \tfrac{19}{320\sigma^2} \cdot (1 - \tfrac{q-p}{\sigma^2} + \tfrac{pq}{\sigma^4}) - \tfrac{q-p}{6\sigma^4} + \tfrac{q-p}{2\sigma^2} - \tfrac{p^2+q^2}{4\sigma^4} - \tfrac{p^3}{6\sigma^6} - \tfrac{p^4}{8\sigma^8} - \tfrac{9p^5}{16\sigma^{10}} \\ &= \tfrac{1}{\sigma^2} \cdot \Big(\tfrac{857}{960} - p - \tfrac{1}{\sigma^2} \cdot (\tfrac{457(1-2p)}{960} + \tfrac{p^2}{2}) - \tfrac{1}{\sigma^4} \cdot (\tfrac{p^3}{6} - \tfrac{19pq}{320}) - \tfrac{p^4}{8\sigma^6} - \tfrac{9p^5}{16\sigma^8}\Big) \\ &\geq \tfrac{1}{\sigma^2} \cdot \Big(\tfrac{857}{960} - p - \tfrac{6}{5} \cdot (\tfrac{457(1-2p)}{960} + \tfrac{p^2}{2}) - \tfrac{36}{25} \cdot \tfrac{p^3}{6} - \tfrac{216p^4}{125 \cdot 8} - \tfrac{9 \cdot 6^4 p^5}{16 \cdot 5^4}\Big) \\ &=: \tfrac{1}{\sigma^2} \cdot i_1(p).\end{aligned}$$

Now i_1 is obviously concave and we have $i_1(0) = \frac{1543}{4800}$ and $i_1(\frac{1}{2}) = \frac{19\,531}{120\,000}$, hence $i_1(p) \geq 0$ for $p \in [0, \frac{1}{2}]$. On the other hand we have

$$\begin{aligned}h(\tfrac{3}{2}\sigma) &= \tfrac{3}{4\sigma} - \tfrac{1}{6\sigma^2} + \tfrac{19}{320\sigma^2} \cdot (1 - \tfrac{3(q-p)}{2\sigma} + \tfrac{9pq}{4\sigma^2}) - \tfrac{27(q-p)}{48\sigma} + \tfrac{3(q-p)}{4\sigma} - \tfrac{9(p^2+q^2)}{16\sigma^2} - \tfrac{27p^3}{48\sigma^3} \\ &\quad - \tfrac{81p^4}{128\sigma^4} - \tfrac{3^7p^5}{2^9\sigma^5} \\ &= \tfrac{1}{\sigma} \cdot \Big(\underbrace{\tfrac{15-6p}{16} - \tfrac{1}{\sigma} \cdot (\tfrac{643}{960} - \tfrac{9pq}{8}) - \tfrac{1}{\sigma^2} \cdot (\tfrac{57(1-2p)}{640} + \tfrac{27p^3}{48}) + \tfrac{1}{\sigma^3} \cdot (\tfrac{171pq}{1280} - \tfrac{81p^4}{128}) - \tfrac{3^7p^5}{2^9\sigma^4}}_{=:s(\sigma)}\Big) \\ &\geq \tfrac{1}{\sigma} \cdot \Big(\tfrac{15-6p}{16} - \tfrac{\sqrt{6}}{\sqrt{5}} \cdot (\tfrac{643}{960} - \tfrac{9pq}{8}) - \tfrac{6}{5} \cdot (\tfrac{57(1-2p)}{640} + \tfrac{27p^3}{48}) + \tfrac{6\sqrt{6}}{5\sqrt{5}} \cdot (\tfrac{171pq}{1280} - \tfrac{81p^4}{128}) \\ &\qquad - \tfrac{36}{25} \cdot \tfrac{3^7p^5}{2^9}\Big) \\ &=: \tfrac{1}{\sigma} \cdot i_2(p).\end{aligned}$$

Above we use in the inequality that $s'(\sigma) = \frac{1}{\sigma^2} \cdot (\frac{643}{960} - \frac{9pq}{8}) + \frac{2}{\sigma^3} \cdot (\frac{57(1-2p)}{640} + \frac{27p^3}{48}) - \frac{3}{\sigma^4} \cdot (\frac{171pq}{1280} - \frac{81p^4}{128}) + \frac{3^7p^5}{2^7\sigma^5} \geq \frac{1}{\sigma^2} \cdot (\frac{643}{960} - \frac{9pq}{8}) - \frac{3}{\sigma^4} \cdot \frac{171pq}{1280} \geq \frac{1}{\sigma^2} \cdot (\frac{643}{960} - \frac{9}{8 \cdot 4} - \frac{171 \cdot 3}{1280 \cdot 4} \cdot \frac{6}{5}) = \frac{10\,303}{38\,400\sigma^2} > 0$ and hence $s(\sigma) \geq s(\sqrt{5/6})$. Here i_2 also is obviously concave with $q = 1 - p$ not constant, and we have

$$\begin{aligned}i_2(0) &= \tfrac{15}{16} - \tfrac{\sqrt{6} \cdot 643}{\sqrt{5} \cdot 960} - \tfrac{6 \cdot 57}{5 \cdot 640} = \tfrac{1329}{1600} - \tfrac{643 \cdot \sqrt{6}}{960 \cdot \sqrt{5}} = 0.0969... \\ i_2(\tfrac{1}{2}) &= \tfrac{3}{4} - \tfrac{\sqrt{6} \cdot 373}{\sqrt{5} \cdot 960} - \tfrac{27}{320} - \tfrac{\sqrt{6} \cdot 189}{\sqrt{5} \cdot 25\,600} - \tfrac{19\,683}{102\,400} = \tfrac{48\,477}{102\,400} - \tfrac{30\,407 \cdot \sqrt{6}}{76\,800 \cdot \sqrt{5}} = 0.0397... \,.\end{aligned}$$

Thus $h(z)$ is even at the boundary positive and thus we have

$$\frac{g(k)}{f(k)} \geq \mathrm{e}^{h(z)} \geq \mathrm{e}^0 = 1$$

and hence in the end (13).

(b) Let $p > 0.5$:

For $k > n$ the claim follows directly from $f(k) = 0 < g(k)$ and the special cases $k = n, n-1, n-2$ we consider later, so let us first assume $k \leq n-3$.

1.) Let $k = np + z \leq n - 3$, implying $nq \geq 3 + z$. We start with the case $k = \lceil np \rceil + 1$, hence $z \in [1,2]$ and $\frac{z}{nq} \leq \frac{z}{z+3} \leq \frac{2}{5}$. Thus we have $\sum_{r=0}^{\infty} (\frac{z}{nq})^r \leq \frac{5}{3}$ and hence $\log(1+\frac{z}{np}) + \log(1-\frac{z}{nq}) \geq \frac{z}{np} - \frac{z^2}{2(np)^2} - \frac{z}{nq} - \frac{z^2}{2(nq)^2} - \sum_{r=3}^{\infty} \frac{z^r}{r(nq)^r} \geq \frac{z(q-p)}{\sigma^2} - \frac{z^2(p^2+q^2)}{2\sigma^4} - \frac{5z^3p^3}{9\sigma^6}$.
Thus we have together with (17) and (16)

$$\begin{aligned}
\log(\tfrac{g(k)}{f(k)}) &\geq -\tfrac{1}{6\sigma 2} + \tfrac{19}{320\sigma^2} + \tfrac{19z(p-q)}{320\sigma^4} + \tfrac{z}{2\sigma^2} + \tfrac{z^3(p-q)}{6\sigma^4} + \tfrac{1}{2}\cdot(\tfrac{z(q-p)}{\sigma^2} - \tfrac{z^2(p^2+q^2)}{2\sigma^4} - \tfrac{5z^3p^3}{9\sigma^6}) \\
&= -\tfrac{1}{6\sigma 2} + \tfrac{19}{320\sigma^2} + \tfrac{z}{2\sigma^2}\cdot(2(1-p) - \tfrac{1}{nq}(-\tfrac{19(1-q/p)}{160} - \tfrac{z^2(1-q/p)}{3} + \tfrac{z(p+q^2/p)}{2} - \tfrac{5z^2p}{9nq})).
\end{aligned}$$

If $-\frac{19(1-q/p)}{160} - \frac{z^2(1-q/p)}{3} + \frac{z(p+q^2/p)}{2} - \frac{5z^2p}{9nq} < 0$, we get with $\frac{z}{2\sigma^2}\cdot(1-2(1-p)) \geq \frac{1}{6\sigma^2}$ the positivity of the right hand side, so we assume $-\frac{19(1-q/p)}{160} - \frac{z^2(1-q/p)}{3} + \frac{z(p+q^2/p)}{2} - \frac{5z^2p}{9nq} \geq 0$, from which with $nq \geq 3 + z \geq 4$ by the assumptions we get

$$\log(\tfrac{g(k)}{f(k)}) \geq -\tfrac{1}{6\sigma 2} + \tfrac{19}{320\sigma^2} + \tfrac{z}{2\sigma^2}\cdot(\underbrace{2(1-p) - \tfrac{1}{4}(-\tfrac{19(1-q/p)}{160} - \tfrac{z^2(1-q/p)}{3} + \tfrac{z(p+q^2/p)}{2} + \tfrac{5z^2p}{9\cdot 4})}_{=:h_2(p)}).$$

Now we have $h_2'(p) = -2 - \frac{1}{4}\cdot(-\frac{19}{160p^2} - \frac{z^2}{3p^2} + \frac{z(1+(p^2-1)/p^2)}{2} + \frac{5z^2}{9\cdot 4}) \leq -2 - \frac{1}{4}\cdot(-\frac{19\cdot 4}{160} - \frac{z^2\cdot 4}{3} - \frac{2z}{2} + \frac{5z^2}{9\cdot 4}) \leq -2 + \frac{19}{160} + \frac{4}{3} + \frac{1}{2} < 0$, hence $h_2(p) \geq h_2(\frac{5}{6})$ and thus

$$\begin{aligned}
\log(\tfrac{g(k)}{f(k)}) &\geq -\tfrac{1}{6\sigma 2} + \tfrac{19}{320\sigma^2} + \tfrac{z}{2\sigma^2}\cdot(\tfrac{1}{3} - \tfrac{1}{4}\cdot(-\tfrac{19\cdot 4}{160\cdot 5} - \tfrac{z^2\cdot 4}{3\cdot 5} + \tfrac{z\cdot 26}{2\cdot 30} + \tfrac{5z^2\cdot 5}{9\cdot 4\cdot 6})) \\
&= -\tfrac{1}{6\sigma 2} + \tfrac{19}{320\sigma^2} + \tfrac{1}{2\sigma^2}\cdot(\tfrac{857z}{2400} - \tfrac{13z^2}{120} + \tfrac{163z^3}{4320}) \\
&\geq -\tfrac{1}{6\sigma 2} + \tfrac{19}{320\sigma^2} + \tfrac{1}{2\sigma^2}\cdot(\tfrac{857}{2400} - \tfrac{13}{120} + \tfrac{163}{4320}) = \tfrac{1553}{43\,200\sigma^2} > 0
\end{aligned}$$

using $\left[\frac{857z}{2400} - \frac{13z^2}{120} + \frac{163z^3}{4320}\right]' = \frac{857}{2400} - \frac{13z}{60} + \frac{163z^2}{1440} \geq \frac{857}{2400} - \frac{13^2/60^2}{4\cdot 163/1440} = \frac{99\,131}{391\,200} > 0$ and $z \geq 1$ at the second inequality.

We now have $g(\lceil np \rceil + 1) \geq f(\lceil np \rceil + 1)$ and because of $f(k)/g(k)$ is decreasing on $\{k \in \mathbb{Z} : np \leq k \leq n\}$ due to Lemma 11.12, this yields (14) for each $np+1 \leq k \leq n-3$.

2.) Let $k = n$ and hence $z = nq$. Then, using (15), we get

$$\begin{aligned}\log(\tfrac{g(n)}{f(n)}) &\geq \log(\tfrac{1}{\sqrt{2\pi}\,\sigma}) - \tfrac{n^2q^2}{2\sigma^2} + \tfrac{nq}{2\sigma^2} - \tfrac{1}{6\sigma^2} - \log(p^n) \\ &= \log(\tfrac{1}{\sqrt{2\pi}}) - \tfrac{1}{2}\log(npq) - \tfrac{nq}{2p} + \tfrac{1}{2p} - \tfrac{1}{6npq} - n\log(p) \\ &=: h_3(n,p).\end{aligned}$$

Now we have $\frac{\mathrm{d}h_3}{\mathrm{d}p} = -\frac{1-2p}{2pq} + \frac{n}{2p^2} - \frac{1}{2p^2} + \frac{1-2p}{6np^2q^2} - \frac{n}{p} \leq -\frac{1-2p}{2pq} + \frac{n}{2p^2} - \frac{n}{p} = \frac{2p-1}{2p^2q} \cdot (p - nq) \leq 0$ since $nq \geq 6q \geq 1$. Thus we have

$$h_3(n,p) \;\geq\; h_3(n,\tfrac{5}{6}) \;=\; \log(\tfrac{1}{\sqrt{2\pi}}) - \tfrac{1}{2}\log(\tfrac{5}{36}) - \tfrac{1}{2}\log(n) - \tfrac{n}{10} + \tfrac{3}{5} - \tfrac{6}{5n} - n\log(\tfrac{5}{6}).$$

Further $\frac{\mathrm{d}^2h_3}{\mathrm{d}n^2}(n,\frac{5}{6}) = \frac{1}{2n^2} - \frac{12}{5n^3} \geq 0$ and $\frac{\mathrm{d}h_3}{\mathrm{d}n}(6,\frac{5}{6}) = -\frac{1}{12} - \frac{1}{10} - \log(\frac{5}{6}) + \frac{6}{5\cdot 36} = 0.0323... > 0$ and hence we have

$$\begin{aligned}h_3(n,p) &\geq h_3(6,\tfrac{5}{6}) \;=\; \log(\tfrac{1}{\sqrt{2\pi}}) - \tfrac{1}{2}\log(\tfrac{5}{36}) - \tfrac{1}{2}\log(6) - \tfrac{6}{10} + \tfrac{3}{5} - \tfrac{1}{5} - 6\log(\tfrac{5}{6}) \\ &= 0.0662... \;>\; 0.\end{aligned}$$

3.) Let $k = n-1$ and hence $z = nq - 1$. From the condition $n - 1 = k \geq np + 1$ it follows firstly $nq \geq 2$. Similar to the above we have again with (15)

$$\begin{aligned}\log(\tfrac{g(n-1)}{f(n-1)}) &\geq \log(\tfrac{1}{\sqrt{2\pi}\,\sigma}) - \tfrac{(nq-1)^2}{2\sigma^2} + \tfrac{nq-1}{2\sigma^2} - \tfrac{1}{6\sigma^2} - \log(nqp^{n-1}) \\ &= \log(\tfrac{1}{\sqrt{2\pi}}) - \tfrac{3}{2}\log(npq) - \tfrac{nq}{2p} + \tfrac{3}{2p} - \tfrac{7}{6npq} - (n-2)\log(p) \\ &=: h_4(n,p).\end{aligned}$$

Now since $nq \geq 2$, we have $q \geq 2/n$ and hence with $n \geq 6$

$$\begin{aligned}\tfrac{\mathrm{d}h_4(n,p)}{\mathrm{d}p} &= -\tfrac{3(1-2p)}{2pq} + \tfrac{n}{2p^2} - \tfrac{3}{2p^2} + \tfrac{7(1-2p)}{6np^2q^2} - \tfrac{n-2}{p} \;\leq\; -\tfrac{3(1-2p)}{2pq} + \tfrac{n}{2p^2} - \tfrac{3}{2p^2} - \tfrac{n-2}{p} \\ &= -\tfrac{1}{p^2q} \cdot \left(\tfrac{3(1-2p)p}{2} + \tfrac{(n-3)q}{2} + (n-2)pq\right) \;\leq\; -\tfrac{1}{p^2q} \cdot \left(-\tfrac{5}{6} + \tfrac{n-3}{n} + (n-2) \cdot \tfrac{5}{36}\right) \\ &\leq -\tfrac{1}{p^2q} \cdot \left(-\tfrac{5}{6} + \tfrac{1}{2} + 4 \cdot \tfrac{5}{36}\right) \;<\; 0.\end{aligned}$$

Thus we have firstly for $n \geq 12$

$$h_4(n,p) \;\geq\; h_4(n,\tfrac{5}{6}) \;=\; \log(\tfrac{1}{\sqrt{2\pi}}) - \tfrac{3}{2}\log(\tfrac{5}{36} \cdot n) - \tfrac{n}{10} + \tfrac{9}{5} - \tfrac{42}{5n} - (n-2)\log(\tfrac{5}{6}).$$

Further we have $\frac{\mathrm{d}^2h_4}{\mathrm{d}n^2}(n,\frac{5}{6}) = \frac{3}{2n^2} - \frac{84}{5n^3} \geq 0$ and $\frac{\mathrm{d}h_4}{\mathrm{d}n}(12,\frac{5}{6}) = -\frac{1}{8} - \frac{1}{10} - \log(\frac{5}{6}) + \frac{42}{5\cdot 144} = 0.0157... > 0$ and hence

$$h_4(n,p) \;\geq\; h_4(12,\tfrac{5}{6}) \;=\; \log(\tfrac{1}{\sqrt{2\pi}}) - \tfrac{3}{2}\log(\tfrac{5}{3}) - \tfrac{6}{5} + \tfrac{9}{5} - \tfrac{7}{10} - 10\log(\tfrac{5}{6}) \;=\; 0.0380... > 0.$$

But if $n \leq 11$ we use $p \leq 1 - \frac{2}{n}$ and then we have

$$\begin{aligned}
h_4(n,p) &\geq h_4(n, 1-\tfrac{2}{n}) \\
&= \log(\tfrac{1}{\sqrt{2\pi}}) - \tfrac{3}{2}\log(2(1-\tfrac{2}{n})) + \tfrac{1}{2(1-2/n)} - \tfrac{7}{12(1-2/n)} - (n-2)\log((1-\tfrac{2}{n})) \\
&= \log(\tfrac{1}{\sqrt{2\pi}}) - \tfrac{3}{2}\cdot\log(2) - (n-\tfrac{1}{2})\cdot\log(1-\tfrac{2}{n}) - \tfrac{1}{12(1-2/n)} \\
&\geq \log(\tfrac{1}{\sqrt{2\pi}}) - \tfrac{3}{2}\cdot\log(2) - (n-\tfrac{1}{2})\cdot(-\tfrac{2}{n} - \tfrac{2}{n^2}) - \tfrac{1}{12(1-2/n)} \\
&= \log(\tfrac{1}{\sqrt{2\pi}}) - \tfrac{3}{2}\cdot\log(2) + 2 + \tfrac{1}{n} - \tfrac{1}{n^2} - \tfrac{1}{12} - \tfrac{1}{6n} - \tfrac{1}{3n(n-2)} \\
&\geq \log(\tfrac{1}{\sqrt{2\pi}}) - \tfrac{3}{2}\cdot\log(2) + 2 + \tfrac{1}{n} - \tfrac{1}{6n} - \tfrac{1}{12} - \tfrac{1}{6n} - \tfrac{1}{12n} \\
&= \log(\tfrac{1}{\sqrt{2\pi}}) - \tfrac{3}{2}\cdot\log(2) + \tfrac{23}{24} + \tfrac{7}{12n} \\
&\geq \log(\tfrac{1}{\sqrt{2\pi}}) - \tfrac{3}{2}\cdot\log(2) + \tfrac{23}{24} + \tfrac{7}{12\cdot 11} \;=\; 0.0110\ldots\,.
\end{aligned}$$

4.) Let $k = n-2$ and hence $z = nq-2$. From the condition $n-2 = k \geq np+1$ it follows $nq \geq 3$. Once again we have with (15)

$$\begin{aligned}
\log(\tfrac{g(k)}{f(k)}) &\geq \log(\tfrac{1}{\sqrt{2\pi}\,\sigma}) - \tfrac{(nq-2)^2}{2\sigma^2} + \tfrac{nq-2}{2\sigma^2} - \tfrac{1}{6\sigma^2} - \log(\tfrac{n(n-1)}{2}q^2p^{n-2}) \\
&= \log(\tfrac{1}{\sqrt{2\pi n}}) - \tfrac{5}{2}\log(pq) - \tfrac{nq}{2p} + \tfrac{5}{2p} - \tfrac{19}{6npq} - \log(\tfrac{n(n-1)}{2}) - (n-4)\log(p) \\
&=: h_5(n,p).
\end{aligned}$$

Now since $nq \geq 3$, we have $q \geq 3/n$ and hence again

$$\begin{aligned}
\tfrac{\mathrm{d}h_5(n,p)}{\mathrm{d}p} &= -\tfrac{5(1-2p)}{2pq} + \tfrac{n}{2p^2} - \tfrac{5}{2p^2} + \tfrac{19(1-2p)}{6np^2q^2} - \tfrac{n-4}{p} \\
&= -\tfrac{5(1-2p)}{2pq} + \tfrac{n(1-2p)}{2p^2} - \tfrac{2(1-2p)}{p^2} + \tfrac{19(1-2p)}{6np^2q^2} - \tfrac{1}{2p^2} \\
&= \tfrac{2p-1}{p^2q}\cdot(2 + \tfrac{p}{2} - \tfrac{19}{6nq} - \tfrac{nq}{2}) - \tfrac{1}{2p^2} \;\leq\; \tfrac{2p-1}{p^2q}\cdot(2 + \tfrac{5}{12} - \tfrac{19}{18} - \tfrac{3}{2}) \;=\; -\tfrac{5(2p-1)}{36p^2q} \;<\; 0
\end{aligned}$$

with $(\frac{19}{6t} + \frac{t}{2})' = \frac{1}{2} - \frac{19}{6t^2} \geq \frac{1}{2} - \frac{19}{6\cdot 9} = \frac{4}{27} > 0$ for $t \geq 3$ in the first inequality.

Thus we have firstly for $n \geq 16$

$$\begin{aligned}
h_5(n,p) &\geq h_5(n, \tfrac{5}{6}) \\
&= \log(\tfrac{1}{\sqrt{2\pi n}}) - \tfrac{5}{2}\log(\tfrac{5}{36}) - \tfrac{n}{10} + 3 - \tfrac{19\cdot 6}{5n} - \log(\tfrac{n(n-1)}{2}) - (n-4)\log(\tfrac{5}{6}).
\end{aligned}$$

Further $\frac{dh_5}{dn}(n, \frac{5}{6}) = -\frac{1}{2n} - \frac{1}{10} + \frac{19\cdot 6}{5n^2} - \frac{1}{n} - \frac{1}{n-1} - \log(\frac{5}{6}) = -\log(\frac{5}{6}) - \frac{1}{10} - \frac{5}{2n} - \frac{1}{n(n-1)} + \frac{19\cdot 6}{5n^2} \geq -\log(\frac{5}{6}) - \frac{1}{10} - \frac{5}{2n} - \frac{16}{15n^2} + \frac{19\cdot 6}{5n^2} = -\log(\frac{5}{6}) - \frac{1}{10} - \frac{5}{2n} + \frac{326}{15n^2} \geq -\log(\frac{5}{6}) - \frac{1}{10} - \frac{25\cdot 15}{16\cdot 326} > 0$ since for the function $j_2(t) := -\frac{5}{2t} + \frac{326}{15t^2}$ for $t \in [1,\infty)$ we have $j_2'(t) = 0 \Leftrightarrow t = \frac{4\cdot 326}{5\cdot 15} =: t_0$ and $j_2''(t_0) = \frac{2\,109\,375}{4\,434\,684\,928} > 0$ from which follows $j_2(t) \geq j_2(t_0) = -\frac{25\cdot 15}{16\cdot 326}$ and hence

$$\begin{aligned}
h_5(n,p) &\geq h_5(16, \tfrac{5}{6}) \;=\; \log(\tfrac{1}{\sqrt{32\pi}}) - \tfrac{5}{2}\log(\tfrac{5}{36}) - \tfrac{16}{10} + 3 - \tfrac{19\cdot 6}{5\cdot 16} - \log(\tfrac{16\cdot 15}{2}) - 12\log(\tfrac{5}{6}) \\
&= 0.0053\ldots \;>\; 0.
\end{aligned}$$

But if $n \le 15$ we use $p \le 1 - \frac{3}{n}$ and then we have

$$\begin{aligned}
h_5(n,p) &\ge h_5(n, 1-\tfrac{3}{n}) \\
&= \log(\tfrac{1}{\sqrt{2\pi n}}) - \tfrac{5}{2}\log(\tfrac{3}{n}\cdot(1-\tfrac{3}{n})) - \tfrac{1}{18(1-3/n)} - \log(\tfrac{n(n-1)}{2}) - (n-4)\log(1-\tfrac{3}{n}) \\
&= \log(\tfrac{1}{\sqrt{2\pi}}) - \tfrac{5\log(3)}{2} - \tfrac{1}{18}\cdot(1+\tfrac{3}{n}+\tfrac{9}{n^2(1-3/n)}) - \log(1-\tfrac{1}{n}) + \log(2) \\
&\quad -(n-\tfrac{3}{2})\log(1-\tfrac{3}{n}) \\
&\ge \log(\tfrac{1}{\sqrt{2\pi}}) - \tfrac{5\log(3)}{2} - \tfrac{1}{18}\cdot(1+\tfrac{3}{n}+\tfrac{18}{n^2}) + \tfrac{1}{n} + \tfrac{1}{2n^2} + \log(2) \\
&\quad -(n-\tfrac{3}{2})\cdot(-\tfrac{3}{n} - \tfrac{9}{2n^2} - \tfrac{9}{n^3}) \\
&= \log(\tfrac{2}{\sqrt{2\pi}}) - \tfrac{5\log(3)}{2} + \tfrac{53}{18} + \tfrac{5}{6n} + \tfrac{7}{4n^2} - \tfrac{27}{2n^3} \\
&\ge \log(\tfrac{2}{\sqrt{2\pi}}) - \tfrac{5\log(3)}{2} + \tfrac{53}{18} + \tfrac{5}{6\cdot 15} + \tfrac{7}{4\cdot 225} - \tfrac{27}{2\cdot 15^3} \;=\; 0.0315... \;>\; 0
\end{aligned}$$

because of $(\frac{5}{6t} + \frac{7}{4t^2} - \frac{27}{2t^3})' = -\frac{1}{t^2}\cdot(\frac{5}{6} + \frac{7}{2t} - \frac{81}{2t^2}) \le -\frac{1}{t^2}\cdot(\frac{5}{6} + \frac{7}{2\cdot 15} - \frac{81}{2\cdot 6^2}) = -\frac{7}{120t^2} < 0$ for $6 \le t \le 15$ in the penultimate inequality. $\square$

Lemma 7.3. *Let $\frac{1}{6} \le p \le \frac{1}{2}$ and $n \ge 6$. Then $\sigma \ge \sqrt{5/6}$ and we have*

$$g(k+1) \;\le\; f(k) \qquad \text{if} \qquad np \le k \le np + \tfrac{3}{2}\cdot\sqrt{npq}\,.$$

For $\frac{1}{2} \le p \le \frac{5}{6}$ and $n \ge 6$, we have in the case $\sigma \ge 2$

$$g(k+1) \;\le\; f(k) \qquad \text{if} \qquad np \le k \le np + \sqrt{\tfrac{3}{2}npq}$$

and in the case $\sqrt{5/6} \le \sigma < 2$

$$g(k+1) \;\le\; f(k) \qquad \text{if} \qquad np \le k \le np + \sqrt{npq}\,.$$

Proof. Let again, like in the previous lemma, $z := k - np \ge 0$. Then we have due to Lemma 11.2 for $p \in [\frac{1}{6}, \frac{5}{6}]$

$$\begin{aligned} g(k+1) &= \Phi(\tfrac{z+1}{\sigma}) - \Phi(\tfrac{z}{\sigma}) \;=\; \Phi(\tfrac{z+1/2}{\sigma} + \tfrac{1}{2\sigma}) - \Phi(\tfrac{z+1/2}{\sigma} - \tfrac{1}{2\sigma}) \\ &\le \tfrac{1}{\sigma}\cdot\tfrac{1}{\sqrt{2\pi}}\cdot \mathrm{e}^{-\frac{(z+1/2)^2}{2\sigma^2}}\cdot \mathrm{e}^{\frac{(z+1/2)^2/\sigma^2-1}{24\sigma^2}+\frac{1}{1440\sigma^4}} \\ &= \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot \mathrm{e}^{-\frac{z^2}{2\sigma^2}-\frac{z}{2\sigma^2}-\frac{1}{6\sigma^2}+\frac{z^2+z}{24\sigma^4}+\frac{1}{90\sigma^4}} \end{aligned}$$

and hence

$$\log(g(k+1)) \;\le\; \log(\tfrac{1}{\sqrt{2\pi}\,\sigma}) - \tfrac{z^2}{2\sigma^2} - \tfrac{z}{2\sigma^2} - \tfrac{1}{6\sigma^2} + \tfrac{z^2+z}{24\sigma^4} + \tfrac{1}{90\sigma^4}. \tag{18}$$

Further we have on the other hand with Lemma 11.4

$$\begin{aligned} \log(f(k)\cdot\sqrt{2\pi}\,\sigma) &= \log(f_{n,\frac{k}{n}}(k)\cdot\sqrt{2\pi}\,\sigma) + \log((\tfrac{np}{k})^k\cdot(\tfrac{nq}{n-k})^{n-k}) \\ &\ge \tfrac{1}{2}\cdot\log(\tfrac{\sigma^2}{\sigma_{k/n}^2}) + \log((\tfrac{np}{k})^k\cdot(\tfrac{nq}{n-k})^{n-k}) - \tfrac{1}{12\sigma_{k/n}^2} \\ &= (k+\tfrac{1}{2})\cdot\log(\tfrac{np}{k}) + (n-k+\tfrac{1}{2})\cdot\log(\tfrac{nq}{n-k}) - \tfrac{1}{12\sigma_{k/n}^2} \\ &= -(np+z+\tfrac{1}{2})\cdot\log(1+\tfrac{z}{np}) - (nq-z+\tfrac{1}{2})\cdot\log(1-\tfrac{z}{nq}) - \tfrac{1}{12\sigma_{k/n}^2}. \end{aligned}$$

Thus in view of (18) and the inequality above it is sufficient to show $h(z,p) \ge 0$ for certain $z \ge 0$ with

$$\begin{aligned} h(z,p) \;:=\; &\tfrac{z^2+z+1/3}{2\sigma^2} - \tfrac{z^2+z+24/90}{24\sigma^4} - (np+z+\tfrac{1}{2})\cdot\log(1+\tfrac{z}{np}) - (nq-z+\tfrac{1}{2})\cdot\log(1-\tfrac{z}{nq}) \\ &-\tfrac{n}{12(np+z)(nq-z)}. \end{aligned}$$

In the next section up to the computations in (19) below we assume $\frac{z}{nq} \leq \frac{2}{3}$. Under this assumption we have

$$\begin{aligned}(nq-z)\cdot\log(1-\tfrac{z}{nq}) &= -\sum_{r=1}^{\infty}\tfrac{1}{r}\cdot\left(\tfrac{z^r}{(nq)^{r-1}}-\tfrac{z^{r+1}}{(nq)^r}\right) = -z+\sum_{r=1}^{\infty}\tfrac{1}{r(r+1)}\cdot\tfrac{z^{r+1}}{(nq)^r}\\ &\leq -z+\tfrac{z^2}{2nq}+\tfrac{z^3}{6(nq)^2}+\tfrac{z^4}{12(nq)^3}+\tfrac{3z^5}{20(nq)^4}\end{aligned}$$

since $\sum_{r=4}^{\infty}\frac{1}{r(r+1)}\cdot\frac{z^{r+1}}{(nq)^r} \leq \frac{z^5}{20(nq)^4}\cdot\sum_{r=0}^{\infty}\frac{z^r}{(nq)^r} \leq \frac{z^5}{20(nq)^4}\cdot\sum_{r=0}^{\infty}(\frac{2}{3})^r = \frac{3z^5}{20(nq)^4}$. We also have

$$(np+z)\cdot\log(1+\tfrac{z}{np}) = np\cdot(1+\tfrac{z}{np})\log(1+\tfrac{z}{np}) \leq z+\tfrac{z^2}{2np}-\tfrac{z^3}{6(np)^2}+\tfrac{z^4}{12(np)^3}$$

since for $j(x) := (1+x)\log(1+x)-(x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12})$ for $x\in[0,\infty)$ we have $j''(x) = -\frac{x^3}{1+x} < 0$, implying $j'(x) \leq j'(0) = 0$ and hence $j(x) \leq j(0) = 0$ applied to $x = \frac{z}{np}$, because $\frac{z}{np} < 1$ holds not necessarily. Further we have

$$\begin{aligned}\tfrac{1}{2}\cdot(\log(1+\tfrac{z}{np})+\log(1-\tfrac{z}{nq})) &\leq \tfrac{z}{2np}-\tfrac{z}{2nq}-\tfrac{z^2}{4(np)^2}-\tfrac{z^2}{4(nq)^2}+\tfrac{z^3}{6(np)^3}-\tfrac{z^3}{6(nq)^3}\\ &= \tfrac{z(q-p)}{2\sigma^2}-\tfrac{z^2(p^2+q^2)}{4\sigma^4}+\tfrac{z^3(q^3-p^3)}{6\sigma^6}\end{aligned}$$

since for $j_2(x) := \log(1+x)-(x-\frac{x^2}{2}+\frac{x^3}{3})$ for $x\in[0,\infty)$ we have $j_2'(x) = -\frac{x^3}{1-x}$ and hence $j_2(x) \leq j_2(0) = 0$ applied again to $x = \frac{z}{np}$.

Hence we have together

$$\begin{aligned}(19)\qquad & (np+z+\tfrac{1}{2})\cdot\log(1+\tfrac{z}{np})+(nq-z+\tfrac{1}{2})\cdot\log(1-\tfrac{z}{nq})\\ \leq\ & z+\tfrac{z^2}{2np}-\tfrac{z^3}{6(np)^2}+\tfrac{z^4}{12(np)^3}-z+\tfrac{z^2}{2nq}+\tfrac{z^3}{6(nq)^2}+\tfrac{z^4}{12(nq)^3}+\tfrac{3z^5}{20(nq)^4}\\ & +\tfrac{z(q-p)}{2\sigma^2}-\tfrac{z^2(p^2+q^2)}{4\sigma^4}+\tfrac{z^3(q^3-p^3)}{6\sigma^6}\\ =\ & \tfrac{z^2}{2\sigma^2}-\tfrac{z^3(q^2-p^2)}{6\sigma^4}+\tfrac{z^4(p^3+q^3)}{12\sigma^6}+\tfrac{3z^5}{20(nq)^4}+\tfrac{z(q-p)}{2\sigma^2}-\tfrac{z^2(p^2+q^2)}{4\sigma^4}+\tfrac{z^3(q^3-p^3)}{6\sigma^6}.\end{aligned}$$

Now we consider the two cases $p\in[\frac{1}{6},\frac{1}{2}]$ and $p\in[\frac{1}{2},\frac{5}{6}]$ separately.

(a) Let $p \le 0.5$.

Due to the conditions we have $0 \le \frac{z}{n} \le \sqrt{\frac{9pq}{4n}} \le \sqrt{\frac{9}{16n}} \le \sqrt{\frac{3}{32}}$, already implying $p + \frac{z}{n} \in [\frac{1}{6}, \frac{1}{2} + \sqrt{\frac{3}{32}}]$ and hence we get with $\frac{\mathrm{d}}{\mathrm{d}p}[\frac{p-p^2}{(p+t)(1-p-t)}] = \frac{t(1-t)(1-2p)}{(p+t)^2(1-p-t)^2} \ge 0$ for $p \in [0, \frac{1}{2}]$ and $t \in [0, 1-p)$ that $\frac{pq}{(p+z/n)(q-z/n)} \le \frac{1/4}{(1/2+z/n)(1/2-z/n)} = \frac{1}{1-4z^2/n^2} \le \frac{1}{1-4\cdot 3/32} = \frac{8}{5}$ holds. Thus it follows firstly

$$\frac{n}{12(np+z)(nq-z)} = \frac{1}{12\sigma^2} \cdot \frac{pq}{(p+z/n)(q-z/n)} \le \frac{1}{12\sigma^2} \cdot \frac{8}{5} \le \frac{1}{12\sigma^2} \cdot \frac{7}{4} = \frac{7}{48\sigma^2}.$$

Further we have $\frac{z}{nq} \le \frac{3}{2} \cdot \sqrt{p/(nq)} \le \frac{3}{2} \cdot \sqrt{1/6} \le \frac{2}{3}$ and thus inequality (19) yields

$$\begin{aligned}
h(z,p) \;\ge\;& \frac{z^2+z+1/3}{2\sigma^2} - \frac{z^2+z+24/90}{24\sigma^4} - \frac{z^2}{2\sigma^2} + \frac{z^3(q^2-p^2)}{6\sigma^4} - \frac{z^4(p^3+q^3)}{12\sigma^6} - \frac{3z^5}{20(nq)^4} \\
& - \frac{7}{48\sigma^2} - \frac{z(q-p)}{2\sigma^2} + \frac{z^2(p^2+q^2)}{4\sigma^4} - \frac{z^3(q^3-p^3)}{6\sigma^6} \\
=\;& \frac{zp}{\sigma^2} + \frac{1}{48\sigma^2} - \frac{z^2+z+24/90}{24\sigma^4} + \frac{z^3(q^2-p^2)}{6\sigma^4} - \frac{z^4(q^3+p^3)}{12\sigma^6} - \frac{3z^5p^4}{20\sigma^8} + \frac{z^2(p^2+q^2)}{4\sigma^4} - \frac{z^3(q^3-p^3)}{6\sigma^6} \\
\ge\;& \frac{z}{6\sigma^2} + \frac{1}{48\sigma^2} - \frac{z^2+z+24/90}{24\sigma^4} - \frac{z^4}{12\sigma^6} \cdot \frac{7}{12} - \frac{3z^5}{2^4\cdot 20\sigma^8} + \frac{z^2}{8\sigma^4} + \frac{z^3(q^2-p^2-(q^3-p^3)/\sigma^2)}{6\sigma^4} \\
\ge\;& \frac{z}{6\sigma^2} + \frac{1}{48\sigma^2} - \frac{z^2+z+24/90}{24\sigma^4} - \frac{z^2}{12\sigma^4} \cdot \frac{7\cdot 9}{12\cdot 4} - \frac{3^5 z}{2^8\cdot 20\sigma^4} + \frac{z^2}{8\sigma^4} - \frac{z^3}{6\cdot 45\sigma^4} \\
\ge\;& \frac{z}{6\sigma^2} - \frac{z}{24\sigma^4} - \frac{3^5 z}{2^8\cdot 20\sigma^4} + \frac{z^2}{\sigma^4} \cdot \left(-\frac{1}{24} + \frac{1}{8} - \frac{1}{12} \cdot \frac{63}{48} - \frac{1}{4\cdot 45}\right) \\
=\;& \frac{z}{\sigma^2} \cdot \left(\frac{1}{6} - \frac{1}{24\sigma^2} - \frac{3^5}{2^8\cdot 20\sigma^2} - \frac{91z}{2880\sigma^2}\right) \\
\ge\;& \frac{z}{\sigma^2} \cdot \left(\frac{1}{6} - \frac{1}{24\cdot 5/6} - \frac{3^5}{2^8\cdot 20\cdot 5/6} - \frac{3\sqrt{6}}{2\sqrt{5}} \cdot \frac{91}{2880}\right) = 0.0077... \ge 0
\end{aligned}$$

due to $q^2 - p^2 - \frac{q^3-p^3}{\sigma^2} = (1-2p) \cdot (1 - \frac{1-pq}{npq}) \ge (1-2p) \cdot (1 - \frac{31/36}{6\cdot 5/36}) = -\frac{1-2p}{30} \ge -\frac{1}{45}$ among others in the third inequality and $\sigma^2 \ge 6 \cdot \frac{5}{36} = \frac{5}{6}$ in the fifth one.

(b) Let $p > 0.5$.

If $\frac{z}{n} \le \sqrt{3/8}\,pq$ holds, then we have

$$\begin{aligned}
\frac{pq}{(p+z/n)(q-z/n)} \;\le\;& \frac{pq}{(p+\sqrt{3/8}\,pq)(q-\sqrt{3/8}\,pq)} = \frac{1}{(1+\sqrt{3/8}\,q)(1-\sqrt{3/8}\,p)} \\
\le\;& \frac{1}{(1+\sqrt{3/8}/6)(1-\sqrt{3/8}\cdot 5/6)} = 1.8530... < \frac{15}{8}
\end{aligned}$$

and also $\frac{z}{nq} \le \sqrt{3/8} \cdot \frac{5}{6} \le \frac{2}{3}$.

Thus inequality (19) yields if additionally $\sigma \geq \frac{5}{3}$ holds

$$
\begin{aligned}
(20)\quad h(z,p) \;\geq\;& \tfrac{zp}{2\sigma^2} + \tfrac{1}{\sigma^2}\cdot\left(\tfrac{1}{6} - \tfrac{15}{96}\right) - \tfrac{z^2+z+24/90}{24\sigma^4} + \tfrac{z^3(q^2-p^2)}{6\sigma^4} - \tfrac{z^4(q^3+p^3)}{12\sigma^6} - \tfrac{3z^5p^4}{20\sigma^8} \\
& + \tfrac{z^2(p^2+q^2)}{4\sigma^4} - \tfrac{z^3(q^3-p^3)}{6\sigma^6} \\
\geq\;& \tfrac{zp}{\sigma^2} - \tfrac{z(p^2-q^2)}{4\sigma^2} - \tfrac{z^2+z}{24\sigma^4} - \tfrac{z^4}{12\sigma^6}\cdot\tfrac{7}{12} - \tfrac{3z^5 5^4}{6^4\cdot 20\sigma^8} + \tfrac{z^2}{8\sigma^4} + \tfrac{1}{\sigma^2}\cdot\left(\tfrac{1}{96} - \tfrac{1}{90\sigma^2}\right) \\
\geq\;& \tfrac{zp}{\sigma^2} - \tfrac{z(p-q)}{4\sigma^2} - \tfrac{z^2+z}{24\sigma^4} - \tfrac{z^2}{12\sigma^4}\cdot\tfrac{7}{12}\cdot\tfrac{3}{2} - \tfrac{3z\cdot 5^4}{6^4\cdot 20\sigma^4}\cdot\tfrac{9}{4} + \tfrac{z^2}{8\sigma^4} \\
=\;& \tfrac{z(1+2p)}{4\sigma^2} + \tfrac{z^2}{96\sigma^4} - \tfrac{157z}{768\sigma^4} \;\geq\; \tfrac{z}{2\sigma^2} - \tfrac{157z}{768\sigma^2}\cdot\tfrac{9}{25} \;=\; \tfrac{2729z}{6400\sigma^2} \;>\; 0.
\end{aligned}
$$

We now distinguish the cases $\sigma \geq 2$, $\frac{5}{3} \leq \sigma < 2$ and $\sigma < \frac{5}{3}$.

1.) Let first $\sigma \geq 2$ and $z \leq \sqrt{3/2}\,\sigma$. Then $\frac{z}{n} \leq \sqrt{3/2}\cdot\frac{\sigma}{n} = \sqrt{3/2}\cdot\frac{pq}{\sigma} \leq \sqrt{3/8}\cdot pq$ and $\sigma \geq \frac{5}{3}$ and hence the computation starting in (20) yields $h(z,p) \geq 0$.

2.) Let now $\frac{5}{3} \leq \sigma < 2$ and $z \leq \sigma$. We also have here $\frac{z}{n} \leq \frac{\sigma}{n} = \frac{pq}{\sigma} \leq \frac{3pq}{5} \leq \sqrt{3/8}\,pq$ and thus computation starting in (20) yields again $h(z,p) \geq 0$.

3.) Let now $\sqrt{5/6} \leq \sigma < \frac{5}{3}$ and $z \leq \sigma$. First of all, the case $k = n$ can not occur, since $k = np + z \leq np + \sqrt{npq} = np + nq\cdot\sqrt{\frac{p}{qn}} < np + nq = n$. Then since $nq - 1 \leq \sigma$ we have $n \leq \frac{\sigma+1}{q} \leq \frac{8}{3q}$ and we also have $n \geq 6 \geq \frac{1}{q}$ according to requirement. Further we have with (18)

$$
\begin{aligned}
\log\left(\tfrac{f(n-1)}{g(n)}\right) \;\geq\;& \log(np^{n-1}q) - \log\left(\tfrac{1}{\sqrt{2\pi}\,\sigma}\right) + \tfrac{z^2}{2\sigma^2} + \tfrac{z}{2\sigma^2} + \tfrac{1}{6\sigma^2} - \tfrac{z^2+z}{24\sigma^4} - \tfrac{1}{90\sigma^4} \\
=\;& (n-2)\log(p) + \tfrac{3}{2}\log(\sigma^2) - \log\left(\tfrac{1}{\sqrt{2\pi}}\right) + \tfrac{nq-1}{2p} + \tfrac{1}{6\sigma^2} - \tfrac{1}{24p^2} + \tfrac{1}{24\sigma^2 p} - \tfrac{1}{90\sigma^4} \\
=:\;& i(n,p).
\end{aligned}
$$

Now $\frac{d^2 i}{dn^2} = -\frac{3}{2n^2} + \frac{1}{3n^3pq} + \frac{1}{12n^3p^2q} - \frac{6}{90n^4p^2q^2} \leq -\frac{3}{2n^2} + \frac{1}{3n^2}\cdot\frac{6}{5} + \frac{1}{12n^2}\cdot\frac{12}{5} < 0$ and further we have

$$
\begin{aligned}
i(\tfrac{1}{q},p) \;=\;& \left(\tfrac{1}{q} - 2\right)\log(p) + \tfrac{3}{2}\log(p) - \log\left(\tfrac{1}{\sqrt{2\pi}}\right) + \tfrac{1}{6p} - \tfrac{1}{90p^2} \\
=\;& \left(\tfrac{1}{q} - \tfrac{1}{2}\right)\log(1-q) - \log\left(\tfrac{1}{\sqrt{2\pi}}\right) + \tfrac{1}{6p} - \tfrac{1}{90p^2} \\
\geq\;& \left(\tfrac{1}{q} - \tfrac{1}{2}\right)\cdot\left(-q - \tfrac{q^2}{2} - \tfrac{2q^3}{3}\right) - \log\left(\tfrac{1}{\sqrt{2\pi}}\right) + \tfrac{1}{6p} - \tfrac{1}{90p^2} \\
=\;& -1 - \tfrac{5}{12}q^2 + \tfrac{q^3}{3} - \log\left(\tfrac{1}{\sqrt{2\pi}}\right) + \tfrac{1}{6p} - \tfrac{1}{90p^2} \\
\geq\;& -1 - \tfrac{5}{12}\cdot\tfrac{1}{4} + \tfrac{1}{3\cdot 8} - \log\left(\tfrac{1}{\sqrt{2\pi}}\right) + \tfrac{1}{6}\cdot\tfrac{6}{5} - \tfrac{1}{90}\cdot\tfrac{36}{25} \\
=\;& 0.0404... \;>\; 0
\end{aligned}
$$

since $\log(1-q) = -q - \frac{q^2}{2} - \frac{q^3}{3} \cdot \sum_{r=0}^{\infty} q^r \geq -q - \frac{q^2}{2} - 2 \cdot \frac{q^3}{3}$ in the first inequality and

$$\begin{aligned}
i(\tfrac{8}{3q}, p) &= (\tfrac{8}{3q} - 2)\log(p) + \tfrac{3}{2}\log(\tfrac{8}{3}p) - \log(\tfrac{1}{\sqrt{2\pi}}) + \tfrac{5/3}{2p} + \tfrac{1}{16p} - \tfrac{1}{24p^2} + \tfrac{1}{64p^2} - \tfrac{1}{640p^2} \\
&= (\tfrac{8}{3q} - \tfrac{1}{2})\log(1-q) + \tfrac{3}{2}\log(\tfrac{8}{3}) - \log(\tfrac{1}{\sqrt{2\pi}}) + \tfrac{43}{48p} - \tfrac{53}{1920p^2} \\
&\geq (\tfrac{8}{3q} - \tfrac{1}{2})(-q - \tfrac{q^2}{2} - \tfrac{2q^3}{3}) + \tfrac{3}{2}\log(\tfrac{8}{3}) - \log(\tfrac{1}{\sqrt{2\pi}}) + \tfrac{5}{6p} + \tfrac{43}{48p} - \tfrac{53}{1920p^2} \\
&= -\tfrac{8}{3} - \tfrac{5}{6}q - \tfrac{55}{36}q^2 + \tfrac{q^3}{3} + \tfrac{3}{2}\log(\tfrac{8}{3}) - \log(\tfrac{1}{\sqrt{2\pi}}) + \tfrac{5}{6p} + \tfrac{43}{48p} - \tfrac{53}{1920p^2} \\
&\geq -\tfrac{8}{3} - \tfrac{5}{6} \cdot \tfrac{1}{2} - \tfrac{55}{36} \cdot \tfrac{1}{4} + \tfrac{1}{3 \cdot 8} + \tfrac{3}{2}\log(\tfrac{8}{3}) - \log(\tfrac{1}{\sqrt{2\pi}}) + \tfrac{43}{48} \cdot \tfrac{6}{5} - \tfrac{53}{1920} \cdot \tfrac{36}{25} \\
&= 0.0018... \ > \ 0.
\end{aligned}$$

Thus we have $i(n,p) > 0$ and hence $g(k+1) \leq f(k)$ in case of $k = n-1$.

We may now assume $k \leq n-2$, hence $z \leq nq - 2$, which implies $\frac{z}{n} \leq q - \frac{2}{n} = q - \frac{2pq}{\sigma^2} \leq q - \frac{18pq}{25}$ and hence $\frac{pq}{(p+z/n)(q-z/n)} \leq \frac{pq}{(1-18/25 \cdot pq) \cdot 18/25pq} = \frac{25}{18(1-18/25 \cdot pq)} \leq \frac{25}{18(1-18/25/4)} = \frac{25 \cdot 100}{18 \cdot 82} = 1.6938... \leq \frac{7}{4}$ and $\frac{z}{nq} \leq 1 - \frac{18}{25}p \leq 1 - \frac{9}{25} \leq \frac{2}{3}$.

Thus we have with (19) similar to the computation starting in (20), only even with $\frac{n}{12(np+z)(nq-z)} \leq \frac{7}{48\sigma^2}$ instead of $\frac{n}{12(np+z)(nq-z)} \leq \frac{15}{96\sigma^2}$ there,

$$\begin{aligned}
h(z,p) &\geq \tfrac{z(1-(q-p))}{2\sigma^2} + \tfrac{1}{\sigma^2} \cdot (\tfrac{1}{6} - \tfrac{7}{48}) - \tfrac{z^2+z+24/90}{24\sigma^4} + \tfrac{z^3(q^2-p^2)}{6\sigma^4} - \tfrac{z^4(q^3+p^3)}{12\sigma^6} - \tfrac{3z^5p^4}{20\sigma^8} \\
&\quad + \tfrac{z^2(p^2+q^2)}{4\sigma^4} - \tfrac{z^3(q^3-p^3)}{6\sigma^6} \\
&\geq \tfrac{z}{2\sigma^2} + \tfrac{z(p-q)}{2\sigma^2} - \tfrac{z(p^2-q^2)}{4\sigma^2} - \tfrac{z^2+z}{24\sigma^4} - \tfrac{z^4}{12\sigma^6} \cdot \tfrac{7}{12} - \tfrac{3z^5 5^4}{6^4 \cdot 20\sigma^8} + \tfrac{z^2}{8\sigma^4} + \tfrac{1}{\sigma^2} \cdot (\tfrac{1}{48} - \tfrac{1}{90\sigma^2}) \\
&\geq \tfrac{z}{2\sigma^2} + \tfrac{z(p-q)}{4\sigma^2} - \tfrac{z^2+z}{24\sigma^4} - \tfrac{z^2}{12\sigma^4} \cdot \tfrac{7}{12} - \tfrac{3z \cdot 5^4}{6^4 \cdot 20\sigma^4} + \tfrac{z^2}{8\sigma^4} \\
&= \tfrac{z(1+2p)}{4\sigma^2} + \tfrac{5z^2}{144\sigma^4} - \tfrac{197z}{1728\sigma^4} \ \geq \ \tfrac{z}{2\sigma^2} - \tfrac{197z}{1728\sigma^2} \cdot \tfrac{6}{5} \ = \ \tfrac{523z}{1440\sigma^2} \ > \ 0.
\end{aligned}$$

□

Lemma 7.4. *Let $n, k \in \mathbb{N}$ and $p \in [\frac{1}{6}, \frac{1}{2}]$. Then if $k > np + \frac{3}{2} \cdot \sqrt{npq}$ we have*

$$|F(k) - G(k)| \;\leq\; c_0 \cdot \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} \tag{21}$$

$$|G(k+1) - F(k)| \;\leq\; c_0 \cdot \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} \tag{22}$$

with $c_0 := \frac{2}{\sqrt{2\pi}\,c_{\mathrm{E}}} - 1 = 0.9473... < 1$.

Proof. Since $G(k) - F(k) \leq G(k+1) - F(k)$ as well as $F(k) - G(k+1) \leq F(k) - G(k)$ it is enough to prove (21) and (22) without absolute values on the left hand side. Further we declare that $\frac{c_0}{2} = (\frac{2}{\sqrt{2\pi}\,c_{\mathrm{E}}} - 1)/2 \leq 0.4737$ and $\frac{3+|p-q|}{6} \geq \frac{1}{2}$.

On $G(k+1) - F(k)$: First, we consider the case that additionally $k \leq np + npq$ holds. Due to Zubkov/Serov (2012) we have with $z := k - np$ and $t := \frac{z}{npq} \leq 1$

$$\begin{aligned}
F(k) \;&\geq\; \Phi\big(sgn(\tfrac{s}{n} - p)\sqrt{2n(\tfrac{s}{n}\log(\tfrac{k}{np}) + \tfrac{n-k}{n}\log(\tfrac{n-k}{n(1-p)}))}\big) \\
&=\; \Phi\big(\sqrt{2n(\tfrac{np+z}{n}\log(1+\tfrac{z}{np}) + \tfrac{nq-z}{n}\log(1-\tfrac{z}{nq}))}\big) \\
&=\; \Phi\big(\sqrt{\tfrac{z^2}{\sigma^2 t^2}\cdot 2n \cdot \tfrac{1}{\sigma^2}\cdot(\tfrac{np+tnpq}{n}\log(1+\tfrac{tnpq}{np}) + \tfrac{nq-tnpq}{n}\log(1-\tfrac{tnpq}{nq}))}\big) \\
&=\; \Phi\big(\tfrac{z}{\sigma}\cdot\sqrt{\tfrac{2}{t^2}\cdot(\tfrac{1+tq}{q}\log(1+tq) + \tfrac{1-tp}{p}\log(1-tp))}\big).
\end{aligned}$$

Now we show $h_1(t) := \frac{1+tq}{q}\log(1+tq) + \frac{1-tp}{p}\log(1-tp) - \frac{t^2}{2}\cdot(1 - \frac{t(1-2p)}{6})^2 \geq 0$. We have

$$\begin{aligned}
h_1''(t) \;&=\; \tfrac{\mathrm{d}}{\mathrm{d}t}\big[\log(1+tq) - \log(1-tp) - t + \tfrac{t^2(1-2p)}{2} - \tfrac{t^3(1-2p)^2}{18}\big] \\
&=\; \tfrac{q}{1+tq} + \tfrac{p}{1-tp} - 1 + t(1-2p) - \tfrac{t^2(1-2p)^2}{6} \\
&=\; \tfrac{-t(1-2p)+t^2pq}{(1+tq)(1-tp)} + t(1-2p) - \tfrac{t^2(1-2p)^2}{6} \\
&=\; \tfrac{t(1-2p)\cdot(t(1-2p)-t^2pq)}{(1+tq)(1-tp)} + \tfrac{t^2pq}{(1+tq)(1-tp)} - \tfrac{t^2(1-2p)^2}{6} \\
&=\; \tfrac{t^2}{(1+tq)(1-tp)}\cdot\big((1-2p)^2 - t(1-2p)pq + pq - \tfrac{(1-2p)^2}{6}\cdot(1+t(1-2p)-t^2pq)\big) \\
&=\; \tfrac{t^2}{(1+tq)(1-tp)}\cdot\big(\tfrac{5}{6}\cdot(1-2p)^2 + pq - t\big((1-2p)pq + \tfrac{(1-2p)^3}{6}\big) + t^2\cdot\tfrac{(1-2p)^2pq}{6}\big) \\
&\geq\; \tfrac{t^2}{(1+tq)(1-tp)}\cdot\big(\tfrac{5}{6}\cdot(1-2p)^2 + pq - \tfrac{1}{p}\cdot\big((1-2p)pq + \tfrac{(1-2p)^3}{6}\big) + \tfrac{1}{p^2}\cdot\tfrac{(1-2p)^2pq}{6}\big) \\
&=\; \tfrac{t^2}{(1+tq)(1-tp)}\cdot\big(\tfrac{5}{6}\cdot(1-2p)^2 + pq - (1-2p)q - \tfrac{(1-2p)^2}{6p}\cdot(1-2p-q)\big) \\
&=\; \tfrac{t^2}{(1+tq)(1-tp)}\cdot\big((1-2p)^2 + p(1-p) - (1-2p)(1-p)\big) \\
&=\; \tfrac{t^2p^2}{(1+tq)(1-tp)} \;>\; 0
\end{aligned}$$

because of $\frac{\mathrm{d}}{\mathrm{d}t}\big[-t\big((1-2p)pq+\frac{(1-2p)^3}{6}\big)+t^2\cdot\frac{(1-2p)^2pq}{6}\big] \leq -(1-2p)pq+\frac{(1-2p)^3}{6}+\frac{2}{p}\cdot\frac{(1-2p)^2pq}{6} = -(1-2p)\cdot(-\frac{1}{6}+\frac{4}{3}p-p^2) \leq -(1-2p)\cdot(-\frac{1}{6}+\frac{4}{3}\cdot\frac{1}{6}-\frac{1}{36}) = -\frac{1-2p}{36} \leq 0$ for $t \leq 1 < \frac{1}{p}$.

From that it follows $h_1'(t) \geq h_1'(0) = 0$ and hence $h_1(t) \geq h_1(0) = 0$. Thus we now have

$$F(k) \;\geq\; \Phi\big(\tfrac{z}{\sigma}\cdot(1-\tfrac{t(1-2p)}{6})\big) \;=\; \Phi\big(\tfrac{z}{\sigma}\cdot(1-\tfrac{z(1-2p)}{6\sigma^2})\big)$$

and hence with $\Phi(y+a)-\Phi(y) \leq a\varphi(y)$ for $y, a > 0$ since φ is decreasing on $[0,\infty)$, $\frac{z}{\sigma} \geq \frac{3}{2}$ and $\frac{z}{\sigma^2} \leq 1$ we have

$$\begin{aligned}
G(k+1)-F(k) &\leq \Phi(\tfrac{z+1}{\sigma}) - \Phi(\tfrac{z}{\sigma}(1-\tfrac{1-2p}{6}\cdot\tfrac{z}{\sigma^2})) \\
&\leq \tfrac{1}{\sigma}\cdot(1+\tfrac{1-2p}{6}\cdot\tfrac{z^2}{\sigma^2})\cdot\varphi\Big(\tfrac{z}{\sigma}\cdot(1-\tfrac{1-2p}{6}\cdot\tfrac{z}{\sigma^2})\Big) \\
&\leq \tfrac{1}{\sigma}\cdot \mathrm{e}^{\frac{1-2p}{6}\cdot\frac{z^2}{\sigma^2}}\cdot\varphi\Big(\tfrac{z}{\sigma}\cdot(1-\tfrac{1-2p}{6})\Big) \\
&= \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot \mathrm{e}^{\frac{z^2}{\sigma^2}\cdot(\frac{1-2p}{6}-\frac{1}{2}\cdot(1-\frac{1-2p}{6})^2)} \\
&= \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot \mathrm{e}^{\frac{z^2}{\sigma^2}\cdot(-\frac{1}{6}-\frac{2}{3}p-\frac{(1-2p)^2}{32})} \\
&\leq \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot \mathrm{e}^{-\frac{z^2}{\sigma^2}\cdot(\frac{1}{6}+\frac{2}{3}p)} \;\leq\; \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot \mathrm{e}^{-\frac{3}{8}-\frac{3}{2}p} \;\leq\; c_0\cdot\tfrac{4-2p}{6\sqrt{2\pi}\,\sigma},
\end{aligned}$$

where in the last step we use that if $i(p) := c_0\cdot\frac{4-2p}{6} - \mathrm{e}^{-\frac{3}{8}-\frac{3}{2}p}$, then we have

$$i'(p) \;=\; -\tfrac{c_0}{3}+\tfrac{3}{2}\cdot\mathrm{e}^{-\frac{3}{8}-\frac{3}{2}p} \;\geq\; -\tfrac{1}{3}+\tfrac{3}{2}\cdot\mathrm{e}^{-\frac{3}{8}-\frac{3}{2\cdot 6}} > 0$$

and hence $i(p) \geq i(\frac{1}{6}) = \frac{11}{18}\cdot c_0 - \mathrm{e}^{-\frac{5}{8}} = 0.0437... > 0$.

Second, we consider the case $k > np + npq$. If $k \geq n$ we have $G(k+1) - F(k) = G(k+1) - 1 < 0$ and hence (22), so we assume $k < n$. Due to Zubkov/Serov (2012) again, we have for $np + npq \leq k < n$

$$F(k) \;\geq\; \Phi\big(\sqrt{2nH(\tfrac{k}{n},p)}\big)$$

with $H(y,p) := y\log\frac{y}{p} + (1-y)\log\frac{1-y}{1-p}$. Further H is increasing with respect to y for $y \geq p$, because we have

$$\begin{aligned}
\tfrac{dH}{dy}(y,p) &= \log\tfrac{y}{p}+\tfrac{1}{p}p-\log\tfrac{1-y}{1-p}+\tfrac{-1}{1-p}(1-p) \;=\; \log\tfrac{y}{p}-\log\tfrac{1-y}{1-p} \;=\; \log\tfrac{y(1-p)}{p(1-y)} \\
&= \log\tfrac{y-yp}{p-yp} \;\geq\; \log 1 \;=\; 0.
\end{aligned}$$

Consequent we get for $np + npq \leq k < n$ that $H(\frac{k}{n},p) \geq H(\frac{np+npq}{n},p)$ holds and additionally with the monotonicity of Φ we have

$$F(k) \;\geq\; \Phi\big(\sqrt{2nH(\tfrac{k}{n},p)}\big) \;\geq \Phi\big(\sqrt{2nH(\tfrac{np+npq}{n},p)}\big).$$

Further we have

$$\begin{aligned} H(\tfrac{np+npq}{n},p) &= (p+pq)\log\tfrac{p+pq}{p} + (1-p-pq)\log\tfrac{1-p-pq}{1-p} \\ &= pq\Big(\tfrac{1+q}{q}\log(1+q)+\tfrac{q}{p}\log(q)\Big) \\ &=: pq\cdot h_2(p), \end{aligned}$$

whereby h_2 is increasing because of

$$\begin{aligned} h_2'(p) &= \tfrac{1}{(1-p)^2}\log(2-p) - \tfrac{1}{1-p} - \tfrac{1}{p^2}\log(1-p) - \tfrac{1}{p} \\ &= \tfrac{1}{q^2}\log(1+q) - \tfrac{1}{q} - \tfrac{1}{p^2}\log(1-p) - \tfrac{1}{p} \\ &= \tfrac{1}{q}(\tfrac{1}{q}\log(1+q)-1) + \tfrac{1}{p}(-\tfrac{1}{p}\log(1-p)-1) \\ &\geq \tfrac{1}{q}(\tfrac{1}{q}\cdot(q-\tfrac{q^2}{2})-1) + \tfrac{1}{p}(-\tfrac{1}{p}\cdot(-\tfrac{1}{p}-\tfrac{p^2}{2}-\tfrac{p^3}{3})-1) \\ &= -\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{p}{3} \;\geq\; 0. \end{aligned}$$

Thus the minimum of h_2 is attained at the smallest p of the domain, which we divide in two parts.

1.Case: Let $p \in [\frac{1}{3}, \frac{1}{2}]$. Then we have $F(k) \geq \Phi(\sqrt{2npq\cdot h_2(\frac{1}{3})}) = \Phi(\sigma\cdot\sqrt{2\cdot h_2(\frac{1}{3})}) \geq \Phi(\sigma\cdot\sqrt{2\cdot 0.466}) \geq \Phi(\sigma\cdot 0.96)$ and hence due to Lemma 11.3 in the third step

$$G(k+1)-F(k) \;\leq\; 1-F(k) \;\leq\; 1-\Phi(\sigma\cdot 0.96) \;\leq\; 0.467\cdot\tfrac{1}{\sqrt{2\pi}\,\sigma} \;\leq\; \tfrac{c_0}{2\sqrt{2\pi}\,\sigma}.$$

2.Case: Let $p \in [\frac{1}{6}, \frac{1}{3}]$. Then $F(k) \geq \Phi(\sigma\cdot\sqrt{2\cdot h_2(\frac{1}{6})}) \geq \Phi(\sigma\cdot\sqrt{2\cdot 0.42}) \geq \Phi(\sigma\cdot 0.9)$ and hence again due to Lemma 11.3

$$G(k+1)-F(k) \;\leq\; 1-F(k) \;\leq\; 1-\Phi(\sigma\cdot 0.9) \;\leq\; \tfrac{1}{2\sqrt{2\pi}\,\sigma} \;\leq\; c_0\cdot\tfrac{4-2p}{6\sqrt{2\pi}\,\sigma}$$

since $c_0\cdot\frac{4-2p}{6} \geq c_0\cdot\frac{4-2/3}{6} = 0.5263... \geq \frac{1}{2}$.

<u>On $F(k)-G(k)$:</u> To make the computation more clearly we make an indexshift of $k \to k-1$, so we show the claim for all $k-1 > np+\frac{3}{2}\sqrt{npq}$, implying $z := k-np \geq \frac{3}{2}\cdot\sigma+1$. If $\sigma < 2$ we have

$$\begin{aligned} F(k-1)-G(k-1) &\leq 1-G(k-1) \;\leq\; 1-\Phi(\tfrac{3}{2}) \;\leq\; 0.067 \leq \tfrac{0.067\cdot 2\cdot\sqrt{2\pi}}{\sqrt{2\pi}\,\sigma} \\ &= \tfrac{0.3359...}{\sqrt{2\pi}\,\sigma} \;\leq\; c_0\cdot\tfrac{1}{2\sqrt{2\pi}\,\sigma}. \end{aligned}$$

Thus we can assume $\sigma \geq 2$.

Let us assume again firstly $k-1 \le np+npq-1$ and hence $z = k-np \le npq$ and $t = \frac{z}{npq} \le 1$. In contrast to above we now estimate $F(k-1)$ upwards. We have with $L(t) := \frac{2}{t^2}\big(\frac{1+tq}{q}\log(1+tq) + \frac{1-tp}{p}\log(1-tp)\big)$ due to Zubkov/Serov (2012) again

$$F(k-1) \ \le \ \Phi\big(\tfrac{z}{\sigma}\sqrt{L(t)}\,\big).$$

Now we estimate L upwards and obtain

$$\begin{aligned}
L(t) &= \tfrac{2}{t^2}\cdot\Big(\tfrac{1+tq}{q}\log(1+tq) + \tfrac{1-tp}{p}\log(1-tp)\Big)\\
&\le \tfrac{2}{t^2}\cdot\Big(\tfrac{1+tq}{q}(tq - \tfrac{t^2q^2}{2} + \tfrac{t^3q^3}{3}) + \tfrac{1-tp}{p}(-tp - \tfrac{t^2p^2}{2} - \tfrac{t^3p^3}{3})\Big)\\
&= 1 - \tfrac{t}{3}(q^2-p^2) + \tfrac{2t^2}{3}(q^3+p^3)\\
&= 1 + \tfrac{t}{3}(2p-1) + \tfrac{2t^2}{3}(q^3+p^3)\\
&\le 1 + \tfrac{t^2}{3}(2p-1+2q^3+2p^3)\\
&\le 1 + \tfrac{t^2}{6} \ \le \ (1+\tfrac{t^2}{12})^2
\end{aligned}$$

since in the penultimate step we use that for $i_2(p) := 2p-1+2q^3+2p^3$ we have $i_2''(p) = 12 > 0$ and hence i_2 convex and additionally from $i_2(\frac{1}{2}) = i_2(\frac{1}{6}) = \frac{1}{2}$ it follows $i_2(p) \le \frac{1}{2}$.

From this it follows that $\sqrt{L(t)} \le 1 + \frac{t^2}{12} = 1 + \frac{z^2}{12\sigma^4}$ and with φ decreasing on $[0,\infty)$ in the second step below we get

$$\begin{aligned}
F(k-1) - G(k-1) &\le \Phi\big(\tfrac{z}{\sigma}(1+\tfrac{z^2}{12\sigma^4})\big) - \Phi\big(\tfrac{z-1}{\sigma}\big) \ \le \ \tfrac{1}{\sigma}\cdot(1+\tfrac{z^3}{12\sigma^4})\cdot\varphi\big(\tfrac{z-1}{\sigma}\big)\\
&\le \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot \mathrm{e}^{\frac{z^3}{12\sigma^4} - \frac{(z-1)^2}{2\sigma^2}} \ \le \ \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot \mathrm{e}^{\frac{(3\sigma/2+1)^3}{12\sigma^4} - \frac{(3\sigma/2)^2}{2\sigma^2}}\\
&\le \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot \mathrm{e}^{\frac{(3+1)^3}{12\cdot 2^4} - \frac{9}{8}} \ = \ \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot \mathrm{e}^{-\frac{19}{24}} \ \le \ 0.46\cdot\tfrac{1}{\sqrt{2\pi}\,\sigma}
\end{aligned}$$

since $\frac{\mathrm{d}}{dz}\big[\frac{z^3}{12\sigma^4} - \frac{(z-1)^2}{2\sigma^2}\big] = \frac{z^2}{4\sigma^4} - \frac{z-1}{\sigma^2} \le \frac{z}{4\sigma^2} - \frac{z-1}{\sigma^2} = -\frac{3}{4}\cdot\frac{z}{\sigma^2} + \frac{1}{\sigma^2} \le -\frac{9}{4}\cdot\frac{1}{\sigma^2} + \frac{1}{\sigma^2} < 0$ with $z \ge \frac{3}{2}\sigma \ge 3$ according to requirement in the fourth and $\sigma \ge 2$ in the fifth inequality.

Finally we consider the case $k-1 > np+npq-1$. In this case we have due to $\sigma > 2$ and Lemma 11.3 in the last inequality

$$F(k-1) - G(k-1) \ \le \ 1 - G(k-1) \ \le \ 1 - \Phi\big(\tfrac{\sigma^2-1}{\sigma}\big) \ \le \ \tfrac{4}{9}\cdot\tfrac{1}{\sqrt{2\pi}\,\sigma}.$$

Thus (21) also follows in this case. $\square$

8. Proof of Proposition 6.2

Let us restate Proposition 6.2 for convenience:

Proposition 6.2. *Let* $\frac{1}{6} \le p \le \frac{5}{6}$ *and* $n \in \mathbb{N}$. *Then we have*

$$\max\Big\{\Delta(\lfloor np \rfloor),\ \Delta(\lfloor np \rfloor -),\ \Delta(\lceil np \rceil),\ \Delta(\lceil np \rceil -)\Big\} \ < \ \tfrac{p^2+q^2}{\sqrt{2\pi}\,\sigma}. \tag{23}$$

In the case $\frac{1}{3} \le p \le \frac{2}{3}$ *we even have*

$$\max\Big\{\Delta(\lfloor np \rfloor),\ \Delta(\lfloor np \rfloor -),\ \Delta(\lceil np \rceil),\ \Delta(\lceil np \rceil -)\Big\} \ < \ \tfrac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}. \tag{24}$$

Proof. As in the proof of Proposition 6.1 we use Lemma 8.1 - 8.5 stated here, but proved below on page 55 - 80, because in this way the structure of the present proof becomes clearer.

We first treat the case of $\frac{1}{6} \le p < \frac{1}{2}$. As already mentioned in chapter 6 on page 29 even (24) holds if $n = 1$ due to Bentkus/Kirsa (1989) for each $p \in (0,1)$, so we can assume $n \ge 2$ if convenient.

We consider now the case of $m = \lfloor np \rfloor = \frac{n-1}{2}$.

Lemma 8.1. *Let* $n \in \mathbb{N}$, $p \in [\frac{1}{6}, \frac{1}{2})$ *with* $n \ge 2$ *odd and* $\lfloor np \rfloor = \frac{n-1}{2}$. *Then we have*

$$\max\Big\{\Delta(\lfloor np \rfloor),\ \Delta(\lfloor np \rfloor -),\ \Delta(\lceil np \rceil),\ \Delta(\lceil np \rceil -)\Big\} \ < \ \tfrac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}.$$

Hence we can assume $m \le \frac{n}{2} - 1$, since we already have $m < \frac{n}{2}$ according to the condition $np < \frac{n}{2}$. The easier cases $\Delta(\lfloor np \rfloor -)$, $\Delta(\lceil np \rceil)$ and $\Delta(\lceil np \rceil -)$ follow from

Lemma 8.2. *Let* $n \in \mathbb{N}$, $p \in [\frac{1}{6}, \frac{1}{2}]$ *with* $m \le \frac{n}{2} - 1$. *Then we have*

$$\max\Big\{\Delta(\lfloor np \rfloor -),\ \Delta(\lceil np \rceil),\ \Delta(\lceil np \rceil -)\Big\} \ < \ \tfrac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}.$$

The case $\Delta(m) = |F(m) - G(m)|$ will be more difficult.

If $G(m) - F(m) > 0$, this case follows from $G(m) - F(m) \le G(m) - F(m-1) \le \Delta(m-)$ and Lemma 8.2 and hence we can assume $|F(m) - G(m)| = F(m) - G(m)$.

It remains to show $\frac{4-2p}{6\sqrt{2\pi}\,\sigma} - F(m) + G(m) \ > \ 0$ for $p \in [\frac{1}{6}, \frac{1}{2})$ and we first assume now $m \ge 2$.

Then, if $\frac{n}{3} \le m+1 \le \frac{n}{2}$ and first if $p \in [\frac{m}{n}, \frac{m}{n} + \frac{1}{2n}]$, hence $np - \lfloor np \rfloor \in [0, \frac{1}{2}]$ we get

Lemma 8.3. *If $n \in \mathbb{N}$, $p \in [\frac{1}{6}, \frac{1}{2})$, such that $m+1 \in [\frac{n}{3}, \frac{n}{2}]$, $m \ge 2$ and $p \in [\frac{m}{n}, \frac{m+1/2}{n}]$, then we have*

$$\mathcal{D} \quad := \quad \frac{4-2p}{6\sqrt{2\pi}\,\sigma} - F(m) + G(m) \quad > \quad 0.$$

If $p \in (\frac{m}{n} + \frac{1}{2n}, \frac{m+1}{n})$, hence $np - \lfloor np \rfloor \in (\frac{1}{2}, 1)$ we also get

Lemma 8.4. *If $n \in \mathbb{N}$, $p \in [\frac{1}{6}, \frac{1}{2})$, such that $m+1 \in [\frac{n}{3}, \frac{n}{2}]$, $m \ge 2$ and $p \in (\frac{m+1/2}{n}, \frac{m+1}{n})$, then we have*

$$\frac{4-2p}{6\sqrt{2\pi}\,\sigma} - F(m) + G(m) \quad > \quad 0.$$

Next we consider $\frac{n}{6} \le m+1 \le \frac{n}{3}$, which implies $p \le \frac{1}{3}$. In this area we are limited to the weaker estimation:

Lemma 8.5. *If $n \in \mathbb{N}$, $p \in [\frac{1}{6}, \frac{1}{2})$, such that $m+1 \in [\frac{n}{6}, \frac{n}{3}]$ and $m \ge 2$, then we have*

$$\frac{\sqrt{10}+3}{6\sqrt{2\pi}} \cdot \frac{p^2+q^2}{\sigma} - F(m) + G(m) \quad > \quad 0.$$

The case $m \le 1$ now follows from

Lemma 8.6. *Let $n \in \mathbb{N}$, $p \in [\frac{1}{6}, \frac{1}{2})$ with $m \le \frac{n}{2} - 1$ and $m \le 1$. Then we have*

$$\begin{array}{llll} (i) & \frac{4-2p}{6\sqrt{2\pi}\,\sigma} - F(m) + G(m) & > 0 & \text{if } p \in [\frac{1}{3}, \frac{1}{2}) \\ (ii) & \frac{\sqrt{10}+3}{6\sqrt{2\pi}} \cdot \frac{p^2+q^2}{\sigma} - F(m) + G(m) & > 0 & \text{if } p \in [\frac{1}{6}, \frac{1}{3}), \end{array}$$

which is proved in chapter 9 from page 87.

Now we turn to the case $p \in (\frac{1}{2}, \frac{5}{6}]$. We use the symmetries

$$F_{n,p}(m) = 1 - F_{n,q}(n-m-1) \quad \text{and} \quad \Phi\left(\tfrac{m-np}{\sigma}\right) = 1 - \Phi\left(\tfrac{(n-m)-nq}{\sigma}\right)$$

and thus we have for $x_0 \in \{\lfloor np \rfloor, \lceil np \rceil\}$

$$\begin{aligned} \Delta(x_0) &= \left|F_{n,p}(x_0) - \Phi\left(\tfrac{x_0-np}{\sigma}\right)\right| = \left|F_{n,q}((n-x_0)-) - \Phi\left(\tfrac{((n-x_0)-)-nq}{\sigma}\right)\right| \\ \Delta(x_0-) &= \left|F_{n,p}(x_0-) - \Phi\left(\tfrac{(x_0-np)-}{\sigma}\right)\right| = \left|F_{n,q}(n-x_0) - \Phi\left(\tfrac{(n-x_0)-nq}{\sigma}\right)\right|, \end{aligned}$$

where $(n-x_0) \in \{\lfloor nq \rfloor, \lceil nq \rceil\}$ and $q \in [\frac{1}{6}, \frac{1}{2})$.

Based on the symmetries of σ and $\frac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}$ respectively $\frac{p^2+q^2}{\sqrt{2\pi}\,\sigma}$ with respect to p, (23) and (24) also follow in case of $p \in (\frac{1}{2}, \frac{5}{6}]$.

Finally, the result of Hipp/Mattner (2007) yields (23) and (24) in case of $p = \frac{1}{2}$. $\square$

We now prove Lemmas 8.1 - 8.5 used above:

Lemma 8.1. *Let $n \in \mathbb{N}$, $p \in [\frac{1}{6}, \frac{1}{2})$ with $n \geq 2$ odd and $\lfloor np \rfloor = \frac{n-1}{2}$. Then we have*

$$\max\Big\{\Delta(\lfloor np \rfloor),\ \Delta(\lfloor np \rfloor -),\ \Delta(\lceil np \rceil),\ \Delta(\lceil np \rceil -)\Big\} \ < \ \tfrac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}.$$

Proof. We have $p = \frac{1}{2} - \frac{y}{n}$ for an $y \in [0, \frac{1}{2}]$ and hence $\frac{3+|p-q|}{6\sqrt{2\pi}\,\sigma} = \frac{4-2p}{6\sqrt{2\pi}\,\sigma} = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot (\frac{1}{2} + \frac{y}{3n})$.

<u>1.) $G(m) - F(m-1)$:</u> According to Lemma 11.4 with $n-1, m$ in place of n, k we have

$$\sqrt{2\pi}\,\sigma \cdot f_{n-1,\frac{1}{2}}(m) \ < \ \sqrt{\tfrac{np(1-p)}{(n-1)/4}} \cdot \mathrm{e}^{-\frac{19}{320(n-1)/4}} \ \leq \ 1 + \tfrac{5}{18(n-1)}, \tag{25}$$

where we use in the second inequality

$$\begin{aligned}
\sqrt{\tfrac{np(1-p)}{(n-1)/4}} \cdot \mathrm{e}^{-\frac{19}{320(n-1)/4}} \ &\leq \ \sqrt{\tfrac{n}{n-1}} \cdot \mathrm{e}^{-\frac{19}{80(n-1)}} \ \leq \ (1 + \tfrac{1}{2(n-1)}) \cdot (1 - \tfrac{19}{80(n-1)} \cdot (1 - \tfrac{19}{2\cdot 80(n-1)})) \\
&\leq \ (1 + \tfrac{1}{2(n-1)}) \cdot (1 - \tfrac{19}{80(n-1)} \cdot \tfrac{301}{320}) \\
&\leq \ 1 + \tfrac{1}{2(n-1)} - \tfrac{2}{9(n-1)} \ = \ 1 + \tfrac{5}{18(n-1)}
\end{aligned}$$

since $e^{-x} \leq 1 - x + \frac{x^2}{2}$ for $x \in \mathbb{R}$ and $\frac{19\cdot 301}{80\cdot 320} \geq \frac{19\cdot 300}{80\cdot 320} = \frac{57}{216} \geq \frac{2}{9}$. Hence with Lemma 11.1 and Lemma 11.13 in the first inequality below we have

$$\begin{aligned}
G(m) - F(m-1) \ &= \ \Phi(-\tfrac{1/2-y}{\sigma}) - \tfrac{1}{2} - (F(m) - \tfrac{1}{2}) + f(m) \\
&\leq \ -\tfrac{1/2-y}{\sqrt{2\pi}\,\sigma} + \tfrac{(1-2y)^3}{12\sqrt{2\pi}\,(n-1)} \\
&\quad\ +f_{n-1,\frac{1}{2}}(m) \cdot (-y + \tfrac{2(n-1)y^3}{3n^2} + 1 - \tfrac{17(n-1)y^2}{12n^2} - \tfrac{1-2y}{n+1}) \\
&< \ \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot \big(y - \tfrac{1}{2} + \tfrac{(1-2y)^3}{12(n-1)} + (1 + \tfrac{5}{18(n-1)}) \cdot (1 - y - \tfrac{1-2y}{n+1} - \tfrac{13(n-1)y^2}{12n^2})\big) \\
&\leq \ \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot \big(y - \tfrac{1}{2} + \tfrac{1-2y}{12(n-1)} + (1 + \tfrac{5}{18(n-1)})(1-y) - \tfrac{1-2y}{2(n-1)} - \tfrac{13(n-1)y^2}{12n^2}\big) \\
&= \ \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot \big(-\tfrac{1}{n-1} \cdot (\tfrac{5(1-2y)}{12} - \tfrac{5(1-y)}{18} + \tfrac{13(n-1)^2y^2}{12n^2} + \tfrac{y(n-1)}{3n}) + \tfrac{1}{2} + \tfrac{y}{3n}\big) \\
&\leq \ \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot \big(-\tfrac{1}{n-1} \cdot (\tfrac{5(1-2y)}{12} - \tfrac{5(1-y)}{18} + \tfrac{13y^2}{12} \cdot \tfrac{4}{9} + \tfrac{2y}{9}) + \tfrac{1}{2} + \tfrac{y}{3n}\big) \\
&= \ \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot \big(-\tfrac{1}{n-1} \cdot (\tfrac{5}{36} - \tfrac{y}{3} + \tfrac{13y^2}{27}) + \tfrac{1}{2} + \tfrac{y}{3n}\big) \\
&\leq \ \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot \big(\tfrac{1}{2} + \tfrac{y}{3n}\big),
\end{aligned}$$

where we used $1 - y - \frac{1-2y}{n+1} - \frac{13(n-1)y^2}{12n^2} \geq 1 - y - \frac{1-2y}{4} - \frac{13\cdot 3y}{12\cdot 16\cdot 2} = \frac{3}{4} - \frac{77y}{128} > 0$ in the penultimate and $\frac{5}{36} - \frac{y}{3} + \frac{13y^2}{27} \geq \frac{5}{36} - \frac{y(1-y)}{3} \geq \frac{5}{36} - \frac{1}{12} = \frac{1}{18} > 0$ in the ultimate inequality.

2.) $F(m)-G(m)$: According to Lemma 11.1 and Lemma 11.13 again in the first and inequality (25) in part 1.) above in the second inequality we get

$$\begin{aligned} F(m)-G(m) &= F(m)-\tfrac{1}{2}-(G(m)-\tfrac{1}{2}) \\ &\le y\cdot f_{n-1,\frac{1}{2}}(m)+\tfrac{1/2-y}{\sqrt{2\pi}\,\sigma} \\ &\le \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot(y\cdot\sqrt{\tfrac{n}{n-1}}\cdot \mathrm{e}^{-\frac{19}{80(n-1)}}+\tfrac{1}{2}-y) \\ &\le \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot\big(\tfrac{1}{2}+\tfrac{y}{3n}-y\cdot(1+\tfrac{1}{3n}-\sqrt{\tfrac{n}{n-1}}\cdot \mathrm{e}^{-\frac{19}{80(n-1)}})\big) \\ &< \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot(\tfrac{1}{2}+\tfrac{y}{3n}) \end{aligned}$$

since in the last inequality we have for $n\ge 5$

$$\begin{aligned} & \log(1+\tfrac{1}{3n})-\log\big(\sqrt{\tfrac{n}{n-1}}\cdot \mathrm{e}^{-\frac{19}{80(n-1)}}\big) \\ = & \log(1+\tfrac{1}{3n})-\tfrac{1}{2}\cdot\log(1+\tfrac{1}{n-1})+\tfrac{19}{80(n-1)} \\ \ge & \tfrac{1}{3n}-\tfrac{1}{18n^2}-\tfrac{1}{2(n-1)}+\tfrac{1}{4(n-1)^2}-\tfrac{1}{6(n-1)^3}+\tfrac{19}{80(n-1)} \\ \ge & \tfrac{4}{3\cdot 5(n-1)}-\tfrac{1}{18(n-1)^2}-\tfrac{1}{2(n-1)}+\tfrac{1}{4(n-1)^2}-\tfrac{1}{24(n-1)^2}+\tfrac{19}{80(n-1)} \\ = & \tfrac{1}{240(n-1)}+\tfrac{11}{72(n-1)^2} > 0 \end{aligned}$$

and if $n=3$ we have $1+\frac{1}{9}-\sqrt{3/2}\cdot \mathrm{e}^{-\frac{19}{160}}=0.0235...>0$.

3.) $G(m+1)-F(m)$: First we can assume $\lceil np\rceil=\lfloor np\rfloor+1$, otherwise we consider $G(m)-F(m-1)$, see 1.) above. We have $G(m+1)-\frac{1}{2}=\Phi(\frac{1/2+y}{\sigma})-\frac{1}{2}\le\frac{1/2+y}{\sigma}$ due to Lemma 11.1 and

$$f_{n-1,\frac{1}{2}}(m) \ \ge\ \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot\sqrt{\tfrac{4np(1-p)}{n-1}}\cdot \mathrm{e}^{-\frac{19}{80(n-1)}} \ \ge\ \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot\sqrt{\tfrac{n+1}{n}}\cdot \mathrm{e}^{-\frac{19}{80(n-1)}} \ >\ \tfrac{1}{\sqrt{2\pi}\,\sigma}$$

due to $\log\big(\sqrt{\frac{n+1}{n}}\big)=\frac{1}{2}\cdot\log(1+\frac{1}{n})\ge\frac{1}{2n}-\frac{1}{4n^2}\ge\frac{1}{2n}\cdot\frac{5}{6}\ge\frac{5}{12(n-1)}\cdot\frac{2}{3}=\frac{5}{18(n-1)}>\frac{19}{80(n-1)}$ and Lemma 11.4.

Thus we have due to Lemma 11.13 in the first inequality below

$$\begin{aligned} G(m+1)-F(m) &= G(m+1)-\tfrac{1}{2}-(F(m)-\tfrac{1}{2}) \\ &\le \tfrac{1/2+y}{\sqrt{2\pi}\,\sigma}-f_{n-1,\frac{1}{2}}(m)\cdot y\cdot(1-\tfrac{2(n-1)y^2}{3n^2}) \\ &< \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot\big(\tfrac{1}{2}+y-y\cdot(1-\tfrac{2(n-1)y^2}{3n^2})\big) \ =\ \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot(\tfrac{1}{2}+\tfrac{2(n-1)y^2}{3n^2}) \\ &\le \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot(\tfrac{1}{2}+\tfrac{y}{3n}). \end{aligned}$$

<u>4.) $F(m+1)-G(m+1)$:</u> We assume analogous to 2.) again $\lceil np\rceil = \lfloor np\rfloor + 1$, since else we consider $F(m)-G(m)$, see 2.) above, and we have

$$\begin{aligned} G(m+1)-\tfrac{1}{2} &= \Phi(\tfrac{1/2+y)}{\sigma}) - \tfrac{1}{2} \geq \tfrac{1/2+y}{\sqrt{2\pi}\,\sigma}\mathrm{e}^{-\frac{(1/2+y)^2}{6\sigma^2}} \geq \tfrac{1/2+y}{\sqrt{2\pi}\,\sigma}\mathrm{e}^{-\frac{2n(1/2+y)^2}{3(n^2-1)}} \\ &\geq \tfrac{1/2+y}{\sqrt{2\pi}\,\sigma} - \tfrac{2n(1/2+y)^3}{3\sqrt{2\pi}\,\sigma(n^2-1)} \end{aligned}$$

due to Lemma 11.1 in the first and $\sigma^2 = n(\frac{1}{2}-\frac{y}{n})(\frac{1}{2}+\frac{y}{n}) = \frac{n}{4}\cdot(1-\frac{4y^2}{n^2}) \geq \frac{n}{4}\cdot(1-\frac{1}{n^2}) = \frac{n^2-1}{4n}$ in the second inequality.

Additionally with Lemma 11.13 in the first and inequality (25) in part 1.) in the second inequality below we get

$$\begin{aligned} F(m+1)-G(m+1) &= F(m)-\tfrac{1}{2}+f(m+1)-(G(m+1)-\tfrac{1}{2}) \\ &\leq f_{n-1,\frac{1}{2}}(m)\cdot(1-\tfrac{1+2y}{n+1}+y) - \tfrac{1/2+y}{\sqrt{2\pi}\,\sigma} + \tfrac{2n(1/2+y)^2}{3\sqrt{2\pi}\,\sigma(n^2-1)} \\ &\leq \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot\big((1+\tfrac{5}{18(n-1)})\cdot(1-\tfrac{1+2y}{n+1}+y)-(\tfrac{1}{2}+y)+\tfrac{2n(1/2+y)^3}{3(n^2-1)}\big) \\ &\leq \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot\big(\tfrac{1}{2}+\tfrac{5(1+y)}{18(n-1)}-\tfrac{1+2y}{n+1}+\tfrac{2n(1/2+y)^3}{3(n^2-1)}\big) \\ &= \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot\Big(\tfrac{1}{2}+\tfrac{y}{3n}-\tfrac{n}{n^2-1}\cdot\big(\tfrac{(n-1)(1+2y)}{n}-\tfrac{5(1+y)(n+1)}{18n} \\ &\qquad -\tfrac{2(1/2+y)^3}{3}+\tfrac{y(n^2-1)}{3n^2}\big)\Big) \\ &\leq \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot\big(\tfrac{1}{2}+\tfrac{y}{3n}-\tfrac{n}{n^2-1}\cdot(\tfrac{2(1+2y)}{3}-\tfrac{5(1+y)\cdot 4}{18\cdot 3}-\tfrac{2(1/2+y)^3}{3}+\tfrac{8y}{27n^2})\big) \\ &< \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot(\tfrac{1}{2}+\tfrac{y}{3n}) \end{aligned}$$

since $h(y) := \frac{2(1+2y)}{3} - \frac{10(1+y)}{27} - \frac{2(1/2+y)^3}{3} + \frac{8y}{27n^2}$ is concave in $y \in [0,\frac{1}{2}]$ and $h(0) = \frac{2}{3}-\frac{10}{27}-\frac{1}{12} = \frac{23}{108} > 0$ as well as $h(\frac{1}{2}) = \frac{4}{3}-\frac{5}{9}-\frac{2}{3}+\frac{4}{27n^2} = \frac{1}{9}+\frac{4}{27n^2} > 0$ in the last inequality.

<u>5.):</u> It follows $\Delta(\lfloor np\rfloor -) < \frac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}$ from 1.) if $G(m) \geq F(m-1)$ and else with $F(m-1)-G(m) \leq F(m)-G(m)$ from 2.) and $\Delta(\lfloor np\rfloor) < \frac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}$ from 2.) if $F(m) \geq G(m)$ and else with $G(m)-F(m) \leq G(m)-F(m-1)$ from 1.).

Analogously follows $\Delta(\lceil np\rceil -) < \frac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}$ from 3.) if $G(m+1) \geq F(m)$ and else with $F(m)-G(m+1) \leq F(m)-G(m)$ from 2.) and $\Delta(\lceil np\rceil) < \frac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}$ from 4.) if $F(m+1) \geq G(m+1)$ and else from 3.).

□

Lemma 8.2. *Let $n \in \mathbb{N}$, $p \in [\frac{1}{6}, \frac{1}{2})$ with $\lfloor np \rfloor \le \frac{n}{2} - 1$. Then we have*

$$\max\Big\{\Delta(\lfloor np\rfloor -),\ \Delta(\lceil np\rceil),\ \Delta(\lceil np\rceil -)\Big\} \ < \ \tfrac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}.$$

Proof. We consider each of the three cases separately.

<u>**1.)** $\Delta(\lceil np\rceil)$:</u>

If $np \in \mathbb{N}$, then $\Delta(\lceil np\rceil) = \Delta(\lfloor np\rfloor) < \frac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}$ due to Lemma 8.3 - 8.5 about the case $\Delta(\lfloor np\rfloor)$, hence we can assume $\lceil np\rceil = \lfloor np\rfloor + 1 = m+1$ again. If $F(m+1) < G(m+1)$, then it would be $G(m+1) - F(m+1) \le G(m+1) - F(m) < \frac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}$ due to 3.) below. Thus we show with $\ell = m+1 = \lfloor np\rfloor + 1$

$$\mathcal{D}_1 \ := \ \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - F(\ell) + G(\ell) \ > \ 0.$$

First we have since $\sigma_\lambda \ge \sigma$ and $p = \frac{\ell-\delta}{n}$

$$\mathcal{D}_1 \ \ge \ \tfrac{4-2\lambda+2\delta/n}{6\sqrt{2\pi}\,\sigma_\lambda} - F_{n,\frac{m+1-\delta}{n}}(\ell) + \Phi(\tfrac{\delta}{\sigma_\lambda}) \ =: \ h(\delta).$$

Further since $(n-\ell)\log(1-\frac{\delta}{n-\ell}) + \ell\log(1-\frac{\delta}{\ell}) \le (n-\ell)(\frac{\delta}{n-\ell} - \frac{\delta^2}{2(n-\ell)^2} + \frac{\delta^3}{3(n-\ell)^3}) - \ell(\frac{\delta}{\ell} + \frac{\delta^2}{2\ell^2} + \frac{\delta^3}{3\ell^3}) = -\frac{\delta^2}{2\sigma_\lambda^2} - \frac{\delta^3((n-\ell)^3-\ell^3}{\ell^3(n-\ell)^3} \le -\frac{\delta^2}{2\sigma_\lambda^2}$ we have

$$\tfrac{f_{n-1,p}(\ell)}{f_{n,\lambda}(\ell)} = (1+\tfrac{\delta}{n-\ell})^{n-\ell-1}\cdot(1-\tfrac{\delta}{\ell})^{\ell} \le (1+\tfrac{\delta}{n-\ell})^{n-\ell}\cdot(1-\tfrac{\delta}{\ell})^{\ell} \le \mathrm{e}^{-\frac{\delta^2}{2\sigma_\lambda^2}}$$

and from Lemma 11.4 and the inequality above it follows

$$\begin{aligned}
h'(\delta) \ &= \ \tfrac{1}{3n\sqrt{2\pi}\,\sigma_\lambda} - f_{n-1,p}(\ell) + \tfrac{1}{\sigma_\lambda}\cdot\varphi(\tfrac{\delta}{\sigma_\lambda}) \\
&= \ \tfrac{1}{\sqrt{2\pi}\,\sigma_\lambda}\cdot\big(\tfrac{1}{3n} - \tfrac{f_{n-1,p}(\ell)}{f_{n,\lambda}(\ell)}\cdot\sqrt{2\pi}\,\sigma_\lambda\cdot f_{n,\lambda}(\ell) + \mathrm{e}^{-\frac{\delta^2}{2\sigma_\lambda^2}}\big) \\
&\ge \ \tfrac{1}{\sqrt{2\pi}\,\sigma_\lambda}\cdot\big(-\mathrm{e}^{-\frac{\delta^2}{2\sigma_\lambda^2}}\cdot\mathrm{e}^{-\frac{19}{320\sigma_\lambda^2}} + \mathrm{e}^{-\frac{\delta^2}{2\sigma_\lambda^2}}\big) \ > \ 0
\end{aligned}$$

and hence according to Lemma 11.9 and $f_{n,\lambda}(\ell) < \frac{1}{\sqrt{2\pi}\,\sigma_\lambda}$ due to Lemma 11.4 we have

$$\mathcal{D}_1 \ \ge \ h(\delta) \ \ge \ h(0) \ = \ \tfrac{4-2\lambda}{6\sqrt{2\pi}\,\sigma_\lambda} - F_{n,\lambda}(\ell) + \tfrac{1}{2} \ \ge \ \tfrac{4-2\lambda}{6\sqrt{2\pi}\,\sigma_\lambda} - f_{n,\lambda}(\ell)\cdot\tfrac{4-2\lambda}{6} \ > \ 0$$

and hence the claim follows.

2.) $\Delta(\lfloor np\rfloor -)$:

We reduce this case to the case $\Delta(\lfloor np\rfloor)$. If $G(m) < F(m-1)$, then $|G(m)-F(m-1)| = F(m-1) - G(m) \;\le\; F(m) - G(m) < \Delta(\lfloor np\rfloor) < \frac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}$ due to Lemma 8.3 - Lemma 8.5, so we consider $G(m) - F(m-1)$ and get the equivalence

$$G(m) - F(m-1) \;\le\; F(m) - G(m) \quad\Leftrightarrow\quad 2F(m) - 2G(m) - f(m) \;\ge\; 0.$$

Thus we show $2F(m) - 2G(m) - f(m) \ge 0$, first for $m \ge 2$, hence $\ell \ge 3$ and reduce it to the case $\Delta(\lfloor np\rfloor)$. We have preliminary by Lemma 11.1 in the second and Lemma 11.4 in the third step

$$\begin{aligned}
\tfrac{\Phi((1-\delta)/\sigma)-1/2}{f_{n,\lambda}(\ell)} \;&\ge\; \tfrac{\Phi((1-\delta)/\sigma_\lambda)-1/2}{f_{n,\lambda}(\ell)} \;\ge\; \tfrac{1-\delta}{\sqrt{2\pi}\,\sigma_\lambda f_{n,\lambda}(\ell)} \cdot \mathrm{e}^{-\frac{(1-\delta)^2}{6\sigma_\lambda^2}} \;\ge\; (1-\delta)\cdot \mathrm{e}^{-\frac{(1-\delta)^2}{6\sigma_\lambda^2}+\frac{19}{320\sigma_\lambda^2}} \\
&\ge\; (1-\delta)\cdot\big(1 - \tfrac{(1-\delta)^2}{6\sigma_\lambda^2} + \tfrac{19}{320\sigma_\lambda^2}\big) \;=\; 1-\delta + \tfrac{1}{\sigma_\lambda^2}\cdot\big(-\tfrac{(1-\delta)^3}{6} + \tfrac{19(1-\delta)}{320}\big).
\end{aligned}$$

Then we have due to the inequality above, Lemma 11.6, Lemma 11.9 and Lemma 11.7 in the third step

$$\begin{aligned}
& 2F_{n,p}(m) - 2G_{n,p}(m) - f_{n,p}(m) \\
=\;& 2\cdot f_{n,\lambda}(\ell)\cdot\Big(\tfrac{F_{n,p}(m)-F_{n,\lambda}(m)}{f_{n,\lambda}(\ell)} + \tfrac{F_{n,\lambda}(\ell)}{f_{n,\lambda}(\ell)} - 1 + \tfrac{\Phi((1-\delta)/\sigma)-1}{f_{n,\lambda}(\ell)} - \tfrac{1}{2}\cdot\tfrac{f_{n,p}(\ell-1)}{f_{n,\lambda}(\ell)}\Big) \\
=\;& 2\cdot f_{n,\lambda}(\ell)\cdot\Big(\tfrac{F_{n,p}(m)-F_{n,\lambda}(m)}{f_{n,\lambda}(\ell)} + \big(\tfrac{\Phi((1-\delta)/\sigma)-1/2}{f_{n,\lambda}(\ell)} - 1\big) + \big(\tfrac{F_{n,\lambda}(\ell)-1/2}{f_{n,\lambda}(\ell)} - \tfrac{1}{2}\big) \\
&\qquad -\tfrac{1}{2}\cdot\big(\tfrac{f_{n,p}(\ell-1)}{f_{n,\lambda}(\ell)} - 1\big)\Big) \\
\ge\;& 2\cdot f_{n,\lambda}(\ell)\cdot\Big(\delta + \tfrac{3\delta^2(1-\lambda)-\delta^3}{6\sigma_\lambda^2} - \delta - \tfrac{(1-\delta)^3}{6\sigma_\lambda^2} + \tfrac{19(1-\delta)}{320\sigma_\lambda^2} + \tfrac{1-2\lambda}{6} - \tfrac{1-2\lambda}{18\sigma_\lambda^2} \\
&\qquad -\tfrac{1}{2\sigma_\lambda^2}\cdot\big(\tfrac{1-2\lambda}{2} + \tfrac{13}{135} - \tfrac{(1-\delta)^2}{2} + \tfrac{(1-\delta)^4}{12}\big)\Big) \\
=\;& 2\cdot f_{n,\lambda}(\ell)\cdot\Big(\tfrac{1-2\lambda}{6} + \tfrac{1}{\sigma_\lambda^2}\cdot\big(-\tfrac{11(1-2\lambda)}{36} - \tfrac{13}{270} + \tfrac{19(1-\delta)}{320} - \tfrac{(1-\delta)^4}{24} + \tfrac{1}{12} + \tfrac{\delta^2(1-2\lambda)}{4}\big)\Big) \\
\ge\;& 2\cdot f_{n,\lambda}(\ell)\cdot\Big(\tfrac{3(1-\lambda)}{\sigma_\lambda^2}\cdot\tfrac{1-2\lambda}{6} + \tfrac{1}{\sigma_\lambda^2}\cdot\big(-\tfrac{11(1-2\lambda)}{36} - \tfrac{13}{270} + \tfrac{1}{12}\big)\Big) \\
=\;& 2\cdot f_{n,\lambda}(\ell)\cdot\tfrac{1}{\sigma_\lambda^2}\cdot\big(\tfrac{31}{135} - \tfrac{8}{9}\lambda + \lambda^2\big) \\
>\;& 0
\end{aligned}$$

because of $\frac{31}{135} - \frac{8}{9}\lambda + \lambda^2 \ge \frac{31}{135} - \frac{8}{9}\lambda(1-\lambda) \ge \frac{31}{135} - \frac{2}{9} = \frac{1}{135} > 0$ in the last inequality.

The remaining case $\ell \le 2$ is proved in Lemma 9.2 on page 86 as a boundary case.

3.) $\Delta(\lceil np\rceil -)$:

If $np \in \mathbb{N}$, then $\Delta(\lceil np\rceil -) = \Delta(\lfloor np\rfloor -) < \frac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}$ due to 2.) above, consequently we show for $m+1 = \lceil np\rceil = \lfloor np\rfloor + 1$

$$|G(m+1) - F(m)| < \tfrac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}.$$

If $F(m) > G(m+1)$, then $F(m) - G(m+1) \le F(m+1) - G(m+1) < \frac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}$ due to 1.), thus it remains to prove

$$\mathcal{D}_2 := \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - G(m+1) + F(m) > 0.$$

Let now again $\delta = m+1-np \in (0,1)$, $\delta < 1$ since $np \notin \mathbb{N}$, $\lambda = \frac{\ell}{n} = \frac{m+1}{n}$ and $e^{S_n^\ell} = \sqrt{2\pi}\,\sigma_\lambda \cdot f_{n,\lambda}(\ell)$ and let first $\ell \ge 2$, then we have due to Lemma 11.1, Lemma 11.6 and Lemma 11.9 in the first inequality below

$$\begin{aligned}
G(m+1) - F(m) &= \Phi(\tfrac{\delta}{\sigma}) - \tfrac12 - (F_{n,p}(m) - F_{n,\lambda}(m)) - (F_{n,\lambda}(\ell) - \tfrac12) + f_{n,\lambda}(\ell)\\
&< \tfrac{\delta}{\sqrt{2\pi}\,\sigma}\cdot e^{-\frac{\delta^2}{6\sigma^2}+\frac{\delta^4}{90\sigma^4}} + f_{n,\lambda}(\ell)\cdot\big(-(\delta + \tfrac{3\delta^2(1-\lambda)-\delta^3}{6\sigma_\lambda^2}) - \tfrac{4-2\lambda}{6} + \tfrac{1-2\lambda}{18\sigma_\lambda^2} + 1\big)\\
&\le \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot\big(\delta - \tfrac{\delta^3}{6\sigma_\lambda^2} + \tfrac{\delta^5}{40\sigma_\lambda^4} + \tfrac{\sigma}{\sigma_\lambda}e^{S_n^\ell}\cdot(\tfrac12 - \delta - \tfrac{3\delta^2(1-\lambda)-\delta^3}{6\sigma_\lambda^2} - \tfrac{1-2\lambda}{6} + \tfrac{1-2\lambda}{18\sigma_\lambda^2})\big)\\
&\le \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot\big(\delta\cdot(1 - \tfrac{\sigma}{\sigma_\lambda}e^{S_n^\ell}) - \tfrac{\delta^3}{6\sigma_\lambda^2} + \tfrac{\delta^5}{40\sigma_\lambda^4} + \tfrac12 - \tfrac{3\delta^2(1-\lambda)-\delta^3}{6\sigma_\lambda^2} - \tfrac{1-2\lambda}{6} + \tfrac{1-2\lambda}{18\sigma_\lambda^2}\big)\\
&= \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot\big(\delta\cdot(1 - \tfrac{\sigma}{\sigma_\lambda}e^{S_n^\ell}) + \tfrac{\delta^5}{40\sigma_\lambda^4} + \tfrac12 - \tfrac{3\delta^2(1-\lambda)}{6\sigma_\lambda^2} - \tfrac{1-2\lambda}{6} + \tfrac{1-2\lambda}{18\sigma_\lambda^2}\big)
\end{aligned}$$

since $e^{-\frac{\delta^2}{6\sigma^2}+\frac{\delta^4}{90\sigma^4}} \le e^{-\frac{\delta^2}{6\sigma_\lambda^2}+\frac{\delta^4}{90\sigma_\lambda^4}} \le 1 - \frac{\delta^2}{6\sigma_\lambda^2} + \frac{\delta^4}{90\sigma_\lambda^4} + \frac{\delta^4}{72\sigma_\lambda^4} \le 1 - \frac{\delta^2}{6\sigma_\lambda^2} + \frac{\delta^4}{40\sigma_\lambda^4}$ in the penultimate and $\frac{\sigma}{\sigma_\lambda}\cdot e^{S_n^\ell} \le e^{S_n^\ell} \le 1$ as well as $\frac12 - \frac{3\delta^2(1-\lambda)-\delta^3}{6\sigma_\lambda^2} - \frac{1-2\lambda}{6} + \frac{1-2\lambda}{18\sigma_\lambda^2} \ge \frac12 - \frac{3(1-\lambda)-1}{6\sigma_\lambda^2} - \frac{1-2\lambda}{6} \ge \frac12 - \frac{1-\lambda}{2(1-\lambda)} + \frac{1}{6(1-\lambda)} - \frac{1-2\lambda}{6} \ge 0$ in the ultimate inequality.

Further we have $\sqrt{1-y} \ge 1 - \frac{7y}{12}$ for $y \in [0, \frac49]$, since $\frac{d^2}{dy^2}\sqrt{1-y} - (1 - \frac{7y}{12}) = -\frac{(1-y)^{-3/2}}{4} < 0$ and $\sqrt{1-\frac49} - 1 + \frac{7\cdot 4}{12\cdot 9} > 0$. Thus we have with $\frac{\sigma}{\sigma_\lambda} = \sqrt{1 - \frac{(n-2k-2)\delta+\delta^2}{(m+1)(n-m-1)}}$ and $\frac{\sigma^2}{\sigma_\lambda^2} = \frac{pq}{\lambda(1-\lambda)} \ge \frac{5/36}{1/4} = \frac59$, implying $\frac{(n-2k-2)\delta+\delta^2}{(m+1)(n-m-1)} = 1 - \frac{\sigma^2}{\sigma_\lambda^2} \le \frac49$ and Lemma 11.4 now

$$\begin{aligned}
\tfrac{\sigma}{\sigma_\lambda}\cdot e^{S_n^\ell} &\ge (1 - \delta\tfrac{7(n-2k-2+\delta)}{12(m+1)(n-m-1)})\cdot e^{-\frac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2}} \ge (1 - \delta\tfrac{7(n-2k-2+\delta)}{12(m+1)(n-m-1)})\cdot(1 - \tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2})\\
&\ge 1 - \tfrac{1}{\sigma_\lambda^2}\cdot(\delta\cdot\tfrac{7(1-2\lambda+\delta/n)}{12} + \tfrac{1-\lambda(1-\lambda)}{12}).
\end{aligned}$$

It follows with $\frac{4-2p}{6} = \frac{1}{2} + \frac{1-2\lambda}{6} + \frac{\delta\lambda(1-\lambda)}{3\sigma_\lambda^2}$ and $\ell \geq 2$

$$\begin{aligned}
\mathcal{D}_2 &> \frac{1}{\sqrt{2\pi}\,\sigma}\Big(\frac{4-2p}{6} - \big(\frac{\delta}{\sigma_\lambda^2}\cdot(\delta\frac{7(1-2\lambda+\delta/n)}{12} + \frac{1-\lambda(1-\lambda)}{12}) + \frac{\delta^5}{40\sigma_\lambda^4} + \frac{1}{2} - \frac{3\delta^2(1-\lambda)}{6\sigma_\lambda^2} - \frac{1-2\lambda}{6} + \frac{1-2\lambda}{18\sigma_\lambda^2}\big)\Big) \\
&= \frac{1}{\sqrt{2\pi}\,\sigma\sigma_\lambda^2}\cdot\big(\frac{(1-2\lambda)(1-\lambda)\cdot\ell}{3} + \frac{\delta\lambda(1-\lambda)}{3} - \frac{7\delta^2(1-2\lambda)}{12} - \frac{7\delta^3\lambda}{12n} - \frac{\delta(1-\lambda(1-\lambda))}{12} \\
&\qquad - \frac{\delta^5}{40(1-\lambda)\ell} + \frac{\delta^2(1-\lambda)}{2} - \frac{1-2\lambda}{18})\big) \\
&\geq \frac{1}{\sqrt{2\pi}\,\sigma\sigma_\lambda^2}\big(\frac{2(1-2\lambda)(1-\lambda)}{3} + \frac{\delta\lambda(1-\lambda)}{3} - \frac{7\delta^2(1-2\lambda)}{12} - \frac{7\delta^2\lambda}{24} - \frac{\delta(1-\lambda(1-\lambda))}{12} - \frac{\delta^2}{40} + \frac{\delta^2(1-\lambda)}{2} - \frac{1-2\lambda}{18})\big) \\
&= \frac{1}{\sqrt{2\pi}\,\sigma\sigma_\lambda^2}\cdot\big(\frac{11-34\lambda+24\lambda^2}{18} + \delta\cdot\frac{5\lambda(1-\lambda)-1}{12} + \delta^2\cdot(-\frac{13}{120} + \frac{3\lambda}{8})\big) \\
&\geq \frac{1}{\sqrt{2\pi}\,\sigma\sigma_\lambda^2}\cdot\big(\frac{11-17+6}{18} + \delta\cdot\frac{5/4-1}{12} + \delta^2\cdot(-\frac{13}{120} + \frac{3}{16})\big) \\
&= \frac{1}{\sqrt{2\pi}\,\sigma\sigma_\lambda^2}\cdot\big(\frac{\delta}{48} + \frac{19\delta^2}{240})\big) \\
&\geq 0
\end{aligned}$$

since $\frac{\mathrm{d}}{\mathrm{d}\lambda}[\frac{11-34\lambda+24\lambda^2}{18} + \delta\cdot\frac{5\lambda(1-\lambda)-1}{12} + \delta^2\cdot(-\frac{13}{120} + \frac{3\lambda}{8})] = -\frac{17}{9} + \frac{24\lambda}{9} + \delta\cdot\frac{5(1-2\lambda)}{12} + \frac{3\delta^2}{8} \leq -\frac{17}{9} + \frac{12}{9} + \frac{3\delta^2}{8} \leq -\frac{5}{9} + \frac{3}{8} \leq 0$ in the penultimate inequality.

The case $\ell \leq 1$, hence $m = 0$, is proved in Lemma 9.1 on page 85 as a boundary case. □

Lemma 8.3. *If $n \in \mathbb{N}$, $p \in [\frac{1}{6}, \frac{1}{2})$, such that $m+1 \in [\frac{n}{3}, \frac{n}{2}]$, $m \geq 2$ and $p \in [\frac{m}{n}, \frac{m+1/2}{n}]$, then we have*

$$\mathcal{D} \ := \ \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - F(m) + G(m) \ > \ 0.$$

Proof. For reminder let $\ell = m+1 \geq 3$, $\lambda = \frac{\ell}{n} \in [\frac{1}{3}, \frac{1}{2}]$, $\sigma_\lambda^2 = n\lambda(1-\lambda)$ as well as $\mathrm{e}^{S_n^\ell} = f_{n,\lambda}(\ell) \cdot \sqrt{2\pi}\,\sigma_\lambda$ and $p = \frac{\ell-\delta}{n}$ with $\delta = \ell - np \in [\frac{1}{2}, 1]$ according to requirement and additionally $S := \frac{\sigma}{\sigma_\lambda}$.

From Lemma 11.9 we already know that

$$F_{n,\lambda}(\ell) - \tfrac{1}{2} \ \leq \ \tfrac{4-2\lambda}{6} \cdot f_{n,\lambda}(\ell) - \tfrac{n-2\ell}{72\ell(n-\ell)} \cdot f_{n,\lambda}(\ell) \ = \ \tfrac{S\mathrm{e}^{S_n^\ell}}{\sqrt{2\pi}\,\sigma} \cdot \big(\tfrac{4-2\lambda}{6} - \tfrac{1-2\lambda}{72\sigma_\lambda^2}\big)$$

holds and due to Lemma 11.1 we have

$$\begin{aligned} G(m) - \tfrac{1}{2} \ &= \ \Phi\big(\tfrac{m-np}{\sigma}\big) - \tfrac{1}{2} \ = \ \tfrac{1}{2} - \Phi\big(\tfrac{np-m}{\sigma}\big) \ = \ \tfrac{1}{2} - \Phi\big(\tfrac{1-\delta}{\sigma}\big) \\ &\geq \ \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot \big(-(1-\delta) + \tfrac{(1-\delta)^3}{6\sigma^2} - \tfrac{(1-\delta)^5}{40\sigma^4}\big). \end{aligned}$$

Now we estimate with the inequalities above the difference $\mathcal{D}$ and get

$$\begin{aligned} (26) \qquad \mathcal{D} \ &= \ \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - [F_{n,\lambda}(\ell) - 0.5] - [F_{n,p}(m) - F_{n,\lambda}(\ell)] + [G(m) - 0.5] \\ &\geq \ \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - \tfrac{S\mathrm{e}^{S_n^\ell}}{\sqrt{2\pi}\,\sigma} \cdot \big(\tfrac{4-2\lambda}{6} - \tfrac{1-2\lambda}{72\sigma_\lambda^2}\big) \\ &\quad -[F_{n,p}(m) - F_{n,\lambda}(\ell)] + \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot \big(-(1-\delta) + \tfrac{(1-\delta)^3}{6\sigma^2} - \tfrac{(1-\delta)^5}{40\sigma^4}\big) \\ &= \ \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot \Big(\tfrac{\delta}{3n} + \tfrac{4-2\lambda}{6}(1 - Se^{S_n^\ell}) + \tfrac{1-2\lambda}{72\sigma_\lambda^2} \cdot Se^{S_n^\ell} \\ &\quad -\big(\tfrac{F_{n,p}(m) - F_{n,\lambda}(m)}{f_{n,\lambda}(\ell)} - 1\big) \cdot Se^{S_n^\ell} - (1-\delta) + \tfrac{(1-\delta)^3}{6\sigma_\lambda^2} \cdot S^2 - \tfrac{(1-\delta)^5}{40\sigma_\lambda^4} \cdot S^4\Big). \end{aligned}$$

The term above becomes smaller, the bigger becomes $Se^{S_n^\ell}$, since due to Lemma 11.6 we have $1 - \frac{F_{n,p}(m) - F_{n,\lambda}(m)}{f_{n,\lambda}(\ell)} \leq 1 - \delta \leq \frac{1}{2}$ and hence

$$-\tfrac{4-2\lambda}{6} + \tfrac{1-2\lambda}{72\sigma_\lambda^2} + 1 - \tfrac{F_{n,p}(m) - F_{n,\lambda}(m)}{f_{n,\lambda}(\ell)} \leq -\tfrac{1-2\lambda}{6} + \tfrac{1-2\lambda}{72\sigma_\lambda^2} \ = \ -\tfrac{1-2\lambda}{6} \cdot \big(1 - \tfrac{1}{12\sigma_\lambda^2}\big) \ \leq \ 0.$$

That is why we determine in the following an upper bound $\overline{Se}$ of $Se^{S_n^\ell}$.

We have due to Lemma 11.4 in the first, $e^{-\delta} \le 1 - \delta + \frac{\delta^2}{2}$ if $\delta \ge 0$ in the second and for $k(\lambda) := (1-\lambda)^2 + \frac{\lambda^3}{1-\lambda} = \frac{1-3\lambda+3\lambda^2}{1-\lambda} = \frac{1}{1-\lambda} - 3\lambda \le \frac{1}{2}$ for $\lambda \in [\frac{1}{3}, \frac{1}{2}]$, since k convex and $k(\frac{1}{3}) = k(\frac{1}{2}) = \frac{1}{2}$ we have $k(\lambda) \le \frac{1}{2}$ in the fourth step

$$
\begin{aligned}
(27)\quad e^{S_n^\ell} &\le \exp\Big(-\tfrac{1}{12}\cdot(\tfrac{1}{\sigma_\lambda^2} - \tfrac{1}{n}) + \tfrac{1}{360}\cdot(\tfrac{1}{\ell^3} + \tfrac{1}{(n-\ell)^3} - \tfrac{1}{n^3})\Big) \\
&\le 1 - \tfrac{1}{12}\cdot(\tfrac{1}{\sigma_\lambda^2} - \tfrac{1}{n}) + \tfrac{1}{360}\cdot(\tfrac{1}{\ell^3} + \tfrac{1}{(n-\ell)^3}) + \tfrac{1}{288}\cdot(\tfrac{1}{\sigma_\lambda^2} - \tfrac{1}{n})^2 \\
&= 1 - \tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2} + \tfrac{1}{\sigma_\lambda^4}\cdot\Big(\tfrac{1}{288}\cdot(1-\lambda(1-\lambda))^2 + \tfrac{1}{360\ell}\cdot((1-\lambda)^2 + \tfrac{\lambda^3}{(1-\lambda)})\Big) \\
&\le 1 - \tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2} + \tfrac{1}{\sigma_\lambda^4}\cdot\Big(\tfrac{1}{288}\cdot(1-\tfrac{2}{9})^2 + \tfrac{1}{360\cdot 3}\cdot\tfrac{1}{2}\Big) \\
&= 1 - \tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2} + \tfrac{1}{\sigma_\lambda^4}\cdot(\tfrac{1}{288}\cdot\tfrac{49}{81} + \tfrac{1}{360\cdot 6}) \\
&\le 1 - \tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2} + \tfrac{1}{\sigma_\lambda^4}\cdot\tfrac{1}{390} \\
&=: E.
\end{aligned}
$$

Further we have

$$
S = \sqrt{\tfrac{(\lambda-\delta/n)(1-\lambda+\delta/n)}{\lambda(1-\lambda)}} = \sqrt{1 - \tfrac{(1-2\lambda)\delta+\delta^2/n}{\sigma_\lambda^2}} \le 1 - \tfrac{(1-2\lambda)\delta+\delta^2/n}{2\sigma_\lambda^2} - \tfrac{(1-2\lambda+\delta/n)^2\delta^2}{8\sigma_\lambda^4} =: \overline{S}
$$

and thus with additionally $1-\lambda(1-\lambda) \le 1 - \frac{2}{9} = \frac{7}{9}$

$$
\begin{aligned}
Se^{S_n^\ell} &\le \overline{S}\cdot E = \Big(1 - \tfrac{(1-2\lambda)\delta+\delta^2/n}{2\sigma_\lambda^2} - \tfrac{(1-2\lambda+\delta/n)^2\delta^2}{8\sigma_\lambda^4}\Big)\cdot\Big(1 - \tfrac{1-\lambda(1-\lambda)}{12} + \tfrac{1}{\sigma_\lambda^4}\cdot\tfrac{1}{390}\Big) \\
&\le 1 - \tfrac{(1-2\lambda)\delta+\delta^2/n}{2\sigma_\lambda^2} - \tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2} - \tfrac{(1-2\lambda+\delta/n)^2\delta^2}{8\sigma_\lambda^4} + \tfrac{7(1-2\lambda+\delta/n)\delta}{9\cdot 24\sigma_\lambda^4} + \tfrac{1}{390\sigma_\lambda^4} + \tfrac{7(1-2\lambda+\delta/n)^2\delta^2}{9\cdot 96\sigma_\lambda^4} \\
&= 1 - \tfrac{1}{\sigma_\lambda^2}\cdot\Big(\tfrac{(1-2\lambda)\delta}{2} + \tfrac{1-\lambda(1-\lambda)}{12}\Big) - \tfrac{1}{\sigma_\lambda^4}\cdot\Big(\tfrac{\delta^2\lambda(1-\lambda)}{2} + \tfrac{(1-2\lambda)^2\delta^2}{8} - \tfrac{7(1-2\lambda)\delta^2}{216} - \tfrac{1}{390}\Big) \\
&\quad + \tfrac{1}{\sigma_\lambda^6}\cdot\Big(-\tfrac{(1-2\lambda)\lambda(1-\lambda)\delta^3}{4} + \tfrac{7\lambda(1-\lambda)\delta^2}{216} + \tfrac{7(1-2\lambda)^2\delta^2}{864}\Big) \\
&\quad + \tfrac{1}{\sigma_\lambda^8}\cdot\Big(-\tfrac{\lambda^2(1-\lambda)^2\delta^4}{8} + \tfrac{7(1-2\lambda)\lambda(1-\lambda)\delta^3}{432} + \tfrac{7\lambda(1-\lambda)\delta^4}{864n}\Big) \\
&\le 1 - \tfrac{1}{\sigma_\lambda^2}\cdot\Big(\tfrac{(1-2\lambda)\delta}{2} + \tfrac{1-\lambda(1-\lambda)}{12}\Big) - \tfrac{1}{\sigma_\lambda^4}\cdot\Big(\tfrac{\delta^2\lambda(1-\lambda)}{2} + \tfrac{(1-2\lambda)^2\delta^2}{8} - \tfrac{7(1-2\lambda)\delta}{216} - \tfrac{1}{390}\Big) \\
&\quad + \tfrac{1}{\sigma_\lambda^6}\cdot\Big(-\tfrac{65(1-2\lambda)\lambda(1-\lambda)\delta^3}{288} + \tfrac{7\lambda(1-\lambda)\delta^2}{216}\Big) \\
&=: \overline{Se}
\end{aligned}
$$

since we have $-\frac{(1-2\lambda)\lambda(1-\lambda)\delta^3}{4} + \frac{7(1-2\lambda)^2\delta^2}{864} = \frac{(1-2\lambda)\lambda(1-\lambda)\delta^3}{4}\cdot(\frac{7(1-2\lambda)}{216\lambda(1-\lambda)\delta} - 1) \le \frac{(1-2\lambda)\lambda(1-\lambda)\delta^3}{4}\cdot(\frac{7/3}{216\cdot 2/9\cdot 1/2} - 1) = -\frac{65(1-2\lambda)\lambda(1-\lambda)\delta^3}{288}$ and $-\frac{\lambda^2(1-\lambda)^2\delta^4}{8} + \frac{7(1-2\lambda)\lambda(1-\lambda)\delta^3}{432} + \frac{7\lambda(1-\lambda)\delta^4}{864n} = \frac{\delta^4\lambda(1-\lambda)}{8}\cdot(-\lambda(1-\lambda) + \frac{7(1-2\lambda)}{54\delta} + \frac{7}{108n}) \le \frac{\delta^4\lambda(1-\lambda)}{8}\cdot(-\frac{2}{9} + \frac{7/3}{54/2} + \frac{7}{108\cdot 6}) = -\frac{\delta^4\lambda(1-\lambda)}{64} \le 0$ in the last inequality.

Thus if we continue the computation starting at (26) and estimate there $Se^{S_n^\ell}$ by $\overline{Se}$ upwards, we receive for $\mathcal{D}$

$$
\begin{aligned}
\mathcal{D} \;\geq\;& \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot\Big(\tfrac{\delta}{3n}+\tfrac{4-2\lambda}{6}(1-\overline{Se})+\tfrac{1-2\lambda}{72\sigma_\lambda^2}\cdot\overline{Se}\\
&-\big(\tfrac{F_{n,p}(m)-F_{n,\lambda}(m)}{f_{n,\lambda}(\ell)}-1\big)\cdot\overline{Se}-(1-\delta)+\tfrac{(1-\delta)^3}{6\sigma_\lambda^2}\cdot\tfrac{1}{S^2}-\tfrac{(1-\delta)^5}{40\sigma_\lambda^4}\cdot\tfrac{1}{S^4}\Big)\\
=\;& \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot\Big(\tfrac{\delta}{3n}+\tfrac{4-2\lambda}{6}(1-\overline{Se})+\tfrac{1-2\lambda}{72\sigma_\lambda^2}\cdot\overline{Se}-\big(\tfrac{F_{n,p}(m)-F_{n,\lambda}(m)}{f_{n,\lambda}(\ell)}-\delta\big)\cdot\overline{Se}\\
&+(1-\delta)\cdot\big(\overline{Se}-1+\tfrac{(1-\delta)^2}{6\sigma_\lambda^2}\cdot\tfrac{1}{S^2}\big)-\tfrac{(1-\delta)^5}{40\sigma_\lambda^4}\cdot\tfrac{1}{S^4}\Big)\\
=\;& \tfrac{1}{\sqrt{2\pi}\,\sigma}\cdot\Big(\underbrace{\tfrac{\delta\lambda(1-\lambda)}{3\sigma_\lambda^2}+\tfrac{4-2\lambda}{6}\cdot\big(\tfrac{\delta(1-2\lambda)}{2\sigma_\lambda^2}+\tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2}\big)+\tfrac{1-2\lambda}{72\sigma_\lambda^2}}_{=:H_1}\\
&\underbrace{-\tfrac{\delta^2(1-\lambda)}{2\sigma_\lambda^2}+\tfrac{\delta^3}{6\sigma_\lambda^2}+(1-\delta)\cdot\big(\tfrac{(1-\delta)^2}{6\sigma_\lambda^2}-\tfrac{\delta(1-2\lambda)}{2\sigma_\lambda^2}-\tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2}\big)}_{=:H_2}\\
&\underbrace{-\big(\tfrac{F_{n,p}(m)-F_{n,\lambda}(m)}{f_{n,\lambda}(\ell)}-\delta\big)\cdot\overline{Se}+\tfrac{\delta^2(1-\lambda)}{2\sigma_\lambda^2}-\tfrac{\delta^3}{6\sigma_\lambda^2}+\tfrac{1-2\lambda}{72\sigma_\lambda^2}\cdot(\overline{Se}-1)}_{=:R_1}\\
&+\underbrace{\big(\tfrac{4-2\lambda}{6}-(1-\delta)\big)\big(1-\overline{Se}-\tfrac{\delta(1-2\lambda)}{2\sigma_\lambda^2}-\tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2)}\big)}_{=:R_2}+\underbrace{\tfrac{(1-\delta)^3}{6\sigma_\lambda^2}\cdot\big(\tfrac{1}{S^2}-1\big)-\tfrac{(1-\delta)^5}{40\sigma_\lambda^4S^4}}_{=:R_3}\Big).
\end{aligned}
$$

First we have with $H := H_1 + H_2$ since $ax^2 - bx \geq -\frac{b^2}{4a}$ for $a, b > 0$ in the inequality below

$$
\begin{aligned}
H \;=\;& \tfrac{1}{\sigma_\lambda^2}\cdot\Big(\tfrac{\delta(1-\lambda)}{3}+\tfrac{4-2\lambda}{6}\cdot\big(\tfrac{\delta(1-2\lambda)}{2}+\tfrac{1-\lambda(1-\lambda)}{12}\big)+\tfrac{(1-2\lambda)}{72}-\tfrac{\delta^2(1-\lambda)}{2}+\tfrac{\delta^3}{6}\\
&+(1-\delta)\cdot\big(\tfrac{(1-\delta)^2}{6}-\tfrac{\delta(1-2\lambda)}{2}-\tfrac{1-\lambda(1-\lambda)}{12}\big)\Big)\\
=\;& \tfrac{1}{\sigma_\lambda^2}\cdot\Big(\delta^2\cdot\big(-\tfrac{1-\lambda}{2}+\tfrac{1-2\lambda}{2}+\tfrac{1}{2}\big)\\
&-\delta\cdot\big(-\tfrac{4-2\lambda}{6}\cdot\tfrac{1-2\lambda}{2}+\tfrac{1-2\lambda}{2}+\tfrac{1}{2}-\tfrac{\lambda(1-\lambda)}{3}-\tfrac{1-\lambda(1-\lambda)}{12}\big)\\
&+\tfrac{(4-2\lambda)(1-\lambda(1-\lambda)}{72}+\tfrac{1-2\lambda}{72}-\tfrac{1-\lambda(1-\lambda)}{12}+\tfrac{1}{6}\Big)\\
=\;& \tfrac{1}{\sigma_\lambda^2}\cdot\Big(\delta^2\cdot\tfrac{1-\lambda}{2}-\delta\cdot\tfrac{7-5\lambda-\lambda^2}{12}+\tfrac{11-2\lambda-2\lambda^3}{72}\Big)\\
\geq\;& \tfrac{1}{\sigma_\lambda^2}\cdot\big(-\big(\tfrac{7-5\lambda-\lambda^2}{12}\big)^2\cdot\tfrac{1}{2(1-\lambda)}+\tfrac{11-2\lambda-2\lambda^3}{72}\big)\\
=\;& \tfrac{1}{\sigma_\lambda^2}\cdot\tfrac{-5+18\lambda-3\lambda^2-18\lambda^3+7\lambda^4}{288(1-\lambda)}\\
=:\;& \tfrac{1}{\sigma_\lambda^2}\cdot h(\lambda).
\end{aligned}
$$

Let now denote $\underline{h}(\lambda)$ the line through $h(\frac{1}{3})$ and $h(\frac{1}{2})$, see Figure 10 on the left, numerical

$$\begin{aligned}\underline{h}(\lambda) &= h(\tfrac{1}{3}) + \tfrac{h(1/2)-h(1/3)}{1/2-1/3}\cdot(\lambda-\tfrac{1}{3}) = \tfrac{7}{15\,552} + 6\cdot(\tfrac{23}{2304}-\tfrac{7}{15\,552})\cdot(\lambda-\tfrac{1}{3})\\ &= \tfrac{7}{15\,552} + \tfrac{593}{10\,368}\cdot(\lambda-\tfrac{1}{3}) = \tfrac{-193+593\lambda}{10\,368}.\end{aligned}$$

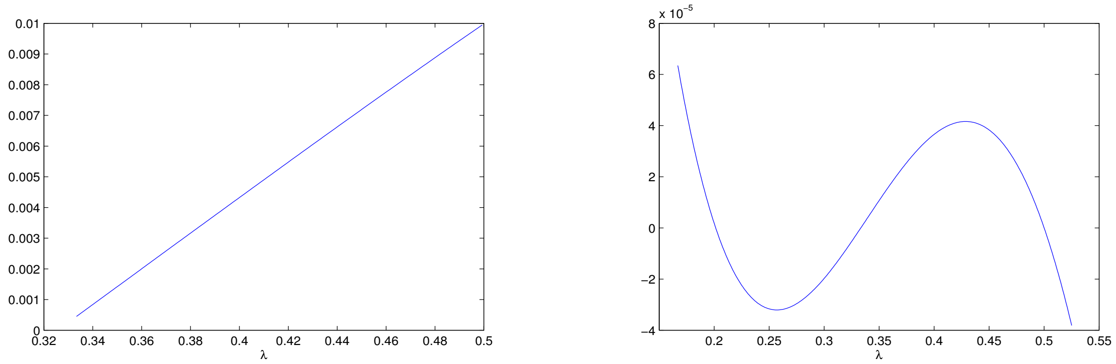

FIGURE 10. Graph of $h(\lambda)$ left and $P(\lambda) = (1-\lambda)\cdot(h(\lambda)-\underline{h}(\lambda))$ right.

For $\lambda \in [\frac{1}{3}, \frac{1}{2}]$ the line $\underline{h}(\lambda)$ is below $h(\lambda)$, see Figure 10 on the right, it holds $h(\lambda) \geq \underline{h}(\lambda)$ because first we have

$$h(\lambda) - \underline{h}(\lambda) = \tfrac{1}{1-\lambda}\cdot\big(\tfrac{-5+18\lambda-3\lambda^2-18\lambda^3+7\lambda^4}{288} - \tfrac{(1-\lambda)(-193+593\lambda)}{10\,368}\big) := \tfrac{1}{1-\lambda}\cdot P(\lambda).$$

We now show that $P \geq 0$ on $[\frac{1}{3}, \frac{1}{2}]$. First P is obviously a polynomial with degree at most 4, so there can exist at most four roots of P on $\mathbb{R}$. By the construction of $\underline{h}$ as a line through $h(\frac{1}{3})$ and $h(\frac{1}{2})$ we have the obvious roots $a_1 = \frac{1}{3}$ and $a_2 = \frac{1}{2}$ of P. Further $P(0) = \frac{13}{10\,368} > 0$ and $P(\frac{1}{4}) = -\frac{7}{221\,184} < 0$ holds and hence there must exist an additional root $a_3 \in [0, \frac{1}{4}]$. Finally we have $P'(\frac{1}{3}) > 0$ and $P'(\frac{1}{2}) < 0$, hence $P(\frac{1}{3}+) > 0$ and $P(\frac{1}{2}-) > 0$ and since there can not be any two other roots, so that we have two changes of sign, P must be non negative on $[\frac{1}{3}, \frac{1}{2}]$.

Thus it follows together with $\underline{h}(\lambda) > 0$ and $\ell \geq 3$

$$H \geq \tfrac{1}{\sigma_\lambda^2}\cdot\underline{h}(\lambda) \geq \tfrac{3}{\ell\cdot\sigma_\lambda^2}\cdot\underline{h}(\lambda) = \tfrac{1}{\sigma_\lambda^4}\cdot 3(1-\lambda)\cdot\tfrac{-193+593\lambda}{10\,368} = \tfrac{1}{\sigma_\lambda^4}\cdot\tfrac{-193+786\lambda-593\lambda^2}{3456}.$$

Now we consider the remainders R_1, R_2 and R_3, but therefor we need the following estimations. Firstly we have with $\delta \in [\frac{1}{2}, 1]$, $\lambda \in [\frac{1}{3}, \frac{1}{2}]$ and $\ell \geq 3$

$$\begin{aligned}
& \tfrac{1}{\sigma_\lambda^6} \cdot \big(\tfrac{\delta^4(1-\lambda)^3}{4} + \tfrac{\delta^5(-13+33\lambda-23\lambda^2)}{60} + \tfrac{\delta^6(7-11\lambda)}{144} - \tfrac{\delta^7}{336}\big) + \tfrac{3}{2\sigma_\lambda^8} \cdot \big(\tfrac{\delta^5(1-\lambda)^4}{5} - \tfrac{\delta^6(1-\lambda)^2(63-110\lambda)}{450}\big) \\
\leq\ & \tfrac{1}{\sigma_\lambda^6} \cdot \big(\tfrac{\delta^4(1-\lambda)^3}{4} + \tfrac{\delta^5(-13+33\lambda-23\lambda^2)}{60} + \tfrac{\delta^6(7-11\lambda)}{144}\big) + \tfrac{1}{2\sigma_\lambda^6(1-\lambda)} \cdot \big(\tfrac{\delta^5(1-\lambda)^4}{5} - \tfrac{\delta^6(1-\lambda)^2(63-110\lambda)}{450}\big) \\
=\ & \tfrac{\delta^4}{\sigma_\lambda^6} \cdot \big(\tfrac{(1-\lambda)^3}{4} + \tfrac{\delta(-13+33\lambda-23\lambda^2)}{60} + \tfrac{\delta^2(7-11\lambda)}{144} + \tfrac{\delta(1-\lambda)^3}{10} - \tfrac{\delta^2(1-\lambda)(63-110\lambda)}{900}\big) \\
=\ & \tfrac{\delta^4}{\sigma_\lambda^6} \cdot \big(\tfrac{(1-\lambda)^3}{4} + \delta \cdot \tfrac{-7+15\lambda-5\lambda^2-6\lambda^3}{60} + \delta^2 \cdot \tfrac{-77+417\lambda-440\lambda^2}{3600}\big) \\
\leq\ & \tfrac{\delta^4}{\sigma_\lambda^6} \cdot \big(\tfrac{(1-\lambda)^3}{4} + \tfrac{-7+15\lambda-5\lambda^2-6\lambda^3}{120} + \tfrac{-77+417\lambda-440\lambda^2}{4\cdot 3600}\big) \\
\leq\ & \tfrac{\delta^4}{\sigma_\lambda^6} \cdot \big(\tfrac{(1-\lambda)^3}{4} - \tfrac{2(1-\lambda)^3}{27}\big) \\
=\ & \tfrac{19\delta^4(1-\lambda)^3}{108\sigma_\lambda^6}.
\end{aligned}$$

We now explain the three inequalities above. In the first inequality we use that $\frac{3}{2\sigma_\lambda^8} = \frac{3}{2\sigma_\lambda^6 \ell(1-\lambda} \leq \frac{1}{2\sigma_\lambda^6(1-\lambda)}$ and that the bracket behind $\frac{3}{2\sigma_\lambda^8}$ must not be negative since else with Lemma 11.6 (ii) we would have a contradiction to Lemma 11.6 (i). In the second inequality we use

$$\begin{aligned}
& \tfrac{\mathrm{d}}{\mathrm{d}\delta}\big(\delta \cdot \tfrac{-7+15\lambda-5\lambda^2-6\lambda^3}{60} + \delta^2 \cdot \tfrac{-77+417\lambda-440\lambda^2}{3600}\big) \ = \ \tfrac{-7+15\lambda-5\lambda^2-6\lambda^3}{60} + \delta \cdot \tfrac{-77+417\lambda-440\lambda^2}{1800} \\
\leq\ & \tfrac{-7+15\lambda-5\lambda^2-6\lambda^3}{60} + \tfrac{-77+417\lambda-440\lambda^2}{1800} \ = \ \tfrac{-287+867\lambda-590\lambda^2-180\lambda^3}{1800} \\
\leq\ & \tfrac{-287+867/2-590/4-180/8}{1800} \ = \ -\tfrac{47}{3600} \ < \ 0,
\end{aligned}$$

where in turn we use $-77+417\lambda-440\lambda^2 = -77+417\lambda(1-\lambda)-23\lambda^2 \geq -77+\frac{417\cdot 2}{9}-\frac{23}{4} = \frac{119}{12} > 0$ and $(867\lambda - 590\lambda^2 - 180\lambda^3)' \geq 867 - 590 - \frac{3}{4}\cdot 180 = 140 > 0$ and $\lambda \leq \frac{1}{2}$, and in the third inequality we use

$$\begin{aligned}
\tfrac{2(1-\lambda)^3}{27} + \tfrac{-7+15\lambda-5\lambda^2-6\lambda^3}{120} + \tfrac{-77+417\lambda-440\lambda^2}{4\cdot 3600} \ &= \ \tfrac{449-2949\lambda+6480\lambda^2-5360\lambda^3}{43\,200} \\
&\leq \ \tfrac{449-2949/3+6480/9-5360/27}{43\,200} \ = \ -\tfrac{169}{583\,200} \ < \ 0
\end{aligned}$$

where in turn we use $(-2949\lambda + 6480\lambda^2 - 5360\lambda^3)' = -2949 + 12\,960\lambda(1-\lambda) - 3120\lambda^2 \leq -2949 + \frac{12\,960}{4} - \frac{3120}{9} = -\frac{167}{3} < 0$ and $\lambda \geq \frac{1}{3}$.

Thus it follows from Lemma 11.6 (ii)

$$\begin{aligned}
\tfrac{F_{n,\frac{\ell-\delta}{n}}(m)-F_{n,\lambda}(m)}{f_{n,\lambda}(\ell)} \ \leq\ & \delta + \tfrac{1}{\sigma_\lambda^2}\cdot\big(\tfrac{\delta^2(1-\lambda)}{2} - \tfrac{\delta^3}{6}\big) + \tfrac{1}{\sigma_\lambda^4}\cdot\big(\tfrac{\delta^3(1-\lambda)^2}{3} - \tfrac{\delta^4(5-7\lambda)}{24} + \tfrac{\delta^5}{40}\big) \\
& + \tfrac{1}{\sigma_\lambda^6}\cdot\big(\tfrac{\delta^4(1-\lambda)^4}{4} + \tfrac{\delta^5(-13+33\lambda-23\lambda^2)}{60} + \tfrac{\delta^6(7-11\lambda)}{144} - \tfrac{\delta^7}{336}\big) \\
& + \tfrac{3}{2\sigma_\lambda^8}\cdot\big(\tfrac{\delta^5(1-\lambda)^4}{5} - \tfrac{\delta^6(1-\lambda)^2(63-110\lambda)}{450}\big) \\
\leq\ & \delta + \tfrac{1}{\sigma_\lambda^2}\cdot\big(\tfrac{\delta^2(1-\lambda)}{2} - \tfrac{\delta^3}{6}\big) + \tfrac{1}{\sigma_\lambda^4}\cdot\big(\tfrac{\delta^3(1-\lambda)^2}{3} - \tfrac{\delta^4(5-7\lambda)}{24} + \tfrac{\delta^5}{40}\big) + \tfrac{19\delta^4(1-\lambda)^3}{108\sigma_\lambda^6}
\end{aligned}$$

Furthermore we need the following estimation

$$\begin{aligned}
& \tfrac{1}{\sigma_\lambda^4}\cdot\big(-\tfrac{(1-2\lambda)^2\delta^2}{8}+\tfrac{7(1-2\lambda)\delta}{216}+\tfrac{1}{390}\big)+\tfrac{1}{\sigma_\lambda^6}\cdot\big(-\tfrac{65(1-2\lambda)\lambda(1-\lambda)\delta^3}{288}+\tfrac{7\lambda(1-\lambda)\delta^2}{216}\big)\\
= & \tfrac{\delta^2}{\sigma_\lambda^4}\cdot\big(-\tfrac{(1-2\lambda)^2}{8}+\tfrac{7(1-2\lambda)}{216\delta}+\tfrac{1}{390\delta^2}-\tfrac{65(1-2\lambda)\delta}{288n}+\tfrac{7}{216n}\big)\\
\le & \tfrac{\delta^2}{\sigma_\lambda^4}\cdot\big(-\tfrac{(1-2\lambda)^2}{8}+\tfrac{7(1-2\lambda)}{108}+\tfrac{2}{195}-\tfrac{65(1-2\lambda)}{2\cdot 288n}+\tfrac{7}{216n}\big)\\
= & \tfrac{\delta^2}{\sigma_\lambda^4}\cdot\big(-\tfrac{1}{8}\cdot(1-2\lambda)^2+\big(\tfrac{7}{108}-\tfrac{65}{2\cdot 288n}\big)\cdot(1-2\lambda)+\tfrac{2}{195}+\tfrac{7}{216n}\big)\\
\le & \tfrac{\delta^2}{\sigma_\lambda^4}\cdot\big(2\cdot\big(\tfrac{7}{108}-\tfrac{65}{2\cdot 288n}\big)^2+\tfrac{2}{195}+\tfrac{7}{216n}\big)\\
= & \tfrac{\delta^2}{\sigma_\lambda^4}\cdot\big(\tfrac{2}{195}+\tfrac{49}{54\cdot 108}-\tfrac{7\cdot 65}{216\cdot 72n}+\tfrac{7}{216n}+\tfrac{65^2}{2\cdot 288^2n^2}\big)\\
\le & \tfrac{\delta^2\lambda(1-\lambda)}{\sigma_\lambda^4}\cdot\tfrac{9}{2}\cdot\big(\tfrac{2}{195}+\tfrac{49}{54\cdot 108}+\tfrac{49}{216\cdot 72}\cdot\tfrac{1}{6}+\tfrac{65^2}{2\cdot 288^2}\cdot\tfrac{1}{36}\big)\;=\;\tfrac{\delta^2\lambda(1-\lambda)}{\sigma_\lambda^4}\cdot\tfrac{285\,971}{3\,194\,880}\\
\le & \tfrac{\delta^2\lambda(1-\lambda)}{\sigma_\lambda^4}\cdot\tfrac{9}{100},
\end{aligned}$$

where in the second inequality above we use $-ay^2+by\le\frac{b^2}{4a}$ for $a,b>0$ with $y=1-2\lambda$ and in the third one $\lambda(1-\lambda)\ge\frac{2}{9}$.

Based on the estimations above we can now estimate $\overline{Se}$ upwards. We have

$$\begin{aligned}
\overline{Se}\;=\;& 1-\tfrac{1}{\sigma_\lambda^2}\cdot\big(\tfrac{(1-2\lambda)\delta}{2}+\tfrac{1-\lambda(1-\lambda)}{12}\big)-\tfrac{1}{\sigma_\lambda^4}\cdot\big(\tfrac{\delta^2\lambda(1-\lambda)}{2}+\tfrac{(1-2\lambda)^2\delta^2}{8}-\tfrac{7(1-2\lambda)\delta}{216}-\tfrac{1}{390}\big)\\
& +\tfrac{1}{\sigma_\lambda^6}\cdot\big(-\tfrac{65(1-2\lambda)\lambda(1-\lambda)\delta^3}{288}+\tfrac{7\lambda(1-\lambda)\delta}{216}\big)\\
\le\;& 1-\tfrac{1}{\sigma_\lambda^2}\cdot\big(\tfrac{(1-2\lambda)\delta}{2}+\tfrac{1-\lambda(1-\lambda)}{12}\big)-\tfrac{1}{\sigma_\lambda^4}\cdot\big(\tfrac{\delta^2\lambda(1-\lambda)}{2}-\tfrac{9\delta^2\lambda(1-\lambda)}{100}\big)\\
=\;& 1-\tfrac{1}{\sigma_\lambda^2}\cdot\big(\tfrac{(1-2\lambda)\delta}{2}+\tfrac{1-\lambda(1-\lambda)}{12}\big)-\tfrac{41\delta^2\lambda(1-\lambda)}{100\sigma_\lambda^4}
\end{aligned}$$

Due to Lemma 11.6 we have $-\big(\frac{F_{n,p}(m)-F_{n,\lambda}(m)}{f_{n,\lambda}(\ell)}-\delta\big)+\frac{1-2\lambda}{72\sigma_\lambda^2}\le-\frac{\delta^2(1-\lambda)}{2\sigma_\lambda^2}+\frac{\delta^3}{6\sigma_\lambda^2}+\frac{1-2\lambda}{72\sigma_\lambda^2}=-\frac{\delta^2}{\sigma_\lambda^2}\cdot\big(\frac{1-\lambda}{2}-\frac{\delta}{6}-\frac{1-2\lambda}{72\delta^2}\big)\le-\frac{\delta^2}{\sigma_\lambda^2}\cdot\big(\frac{1}{4}-\frac{1}{6}-\frac{1}{3\cdot 18}\big)\le 0$, consequent R_1 becomes smaller, if we estimate $\overline{Se}$ upwards and hence we finally have

$$\begin{aligned}
R_1\;=\;& -\overline{Se}\cdot\big(\tfrac{F_{n,p}(m)-F_{n,\lambda}(m)}{f_{n,\lambda}(\ell)}-\delta\big)+\tfrac{\delta^2(1-\lambda)}{2\sigma_\lambda^2}-\tfrac{\delta^3}{6\sigma_\lambda^2}+\tfrac{1-2\lambda}{72\sigma_\lambda^2}\cdot(\overline{Se}-1)\\
\ge\;& \Big(\underbrace{\tfrac{1}{\sigma_\lambda^2}\cdot\big(\tfrac{\delta^2(1-\lambda)}{2}-\tfrac{\delta^3}{6\sigma_\lambda^2}\big)}_{=:A_1}+\underbrace{\tfrac{1}{\sigma_\lambda^4}\cdot\big(\tfrac{\delta^3(1-\lambda)^2}{3}-\tfrac{\delta^4(5-7\lambda)}{24}+\tfrac{\delta^5}{40}\big)}_{=:A_2}+\underbrace{\tfrac{19\delta^4(1-\lambda)^3}{108\sigma_\lambda^6}}_{=:A_3}\Big)\cdot\\
& \cdot\Big(-1+\underbrace{\tfrac{1}{\sigma_\lambda^2}\cdot\big(\tfrac{\delta(1-2\lambda)}{2}+\tfrac{1-\lambda(1-\lambda)}{12}\big)}_{=:B_1}+\underbrace{\tfrac{41\delta^2\lambda(1-\lambda)}{100\sigma_\lambda^4}}_{=:B_2}\Big)+\tfrac{\delta^2(1-\lambda)}{2\sigma_\lambda^2}-\tfrac{\delta^3}{6\sigma_\lambda^2}\\
& \underbrace{-\tfrac{1-2\lambda}{72\sigma_\lambda^4}\cdot\big(\tfrac{\delta(1-2\lambda)}{2}+\tfrac{1-\lambda(1-\lambda)}{12}\big)}_{=:C_2}\underbrace{-\tfrac{41\delta^2\lambda(1-\lambda)(1-2\lambda)}{7200\sigma_\lambda^6}}_{=:C_3}\\
\ge\;& A_1\cdot B_1-A_2+A_1\cdot B_2+A_2\cdot B_1-A_3-C_2-C_3.
\end{aligned}$$

Further we have now

$$
\begin{aligned}
R_2 &= \big(\tfrac{4-2\lambda}{6} - (1-\delta)\big)\cdot\big(1-\overline{Se}-\tfrac{\delta(1-2\lambda)}{2\sigma_\lambda^2}-\tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2}\big)\\
&= \tfrac{3\delta-1-\lambda}{3}\cdot\Big(\tfrac{1}{\sigma_\lambda^4}\big(\tfrac{\delta^2\lambda(1-\lambda)}{2}+\tfrac{(1-2\lambda)^2\delta^2}{8}-\tfrac{7(1-2\lambda)\delta^2}{216}-\tfrac{1}{390}\big)+\tfrac{1}{\sigma_\lambda^6}\big(\tfrac{65(1-2\lambda)\lambda(1-\lambda)\delta^3}{288}-\tfrac{7\lambda(1-\lambda)\delta}{216}\big)\Big)\\
&= \underbrace{\tfrac{3\delta-1-\lambda}{3\sigma_\lambda^4}\cdot\big(\tfrac{\delta^2\lambda(1-\lambda)}{2}+\tfrac{(1-2\lambda)^2\delta^2}{8}-\tfrac{7(1-2\lambda)\delta}{216}-\tfrac{1}{390}\big)}_{=:D_2}+\underbrace{\tfrac{(3\delta-1-\lambda)\lambda(1-\lambda)}{3\sigma_\lambda^6}\cdot\big(\tfrac{65\delta^3(1-2\lambda)}{288}-\tfrac{7\delta^2}{216}\big)}_{=:D_3}.
\end{aligned}
$$

Finally we have with $\frac{1}{S^2} = 1+\frac{(n-2\ell+\delta)\delta}{(\ell-\delta)(n-\ell+\delta)} \geq 1+\frac{(n-2\ell+\delta)\delta}{\ell(n-\ell)} = 1+\frac{\delta(1-2\lambda+\delta/n)}{\sigma_\lambda^2}$

$$
\begin{aligned}
R_3 &= \tfrac{(1-\delta)^3}{6\sigma_\lambda^2}\cdot\big(\tfrac{1}{S^2}-1\big)-\tfrac{(1-\delta)^5}{40\sigma_\lambda^4}\cdot\tfrac{1}{S^4}\\
&\geq \tfrac{(1-\delta)^3}{6\sigma_\lambda^2}\cdot\tfrac{(1-2\lambda+\delta/n)\delta}{\sigma_\lambda^2}-\tfrac{(1-\delta)^5}{40\sigma_\lambda^4}\cdot\big(1+\tfrac{(1-2\lambda+\delta/n)\delta}{\sigma_\lambda^2}\big)^2\\
&= \tfrac{(1-\delta)^3}{6\sigma_\lambda^2}\cdot\tfrac{(1-2\lambda)\delta}{\sigma_\lambda^2}-\tfrac{(1-\delta)^5}{40\sigma_\lambda^4}+\tfrac{(1-\delta)^3\delta^2}{6n\sigma_\lambda^4}-\tfrac{(1-\delta)^5}{40\sigma_\lambda^4}\cdot\big(\tfrac{2(1-2\lambda+\delta/n)\delta}{\sigma_\lambda^2}+\tfrac{(1-2\lambda+\delta/n)^2\delta^2}{\sigma_\lambda^4}\big)\\
&\geq \tfrac{(1-\delta)^3}{6\sigma_\lambda^2}\cdot\tfrac{(1-2\lambda)\delta}{\sigma_\lambda^2}-\tfrac{(1-\delta)^5}{40\sigma_\lambda^4}+\tfrac{(1-\delta)^3\delta^2}{n\sigma_\lambda^4}\cdot\Big(\tfrac{1}{6}-\tfrac{1-2\lambda+\delta/n}{40\lambda(1-\lambda)}-\tfrac{(1-2\lambda+\delta/n)^2}{160\sigma_\lambda^2\lambda(1-\lambda)}\Big)\\
&\geq \tfrac{(1-\delta)^3\delta(1-2\lambda)}{6\sigma_\lambda^4}-\tfrac{(1-\delta)^5}{40\sigma_\lambda^4} \;=:\; E_2
\end{aligned}
$$

since due to $\frac{\delta}{n} = \frac{\delta\lambda}{\ell} \leq \frac{\lambda}{3}$ and $\sigma_\lambda^2 = \ell(1-\lambda) \geq 3\cdot\frac{1}{2}$ we have $\frac{1}{6}-\frac{1-2\lambda+\delta/n}{40\lambda(1-\lambda)}-\frac{(1-2\lambda+\delta/n)^2}{160\sigma_\lambda^2\lambda(1-\lambda)} \geq \frac{1}{6}-\frac{1-5\lambda/3}{40\cdot 2/9}-\frac{(1-5\lambda/3)^2}{160\cdot 3/2\cdot 2/9} = \frac{1}{6}-\frac{1}{20}-\frac{1}{270} > 0$ in the last inequality.

Thus we receive together with $R := R_1+R_2+R_3$

$$
R \;\geq\; A_1\cdot B_1 - A_2 - C_2 + D_2 + E_2 + A_1\cdot B_2 + A_2\cdot B_1 - A_3 - C_3 + D_3.
$$

We have $A_2 = \frac{\delta^3(1-\lambda)^2}{\sigma_\lambda^4}\cdot\big(\frac{1}{3}-\frac{\delta(5-7\lambda)}{24(1-\lambda)^2}+\frac{\delta^2}{40(1-\lambda)^2}\big) \geq \frac{\delta^3(1-\lambda)^2}{\sigma_\lambda^4}\cdot\big(\frac{1}{3}-\frac{5-7\lambda}{24(1-\lambda)^2}+\frac{1}{40(1-\lambda)^2}\big) \geq \frac{\delta^3(1-\lambda)^2}{\sigma_\lambda^4}\cdot\big(\frac{1}{3}-\frac{5-7/3}{24\cdot(2/3)^2}+\frac{1}{40\cdot(2/3)^2}\big) = \frac{67}{480}\cdot\frac{\delta^3(1-\lambda)^2}{\sigma_\lambda^4}$ where we use $\frac{\mathrm{d}}{\mathrm{d}\delta}[-\frac{\delta(5-7\lambda)}{24(1-\lambda)^2}+\frac{\delta^2}{40(1-\lambda)^2}] = -\frac{5-7\lambda}{24}+\frac{\delta}{20} \leq -\frac{5-7/2}{24}+\frac{1}{20} = -\frac{1}{80} < 0$ in the first and $\frac{\mathrm{d}}{\mathrm{d}\lambda}[-\frac{5-7\lambda}{24(1-\lambda)^2}+\frac{1}{40(1-\lambda)^2}] = \frac{7(1-\lambda)^2-2(5-7\lambda)(1-\lambda)}{24(1-\lambda)^4}+\frac{1}{20(1-\lambda)^3} = \frac{1}{(1-\lambda)^3}\cdot\frac{-9+85\lambda-70\lambda^2}{120} \geq \frac{1}{(1-\lambda)^3}\cdot\frac{-9+85/3-70/4}{120} = \frac{1}{(1-\lambda)^3}\cdot\frac{11}{720} > 0$ in the second inequality.

Further we have $B_1 = \frac{1}{\sigma_\lambda^2}\cdot\big(\frac{\delta(1-2\lambda)}{2}+\frac{1-\lambda(1-\lambda)}{12}\big) \geq \frac{1}{\sigma_\lambda^2}\cdot\big(\frac{\delta(1-2\lambda)}{2}+\frac{1}{16}\big)$, since $1-\lambda(1-\lambda) \geq \frac{3}{4}$ and thus we have for the terms of order n^{-3}:

$$
\begin{aligned}
& A_1 \cdot B_2 + A_2 \cdot B_1 - A_3 - C_3 + D_3 \\
\geq\ & \left(\tfrac{\delta^2(1-\lambda)}{2} - \tfrac{\delta^3}{6}\right) \cdot \tfrac{41\delta^2\lambda(1-\lambda)}{100\sigma_\lambda^6} + \tfrac{67}{480} \cdot \tfrac{\delta^3(1-\lambda)^2}{\sigma_\lambda^2} \cdot \left(\tfrac{\delta(1-2\lambda)}{2\sigma_\lambda^2} + \tfrac{1}{16\sigma_\lambda^2}\right) - \tfrac{19\delta^4(1-\lambda)^3}{108\sigma_\lambda^6} \\
& -\tfrac{1-2\lambda}{72\sigma_\lambda^2} \cdot \tfrac{41\delta^2\lambda(1-\lambda)}{100\sigma_\lambda^4} + \tfrac{(3\delta-1-\lambda)\lambda(1-\lambda)}{3\sigma_\lambda^6} \cdot \left(\tfrac{65\delta^3(1-2\lambda)}{288} - \tfrac{7\delta^2}{216}\right) \\
=\ & \tfrac{\delta^4(1-\lambda)}{24\sigma_\lambda^6} \cdot \Big(\left(\tfrac{(1-\lambda)}{2} - \tfrac{\delta}{6}\right) \cdot \tfrac{6\cdot 41\lambda}{25} + \tfrac{67(1-\lambda)}{20} \cdot \left(\tfrac{(1-2\lambda)}{2} + \tfrac{1}{16\delta}\right) - \tfrac{38(1-\lambda)^2}{9} \\
& -\tfrac{1-2\lambda}{3} \cdot \tfrac{41\lambda}{100\delta^2} + \tfrac{(3\delta-1-\lambda)\lambda}{3} \cdot \left(\tfrac{65(1-2\lambda)}{12} - \tfrac{7}{9\delta^2}\right)\Big) \\
=\ & \tfrac{\delta^4(1-\lambda)}{24\sigma_\lambda^6} \cdot \Big(- \delta \cdot \tfrac{41\lambda}{25} + \tfrac{123\lambda(1-\lambda)}{25} + \tfrac{67(1-2\lambda)(1-\lambda)}{40} - \tfrac{38(1-\lambda)^2}{9} + \tfrac{65\lambda(1-2\lambda)}{12} \\
& + \tfrac{1}{\delta} \cdot \left(\tfrac{67(1-\lambda)}{320} - \tfrac{7\lambda}{9} - \tfrac{65\lambda(1-2\lambda)(1+\lambda)}{36}\right) + \tfrac{1}{\delta^2} \cdot \left(-\tfrac{41\lambda(1-2\lambda)}{300} + \tfrac{7\lambda(1+\lambda)}{27}\right)\Big) \\
=\ & \tfrac{\delta^4(1-\lambda)}{24\sigma_\lambda^6} \cdot \Big(- \tfrac{41\lambda\delta}{25} + \tfrac{-4585+24\,761\lambda-29\,926\lambda^2}{1800} + \tfrac{603-8043\lambda+5200\lambda^2+10\,400\lambda^3}{2880\delta} + \tfrac{331\lambda+1438\lambda^2}{2700\delta^2}\Big) \\
=:\ & \tfrac{\delta^4(1-\lambda)}{24\sigma_\lambda^6} \cdot r_3(\delta, \lambda).
\end{aligned}
$$

Now we want to estimate the function r_3 downwards. Since $\frac{\mathrm{d}^3 r_3(\delta,\lambda)}{\mathrm{d}\delta \mathrm{d}\lambda^2} = -\frac{2\cdot 5200+6\lambda\cdot 10\,400}{2880\delta^2} - \frac{2\cdot 1438}{1350\delta^3} < 0$ we have

$$
\begin{aligned}
\tfrac{\mathrm{d}^2 r_3}{\mathrm{d}\delta \mathrm{d}\lambda}(\delta, \lambda) \ &\leq\ \tfrac{\mathrm{d}r_3^2}{\mathrm{d}\delta\mathrm{d}\lambda}(\delta, \tfrac{1}{3}) \ =\ -\tfrac{41}{25} - \tfrac{-8043+2\cdot 5200/3+3\cdot 10\,400/9}{2880\delta^2} - \tfrac{331+2\cdot 1438/3}{2700\delta^3/2} \\
&=\ -\tfrac{41}{25} + \tfrac{1}{\delta^2} \cdot \left(\tfrac{3329}{8640} - \tfrac{1883}{4050\delta}\right) \ \leq\ -\tfrac{41}{25} + \tfrac{1}{\delta^2} \cdot \left(\tfrac{3329}{8640} - \tfrac{1883}{4050}\right) \\
&=\ -\tfrac{41}{25} - \tfrac{10\,321}{129\,600\delta^2} \ <\ 0
\end{aligned}
$$

and thus we have

$$
\begin{aligned}
\tfrac{\mathrm{d}r_3}{\mathrm{d}\delta}(\delta, \lambda) \ &\leq\ \tfrac{\mathrm{d}r_3(\delta,1/3)}{\mathrm{d}\delta} \ =\ -\tfrac{41/3}{25} - \tfrac{603-8043/3+5200/9+10\,400/27}{2880\delta^2} - \tfrac{331/3+1438/9}{1350\delta^3} \\
&=\ -\tfrac{41}{75} + \tfrac{15053}{38\,800\delta^2} - \tfrac{2341}{12\,150\delta^3} \ \leq\ -\tfrac{41}{75} + \tfrac{2}{5\delta^2} - \tfrac{1}{6\delta^3} \ \leq\ -\tfrac{41}{75} + \tfrac{4}{27} \cdot \tfrac{8/125}{1/36} \\
&=\ -\tfrac{41}{75} + \tfrac{128}{375} \ =\ -\tfrac{77}{375} \ <\ 0
\end{aligned}
$$

since for $k(t) = \frac{b}{t^2} - \frac{c}{t^3}$ holds $k'(t) = 0 \Leftrightarrow t = \frac{3c}{2b}$, $k''(\frac{3c}{2b}) = -\frac{(2b)^5}{(3c)^4} < 0$ and $k(\frac{3c}{2b}) = \frac{4}{27} \cdot \frac{b^3}{c^2}$ in the third inequality above.

Thus we have $r_3(\delta, \lambda) \geq r_3(1, \lambda)$, numerical

$$
\begin{aligned}
r_3(\delta, \lambda) \ &\geq\ -\tfrac{41\lambda}{25} + \tfrac{-4585+24\,761\lambda-29\,926\lambda^2}{1800} + \tfrac{603-8043\lambda+5200\lambda^2+10\,400\lambda^3}{2880} + \tfrac{331\lambda+1438\lambda^2}{2700} \\
&=\ -\tfrac{6733}{2880} + \tfrac{408\,067\lambda}{43\,200} - \tfrac{9644\lambda^2}{675} + \tfrac{65\lambda^3}{18} \ =:\ g_3(\lambda) \\
&\geq\ -\tfrac{1+\lambda}{2} \ =:\ h_3(\lambda)
\end{aligned}
$$

since $(g_3 - h_3)''(\lambda) = -\frac{2\cdot 9644}{675} + \frac{6\cdot 65\lambda}{18} \leq -\frac{2\cdot 9644}{675} + \frac{3\cdot 65}{18} = -\frac{23\,951}{1350} < 0$ and

$$
\begin{aligned}
(g_3 - h_3)(\tfrac{1}{2}) \ &=\ -\tfrac{6733}{2880} + \tfrac{408\,067/2}{43\,200} - \tfrac{9644/4}{675} + \tfrac{65/8}{18} + \tfrac{3}{4} \ =\ \tfrac{47}{3200} \\
(g_3 - h_3)(\tfrac{1}{3}) \ &=\ -\tfrac{6733}{2880} + \tfrac{408\,067/3}{43\,200} - \tfrac{9644/9}{675} + \tfrac{65/27}{18} + \tfrac{2}{3} \ =\ \tfrac{923}{38\,880}.
\end{aligned}
$$

Thus we have together with $r_3(\delta,\lambda) \geq -\frac{1+\lambda}{2}$ and $\frac{1-\lambda}{\sigma_\lambda^2} = \frac{1}{\ell} \leq \frac{1}{3}$

$$A_1 \cdot B_2 + A_2 \cdot B_1 - A_3 - C_3 + D_3 \;\;\geq\;\; -\frac{\delta^4(1-\lambda)}{\sigma_\lambda^6} \cdot \frac{1+\lambda}{48} \;\;\geq\;\; -\frac{\delta^4}{\sigma_\lambda^4} \cdot \frac{1+\lambda}{144}$$

and hence finally

$$\begin{aligned}
R \;\;&\geq\;\; A_1 \cdot B_1 - A_2 - C_2 + D_2 + E_2 - \frac{\delta^4}{\sigma_\lambda^4} \cdot \frac{1+\lambda}{144} \\
&= \frac{1}{\sigma_\lambda^4} \cdot \Big(\big(\frac{\delta^2(1-\lambda)}{2} - \frac{\delta^3}{6}\big) \cdot \big(\frac{\delta(1-2\lambda)}{2} + \frac{1-\lambda(1-\lambda)}{12}\big) - \big(\frac{\delta^3(1-\lambda)^2}{3} - \frac{\delta^4(5-7\lambda)}{24} + \frac{\delta^5}{40}\big) \\
&\quad - \frac{1-2\lambda}{72} \cdot \big(\frac{1-2\lambda}{2}\delta + \frac{1-\lambda(1-\lambda)}{12}\big) + \frac{3\delta-1-\lambda}{3} \cdot \big(\frac{\delta^2\lambda(1-\lambda)}{2} + \frac{(1-2\lambda)^2\delta^2}{8} - \frac{7(1-2\lambda)\delta}{216} - \frac{1}{390}\big) \\
&\quad + \frac{(1-2\lambda)\delta(1-\delta)^3}{6} - \frac{(1-\delta)^5}{40} - \frac{\delta^4(1+\lambda)}{144}\big)\Big) \\
&= \frac{1}{\sigma_\lambda^4} \cdot \Big(- \frac{(1-2\lambda)(1-\lambda(1-\lambda))}{72 \cdot 12} + \frac{1+\lambda}{3 \cdot 390} - \frac{1}{40} + \delta \cdot \big(-\frac{(1-2\lambda)^2}{144} + \frac{7(1-2\lambda)(1+\lambda)}{216 \cdot 3} - \frac{1}{390} + \frac{1}{8} + \frac{1-2\lambda}{6}\big) \\
&\quad + \delta^2 \cdot \big(\frac{(1-\lambda)(1-\lambda(1-\lambda))}{24} - \frac{7(1-2\lambda)}{216} - \frac{(1+\lambda)\lambda(1-\lambda)}{6} - \frac{(1-2\lambda)^2(1+\lambda)}{24} - \frac{1}{4} - \frac{1-2\lambda}{2}\big) \\
&\quad + \delta^3 \cdot \big(\frac{(1-2\lambda)(1-\lambda)}{4} - \frac{1-\lambda(1-\lambda)}{6 \cdot 12} - \frac{(1-\lambda)^2}{3} + \frac{\lambda(1-\lambda)}{2} + \frac{(1-2\lambda)^2}{8} + \frac{1}{4} + \frac{1-2\lambda}{2}\big) \\
&\quad + \delta^4 \cdot \big(\frac{1-\lambda}{8} - \frac{1}{8} - \frac{1-2\lambda}{6} - \frac{1+\lambda}{144}\big)\Big) \\
&= \frac{1}{\sigma_\lambda^4} \cdot \Big(- \frac{1421}{56\,160} + \frac{9\lambda}{2080} - \frac{\lambda^2}{288} + \frac{\lambda^3}{432} + \delta \cdot \big(\frac{24\,679}{84\,240} - \frac{205\lambda}{648} - \frac{4\lambda^2}{81}\big) \\
&\quad + \delta^2 \cdot \big(-\frac{169}{216} + \frac{203\lambda}{216} + \frac{\lambda^2}{12} - \frac{\lambda^3}{24}\big) + \delta^3 \cdot \big(\frac{7}{9} - \frac{77\lambda}{72} + \frac{11\lambda^2}{72}\big) + \delta^4 \cdot \big(-\frac{25}{144} + \frac{29\lambda}{144}\big)\Big) \\
&\geq \frac{1}{\sigma_\lambda^4} \cdot \Big(- \frac{(1-\delta)^4}{126} - \frac{1421}{56\,160} + \frac{9\lambda}{2080} - \frac{\lambda^2}{288} + \frac{\lambda^3}{432} + \delta \cdot \big(\frac{24\,679}{84\,240} - \frac{205\lambda}{648} - \frac{4\lambda^2}{81}\big) \\
&\quad + \delta^2 \cdot \big(-\frac{169}{216} + \frac{203\lambda}{216} + \frac{\lambda^2}{12} - \frac{\lambda^3}{24}\big) + \delta^3 \cdot \big(\frac{7}{9} - \frac{77\lambda}{72} + \frac{11\lambda^2}{72}\big) + \delta^4 \cdot \big(-\frac{25}{144} + \frac{29\lambda}{144}\big)\Big) \\
&=: \frac{1}{\sigma_\lambda^4} \cdot r(\delta,\lambda).
\end{aligned}$$

We only used in the last inequality above $0 \leq \frac{(1-\delta)^4}{126}$. With this addditional negative term $-\frac{(1-\delta)^4}{126}$ we receive the function $r(\delta,\lambda)$, which is increasing with respect to δ, while without this term we would not have this monotonicity for δ near to $\frac{1}{2}$ and λ near to $\frac{1}{3}$.

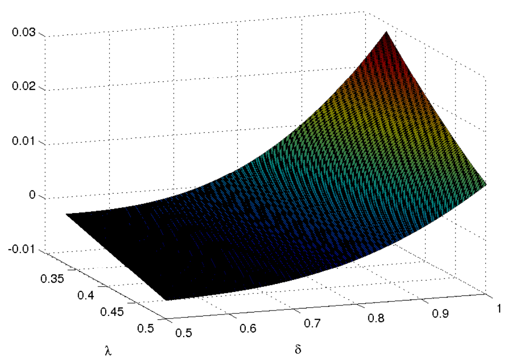

FIGURE 11. Graph of $r(\delta,\lambda)$

Now $\frac{dr}{d\delta}(\delta,\lambda) \geq 0$ holds, because first we have

$$\begin{aligned}
\frac{\mathrm{d}^3 r}{\mathrm{d}\delta^2 d\lambda} &= 2\cdot(\tfrac{203}{216}+\tfrac{\lambda}{6}-\tfrac{\lambda^2}{8}) + 6\delta\cdot(-\tfrac{77}{72}+\tfrac{11\lambda}{36}) + 12\delta^2\cdot\tfrac{29}{144} \\
&= \tfrac{203}{108}+\tfrac{\lambda}{3}-\tfrac{\lambda^2}{4}+\delta\cdot(-\tfrac{77}{12}+\tfrac{11\lambda}{6})+\delta^2\cdot\tfrac{29}{12} \\
&\leq \tfrac{203}{108}+\tfrac{1}{6}-\tfrac{1}{8}+\delta\cdot(-\tfrac{77}{12}+\tfrac{11}{12})+\delta^2\cdot\tfrac{29}{12} \\
&= \tfrac{415}{216}-\tfrac{11}{2}\delta+\tfrac{29}{12}\delta^2 \;\leq\; \tfrac{415}{216}-\tfrac{11}{4}+\tfrac{29}{48} \;=\; -\tfrac{97}{432} \;<\; 0
\end{aligned}$$

and thus second with $\lambda \leq \frac{1}{2}$ and $\frac{27\delta}{16} - \frac{7\delta^2}{8} \geq \min\{\frac{27/2}{16} - \frac{7/4}{8}, \frac{27}{16} - \frac{7}{8}\} = \frac{5}{8}$

$$\begin{aligned}
\frac{\mathrm{d}^2 r}{\mathrm{d}\delta^2}(\delta,\lambda) &\geq \frac{\mathrm{d}^2 r}{\mathrm{d}\delta^2}(\delta,\tfrac{1}{2}) \\
&= -\tfrac{12(1-\delta)^2}{126}-\tfrac{169}{108}+\tfrac{203}{216}+\tfrac{1}{12\cdot 2}-\tfrac{1}{24\cdot 4}+\delta\cdot(\tfrac{14}{3}-\tfrac{77}{24}+\tfrac{11}{48})+\delta^2\cdot(-\tfrac{25}{12}+\tfrac{29}{24}) \\
&= -\tfrac{2(1-\delta)^2}{21}-\tfrac{19}{32}+\delta\cdot\tfrac{27}{16}-\delta^2\cdot\tfrac{7}{8} \;\geq\; -\tfrac{2/4}{21}-\tfrac{19}{32}+\tfrac{5}{8} \;=\; -\tfrac{1}{42}+\tfrac{1}{32} \;>\; 0,
\end{aligned}$$

from which finally third with $\delta \geq \frac{1}{2}$ follows

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}\delta}(\delta,\lambda) &\geq \frac{\mathrm{d}r}{\mathrm{d}\delta}(\tfrac{1}{2},\lambda) \\
&= \tfrac{4/8}{126}+\tfrac{24\,679}{84\,240}-\tfrac{205\lambda}{648}-\tfrac{4\lambda^2}{81}-\tfrac{169}{216}+\tfrac{203\lambda}{216}+\tfrac{\lambda^2}{12}-\tfrac{\lambda^3}{24} \\
&\quad +\tfrac{3}{4}\cdot(\tfrac{7}{9}-\tfrac{77\lambda}{72}+\tfrac{11\lambda^2}{72})+\tfrac{1}{2}\cdot(-\tfrac{25}{144}+\tfrac{29\lambda}{144}) \\
&= \tfrac{1}{252}+\tfrac{1193}{168\,480}-\tfrac{101}{1296}\cdot\lambda+\tfrac{385}{2592}\cdot\lambda^2-\tfrac{\lambda^3}{24} \\
&\geq \tfrac{1}{252}+\tfrac{1193}{168\,480}-\tfrac{101}{1296\cdot 3}+\tfrac{385}{2592\cdot 9}-\tfrac{1}{24\cdot 27} \;=\; \tfrac{1}{252}-\tfrac{373}{94\,770} \;=\; \tfrac{43}{1\,326\,780} \;>\; 0
\end{aligned}$$

since $(-\frac{101\lambda}{1296}+\frac{385\lambda^2}{2592}-\frac{\lambda^3}{24})' = -\frac{101}{1296}+\frac{385\lambda}{1296}-\frac{\lambda^2}{8} \geq -\frac{101}{1296}+\frac{385}{1296\cdot 3}-\frac{1}{8\cdot 9} = \frac{7}{972} > 0$.

Thus we have $R \geq \frac{1}{\sigma_\lambda^4}\cdot r(\frac{1}{2},\lambda)$ and with $H \geq \frac{1}{\sigma_\lambda^4}\cdot(-\frac{386}{6912}+\frac{1572}{6912}\lambda-\frac{1186}{6912}\lambda^2)$ due to the previous part of this proof we have

$$\begin{aligned}
H+R &\geq \tfrac{1}{\sigma_\lambda^4}\cdot\Big(-\tfrac{386}{6912}+\tfrac{1572\lambda}{6912}-\tfrac{1186\lambda^2}{6912}-\tfrac{(1-1/2)^4}{126}-\tfrac{1421}{56\,160}+\tfrac{9\lambda}{2080}-\tfrac{\lambda^2}{288}+\tfrac{\lambda^3}{432} \\
&\qquad +\tfrac{1}{2}\cdot(\tfrac{24\,679}{84\,240}-\tfrac{205\lambda}{648}-\tfrac{4\lambda^2}{81})+\tfrac{1}{4}\cdot(-\tfrac{169}{216}+\tfrac{203\lambda}{216}+\tfrac{\lambda^2}{12}-\tfrac{\lambda^3}{24}) \\
&\qquad +\tfrac{1}{8}\cdot(\tfrac{7}{9}-\tfrac{77\lambda}{72}+\tfrac{11\lambda^2}{72})+\tfrac{1}{16}\cdot(-\tfrac{25}{144}+\tfrac{29\lambda}{144})\Big) \\
&= \tfrac{1}{\sigma_\lambda^4}\cdot\big(-\tfrac{418\,849}{9\,434\,880}+\lambda\cdot\tfrac{252\,637}{1\,347\,840}-\lambda^2\cdot\tfrac{1657}{10\,368}-\lambda^3\cdot\tfrac{7}{864}\big) \\
&\geq \tfrac{1}{\sigma_\lambda^4}\cdot\big(-\tfrac{418\,849}{9\,434\,880}+\tfrac{252\,637}{1\,347\,840}\cdot\tfrac{1}{3}-\tfrac{1657}{10\,368}\cdot\tfrac{1}{9}-\tfrac{7}{864}\cdot\tfrac{1}{27}\big) \;=\; \tfrac{1}{\sigma_\lambda^4}\cdot\tfrac{1193}{42\,456\,960}
\end{aligned}$$

since $(\frac{252\,637\lambda}{1\,347\,840}-\frac{1657\lambda^2}{10\,368}-\frac{7\lambda^3}{864})' \geq \frac{252\,637}{1\,347\,840}-\frac{1657}{10\,368}-\frac{3}{4}\cdot\frac{7}{864} = \frac{9\,679}{449\,280} > 0$ and $\lambda \geq \frac{1}{3}$ the last inequality. Thus it follows

$$\mathcal{D} \;\geq\; \frac{1}{\sqrt{2\pi}\,\sigma}\cdot(H+R) \;\geq\; \frac{1}{\sqrt{2\pi}\,\sigma}\cdot\frac{1}{\sigma_\lambda^2}\cdot\frac{1193}{42\,456\,960} \;>\; 0$$

and hence Lemma 8.3. □

Lemma 8.4. *If $n \in \mathbb{N}$, $p \in [\frac{1}{6}, \frac{1}{2})$, such that $m+1 \in [\frac{n}{3}, \frac{n}{2}]$, $m \geq 2$ and $p \in (\frac{m+1/2}{n}, \frac{m+1}{n})$, then we have*

$$\tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - F(m) + G(m) \quad > \quad 0.$$

Proof. For given $n \in \mathbb{N}$ and $p \in [\frac{1}{6}, \frac{1}{2}]$ we have a unique $m = \lfloor np \rfloor$. Let us now, for the given n and m, consider $\mathcal{D}$, defined by

$$\mathcal{D}(p) \;\; := \;\; \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - F_{n,p}(m) + G_{n,p}(m)$$

as a function of p on $[\frac{m}{n}, \frac{m+1}{n})$.

Let now $\delta \in (0, \frac{1}{2})$ with $p = \frac{m+1-\delta}{n}$. Due to Lemma 8.3 we already know that $\mathcal{D}(\frac{m+\delta}{n}) \geq 0$ holds.

Let in the following denote $\varsigma_y := \sigma_{\frac{m+y}{n}} = \sqrt{(m+y)(n-m-y)/n}$ for $y \in [0,1]$, in particular $\sigma_p = \varsigma_{1-\delta}$. If we show

$$\sigma_p \cdot \mathcal{D}(p) \;=\; \varsigma_{1-\delta} \cdot \mathcal{D}(\tfrac{m+1-\delta}{n}) \;\;\geq\;\; \varsigma_\delta \cdot \mathcal{D}(\tfrac{m+\delta}{n}) \tag{28}$$

it follows $\mathcal{D}(p) = \frac{1}{\varsigma_{1-\delta}} \cdot \varsigma_{1-\delta} \cdot \mathcal{D}(\frac{m+1-\delta}{n}) \geq \frac{\varsigma_\delta}{\varsigma_{1-\delta}} \cdot \mathcal{D}(\frac{m+\delta}{n}) \geq 0$ and hence Lemma 8.4.

We start with

$$\begin{aligned}
& \varsigma_{1-\delta} \cdot \mathcal{D}(\tfrac{m+1-\delta}{n}) - \varsigma_\delta \cdot \mathcal{D}(\tfrac{m+\delta}{n}) \\
= \;& \varsigma_\delta \cdot F_{n,\frac{m+\delta}{n}}(m) - \varsigma_{1-\delta} \cdot F_{n,\frac{m+1-\delta}{n}}(m) + \varsigma_{1-\delta} \cdot \Phi(-\tfrac{1-\delta}{\varsigma_{1-\delta}}) - \varsigma_\delta \cdot \Phi(-\tfrac{\delta}{\varsigma_\delta}) \\
= \;& \varsigma_{1-\delta} \cdot (F_{n,\frac{m+\delta}{n}}(m) - F_{n,\frac{m+1-\delta}{n}}(m)) + \varsigma_{1-\delta} \cdot (\Phi(-\tfrac{1-\delta}{\varsigma_{1-\delta}}) - \Phi(-\tfrac{\delta}{\varsigma_\delta})) \\
& -(F_{n,\frac{m+\delta}{n}}(m) - \Phi(-\tfrac{\delta}{\varsigma_{1-\delta}}) \cdot (\varsigma_{1-\delta} - \varsigma_\delta) \\
\geq \;& \varsigma_{1-\delta} \cdot (F_{n,\frac{m+\delta}{n}}(m) - F_{n,\frac{m+1-\delta}{n}}(m)) + \varsigma_{1-\delta} \cdot (\Phi(-\tfrac{1-\delta}{\varsigma_{1-\delta}}) - \Phi(-\tfrac{\delta}{\varsigma_\delta})) \\
& -\tfrac{4-2(m+\delta)/n}{6\sqrt{2\pi}\,\varsigma_\delta} \cdot (\varsigma_{1-\delta} - \varsigma_\delta)
\end{aligned}$$

since $\frac{m+\delta}{n} \in [\frac{m}{n}, \frac{m+1/2}{n}]$ and hence we can apply Lemma 8.3 in the last inequality.

Further due to Lemma 11.1 with $h := \frac{1-\delta}{\varsigma_{1-\delta}} - \frac{\delta}{\varsigma_\delta} > 0$, since $\frac{\mathrm{d}}{\mathrm{d}\delta}\frac{\delta}{\varsigma_\delta} = \frac{1}{\varsigma_\delta}\cdot(1 - \frac{\delta(n-2k-2\delta)}{2(m+\delta)(n-m-\delta)}) \geq 0$ and $1-\delta > \delta$, we have

$$\begin{aligned}
\Phi(-\tfrac{1-\delta}{\varsigma_{1-\delta}}) - \Phi(-\tfrac{\delta}{\varsigma_\delta}) &= \Phi(\tfrac{\delta}{\varsigma_\delta}) - \Phi(\tfrac{1-\delta}{\varsigma_{1-\delta}})] \\
&= \Phi\big(\tfrac{1-\delta}{2\varsigma_{1-\delta}} + \tfrac{\delta}{2\varsigma_\delta} - \tfrac{1}{2}\cdot(\tfrac{1-\delta}{\varsigma_{1-\delta}} - \tfrac{\delta}{\varsigma_\delta})\big) - \Phi\big(\tfrac{1-\delta}{2\varsigma_{1-\delta}} + \tfrac{\delta}{2\varsigma_\delta} + \tfrac{1}{2}\cdot(\tfrac{1-\delta}{\varsigma_{1-\delta}} - \tfrac{\delta}{\varsigma_\delta})\big) \\
&\geq -h\cdot\varphi(\tfrac{1-\delta}{2\varsigma_{1-\delta}} + \tfrac{\delta}{2\varsigma_\delta})\cdot\exp\big(((\tfrac{1-\delta}{2\varsigma_{1-\delta}} + \tfrac{\delta}{2\varsigma_\delta})^2 - 1)\cdot\tfrac{h^2}{24} + \tfrac{h^4}{1440}\big) \\
&\geq -h\cdot\varphi(\tfrac{1-\delta}{2\varsigma_{1-\delta}} + \tfrac{\delta}{2\varsigma_\delta})\cdot\exp\big((\tfrac{1}{4} - 1)\cdot\tfrac{h^2}{24} + \tfrac{h^2}{1440}\cdot\tfrac{2}{3}\big) \\
&\geq -h\cdot\varphi(\tfrac{1-\delta}{2\varsigma_{1-\delta}} + \tfrac{\delta}{2\varsigma_\delta}) \\
&= -\tfrac{1}{\sqrt{2\pi}}\cdot(\tfrac{1-\delta}{\varsigma_{1-\delta}} - \tfrac{\delta}{\varsigma_\delta})\cdot\exp\big(-\tfrac{1}{8}\cdot(\tfrac{1-\delta}{\varsigma_{1-\delta}} + \tfrac{\delta}{\varsigma_\delta})^2\big) \\
&\geq -\tfrac{1}{\sqrt{2\pi}}\cdot(\tfrac{1-\delta}{\varsigma_{1-\delta}} - \tfrac{\delta}{\varsigma_\delta})\cdot\exp(-\tfrac{1}{8\varsigma_1^2})
\end{aligned}$$

since $\frac{1-\delta}{2\varsigma_{1-\delta}} + \frac{\delta}{2\varsigma_\delta} \leq \frac{1-\delta}{2\varsigma_\delta} + \frac{\delta}{2\varsigma_\delta} = \frac{1}{2\varsigma_\delta} \leq \frac{1}{2\varsigma_0} \leq \frac{1}{2}$ and $h \leq \frac{1}{\varsigma_1} \leq \sqrt{2/3}$ in the second and $\frac{1-\delta}{\varsigma_{1-\delta}} + \frac{\delta}{\varsigma_\delta} \geq \frac{1-\delta}{\varsigma_{1-\delta}} + \frac{\delta}{\varsigma_{1-\delta}} = \frac{1}{\varsigma_{1-\delta}} \geq \frac{1}{\varsigma_1}$ in the last inequality.

Due to Lemma 11.6 part (iv) and Lemma 11.4 we further have

$$\begin{aligned}
\tfrac{F_{n,\frac{m+\delta}{n}}(m) - F_{n,\frac{m+1-\delta}{n}}(m)}{1-2\delta} &\geq f_{n,\lambda}(\ell)\cdot(1 + \tfrac{1}{2\ell} - \tfrac{n}{6\ell(n-\ell)}) \\
&\geq \tfrac{1}{\sqrt{2\pi}\,\varsigma_1}\cdot(1 + \tfrac{1}{2\ell} - \tfrac{n}{6\ell(n-\ell)})\cdot\mathrm{e}^{-\frac{1}{12\varsigma_1^2} + \frac{1}{12n}}.
\end{aligned}$$

Thus we have from the inequalities above

$$\begin{aligned}
&\varsigma_{1-\delta}\cdot\mathcal{D}(\tfrac{m+1-\delta}{n}) - \varsigma_\delta\mathcal{D}(\tfrac{m+\delta}{n}) \\
\geq\;& \varsigma_{1-\delta}\cdot(F_{n,\frac{m+\delta}{n}}(m) - F_{n,\frac{m+1-\delta}{n}}(m)) + \varsigma_{1-\delta}\cdot(\Phi(-\tfrac{1-\delta}{\varsigma_{1-\delta}}) - \Phi(-\tfrac{\delta}{\varsigma_\delta})) \\
&-\tfrac{4-2(m+\delta)/n}{6\sqrt{2\pi}\,\varsigma_\delta}\cdot(\varsigma_{1-\delta} - \varsigma_\delta) \\
\geq\;& \tfrac{1-2\delta}{\sqrt{2\pi}}\cdot\tfrac{\varsigma_{1-\delta}}{\varsigma_1}\cdot(1 + \tfrac{1}{2\ell} - \tfrac{n}{6\ell(n-\ell)})\cdot\mathrm{e}^{-\frac{1}{12\varsigma_1^2} + \frac{1}{12n}} - \tfrac{\varsigma_{1-\delta}}{\sqrt{2\pi}}\cdot(\tfrac{1-\delta}{\varsigma_{1-\delta}} - \tfrac{\delta}{\varsigma_\delta})\cdot\mathrm{e}^{-\frac{1}{8\varsigma_1^2}} \\
&-\tfrac{4-2(m+\delta)/n}{6\sqrt{2\pi}\,\varsigma_\delta}\cdot(\varsigma_{1-\delta} - \varsigma_\delta) \\
=\;& \tfrac{1-2\delta}{\sqrt{2\pi}}\Big(\tfrac{\varsigma_{1-\delta}}{\varsigma_1}(1 + \tfrac{1}{2\ell} - \tfrac{n}{6\ell(n-\ell)})\cdot\mathrm{e}^{-\frac{1}{12\varsigma_1^2} + \frac{1}{12n}} - \tfrac{1-\delta-\delta\cdot\varsigma_{1-\delta}/\varsigma_\delta}{1-2\delta}\cdot\mathrm{e}^{-\frac{1}{8\varsigma_1^2}} - \tfrac{4-2(m+\delta)/n}{6(1-2\delta)}(\tfrac{\varsigma_{1-\delta}}{\varsigma_\delta} - 1)\Big).
\end{aligned}$$

Finally $\frac{\varsigma_{1-\delta}}{\varsigma_\delta} = \sqrt{1+\frac{(n-2k-1)(1-2\delta)}{(m+\delta)(n-m-\delta)}} \leq 1+\frac{(n-2k-1)(1-2\delta)}{2(m+\delta)(n-m-\delta)}$ holds and with $\delta\cdot \mathrm{e}^{-\frac{1}{8\sigma_1^2}} \leq \delta \leq \frac{1}{2} \leq \frac{4-2(m+\delta)/n}{6}$ we have with $1-2\delta>0$

$$\begin{aligned}
& \tfrac{\sqrt{2\pi}}{1-2\delta}\cdot\big(\varsigma_{1-\delta}\cdot\mathcal{D}(\tfrac{m+1-\delta}{n}) - \varsigma_\delta\mathcal{D}(\tfrac{m+\delta}{n})\big)\\
\geq\ & \tfrac{\varsigma_{1-\delta}}{\varsigma_1}\cdot(1+\tfrac{1}{2\ell}-\tfrac{n}{6\ell(n-\ell)})\cdot \mathrm{e}^{-\frac{1}{12\varsigma_1^2}+\frac{1}{12n}} - (1-\delta-(1+\tfrac{(n-2k-1)(1-2\delta)}{2(m+\delta)(n-m-\delta)})\cdot\delta)/(1-2\delta)\cdot \mathrm{e}^{-\frac{1}{8\varsigma_1^2}}\\
& -\tfrac{4-2(m+\delta)/n}{6(1-2\delta)}\cdot((1+\tfrac{(n-2k-1)(1-2\delta)}{2(m+\delta)(n-m-\delta)})-1)\\
=\ & \tfrac{\varsigma_{1-\delta}}{\varsigma_1}\cdot(1+\tfrac{1}{2\ell}-\tfrac{n}{6\ell(n-\ell)})\cdot \mathrm{e}^{-\frac{1}{12\varsigma_1^2}+\frac{1}{12n}} - (1-\tfrac{(n-2k-1)}{2(m+\delta)(n-m-\delta)}\cdot\delta)\cdot \mathrm{e}^{-\frac{1}{8\varsigma_1^2}}\\
& -\tfrac{4-2(m+\delta)/n}{6}\cdot\tfrac{(n-2k-1)}{2(m+\delta)(n-m-\delta)}\\
=\ & \tfrac{\varsigma_{1-\delta}}{\varsigma_1}\cdot(1+\tfrac{1-\lambda}{2\varsigma_1^2}-\tfrac{1}{6\varsigma_1^2})\cdot \mathrm{e}^{-\frac{1}{12\varsigma_1^2}+\frac{1}{12n}} - \mathrm{e}^{-\frac{1}{8\varsigma_1^2}} - (\tfrac{4-2\lambda}{6}+\tfrac{1-\delta}{3n}-\delta\cdot \mathrm{e}^{-\frac{1}{8\varsigma_1^2}})\cdot\tfrac{(n-2\ell+1)}{2(m+\delta)(n-m-\delta)}\\
\geq\ & (1-\tfrac{33(n-2\ell+1/2)\delta}{64\ell(n-\ell)})\cdot(1+\tfrac{1-\lambda}{2\varsigma_1^2}-\tfrac{1}{6\varsigma_1^2})\cdot \mathrm{e}^{-\frac{1}{12\varsigma_1^2}+\frac{1}{12n}} - \mathrm{e}^{-\frac{1}{8\varsigma_1^2}}\\
& -(\tfrac{4-2\lambda}{6}+\tfrac{1}{3n}-\delta\cdot \mathrm{e}^{-\frac{1}{8\varsigma_1^2}})\cdot\tfrac{9(n-2\ell+1)}{14\ell(n-\ell)}\\
=:\ & h(\delta).
\end{aligned}$$

Here we use in the last inequality above firstly that

$$\begin{aligned}
\tfrac{\varsigma_{1-\delta}}{\varsigma_1} &= \sqrt{1-\tfrac{(n-2\ell)\delta+\delta^2}{\ell(n-\ell)}} = 1-\tfrac{(n-2a)\delta+\delta^2}{2\ell(n-\ell)}+\sum_{r=2}^{\infty}\tfrac{(2r-3)!!}{(2r)!!}\cdot[\tfrac{(n-2\ell)\delta+\delta^2}{\ell(n-\ell)}]^r\\
&\geq 1-\tfrac{(n-2\ell)\delta+\delta^2}{\ell(n-\ell)}\cdot(\tfrac{1}{2}+\tfrac{1}{8}\sum_{r=1}^{\infty}[\tfrac{(n-2\ell)\delta+\delta^2}{\ell(n-\ell)}]^r)\\
&\geq 1-\tfrac{(n-2\ell)\delta+\delta/2}{\ell(n-\ell)}\cdot(\tfrac{1}{2}+\tfrac{1}{8}\sum_{r=1}^{\infty}(\tfrac{1}{9})^r) = 1-\tfrac{(n-2\ell+1/2)\delta}{\ell(n-\ell)}\cdot\tfrac{33}{64}
\end{aligned}$$

since $\frac{(n-2\ell)\delta+\delta^2}{\ell(n-\ell)} \leq \frac{1-2\lambda}{6(1-\lambda)}+\frac{1}{12(n-\ell)} \leq \frac{1}{12}+\frac{1}{36}=\frac{1}{9}$ and $\frac{(2r-3)!!}{(2r)!!}\leq\frac{1}{8}$ for $r\geq 2$ and secondly that $\frac{\ell(n-\ell)}{(m+\delta)(n-m-\delta)} \leq \frac{(m+1)(n-m-1)}{m(n-m)} \leq \frac{(m+1)\cdot 2(m+1)}{m(2(m+1)+1)} = \frac{(m+1)^2}{m(m+3/2)} \leq \frac{9}{2\cdot 7/2}=\frac{9}{7}$ since $n-m-1\leq 2(m+1)$ and $m+1\geq 3$.

Further we have

$$\begin{aligned}
h'(\delta) &= -\tfrac{33(n-2\ell+1/2)}{64\ell(n-\ell)}) \cdot (1+\tfrac{1-\lambda}{2\varsigma_1^2}-\tfrac{1}{6\varsigma_1^2}) \cdot \mathrm{e}^{-\frac{1}{12\varsigma_1^2}+\frac{1}{12n}} + \mathrm{e}^{-\frac{1}{8\varsigma_1^2}} \cdot \tfrac{9(n-2\ell+1)}{14\ell(n-\ell)} \\
&\geq \tfrac{n-2\ell+1}{\ell(n-\ell)} \cdot \mathrm{e}^{-\frac{1}{8\varsigma_1^2}} \cdot \big(\tfrac{9}{14}-\tfrac{33}{64}\cdot(1+\tfrac{1-\lambda}{2\varsigma_1^2}-\tfrac{1}{6\varsigma_1^2}) \cdot \mathrm{e}^{\frac{1}{8\varsigma_1^2}-\frac{1}{12\sigma_1^2}+\frac{1}{12n}}\big) \\
&= \tfrac{n-2\ell+1}{\ell(n-\ell)} \cdot \mathrm{e}^{-\frac{1}{8\varsigma_1^2}} \cdot \big(\tfrac{9}{14}-\tfrac{33}{64}\cdot(1+\tfrac{1}{2a}\cdot(1-\tfrac{1}{3(1-\lambda)})) \cdot \mathrm{e}^{\frac{1}{24\varsigma_1^2}+\frac{1}{12n}}\big) \\
&\geq \tfrac{n-2\ell+1}{\ell(n-\ell)} \cdot \mathrm{e}^{-\frac{1}{8\varsigma_1^2}} \cdot \big(\tfrac{9}{14}-\tfrac{33}{64}\cdot(1+\tfrac{1}{6}\cdot(1-\tfrac{1}{2})) \cdot \mathrm{e}^{\frac{1}{36}+\frac{1}{72}}\big) \\
&= \tfrac{n-2\ell+1}{\ell(n-\ell)} \cdot \mathrm{e}^{-\frac{1}{8\varsigma_1^2}} \cdot \big(\tfrac{9}{14}-\tfrac{33\cdot 13}{64\cdot 12}\cdot \mathrm{e}^{\frac{1}{24}}\big) \;=\; \tfrac{n-2\ell+1}{\ell(n-\ell)} \cdot \mathrm{e}^{-\frac{1}{8\varsigma_1^2}} \cdot 0.0605... \;\geq\; 0
\end{aligned}$$

since $\varsigma_1^2 \geq 3(1-\lambda) \geq 3/2$ and $n \geq 2\ell \geq 6$ and hence

$$\begin{aligned}
h(0) &= (1+\tfrac{1-\lambda}{2\varsigma_1^2}-\tfrac{1}{6\varsigma_1^2}) \cdot \mathrm{e}^{-\frac{1}{12\varsigma_1^2}+\frac{1}{12n}} - \mathrm{e}^{-\frac{1}{8\varsigma_1^2}} - (\tfrac{4-2\lambda}{6}+\tfrac{1}{3n}) \cdot \tfrac{9(n-2\ell+1)}{14\ell(n-\ell)} \\
&\geq (1+\tfrac{1-\lambda}{2\varsigma_1^2}-\tfrac{1}{6\varsigma_1^2}) \cdot (1-\tfrac{1}{12\varsigma_1^2}+\tfrac{1}{12n}) - (1-\tfrac{1}{8\varsigma_1^2}+\tfrac{1}{2\cdot 8^2\varsigma_1^4}) - (\tfrac{4-2\lambda}{6}+\tfrac{1}{18}) \cdot \tfrac{9(n-2\ell+1)}{14\ell(n-\ell)} \\
&= \tfrac{1-\lambda}{2\varsigma_1^2}-\tfrac{1}{6\varsigma_1^2}-\tfrac{1}{12\varsigma_1^2}+\tfrac{1}{12n}-(\tfrac{1-\lambda}{2\varsigma_1^2}-\tfrac{1}{6\varsigma_1^2})\cdot(\tfrac{1}{12\varsigma_1^2}-\tfrac{1}{12n})+\tfrac{1}{8\varsigma_1^2}-\tfrac{1}{2\cdot 8^2\varsigma_1^4}) \\
&\quad -(\tfrac{4-2\lambda}{6}+\tfrac{1}{18}) \cdot \tfrac{9(n-2\ell+1)}{14\ell(n-\ell)} \\
&= \tfrac{1}{\varsigma_1^2} \cdot \big(\tfrac{1-\lambda}{2}-\tfrac{1}{6}-\tfrac{1}{12}+\tfrac{\lambda(1-\lambda)}{12}+\tfrac{1}{8}-(\tfrac{4-2\lambda}{6}+\tfrac{1}{18})\cdot\tfrac{9(1-2\lambda)}{14}\big) \\
&\quad +\tfrac{1}{\varsigma_1^4} \cdot \big(-(\tfrac{1-\lambda}{2}-\tfrac{1}{6})\cdot(\tfrac{1}{12}-\tfrac{\lambda(1-\lambda)}{12})-\tfrac{1}{128}-(\tfrac{4-2\lambda}{6}+\tfrac{1}{18})\cdot\tfrac{9\lambda(1-\lambda)}{14}\big) \\
&= \tfrac{1}{\varsigma_1^2} \cdot \big(\tfrac{3}{8}-\tfrac{5}{12}\lambda-\tfrac{\lambda^2}{12}-\tfrac{9}{14}\cdot(\tfrac{13}{18}-\tfrac{16}{9}\lambda+\tfrac{2}{3}\lambda^2)\big) \\
&\quad +\tfrac{1}{\varsigma_1^4} \cdot \big(-\tfrac{1-\lambda(1-\lambda)}{24}\cdot(\tfrac{2}{3}-\lambda)-\tfrac{1}{128}-\tfrac{9\lambda(1-\lambda)}{14}\cdot(\tfrac{13}{18}-\tfrac{\lambda}{3})\big) \\
&= \tfrac{1}{\varsigma_1^2} \cdot \big(-\tfrac{5}{56}+\lambda\cdot\tfrac{61}{84}-\lambda^2\cdot\tfrac{43}{84}\big)+\tfrac{1}{\sigma_1^4}\cdot\big(-\tfrac{41}{1155}-\lambda\cdot\tfrac{199}{504}+\lambda^2\cdot\tfrac{307}{504}-\lambda^3\cdot\tfrac{29}{108}\big) \\
&\geq \tfrac{1}{\varsigma_1^2} \cdot \Big(-\tfrac{5}{56}+\lambda\cdot\tfrac{61}{84}-\lambda^2\cdot\tfrac{43}{84}+\tfrac{2}{3}\cdot\big(-\tfrac{41}{1155}-\lambda\cdot\tfrac{199}{504}+\lambda^2\cdot\tfrac{307}{504}-\lambda^3\cdot\tfrac{29}{108}\big)\Big) \\
&= \tfrac{1}{\varsigma_1^2} \cdot \big(-\tfrac{1367}{12096}+\tfrac{25\lambda}{54}-\tfrac{20\lambda^2}{189}-\tfrac{29\lambda^3}{252}\big) \\
&\geq \tfrac{1}{\varsigma_1^2} \cdot \big(-\tfrac{1367}{12096}+\tfrac{25/3}{54}-\tfrac{20/4}{189}-\tfrac{29/8}{252}\big) \;=\; \tfrac{1}{\varsigma_1^2}\cdot\tfrac{17}{36288} \;>\; 0.
\end{aligned}$$

Thus it follows together $\varsigma_{1-\delta} \cdot \mathcal{D}(\frac{m+1-\delta}{n}) - \varsigma_\delta \cdot \mathcal{D}(\frac{m+\delta}{n}) \geq \frac{1-2\delta}{\sqrt{2\pi}} \cdot h(\delta) \geq \frac{1-2\delta}{\sqrt{2\pi}} \cdot h(0) > 0$ and hence (28). □

Lemma 8.5. *If $n \in \mathbb{N}$, $p \in [\frac{1}{6}, \frac{1}{2})$, such that $m+1 \in [\frac{n}{6}, \frac{n}{3}]$ and $m \geq 2$, then we have*

$$\mathcal{D} \quad := \quad \tfrac{\sqrt{10}+3}{6\sqrt{2\pi}} \cdot \tfrac{p^2+q^2}{\sigma} - F(m) + G(m) \quad > \quad 0.$$

Proof. Let in the following as usual $\ell = m+1 \geq 3$, $\lambda = \frac{\ell}{n} \in [\frac{1}{6}, \frac{1}{3}]$, $\sigma_\lambda^2 = n\lambda(1-\lambda)$, $\mathrm{e}^{S_n^\ell} = f_{n,\lambda}(\ell) \cdot \sqrt{2\pi}\,\sigma_\lambda$ and $p = \frac{\ell-\delta}{n}$ with $\delta \in (0,1]$ and additionally $S := \frac{\sigma}{\sigma_\lambda}$. Analogous to the proof of Lemma 8.3 we get for the difference $\mathcal{D}$ due to Lemma 11.9 and 11.1 and $p^2+q^2 = \lambda^2 + (1-\lambda)^2 + \frac{2\delta(1-2\lambda)}{n} + \frac{2\delta^2}{n^2}$ in the inequality below

$$\begin{aligned}
\mathcal{D} \;&=\; \tfrac{4-2\lambda}{6\sqrt{2\pi}\,\sigma} - [F_{n,\lambda}(\ell) - 0.5] - [F_{n,p}(m) - F_{n,\lambda}(\ell)] + [G(p) - 0.5] \\
&\quad + \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot \Big(\tfrac{\sqrt{10}+3}{6}(p^2+q^2) - \tfrac{4-2\lambda}{6} \Big) \\
&\geq\; \tfrac{4-2\lambda}{6\sqrt{2\pi}\,\sigma} - \tfrac{4-2\lambda}{6} \cdot \mathrm{e}^{S_n^\ell} \cdot S \cdot \tfrac{1}{\sqrt{2\pi}\,\sigma} + \mathrm{e}^{S_n^\ell} \cdot S \cdot \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot \tfrac{n-2\ell}{72\ell(n-\ell)} \\
&\quad - [F_{n,p}(m) - F_{n,\lambda}(\ell)] + \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot \big(-(1-\delta) + \tfrac{(1-\delta)^3}{6\sigma^2} - \tfrac{(1-\delta)^5}{40\sigma^4}\big) \\
&\quad + \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot \Big(\tfrac{\sqrt{10}+3}{6}(\lambda^2 + (1-\lambda)^2) + \tfrac{2\delta(1-2\lambda)}{n} + \tfrac{2\delta^2}{n^2} - \tfrac{4-2\lambda}{6} \Big) \\
&=\; \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot \Big(\tfrac{4-2\lambda}{6}(1 - Se^{S_n^\ell}) + \tfrac{\sqrt{10}+3}{6}(\lambda^2 + (1-\lambda)^2) - \tfrac{2-\lambda}{3} + \tfrac{\lambda}{3} + \tfrac{2\delta(1-2\lambda)}{n} + \tfrac{2\delta^2}{n^2} \\
&\quad + \tfrac{1-2\lambda}{72\sigma_\lambda^2} \cdot Se^{S_n^\ell} - \big(\tfrac{F_{n,p}(m) - F_{n,\lambda}(m)}{f_{n,\lambda}(\ell)} - 1\big) \cdot Se^{S_n^\ell} - (1-\delta) + \tfrac{(1-\delta)^3}{6\sigma_\lambda^2} \cdot S^2 - \tfrac{(1-\delta)^5}{40\sigma_\lambda^4} \cdot S^4 \Big) \\
&=\; \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot \Big(\underbrace{\tfrac{\sqrt{10}+3}{6}(\lambda^2 + (1-\lambda)^2) - \tfrac{2-\lambda}{3}}_{=:G} + \underbrace{\tfrac{\delta}{3n} + \tfrac{4-2\lambda}{6} \cdot \big(\tfrac{\delta(1-2\lambda)}{2\sigma_\lambda^2} + \tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2}\big) + \tfrac{1-2\lambda}{72\sigma_\lambda^2}}_{=:H_1} \\
&\quad \underbrace{- \tfrac{\delta^2(1-\lambda)}{2\sigma_\lambda^2} + \tfrac{\delta^3}{6\sigma_\lambda^2} + (1-\delta) \cdot \big(\tfrac{(1-\delta)^2}{6\sigma_\lambda^2} - \tfrac{\delta(1-2\lambda)}{2\sigma_\lambda^2} - \tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2}\big)}_{=:H_2} \\
&\quad + \underbrace{\big((\delta - \tfrac{1+\lambda}{3}) \cdot (1 - Se^{S_n^\ell} - \tfrac{\delta(1-2\lambda)}{2\sigma_\lambda^2} - \tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2}\big) + \tfrac{1-2\lambda}{72\sigma_\lambda^2} \cdot (Se^{S_n^\ell} - 1)}_{=R_1} \\
&\quad \underbrace{-\big(\tfrac{F_{n,p}(m) - F_{n,\lambda}(m)}{f_{n,\lambda}(\ell)} - \delta\big) \cdot Se^{S_n^\ell} + \tfrac{\delta^2(1-\lambda)}{2\sigma_\lambda^2} - \tfrac{\delta^3}{6\sigma_\lambda^2}}_{=:R_2} \\
&\quad + \underbrace{\tfrac{(1-\delta)^3}{6\sigma_\lambda^2} \cdot \big(\tfrac{1}{S^2} - 1\big) - \tfrac{(1-\delta)^5}{40\sigma_\lambda^4} \cdot \tfrac{1}{S^4} + \tfrac{\delta}{n} \cdot (2(1-2\lambda) - \tfrac{1}{3}) + \tfrac{2\delta^2}{n^2}}_{=:R_3}.
\end{aligned}$$

The term $H := H_1 + H_2$ was already estimated in the proof of Lemma 8.3 on page 65, we have due to there

$$\begin{aligned}
H \;&\geq\; \tfrac{1}{\sigma_\lambda^2} \cdot \tfrac{-5+18\lambda-3\lambda^2-18\lambda^3+7\lambda^4}{288(1-\lambda)} \;=\; \tfrac{1}{\ell} \cdot \tfrac{-5+18\lambda-3\lambda^2-18\lambda^3+7\lambda^4}{288(1-\lambda)^2} \\
&=\; \tfrac{1}{\ell} \cdot \big(\tfrac{-5+8\lambda+18\lambda^2}{288} + \tfrac{10\lambda^3-11\lambda^4}{288(1-\lambda)^2} \big).
\end{aligned}$$

Consequently we have with $\frac{\sqrt{10}+3}{6} = \frac{1+\sqrt{1+1/9}}{2} \geq \frac{1+1+1/9-1/8/9^2}{2} = \frac{1331}{1296}$ and $\ell \geq 3$ at the second step below

$$\begin{array}{rcl} G+H & \geq & \frac{\sqrt{10}+3}{6}\cdot(\lambda^2+(1-\lambda)^2) - \frac{2-\lambda}{3} + \frac{1}{\ell}\cdot\big(\frac{-5+8\lambda+18\lambda^2}{288} + \frac{10\lambda^3-11\lambda^4}{288(1-\lambda)^2}\big) \\ & \geq & \frac{1}{\ell}\cdot\big(3\cdot(\frac{1331}{1296}\cdot(\lambda^2+(1-\lambda)^2) - \frac{2-\lambda}{3}) + \frac{-5+8\lambda+18\lambda^2}{288} + \frac{10\lambda^3-11\lambda^4}{288(1-\lambda)^2}\big) \\ & =: & \frac{1}{\ell}\cdot h(\lambda). \end{array}$$

First we have, using $\lambda \leq \frac{1}{3}$ and hence $(30\lambda^2 - 54\lambda^3)' = 60\lambda - 54\cdot 3\lambda^2 \geq 60\lambda - 54\lambda > 0$ at the third step below

$$\begin{array}{rcl} h'(\lambda) & = & 3\cdot(\frac{1331}{1296}\cdot(2\lambda - 2(1-\lambda)) + \frac{1}{3}) + \frac{8+36\lambda}{288} + \frac{(30\lambda^2-44\lambda^3)(1-\lambda)+2(10\lambda^3-11\lambda^4)}{288(1-\lambda)^3} \\ & = & -\frac{1331}{216}(1-2\lambda) + 1 + \frac{1}{36} + \frac{\lambda}{8} + \frac{30\lambda^2-54\lambda^3+22\lambda^4}{288(1-\lambda^3)} \\ & \leq & -\frac{1331}{216}\cdot\frac{1}{3} + 1 + \frac{1}{36} + \frac{1}{8\cdot 3} + \frac{30/9-54/27+22/81}{288}\cdot\frac{3^3}{2^3} \\ & = & -\frac{683}{648} + \frac{5}{72} + \frac{90-54+22/3}{288\cdot 2^3} \;=\; -\frac{319}{324} + \frac{65}{3456} \;=\; -\frac{10013}{10368} \;\leq\; -\frac{9}{10}. \end{array}$$

It follows that $h(\lambda) \geq h(\frac{1}{3}) + \frac{9}{10}\cdot(\frac{1}{3}-\lambda) \geq h(\frac{1}{3}) = \frac{1421}{31104} > 0.0456$ and hence

$$G+H \;\geq\; \frac{1}{\ell}\cdot\big(0.0456 + \frac{9}{10}\cdot(\frac{1}{3}-\lambda)\big) \;\geq\; \frac{1}{\ell}\cdot 0.0456.$$

Now we show in the following $R_1 + R_2 + R_3 \geq -\frac{0.02}{\ell}$, from which with the inequality above the claim in Lemma 8.5 follows.

We consider the remainders $R_1 + R_2 + R_3$. We have $1 - \frac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2} \leq \mathrm{e}^{S_n^\ell} \leq 1$ due to Lemma 11.4 and with $1 \geq S^2 = \frac{\sigma^2}{\sigma_\lambda^2} \geq \frac{\sigma^2_{(\ell-1)/n}}{\sigma_\lambda^2} = \frac{\ell-1)(n-\ell+1)}{\ell(n-\ell)} \geq \frac{\ell-1}{\ell} \geq \frac{2}{3}$ we get $\frac{\mathrm{d}}{\mathrm{d}S}\big[\frac{(1-\delta)^3}{6\sigma_\lambda^2 S^2} - \frac{(1-\delta)^5}{40\sigma_\lambda^4 S^4}\big] = -\frac{1-\delta)^3}{\sigma_\lambda^2 S^3}\cdot(\frac{1}{3} - \frac{(1-\delta)^2}{10\sigma_\lambda^2 S^2}) \leq 0$ and hence $\frac{(1-\delta)^3}{6\sigma_\lambda^2 S^2} - \frac{(1-\delta)^5}{40\sigma_\lambda^4 S^4} \geq \frac{(1-\delta)^3}{6\sigma_\lambda^2} - \frac{(1-\delta)^5}{40\sigma_\lambda^4}$. Further with $S\cdot\mathrm{e}^{S_n^\ell} \leq 1$ and Lemma 11.6 (ii) and (vi) we have

$$(\tfrac{F_{n,p}(m)-F_{n,\lambda}(m)}{f_{n,\lambda}(\ell)} - \delta)\cdot S e^{S_n^\ell} \;\leq\; (\tfrac{F_{n,p}(m)-F_{n,\lambda}(m)}{f_{n,\lambda}(\ell)} - \delta) \;\leq\; \tfrac{\delta^2(1-\lambda)}{2\sigma_\lambda^2} - \tfrac{\delta^3}{6\sigma_\lambda^2} + \tfrac{\delta^3(1-\lambda)^2}{3\sigma_\lambda^4} - \tfrac{\delta^4(1-\lambda)}{16\sigma_\lambda^4}.$$

Thus we get together

$$\begin{array}{rcl} R_1+R_2+R_3 & \geq & -(\frac{1+\lambda}{3} - \delta)\cdot(1 - Se^{S_n^\ell} - \frac{\delta(1-2\lambda)}{2\sigma_\lambda^2} - \frac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2}) \\ & & +\frac{1-2\lambda}{72\sigma_\lambda^2}\cdot(S\cdot(1 - \frac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2}) - 1) \\ & & -\frac{\delta^3(1-\lambda)^2}{3\sigma_\lambda^4} + \frac{\delta^4(1-\lambda)}{16\sigma_\lambda^2} - \frac{(1-\delta)^5}{40\sigma_\lambda^4} + \frac{\delta}{n}\cdot(\frac{5}{3} - 4\lambda) + \frac{2\delta^2}{n^2} \\ & =: & R(\delta). \end{array}$$

Further we have $\frac{dS}{\mathrm{d}\delta} = -\frac{1}{2S} \cdot \frac{1-2\lambda+2\delta/n}{\sigma_\lambda^2} < 0$ and hence we have with $\frac{1}{n} = \frac{\ell}{n^2\lambda} \geq \frac{3}{n^2\lambda}$ and $\mathrm{e}^{S_n^\ell} \leq 1 - \frac{1}{12} \cdot (\frac{1}{\sigma_\lambda^2} - \frac{1}{n}) + \frac{1}{390\sigma_\lambda^4}$ due to the computation starting in (27) on page 63 in the second step below

$$\begin{aligned}
R'(\delta) &= 1 - Se^{S_n^\ell} - \tfrac{\delta(1-2\lambda)}{2\sigma_\lambda^2} - \tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2} \\
&\quad -(\tfrac{1+\lambda}{3} - \delta) \cdot (\tfrac{1}{2S} \cdot \tfrac{1-2\lambda+2\delta/n}{\sigma_\lambda^2}\mathrm{e}^{S_n^\ell} - \tfrac{1-2\lambda}{2\sigma_\lambda^2}) - \tfrac{1-2\lambda}{72\sigma_\lambda^2} \cdot (1 - \tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2}) \cdot \tfrac{1}{2S} \cdot \tfrac{1-2\lambda+2\delta/n}{\sigma_\lambda^2} \\
&\quad -\tfrac{\delta^2(1-\lambda)^2}{\sigma_\lambda^4} + \tfrac{\delta^3(1-\lambda)}{4\sigma_\lambda^4} + \tfrac{(1-\delta)^4}{8\sigma_\lambda^4} + \tfrac{1}{n} \cdot (\tfrac{5}{3} - 4\lambda) + \tfrac{4\delta}{n^2} \\
&\geq 1 - (1 - \tfrac{\delta(1-2\lambda)+\delta/n}{2\sigma_\lambda^2}) \cdot (1 - \tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2}) - \tfrac{1}{390\sigma_\lambda^4} - \tfrac{\delta(1-2\lambda)}{2\sigma_\lambda^2} - \tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2} \\
&\quad -(\tfrac{1+\lambda}{3} - \delta) \cdot (\tfrac{1}{2S} \cdot \tfrac{1-2\lambda+2\delta/n}{\sigma_\lambda^2)}\mathrm{e}^{S_n^\ell} - \tfrac{1-2\lambda}{2\sigma_\lambda^2}) - \tfrac{1-2\lambda}{72\sigma_\lambda^2} \cdot \tfrac{1}{2S} \cdot \tfrac{1-2\lambda+2\delta/n}{\sigma_\lambda^2} \\
&\quad -\tfrac{\delta^2(1-\lambda)^2}{\sigma_\lambda^4} + \tfrac{\delta^3(1-\lambda)}{4\sigma_\lambda^4} + \tfrac{(1-\delta)^4}{8\sigma_\lambda^4} + \tfrac{3}{n^2\lambda} \cdot (\tfrac{5}{3} - 4\lambda) + \tfrac{4\delta}{n^2} \\
&= \tfrac{\delta^2}{2n\sigma_\lambda^2} - \tfrac{\delta(1-2\lambda+\delta/n)}{2\sigma_\lambda^2} \cdot \tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2} - \tfrac{1}{390\sigma_\lambda^4} - (\tfrac{1+\lambda}{3} - \delta) \cdot (\tfrac{\delta}{n\sigma_\lambda^2} + (\tfrac{\mathrm{e}^{S_n^\ell}}{S} - 1) \cdot \tfrac{1-2\lambda+2\delta/n}{2\sigma_\lambda^2}) \\
&\quad -\tfrac{1-2\lambda}{72\sigma_\lambda^2} \cdot \tfrac{1}{2S} \cdot \tfrac{1-2\lambda+2\delta/n}{\sigma_\lambda^2} - \tfrac{\delta^2(1-\lambda)^2}{\sigma_\lambda^4} + \tfrac{\delta^3(1-\lambda)}{4\sigma_\lambda^4} + \tfrac{(1-\delta)^4}{8\sigma_\lambda^4} + \tfrac{3}{n^2\lambda} \cdot (\tfrac{5}{3} - 4\lambda) + \tfrac{4\delta}{n^2} \\
&= \tfrac{1}{\sigma_\lambda^4}\Big(\underbrace{\tfrac{3\delta^2\lambda(1-\lambda)}{2} - \delta^2(1-\lambda)^2 - \tfrac{\delta\lambda(1-\lambda)(1+\lambda)}{3} + \tfrac{\delta^3(1-\lambda)}{4} + \tfrac{(1-\delta)^4}{8} + 3\lambda(1-\lambda)^2(\tfrac{5}{3}-4\lambda)+4\delta\lambda^2(1-\lambda)^2}_{=:r_1(\delta,\lambda)} \\
&\quad \underbrace{-\tfrac{\delta(1-2\lambda+\delta/n)}{2} \cdot \tfrac{1-\lambda(1-\lambda)}{12} - \tfrac{1}{390} - \sigma_\lambda^2(\tfrac{1+\lambda}{3} - \delta)(\tfrac{\mathrm{e}^{S_n^\ell}}{S} - 1) \cdot \tfrac{1-2\lambda+2\delta/n}{2} - \tfrac{(1-2\lambda)(1-2\lambda+2\delta/n)}{144S}}_{=:r_2(\delta)} \Big).
\end{aligned}$$

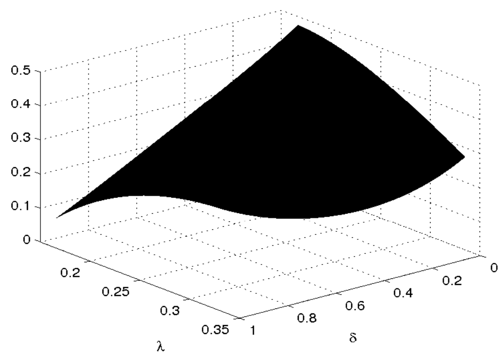

FIGURE 12. Graph of $r_1(\delta, \lambda)$

Now we consider $r_1(\delta, \lambda)$. This function is concave with respect to λ since

$$\begin{aligned}
\tfrac{\mathrm{d}^2 r_1}{\mathrm{d}\lambda^2} &= -3\delta^2 - 2\delta^2 + 2\delta\lambda + 3(-\tfrac{44}{3} + 58\lambda - 48\lambda^2) + 4\delta(2 - 12\lambda + 12\lambda^2) \\
&= -5\delta^2 + 3(-\tfrac{44}{3} + 58\lambda - 48\lambda^2) + 8\delta(1 - 6\lambda(1-\lambda)) + 2\delta\lambda \\
&\leq -5\delta^2 + 3(-\tfrac{44}{3} + \tfrac{58}{3} - \tfrac{48}{9}) + 8\delta(1 - 6 \cdot \tfrac{5}{36}) + \tfrac{2}{3} \cdot \delta \;=\; -5\delta^2 - 2 + 2\delta \\
&\leq 0.
\end{aligned}$$

Thus $r_1(\delta, \lambda)$ becomes with respect to λ minimal at the boundary of the domain of λ, namely at the points $\lambda = \frac{1}{6}$ or $\lambda = \frac{1}{3}$.

For those we get

$$\begin{aligned} r_1(\delta, \tfrac{1}{6}) &= \tfrac{3\delta^2}{2} \cdot \tfrac{5}{36} - \delta^2 \cdot \tfrac{25}{36} - \tfrac{\delta}{3} \cdot \tfrac{35}{216} + \tfrac{\delta^3}{4} \cdot \tfrac{5}{6} + \tfrac{(1-\delta)^4}{8} + 3 \cdot \tfrac{25}{216} + 4\delta \cdot \tfrac{25}{36^2} \\ &= \tfrac{25}{72} + \tfrac{1}{8} - \delta \cdot (\tfrac{35}{648} + \tfrac{1}{2} - \tfrac{25}{324}) + \delta^2 \cdot (\tfrac{5}{24} - \tfrac{25}{36} + \tfrac{3}{4}) + \delta^3 \cdot (\tfrac{5}{24} - \tfrac{1}{2}) + \tfrac{\delta^4}{8} \\ &= \tfrac{34}{72} - \delta \cdot \tfrac{103}{216} + \delta^2 \cdot \tfrac{19}{72} - \delta^3 \cdot \tfrac{7}{24} + \tfrac{\delta^4}{8} \\ &\geq \tfrac{34}{72} - \tfrac{103}{216} + \tfrac{19}{72} - \tfrac{7}{24} + \tfrac{1}{8} \;=\; \tfrac{5}{54} \end{aligned}$$

since $\frac{\mathrm{d}}{\mathrm{d}\delta}(-\delta \cdot \frac{103}{216} + \delta^2 \cdot \frac{19}{72} - \delta^3 \cdot \frac{7}{24} + \frac{\delta^4}{8}) = -\frac{103}{216} + \delta \cdot \frac{19}{36} - \delta^2 \cdot \frac{7}{8} + \frac{\delta^3}{2} \leq -\frac{103}{216} + \delta \cdot \frac{19}{36} - \delta^2 \cdot \frac{3}{8} = -\frac{103}{216} + \delta(1-\delta) \cdot \frac{3}{8} + \delta \cdot \frac{11}{72} \leq -\frac{103}{216} + \frac{1}{4} \cdot \frac{3}{8} + \frac{11}{72} = -\frac{70}{216} + \frac{3}{32} < 0$ and

$$\begin{aligned} r_1(\delta, \tfrac{1}{3}) &= \tfrac{3\delta^2}{2} \cdot \tfrac{2}{9} - \delta^2 \cdot \tfrac{4}{9} - \tfrac{\delta}{3} \cdot \tfrac{8}{27} + \tfrac{\delta^3}{4} \cdot \tfrac{2}{3} + \tfrac{(1-\delta)^4}{8} + 3 \cdot \tfrac{4}{81} + 4\delta \cdot \tfrac{4}{81} \\ &\geq \tfrac{\delta^2}{3} - \delta^2 \cdot \tfrac{4}{9} - \delta \cdot \tfrac{8}{81} + \tfrac{4}{27} + \delta \cdot \tfrac{16}{81} \\ &= \tfrac{4}{27} + \delta \cdot \tfrac{8}{81} - \tfrac{\delta^2}{9} \;\geq\; \tfrac{4}{27} + \delta \cdot \tfrac{8}{81} - \tfrac{\delta}{9} \;=\; \tfrac{4}{27} - \delta \cdot \tfrac{1}{81} \\ &\geq \tfrac{4}{27} - \tfrac{1}{81} \;=\; \tfrac{11}{81} \;>\; \tfrac{5}{54}. \end{aligned}$$

and hence $r_1(\delta, \lambda) \geq \frac{5}{54}$.

In the following we show $r_2(\delta) \geq -\frac{3}{54}$.

We consider the term $-\sigma_\lambda^2 \cdot (\frac{1+\lambda}{3} - \delta) \cdot (\frac{\mathrm{e}^{S_n^\ell}}{S} - 1) \cdot \frac{1-2\lambda+2\delta/n}{2}$ in two cases.

First, let $\delta \leq \frac{1+\lambda}{3}$, hence $\delta < \frac{4}{9}$, then we have with $\frac{1}{n} = \frac{\lambda}{\ell} \leq \frac{\lambda}{3}$ and $\mathrm{e}^{S_n^\ell} \leq 1$

$$\begin{aligned} -(\tfrac{1+\lambda}{3} - \delta) \cdot \sigma_\lambda^2 \cdot (\tfrac{\mathrm{e}^{S_n^\ell}}{S} - 1) \cdot \tfrac{1-2\lambda+2\delta/n}{2} &\geq -(\tfrac{1+\lambda}{3} - \delta) \cdot \sigma_\lambda^2 \cdot (\tfrac{1}{S} - 1) \cdot \tfrac{1-2\lambda+2\lambda/3}{2} \\ &\geq -(\tfrac{1+\lambda}{3} - \delta) \cdot \tfrac{1-2\lambda+\lambda/6}{2} \cdot \tfrac{11}{10} \cdot \delta \cdot \tfrac{1-2\lambda+2\lambda/3}{2} \\ &\geq -\delta \cdot (\tfrac{4}{9} - \delta) \cdot \tfrac{1-1/3+1/36}{2} \cdot \tfrac{11}{10} \cdot \tfrac{1-1/3+1/9}{2} \\ &= -(\tfrac{4}{9}\delta - \delta^2) \cdot \tfrac{25}{72} \cdot \tfrac{11}{10} \cdot \tfrac{7}{18} \\ &\geq -\tfrac{4}{81} \cdot \tfrac{25}{72} \cdot \tfrac{11}{10} \cdot \tfrac{7}{18} \;=\; -\tfrac{1}{54} \cdot \tfrac{385}{972} \;>\; -\tfrac{1}{54} \end{aligned}$$

since $\frac{1}{S} = \sqrt{1 + \frac{\delta(n-2\ell+\delta)}{(\ell-\delta)(n-\ell+\delta)}} \leq 1 + \frac{\delta(n-2\ell+\delta)}{2(\ell-\delta)(n-\ell+\delta)} = 1 + \frac{\delta(n-2\ell+\delta)}{2\ell(n-\ell)} \cdot \frac{\ell(n-\ell)}{(\ell-\delta)(n-\ell+\delta)} \leq 1 + \frac{\delta(n-2\ell+1/2)}{2\ell(n-\ell)} \cdot \frac{3 \cdot 6}{(3-\delta)(6+\delta)} \leq 1 + \sigma_\lambda^2 \cdot \frac{\delta(1-2\lambda+1/(2n))}{2} \cdot \frac{3 \cdot 6}{(3-4/9)(6+4/9)} \leq 1 + \frac{\delta(1-2\lambda+\lambda/6)}{2\sigma_\lambda^2} \cdot \frac{11}{10}$ in the first inequality.

Second, let $\delta \geq \frac{1+\lambda}{3}$, hence $\delta \geq \frac{7}{18}$, then we have with $S \leq 1$ in the first and Lemma 11.4 in the second step

$$\begin{aligned} -(\tfrac{1+\lambda}{3} - \delta) \cdot \sigma_\lambda^2 \cdot (\tfrac{e^{S_n^\ell}}{S} - 1) \cdot \tfrac{1-2\lambda+2\delta/n}{2} &\geq (\delta - \tfrac{1+\lambda}{3}) \cdot \sigma_\lambda^2 \cdot (e^{S_n^\ell} - 1) \cdot \tfrac{1-2\lambda+2\delta/n}{2} \\ &\geq -(\delta - \tfrac{1+\lambda}{3}) \cdot \tfrac{1-\lambda(1-\lambda)}{12} \cdot \tfrac{1-2\lambda+2\lambda/3}{2} \\ &\geq -(1 - \tfrac{7}{18}) \cdot \tfrac{1-5/36}{12} \cdot \tfrac{1-1/3+1/9}{2} = -\tfrac{11}{18} \cdot \tfrac{31}{12\cdot 36} \cdot \tfrac{7}{18} \\ &= -\tfrac{1}{54} \cdot \tfrac{11\cdot 31\cdot 7}{4\cdot 36\cdot 18} = -\tfrac{1}{54} \cdot \tfrac{2387}{2592} > -\tfrac{1}{54}. \end{aligned}$$

Based on these two cases with $\frac{1}{S} = \sqrt{1 + \frac{\delta(n-2\ell+\delta)}{(\ell-\delta)(n-\ell+\delta)}} \leq 1 + \frac{\delta(n-2\ell+\delta)}{2(\ell-\delta)(n-\ell+\delta)}$ and $\frac{1}{n} \leq \frac{\lambda}{3}$ we have

$$\begin{aligned} r_2(\delta) &\geq -\tfrac{(1-2\lambda+\delta/n)\delta}{2} \cdot \tfrac{1-\lambda(1-\lambda)}{12} - \tfrac{1}{390} - \tfrac{1}{54} - \tfrac{1-2\lambda}{72} \cdot \tfrac{1-2\lambda+2\delta/n}{2S} \\ &\geq -\tfrac{1-2\lambda+\lambda/3}{2} \cdot \tfrac{1-5/36}{12} - \tfrac{1}{390} - \tfrac{1}{54} - \tfrac{1-2\lambda}{72} \cdot \tfrac{1-2\lambda+2\lambda/3}{2} \cdot (1 + \tfrac{(n-2\ell+\delta)\delta}{2(\ell-\delta)(n-\ell+\delta)}) \\ &\geq -\tfrac{1-1/3+1/18}{2} \cdot \tfrac{31}{12\cdot 36} - \tfrac{1}{390} - \tfrac{1}{54} - \tfrac{1-1/3}{72} \cdot \tfrac{1-1/3+1/9}{2} \cdot (1 + \tfrac{1}{2(\ell-\delta)}) \\ &\geq -\tfrac{13}{36} \cdot \tfrac{31}{12\cdot 36} - \tfrac{1}{390} - \tfrac{1}{54} - \tfrac{1}{108} \cdot \tfrac{7}{18} \cdot (1 + \tfrac{1}{2\cdot 2}) = -\tfrac{13\cdot 31}{12\cdot 36^2} - \tfrac{1}{390} - \tfrac{1}{54} - \tfrac{7\cdot 5}{108\cdot 18\cdot 4} \\ &= -\tfrac{1}{54} \cdot (\tfrac{13\cdot 31}{12\cdot 24} + \tfrac{9}{65} + 1 + \tfrac{35}{2\cdot 18\cdot 4}) \geq -\tfrac{1}{54} \cdot (\tfrac{3}{2} + \tfrac{1}{4} + 1 + \tfrac{1}{4}) = -\tfrac{3}{54} \end{aligned}$$

and thus $R'(\delta) \geq \frac{1}{\sigma_\lambda^4} \cdot (r_1(\delta, \lambda) + r_2(\delta)) \geq \frac{1}{\sigma_\lambda^4} \cdot (\frac{5}{54} - \frac{3}{54}) > 0$ and hence with Lemma 11.4 we have

$$\begin{aligned} R(\delta) &\geq R(0) = -\tfrac{1+\lambda}{3} \cdot (1 - e^{S_n^\ell} - \tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2}) - \tfrac{1-2\lambda}{72\sigma_\lambda^2} \cdot \tfrac{1-\lambda(1-\lambda)}{12\sigma_\lambda^2} - \tfrac{1}{40\sigma_\lambda^4} \\ &\geq -\tfrac{1-2\lambda}{72\sigma_\lambda^2} \cdot \tfrac{1}{12\sigma_\lambda^2} - \tfrac{1}{40\sigma_\lambda^4} = -\tfrac{1}{\sigma_\lambda^4} \cdot (\tfrac{1-2\lambda}{72\cdot 12} + \tfrac{1}{40}) \\ &\geq -\tfrac{1}{\ell} \cdot \tfrac{3}{4} \cdot (\tfrac{1}{72\cdot 12} + \tfrac{1}{40}) \geq -\tfrac{1}{50\ell} \end{aligned}$$

since $\sigma_\lambda^4 = \ell^2(1-\lambda)^2 \geq 3 \cdot \ell \cdot \frac{4}{9} = \ell \cdot \frac{4}{3}$.

Thus it follows

$$\begin{aligned} \mathcal{D} &\geq G + H + R_1 + R_2 + R_3 \geq \tfrac{1}{\ell} \cdot (0.0456 + \tfrac{9}{10} \cdot (\tfrac{1}{3} - \lambda) - \tfrac{1}{50}) \\ &= \tfrac{1}{\ell} \cdot (0.0256 + \tfrac{9}{10} \cdot (\tfrac{1}{3} - \lambda)) \geq \tfrac{1}{\ell} \cdot 0.0256 > 0 \end{aligned}$$

and hence Lemma 8.5. □

9. Proofs of the boundary cases in Propositions 6.1 and 6.2

In this chapter we proof the cases in Proposition 6.1 and 6.2 where n respectively ℓ are very small. We start with

Lemma 7.5. *Let $n \in \mathbb{N},\ n \leq 5$. If $p \in [\frac{1}{2}, \frac{5}{6}]$, then*

$$\begin{array}{rcl} f(k) & \leq & g(k) \quad \text{if} \quad np+1 \leq k \\ g(k+1) & \leq & f(k) \quad \text{if} \quad np \leq k \leq np + \frac{8}{9} \cdot \sqrt{npq} \end{array}$$

and if $p \in [\frac{1}{6}, \frac{1}{2})$, then

$$\begin{array}{rcl} f(k) & \leq & g(k) \quad \text{if} \quad np+1 \leq k \leq np+npq \\ g(k+1) & \leq & f(k) \quad \text{if} \quad np \ \leq k \leq np+npq \end{array}$$

and if $p \in [\frac{1}{6}, \frac{1}{2}]$and $k > np + npq$, then with $c_0 := \frac{2}{\sqrt{2\pi}\, c_{\mathrm{E}}} - 1$

$$|F(k) - G(k)| \ \leq \ c_0 \cdot \tfrac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}, \qquad |G(k+1) - F(k)| \ \leq \ c_0 \cdot \tfrac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}.$$

Proof. We consider all cases separately.

a) First let $p \geq \frac{1}{2}$. We show $f(k) \leq g(k)$ if $k \geq np+1$.

1.) Let $k = n$. Then we have due to the proof of Lemma 7.2 (there in (b), 2.) on page 40)

$$\begin{array}{rcl} \log(g(k)) - \log(f(k)) & \geq & \log(\frac{1}{\sqrt{2\pi}}) - \frac{1}{2}\log(npq) - \frac{nq}{2p} + \frac{1}{2p} - \frac{1}{6npq} - n\log(p) \\ & =: & h_1(n,p) \end{array}$$

and we have $\frac{\mathrm{d}h_1}{\mathrm{d}p} \leq \frac{1-2p}{2p^2q} \cdot (nq-1) \leq 0$, since $nq = n - np \geq n - (k-1) = 1$. Thus h_1 decreases with respect to p. Since $nq \geq 1$ it follows the condition $q \geq \frac{1}{n}$ and hence only the following cases are possible: $n = 5$ and $p \leq \frac{4}{5}$; $n = 4$ and $p \leq \frac{3}{4}$; $n = 3$ and $p \leq \frac{2}{3}$; $n = 2$ and $p = \frac{1}{2}$. In these cases we have $h_1(5, \frac{4}{5}) = 0.100\ldots$, $h_1(4, \frac{3}{4}) = 0.153\ldots$ $h_1(3, \frac{2}{3}) = 0.250\ldots$ and $h_1(2, \frac{1}{2}) = 0.480\ldots$.

2.) Let $k = n-1$ and hence $nq \geq 2$ implying $q \geq \frac{2}{n}$. This can only occur, if $n = 5,\ p \leq \frac{3}{5}$. We have again due to the proof of Lemma 7.2 (this time there in (b) 3.) on page 40)

$$\begin{aligned}\log(g(k)) - \log(f(k)) &\geq \log(\tfrac{1}{\sqrt{2\pi}}) - \tfrac{3}{2}\log(npq) - \tfrac{nq}{2p} + \tfrac{3}{2p} - \tfrac{7}{6npq} - (n-2)\log(p) \\ &=: h_2(n,p)\end{aligned}$$

and $\frac{dh_2}{dp} \leq -\frac{1}{p^2q} \cdot (\frac{3}{2}(1-2p)p + \frac{n-3}{2}q + (n-2)pq) \leq -\frac{1}{p^2q} \cdot (\frac{3}{2} \cdot \frac{3}{25} + \frac{n-3}{n} + (n-2) \cdot \frac{6}{25}) \leq -\frac{1}{p^2q} \cdot (-\frac{9}{50} + \frac{2}{5} + \frac{18}{25}) < 0$. Thus we have $h_2(5,p) \geq h_2(5,\frac{3}{5}) = 0.201....$

3.) The case $k \leq n-2$ can not occur, since else would be $q \geq \frac{3}{n} \geq \frac{3}{5}$ and the case $k > n$ is clear, since then we have $f(k) = 0$.

b) Let again $p > \frac{1}{2}$. We now show $g(k+1) \leq f(k)$ if $np \leq k \leq np + \frac{8}{9} \cdot \sqrt{npq}$.

Here we have $\frac{8}{9} \cdot \sigma \leq \frac{8}{9} \cdot \sqrt{5/4} \leq 1$ and hence $np \leq k \leq np+1$. Further we have with $z := k - np$ analogous to (18) in the proof of Lemma 7.3 on page 43 due to Lemma 11.1 now $\log(g(k+1)) \leq \log(\frac{1}{\sqrt{2\pi}}) - \frac{z^2+z+1/3}{2\sigma^2} + \frac{z^2+z+24/90}{24\sigma^4}$.

1.) Let $k = n$ and hence $q \leq \frac{1}{n}$ and $z = nq$. Then with $\frac{5n}{36} \leq \sigma^2 \leq \frac{n-1}{n}$ we have

$$\begin{aligned}\log(\tfrac{f(n)}{g(n+1)}) &\geq \log(p^n) - \log(\tfrac{1}{\sqrt{2\pi}\,\sigma}) + \tfrac{n^2q^2+nq+1/3}{2npq} - \tfrac{n^2q^2+nq+24/90}{24n^2p^2q^2} \\ &= n\log(p) + \tfrac{\log(npq)}{2} - \log(\tfrac{1}{\sqrt{2\pi}}) + \tfrac{nq}{2p} + \tfrac{1}{2p} - \tfrac{1}{24p^2} + \tfrac{1}{6\sigma^2} - \tfrac{1}{24p\sigma^2} - \tfrac{1}{90n^2p^2q^2} \\ &\geq n\log(p) + \tfrac{\log(npq)}{2} - \log(\tfrac{1}{\sqrt{2\pi}}) \\ &\quad + \tfrac{nq}{2p} + \tfrac{1}{2p} - \tfrac{1}{24p^2} + \tfrac{n}{6(n-1)} - \tfrac{n}{24p(n-1)} - \tfrac{36^2}{90 \cdot 25n^2} \\ &\geq n\log(1-\tfrac{1}{n}) + \tfrac{\log(\frac{5n}{36})}{2} - \log(\tfrac{1}{\sqrt{2\pi}}) \\ &\quad + \tfrac{n}{2(n-1)} + \tfrac{3}{5} - \tfrac{3}{50} + \tfrac{n}{6(n-1)} - \tfrac{n}{20(n-1)} - \tfrac{36^2}{90 \cdot 25n^2} \\ &=: h_3(n)\end{aligned}$$

since $\frac{\mathrm{d}}{\mathrm{d}p}(n\log(p) + \frac{nq}{2p}) = \frac{n}{p} - \frac{n}{2p^2} > 0$, $\frac{\mathrm{d}}{\mathrm{d}p}(\frac{1}{2p} - \frac{1}{24p^2} - \frac{n}{24p(n-1)}) = -\frac{1}{2p^2} + \frac{1}{12p^3} + \frac{n}{24(n-1)p^2} \leq 0$ and $\frac{n-1}{n} \leq p \leq \frac{5}{6}$ in the third inequality. Finally we have $h_3(2) = 0.521...$, $h_3(3) = 0.665...$, $h_3(4) = 0.800...$ and $h_3(5) = 0.908....$

2.) Let $k = n-1$, hence $\frac{1}{n} \le q \le \frac{2}{n}$ and $z = nq - 1$ and $\frac{n-1}{n} \le \sigma^2 \le \frac{n}{4}$. Thus we have due to Lemma 11.1

$$\begin{aligned}
\log(\tfrac{f(n)}{g(n+1)}) &\ge \log(nqp^{n-1}) - \log(\tfrac{1}{\sqrt{2\pi}\,\sigma}) + \tfrac{n^2q^2-nq+1/3}{2npq} - \tfrac{n^2q^2-nq+24/90}{24n^2p^2q^2} \\
&= (n-2)\log(p) + \tfrac{3\log(npq)}{2} - \log(\tfrac{1}{\sqrt{2\pi}}) \\
&\quad + \tfrac{nq}{2p} - \tfrac{1}{2p} - \tfrac{1}{24p^2} + \tfrac{1}{6\sigma^2} + \tfrac{1}{24p\sigma^2} - \tfrac{1}{90n^2p^2q^2} \\
&\ge (n-2)\log(p) + \tfrac{3\log(npq)}{2} - \log(\tfrac{1}{\sqrt{2\pi}}) + \tfrac{nq}{2p} - \tfrac{1}{2p} - \tfrac{1}{24p^2} + \tfrac{2}{3n} + \tfrac{1}{6pn} - \tfrac{16}{90n^2} \\
&=: h_4(n,p).
\end{aligned}$$

Now we have $\frac{dh_4}{dp} = \frac{n}{p} - \frac{2}{p} + \frac{3(1-2p)}{2pq} - \frac{n}{2p^2} + \frac{1}{2p^2} + \frac{1}{12p^3} - \frac{1}{6np^2} = \frac{(1-2p)}{p^2q}\cdot(\frac{3}{2}p - \frac{n}{2}q) - \frac{2}{p}\cdot(1 - \frac{1}{4p} - \frac{1}{24p^2} + \frac{1}{12np}) \le \frac{(1-2p)}{p^2q}\cdot(\frac{3}{2}p - \frac{5}{2}q) - \frac{2}{p}\cdot(1 - \frac{1}{2} - \frac{1}{6} + \frac{1}{30}) \le \frac{1}{32p^2q} - \frac{2}{3p} = -\frac{1}{p}\cdot(\frac{2}{3} - \frac{1}{32pq}) < 0$ since $(1-2p)((\frac{3}{2}p - \frac{5}{2}q) = -8p^2 + 9p - \frac{5}{2} \le \frac{9^2}{4\cdot 8} - \frac{5}{2} = \frac{1}{32}$.

Finally we have $h_4(5, \frac{4}{5}) = 0.017...$, $h_4(4, \frac{3}{4}) = 0.049...$ and $h_4(3, \frac{2}{3}) = 0.097...$. The case $n = 2$ can not occur since $q \ge \frac{1}{n} = \frac{1}{2}$.

3.) Let $k = n-2$, hence $\frac{2}{n} \le q \le \frac{3}{n}$ and $z = nq - 1$. This is only possible if $n = 5$ and $p \in (\frac{1}{2}, \frac{3}{5}]$, since else it would be $q \ge \frac{1}{2}$. With $z^2 + z = (5q)^2 - 15q + 2$ we now have

$$\begin{aligned}
\log(\tfrac{f(3)}{g(4)}) &\ge \log(10p^3q^2) - \log(\tfrac{1}{\sqrt{10\pi pq}}) + \tfrac{5q}{2p} - \tfrac{3}{2p} - \tfrac{1}{24p^2} + \tfrac{7}{30pq} + \tfrac{1}{40p^2q} - \tfrac{1}{12\cdot 25p^2q^2} - \tfrac{1}{90\cdot 25p^2q^2} \\
&=: h_5(p).
\end{aligned}$$

From $h_5'(p) = \frac{3}{p} - \frac{2}{q} + \frac{1-2p}{2pq} - \frac{5}{2p^2} + \frac{3}{2p^2} + \frac{1}{12p^3} - \frac{7(1-2p)}{30p^2q^2} - \frac{2p-3p^2}{40(p^2q)^2} + \frac{2(1-2p)}{12\cdot 25(p^2q^2)^2} + \frac{1-2p}{90\cdot 25(p^2q^2)^2} \le -\frac{1}{p^2} + \frac{1}{12p^3} - \frac{7(1-2p)}{30p^2q^2} \le -\frac{1}{p^2} + \frac{1}{6p^2} - \frac{7/5}{30p^2\cdot 4/25} = -\frac{1}{p^2}\cdot(1 - \frac{1}{6} - \frac{7}{24}) < 0$ and $h_5(\frac{3}{5}) = 0.078... > 0$ this case follows.

4.) The case $k \le n-3$ can not occur, since else $q \ge \frac{3}{n} > \frac{1}{2}$, and neither can $k > n$ occur, since $np + \frac{8}{9}\sqrt{npq} < n + \frac{8}{9}\sqrt{5/4} < n+1$.

c) Let now $p < \frac{1}{2}$ and we show $f(k) \le g(k)$ for $np + 1 \le k \le np + npq$.

If $\sigma^2 \ge \frac{5}{6}$ then we have $f(k) \le g(k)$ due to Lemma 7.2 for $k \le np + \frac{3}{2}\sigma$ and since $\frac{3}{2}\sigma \ge \frac{3}{2}\sqrt{4/5}\,\sigma^2 \ge \sigma^2$ also for $k \le np + \sigma^2$. Thus we may assume $\sigma^2 \le \frac{5}{6} < 1$, hence $np + npq < np + 1$ and thus the requirement is not satisfied by any $k \in \mathbb{N}$.

d) Let again $p < \frac{1}{2}$ and we now show $g(k+1) \le f(k)$, $np \le k \le np + npq$.

Here we have $\frac{z}{n} \le pq \le \frac{1}{4}$ and $\frac{z}{nq} \le \frac{1}{2}$ and hence $\frac{pq}{(p+z/n)(q-z/n)} \le \frac{1/4}{3/16} = \frac{4}{3}$ and $\sum_{r=0}^{\infty} \frac{z^r}{(nq)^r} \le 2$ and thus due to (a) in the proof of Lemma 7.3 on page 45 with $z \le \sigma^2 \le \sqrt{5/4}\,\sigma$ and first if $\sigma^2 \ge \frac{5}{12}$

$$\begin{aligned}
h(z,p) \;\ge\;& \tfrac{z^2+z+1/3}{2\sigma^2} - \tfrac{z^2+z+24/90}{24\sigma^4} - \big(\tfrac{z^2}{2np} - \tfrac{z^3}{6(np)^2} + \tfrac{z^4}{12(np)^3}\big) - \big(\tfrac{z^2}{2nq} + \tfrac{z^3}{6(nq)^2} + \tfrac{z^4}{12(nq)^3} + \tfrac{2z^5}{20(nq)^4}\big) \\
& - \tfrac{1}{9\sigma^2} - \big(\tfrac{z(q-p)}{2\sigma^2} - \tfrac{z^2(p^2+q^2)}{4\sigma^4} + \tfrac{z^3(q^3-p^3)}{\sigma^6}\big) \\
=\;& \tfrac{z(1-(q-p))}{2\sigma^2} + \tfrac{1}{18\sigma^2} - \tfrac{1}{90\sigma^4} - \tfrac{z^2+z}{24\sigma^4} + \tfrac{z^3(q^2-p^2)}{6\sigma^4} - \tfrac{z^4(q^3+p^3)}{12\sigma^6} - \tfrac{z^5p^4}{10\sigma^8} + \tfrac{z^2(p^2+q^2)}{4\sigma^4} - \tfrac{z^3(q^3-p^3)}{6\sigma^6} \\
\ge\;& \tfrac{z(1-(q-p))}{2\sigma^2} + \tfrac{1}{18\sigma^2} - \tfrac{12}{90\cdot 5\sigma^2} - \tfrac{z^2+z}{24\sigma^4} - \tfrac{z^4}{12\sigma^6}\cdot\tfrac{7}{12} - \tfrac{z^5}{10\cdot 2^4\sigma^8} + \tfrac{z^2}{8\sigma^4} - \tfrac{z^3(q^3-p^3-(q-p)\cdot 5/12)}{6\sigma^6} \\
\ge\;& \tfrac{z}{6\sigma^2} + \tfrac{1}{36\sigma^2} - \tfrac{z^2+z}{24\sigma^4} - \tfrac{z^2}{12\sigma^4}\cdot\tfrac{7}{12}\cdot\tfrac{5}{4} - \tfrac{z^2}{10\cdot 2^4\sigma^4}\cdot\tfrac{5}{4} + \tfrac{z^2}{8\sigma^4} - \tfrac{z(q^3-p^3-(q-p)\cdot 5/12)}{6\sigma^2} \\
\ge\;& \tfrac{z}{6\sigma^2} + \tfrac{1}{36\sigma^2} - \tfrac{z}{24\sigma^4} + \tfrac{z^2}{\sigma^4}\cdot\big(\tfrac{1}{8} - \tfrac{1}{12}\cdot\tfrac{7}{12}\cdot\tfrac{5}{4} - \tfrac{1}{10\cdot 2^4}\cdot\tfrac{5}{4}\big) - \tfrac{z\cdot 8/27}{6\sigma^2} \\
\ge\;& \tfrac{z}{36\sigma^2}\cdot\tfrac{4}{5} + \tfrac{z}{6\sigma^2} - \tfrac{z}{24\sigma^4} - \tfrac{z\cdot 4}{81\sigma^2} \;\ge\; \tfrac{z}{45\sigma^2} + \tfrac{z}{6\sigma^2} - \tfrac{12z}{24\cdot 5\sigma^2} - \tfrac{z\cdot 4}{81\sigma^2} \\
=\;& \tfrac{z}{\sigma^2}\cdot\big(\tfrac{1}{6} + \tfrac{1}{45} - \tfrac{1}{10} - \tfrac{4}{81}\big) \;\ge\; 0.
\end{aligned}$$

If $\sigma^2 < \frac{5}{12}$, we have $n = 2$ and $p \le 0.3$. So that $[np, np+\sigma^2]$ contains an integer we must have $np \ge 1 - \sigma^2 \ge \frac{7}{12}$ and hence $p \ge \frac{7}{24}$ and thus we have $p \in [\frac{7}{24}, \frac{3}{10}]$ and $g(k) - f(k) = \Phi(\frac{1-2p}{\sigma}) - \Phi(\frac{-2p}{\sigma}) - 2pq \ge \Phi(\frac{1-0.6}{\sqrt{2\cdot 21/100}}) - \Phi(\frac{-0.6}{\sqrt{2\cdot 21/100}}) - 2\cdot\frac{21}{100} = 0.134... > 0$.

e) Let $k > np + npq$ and $p \le \frac{1}{2}$. Then Lemma 11.3 yields

$$F(k) - G(k) \;\le\; 1 - G(k) \;\le\; 1 - \Phi(\sigma) \;\le\; \frac{0.467}{\sqrt{2\pi}\,\sigma} \;\le\; c_0 \cdot \frac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}$$

and further due to the proof of Lemma 7.4 in the case $k \ge np + npq$ we have

$$G(k+1) - F(k) \;\le\; c_0 \cdot \frac{3+|p-q|}{6\sqrt{2\pi}\,\sigma}$$

because in that proof, there were no conditions on n.

With $G(k) - F(k) \le G(k+1) - F(k)$ and $F(k) - G(k+1) \le F(k) - G(k)$ we also have $\max\{|F(k) - G(k)|, |G(k+1) - F(k)|\} \le c_0$. □

Lemma 9.1. *Let $n \in \mathbb{N}$, $p \in [\frac{1}{6}, \frac{1}{2}]$ with $m = \lfloor np \rfloor = 0$ and $m \leq \frac{n}{2} - 1$. Then*

$$\tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - G(m+1) + F(m) \quad > \quad 0.$$

Proof. It follows from the conditions that $2 \leq n \leq 5$ and $p \in [\frac{1}{6}, \frac{1}{n})$. First, if we assume $(n - \frac{1}{3})p \geq \frac{1}{3}$, then we have

$$\begin{aligned}
\tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - G(m+1) + F(m) &= \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - \Phi(\tfrac{1-np}{\sigma}) + (1-p)^n \\
&= \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - (\Phi(\tfrac{1-np}{\sigma}) - \tfrac{1}{2}) - \tfrac{1}{2} + (1-p)^n \\
&\geq \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - \tfrac{1-np}{\sqrt{2\pi}\,\sigma} - \tfrac{1}{2} + (1-p)^n \\
&\geq \tfrac{4-2p}{6\sqrt{2\pi(n-1)/n}} - \tfrac{1-np}{\sqrt{2\pi(n-1)/n}} - \tfrac{1}{2} + (1-p)^n \quad =: \; h(p)
\end{aligned}$$

since Lemma 11.1 in the first inequality and $\sigma^2 \leq n \cdot \frac{1}{n} \cdot (1 - \frac{1}{n}) = \frac{n-1}{n}$ and $\frac{4-2p}{6} - (1-np) = (n - \frac{1}{3})p - \frac{1}{3} \geq 0$ in the second one. We now have $h''(p) = n(n-1)(1-p)^{n-2} > 0$ and

$$\begin{aligned}
h'(p) \quad &= \quad -\tfrac{1}{3\sqrt{2\pi(n-1)/n}} + \tfrac{n}{\sqrt{2\pi(n-1)/n}} - n(1-p)^{n-1} \; = \; 0 \\
\Leftrightarrow \; p \; &= \; 1 - \big(\tfrac{1-1/(3n)}{\sqrt{2\pi(n-1)/n}}\big)^{1/(n-1)} \quad =: \; p_0
\end{aligned}$$

and hence $h(p) \geq h(p_0) \in \{0.031..., 0.067..., 0.085..., 0.095...\}$, for $n = 2, 3, 4, 5$ specifically calculated.

Let us now assume $(n - \frac{1}{3})p < \frac{1}{3}$, hence $n = 2$ and $p \in [\frac{1}{6}, \frac{1}{5})$. Then with $\sigma^2 \geq n \cdot \frac{5}{36} = \frac{5}{18}$ we have analogous to the above

$$\begin{aligned}
\tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - G(m+1) + F(m) &\geq \tfrac{4-2p}{6\sqrt{2\pi \cdot 5/18}} - \tfrac{1-2p}{\sqrt{2\pi \cdot 5/18}} - \tfrac{1}{2} + (1-p)^2 \\
&= \tfrac{1}{2\sqrt{2\pi \cdot 5/18}} + (1-2p) \cdot (1 - \tfrac{5}{6\sqrt{2\pi \cdot 5/18}}) - \tfrac{1}{2} + p^2 \\
&\geq \tfrac{1}{2\sqrt{2\pi \cdot 5/18}} + (1 - \tfrac{2}{5}) \cdot (1 - \tfrac{5}{6\sqrt{2\pi \cdot 5/18}}) - \tfrac{1}{2} + \tfrac{1}{36} \\
&= 0.127... \; > \; 0.
\end{aligned}$$

□

Lemma 9.2. *Let $n \in \mathbb{N}$, $p \in [\frac{1}{6}, \frac{1}{2}]$ with $m = \lfloor np \rfloor \le 1$ and $m \le \frac{n}{2} - 1$. Then*

$$\tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - G(m) + F(m-1) \;>\; 0.$$

Proof. Let first $m = 1$, implying $4 \le n \le 11$ and $p \in [\frac{1}{n}, \frac{2}{n}) \cap [\frac{1}{6}, \frac{1}{2}]$. Analogous to Lemma 9.1 we have due to Lemma 11.1

$$\begin{aligned}
\tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - G(m) + F(m-1) \;&=\; \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} + (\Phi(\tfrac{np-1}{\sigma}) - \tfrac{1}{2}) - \tfrac{1}{2} + (1-p)^n \\
&\ge\; \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} + \tfrac{np-1}{\sqrt{2\pi}\,\sigma} - \tfrac{(np-1)^3}{6\sqrt{2\pi}\,\sigma^3} - \tfrac{1}{2} + (1-p)^n \\
&\ge\; \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} + \tfrac{np-1}{\sqrt{2\pi}\,\sigma} - \tfrac{np-1}{6\sqrt{2\pi}\,\sigma} - \tfrac{1}{2} + (1-p)^n \\
&\ge\; \tfrac{4-2p}{6\sqrt{2\pi\cdot 2(n-2)/n}} + \tfrac{5(np-1)}{6\sqrt{2\pi\cdot 2(n-2)/n}} - \tfrac{1}{2} + (1-p)^n \\
&=:\; h(p)
\end{aligned}$$

because of $\frac{(np-1)^2}{npq} \le \frac{1}{2(n-2)/n} \le 1$ since $np \le 2$ and $n \ge 4$ in the penultimate and $\sigma^2 \le n \cdot \frac{2}{n} \cdot (1 - \frac{2}{n}) = \frac{2(n-2)}{n}$ in the ultimate inequality. Further we have $h''(p) > 0$ and

$$\begin{aligned}
h'(p) \;&=\; -\tfrac{1}{3\sqrt{2\pi\cdot 2(n-2)/n}} + \tfrac{5n}{6\sqrt{2\pi\cdot 2(n-2)/n}} - n(1-p)^{n-1} \;=\; 0 \\
\Leftrightarrow\; p \;&=\; 1 - \big(\tfrac{5/6 - 1/(3n)}{\sqrt{2\pi\cdot 2(n-2)/n}}\big)^{1/(n-1)} \;=:\; p_0
\end{aligned}$$

and thus we obtain for each $n = 5, ..., 11$ specifically calculated
$h(p) \ge h(p_0) \in \{0.300..., 0.023..., 0.021..., 0.021..., 0.021..., 0.021..., 0.021..., 0.022...\}$.

Let now $m = 0$, hence $2 \le n \le 5$ and $p \in [\frac{1}{6}, \frac{1}{n}]$. Then we have

$$\begin{aligned}
\tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - G(m) + F(m-1) \;&=\; \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - \Phi(-\tfrac{np}{\sigma}) \;=\; \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} + (\Phi(\tfrac{np}{\sigma}) - \tfrac{1}{2}) - \tfrac{1}{2} \\
&\ge\; \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} + \tfrac{np}{\sqrt{2\pi}\,\sigma} - \tfrac{(np)^3}{6\sqrt{2\pi}\,\sigma^3} - \tfrac{1}{2} \\
&=\; \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot (\tfrac{4-2p}{6} + np - \tfrac{n^2p^2}{6q}) - \tfrac{1}{2} \\
&\ge\; \tfrac{1}{\sqrt{2\pi(n-1)/n}} \cdot (\tfrac{4-2p}{6} + np - \tfrac{n^2p^2}{6q}) - \tfrac{1}{2} \;=:\; i(p)
\end{aligned}$$

since $\sigma^2 \le \frac{n-1}{n}$. Now we have $i''(p) < 0$ since $(\frac{p^2}{q})'' = \frac{2}{q^2} + \frac{2p}{q^3} > 0$, hence i concave and at the boundary points we have

$$\begin{aligned}
i(\tfrac{1}{6}) \;&=\; \tfrac{1}{\sqrt{2\pi(n-1)/n}} \cdot (\tfrac{4-1/3}{6} + \tfrac{n}{6} - \tfrac{n^2}{36\cdot 5}) - \tfrac{1}{2} \;\in\; \{0.020..., 0.018..., 0.047..., 0.082...\} \\
i(\tfrac{1}{n}) \;&=\; \tfrac{1}{\sqrt{2\pi(n-1)/n}} \cdot (\tfrac{4-2/n}{6} + 1 - \tfrac{n}{6(n-1)}) - \tfrac{1}{2} \;\ge\; \tfrac{1}{\sqrt{2\pi\cdot 5/6}} \cdot (\tfrac{1}{2} + 1 - \tfrac{1}{3}) - \tfrac{1}{2} \;=\; 0.0099.
\end{aligned}$$

$\square$

Lemma 8.6. *Let $n \in \mathbb{N}$, $p \in [\frac{1}{6}, \frac{1}{2}]$ with $m \le \frac{n}{2} - 1$ and $m \le 1$. Then we have*

$$(i) \qquad \mathcal{D}_1 \ := \ \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - F(m) + G(m) \ > \ 0\ , \qquad \textit{if}\ \ p \in [\tfrac{1}{3}, \tfrac{1}{2})$$

$$(ii) \qquad \mathcal{D}_2 \ := \ \tfrac{\sqrt{10}+3}{6\sqrt{2\pi}} \cdot \tfrac{p^2+q^2}{\sigma} - F(m) + G(m) \ > \ 0\ , \qquad \textit{if}\ \ p \in [\tfrac{1}{6}, \tfrac{1}{3}).$$

Proof. (i): We consider all with the conditions possible cases of m and n. The only cases are:

1.) $m = 0,\ n = 2;$
2.) $m = 1,\ n = 4, 5.$

On 1.): Let $m = 0$ and $n = 2$. Then $p \in [\frac{1}{3}, \frac{1}{2})$ and due to Lemma 11.1 and $1 \le \frac{(2p)^2}{\sigma^2} \le 2$ as $\frac{2}{3} \le \sigma \le \sqrt{1/2}$ we have

$$\begin{aligned}
\mathcal{D}_1 \ &= \ \tfrac{4-2p}{6\sigma} - F_{2,p}(0) + G_{2,p}(0) \ = \ \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - (1-p)^2 + \Phi(\tfrac{-2p}{\sigma}) \\
&\ge \ \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - (1-p)^2 + \tfrac{1}{2} - \tfrac{2p}{\sqrt{2\pi}\,\sigma} + \tfrac{(2p)^3}{6\sigma^3\sqrt{2\pi}} - \tfrac{(2p)^5}{40\sigma^5\sqrt{2\pi}} \\
&\ge \ \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - (1-p)^2 + \tfrac{1}{2} - \tfrac{2p}{\sqrt{2\pi}\,\sigma} + \tfrac{(2p)^3}{6\sigma^3\sqrt{2\pi}} \cdot (1 - \tfrac{6\cdot 2}{40}) \\
&\ge \ \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - (1-p)^2 + \tfrac{1}{2} - \tfrac{2p}{\sqrt{2\pi}\,\sigma} \cdot (1 - \tfrac{1}{6} \cdot (1 - \tfrac{3}{10})) \\
&\ge \ \tfrac{4-2p}{6\sqrt{2\pi}\sqrt{1/2}} - (1-p)^2 + \tfrac{1}{2} - \tfrac{2p}{\sqrt{2\pi}\cdot 2/3} \cdot (1 - \tfrac{7}{60}) \ =: \ h_1(p).
\end{aligned}$$

Now obviously we have $h_1''(p) = -2 < 0$ and hence h_1 minimal at the boundary. There we have

$$\begin{aligned}
h_1(\tfrac{1}{3}) \ &= \ \tfrac{4-2/3}{6\sqrt{2\pi}\sqrt{1/2}} - (\tfrac{2}{3})^2 + \tfrac{1}{2} - \tfrac{2/3}{\sqrt{2\pi}\cdot 2/3} \cdot (1 - \tfrac{7}{60}) \ = \ 0.036... > 0 \\
h_1(\tfrac{1}{2}) \ &= \ \tfrac{4-1}{6\sqrt{2\pi}\sqrt{1/2}} - (\tfrac{1}{2})^2 + \tfrac{1}{2} - \tfrac{1}{\sqrt{2\pi}\cdot 2/3} \cdot (1 - \tfrac{7}{60}) \ = \ 0.033... > 0
\end{aligned}$$

and hence $\mathcal{D}_1 \ge h_1(p) \ge 0.0337 > 0$.

On 2.): Let $n = 4$. Then we have again $p \in [\frac{1}{3}, \frac{1}{2})$ and Lemma 11.1 yields again with $\sigma \le 1$

$$\begin{aligned}
\mathcal{D}_1 \ &= \ \tfrac{4-2p}{6\sigma} - F_{4,p}(1) + G_{4,p}(1) \ = \ \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - ((1-p)^4 + 4p(1-p)^3) + \Phi(\tfrac{1-4p}{\sigma}) \\
&\ge \ \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - ((1-p)^4 + 4p(1-p)^3) + \tfrac{1}{2} - \tfrac{4p-1}{\sqrt{2\pi}\,\sigma} + \tfrac{(4p-1)^3}{6\sigma^3\sqrt{2\pi}} - \tfrac{(4p-1)^5}{40\sigma^5\sqrt{2\pi}} \\
&\ge \ \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - ((1-p)^4 + 4p(1-p)^3) + \tfrac{1}{2} - \tfrac{4p-1}{\sqrt{2\pi}\,\sigma} + \tfrac{17(4p-1)^3}{120\sigma\sqrt{2\pi}}.
\end{aligned}$$

Here we used $\frac{(4p-1)^3}{6\sigma^3\sqrt{2\pi}} - \frac{(4p-1)^5}{40\sigma^5\sqrt{2\pi}} = \frac{(4p-1)^3}{\sigma\sqrt{2\pi}} \cdot (\frac{1}{6\sigma^2} - \frac{(4p-1)^2}{40\sigma^4}) \ge \frac{(4p-1)^3}{\sigma\sqrt{2\pi}} \cdot (\frac{1}{6\sigma^2} - \frac{1}{40\sigma^4}) \ge \frac{(4p-1)^3}{\sigma\sqrt{2\pi}} \cdot (\frac{1}{6} - \frac{1}{40}) = \frac{17(4p-1)^3}{120\sqrt{2\pi}\,\sigma}$ since $(\frac{1}{6\sigma^2} - \frac{1}{40\sigma^4})' = -\frac{1}{3\sigma^3} + \frac{1}{10\sigma^5} \le -\frac{1}{3\sigma^3} + \frac{9}{80\sigma^3} < 0$ and $\sigma^2 \ge 4 \cdot \frac{2}{9}$ in the last inequality.

Let $p \in [\frac{1}{3}, \frac{3}{8}]$. Then with $\frac{4-2p}{6} \geq \frac{1}{2} \geq (4p-1)$ and $\sigma \leq \sqrt{15/16}$ we have

$$\begin{aligned} \mathcal{D}_1 &\geq \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot (\tfrac{4-2p}{6} - (4p-1) + \tfrac{(1/3)^3}{6} \cdot \tfrac{17}{20}) - ((1-p)^4 + 4p(1-p)^3) + \tfrac{1}{2} \\ &\geq \tfrac{1}{\sqrt{2\pi \cdot 15/16}} \cdot (\tfrac{4-2p}{6} - (4p-1) + \tfrac{17}{27 \cdot 120}) - ((1-p)^4 + 4p(1-p)^3) + \tfrac{1}{2} \\ &=: h_2(p). \end{aligned}$$

Now we have again $h_2''(p) = -12(1-p)(3p-1) \leq 0$ and hence h_2 concave and from $h_2(\frac{1}{3}) = 0.001...$ and $h_2(\frac{3}{8}) = 0.0005...$ this case follows.

Let next $p \in [\frac{3}{8}, \frac{1}{2}]$. Then with $\sqrt{15/16} \leq \sigma \leq 1$ and $\frac{1}{3} - \frac{p}{3} - (4p-1) + \frac{(4p-1)^3}{6} \cdot \frac{17}{20} < \frac{1}{3} - \frac{1}{9} - \frac{1}{2} + \frac{1}{6} < 0$ we have

$$\begin{aligned} \mathcal{D}_1 &\geq \tfrac{1}{3\sqrt{2\pi}\,\sigma} + \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot (\tfrac{1}{3} - \tfrac{2p}{6} - (4p-1) + \tfrac{17(4p-1)^3}{120}) - ((1-p)^4 + 4p(1-p)^3) + \tfrac{1}{2} \\ &\geq \tfrac{1}{3\sqrt{2\pi}} + \tfrac{1}{\sqrt{2\pi \cdot 15/16}} \cdot (\tfrac{1}{3} - \tfrac{p}{3} - (4p-1) + \tfrac{(4p-1)^3}{6} \cdot \tfrac{17}{20}) - ((1-p)^4 + 4p(1-p)^3) + \tfrac{1}{2} \\ &=: h_3(p). \end{aligned}$$

Now we have

$$\begin{aligned} h_3''(p) &= \tfrac{1}{\sqrt{2\pi \cdot 15/16}} \cdot 12 \cdot 8 \cdot (4p-1) \cdot \tfrac{17}{120} - 12(1-p)(3p-1) \\ &= (3p-1) \cdot (\tfrac{1}{\sqrt{2\pi \cdot 15/16}} \cdot \tfrac{68}{5} - 12(1-p) + p\tfrac{1}{\sqrt{2\pi \cdot 15/16}} \cdot \tfrac{68}{5} \\ &\geq -\tfrac{1}{2} \cdot (12 \cdot \tfrac{5}{8} - \tfrac{1}{\sqrt{2\pi \cdot 15/16}} \cdot \tfrac{68}{5}) + \tfrac{1}{\sqrt{2\pi \cdot 15/16}} \cdot \tfrac{68}{5} \cdot \tfrac{3}{8} = 1.153... > 0 \end{aligned}$$

and hence $h_3'(p) \geq h_3'(\frac{3}{8}) = \frac{1}{\sqrt{2\pi \cdot 15/16}} \cdot (-\frac{1}{3} - 4 + \frac{12 \cdot (4 \cdot 3/8 - 1)^2}{6} \cdot \frac{17}{20}) - 12 \cdot \frac{3}{8}(1 - \frac{3}{8})^2) = 0.147... > 0$ and thus we have

$$\mathcal{D}_1 \;\geq\; h_3(p) \;\geq\; h_3(\tfrac{3}{8}) = 0.001... \;>\; 0.$$

Second, let now $n = 5$ and hence $p \in [\frac{1}{3}, \frac{2}{5})$. Analogous to above we have with $\sigma^2 \in [\frac{10}{9}, \frac{6}{5})$ and $5p - 1 - \frac{4-2p}{6} \geq \frac{1}{9} \geq \frac{(5p-1) \cdot 4/9}{6 \cdot 6/5} \cdot \frac{17}{20}$

$$\begin{aligned} \mathcal{D}_1 &= \tfrac{4-2p}{\sqrt{2\pi}\,\sigma} - F_{5,p}(1) + G_{5,p}(1) \\ &\geq \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - ((1-p)^5 + 5p(1-p)^4) + \tfrac{1}{2} - \tfrac{5p-1}{\sqrt{2\pi}\,\sigma} + \tfrac{(5p-1)^3}{6\sigma^3\sqrt{2\pi}} - \tfrac{(5p-1)^5}{40\sigma^5\sqrt{2\pi}} \\ &\geq \tfrac{4-2p}{6\sqrt{2\pi}\,\sigma} - ((1-p)^5 + 5p(1-p)^4) + \tfrac{1}{2} - \tfrac{5p-1}{\sqrt{2\pi}\,\sigma} + \tfrac{(5p-1) \cdot 4/9}{6\sigma\sqrt{2\pi} \cdot 6/5} \cdot (1 - \tfrac{6}{40}) \\ &\geq -\tfrac{1}{\sqrt{2\pi \cdot 10/9}} \cdot ((5p-1) - \tfrac{(5p-1) \cdot 4/9}{6 \cdot 6/5} \cdot \tfrac{17}{20} - \tfrac{4-2p}{6}) - ((1-p)^5 + 5p(1-p)^4) + \tfrac{1}{2} \\ &=: h_4(p). \end{aligned}$$

Now we have again $h_4''(p) = -20(1-p)^2(4p-1) < 0$ and with $h_4(\frac{1}{3}) = 0.010...$ as well as $h_4(\frac{2}{5}) = 0.006...$ this case also follows.

(ii): Under the given conditions only the following cases are possible:

3.) $m = 0,\ n = 2, 3, 4, 5;$
4.) $m = 1,\ n = 4, 5, 6, 7, 8, 9, 10, 11.$

On 3.): Let $m = 0$ and $n = 2$. Then $p \in [\frac{1}{6}, \frac{1}{3})$ and due to Lemma 11.1 we have with $\frac{2}{5} \leq \frac{(2p)^2}{\sigma^2} = \frac{2p}{q} \leq 1$ analogous to 1.) in part (i)

$$\begin{aligned}\mathcal{D}_2 &\geq \tfrac{\sqrt{10}+3}{6} \cdot \tfrac{p^2+q^2}{\sqrt{2\pi}\,\sigma} - (1-p)^2 + \tfrac{1}{2} - \tfrac{2p}{\sqrt{2\pi}\,\sigma} + \tfrac{(2p)^3}{6\sigma^3\sqrt{2\pi}} - \tfrac{(2p)^5}{40\sigma^5\sqrt{2\pi}} \\ &\geq \tfrac{\sqrt{10}+3}{6} \cdot \tfrac{p^2+q^2}{\sqrt{2\pi}\,\sigma} - (1-p)^2 + \tfrac{1}{2} - \tfrac{2p}{\sqrt{2\pi}\,\sigma} + \tfrac{(2p)^3}{6\sigma^3\sqrt{2\pi}} \cdot (1 - \tfrac{6}{40}) \\ &\geq \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot (\tfrac{\sqrt{10}+3}{6} \cdot (p^2+q^2) - 2p + \tfrac{17}{120}p \cdot \tfrac{2}{5}) - (1-p)^2 + \tfrac{1}{2}.\end{aligned}$$

1.Case: If $\frac{\sqrt{10}+3}{6} \cdot (p^2+q^2) - 2p + \frac{17}{120}p \cdot \frac{2}{5} > 0$, then with $\sigma \leq \frac{2}{3}$ we have

$$\mathcal{D}_2 \;\geq\; \tfrac{1}{\sqrt{2\pi}\cdot 2/3} \cdot (\tfrac{\sqrt{10}+3}{6} \cdot (p^2+q^2) - 2p + \tfrac{17}{120}p \cdot \tfrac{2}{5}) - (1-p)^2 + \tfrac{1}{2} \;=:\; h_5(p).$$

Now we have $h_5''(p) = 2 \cdot \frac{\sqrt{10}+3}{6} \cdot \frac{1}{\sqrt{2\pi}\cdot 2/3} - 2 = 0.458.... > 0$ and hence $h_5'(p) \leq h_5'(\frac{1}{3}) = \frac{1}{\sqrt{2\pi}\,2/3} \cdot (\frac{\sqrt{10}+3}{6} \cdot (-2+4/3) - 2 + \frac{17}{120} \cdot \frac{2}{5}) + 2(1-1/3) = -0.239... < 0$ and thus we have $h_5(p) \geq h_5(\frac{1}{3}) = 0.009... > 0$.

2.Case: If $\frac{\sqrt{10}+3}{6} \cdot (p^2+q^2) - 2p + \frac{17}{120}p \cdot \frac{2}{5} < 0$, which implies $p \geq \frac{1}{4}$, since else already $p^2 + q^2 - 2p > \frac{5}{8} - \frac{1}{2} > 0$, then with $\sigma^2 \geq \frac{3}{8}$ we have

$$\mathcal{D}_2 \;\geq\; \tfrac{1}{\sqrt{2\pi}\cdot 3/8} \cdot (\tfrac{\sqrt{10}+3}{6} \cdot (p^2+q^2) - 2p + \tfrac{17}{120}p \cdot \tfrac{2}{5}) - (1-p)^2 + \tfrac{1}{2} \;=:\; h_6(p)$$

and analogous to the first case $h_6''(p) = 2\cdot \frac{\sqrt{10}+3}{6} \cdot \frac{1}{\sqrt{2\pi}\cdot 3/8} - 2 = 0.674.... > 0$, consequently $h_6'(p) \leq h_6'(\frac{1}{3}) = \frac{1}{\sqrt{2\pi}\cdot 3/8} \cdot (\frac{\sqrt{10}+3}{6} \cdot (-2+4/3) - 2 + \frac{17}{120} \cdot \frac{2}{5}) + 2(1-1/3) = -0.378... < 0$ and hence $h_6(p) \geq h_6(\frac{1}{3}) = 0.005... > 0$.

Let now $m = 0$ and $n \in \{3, 4, 5\}$ and hence $p \in [\frac{1}{6}, \frac{1}{n})$. Then we have again like in (i) with $\frac{n}{5} \leq \frac{(np)^2}{\sigma^2} \leq \frac{3}{2}$

$$\begin{aligned}\mathcal{D}_2 &\geq \tfrac{\sqrt{10}+3}{6} \cdot \tfrac{p^2+q^2}{\sqrt{2\pi}\,\sigma} - (1-p)^n + \tfrac{1}{2} - \tfrac{2p}{\sqrt{2\pi}\,\sigma} + \tfrac{(np)^3}{6\sigma^3\sqrt{2\pi}} \cdot (1 - \tfrac{6\cdot 9}{40\cdot 4}) \\ &\geq \tfrac{1}{\sqrt{2\pi}\,\sigma} \cdot (\tfrac{\sqrt{10}+3}{6} \cdot (p^2+q^2) - 2p + n^2 p \cdot \tfrac{1}{30} \cdot \tfrac{53}{80}) - (1-p)^n + \tfrac{1}{2}.\end{aligned}$$

1.Case: Let $y := \frac{\sqrt{10}+3}{6} \cdot (p^2+q^2) - np + n^2 p \cdot \frac{1}{30} \cdot \frac{53}{80} > 0$. Since $\sigma^2 \leq (1-\frac{1}{n})$ we have

$$\mathcal{D}_2 \;\geq\; \frac{1}{\sqrt{2\pi \cdot (1-1/n)}} \cdot \left(\frac{\sqrt{10}+3}{6} \cdot (p^2+q^2) - np + n^2 p \cdot \frac{1}{30} \cdot \frac{53}{80}\right) - (1-p)^n + \frac{1}{2} \;=:\; h_7(n,p).$$

Because of $\frac{\mathrm{d}^2 h_7}{\mathrm{d}p^2} = 4 \cdot \frac{\sqrt{10}+3}{6\sqrt{2\pi\cdot(1-1/n)}} - n(n-1)(1-p)^{n-2} \leq 4 \cdot \frac{\sqrt{10}+3}{6\sqrt{2\pi\cdot 2/3}} - 6(1-\frac{1}{3}) = -1.992... < 0$ and $h_7(3,\frac{1}{6}) = 0.0566...$, $h_7(4,\frac{1}{6}) = 0.079...$, $h_7(5,\frac{1}{6}) = 0.098...$ and $h_7(3,\frac{1}{3}) = 0.026...$, $h_7(4,\frac{1}{4}) = 0.059...$, $h_7(5,\frac{1}{5}) = 0.084...$ (ii) holds in this case.

2.Case: Let $y \leq 0$ and hence $p \geq \frac{(\sqrt{10}+3)/6\cdot(p^2+q^2)}{n-n^2\cdot 53/30/80} \geq \frac{(\sqrt{10}+3)/6\cdot(1/n^2+(n-1)^2/n^2)}{n-n^2\cdot 53/30/80} \geq \frac{1}{5}$. Then we have $\sigma \geq \sqrt{n\cdot 4/25} =: \varsigma_n$ and thus

$$\mathcal{D}_2 \;\geq\; \frac{1}{\sqrt{2\pi}\,\varsigma_n} \cdot \left(\frac{\sqrt{10}+3}{6} \cdot (p^2+q^2) - np + n^2 p \cdot \frac{1}{30} \cdot \frac{53}{80}\right) - (1-p)^n + \frac{1}{2} \;=:\; h_8(n,p).$$

Here we have again $\frac{\mathrm{d}^2 h_8}{\mathrm{d}p^2} = 4\cdot\frac{\sqrt{10}+3}{6\sqrt{2\pi}\,\varsigma_n} - n(n-1)(1-p)^{n-2} \leq 4\cdot\frac{\sqrt{10}+3}{6\sqrt{2\pi}\,\varsigma_3} - n(n-1)(\frac{n-1}{n})^{n-2} \leq 4 \cdot \frac{\sqrt{10}+3}{6\sqrt{2\pi\cdot 12/25}} - 6(1-\frac{1}{3}) < 0$ and (ii) follows from $h_8(3,\frac{1}{6}) = 0.058...$, $h_8(4,\frac{1}{6}) = 0.084...$, $h_8(5,\frac{1}{6}) = 0.098...$ and $h_8(3,\frac{1}{3}) = 0.022...$, $h_8(4,\frac{1}{4}) = 0.049...$, $h_8(5,\frac{1}{5}) = 0.077...$.

<u>On 4.):</u> Let $m=1$ and $n \in \{4,5,6,7,8,9,10,11\}$. We divide this into the subcases

(a) $n = 5,6,7,8$, $p \in [\frac{1}{n}, \frac{3}{2n})$, $\quad n = 4$, $p \in [\frac{1}{n}, \frac{1}{3})$;
(b) $n = 6,7,8,9,10,11$, $p \in [\frac{3}{2n}, \frac{2}{n}]$, $\quad n = 5$, $p \in [\frac{3}{2n}, \frac{1}{3})$.

On (b): Due to Lemma 11.1 and $\frac{(np-1)^2}{npq} \leq \frac{1}{2(1-2/n)} \leq \frac{3}{4}$ we have

$$\begin{aligned}
\mathcal{D}_2 &\geq \frac{\sqrt{10}+3}{6} \cdot \frac{p^2+q^2}{\sqrt{2\pi}\,\sigma} - F_{n,p}(1) + \frac{1}{2} - \frac{np-1}{\sqrt{2\pi}\,\sigma} + \frac{(np-1)^3}{6\sigma^3\sqrt{2\pi}} \cdot \left(1 - \frac{6}{40}\cdot\frac{3}{4}\right) \\
&= \frac{(\sqrt{10}+3)/6\cdot(p^2+q^2)-\frac{1}{2}}{\sqrt{2\pi}\,\sigma} - F_{n,p}(1) + \frac{1}{2} - \frac{np-1-1/2}{\sqrt{2\pi}\,\sigma} + \frac{(np-1)^3}{6\sigma^3\sqrt{2\pi}} \cdot \frac{71}{80} \\
&\geq \frac{(\sqrt{10}+3)/6\cdot(p^2+q^2)-\frac{1}{2}}{\sqrt{2\pi}\sqrt{2(1-2/n)}} - F_{n,p}(1) + \frac{1}{2} - \frac{np-1-1/2}{\sqrt{2\pi}\sqrt{3/2(1-3/2/n)}} + \frac{(np-1)^2}{12\sqrt{2(1-2/n)}^3\sqrt{2\pi}} \cdot \frac{71}{80} \\
&=: h_9(p)
\end{aligned}$$

since $3/2(1-3/2/n) \leq \sigma^2 \leq 2(1-2/n)$ and $np-1-1/2 \geq 0$. Now we have

$$\begin{aligned}
h_9'''(p) &= -F_{n,p}'''(1) = n(n-1)(n-2)\cdot(-2f_{n-3,p}(0) + f_{n-3,p}(1)) \\
&= n(n-1)(n-2)(1-p)^{n-4}\cdot(-2+(n-1)p) \;<\; 0
\end{aligned}$$

and hence $h_9'(p)$ concave.

Further we have

$$\begin{aligned} h_9'(\tfrac{2}{n}) &= \frac{(\sqrt{10}+3)/6\cdot(-2+8/n)}{\sqrt{2\pi}\sqrt{2(1-2/n)}} + nf_{n-1,\frac{2}{n}}(1) - \frac{n}{\sqrt{2\pi}\sqrt{3/2(1-3/2/n)}} + \frac{2n}{12\sqrt{2(1-2/n)}^3\sqrt{2\pi}}\cdot\frac{71}{80} \\ &\in \{-0.143..., -0.288..., -0.393..., -0.467..., -0.547..., -0.610..., -0.667...\} \end{aligned}$$

and also

$$h_9'(\tfrac{3}{2n}) \in \{-0.075..., -0.123..., -0.148..., -0.159..., -0.163..., -0.162..., -0.157...\}.$$

Thus we have $h_9'(p) < 0$ and hence $h_9(p) \geq h_9(\frac{2}{n})$ if $n > 5$ and further

$$\begin{aligned} h_9(\tfrac{2}{n}) &= \frac{(\sqrt{10}+3)/6\cdot((2/n)^2+(1-2/n)^2)-1/2}{\sqrt{2\pi}\sqrt{2(1-2/n)}} - F_{n,\frac{2}{n}}(1) + \frac{1}{2} \\ &\quad - \frac{1/2}{\sqrt{2\pi}\sqrt{3/2(1-3/2/n)}} + \frac{1}{12\sqrt{2(1-2/n)}^3\sqrt{2\pi}}\cdot\frac{71}{80} \\ &\quad \in \{0.004\ldots, 0.009\ldots, 0.014\ldots, 0.019\ldots, 0.024\ldots, 0.029\ldots.\}. \end{aligned}$$

If $n = 5$ we have $h_9(\frac{1}{3}) = 0.005... > 0$ and hence (b).

On (a): Due to Lemma 11.1 and $\sigma \leq \sqrt{\frac{3}{2}\cdot(1-\frac{3}{2n})}$ as $p^2 - q^2 \geq \frac{1}{2} \geq np - 1$ we have again

$$\begin{aligned} \mathcal{D}_2 &\geq \frac{\sqrt{10}+3}{6}\cdot\frac{p^2+q^2}{\sqrt{2\pi}\,\sigma} - F_{n,p}(1) + \frac{1}{2} - \frac{np-1}{\sqrt{2\pi}\,\sigma} \\ &\geq \frac{\sqrt{10}+3}{6}\cdot\frac{p^2+q^2}{\sqrt{2\pi}\sqrt{3/2\cdot(1-3/(2n))}} - F_{n,p}(1) + \frac{1}{2} - \frac{np-1}{\sqrt{2\pi}\sqrt{3/2\cdot(1-3/(2n))}} =: h_{10}(p). \end{aligned}$$

Analogous to the above we have $h_{10}'''(p) < 0$ and further $h_{10}'(\frac{1}{6}) = \frac{\sqrt{10}+3}{6}\cdot\frac{-2+4/6}{\sqrt{2\pi}\sqrt{3/2\cdot(1-3/(2n))}} - nf_{n-1,\frac{1}{6}}(1) - \frac{n}{\sqrt{2\pi}\sqrt{3/2\cdot(1-3/(2n))}} \in \{-0.823..., -0.550..., -0.360..., -0.262..., -0.260...\}$ and $h_{10}'(\frac{3}{2n}) \in \{-0.101\ldots, -0.208\ldots, -0.270\ldots, -0.308\ldots, -0.334...\}$, thus we have $h_{10}'(p) < 0$ and if $n = 4$ (then we have $\frac{3}{2n} > \frac{1}{3}$) we have $h_{10}(\frac{1}{3}) = 0.005...$ and if $n = 5, 6, 7, 8$ we have $h_{10}(\frac{3}{2n}) \in \{0.0009\ldots, 0.019\ldots, 0.028\ldots, 0.036...\}$ and hence (ii) in this case. $\square$

10. Proof of Theorem 2

Let us restate Theorem 2 for convenience:

Theorem 2. *Let $n \in \mathbb{N}$, $n \geq 6$ and $p \in [\frac{1}{6}, \frac{5}{6}]$ and X binomially distributed with parameters n and p and hence $\mu := \mathbb{E}X = np$ and $\sigma^2 := \mathbb{E}(X-\mu)^2 = np(1-p)$. Let further denote $\mathrm{Bi}_{n,p}$ the distribution of X and $\mathrm{N}_{\mu,\sigma^2}$ the normal distribution with mean μ and variance σ^2. Then we have*

$$\sup_{I \subset \mathbb{R} \text{ interval}} |\mathrm{Bi}_{n,p}(I) - \mathrm{N}_{\mu,\sigma^2}(I)| \quad < \quad \tfrac{2}{\sqrt{2\pi}} \cdot \tfrac{p^2+q^2}{\sigma}.$$

Proof. First of all we have

$$\sup_{I \subset \mathbb{R} \text{ interval}} |\mathrm{Bi}_{n,p}(I) - \mathrm{N}_{\mu,\sigma^2}(I)| \quad = \quad \sup_{-\infty < a < b < \infty} |\mathrm{Bi}_{n,p}((a,b]) - \mathrm{N}_{\mu,\sigma^2}((a,b])|$$

since $\mathrm{N}_{\mu,\sigma^2}$ is continuous and since we have due to the continuity from above or from below

$$\begin{aligned} \mathrm{Bi}_{n,p}([a,b]) &= \lim_{k\to\infty} \mathrm{Bi}_{n,p}((a-\tfrac{1}{k}, b]) \\ \mathrm{Bi}_{n,p}((a,b)) &= \lim_{k\to\infty} \mathrm{Bi}_{n,p}((a, b-\tfrac{1}{k}]) \\ \mathrm{Bi}_{n,p}([a,b)) &= \lim_{k\to\infty} \mathrm{Bi}_{n,p}((a-\tfrac{1}{k}, b-\tfrac{1}{k}]). \end{aligned}$$

Thus we consider for $a, b \in \mathbb{R}$ with $a < b$

$$\begin{aligned} |\mathrm{Bi}_{n,p}((a,b]) - \mathrm{N}_{\mu,\sigma^2}((a,b])| &= |F(b) - F(a) - (G(b) - G(a))| \\ &= |F(b) - G(b) + G(a) - F(a)| \\ &\leq \sup_{s,t\in\mathbb{R}} \big(F(s) - G(s) + G(t) - F(t)\big) \\ &= \sup_{u,v\in\mathbb{Z}} \big(F(u) - G(u) + G(v) - F(v-1)\big) \end{aligned}$$

since $F(s) - G(s) \leq F(\lfloor s \rfloor) - G(\lfloor s \rfloor)$ and $G(t) - F(t) \leq G(\lceil t \rceil) - F(\lceil t \rceil - 1)$ for $s, t \in \mathbb{R}$ in the last equality and $-\big(F(b) - G(b) + G(a) - F(a)\big) = F(a) - G(a) + G(b) - F(b)$ in the inequality.

Thus we show for all $u, v \in \mathbb{Z}$ in the following

$$F(u) - G(u) + G(v) - F(v-1) \quad < \quad \tfrac{2}{\sqrt{2\pi}} \cdot \tfrac{p^2+q^2}{\sigma}. \tag{29}$$

Due to Lemma 7.2 - 7.4 and the conclusions in the proof of Proposition 6.1 we have listed again the results for $n \geq 6,\ k \in \mathbb{N}$ and first if $p \leq \frac{1}{2}$

$$\begin{array}{rcll} f(k) & \leq & g(k)\ , & np+1 \leq k \leq np + \frac{3}{2}\sigma \\ g(k+1) & \leq & f(k)\ , & np \leq k \leq np + \frac{3}{2}\sigma \\ |F(k)-G(k)| & \leq & (2-\frac{\sqrt{10}+3}{6\sqrt{2\pi}})\cdot \frac{p^2+q^2}{\sigma}\ , & k > np + \frac{3}{2}\sigma \\ |G(k+1)-F(k)| & \leq & (2-\frac{\sqrt{10}+3}{6\sqrt{2\pi}})\cdot \frac{p^2+q^2}{\sigma}\ , & k > np + \frac{3}{2}\sigma \end{array}$$

and if $p \geq \frac{1}{2}$

$$\begin{array}{rcll} f(k) & \leq & g(k)\ , & np+1 \leq k \\ g(k+1) & \leq & f(k)\ , & np \leq k \leq np + \sqrt{\frac{3}{2}}\sigma,\ \sigma \geq \frac{3}{2} \\ g(k+1) & \leq & f(k)\ , & np \leq k \leq np + \frac{8}{9}\sigma,\ \sigma < \frac{3}{2} \\ |G(k+1)-F(k)| & \leq & (2-\frac{\sqrt{10}+3}{6\sqrt{2\pi}})\cdot \frac{p^2+q^2}{\sigma}\ , & np + \sqrt{\frac{3}{2}}\sigma,\ \sigma \geq \frac{3}{2} \\ |G(k+1)-F(k)| & \leq & (2-\frac{\sqrt{10}+3}{6\sqrt{2\pi}})\cdot \frac{p^2+q^2}{\sigma}\ , & np + \frac{8}{9}\sigma,\ \sigma < \frac{3}{2}. \end{array}$$

If $u, v \in \mathbb{N}$ are such that above the case $F(u) - G(u) \leq (2-\frac{\sqrt{10}+3}{6\sqrt{2\pi}})\cdot \frac{p^2+q^2}{\sigma}$ or $G(v) - F(v-1) \leq (2-\frac{\sqrt{10}+3}{6\sqrt{2\pi}})\cdot \frac{p^2+q^2}{\sigma}$ occurs it follows from $G(u) - F(u-1) < \frac{\sqrt{10}+3}{6\sqrt{2\pi}}\cdot \frac{p^2+q^2}{\sigma}$ or $F(v) - G(v) < \frac{\sqrt{10}+3}{6\sqrt{2\pi}}\cdot \frac{p^2+q^2}{\sigma}$ due to Theorem 1 inequality (29), so we may exclude these cases.

Further if we have $k < \lfloor np \rfloor$, then $n - s > \lceil nq \rceil$ and hence due to the above

$$\begin{array}{rcl} F(k) - G(k) & = & 1 - F_{n,q,}(n-k-1) - (1 - \Phi(\frac{n-k-nq}{\sigma})) = \Phi(\frac{n-k-nq}{\sigma}) - F_{n,q}(n-k-1) \\ & \leq & \Phi(\frac{\lceil nq \rceil - nq}{\sigma}) - F_{n,q}(\lceil nq \rceil - 1) = F(\lfloor np \rfloor) - G(\lfloor np \rfloor) \end{array}$$

since $n - \lceil nq \rceil = \lfloor np \rfloor$ (or it follows directly (29) with Theorem 1, see above). Analogously we have

$$\begin{array}{rcl} G(k) - F(k-1) & = & 1 - \Phi(\frac{n-k-nq}{\sigma}) - (1 - F_{n,q}(n-k)) = F_{n,q}(n-k) - \Phi(\frac{n-k-nq}{\sigma}) \\ & \leq & F_{n,q}(\lceil nq \rceil) - \Phi(\frac{\lceil nq \rceil - nq}{\sigma}) = G(\lfloor np \rfloor) - F(\lfloor np \rfloor - 1). \end{array}$$

Thus it only remains to consider $u, v \in \{\lfloor np \rfloor, \lceil np \rceil\}$. We first assume $m = \lfloor np \rfloor \neq \lceil np \rceil = m+1$ and receive the four possibilities

$$F(u)-G(u)+G(v)-F(v-1) \in \Big\{f(m), f(m+1), g(m+1), f(m+1)+f(m)-g(m+1)\Big\}.$$

Now we study the elements separately and Lemma 11.14 yields

$$\begin{aligned} f(m)+f(m+1)-g(m+1) &< \tfrac{2}{\sqrt{2\pi}} \cdot \tfrac{p^2+q^2}{\sigma} \\ f(m),\ f(m+1),\ g(m+1) &< \tfrac{2}{\sqrt{2\pi}} \cdot \tfrac{p^2+q^2}{\sigma} \end{aligned}$$

and thus $F(u) - G(u) + G(v) - F(v-1) < \frac{2}{\sqrt{2\pi}} \cdot \frac{p^2+q^2}{\sigma}$ and hence we get (29) in this case.

Is however finally $\lfloor np \rfloor = \lceil np \rceil = np$, it follows $u = v = np$ and we have

$$F(u) - G(u) + G(v) - F(v-1) \;=\; f(np) \;<\; \tfrac{1}{\sqrt{2\pi}\,\sigma} \;\leq\; \tfrac{2}{\sqrt{2\pi}} \cdot \tfrac{p^2+q^2}{\sigma}$$

since Lemma 11.4 and hence (29) also in this case. □

11. Auxiliary lemmas

In the following series of lemmas, Lemmas 11.1 - 11.3 are on the normal distribution, Lemmas 11.4 - 11.11 on the binomial distribution and Lemmas 11.12 - 11.14 are on both distributions.

Further Lemmas 11.4, 11.6 and 11.9 are the main ones, Lemmas 11.7, 11.10, 11.13 and 11.14 yield less important results, while Lemmas 11.1, 11.2, 11.3, 11.5, 11.8, 11.11 and 11.12 are of auxiliary character.

Finally Lemmas 11.5 and 11.8 are only used in the present chapter and Lemma 11.11 is not used in the present work at all.

Lemma 11.1. *If $x > 0$, then*

$$x\mathrm{e}^{-\frac{x^2}{6}} \;<\; \sqrt{2\pi}\,(\Phi(x) - \tfrac{1}{2}) \;<\; x\mathrm{e}^{-\frac{x^2}{6}+\frac{x^4}{90}}.$$

Proof. For the proof see Mattner/Schulz (2014, Lemma 3.2). □

Lemma 11.2. *If $x, h \in \mathbb{R}$ with $h \neq 0$, then*

$$\mathrm{e}^{\frac{(x^2-1)h^2}{24}-\frac{x^4h^4}{960}} \;<\; \frac{\Phi(x+h/2)-\Phi(x-h/2)}{h\varphi(x)} \;<\; \mathrm{e}^{\frac{(x^2-1)h^2}{24}+\frac{h^4}{1440}}.$$

Proof. For the proof see Mattner/Schulz (2014, Lemma 3.1). □

Lemma 11.3.

(i) If $x \geq 2$*, then*

$$1 - \Phi(\tfrac{x^2-1}{x}) \;\leq\; \tfrac{1}{2\sqrt{2\pi}} \cdot \tfrac{1}{x}.$$

(ii) If $x > 0$*,* $c_1 \geq 0.9$ *and* $c_2 \geq 0.96$*, then*

$$1 - \Phi(c_1 \cdot x) \leq \tfrac{1}{2\sqrt{2\pi}} \cdot \tfrac{1}{x} \quad , \quad 1 - \Phi(c_2 \cdot x) \leq \tfrac{0.467}{\sqrt{2\pi}} \cdot \tfrac{1}{x}.$$

Proof. on (i): Due to Gordon (1941) for the Mills ratio $r(t) := \frac{1-\Phi(t)}{\varphi(t)}$ we have $r(t) < \frac{1}{t}$, $\forall t > 0$. Thus it follows if $x \geq 2$

$$\begin{aligned} \sqrt{2\pi}\, x \cdot (1 - \Phi(\tfrac{x^2-1}{x})) \;&\leq\; \sqrt{2\pi}\, x \cdot \tfrac{\varphi((x^2-1)/x)}{(x^2-1)/x} \;=\; \tfrac{x^2}{x^2-1} \cdot \mathrm{e}^{-\frac{1}{2}\cdot(\frac{x^2-1}{x})^2} \\ &\leq\; \tfrac{4}{3} \cdot \mathrm{e}^{-\frac{1}{2}\cdot(\frac{3}{2})^2} \;=\; \tfrac{4}{3} \cdot \mathrm{e}^{-\frac{9}{8}} \;=\; 0.43... \;<\; \tfrac{1}{2}. \end{aligned}$$

on (ii): Let $h(x) := \sqrt{2\pi}\, x \cdot (1 - \Phi(c \cdot x)$ for $c \in \mathbb{R}$. Since h is decreasing with respect to c, it is enough to show $h(x) \leq \frac{1}{2}$ for $c = 0.9$ and $h(x) \leq 0.467$ for $c = 0.96$.

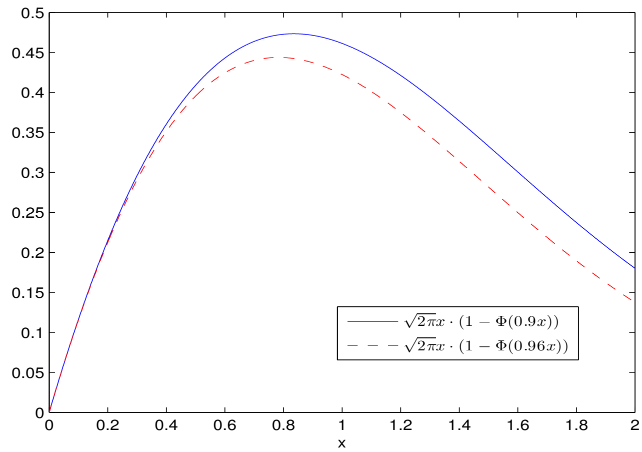

FIGURE 13. Graph of $h(x)$ for $c = 0.9$ and $c = 0.96$

If $c \in \{0.9, 0.96\}$ and $x \geq \frac{1}{c}$, then we have by the estimation on the Mills ratio above

$$\begin{aligned} h'(x) \;&=\; \sqrt{2\pi} \cdot (1 - \Phi(c \cdot x)) - \sqrt{2\pi}\, cx \cdot \varphi(c \cdot x) \;\leq\; \sqrt{2\pi} \cdot \tfrac{1}{cx}\varphi(c \cdot x) - \sqrt{2\pi}\, cx \cdot \varphi(c \cdot x) \\ &\leq\; 0 \end{aligned}$$

and hence $h(x) \leq h(\frac{1}{c}) = \frac{\sqrt{2\pi}}{c} \cdot (1 - \Phi(1)) \leq \frac{\sqrt{2\pi}}{0.9} \cdot (1 - \Phi(1)) = 0.4419\ldots$.

Let now $c \in \{0.9, 0.96\}$ and $x < \frac{1}{c}$, then Lemma 11.1 yields

$$1 - \Phi(c \cdot x) \;=\; \tfrac{1}{2} - (\Phi(xc) - \tfrac{1}{2}) \;\leq\; \tfrac{1}{2} - \tfrac{xc}{\sqrt{2\pi}} \mathrm{e}^{-\frac{x^2c^2}{6}} \;\leq\; \tfrac{1}{2} - \tfrac{xc}{\sqrt{2\pi}} + \tfrac{x^3c^3}{6\sqrt{2\pi}}$$

and hence

$$h(x) \;\leq\; \tfrac{\sqrt{2\pi}}{2} \cdot x - cx^2 + \tfrac{c^3x^4}{6} \;\leq\; \tfrac{\sqrt{2\pi}}{2} \cdot x - cx^2 + \tfrac{c^2x^3}{6} \;=:\; i(x).$$

Since $i''(x) = -2c + c^2x < 0$, so i is concave, and $i'(x) = 0 \Leftrightarrow x = \frac{2}{c} - \sqrt{(\frac{2}{c})^2 - \frac{\sqrt{2\pi}}{c^2}} =: x_0$ we have $h(x) \leq i(x) \leq i(x_0)$. In the case $c = 0.9$ we have $i(x_0) = 0.4980... < \frac{1}{2}$ and in the case $c = 0.96$ we have $i(x_0) = 0.4669...$. □

Lemma 11.4. *Let $n, k \in \mathbb{N}$ with $k \geq 1$ and $n-k \geq 1$ and let $\kappa := \frac{k}{n}$ and $\sigma_\kappa^2 = n\kappa(1-\kappa)$.*

(i) Then we have

$$\frac{1}{\sigma_\kappa\sqrt{2\pi}} \cdot \mathrm{e}^{-\frac{1}{12\sigma_\kappa^2}} \;<\; f_{n,\kappa}(k) \;<\; \frac{1}{\sigma_\kappa\sqrt{2\pi}} \cdot \mathrm{e}^{-\frac{19}{320\sigma_\kappa^2}}.$$

More precisely we even have

$$\mathrm{e}^{-\frac{1}{12\sigma_\kappa^2}+\frac{1}{12n}} \;<\; f_{n,\kappa}(k)\cdot\sqrt{2\pi}\,\sigma_\kappa \;<\; \mathrm{e}^{-\frac{1}{12\sigma_\kappa^2}+\frac{1}{12n}+\frac{1}{360}\cdot\left(\frac{1}{k^3}+\frac{1}{(n-k)^3}-\frac{1}{n^3}\right)}.$$

(ii) Further we have under the additional assumption $k \leq \frac{n-1}{2}$

$$\sqrt{\frac{(k+1)(n-k-1)}{k(n-k)}} \cdot \mathrm{e}^{-\frac{1}{12}\cdot\left(\frac{1}{k}-\frac{1}{k+1}+\frac{1}{n-k}-\frac{1}{n-k-1}\right)} \;\leq\; \frac{f_{n,k/n}(k)}{f_{n,(k+1)/n}(k+1)} \;\leq\; \sqrt{\frac{(k+1)(n-k-1)}{k(n-k)}}.$$

Proof. Due to Olver (1974) [35, p.294] for all $s \in \mathbb{N}$ exists a $\theta_s \in (0,1)$ such that if $x \geq 1$ we have with $R_s(x) := \frac{B_{2s}}{2s(2s-1)} \cdot \frac{\theta_s}{x^{2s-1}}$

$$\log\big(\Gamma(x+1)\big) \;=\; \big(x+\tfrac{1}{2}\big)\log x - x + \tfrac{1}{2}\log(2\pi) + \sum_{i=1}^{s-1}\frac{B_{2i}}{2i(2i-1)x^{2i-1}} + R_s(x),$$

where B_s denote the s-th Bernoulli number , so $B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42}, B_8 = -\frac{1}{30}$, and Γ the Gamma function. It follows

$$\begin{aligned}
\log\big(f_{n,\frac{k}{n}}(k)\big) \;&=\; \log\Big(\tfrac{n!}{k!(n-k)!}\cdot\big(\tfrac{k}{n}\big)^k\cdot\big(\tfrac{n-k}{n}\big)^{n-k}\Big)\\
&=\; \log\Big(\tfrac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}\Big) + k\log\big(\tfrac{k}{n}\big) + (n-k)\log\big(\tfrac{n-k}{n}\big)\\
&=\; \big(n+\tfrac{1}{2}\big)\log n - n + \frac{1}{2}\log(2\pi) + \sum_{i=1}^{s-1}\tfrac{B_{2i}}{2i(2i-1)n^{2i-1}}\\
&\quad +R_s(n) - (k+\tfrac{1}{2})\log k + k - \tfrac{1}{2}\log(2\pi)\\
&\quad -\sum_{i=1}^{s-1}\tfrac{B_{2i}}{2i(2i-1)k^{2i-1}} - R_s(k) - (n-k+\frac{1}{2})\log(n-k) + (n-k)\\
&\quad -\tfrac{1}{2}\log(2\pi) - \sum_{i=1}^{s-1}\tfrac{B_{is}}{2i(2i-1)(n-k)^{2i-1}} - R_s(n-k)\\
&\quad +k\log k - k\log n + (n-k)\log(n-k) - (n-k)\log n\\
&=\; \sum_{i=1}^{s-1}\tfrac{B_{2i}}{2i(2i-1)}\left(\tfrac{1}{n^{2i-1}} - \tfrac{1}{k^{2i-1}} - \tfrac{1}{(n-k)^{2i-1}}\right)\\
&\quad +R_s(n) - R_s(k) - R_s(n-k) - \tfrac{1}{2}\log(2\pi) + \tfrac{1}{2}\log\big(\tfrac{n}{k(n-k)}\big)\\
&=\; -\log(\sigma_\kappa\sqrt{2\pi}) + \sum_{i=1}^{s-2}\tfrac{B_{2i}}{2i(2i-1)}\left(\tfrac{1}{n^{2i-1}} - \tfrac{1}{k^{2i-1}} - \tfrac{1}{(n-k)^{2i-1}}\right)\\
&\quad + \underbrace{\tfrac{B_{2(s-1)}}{2(s-1)(2s-3)}\cdot\big(\tfrac{1}{n^{2s-3}} - \tfrac{1}{k^{2s-3}} - \tfrac{1}{(n-k)^{2s-3}}\big) + R_s(n) - R_s(k) - R_s(n-k)}_{=:Q_s}.
\end{aligned}$$

Now it holds $Q_4 < 0$ because of

$$\begin{aligned}
Q_4 &= \tfrac{B_6}{30}\big(\tfrac{1}{n^5}-\tfrac{1}{k^5}-\tfrac{1}{(n-k)^5}\big)+\tfrac{B_8}{56}\big(\tfrac{\Theta_4^n}{n^7}-\tfrac{\Theta_4^k}{k^7}-\tfrac{\Theta_4^{n-k}}{(n-k)^7}\big)\\
&= \tfrac{1}{n^5}\Big(\tfrac{1}{42\cdot 30}\big(1-\tfrac{n^5}{k^5}-\tfrac{n^5}{(n-k)^5}\big)-\tfrac{1}{56\cdot 30}\big(\tfrac{\Theta_4^n}{n^2}-\tfrac{\Theta_4^k n^5}{k^7}-\tfrac{\Theta_4^{n-k}n^5}{(n-k)^7}\big)\Big)\\
&\le \tfrac{1}{30n^5}\Big(\tfrac{1}{42}\big(1-\tfrac{n^5}{k^5}-\tfrac{n^5}{(n-k)^5}\big)-\tfrac{1}{56}\big(-\tfrac{n^5}{k^7}-\tfrac{n^5}{(n-k)^7}\big)\Big)\\
&= \tfrac{1}{30\cdot 42n^5}\Big(1-\tfrac{n^5}{k^5}\big(1-\tfrac{3}{4k^2}\big)-\tfrac{n^5}{(n-k)^5}\big(1-\tfrac{3}{4(n-k)^2}\big)\Big)\\
&\le \tfrac{1}{30\cdot 42n^5}\Big(1-\tfrac{1}{4}\cdot\big(\tfrac{n^5}{k^5}+\tfrac{n^5}{(n-k)^5}\big)\Big) \;\le\; \tfrac{1}{30\cdot 42\cdot n^5}\Big(1-\tfrac{2^5}{4}\Big) \;<\; 0
\end{aligned}$$

and hence for the estimation upwards we have with $\kappa = \frac{k}{n}$ and $\kappa(1-\kappa) \le \frac{1}{4}$

$$\begin{aligned}
\log\big(f_{n,\frac{k}{n}}(k)\big) &< -\log(\sigma_\kappa\sqrt{2\pi}) + \tfrac{1}{12}\big(\tfrac{1}{n}-\tfrac{1}{k}-\tfrac{1}{n-k}\big)-\tfrac{1}{360}\big(\tfrac{1}{n^3}-\tfrac{1}{k^3}-\tfrac{1}{(n-k)^3}\big)\\
&= -\log(\sigma_\kappa\sqrt{2\pi}) + \tfrac{1}{12\sigma_\kappa^2}\cdot\big(\kappa(1-\kappa)-1-\tfrac{\kappa^2(1-\kappa)^2}{30k(n-k)}+\tfrac{1-\kappa}{30k^2}+\tfrac{\kappa}{30(n-k)^2}\big)\\
&\le -\log(\sigma_\kappa\sqrt{2\pi}) + \tfrac{1}{12\sigma_\kappa^2}\cdot\big(\kappa(1-\kappa)-1+\tfrac{1-\kappa}{30}+\tfrac{\kappa}{30}\big)\\
&\le -\log(\sigma_\kappa\sqrt{2\pi}) + \tfrac{1}{12\sigma_\kappa^2}\cdot\big(\tfrac{1}{4}-1+\tfrac{1}{30}\big)\\
&= -\log(\sigma_\kappa\sqrt{2\pi}) - \tfrac{43}{720\sigma_\kappa^2}\\
&\le -\log(\sigma_\kappa\sqrt{2\pi}) - \tfrac{19}{320\sigma_\kappa^2}.
\end{aligned}$$

On the other hand we have for the estimation downwards

$$\begin{aligned}
Q_3 &= \tfrac{B_4}{12}\big(\tfrac{1}{n^3}-\tfrac{1}{k^3}-\tfrac{1}{(n-k)^3}\big)+\tfrac{B_6}{30}\big(\tfrac{\Theta_3^n}{n^5}-\tfrac{\Theta_3^k}{k^5}-\tfrac{\Theta_3^{n-k}}{(n-k)^5}\big)\\
&= \tfrac{1}{n^3}\Big(-\tfrac{1}{12\cdot 30}\big(1-\tfrac{n^3}{k^3}-\tfrac{n^3}{(n-k)^3}\big)+\tfrac{1}{42\cdot 30}\big(\tfrac{\Theta_4^n}{n^2}-\tfrac{\Theta_4^k n^3}{k^5}-\tfrac{\Theta_4^{n-k}n^3}{(n-k)^5}\big)\Big)\\
&\ge \tfrac{1}{12\cdot 30n^3}\Big(-1+\tfrac{n^3}{k^3}+\tfrac{n^3}{(n-k)^3}-\tfrac{2}{7}\big(\tfrac{n^3}{k^5}+\tfrac{n^3}{(n-k)^5}\big)\Big)\\
&\ge \tfrac{1}{12\cdot 30n^3}\Big(-1+\big(\tfrac{n^3}{k^3}+\tfrac{n^3}{(n-k)^3}\big)\cdot\tfrac{5}{7}\Big)\\
&\ge \tfrac{1}{12\cdot 30n^3}\Big(-1+\tfrac{2^3\cdot 5}{7}\Big) \;>\; 0
\end{aligned}$$

and hence

$$\begin{aligned}
\log\big(f_{n,\frac{k}{n}}(k)\big) &> -\log\big(\sigma_\kappa\sqrt{2\pi}\big)+\tfrac{1}{12}\big(\tfrac{1}{n}-\tfrac{1}{k}-\tfrac{1}{n-k}\big)\\
&= -\log\big(\sigma_\kappa\sqrt{2\pi}\big)-\tfrac{1}{12\sigma_\kappa^2}+\tfrac{1}{12n}\\
&\ge -\log(\sigma_\kappa\sqrt{2\pi})-\tfrac{1}{12\sigma_\kappa^2}.
\end{aligned}$$

On part (ii): In the case $k = n-k-1$ we have $1 = 1 = 1$ in the inequality in (ii), so we may assume $k \leq n/2-1$. We further have with $k = tn$, $t \in [\frac{1}{n}, \frac{1}{2} - \frac{1}{n}]$ and the notations above

$$\begin{array}{rl} & \log\left(\frac{f_{n,k/n}(k)}{f_{n,(k+1)/n}(k+1)}\right) - \log\left(\sqrt{\frac{(k+1)(n-k-1)}{k(n-k)}} \cdot \mathrm{e}^{-\frac{1}{12}\cdot(\frac{1}{k}-\frac{1}{k+1}+\frac{1}{n-k}-\frac{1}{n-k-1})}\right) \\ = & \frac{1}{360}\cdot(\frac{1}{k^3} - \frac{1}{(k+1)^3} + \frac{1}{(n-k)^3} - \frac{1}{(n-k-1)^3}) \\ & +R_3(k) + R_3(n-k) - R_3(k+1) - R_3(n-k-1) \\ \geq & \frac{1}{360}\cdot(\frac{1}{k^3} - \frac{1}{(k+1)^3} + \frac{1}{(n-k)^3} - \frac{1}{(n-k-1)^3}) - \frac{1}{1260}\cdot(\frac{1}{k^5} + \frac{1}{(n-k)^5}) \\ = & \frac{1}{360n^3}\cdot\left(\frac{1}{t^3} - \frac{1}{(t+1/n)^3} + \frac{1}{(1-t)^3} - \frac{1}{(1-t-1/n)^3} - \frac{2}{7n^2}\cdot(\frac{1}{t^5} + \frac{1}{(1-t)^5})\right) \\ =: & \frac{1}{360n^3}\cdot h(t). \end{array}$$

We now have

$$\begin{array}{rcl} h'(t) & = & -\frac{3}{t^4} + \frac{3}{(t+1/n)^4} - \frac{3}{(1-t)^4} - \frac{3}{(1-t-1/n)^4} + \frac{10}{7n^2}\cdot(\frac{1}{t^6} - \frac{1}{(1-t)^6}) \ \leq \frac{3}{t^4} + \frac{3}{(t+1/n)^4} + \frac{10}{7n^2}\cdot\frac{1}{t^6} \\ & = & -\frac{3}{(t+1/n)^4}\cdot((\frac{t+1/n}{t})^4\cdot(1-\frac{10}{21n^2t^2}) - 1) \ \leq \ -\frac{3}{(t+1/n)^4}\cdot((1+\frac{4}{nt})\cdot(1-\frac{10}{21nt}) - 1) \\ & \leq & 0 \end{array}$$

and hence

$$h(t) \ \geq \ f(\tfrac{1}{2} - \tfrac{1}{n}) \ = \ \frac{1}{(1/2-1/n)^3} + \frac{1}{(1/2+1/n)^3} - 16 - \frac{2}{7n^2}\cdot(\frac{1}{(1/2-1/n)^5} + \frac{1}{(1/2+1/n)^5}) \ =: \ i(n).$$

Because of the condition $1 \leq k \leq n/2-1$ it must be $n \geq 4$ and thus we have, treating n as a continuous variable

$$\begin{array}{rcl} i'(n) & = & -\frac{3}{n^2}\cdot(\frac{1}{(1/2-1/n)^4} - \frac{1}{(1/2+1/n)^4}) + \frac{4}{7n^3}\cdot(\frac{1}{(1/2-1/n)^5} + \frac{1}{(1/2+1/n)^5}) \\ & & -\frac{10}{7n^2}\cdot(\frac{1}{(1/2-1/n)^6} - \frac{1}{(1/2+1/n)^6}) \\ & \leq & -\frac{3}{n^2(1/2-1/n)^4(1/2+1/n)^4}\cdot((\frac{1}{2}+\frac{1}{n})^4 - (\frac{1}{2}-\frac{1}{n})^4 - \frac{4}{21n}\cdot(\frac{(1/2+1/n)^4}{(1/2-1/n)} + \frac{1}{2^3})) \\ & \leq & -\frac{3}{n^2(1/2-1/n)^4(1/2+1/n)^4}\cdot(\frac{4}{2^3n} - \frac{4}{21n}\cdot(\frac{(3/4)^4}{1/4} + \frac{1}{2^3})) = -\frac{3}{n^3(1/2-1/n)^4(1/2+1/n)^4}\cdot\frac{79}{336} \\ & < & 0 \end{array}$$

and hence finally $h(t) \geq i(n) \geq \lim_{n\to\infty} i(n) = 0$.

On the other hand we have analogously to the above

$$\begin{array}{rl} & \log\left(\frac{f_{n,k/n}(k)}{f_{n,(k+1)/n}(k+1)}\right) - \log\left(\sqrt{\frac{(k+1)(n-k-1)}{k(n-k)}}\right) \\ = & \frac{1}{12}\cdot(\frac{1}{k}-\frac{1}{k+1}+\frac{1}{n-k}-\frac{1}{n-k-1}) \\ & -R_2(k)-R_2(n-k)+R_2(k+1)+R_2(n-k-1) \\ \geq & -\frac{1}{12}\cdot(\frac{1}{k}-\frac{1}{k+1}-\frac{1}{n-k}+\frac{1}{n-k-1})+\frac{1}{360}\cdot(\frac{1}{k^3}+\frac{1}{(n-k)^3}) \\ = & -\frac{1}{12n}\cdot\left(\frac{1}{t}-\frac{1}{t+1/n}+\frac{1}{1-t}-\frac{1}{1-t-1/n}-\frac{1}{30n^2}\cdot(\frac{1}{t^3}+\frac{1}{(1-t)^3})\right) \\ =: & -\frac{1}{12n^3}\cdot h_2(t). \end{array}$$

We now have

$$\begin{array}{rcl} h_2'(t) & = & -\frac{1}{t^2}+\frac{1}{(t+1/n)^2}+\frac{1}{(1-t)^2}-\frac{1}{(1-t-1/n)^2}+\frac{1}{30n^2}\cdot(\frac{3}{t^4}-\frac{3}{(1-t)^4}) \;\leq \frac{1}{t^2}+\frac{1}{(t+1/n)^2}+\frac{1}{10n^2}\cdot\frac{1}{t^4} \\ & = & -\frac{1}{(t+1/n)^2}\cdot((\frac{t+1/n}{t})^2\cdot(1-\frac{1}{10n^2t^2})-1) \;\leq\; -\frac{1}{(t+1/n)^2}\cdot((1+\frac{2}{nt})\cdot(1-\frac{1}{10nt})-1) \\ & \leq & 0 \end{array}$$

and hence

$$h_2(t) \;\geq\; f(\tfrac{1}{2}-\tfrac{1}{n}) \;=\; \frac{1}{1/2-1/n}+\frac{1}{1/2+1/n}-4-\frac{1}{30n^2}\cdot(\frac{1}{(1/2-1/n)^3}+\frac{1}{(1/2+1/n)^3}) \;=:\; i_2(n).$$

Because of the condition $1 \leq k \leq n/2-1$ it must be $n \geq 4$ and thus we have, treating n as a continuous variable

$$\begin{array}{rcl} i_2'(n) & = & -\frac{1}{n^2}\cdot(\frac{1}{(1/2-1/n)^2}-\frac{1}{(1/2+1/n)^2})+\frac{1}{15n^3}\cdot(\frac{1}{(1/2-1/n)^3}+\frac{1}{(1/2+1/n)^3}) \\ & & -\frac{1}{10n^2}\cdot(\frac{1}{(1/2-1/n)^4}-\frac{1}{(1/2+1/n)^4}) \\ & \leq & -\frac{1}{n^2(1/2-1/n)^2(1/2+1/n)^2}\cdot((\frac{1}{2}+\frac{1}{n})^2-(\frac{1}{2}-\frac{1}{n})^2-\frac{1}{15n}\cdot(\frac{(1/2+1/n)^2}{(1/2-1/n)}+\frac{1}{2})) \\ & \leq & -\frac{1}{n^2(1/2-1/n)^2(1/2+1/n)^2}\cdot(\frac{2}{n}-\frac{1}{15n}\cdot(\frac{(3/4)^2}{1/4}+\frac{1}{2}))=-\frac{1}{n^3(1/2-1/n)^2(1/2+1/n)^2}\cdot\frac{109}{60} \\ & < & 0 \end{array}$$

and hence finally $h_2(t) \geq i_2(n) \geq \lim\limits_{n\to\infty} i_2(n) = 0$. □

Lemma 11.5. *Let $n, k \in \mathbb{N}$ with $\kappa := \frac{k}{n} \in [\frac{1}{6}, \frac{1}{2}]$. Further for $t \in [0,1]$ let*

$$h(t) \ := \ (1 - \tfrac{t}{k})^{k-1} \cdot (1 + \tfrac{t}{n-k})^{n-k}.$$

(i) If $k \geq 2$, then

$$h(1) \ \geq \ 1 + \tfrac{1-2\kappa}{2\sigma_\kappa^2} + \tfrac{7-20\kappa+24\kappa^2}{24\sigma_\kappa^4} + \tfrac{9-34\kappa+52\kappa^2-48\kappa^3}{48\sigma_\kappa^6} \ =: \ h_6.$$

(ii) If $k \geq 3$, then

$$h(1) \ \leq \ h_6 + \tfrac{15-24\kappa}{80\sigma_\kappa^8}.$$

(iii) If $x \in [0,1]$ and $k \geq 2$, then

$$\begin{aligned}\int_0^x h(t)\mathrm{d}t \ &\geq \ x + \tfrac{1}{\sigma_\kappa^2} \cdot \big(\tfrac{x^2(1-\kappa)}{2} - \tfrac{x^3}{6}\big) + \tfrac{1}{\sigma_\kappa^4} \cdot \big(\tfrac{x^3(1-\kappa)^2}{3} - \tfrac{x^4(5-7\kappa)}{24} + \tfrac{x^5}{40}\big) \\ &\quad + \tfrac{1}{\sigma_\kappa^6} \cdot \big(\tfrac{x^4(1-\kappa)^3}{4} + \tfrac{x^5(-13+33\kappa-23\kappa^2)}{60} + \tfrac{x^6(7-11\kappa)}{144} - \tfrac{x^7}{336}\big) \\ &=: \ H_6(x).\end{aligned}$$

(iv) If $x \in [0,1]$ and $k \geq 3$, then

$$\int_0^x h(t)\mathrm{d}t \ \leq \ H_6(x) + \tfrac{3}{2\sigma_\kappa^8} \cdot \big(\tfrac{x^5(1-\kappa)^4}{5} - \tfrac{x^6(1-\kappa)^2(63-110\kappa)}{450}\big).$$

(v) If $k \geq 2$, then

$$h(t) \ \geq \ 1 + \tfrac{t}{\sigma_\kappa^2} \cdot \big(1 - \kappa - \tfrac{t}{2}\big).$$

Proof. First we consider the estimations downwards. To this we have

$$\begin{aligned} & \log\big((1 - \tfrac{t}{k})^{k-1} \cdot (1 + \tfrac{t}{n-k})^{n-k}\big) \\ = \ & -\sum_{r=1}^{\infty} \tfrac{t^r}{rk^{r-1}} + \sum_{r=1}^{\infty} \tfrac{t^r}{rk^r} + \sum_{r=1}^{\infty} \tfrac{(-1)^{r+1}t^r}{r(n-k)^{r-1}} \\ = \ & \sum_{r=1}^{\infty} \tfrac{t^r}{k^r} \cdot (\tfrac{1}{r} - \tfrac{t}{r+1}) + \sum_{r=2}^{\infty} \tfrac{(-1)^{r+1}t^r}{r(n-k)^{r-1}} \\ \geq \ & \tfrac{t}{k} \cdot (1 - \tfrac{t}{2}) + \tfrac{t^2}{k^2} \cdot (\tfrac{1}{2} - \tfrac{t}{3}) + \tfrac{t^3}{k^3} \cdot (\tfrac{1}{3} - \tfrac{t}{4}) + \tfrac{t^4}{k^4} \cdot (\tfrac{1}{4} - \tfrac{t}{5}) - \tfrac{t^2}{2(n-k)} + \tfrac{t^3}{3(n-k)^2} - \tfrac{t^4}{4(n-k)^3} \\ = \ & \tfrac{t(1-\kappa-t/2)}{\sigma_\kappa^2} + \tfrac{t^2}{\sigma_\kappa^4} \cdot \big(\tfrac{(1-\kappa)^2}{2} - \tfrac{t(1-2\kappa)}{3}\big) + \tfrac{t^3}{\sigma_\kappa^6} \cdot \big(\tfrac{(1-\kappa)^3}{3} - \tfrac{t((1-\kappa)^3+\kappa^3)}{4}\big) + \tfrac{t^4(1-\kappa)^4}{\sigma_\kappa^8} \cdot (\tfrac{1}{4} - \tfrac{t}{5}). \end{aligned}$$

Thus, since $\mathrm{e}^y \geq 1 + y + \frac{z_1^2}{2} + \frac{z_2^3}{6}$ if $y \geq z_1, z_2 \geq 0$, we have

$$
\begin{aligned}
h(t) \;\ge\;& 1+\tfrac{t}{\sigma_\kappa^2}\cdot(1-\kappa-\tfrac{t}{2})+\tfrac{t^2}{\sigma_\kappa^4}\cdot(\tfrac{(1-\kappa)^2}{2}-\tfrac{t(1-2\kappa)}{3})+\tfrac{t^3}{\sigma_\kappa^6}\cdot(\tfrac{(1-\kappa)^3}{3}-\tfrac{t((1-\kappa)^3+\kappa^3)}{4})\\
&+\tfrac{t^4(1-\kappa)^4}{\sigma_\kappa^8}\cdot(\tfrac{1}{4}-\tfrac{t}{5})+\tfrac{1}{2}\cdot\Big(\tfrac{t^2}{\sigma_\kappa^4}\cdot(1-\kappa-\tfrac{t}{2})^2+\tfrac{2t^3}{\sigma_\kappa^6}\cdot(1-\kappa-\tfrac{t}{2})\cdot(\tfrac{(1-\kappa)^2}{2}-\tfrac{t(1-2\kappa)}{3})\\
&+\tfrac{t^4}{\sigma_\kappa^8}\cdot\big((\tfrac{(1-\kappa)^2}{2}-\tfrac{t(1-2\kappa)}{3})^2+2\cdot(1-\kappa-\tfrac{t}{2})\cdot(\tfrac{(1-\kappa)^3}{3}-\tfrac{t((1-\kappa)^3+\kappa^3)}{4})\big)\\
&+2\cdot\tfrac{t^5}{\sigma_\kappa^{10}}\cdot(\tfrac{(1-\kappa)^2}{2}-\tfrac{t(1-2\kappa)}{3})\cdot(\tfrac{(1-\kappa)^3}{3}-\tfrac{t((1-\kappa)^3+\kappa^3)}{4})\Big)+\tfrac{1}{6}\cdot\tfrac{t^3}{\sigma_\kappa^6}\cdot(1-\kappa-\tfrac{t}{2})^3.
\end{aligned}
$$

If we have $\frac{(1-\kappa)^3}{3}-\frac{t((1-\kappa)^3+\kappa^3)}{4}>0$, then the terms $\frac{t^4}{\sigma_\kappa^8}$ and $\frac{t^5}{\sigma_\kappa^{10}}$ are obviously positive and we may estimate them downwards by zero. Thus we can assume in the following $\frac{(1-\kappa)^3}{3}-\frac{t((1-\kappa)^3+\kappa^3)}{4}<0$. Further $\frac{t^5}{\sigma_\kappa^{10}}\le\frac{t^5}{2(1-\kappa)\sigma_\kappa^8}\le\frac{t^4}{\sigma_\kappa^8}$ holds, wherefore in the following we discuss the positivity of the $R(t,\kappa)$ below, defined by

$$
\begin{aligned}
R(t,\kappa) \;:=\;& \tfrac{(1-\kappa)^4}{4}-\tfrac{t(1-\kappa)^4}{5}+\tfrac{1}{2}\cdot(\tfrac{(1-\kappa)^2}{2}-\tfrac{t(1-2\kappa)}{3})^2+(1-\kappa-\tfrac{t}{2})\cdot(\tfrac{(1-\kappa)^3}{3}-\tfrac{t((1-\kappa)^3+\kappa^3)}{4})\\
&+(\tfrac{(1-\kappa)^2}{2}-\tfrac{t(1-2\kappa)}{3})\cdot(\tfrac{(1-\kappa)^3}{3}-\tfrac{t((1-\kappa)^3+\kappa^3)}{4}).
\end{aligned}
$$

We now show that $\frac{dR(t,\kappa)}{\mathrm{d}t}<0$ and hence $R(t,\kappa)\ge R(1,\kappa)$. We have with $t\le 1$

$$
\begin{aligned}
\tfrac{dR(t,\kappa)}{\mathrm{d}t} \;=\;& -\tfrac{(1-\kappa)^4}{5}-\tfrac{1-2\kappa}{3}\cdot(\tfrac{(1-\kappa)^2}{2}-t\cdot\tfrac{1-2\kappa}{3})-\tfrac{(1-\kappa)^3}{6}-\tfrac{(1-\kappa)((1-\kappa)^3+\kappa^3)}{4}+t\cdot\tfrac{(1-\kappa)^3+\kappa^3}{4}\\
&-\tfrac{(1-2\kappa)(1-\kappa)^3}{9}-\tfrac{(1-\kappa)^2((1-\kappa)^3+\kappa^3)}{8}+t\cdot\tfrac{(1-2\kappa)((1-\kappa)^3+\kappa^3)}{6}\\
\le\;& -\tfrac{(1-\kappa)^4}{5}-\tfrac{1-2\kappa}{3}\cdot(\tfrac{(1-\kappa)^2}{2}-\tfrac{1-2\kappa}{3})-\tfrac{(1-\kappa)^3}{6}-\tfrac{(1-\kappa)((1-\kappa)^3+\kappa^3)}{4}+\tfrac{(1-\kappa)^3+\kappa^3}{4}\\
&-\tfrac{(1-2\kappa)(1-\kappa)^3}{9}-\tfrac{(1-\kappa)^2((1-\kappa)^3+\kappa^3)}{8}+\tfrac{(1-2\kappa)((1-\kappa)^3+\kappa^3)}{6}\\
=\;& -\tfrac{59}{120}+\tfrac{763\kappa}{360}-\tfrac{323\kappa^2}{90}+\tfrac{1063\kappa^3}{360}-\tfrac{287\kappa^4}{360}.
\end{aligned}
$$

Further we have $\frac{\mathrm{d}^3R(1,\kappa)}{dtd\kappa^2}=-\frac{323}{45}+\frac{1063\kappa}{60}-\frac{287\kappa^2}{30}\le-\frac{323}{45}+\frac{1063}{2\cdot 60}-\frac{287}{4\cdot 30}=-\frac{32}{45}$ and hence $\frac{\mathrm{d}^2R(1,\kappa)}{dtd\kappa}\ge\frac{\mathrm{d}^2R(1,1/2)}{dtd\kappa}=\frac{763\kappa}{360}-\frac{323}{90}+\frac{3\cdot 1063}{4\cdot 360}-\frac{4\cdot 287}{8\cdot 360}=\frac{44}{1440}>0$ from which finally $\frac{dR(1,\kappa)}{\mathrm{d}t}\le\frac{dR(1,1/2)}{\mathrm{d}t}=-\frac{59}{120}+\frac{763}{2\cdot 360}-\frac{323}{4\cdot 90}+\frac{1063}{8\cdot 360}-\frac{287}{16\cdot 360}=-\frac{19}{1920}<0$ follows. Thus we have

$$
\begin{aligned}
R(t,\kappa) \;\ge\;& \tfrac{(1-\kappa)^4}{20}+\tfrac{1}{2}\cdot(\tfrac{(1-\kappa)^2}{2}-\tfrac{1-2\kappa}{3})^2+\tfrac{1-2\kappa}{2}\cdot(\tfrac{(1-\kappa)^3}{3}-\tfrac{(1-\kappa)^3+\kappa^3}{4})\\
&+(\tfrac{(1-\kappa)^2}{2}-\tfrac{1-2\kappa}{3})\cdot(\tfrac{(1-\kappa)^3}{3}-\tfrac{(1-\kappa)^3+\kappa^3}{4})\\
=\;& \tfrac{43}{360}-\tfrac{8\kappa}{15}+\tfrac{353\kappa^2}{360}-\tfrac{377\kappa^3}{360}+\tfrac{67\kappa^4}{90}-\tfrac{\kappa^5}{6} \;=:\; r(\kappa).
\end{aligned}
$$

Furthermore $r''(\kappa)=\frac{353}{180}-\frac{377\kappa}{60}+\frac{67\cdot 2\cdot\kappa^2}{15}-\frac{10\kappa^3}{3}=\frac{353(1-\kappa)^3}{180}+\kappa\cdot\frac{-72+549\kappa-247\kappa^2}{180}\ge\frac{353(1-\kappa)^3}{180}+\kappa\cdot\frac{-72+549/6-247/36}{180}=\frac{353(1-\kappa)^3}{180}+\kappa\cdot\frac{91}{1296}>0$ yields $r'(\kappa)\le r'(\frac{1}{2})=-\frac{8}{15}+\frac{2\cdot 353}{2\cdot 360}-\frac{3\cdot 377}{4\cdot 360}+\frac{4\cdot 67}{8\cdot 90}-\frac{5}{16\cdot 6}=\frac{-13}{720}<0$ and hence we have finally

$$
\begin{aligned}
R(t,\kappa) \;\ge\;& R(1,\kappa) \;=\; r(\kappa) \;\ge\; r(\tfrac{1}{2})\\
=\;& \tfrac{43}{360}-\tfrac{4}{15}+\tfrac{353}{4\cdot 360}-\tfrac{377}{8\cdot 360}+\tfrac{67}{16\cdot 90}-\tfrac{1}{32\cdot 6} \;=\; \tfrac{1}{120} \;>\; 0.
\end{aligned}
$$

Thus we can estimate the terms of $\frac{1}{\sigma_\kappa^8}$ and $\frac{1}{\sigma_\kappa^{10}}$ downwards by zero and receive

$$
\begin{aligned}
h(t) \;\geq\;& 1 + \tfrac{t}{\sigma_\kappa^2}\cdot\left(1-\kappa-\tfrac{t}{2}\right) + \tfrac{t^2}{\sigma_\kappa^4}\cdot\left(\tfrac{(1-\kappa)^2}{2} - \tfrac{t(1-2\kappa)}{3}\right) + \tfrac{t^3}{\sigma_\kappa^6}\cdot\left(\tfrac{(1-\kappa)^3}{3} - \tfrac{t((1-\kappa)^3+\kappa^3)}{4}\right)\\
&+\tfrac{1}{2}\cdot\Big(\tfrac{t^2}{\sigma_\kappa^4}\cdot\left(1-\kappa-\tfrac{t}{2}\right)^2 + \tfrac{2t^3}{\sigma_\kappa^6}\cdot\left(1-\kappa-\tfrac{t}{2}\right)\cdot\left(\tfrac{(1-\kappa)^2}{2} - \tfrac{t(1-2\kappa)}{3}\right)\Big) + \tfrac{1}{6}\cdot\tfrac{t^3}{\sigma_\kappa^6}\cdot\left(1-\kappa-\tfrac{t}{2}\right)^3\\
=\;& 1 + \tfrac{t}{\sigma_\kappa^2}\cdot\left(1-\kappa-\tfrac{t}{2}\right) + \tfrac{t^2}{\sigma_\kappa^4}\cdot\left((1-\kappa)^2 - \tfrac{t(5-7\kappa)}{6} + \tfrac{t^2}{8}\right)\\
&+\tfrac{t^3}{\sigma_\kappa^6}\cdot\left((1-\kappa)^3 + \tfrac{t(-13+33\kappa-23\kappa^2)}{12} + \tfrac{t^2(7-11\kappa)}{24} - \tfrac{t^3}{48}\right).
\end{aligned}
$$

If we set $t = 1$, we get

$$
\begin{aligned}
h(1) \;\geq\;& 1 + \tfrac{1-2\kappa}{2\sigma_\kappa^2} + \tfrac{1}{\sigma_\kappa^4}\cdot\left((1-\kappa)^2 - \tfrac{(5-7\kappa)}{6} + \tfrac{1}{8}\right) + \tfrac{1}{\sigma_\kappa^6}\cdot\left((1-\kappa)^3 + \tfrac{-13+33\kappa-23\kappa^2}{12} + \tfrac{7-11\kappa}{24} - \tfrac{1}{48}\right)\\
=\;& 1 + \tfrac{1-2\kappa}{2\sigma_\kappa^2} + \tfrac{7-20\kappa+24\kappa^2}{24\sigma_\kappa^4} + \tfrac{9-34\kappa+52\kappa^2-48\kappa^3}{48\sigma_\kappa^6}
\end{aligned}
$$

and thus part (i). If we however integrate, we get for $0 \leq x \leq 1$

$$
\begin{aligned}
\int_0^x h(t)\mathrm{d}t \;\geq\;& x + \tfrac{1}{\sigma_\kappa^2}\cdot\left(\tfrac{x^2(1-\kappa)}{2} - \tfrac{x^3}{6}\right) + \tfrac{1}{\sigma_\kappa^4}\cdot\left(\tfrac{x^3(1-\kappa)^2}{3} - \tfrac{x^4(5-7\kappa)}{24} + \tfrac{x^5}{40}\right)\\
&+\tfrac{1}{\sigma_\kappa^6}\cdot\left(\tfrac{x^4(1-\kappa)^3}{4} + \tfrac{x^5(-13+33\kappa-23\kappa^2)}{60} + \tfrac{x^6(7-11\kappa)}{144} - \tfrac{x^7}{336}\right)
\end{aligned}
$$

and thus the claim of part (iii).

Further to prove (v), we have

$$
\begin{aligned}
\tfrac{(1-\kappa)^3}{4} + \tfrac{t(-13+33\kappa-23\kappa^2)}{12} + \tfrac{t^2(7-11\kappa)}{24} - \tfrac{t^3}{48} \;\geq\;& \tfrac{(1-\kappa^3)}{4} + \tfrac{-13+33\kappa-23\kappa^2}{12} + \tfrac{7-11\kappa}{24} - \tfrac{1}{48}\\
=\;& \tfrac{9-34\kappa+52\kappa^2-48\kappa^3}{48}\\
\geq\;& \tfrac{9-17+13-6}{48} \;=\; -\tfrac{1}{48}
\end{aligned}
$$

since $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t(-13+33\kappa-23\kappa^2)}{12} + \frac{t^2(7-11\kappa)}{24}\right) \leq \frac{-13+33\kappa-23\kappa^2}{12} + \frac{7-11\kappa}{12} = \frac{-6+22\kappa-23\kappa^2}{12} \leq \frac{-6+22\kappa(1-\kappa)}{12} \leq \frac{-6+22/4}{12} = -\frac{1}{24} < 0$ in the first and $\frac{\mathrm{d}}{\mathrm{d}\kappa}(-34\kappa + 52\kappa^2 - 48\kappa^3) = -34 + 104\kappa - 144\kappa^3 \leq -34 + 104\kappa(1-\kappa) \leq -34 + \frac{104}{4} = -8 < 0$ in the second inequality.

Further with $\sigma_\kappa^2 \geq 2(1-\kappa) \geq 1$ we have

$$
(1-\kappa)^2 - \tfrac{t(5-7\kappa)}{6} + \tfrac{t^2}{8} \;\geq\; (1-\kappa)^2 - \tfrac{5-7\kappa}{6} + \tfrac{1}{8} \;=\; \tfrac{7-20\kappa+24\kappa}{24} \geq \tfrac{7-20\kappa(1-\kappa)}{24} \;\geq\; \tfrac{7-5}{24} \;=\; \tfrac{1}{12}
$$

since $\frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{t(5-7\kappa)}{6} + \frac{t^2}{8}\right) = -\frac{5-7\kappa}{6} + \frac{t}{4} \leq -\frac{5-7/2}{6} + \frac{1}{4} = 0$ in the first inequality.

The estimations above yield

$$\begin{aligned}
h(t) &\geq 1+\tfrac{t}{\sigma_\kappa^2}\cdot\big(1-\kappa-\tfrac{t}{2}\big)+\tfrac{t^2}{\sigma_\kappa^4}\cdot\big((1-\kappa)^2-\tfrac{t(5-7\kappa)}{6}+\tfrac{t^2}{8}\big)\\
&\quad+\tfrac{t^3}{\sigma_\kappa^6}\cdot\big((1-\kappa)^3+\tfrac{t(-13+33\kappa-23\kappa^2)}{12}+\tfrac{t^2(7-11\kappa)}{24}-\tfrac{t^3}{48}\big)\\
&\geq 1+\tfrac{t}{\sigma_\kappa^2}\cdot\big(1-\kappa-\tfrac{t}{2}\big)+\tfrac{t^2}{12\sigma_\kappa^4}-\tfrac{t^3}{48\sigma_\kappa^6}\big)\\
&\geq 1+\tfrac{t}{\sigma_\kappa^2}\cdot\big(1-\kappa-\tfrac{t}{2}\big)
\end{aligned}$$

and hence part (v) follows.

In this section we prove the estimations upwards with $k \geq 3$. First we can write $f(t)=\frac{k}{k-t}\cdot(1-\frac{t}{k})^k\cdot(1+\frac{t}{n-k})^{n-k}$ and with $n-k\geq k$ we have

$$\begin{aligned}
\log\big((1-\tfrac{t}{k})^k\cdot(1+\tfrac{t}{n-k})^{n-k}\big) &= -k\cdot\sum_{r=1}^{\infty}\tfrac{t^r}{rk^r}+(n-k)\cdot\sum_{r=1}^{\infty}\tfrac{(-1)^{r+1}t^r}{r(n-k)^r}\\
&= -\sum_{r=1}^{\infty}\tfrac{t^r}{r}\cdot\big(\tfrac{1}{k^{r-1}}+\tfrac{(-1)^r}{(n-k)^{r-1}}\big)\\
&\leq -\tfrac{t^2}{2k}-\tfrac{t^2}{2(n-k)}-\tfrac{t^3}{3k^2}+\tfrac{t^3}{3(n-k)^3}-\tfrac{t^4}{4k^4}-\tfrac{t^4}{4(n-k)^4}-\tfrac{t^5}{5k^4}+\tfrac{t^5}{5(n-k)^4}\\
&= -\tfrac{t^2}{2\sigma_\kappa^2}-\tfrac{t^3(1-2\kappa)}{3\sigma_\kappa^4}-\tfrac{t^4\cdot((1-\kappa)^3+\kappa^3)}{4\sigma_\kappa^6}-\tfrac{t^5\cdot((1-\kappa)^4-\kappa^4)}{5\sigma_\kappa^8}
\end{aligned}$$

and with $\mathrm{e}^{-y}\leq 1-y+\frac{y^2}{2}-\frac{y^3}{6}+\frac{y^4}{24}\leq 1-y+\frac{y^2}{2}-\frac{z^3}{6}+\frac{z^4}{24}$ if $0\leq z\leq y\leq 1$ we now have

$$\begin{aligned}
(1-\tfrac{t}{k})^k\cdot(1+\tfrac{t}{n-k})^{n-k} &\leq 1-\tfrac{t^2}{2\sigma_\kappa^2}-\tfrac{t^3(1-2\kappa)}{3\sigma_\kappa^4}-\tfrac{t^4\cdot((1-\kappa)^3+\kappa^3)}{4\sigma_\kappa^6}-\tfrac{t^5\cdot((1-\kappa)^4-\kappa^4)}{5\sigma_\kappa^8}\\
&\quad+\tfrac{1}{2}\cdot\big(\tfrac{t^2}{2\sigma_\kappa^2}+\tfrac{t^3(1-2\kappa)}{3\sigma_\kappa^4}+\tfrac{t^4\cdot(19(1-\kappa)^3+11\kappa^3)}{60\sigma_\kappa^6}\big)^2-\tfrac{t^6}{6\cdot 8\sigma_\kappa^6}+\tfrac{t^8}{24\cdot 16\sigma_\kappa^8}\\
&= 1-\tfrac{t^2}{2\sigma_\kappa^2}-\tfrac{1}{\sigma_\kappa^4}\cdot\big(\tfrac{t^3(1-2\kappa)}{3}-\tfrac{t^4}{8}\big)-\tfrac{1}{\sigma_\kappa^6}\cdot\big(\tfrac{t^4((1-\kappa)^3+\kappa^3)}{4}-\tfrac{t^5(1-2\kappa)}{6}+\tfrac{t^6}{48}\big)\\
&\quad-\tfrac{1}{\sigma_\kappa^8}\cdot\big(\tfrac{t^5\cdot((1-\kappa)^4-\kappa^4)}{5}-\tfrac{t^6(1-2\kappa)^2}{18}-\tfrac{t^6\cdot(19(1-\kappa)^3+11\kappa^3)}{120}-\tfrac{t^8}{768}\\
&\qquad-\tfrac{t^7(1-2\kappa)(19(1-\kappa)^3+11\kappa^3)}{180a\cdot(1-\kappa)}-\tfrac{t^8\cdot(19(1-\kappa)^3+11\kappa^3)^2}{2\cdot 60^2k^2\cdot(1-\kappa)^2}\big)\\
&\leq 1-\tfrac{t^2}{2\sigma_\kappa^2}-\tfrac{1}{\sigma_\kappa^4}\cdot\big(\tfrac{t^3(1-2\kappa)}{3}-\tfrac{t^4}{8}\big)-\tfrac{1}{\sigma_\kappa^6}\cdot\big(\tfrac{t^4((1-\kappa)^3+\kappa^3)}{4}-\tfrac{t^5(1-2\kappa)}{6}+\tfrac{t^6}{48}\big)\\
&\quad-\tfrac{1}{\sigma_\kappa^8}\cdot\big(\tfrac{t^5\cdot((1-\kappa)^4-\kappa^4)}{5}-\tfrac{t^6(1-2\kappa)^2}{18}-\tfrac{t^6\cdot(19(1-\kappa)^3+11\kappa^3)}{120}-\tfrac{t^8}{768}\\
&\qquad-\tfrac{19t^7(1-2\kappa)^2}{540}-\tfrac{19\cdot t^8\cdot(19(1-\kappa)^3+11\kappa^3)}{64\,800}\big).
\end{aligned}$$

Thus from above, $\frac{k}{k-t} = \sum\limits_{r=0}^{m} (\frac{t}{k})^r + \frac{t^{m+1}}{k^m(k-t)}$ and $\frac{t}{k} = \frac{t(1-\kappa)}{\sigma_\kappa^2}$ follows

$$\begin{aligned}
h(t) \;\leq\;& 1 + \tfrac{t}{k} + \tfrac{t^2}{k^2} + \tfrac{t^3}{k^3} + \tfrac{t^4}{k^3(k-t)} - (1 + \tfrac{t}{k} + \tfrac{t^2}{k^2} + \tfrac{t^3}{k^2(k-t)}) \cdot \tfrac{t^2}{2\sigma_\kappa^2} \\
& -(1 + \tfrac{t}{k} + \tfrac{t^2}{k(k-t)}) \cdot \tfrac{1}{\sigma_\kappa^4} \cdot (\tfrac{t^3(1-2\kappa)}{3} - \tfrac{t^4}{8}) \\
& -(1 + \tfrac{t}{(k-t)}) \cdot \tfrac{1}{\sigma_\kappa^6} \cdot (\tfrac{t^4((1-\kappa)^3+\kappa^3)}{4} - \tfrac{t^5(1-2\kappa)}{6} + \tfrac{t^6}{48}) - \tfrac{k}{\sigma_\kappa^8(k-t)} \cdot \Big(\tfrac{t^5((1-\kappa)^4-\kappa^4)}{5} - \tfrac{t^6(1-2\kappa)^2}{18} \\
& \quad -\tfrac{t^6(19(1-\kappa)^3+11\kappa^3)}{120} - \tfrac{t^8}{768} - \tfrac{19t^7(1-2\kappa)^2}{540} - \tfrac{19t^8\cdot(19(1-\kappa)^3+11\kappa^3)}{64\,800}\Big) \\
=\;& 1 + \tfrac{1}{\sigma_\kappa^2} \cdot \big(t(1-\kappa) - \tfrac{t^2}{2}\big) + \tfrac{1}{\sigma_\kappa^4} \cdot \big(t^2(1-\kappa)^2 - \tfrac{t^3(1-\kappa)}{2} - \tfrac{t^3(1-2\kappa)}{3} + \tfrac{t^4}{8}\big) \\
& + \tfrac{1}{\sigma_\kappa^6} \cdot \big(t^3(1-\kappa)^3 - \tfrac{t^4(1-\kappa)^2}{2} - \tfrac{t^4(1-2\kappa)(1-\kappa)}{3} + \tfrac{t^5(1-\kappa)}{8} - \tfrac{t^4((1-\kappa)^3+\kappa^3)}{4} + \tfrac{t^5(1-2\kappa)}{6} - \tfrac{t^6}{48}\big) \\
& + \tfrac{k}{\sigma_\kappa^8(k-t)} \cdot \Big(t^4(1-\kappa)^4 - \tfrac{t^5(1-\kappa)^3}{2} - \tfrac{t^5(1-2\kappa)(1-\kappa)}{3} + \tfrac{t^6(1-\kappa)^2}{8} - \tfrac{t^5((1-\kappa)^3+\kappa^3)(1-\kappa)}{4} \\
& \qquad + \tfrac{t^6(1-2\kappa)(1-\kappa)}{6} - \tfrac{t^7(1-\kappa)}{48} - \tfrac{t^5((1-\kappa)^4-\kappa^4)}{5} + \tfrac{t^6(1-2\kappa)^2}{18} \\
& \qquad + \tfrac{t^6(19(1-\kappa)^3+11\kappa^3)}{120} + \tfrac{t^8}{768} + \tfrac{19t^7(1-2\kappa)^2}{540} + \tfrac{19t^8\cdot(19(1-\kappa)^3+11\kappa^3)}{64\,800}\Big) \\
=\;& 1 + \tfrac{1}{\sigma_\kappa^2} \cdot \big(t(1-\kappa) - \tfrac{t^2}{2}\big) + \tfrac{1}{\sigma_\kappa^4} \cdot \big(t^2(1-\kappa)^2 - \tfrac{t^3(5-7\kappa)}{6} + \tfrac{t^4}{8}\big) \\
& + \tfrac{1}{\sigma_\kappa^6} \cdot \big(t^3(1-\kappa)^3 + \tfrac{t^4(-13+33\kappa-23\kappa^2)}{12} + \tfrac{t^5(7-11\kappa)}{24} - \tfrac{t^6}{48}\big) \\
& + \tfrac{k}{\sigma_\kappa^8(k-t)} \cdot \Big(t^4(1-\kappa)^4 + \tfrac{t^5(-77+258\kappa-292\kappa^2+123\kappa^3)}{60} + \tfrac{t^6(182-521\kappa+416\kappa^2-24\kappa^3)}{360} \\
& \qquad + \tfrac{t^7(31-259\kappa+304\kappa^2)}{2160} + \tfrac{t^8}{768} + \tfrac{19t^8\cdot(19(1-\kappa)^3+11\kappa^3)}{64\,800}\Big) \\
\leq\;& 1 + \tfrac{1}{\sigma_\kappa^2} \cdot \big(t(1-\kappa) - \tfrac{t^2}{2}\big) + \tfrac{1}{\sigma_\kappa^4} \cdot \big(t^2(1-\kappa)^2 - \tfrac{t^3(5-7\kappa)}{6} + \tfrac{t^4}{8}\big) \\
& + \tfrac{1}{\sigma_\kappa^6} \cdot \big(t^3(1-\kappa)^3 + \tfrac{t^4(-13+33\kappa-23\kappa^2)}{12} + \tfrac{t^5(7-11\kappa)}{24} - \tfrac{t^6}{48}\big) \\
& + \tfrac{3}{2\sigma_\kappa^8} \cdot \Big(t^4(1-\kappa)^4 + \tfrac{t^5(-77+258\kappa-292\kappa^2+123\kappa^3)}{60} + \tfrac{t^6(182-521\kappa+416\kappa^2-24\kappa^3)}{360} \\
& \qquad + \tfrac{t^7(31-259\kappa+304\kappa^2)}{2160} + \tfrac{t^8}{768} + \tfrac{19t^8\cdot(19(1-\kappa)^3+11\kappa^3)}{64\,800}\Big)
\end{aligned}$$

since we know by the estimations downwards that the interior of the bracket of $\frac{1}{\sigma_\kappa^8}$ must be positive, because else we would have a contradiction, and thus we can use the estimate $\frac{k}{k-t} \leq \frac{k}{k-1} \leq \frac{3}{2}$.

If we now set $t = 1$ for the special case, we get

$$\begin{aligned} h(1) &\le 1 + \tfrac{1}{\sigma_\kappa^2}\cdot\left(1-\kappa-\tfrac{1}{2}\right) + \tfrac{1}{\sigma_\kappa^4}\cdot\left((1-\kappa)^2 - \tfrac{5-7\kappa}{6} + \tfrac{1}{8}\right) \\ &\quad + \tfrac{1}{\sigma_\kappa^6}\cdot\left((1-\kappa)^3 + \tfrac{-13+33\kappa-23\kappa^2}{12} + \tfrac{7-11\kappa}{24} - \tfrac{1}{48}\right) + \tfrac{k}{\sigma_\kappa^8(k-1)}\cdot\Big((1-\kappa)^4 \\ &\quad + \tfrac{-77+258\kappa-292\kappa^2+123\kappa^3}{60} + \tfrac{182-521\kappa+416\kappa^2-24\kappa^3}{360} + \tfrac{31-259\kappa+304\kappa^2}{2160} + \tfrac{1}{768} + \tfrac{19\cdot(19(1-\kappa)^3+11\kappa^3)}{64\,800}\Big) \\ &= 1 + \tfrac{1-2\kappa}{2\sigma_\kappa^2} + \tfrac{7-20\kappa+24\kappa^2}{24\sigma_\kappa^4} + \tfrac{9-14\kappa+52\kappa^2-48\kappa^3}{48\sigma_\kappa^6} \\ &\quad + \tfrac{k}{\sigma_\kappa^8(k-1)}\cdot\left(\tfrac{126\,203}{518\,400} - \tfrac{27\,731\kappa}{21\,600} + \tfrac{52\,841\kappa^2}{21\,600} - \tfrac{8177\kappa^3}{4050} + \kappa^4\right) \\ &\le 1 + \tfrac{1-2\kappa}{2\sigma_\kappa^2} + \tfrac{7-20\kappa+24\kappa^2}{24\sigma_\kappa^4} + \tfrac{9-14\kappa+52\kappa^2-48\kappa^3}{48\sigma_\kappa^6} + \tfrac{k}{\sigma_\kappa^8(k-1)}\cdot\tfrac{5-8\kappa}{40} \\ &\le 1 + \tfrac{1-2\kappa}{2\sigma_\kappa^2} + \tfrac{7-20\kappa+24\kappa^2}{24\sigma_\kappa^4} + \tfrac{9-14\kappa+52\kappa^2-48\kappa^3}{48\sigma_\kappa^6} + \tfrac{1}{\sigma_\kappa^8}\cdot\tfrac{15-24\kappa}{80} \end{aligned}$$

since $i''(\kappa) := \left(\frac{126\,203}{518\,400} - \frac{27\,731\kappa}{21\,600} + \frac{52\,841\kappa^2}{21\,600} - \frac{8177\kappa^3}{4050} + \kappa^4\right)'' = \frac{52\,841}{10\,800} - \frac{8177\kappa}{675} + 12\kappa^2 = \frac{52\,841}{10\,800} - 12\kappa(1-\kappa) - \frac{77\kappa}{675} \ge \frac{52\,841}{10\,800} - 3 - \frac{77}{1350} = \frac{793}{432} > 0$ and $i(\frac{1}{6}) = \frac{1\,243\,639}{13\,996\,800} \le \frac{11}{120} = \frac{5-8/6}{40}$ as well as $i(\frac{1}{2}) = \frac{803}{34\,560} \le \frac{1}{40} = \frac{5-8/2}{40}$ in the penultimate and $\frac{k}{k-1} \le \frac{3}{2}$ in the ultimate inequality and hence part (ii).

We now go a step back and consider $H(x) := \int_0^x h(t)\mathrm{d}t$. To this integrating the right hand side of the inequality of $f(t)$ above yields

$$\begin{aligned} H(x) &\le x + \tfrac{1}{\sigma_\kappa^2}\cdot\left(\tfrac{x^2(1-\kappa)}{2} - \tfrac{x^3}{6}\right) + \tfrac{1}{\sigma_\kappa^4}\cdot\left(\tfrac{x^3(1-\kappa)^2}{3} - \tfrac{x^4(5-7\kappa)}{24} + \tfrac{x^5}{40}\right) + \tfrac{1}{\sigma_\kappa^6}\cdot\Big(\tfrac{x^4(1-\kappa)^3}{4} \\ &\quad + \tfrac{x^5(-13+33\kappa-23\kappa^2)}{60} + \tfrac{x^6(7-11\kappa)}{144} - \tfrac{x^7}{336}\Big) + \tfrac{3}{2\sigma_\kappa^8}\cdot\Big(\tfrac{x^5(1-\kappa)^4}{5} + \tfrac{x^6(-77+258\kappa-292\kappa^2+123\kappa^3)}{360} \\ &\quad + \tfrac{x^7(182-521\kappa+416\kappa^2-24\kappa^3)}{7\cdot 360} + \tfrac{x^8(31-259\kappa+304\kappa^2)}{8\cdot 2160} + \tfrac{x^9}{9\cdot 768} + \tfrac{19x^9\cdot(19(1-\kappa)^3+11\kappa^3)}{9\cdot 64\,800}\Big). \end{aligned}$$

If we set $i_2(x) := \frac{-77+258\kappa-292\kappa^2+123\kappa^3}{360} + \frac{x(182-521\kappa+416\kappa^2-24\kappa^3)}{7\cdot 360} + \frac{x^2(31-259\kappa+304\kappa^2)}{8\cdot 2160} + \frac{x^3}{9\cdot 768} + \frac{19x^3\cdot(19(1-\kappa)^3+11\kappa^3)}{9\cdot 64800}$ we can find

$$\begin{aligned} i_2'(x) &\ge \tfrac{182-521\kappa+416\kappa^2-24\kappa^3}{7\cdot 360} + \tfrac{2x(31-259\kappa+304\kappa^2)}{8\cdot 2160} \\ &\ge \tfrac{182-521\kappa+416\kappa^2-24\kappa^3}{7\cdot 360} + \tfrac{31-259\kappa+304\kappa^2}{4\cdot 2160} \quad =: \; j(\kappa) \end{aligned}$$

since $31 - 259\kappa + 304\kappa^2 \le 31 - \frac{259}{6} + \frac{304}{36} = -\frac{67}{18} < 0$ in the last inequality.

Now $j''(\kappa) \ge j''(\frac{1}{2}) = \frac{416}{7\cdot 180} - \frac{24}{7\cdot 120} + \frac{304}{2\cdot 2160} = \frac{703}{1890}$ yields $j'(\kappa) \le h'(\frac{1}{2}) = \frac{-521+416-18}{7\cdot 360} + \frac{-259+304}{4\cdot 2160} = -\frac{-293}{6720} < 0$ and hence $j(\kappa) \ge j(\frac{1}{2}) = \frac{17}{2688} > 0$ holds.

Thus we have $i_2'(x) \geq 0$ and hence

$$
\begin{aligned}
i_2(x) &\leq \tfrac{-77+258\kappa-292\kappa^2+123\kappa^3}{360} + \tfrac{182-521\kappa+416\kappa^2-24\kappa^3}{7\cdot 360} + \tfrac{31-259\kappa+304\kappa^2}{8\cdot 2160} + \tfrac{1}{9\cdot 768} + \tfrac{19\cdot(19(1-\kappa)^3+11\kappa^3)}{9\cdot 64\,800} \\
&= -\tfrac{649\,027}{4\,665\,600} + \tfrac{2\,683\,907\kappa}{5\,443\,200} - \tfrac{852\,653\kappa^2}{1\,360\,800} + \tfrac{338\,719\kappa^3}{1\,020\,600} \\
&\leq -0.139 + 0.494\kappa - 0.625\kappa^2 + 0.332\kappa^3 \;\leq\; -0.14 + 0.5\kappa - 0.625\kappa^2 + \tfrac{1}{3}\cdot\kappa^3 \\
&= -\tfrac{7}{50} + \tfrac{\kappa}{2} - \tfrac{5\kappa^2}{8} + \tfrac{\kappa^3}{3} \;\leq\; -\tfrac{7}{50} + \tfrac{118\kappa}{225} - \tfrac{283\kappa^2}{450} + \tfrac{11\kappa^3}{45} \;=\; -\tfrac{(1-\kappa)^2(63-110\kappa)}{450}
\end{aligned}
$$

since $0.001 - 0.006\kappa \leq 0$ in the penultimate and $\frac{118\kappa}{225} - \frac{283\kappa^2}{450} + \frac{11\kappa^3}{45} - (\frac{\kappa}{2} - \frac{5\kappa^2}{8} + \frac{\kappa^3}{3}) = \kappa\cdot\frac{44-7\kappa-160\kappa^2}{1800} \geq \kappa\cdot\frac{44-7/2-40}{1800} = \frac{\kappa}{3600} > 0$ in the ultimate inequality.

Thus it follows

$$
\begin{aligned}
H(x) \;\leq\; & x + \tfrac{1}{\sigma_\kappa^2}\cdot\big(\tfrac{x^2(1-\kappa)}{2} - \tfrac{x^3}{6}\big) + \tfrac{1}{\sigma_\kappa^4}\cdot\big(\tfrac{x^3(1-\kappa)^2}{3} - \tfrac{x^4(5-7\kappa)}{24} + \tfrac{x^5}{40}\big) + \tfrac{1}{\sigma_\kappa^6}\cdot\big(\tfrac{x^4(1-\kappa)^3}{4} \\
& + \tfrac{x^5(-13+33\kappa-23\kappa^2)}{60} + \tfrac{x^6(7-11\kappa)}{144} - \tfrac{x^7}{336}\big) + \tfrac{3}{2\sigma_\kappa^8}\cdot\big(\tfrac{x^5(1-\kappa)^4}{5} - \tfrac{x^6(1-\kappa)^2(63-110\kappa)}{450}\big)
\end{aligned}
$$

and hence part (iv). □

Lemma 11.6. *Let $n, k \in \mathbb{N}$ with $\kappa := \frac{k}{n} \in [\frac{1}{6}, \frac{1}{2}]$, and for $x \in [0,1]$ let*

$$\mathcal{F}(x) \quad := \quad \tfrac{F_{n,(k-x)/n}(k-1) - F_{n,\kappa}(k-1)}{f_{n,\kappa}(k)}.$$

(i) If $k \geq 2$ we have

$$\begin{aligned} \mathcal{F}(x) \quad &\geq \quad x + \tfrac{1}{\sigma_\kappa^2} \cdot \big(\tfrac{x^2(1-\kappa)}{2} - \tfrac{x^3}{6}\big) + \tfrac{1}{\sigma_\kappa^4} \cdot \big(\tfrac{x^3(1-\kappa)^2}{3} - \tfrac{x^4(5-7\kappa)}{24} + \tfrac{x^5}{40}\big) \\ &\qquad + \tfrac{1}{\sigma_\kappa^6} \cdot \big(\tfrac{x^4(1-\kappa)^3}{4} + \tfrac{x^5(-13+33\kappa-23\kappa^2)}{60} + \tfrac{x^6(7-11\kappa)}{144} - \tfrac{x^7}{336}\big) \\ &=: \quad \mathcal{F}_6(x). \end{aligned}$$

(ii) If $k \geq 3$ we have

$$\mathcal{F}(x) \quad \leq \quad \mathcal{F}_6(x) + \tfrac{3}{2\sigma_\kappa^8} \cdot \big(\tfrac{x^5(1-\kappa)^4}{5} - \tfrac{x^6(1-\kappa)^2(63-110\kappa)}{450}\big).$$

If additionally $\kappa \in [\frac{1}{6}, \frac{1}{3}]$, we also have

$$\mathcal{F}(x) \quad \leq \quad x + \tfrac{1}{\sigma_\kappa^2} \cdot \big(\tfrac{x^2(1-\kappa)}{2} - \tfrac{x^3}{6}\big) + \tfrac{1}{\sigma_\kappa^4} \cdot \big(\tfrac{x^3(1-\kappa)^2}{3} - \tfrac{x^4(1-\kappa)}{16}\big).$$

(iii) In the special case $x = 1$ we have if $k \geq 2$

$$\mathcal{F}(1) \quad \geq \quad 1 + \tfrac{2-3\kappa}{6\sigma_\kappa^2} + \tfrac{18-45\kappa+40\kappa^2}{120\sigma_\kappa^4} + \tfrac{398-1393\kappa+1848\kappa^2-1260\kappa^3}{5040\sigma_\kappa^6} \quad =: \ \mathcal{F}_6.$$

(iv) In the special case $x = 1$ we have if $k \geq 3$

$$\mathcal{F}(1) \quad \leq \quad \mathcal{F}_6 + \tfrac{(1-\kappa)(27-97\kappa+160\kappa^2-90\kappa^3)}{300\sigma_\kappa^8}.$$

(v) If $x \in [0, \frac{1}{2}]$ and if $k \geq 1$ we have

$$\mathcal{F}(1-x) - \mathcal{F}(x) \quad \geq \quad (1-2x) \cdot (1 + \tfrac{2-3\kappa}{6\sigma_\kappa^2}).$$

(vi) We have with no further restrictions

$$\mathcal{F}(x) \quad \geq \quad x + \tfrac{3x^2(1-\kappa)-x^3}{6\sigma_\kappa^2} \quad \geq \quad x.$$

Proof. on (i): Using $\frac{\mathrm{d}}{\mathrm{d}p}F_{n,p}(k) = -nf_{n-1,p}(k)$ the fundamental theorem of calculus yields

$$\begin{aligned}
\frac{F_{n,\frac{k-x}{n}}(k-1)-F_{n,\kappa}(k-1)}{f_{n,\kappa}(k)} &= \int\limits_{\kappa}^{(k-x)/n} -n\cdot\frac{f_{n-1,p}(k-1)}{f_{n,\kappa}(k)}\mathrm{d}p \;=\; n\cdot\int\limits_{\kappa-x/n}^{\kappa}\frac{f_{n-1,p}(k-1)}{f_{n,\kappa}(k)}\mathrm{d}p \\
&= n\cdot\int\limits_{\kappa-x/n}^{\kappa}(\tfrac{p}{\kappa})^{k-1}\cdot(\tfrac{1-p}{1-\kappa})^{n-k}\mathrm{d}p \;=\; n\cdot\int\limits_{-x/n}^{0}(\tfrac{\kappa+t}{\kappa})^{k-1}\cdot(\tfrac{1-\kappa-t}{1-\kappa})^{n-k}\mathrm{d}t \\
&= \int\limits_{0}^{x}(1-\tfrac{t}{k})^{k-1}\cdot(1+\tfrac{t}{1-\kappa})^{n-k}\mathrm{d}t.
\end{aligned}$$

With this representation we now get

$$\mathcal{F}(x) \;=\; \int\limits_{0}^{x}(1-\tfrac{t}{k})^{k-1}\cdot(1+\tfrac{t}{n-k})^{n-k}\mathrm{d}t \tag{30}$$

and then Lemma 11.5 (iii) yields the claim in (i).

on (ii): Analogous to (i) Lemma 11.5 (iv) yields the first statement in (ii). For the second statement let $\kappa\in[\frac{1}{6},\frac{1}{3}]$. Then with $k\geq 3$, which implies $\sigma_\kappa^2\geq 3(1-\kappa)$, we have

$$\begin{aligned}
&\tfrac{x^4(1-\kappa)}{16\sigma_\kappa^4}+\tfrac{1}{\sigma_\kappa^4}\cdot\big(-\tfrac{x^4(5-7\kappa)}{24}+\tfrac{x^5}{40}\big)+\tfrac{1}{\sigma_\kappa^6}\cdot\big(\tfrac{x^4(1-\kappa)^3}{4}+\tfrac{x^5(-13+33\kappa-23\kappa^2)}{60}+\tfrac{x^6(7-11\kappa)}{144}-\tfrac{x^7}{336}\big)\\
&+\tfrac{3}{2\sigma_\kappa^8}\cdot\big(\tfrac{x^5(1-\kappa)^4}{5}-\tfrac{x^6(1-\kappa)^2(63-110\kappa)}{450}\big)\\
\leq\;&\tfrac{3(1-\kappa)}{\sigma_\kappa^6}\cdot\big(\tfrac{x^4(1-\kappa)}{16}-\tfrac{x^4(5-7\kappa)}{24}+\tfrac{x^5}{40}\big)+\tfrac{1}{\sigma_\kappa^6}\cdot\big(\tfrac{x^4(1-\kappa)^3}{4}+\tfrac{x^5(-13+33\kappa-23\kappa^2)}{60}+\tfrac{x^6(7-11\kappa)}{144}\big)\\
&+\tfrac{3}{2\cdot3(1-\kappa)}\cdot\big(\tfrac{x^5(1-\kappa)^4}{5}-\tfrac{x^6(1-\kappa)^2(63-110\kappa)}{450}\big)\\
=\;&\tfrac{1}{\sigma_\kappa^6}\cdot\Big(x^4\cdot\tfrac{-3+6\kappa+\kappa^2-4\kappa^3}{16}+x^5\cdot\tfrac{-5+21\kappa-10\kappa^2-12\kappa^3}{120}+x^6\cdot\tfrac{-77+417\kappa-440\kappa^2}{3600}\Big)\\
\leq\;&\tfrac{1}{\sigma_\kappa^6}\cdot\Big(x^5\cdot\big(\tfrac{-3+6\kappa+\kappa^2-4\kappa^3}{16}+\tfrac{-5+21\kappa-10\kappa^2-12\kappa^3}{120}\big)+x^6\cdot\tfrac{-77+417\kappa-440\kappa^2}{3600}\Big)\\
=\;&\tfrac{1}{\sigma_\kappa^6}\cdot\Big(x^5\cdot\tfrac{-55+132\kappa-5\kappa^2-84\kappa^3}{240}+x^6\cdot\tfrac{-77+417\kappa-440\kappa^2}{3600}\Big)\\
\leq\;&\tfrac{x^6}{\sigma_\kappa^6}\cdot\big(\tfrac{-55+132\kappa-5\kappa^2-84\kappa^3}{240}+\tfrac{-77+417\kappa-440\kappa^2}{3600}\big)\;=\;\tfrac{x^6}{\sigma_\kappa^6}\cdot\tfrac{-902+2397\kappa-515\kappa^2-1260\kappa^3}{3600}\\
\leq\;&\tfrac{x^6}{\sigma_\kappa^6}\cdot\tfrac{-902+2397/3}{3600}\;=\;-\tfrac{103x^6}{3600\sigma_\kappa^6}\;<\;0
\end{aligned}$$

since $\frac{x^4(1-\kappa)}{16}-\frac{x^4(5-7\kappa)}{24}+\frac{x^5}{40}\leq x^4\cdot(\frac{1-\kappa}{16}-\frac{5-7\kappa}{24}+\frac{1}{40})=x^4\cdot\frac{-29+55\kappa}{240}\leq x^4\cdot\frac{-29+55/3}{240}<0$ and $\frac{x^5(1-\kappa)^4}{5}-\frac{x^6(1-\kappa)^2(63-110\kappa)}{450}\geq x^5\cdot(\frac{(1-\kappa)^4}{5}-\frac{(1-\kappa)^2(63-110\kappa)}{450})=\frac{x^5(1-\kappa)^2}{5}\cdot\frac{27-70\kappa+90\kappa^2}{90}\geq\frac{x^5(1-\kappa)^2}{5}\cdot\frac{27-70\kappa(1-\kappa)}{90}\geq\frac{x^5(1-\kappa)^2}{5}\cdot\frac{27-70\cdot 2/9}{90}>0$ in the first inequality.

on (iii): Lemma 11.5 (iii) used with $x = 1$ yields

$$\begin{aligned}\mathcal{F}(x) &\geq 1 + \tfrac{1}{\sigma_\kappa^2}\cdot\big(\tfrac{1-\kappa}{2} - \tfrac{1}{6}\big) + \tfrac{1}{\sigma_\kappa^4}\cdot\big(\tfrac{(1-\kappa)^2}{3} - \tfrac{5-7\kappa}{24} + \tfrac{1}{40}\big)\\ &\quad + \tfrac{1}{\sigma_\kappa^6}\cdot\big(\tfrac{(1-\kappa)^3}{4} + \tfrac{-13+33\kappa-23\kappa^2}{60} + \tfrac{7-11\kappa}{144} - \tfrac{1}{336}\big)\\ &= 1 + \tfrac{2-3\kappa}{6\sigma_\kappa^2} + \tfrac{18-45\kappa+40\kappa^2}{120\sigma_\kappa^4} + \tfrac{398-1393\kappa+1848\kappa^2-1260\kappa^3}{5040\sigma_\kappa^6}.\end{aligned}$$

on (iv): Lemma 11.5 (iv) used with $x = 1$ yields

$$\begin{aligned}\mathcal{F}(x) &\leq 1 + \tfrac{1}{\sigma_\kappa^2}\cdot\big(\tfrac{1-\kappa}{2} - \tfrac{1}{6}\big) + \tfrac{1}{\sigma_\kappa^4}\cdot\big(\tfrac{(1-\kappa)^2}{3} - \tfrac{5-7\kappa}{24} + \tfrac{1}{40}\big) + \tfrac{1}{\sigma_\kappa^6}\cdot\big(\tfrac{(1-\kappa)^3}{4}\\ &\quad + \tfrac{-13+33\kappa-23\kappa^2}{60} + \tfrac{7-11\kappa}{144} - \tfrac{1}{336}\big) + \tfrac{3}{2\sigma_\kappa^8}\cdot\big(\tfrac{(1-\kappa)^4}{5} - \tfrac{(1-\kappa)^2(63-110\kappa)}{450}\big)\\ &= 1 + \tfrac{2-3\kappa}{6\sigma_\kappa^2} + \tfrac{18-45\kappa+40\kappa^2}{120\sigma_\kappa^4} + \tfrac{398-1393\kappa+1848\kappa^2-1260\kappa^3}{5040\sigma_\kappa^6} + \tfrac{(1-\kappa)\cdot(27-97\kappa+160\kappa^2-90\kappa^3)}{300\sigma_\kappa^8}.\end{aligned}$$

on (v): Due to Lemma 11.5 (v) we have if $k \geq 2$ that $(1 - \frac{t}{a})^{k-1}\cdot(1 + \frac{t}{n-k})^{n-k} \geq 1 + \frac{t}{\sigma_\kappa^2}\cdot(1 - \kappa - \frac{t}{2})$ and hence if $x \in [0, \frac{1}{2}]$

$$\begin{aligned}\mathcal{F}(1-x) - \mathcal{F}(x) &= \int_x^{1-x} (1 - \tfrac{t}{a})^{k-1}\cdot(1 + \tfrac{t}{n-k})^{n-k}\mathrm{d}t \;\geq\; \int_x^{1-x} 1 + \tfrac{t}{\sigma_\kappa^2}\cdot(1 - \kappa - \tfrac{t}{2})\mathrm{d}t\\ &= \big[t + \tfrac{1}{\sigma_\kappa^2}\cdot(\tfrac{t^2(1-\kappa)}{2} - \tfrac{t^3}{6})\big]_x^{1-x}\\ &= 1 - 2x + \tfrac{1}{\sigma_\kappa^2}\cdot(\tfrac{(1-2x)\cdot(1-\kappa)}{2} - \tfrac{1-3x+3x^2+2x^3}{6})\\ &= (1-2x)\cdot(1 + \tfrac{1}{\sigma_\kappa^2}\cdot(\tfrac{1-\kappa}{2} - \tfrac{1-x+x^2}{6}))\\ &\geq (1-2x)\cdot(1 + \tfrac{1}{\sigma_\kappa^2}\cdot(\tfrac{1-\kappa}{2} - \tfrac{1}{6})) \;=\; (1-2x)\cdot(1 + \tfrac{2-3\kappa}{6\sigma_\kappa^2}).\end{aligned}$$

If $k = 1$ we have $(1 - \frac{t}{a})^{k-1}\cdot(1 + \frac{t}{n-k})^{n-k} = (1 + \frac{t}{n-1})^{n-1} \geq 1 + t = 1 + \frac{t(1-\kappa)}{\sigma_\kappa^2} \geq 1 + \frac{t}{\sigma_\kappa^2}\cdot(1 - \kappa - \frac{t}{2})$ and hence due to the above (v) also in this case.

on (vi): The proof of (v) above yields

$$\mathcal{F}(x) \;\geq\; \int_0^x 1 + \tfrac{t}{\sigma_\kappa^2}\cdot(1 - \kappa - \tfrac{t}{2})\mathrm{d}t \;=\; x + \tfrac{1}{\sigma_\kappa^2}\cdot(\tfrac{x^2(1-\kappa)}{2} - \tfrac{x^3}{6}) \;=\; x + \tfrac{3x^2(1-\kappa)-x^3}{6\sigma_\kappa^2}.$$

□

Lemma 11.7. *Let $n, k \in \mathbb{N}$ with $\kappa := \frac{k}{n} \in [\frac{1}{6}, \frac{1}{2}]$.*

(i) If $k \geq 2$, then we have

$$\frac{f_{n,\frac{k-1}{n}}(k-1)}{f_{n,\kappa}(k)} \quad \geq \quad 1 + \frac{1-2\kappa}{2\sigma_\kappa^2} + \frac{7-20\kappa+24\kappa^2}{24\sigma_\kappa^4} + \frac{13-30\kappa}{96\sigma_\kappa^6}.$$

(ii) If $k \geq 3$, then we have

$$\frac{f_{n,\frac{k-1}{n}}(k-1)}{f_{n,\kappa}(k)} \quad \leq \quad 1 + \frac{1-2\kappa}{2\sigma_\kappa^2} + \frac{7-20\kappa+24\kappa^2}{24\sigma_\kappa^4} + \frac{60-179\kappa+251\kappa^2-250\kappa^3}{240\sigma_\kappa^6}.$$

(iii) If $k \geq 3$ and $x \in [0, 1]$, then we have

$$\frac{f_{n,\frac{k-x}{n}}(k-1)}{f_{n,\kappa}(k)} \quad \leq \quad 1 + \frac{1}{\sigma_\kappa^2} \cdot \left(\frac{1-2\kappa}{2} + \frac{13}{135} - \frac{(1-x)^2}{2} + \frac{(1-x)^4}{12}\right).$$

Proof. on (i): Due to Lemma 11.5 part (i) in the first inequality we have

$$\begin{aligned}
\frac{f_{n,\frac{k-1}{n}}(k-1)}{f_{n,\kappa}(k)} &= \frac{k}{n-k+1} \cdot \left(\frac{k-1}{k}\right)^{k-1} \cdot \frac{n}{k} \cdot \left(\frac{n-k+1}{n-k}\right)^{n-k} \cdot \frac{n-k+1}{n} = \left(1 - \frac{1}{k}\right)^{k-1} \cdot \left(1 + \frac{1}{n-k}\right)^{n-k} \\
&\geq 1 + \frac{1-2\kappa}{2\sigma_\kappa^2} + \frac{7-20\kappa+24\kappa^2}{24\sigma_\kappa^4} + \frac{9-34\kappa+52\kappa^2-48\kappa^3}{48\sigma_\kappa^6} \\
&\geq 1 + \frac{1-2\kappa}{2\sigma_\kappa^2} + \frac{7-20\kappa+24\kappa^2}{24\sigma_\kappa^4} + \frac{13-30\kappa}{96\sigma_\kappa^6},
\end{aligned}$$

where in the last inequality we used that for

$$h(\kappa) := 9 - 34\kappa + 52\kappa^2 - 48\kappa^3 - \frac{13}{2} + 15\kappa$$

we have $h'(\kappa) = -19 + 104\kappa - 144\kappa^2 \leq -19 + \frac{104^2}{4 \cdot 144} = -19 + \frac{169}{9} = -\frac{2}{9} < 0$ and $h(\kappa) \geq h(\frac{1}{2}) = 9 - 17 + 13 - 6 - 13/2 + 15/2 = 0$ and hence $h(\kappa) \geq 0$.

on (ii): Due to Lemma 11.5, this time part (ii), we have analogous to the above

$$\begin{aligned}
\frac{f_{n,\frac{k-1}{n}}(k-1)}{f_{n,\kappa}(k)} &= \left(1 - \frac{1}{k}\right)^{k-1} \cdot \left(1 + \frac{1}{n-k}\right)^{n-k} \\
&\leq 1 + \frac{1-2\kappa}{2\sigma_\kappa^2} + \frac{7-20\kappa+24\kappa^2}{24\sigma_\kappa^4} + \frac{9-34\kappa+52\kappa^2-48\kappa^3}{48\sigma_\kappa^6} + \frac{15-24\kappa}{80\sigma_\kappa^8} \\
&\leq 1 + \frac{1-2\kappa}{2\sigma_\kappa^2} + \frac{7-20\kappa+24\kappa^2}{24\sigma_\kappa^4} + \frac{9-34\kappa+52\kappa^2-48\kappa^3}{48\sigma_\kappa^6} + \frac{5-8\kappa}{80(1-\kappa)\sigma_\kappa^6} \\
&\leq 1 + \frac{1-2\kappa}{2\sigma_\kappa^2} + \frac{7-20\kappa+24\kappa^2}{24\sigma_\kappa^4} + \frac{9-34\kappa+52\kappa^2-48\kappa^3}{48\sigma_\kappa^6} + \frac{5-3\kappa-3\kappa^2-10\kappa^3/3}{80\sigma_\kappa^6} \\
&= 1 + \frac{1-2\kappa}{2\sigma_\kappa^2} + \frac{7-20\kappa+24\kappa^2}{24\sigma_\kappa^4} + \frac{60-179\kappa+251\kappa^2-250\kappa^3}{240\sigma_\kappa^6}
\end{aligned}$$

since $\frac{5-8\kappa}{1-\kappa} = 5 - 3\kappa - 3\kappa^2 - \frac{3\kappa^3}{1-\kappa} \leq 5 - 3\kappa - 3\kappa^2 - \frac{10\kappa^3}{3}$ in the last inequality.

on (iii): First we have $h_2(\kappa) := \frac{7-20\kappa+24\kappa^2}{72(1-\kappa)} + \frac{60-179\kappa+251\kappa^2-250\kappa^3}{240\cdot 9(1-\kappa)^2} = \frac{11}{72(1-\kappa)} - \frac{1}{18} - \frac{\kappa}{3} + \frac{309-427\kappa}{2160(1-\kappa)^2} - \frac{249}{2160} - \frac{250\kappa}{2160} \leq \frac{13}{135}$, since

$$\begin{aligned} h_2''(\kappa) &= \frac{22}{72(1-\kappa)^3} + \frac{309\cdot 6}{2160(1-\kappa)^4} - \frac{427\cdot(4+2\kappa)}{2160(1-\kappa)^4} = \frac{1}{(1-\kappa)^4}\cdot\left(\frac{11(1-\kappa)}{36} + \frac{309}{360} - \frac{427(2+\kappa)}{1080}\right) \\ &\geq \frac{1}{(1-\kappa)^4}\cdot\left(\frac{11}{72} + \frac{309}{360} - \frac{427}{1080}\cdot\frac{5}{2}\right) = \frac{49}{2160} > 0 \end{aligned}$$

and $h_2(\frac{1}{6}) = \frac{7793}{81\,000} \leq \frac{7800}{81\,000} = \frac{13}{135}$ as well as $h_2(\frac{1}{2}) = \frac{47}{540} = \frac{7050}{81\,000} < h_2(\frac{1}{6})$.

Thus we have with part (ii), $k \geq 3$ and $7 - 20\kappa + 24\kappa^2 \geq 7 - 20\kappa(1-\kappa) \geq 2 > 0$ and $60 - 179\kappa + 251\kappa^2 - 250\kappa^3 \geq 60 - \frac{179}{2} + \frac{251}{4} - \frac{250}{8} = 2 > 0$

$$\begin{aligned} \frac{f_{n,\frac{k-1}{n}}(k-1)}{f_{n,\kappa}(k)} &\leq 1 + \frac{1-2\kappa}{2\sigma_\kappa^2} + \frac{7-20\kappa+24\kappa^2}{24\sigma_\kappa^4} + \frac{60-179\kappa+251\kappa^2-250\kappa^3}{240\sigma_\kappa^6} \\ &\leq 1 + \frac{1-2\kappa}{2\sigma_\kappa^2} + \frac{1}{\sigma_\kappa^2}\cdot\left(\frac{7-20\kappa+24\kappa^2}{72(1-\kappa)} + \frac{60-179\kappa+251\kappa^2-250\kappa^3}{2160(1-\kappa)^2}\right) \\ &\leq 1 + \frac{1-2\kappa}{2\sigma_\kappa^2} + \frac{1}{\sigma_\kappa^2}\cdot\frac{13}{145} = 1 + \frac{1}{\sigma_\kappa^2}\cdot\left(\frac{1-2\kappa}{2} + \frac{13}{135}\right). \end{aligned}$$

On the other hand we also have for $y := 1 - x \in [0,1]$ and

$$\frac{f_{n,\frac{k-x}{n}}(k-1)}{f_{n,\frac{k-1}{n}}(k-1)} = \left(\frac{k-x}{k-1}\right)^{k-1}\cdot\left(\frac{n-k+x}{n-k+1}\right)^{n-k+1} = \left(1+\frac{y}{k-1}\right)^{k-1}\cdot\left(1-\frac{y}{n-k+1}\right)^{n-k+1} =: h_3(y)$$

that $\log(h_3(y)) \leq -\frac{y^2}{2\sigma_\kappa^2}$, since

$$\begin{aligned} \big[\log(h_3(y)\big]' &= \left(1+\frac{y}{k-1}\right)^{-1} - \left(1-\frac{y}{n-k+1}\right)^{-1} = \frac{k-1}{k-1+y} - \frac{n-k+1}{n-k+1-y} = -\frac{ny}{(k-1+y)(n-k+1-y)} \\ &\leq -\frac{y}{2\sigma_\kappa^2} \end{aligned}$$

and $\log(h_3(0)) = \log(1) = 0$. Thus we have with $\sigma_\kappa^2 \geq 3(1-\kappa) \geq \frac{3}{2}$

$$\frac{f_{n,\frac{k-x}{n}}(k-1)}{f_{n,\frac{k-1}{n}}(k-1)} \leq \mathrm{e}^{-\frac{(1-x)^2}{2\sigma_\kappa^2}} \leq 1 - \frac{(1-x)^2}{2\sigma_\kappa^2} + \frac{(1-x)^4}{8\sigma_\kappa^4} \leq 1 - \frac{(1-x)^2}{2\sigma_\kappa^2} + \frac{(1-x)^4}{12\sigma_\kappa^2}$$

and hence together

$$\begin{aligned} \frac{f_{n,\frac{k-x}{n}}(k-1)}{f_{n,\kappa}(k)} &\leq \left(1 + \frac{1}{\sigma_\kappa^2}\cdot\left(\frac{1-2\kappa}{2} + \frac{13}{135}\right)\right)\cdot\left(1 - \frac{1}{\sigma_\kappa^2}\cdot\left(\frac{(1-x)^2}{2} - \frac{(1-x)^4}{12}\right)\right) \\ &\leq 1 + \frac{1}{\sigma_\kappa^2}\cdot\left(\frac{1-2\kappa}{2} + \frac{13}{135} - \frac{(1-x)^2}{2} + \frac{(1-x)^4}{12}\right). \end{aligned}$$

□

Lemma 11.8. *Let $n, k \in \mathbb{N}$ with $k = \frac{n-1}{2} \geq 1$, hence $n \geq 3$ odd. Then we have*

$$\tfrac{1}{2} + \tfrac{1}{6n} - \tfrac{2}{15(n-1)(n+1)} \quad \leq \quad \frac{F_{n,\frac{k}{n}}(k) - 1/2}{f_{n,\frac{k}{n}}(k)} \quad \leq \quad \tfrac{1}{2} + \tfrac{1}{6n} - \tfrac{2}{15n(n+1)}.$$

Proof. First we have $F_{n,\frac{k}{n}}(k) = F_{n,\frac{k+1-1}{n}}(k)$ and $\frac{1}{2} = F_{n,\frac{1}{2}}(k) = F_{n,\frac{k+1-1/2}{n}}(k)$ and hence due to the proof of Lemma 11.6 with $k = n - k - 1$

$$\begin{aligned} F_{n,\frac{k}{n}}(k) - \tfrac{1}{2} \quad &= \int_{1/2}^{1} (1 + \tfrac{t}{k+1})^k (1 - \tfrac{t}{n-k-1})^{n-k-1} \mathrm{d}t \;=\; \int_{1/2}^{1} \big((1 + \tfrac{t}{k+1})(1 - \tfrac{t}{n-k-1})\big)^k \mathrm{d}t \\ &= \int_{1/2}^{1} (1 + \tfrac{t(1-t)}{k(k+1)})^k \mathrm{d}t. \end{aligned}$$

Since $(1+y)^k \geq 1 + ky + \frac{k(k-1)}{2} y^2$ if $y \geq 0$ by the binomial theorem we have

$$(1 + \tfrac{t(1-t)}{k(k+1)})^k \quad \geq \quad 1 + \tfrac{t(1-t)}{k+1} + \tfrac{t^2(1-t)^2(k-1)}{2k(k+1)^2}$$

and thus with $k + 1 = \frac{n+1}{2}$ it follows

$$\begin{aligned} F_{n,\frac{k}{n}}(k) - \tfrac{1}{2} \quad &\geq \int_{1/2}^{1} 1 + \tfrac{t(1-t)}{k+1} + \tfrac{t^2(1-t)^2(k-1)}{2k(k+1)^2} \mathrm{d}t \\ &= \Big[t + \tfrac{1}{k+1} \cdot (\tfrac{t^2}{2} - \tfrac{t^3}{3}) + \tfrac{k-1}{2k(k+1)^2} \cdot (\tfrac{t^3}{3} - \tfrac{t^4}{2} + \tfrac{t^5}{5}) \Big]_{1/2}^{1} \\ &= \tfrac{1}{2} + \tfrac{1}{k+1} \cdot (\tfrac{3}{8} - \tfrac{7}{24}) + \tfrac{k-1}{2k(k+1)^2} \cdot (\tfrac{7}{24} - \tfrac{15}{32} + \tfrac{31}{160}) \\ &= \tfrac{1}{2} + \tfrac{1}{12(k+1)} + \tfrac{k-1}{120k(k+1)^2} \\ &= \tfrac{1}{2} + \tfrac{1}{6n} - \tfrac{1}{6n(n+1)} + \tfrac{1}{30(n+1)^2} \cdot (1 - \tfrac{1}{k}) \\ &= \tfrac{1}{2} + \tfrac{1}{6n} - \tfrac{1}{6n(n+1)} + \tfrac{1}{30n(n+1)} + \tfrac{1}{30n(n+1)^2} - \tfrac{1}{15(n-1)(n+1)^2} \\ &= \tfrac{1}{2} + \tfrac{1}{6n} - \tfrac{2}{15(n-1)(n+1)} + \tfrac{2}{15n(n-1)(n+1)} + \tfrac{1}{30n(n+1)^2} - \tfrac{1}{15(n-1)(n+1)^2} \\ &\geq \tfrac{1}{2} + \tfrac{1}{6n} - \tfrac{2}{15(n-1)(n+1)} + \tfrac{2}{15n(n-1)(n+1)} - \tfrac{1}{15(n-1)n(n+1)} \\ &\geq \tfrac{1}{2} + \tfrac{1}{6n} - \tfrac{2}{15(n-1)(n+1)} \end{aligned}$$

and hence the estimation downwards.

On the other hand we have with $k\log(1+\frac{t(1-t)}{k(k+1)}) \leq 1+\frac{t(1-t)}{k+1}-\frac{t^2(1-t)^2}{2k(k+1)^2}+\frac{t^3(1-t)^3}{k^2(k+1)^3}$

$$\begin{aligned}
\big(1+\tfrac{t(1-t)}{k(k+1)}\big)^k &\leq 1+\tfrac{t(1-t)}{k+1}-\tfrac{t^2(1-t)^2}{2k(k+1)^2}+\tfrac{t^3(1-t)^3}{3k^2(k+1)^3}+\tfrac{t^2(1-t)^2}{2(k+1)^2}+\tfrac{t^3(1-t)^3}{6(k+1)^3}+\tfrac{1}{24}\cdot\sum_{r=0}^{\infty}(\tfrac{t(1-t)}{k+1})^r \\
&= 1+\tfrac{t(1-t)}{k+1}+\tfrac{t^2(1-t)^2}{2(k+1)^2} \\
&\quad -\tfrac{t^2(1-t)^2}{(k+1)^3}\cdot\Big(\tfrac{k+1}{2k}-\tfrac{t(1-t)}{3k^2}-\tfrac{t(1-t)}{6}-\tfrac{t^2(1-t)^2}{24(k+1)}\cdot\sum_{r=0}^{\infty}(\tfrac{t(1-t)}{k+1})^r\Big) \\
&\leq 1+\tfrac{t(1-t)}{k+1}+\tfrac{t^2(1-t)^2}{k(k+1)^2}-\tfrac{t^2(1-t)^2}{(k+1)^3}\cdot\Big(\tfrac{k+1}{2k}-\tfrac{1}{12k}-\tfrac{1}{24}-\tfrac{1}{16\cdot 24k}\cdot\sum_{r=0}^{\infty}(\tfrac{1}{4})^r\Big) \\
&= 1+\tfrac{t(1-t)}{k+1}+\tfrac{t^2(1-t)^2}{2(k+1)^2}-\tfrac{t^2(1-t)^2}{(k+1)^3}\cdot(\tfrac{11}{24}+\tfrac{119}{288k}) \\
&\leq 1+\tfrac{t(1-t)}{k+1}+\tfrac{t^2(1-t)^2}{2(k+1)^2}
\end{aligned}$$

and thus it follows similarly as above

$$\begin{aligned}
F_{n,\frac{k}{n}}(k)-\tfrac{1}{2} &\leq \int_{1/2}^{1} 1+\tfrac{t(1-t)}{k+1}+\tfrac{t^2(1-t)^2}{2(k+1)^2}\mathrm{d}t = \tfrac{1}{2}+\tfrac{1}{12(k+1)}+\tfrac{1}{120(k+1)^2} \\
&= \tfrac{1}{2}+\tfrac{1}{6(n+1)}+\tfrac{1}{30(n+1)^2} = \tfrac{1}{2}+\tfrac{1}{6n}-\tfrac{1}{6n(n+1)}+\tfrac{1}{30(n+1)^2} \\
&\leq \tfrac{1}{2}+\tfrac{1}{6n}-\tfrac{2}{15n(n+1)}.
\end{aligned}$$

□

Lemma 11.9. *Let $k, n \in \mathbb{N}$ with $\frac{k}{n} \in [\frac{1}{6}, \frac{1}{2}]$. Then we have*

$$\frac{4-2\cdot k/n}{6} - \frac{n-2k}{18k(n-k)} \;\le\; \frac{F_{n,k/n}(k)-1/2}{f_{n,k/n}(k)} \;\le\; \frac{4-2\cdot k/n}{6} - \frac{n-2k}{72k(n-k)}. \tag{31}$$

Proof. We divide the following proof into two steps. In the first one we show (31) in case of $k = \lfloor \frac{n}{2} \rfloor$, and in the second one by means of monotonicity also for the $\frac{n}{6} \le k < \lfloor \frac{n}{2} \rfloor$. This proof uses Lemmas 11.4, 11.6, 11.7, 11.8, where Lemmas 11.6 and 11.7 are in turn based on Lemma 11.5.

Proof if $k = \lfloor \frac{n}{2} \rfloor$

1.) Let n even and thus $k = n/2$, hence $k/n = 1/2$. This case is obvious, since $F_{n,\frac{1}{2}}$ is symmetrical to $\frac{n}{2}$ it follows

$$F_{n,\frac{1}{2}}(\tfrac{n}{2}) \;=\; \tfrac{1}{2} + \tfrac{1}{2}\cdot f_{n,\frac{1}{2}}(\tfrac{n}{2}) \;=\; \tfrac{1}{2} + \tfrac{4-2\cdot k/n}{6}\cdot f_{n,\frac{1}{2}}(\tfrac{n}{2})$$

and hence with $n - 2k = 0$ the claim.

2.) Let now n odd and hence $k = (n-1)/2 > 0$. Then $n = 2k+1 \ge 3$ and hence

$$\begin{aligned}\tfrac{4-2\cdot k/n}{6} - \tfrac{n-2k}{72k(n-k)} \;&=\; \tfrac{2}{3} - \tfrac{n-1}{6n} - \tfrac{1}{18(n-1)(n+1)} \;=\; \tfrac{1}{2} + \tfrac{1}{6n} - \tfrac{1}{18n(n+1)}\cdot\tfrac{n}{n-1}\\ &\ge\; \tfrac{1}{2} + \tfrac{1}{6n} - \tfrac{1}{12n(n+1)}\end{aligned}$$

and due to Lemma 11.8 on page 116 we have

$$\frac{F_{n,\frac{k}{n}}(k)-1/2}{f_{n,\frac{k}{n}}(k)} \;\le\; \tfrac{1}{2} + \tfrac{1}{6n} - \tfrac{2}{15n(n+1)} \;\le\; \tfrac{1}{2} + \tfrac{1}{6n} - \tfrac{1}{12n(n+1)}$$

and hence the upper bound.

If we now estimate downwards, Lemma 11.8 yields

$$\frac{F_{n,\frac{k}{n}}(k)-1/2}{f_{n,\frac{k}{n}}(k)} \;\ge\; \tfrac{1}{2} + \tfrac{1}{6n} - \tfrac{2}{15(n-1)(n+1)} \;\ge\; \tfrac{1}{2} + \tfrac{1}{6n} - \tfrac{2}{9(n-1)(n+1)} \;=\; \tfrac{4-2\cdot k/n}{6} - \tfrac{n-2k}{18k(n-k)}.$$

(Primarily the case $k = (n-1)/2$ was proved by means of an estimation of a representation of the symmetric incomplete Beta-function $I_x(a,a)$ due to Temme (1982, 3.3.1), applied to $F_{n,p}(k) = I_{1-p}(n-k, k+1)$ and using $n - k + k + 1$. However this kind of proof was much more awkward than the proof above and so it was replaced by the easier one.)

Proof if $\frac{n}{6} \leq k < \lfloor \frac{n}{2} \rfloor$

In the section above we showed (31) in case of $k = \lfloor \frac{n}{2} \rfloor$ and with the monotonicity of the differences we now show it also in case of $k < \lfloor \frac{n}{2} \rfloor$.

We first assume $k + 1 \geq 4$ and show

$$\underbrace{F_{n,\frac{k}{n}}(k) - \tfrac{1}{2}}_{=:F_k} \quad \leq \quad \underbrace{\tfrac{4-2k/n}{6} \cdot f_{n,\frac{k}{n}}(k)}_{=:D_k} - \underbrace{\tfrac{n-2k}{72k(n-k)} \cdot f_{n,\frac{k}{n}}(k)}_{=:E_k}.$$

We prove that $D_k - E_k - F_k$ becomes minimal at the point $k = \lfloor n/2 \rfloor$, see Figure 14, so we show

$$D_k - E_k - F_k \;>\; D_{k+1} - E_{k+1} - F_{k+1} \qquad \text{for} \qquad \tfrac{n}{6} \leq k \leq \tfrac{n}{2} - 1$$

from which with the previous part for $k = \lfloor \frac{n}{2} \rfloor$ the right inequality in (31) also for $\frac{n}{6} \leq k \leq \frac{n}{2} - 1$ follows. We consider the difference of the both sides and get

$$\begin{aligned} & D_k - E_k - F_k - (D_{k+1} - E_{k+1} - F_{k+1}) \\ = \; & D_k - D_{k+1} - E_k + E_{k+1} + F_{n\frac{k+1}{n}}(k+1) - F_{n\frac{k}{n}}(k) \\ = \; & D_k - D_{k+1} - (E_k - E_{k+1}) + f_{n,\frac{k+1}{n}}(k+1) + F_{n,\frac{k+1}{n}}(k) - F_{n,\frac{k}{n}}(k). \end{aligned}$$

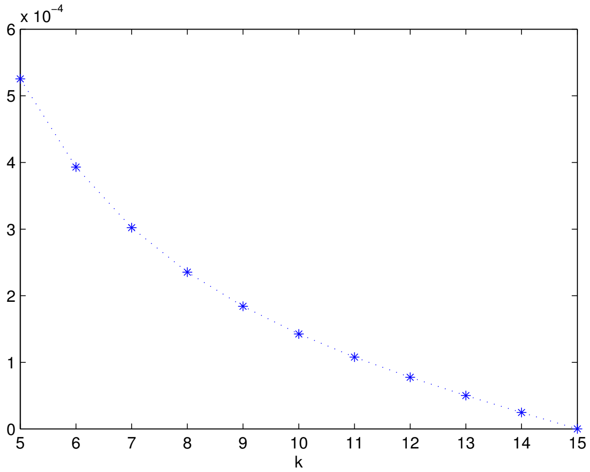

FIGURE 14. Graph of $k \mapsto D_k - E_k - F_k$ for $n = 30$.

First, we have due to Lemma 11.7 with $\alpha := \frac{k+1}{n}$ and hence $\sigma_\alpha^2 = n\alpha(1-\alpha) = \frac{(k+1)(n-k-1)}{n}$

$$\begin{aligned}
\frac{D_k - D_{k+1}}{f_{n,\alpha}(k+1)} &= \frac{4-2k/n}{6} \cdot \frac{f_{n,k/n}(k)}{f_{n,\alpha}(k+1)} - \frac{4-2(k+1)/n}{6} = \frac{1}{3n} + \frac{4-2k/n}{6} \cdot \left(\frac{f_{n,k/n}(k)}{f_{n,\alpha}(k+1)} - 1\right) \\
&= \frac{\alpha(1-\alpha)}{3\sigma_\alpha^2} + \left(\frac{2-\alpha}{3} + \frac{\alpha(1-\alpha)}{3\sigma_\alpha^2}\right) \cdot \left(\frac{f_{n,k/n}(k)}{f_{n,\alpha}(k+1)} - 1\right) \\
&\geq \frac{\alpha(1-\alpha)}{3\sigma_\alpha^2} + \left(\frac{2-\alpha}{3} + \frac{\alpha(1-\alpha)}{3\sigma_\alpha^2}\right) \cdot \left(\frac{1-2\alpha}{2\sigma_\alpha^2} + \frac{7-20\alpha+24\alpha^2}{24\sigma_\alpha^2} + \frac{13-30\alpha}{96\sigma_\alpha^6}\right) \\
&= \frac{1}{\sigma_\alpha^2} \cdot \left(\frac{\alpha(1-\alpha)}{3} + \frac{(1-2\alpha)(2-\alpha)}{6}\right) + \frac{1}{\sigma_\alpha^4} \cdot \left(\frac{(7-20\alpha+24\alpha^2)(2-\alpha)}{72} + \frac{(1-2\alpha)\alpha(1-\alpha)}{6}\right) \\
&\quad + \frac{1}{\sigma_\alpha^6} \cdot \left(\frac{(13-30\alpha)(2-\alpha)}{288} + \frac{(7-20\alpha+24\alpha^2)\alpha(1-\alpha)}{72}\right) + \frac{1}{\sigma_\alpha^8} \cdot \frac{(13-30\alpha)\alpha(1-\alpha)}{288} \\
&= \frac{2-3\alpha}{6\sigma_\alpha^2} + \frac{14-35\alpha+32\alpha^2}{72\sigma_\alpha^4} + \frac{26-45\alpha-78\alpha^2+176\alpha^3-96\alpha^4}{288\sigma_\alpha^6} + \frac{(13\alpha-30\alpha^2)(1-\alpha)}{288\sigma_\alpha^8}.
\end{aligned}$$

Second, we consider $E_k - E_{k+1}$. If $k = 3$ we have

$$\frac{(k+1)(n-k-1)}{k(n-k)} = \frac{4(n-3-1)}{3(n-3)} \leq \frac{4(18-3-1)}{3(18-3)} = \frac{56}{45}$$

since if $n > 18$ it follows $\frac{k}{n} < \frac{3}{18} = \frac{1}{6}$, inconsistent to the requirement. If $k = 4$ we have $\frac{(k+1)(n-k-1)}{k(n-k)} \leq \frac{5(24-4-1)}{4(24-4)} = \frac{19}{16} \leq \frac{56}{45}$ with the same argument and if $k \geq 5$ one finds directly $\frac{(k+1)(n-k-1)}{k(n-k)} \leq \frac{6}{5} \leq \frac{56}{45}$. Since $\frac{(k+1)(n-k-1)}{k(n-k)} = 1 + \frac{n-2k-1}{k(n-k)}$ it follows $\frac{n-2k-1}{k(n-k)} \leq \frac{11}{45}$.

Next we get for $0 \leq t \leq \frac{11}{45}$ that $(1+t)^{3/2} \leq 1 + \frac{159}{100}t$, since for $g(t) := 1 + \frac{159}{100}t - (1+t)^{3/2}$ holds $g''(t) = -\frac{3}{4}(1+t)^{-1/2} < 0$, thus g is concave and together with $g(0) = 0$ and $g(\frac{11}{45}) = 0.0004... > 0$ we get $g(t) \geq 0$ on $[0, \frac{11}{45}]$.

From Lemma 11.4 follows $\frac{f_{n,k/n}(k)}{f_{n,\alpha}(k+1)} \leq \sqrt{1 + \frac{n-2k-1}{k(n-k)}}$ and hence with $\frac{159}{100} \cdot \frac{56}{45} = \frac{2226}{1125} \leq 2$

$$\begin{aligned}
\frac{(k+1)(n-k-1)}{k(n-k)} \cdot \frac{f_{n,k/n}(k)}{f_{n,\alpha}(k+1)} - 1 &\leq \left(1 + \frac{n-2k-1}{k(n-k)}\right)^{3/2} - 1 \\
&\leq \frac{159}{100} \cdot \frac{n-2k-1}{k(n-k)} = \frac{159}{100} \cdot \frac{1-2\alpha+1/n}{n\alpha(1-\alpha)} + \frac{159}{100} \cdot \frac{(1-2\alpha+1/n)^2}{n^2\alpha^2(1-\alpha)^2} \cdot \frac{(k+1)(n-k-1)}{k(n-k)} \\
&\leq \frac{8}{5} \cdot \frac{1-2\alpha+1/n}{\sigma_\alpha^2} + 2 \cdot \frac{(1-2\alpha+1/n)^2}{\sigma_\alpha^4}.
\end{aligned}$$

With this preparations we now get with $k + 1 \geq 4$

$$
\begin{aligned}
\tfrac{E_k-E_{k+1}}{f_{n,\alpha}(k+1)} &= \tfrac{n-2k}{72k(n-k)}\cdot\tfrac{f_{n,k/n}(k)}{f_{n,\alpha}(k+1)} - \tfrac{n-2k-2}{72(k+1)(n-k-1)}\\
&= \tfrac{1}{72\sigma_\alpha^2}\cdot\big(\tfrac{2}{n} + \big(\tfrac{(k+1)(n-k-1)}{k(n-k)}\cdot\tfrac{f_{m,k/n}(k)}{f_{n,\alpha}(k+1)} - 1\big)\cdot\tfrac{n-2k}{n}\big)\\
&\le \tfrac{1}{72\sigma_\alpha^2}\cdot\big(\tfrac{2}{n} + \big(\tfrac{8}{5}\cdot\tfrac{1-2\alpha+1/n}{\sigma_\alpha^2} + 2\cdot\tfrac{(1-2\alpha+1/n)^2}{\sigma_\alpha^4}\big)\cdot(1-2\alpha+\tfrac{2}{n})\big)\\
&= \tfrac{1}{\sigma_\alpha^4}\cdot\big(\tfrac{\alpha(1-\alpha)}{36} + \tfrac{(1-2\alpha)^2+3/n(1-2\alpha)+2/n^2}{45} + \tfrac{(1-2\alpha)^3+4/n(1-2\alpha)^2+5/n^2(1-2\alpha)+2/n^3}{36\sigma_\alpha^2}\big)\\
&= \tfrac{1}{\sigma_\alpha^4}\cdot\big(\tfrac{\alpha(1-\alpha)}{36} + \tfrac{(1-2\alpha)^2}{45}\big) + \tfrac{1}{\sigma_\alpha^6}\cdot\big(\tfrac{\alpha(1-\alpha)(1-2\alpha)}{15} + \tfrac{(1-2\alpha)^3}{36}\big)\\
&\quad + \tfrac{1-\alpha}{\sigma_\alpha^8}\cdot\big(\tfrac{2\alpha^2(1-\alpha)}{45} + \tfrac{\alpha(1-2\alpha)^2}{9} + \tfrac{5\alpha^2(1-2\alpha)}{36a} + \tfrac{\alpha^3}{18a^2}\big)\\
&\le \tfrac{1}{\sigma_\alpha^4}\cdot\big(\tfrac{\alpha(1-\alpha)}{36} + \tfrac{(1-2\alpha)^2}{45}\big) + \tfrac{1}{\sigma_\alpha^6}\cdot\big(\tfrac{\alpha(1-\alpha)(1-2\alpha)}{15} + \tfrac{(1-2\alpha)^3}{36}\big)\\
&\quad + \tfrac{1-\alpha}{\sigma_\alpha^8}\cdot\big(\tfrac{2\alpha^2(1-\alpha)}{45} + \tfrac{\alpha(1-2\alpha)^2}{9} + \tfrac{5\alpha^2(1-2\alpha)}{144} + \tfrac{\alpha^3}{288}\big)\\
&= \tfrac{4-11\alpha+11\alpha^2}{180\sigma_\alpha^4} + \tfrac{5-18\alpha+24\alpha^2-16\alpha^3}{180\sigma_\alpha^6} + \tfrac{(1-\alpha)\cdot(160\alpha-526\alpha^2+481\alpha^3)}{1440\sigma_\alpha^8}.
\end{aligned}
$$

Third, it follows by Lemma 11.6 (iii)

$$
\tfrac{F_k-F_{k+1}}{f_{n,\alpha}(k+1)} \le 1 + \tfrac{2-3\alpha}{6\sigma_\alpha^2} + \tfrac{18-45\alpha+40\alpha^2}{120\sigma_\alpha^4} + \tfrac{398-1393\alpha+1848\alpha^2-1260\alpha^3}{5040\sigma_\alpha^6} + \tfrac{(1-\alpha)\cdot(27-97\alpha+160\alpha^2-90\alpha^3)}{300\sigma_\alpha^8}.
$$

We now summarise the results of $F_k,\ D_k$ and E_k and get

$$
\begin{aligned}
&\tfrac{D_k-D_{k+1}-(E_k-E_{k+1})+f_{n,\alpha}(k+1)-(F_k-F_{k+1})}{f_{n,\alpha}(k+1)}\\
\ge\ & \tfrac{2-3\alpha}{6\sigma_\alpha^2} + \tfrac{14-35\alpha+32\alpha^2}{72\sigma_\alpha^4} + \tfrac{26-45\alpha-78\alpha^2+176\alpha^3-96\alpha^4}{288\sigma_\alpha^6} + \tfrac{(13\alpha-30\alpha^2)(1-\alpha)}{288\sigma_\alpha^8}\\
&-\Big(\tfrac{4-11\alpha+11\alpha^2}{180\sigma_\alpha^4} + \tfrac{5-18\alpha+24\alpha^2-16\alpha^3}{180\sigma_\alpha^6} + \tfrac{(1-\alpha)\cdot(160\alpha-526\alpha^2+481\alpha^3)}{1440\sigma_\alpha^8}\Big) + 1\\
&-\Big(1 + \tfrac{2-3\alpha}{6\sigma_\alpha^2} + \tfrac{18-45\alpha+40\alpha^2}{120\sigma_\alpha^4} + \tfrac{398-1393\alpha+1848\alpha^2-1260\alpha^3}{5040\sigma_\alpha^6} + \tfrac{(1-\alpha)\cdot(27-97\alpha+160\alpha^2-90\alpha^3)}{300\sigma_\alpha^8}\Big)\\
=\ & \tfrac{4-9\alpha+9\alpha^2}{180\sigma_\alpha^4} + \tfrac{1}{\sigma_\alpha^6}\cdot\big(-\tfrac{83}{5040} + \tfrac{317\alpha}{1440} - \tfrac{37\alpha^2}{48} + \tfrac{19\alpha^3}{20} - \tfrac{\alpha^4}{3}\big) + \tfrac{1-\alpha}{\sigma_\alpha^8}\cdot\big(-\tfrac{9}{100} + \tfrac{1853\alpha}{7200} - \tfrac{49\alpha^2}{180} - \tfrac{49\alpha^3}{1440}\big)\\
\ge\ & \tfrac{1}{\sigma_\alpha^6}\cdot\Big(\tfrac{4(1-\alpha)(4-9\alpha+9\alpha^2)}{180} - \tfrac{83}{5040} + \tfrac{317\alpha}{1440} - \tfrac{37\alpha^2}{48} + \tfrac{19\alpha^3}{20} - \tfrac{\alpha^4}{3} + \tfrac{1}{4}\big(-\tfrac{9}{100} + \tfrac{1853\alpha}{7200} - \tfrac{49\alpha^2}{180} - \tfrac{49\alpha^3}{1440}\big)\Big)\\
=\ & \tfrac{1}{\sigma_\alpha^6}\cdot\Big(\tfrac{629}{12\,600} - \tfrac{127\alpha}{28\,800} - \tfrac{79\alpha^2}{180} + \tfrac{4271\alpha^3}{5760} - \tfrac{\alpha^4}{3}\Big)\\
\ge\ & \tfrac{1}{\sigma_\alpha^6}\cdot\Big(\tfrac{629}{12\,600} - \tfrac{127}{28\,800\cdot 2} - \tfrac{79}{180\cdot 4} + \tfrac{4271}{5760\cdot 8} - \tfrac{1}{3\cdot 16}\Big) = \tfrac{1}{\sigma_\alpha^6}\cdot\tfrac{15\,881}{1\,612\,800} > 0
\end{aligned}
$$

since $4-9\alpha(1-\alpha) \ge 4-\frac{9}{4} > 0$ and $-\frac{9}{100} + \frac{1853\alpha}{7200} - \frac{49\alpha^2}{180} - \frac{49\alpha^3}{1440} \le -\frac{9}{100} + \frac{1853\alpha(1-\alpha)}{7200} \le -\frac{9}{100} + \frac{1853}{7200\cdot 4} = -\frac{739}{28\,800} < 0$ in the second inequality and $(-\frac{127\alpha}{28\,800} - \frac{79\alpha^2}{180} + \frac{4271\alpha^3}{5760} - \frac{\alpha^4}{3})' = -\frac{127}{28\,800} - \frac{79\alpha}{90} + \frac{4271\alpha^3}{1920} - \frac{4\alpha^3}{3} = -\frac{127}{28\,800} - \frac{79\alpha(1-\alpha)}{90} + \frac{7757\alpha^2(1-\alpha)}{5760} + \frac{77\alpha^3}{5760} \le -\frac{127}{28\,800} - \frac{79\alpha(1-\alpha)}{90} + \frac{7757\alpha(1-\alpha)}{11\,520} + \frac{77}{5760\cdot 8} = -\frac{631}{230\,400} - \frac{157\alpha(1-\alpha)}{768} < 0$ in the third inequality.

Thus the right inequaliy in (31) also holds for $\frac{n}{6} \le k < \lfloor\frac{n}{2}\rfloor$ in the case $k \ge 3$.

It remains the case $k \leq 2$. Since $k \geq \frac{n}{6}$ and since we finished $k = \lfloor \frac{n}{2} \rfloor$ already in the first part, remain if $k = 2$ still $n = 6, 7, 8, 9, 10, 11, 12$ and if $k = 1$ we still have to consider $n = 4, 5, 6$. For those we have

$$\begin{array}{rcccccl}
F_{6,\frac{2}{6}}(2) - \frac{1}{2} & = & 0.1804... & \leq & 0.1818... & = & (\frac{4-2\cdot 2/6}{6} - \frac{2}{72\cdot 2\cdot 4}) \cdot f_{6,\frac{2}{6}}(2) \\
F_{7,\frac{2}{7}}(2) - \frac{1}{2} & = & 0.1792... & \leq & 0.1808... & = & (\frac{4-2\cdot 2/7}{6} - \frac{3}{72\cdot 2\cdot 5}) \cdot f_{7,\frac{2}{7}}(2) \\
F_{8,\frac{2}{8}}(2) - \frac{1}{2} & = & 0.1785... & \leq & 0.1802... & = & (\frac{4-2\cdot 2/8}{6} - \frac{4}{72\cdot 2\cdot 6}) \cdot f_{8,\frac{2}{8}}(2) \\
F_{9,\frac{2}{9}}(2) - \frac{1}{2} & = & 0.1781... & \leq & 0.1799... & = & (\frac{4-2\cdot 2/9}{6} - \frac{5}{72\cdot 2\cdot 7}) \cdot f_{9,\frac{2}{9}}(2) \\
F_{10,\frac{2}{10}}(2) - \frac{1}{2} & = & 0.1778... & \leq & 0.1796... & = & (\frac{4-2\cdot 2/10}{6} - \frac{6}{72\cdot 2\cdot 8}) \cdot f_{10,\frac{2}{10}}(2) \\
F_{11,\frac{2}{11}}(2) - \frac{1}{2} & = & 0.1776... & \leq & 0.1794... & = & (\frac{4-2\cdot 2/11}{6} - \frac{7}{72\cdot 2\cdot 9}) \cdot f_{11,\frac{2}{11}}(2) \\
F_{12,\frac{2}{12}}(2) - \frac{1}{2} & = & 0.1774... & \leq & 0.1793... & = & (\frac{4-2\cdot 2/12}{6} - \frac{8}{72\cdot 2\cdot 10}) \cdot f_{12,\frac{2}{12}}(2)
\end{array}$$

as well as

$$\begin{array}{rcccccl}
F_{4,\frac{1}{4}}(1) - \frac{1}{2} & = & 0.2383... & \leq & 0.2422... & = & (\frac{4-2\cdot 1/4}{6} - \frac{2}{72\cdot 3}) \cdot f_{4,\frac{1}{4}}(1) \\
F_{5,\frac{1}{5}}(1) - \frac{1}{2} & = & 0.2373... & \leq & 0.2415... & = & (\frac{4-2\cdot 1/5}{6} - \frac{3}{72\cdot 4}) \cdot f_{5,\frac{1}{5}}(1) \\
F_{6,\frac{1}{6}}(1) - \frac{1}{2} & = & 0.2368... & \leq & 0.2411... & = & (\frac{4-2\cdot 1/6}{6} - \frac{4}{72\cdot 5}) \cdot f_{6,\frac{1}{6}}(1).
\end{array}$$

Thus it follows completely the right inequality in (31).

Next we show for $\frac{n}{6} \leq k \leq \frac{n}{2} - 1$

$$\underbrace{F_{n,\frac{k}{n}}(k) - \tfrac{1}{2}}_{=:F_k} \geq \underbrace{\tfrac{4-2k/n}{6} \cdot f_{n,\frac{k}{n}}(k)}_{=:D_k} - \underbrace{\tfrac{n-2k}{18k(n-k)} \cdot f_{n,\frac{k}{n}}(k)}_{=:C_k}$$

by estimating for again $k + 1 \geq 3$

$$\begin{aligned}
& D_k - C_k - F_k - (D_{k+1} - C_{k+1} - F_{k+1}) \\
= \; & D_k - D_{k+1} - C_k + C_{k+1} + F_{n,\frac{k+1}{n}}(k+1) - F_{n,\frac{k}{n}}(k) \\
= \; & D_k - D_{k+1} - (C_k - C_{k+1}) + f_{n,\frac{k+1}{n}}(k+1) + F_{n,\frac{k+1}{n}}(k) - F_{n,\frac{k}{n}}(k)
\end{aligned}$$

this time smaller than zero, so that the difference $D_k - F_k - C_k$ becomes smaller, the smaller becomes k. Therefor we have with Lemma 11.7 similar to the part above

$$
\begin{aligned}
\tfrac{D_k-D_{k+1}}{f_{n,\alpha}(k+1)} &= \tfrac{\alpha(1-\alpha)}{3\sigma_\alpha^2}+\big(\tfrac{2-\alpha}{3}+\tfrac{\alpha(1-\alpha)}{3\sigma_\alpha^2}\big)\cdot\big(\tfrac{f_{n,k/n}(k)}{f_{n,\alpha}(k+1)}-1\big)\\
&\le \tfrac{\alpha(1-\alpha)}{3\sigma_\alpha^2}+\big(\tfrac{2-\alpha}{3}+\tfrac{\alpha(1-\alpha)}{3\sigma_\alpha^2}\big)\cdot\big(\tfrac{1-2\alpha}{2\sigma_\alpha^2}+\tfrac{7-20\alpha+24\alpha^2}{24\sigma_\alpha^4}+\tfrac{60-179\alpha+251\alpha^2-250\alpha^3}{240\sigma_\alpha^6}\big)\\
&= \tfrac{2-3\alpha}{6\sigma_\alpha^2}+\tfrac{14-35\alpha+32\alpha^2}{72\sigma_\alpha^4}+\tfrac{120-348\alpha+411\alpha^2-311\alpha^3+10\alpha^4}{720\sigma_\alpha^6}+\tfrac{\alpha(1-\alpha)(60-179\alpha+251\alpha^2-250\alpha^3)}{720\sigma_\alpha^8}.
\end{aligned}
$$

Further we have $\frac{(k+1)(n-k-1)}{k(n-k)}=1+\frac{n-2k-1}{k(n-k)}=1+\frac{n-2k-2}{(k+1)(n-k-1)}+\frac{1}{(k+1)(n-k-1)}+\frac{(n-2k-1)^2}{k(n-k)(k+1)(n-k-1)}\ge 1+\frac{1-2\alpha}{\sigma_\alpha^2}+\frac{\alpha(1-\alpha)}{\sigma_\alpha^4}+\frac{(1-2\alpha)^2}{\sigma_\alpha^4}=1+\frac{1-2\alpha}{\sigma_\alpha^2}+\frac{1-3\alpha+3\alpha^2}{\sigma_\alpha^4}$ and hence with Lemma 11.7 and $\frac{13-30\alpha}{\sigma_\alpha^2}\ge-\frac{2}{\sigma_\alpha^2}\ge-\frac{2}{3(1-\alpha)}\ge-\frac{4}{3}$ we get

$$
\begin{aligned}
\tfrac{(k+1)(n-k-1)}{k(n-k)}\cdot\tfrac{f_{n,k/n}(k)}{f_{n,\alpha}(k+1)} &\ge \big(1+\tfrac{1-2\alpha}{\sigma_\alpha^2}+\tfrac{1-3\alpha+3\alpha^2}{\sigma_\alpha^4}\big)\cdot\big(1+\tfrac{1-2\alpha}{2\sigma_\alpha^2}+\tfrac{7-20\alpha+24\alpha^2}{24\sigma_\alpha^4}+\tfrac{13-30\alpha}{96\sigma_\alpha^6}\big)\\
&\ge \big(1+\tfrac{1-2\alpha}{\sigma_\alpha^2}+\tfrac{1-3\alpha+3\alpha^2}{\sigma_\alpha^4}\big)\cdot\big(1+\tfrac{1-2\alpha}{2\sigma_\alpha^2}+\tfrac{5-15\alpha+18\alpha^2}{18\sigma_\alpha^4}\big)\\
&\ge 1+\tfrac{3(1-2\alpha)}{2\sigma_\alpha^2}+\tfrac{1}{\sigma_\alpha^4}\cdot\big(\tfrac{5-15\alpha+18\alpha^2}{18}+\tfrac{(1-2\alpha)^2}{2}+1-3\alpha+3\alpha^2\big)\\
&\quad+\tfrac{1}{\sigma_\alpha^6}\cdot\big(\tfrac{(1-2\alpha)(1-3\alpha+3\alpha^2)}{2}+\tfrac{(5-15\alpha+18\alpha^2)(1-2\alpha)}{18}\big)\\
&= 1+\tfrac{3(1-2\alpha)}{2\sigma_\alpha^2}+\tfrac{32-105\alpha+108\alpha^2}{18\sigma_\alpha^4}+\tfrac{14-70\alpha+129\alpha^2-90\alpha^3}{18\sigma_\alpha^6}
\end{aligned}
$$

and thus we have

$$
\begin{aligned}
\tfrac{C_k-C_{k+1}}{f_{n,\alpha}(k+1)} &= \tfrac{n-2k}{18k(n-k)}\cdot\tfrac{f_{n,k/n}(k)}{f_{n,\alpha}(k+1)}-\tfrac{n-2k-2}{72(k+1)(n-k-1)}\\
&= \tfrac{1}{18\sigma_\alpha^2}\cdot\Big(\tfrac{2\alpha(1-\alpha)}{\sigma_\alpha^2}+\big(\tfrac{(k+1)(n-k-1)}{k(n-k)}\cdot\tfrac{f_{n,\frac{k}{n}}(k)}{f_{n,\alpha}(k+1)}-1\big)\cdot\big(1-2\alpha+\tfrac{2\alpha(1-\alpha)}{\sigma_\alpha^2}\big)\Big)\\
&\ge \tfrac{1}{18\sigma_\alpha^2}\cdot\Big(\tfrac{2\alpha(1-\alpha)}{\sigma_\alpha^2}+\big(\tfrac{3(1-2\alpha)}{2\sigma_\alpha^2}+\tfrac{32-105\alpha+108\alpha^2}{18\sigma_\alpha^4}+\tfrac{14-70\alpha+129\alpha^2-90\alpha^3}{18\sigma_\alpha^6}\big)\big(1-2\alpha+\tfrac{2\alpha(1-\alpha)}{\sigma_\alpha^2}\big)\Big)\\
&\ge \tfrac{1}{18\sigma_\alpha^2}\cdot\Big(\tfrac{2\alpha(1-\alpha)}{\sigma_\alpha^2}+\tfrac{3(1-2\alpha)^2}{2\sigma_\alpha^2}+\tfrac{1}{\sigma_\alpha^4}\cdot\big(3\alpha(1-\alpha)(1-2\alpha)+\tfrac{(1-2\alpha)(32-105\alpha+108\alpha^2}{18}\big)\\
&\quad+\tfrac{1}{\sigma_\alpha^6}\cdot\big(\tfrac{\alpha(1-\alpha)(32-105\alpha+108\alpha^2)}{9}+\tfrac{(1-2\alpha)(14-70\alpha+129\alpha^2-90\alpha^3}{18)}\big)\Big)\\
&= \tfrac{3-8\alpha+8\alpha^2}{36\sigma_\alpha^4}+\tfrac{32-115\alpha+156\alpha^2-108\alpha^3}{324\sigma_\alpha^6}+\tfrac{14-34\alpha-5\alpha^2+78\alpha^3-36\alpha^4}{324\sigma_\alpha^8}.
\end{aligned}
$$

Further Lemma 11.6 yields

$$
\tfrac{F_{n,k/n}(k)-F_{n,\alpha}(k+1)}{f_{n,\alpha}(k+1)} \ge 1+\tfrac{2-3\alpha}{6\sigma_\alpha^2}+\tfrac{18-45\alpha+40\alpha^2}{120\sigma_\alpha^4}+\tfrac{398-1393\alpha+1848\alpha^2-1260\alpha^3}{5040\sigma_\alpha^6}
$$

and thus we receive together

$$
\begin{aligned}
&\tfrac{D_k - D_{k+1} - (C_k - C_{k+1}) + f_{n,\alpha}(k+1) - (F_{n,k/n}(k) - F_{n,\alpha}(k+1))}{f_{n,\alpha}(k+1)} \\
\leq\ & \tfrac{2-3\alpha}{6\sigma_\alpha^2} + \tfrac{14-35\alpha+32\alpha^2}{72\sigma_\alpha^4} + \tfrac{120-348\alpha+411\alpha^2-311\alpha^3+10\alpha^4}{720\sigma_\alpha^6} + \tfrac{\alpha(1-\alpha)(60-179\alpha+251\alpha^2-250\alpha^3)}{720\sigma_\alpha^8} \\
& -\big(\tfrac{3-8\alpha+8\alpha^2}{36\sigma_\alpha^4} + \tfrac{32-115\alpha+156\alpha^2-108\alpha^3}{324\sigma_\alpha^6} + \tfrac{14-34\alpha-5\alpha^2+78\alpha^3-36\alpha^4}{324\sigma_\alpha^8}\big) \\
& +1 - \big(1 + \tfrac{2-3\alpha}{6\sigma_\alpha^2} + \tfrac{18-45\alpha+40\alpha^2}{120\sigma_\alpha^4} + \tfrac{398-1393\alpha+1848\alpha^2-1260\alpha^3}{5040\sigma_\alpha^6}\big) \\
=\ & -\tfrac{7-20\alpha+20\alpha^2}{180\sigma_\alpha^4} - \tfrac{502-6713\alpha+12\,579\alpha^2-6867\alpha^3-630\alpha^4}{45\,360\sigma_\alpha^6} - \tfrac{280-1220\alpha+2051\alpha-2310\alpha^3+3789\alpha^4-2250\alpha^5}{6480\sigma_\alpha^8} \\
\leq\ & -\tfrac{7-20\alpha+20\alpha^2}{180\sigma_\alpha^4} - \tfrac{502-6713\alpha+12\,579\alpha^2-6867\alpha^3-630\alpha^4}{45\,360\sigma_\alpha^6} \\
\leq\ & -\tfrac{3(1-\alpha)(7-20\alpha+20\alpha^2)}{180\sigma_\alpha^6} - \tfrac{502-6713\alpha+12\,579\alpha^2-6867\alpha^3-630\alpha^4}{45\,360\sigma_\alpha^6} \\
=\ & \tfrac{1}{\sigma_\alpha^6}\cdot\big(-\tfrac{2897}{22\,680} + \tfrac{775\alpha}{1296} - \tfrac{2039\alpha^2}{2160} + \tfrac{349\alpha^3}{720} + \tfrac{\alpha^4}{72}\big) \\
=:\ & \tfrac{1}{\sigma_\alpha^6}\cdot h(\alpha)
\end{aligned}
$$

since $\sigma_\alpha^2 \geq 3(1-\alpha)$ in the last inequality and

$$
\begin{aligned}
& 280 - 1220\alpha + 2051\alpha - 2310\alpha^3 + 3789\alpha^4 - 2250\alpha^5 \\
=\ & (1-\alpha)(1-2\alpha)(280 - 380\alpha + 351\alpha^2 - 497\alpha^3) + 1596(1-\alpha)\alpha^4 + 340\alpha^5 \\
\geq\ & (1-\alpha)(1-2\alpha)(280 - \tfrac{380}{2} + \tfrac{351}{4} - \tfrac{497}{8}) \ = \ \tfrac{925}{8}\cdot(1-\alpha)(1-2\alpha) \ > \ 0
\end{aligned}
$$

in the penultimate inequality. Because of

$$
\begin{aligned}
h'(\alpha) \ &= \ \tfrac{775}{1296} - \tfrac{2039\alpha}{1080} + \tfrac{349\alpha^2}{240} + \tfrac{\alpha^3}{18} \ \geq \ \tfrac{775}{1296} - \tfrac{2039}{2\cdot 1080} + \tfrac{349}{4\cdot 240} + \tfrac{1}{8\cdot 18} \\
&= \ \tfrac{127}{5184} \ > \ 0
\end{aligned}
$$

we get $h(\alpha) \leq f(\frac{1}{2}) = -\frac{2897}{22\,680} + \frac{775}{2\cdot 1296} - \frac{2039}{4\cdot 2160} + \frac{349}{8\cdot 720} + \frac{1}{16\cdot 72} = -\frac{11}{3360} < 0$.

Hereby it follows finally $D_{k+1} - C_{k+1} - F_{k+1} \ \leq \ D_k - C_k - F_k$ and hence with the first part of the proof for the case $k = \lfloor \frac{n}{2} \rfloor$

$$
D_k - F_k - C_k \ \ \leq \ \ D_{\lfloor n/2 \rfloor} - F_{\lfloor n/2 \rfloor} - C_{\lfloor n/2 \rfloor} \ \leq \ 0
$$

and hence the left inequality in (31) in case of $k \geq 2$. If $k = 1$ we finally have

$$
\begin{aligned}
F_{4,\frac{1}{4}}(1) - \tfrac{1}{2} \ &= \ 0.2383... \ \geq \ 0.2305... \ = \ (\tfrac{4-2/4}{6} - \tfrac{2}{18\cdot 3})\cdot f_{4,\frac{1}{4}}(1) \\
F_{5,\frac{1}{5}}(1) - \tfrac{1}{2} \ &= \ 0.2373... \ \geq \ 0.2287... \ = \ (\tfrac{4-2/5}{6} - \tfrac{3}{18\cdot 4})\cdot f_{5,\frac{1}{5}}(1) \\
F_{6,\frac{1}{6}}(1) - \tfrac{1}{2} \ &= \ 0.2368... \ \geq \ 0.2277... \ = \ (\tfrac{4-2/6}{6} - \tfrac{4}{18\cdot 5})\cdot f_{6,\frac{1}{6}}(1).
\end{aligned}
$$

□

Lemma 11.10. *Let $k, n \in \mathbb{N}$ with $\frac{n}{6} \leq k \leq \frac{5n}{6} - 1$. For the $p_{n,k}$ unique defined by $F(k, n, p_{n,k}) = \frac{1}{2}$ we have*

$$p_{n,k} \quad \in \quad [\tfrac{k}{n} + \tfrac{1}{2n} + \tfrac{1-2(k+1)/n}{6n}, \tfrac{k}{n} + \tfrac{1}{2n} + \tfrac{1-2k/n}{6n}].$$

Proof. Existence and uniqueness of the $p_{n,k}$ follows from $\frac{\mathrm{d}}{\mathrm{d}p}F_{n,p}(k) = -nf_{n-1,p}(k) < 0$ and $F_{n,0}(k) = 1$ as $F_{n,1}(k) = 0$. Let now $\kappa := \frac{k}{n}$ and hence $\sigma_\kappa^2 := \frac{k(n-k)}{n}$ and first $k \leq \frac{n}{2}$. Then we have for $p_1 := \frac{k}{n} + \frac{1}{2n} + \frac{1-2k/n}{6n}$ the Fundamental theorem of calculus and Lemma 11.9 yield

$$\begin{aligned} F_{n,p_1}(k) &= F_{n,\kappa}(k) + \int_\kappa^{p_1} -n \cdot f_{n-1,p}(k)\mathrm{d}p \\ &\leq \tfrac{1}{2} + f_{n,\kappa}(k) \cdot \big(\tfrac{4-2\kappa}{6} - \tfrac{1-2\kappa}{72\sigma_\kappa^2} - n \cdot \int_\kappa^{p_1} \tfrac{f_{n-1,p}(k)}{f_{n,\kappa}(k)}\mathrm{d}p\big). \end{aligned}$$

Further we have

$$\begin{aligned} n\int_\kappa^{p_1} \tfrac{f_{n-1,p}(k)}{f_{n,\kappa}(k)}\mathrm{d}p &= n\int_\kappa^{p_1} (\tfrac{np}{k})^k \cdot (\tfrac{n(1-p)}{n-k})^{n-k-1}\mathrm{d}p = \int_0^{1/2+(1-2\kappa)/6} (\tfrac{t+k}{k})^k \cdot (\tfrac{-t+n-k}{n-k})^{n-k-1}\mathrm{d}t \\ &= \int_0^{(4-2\kappa)/6} (1 + \tfrac{t}{k})^k \cdot (1 - \tfrac{t}{n-k})^{n-k-1}\mathrm{d}t \geq \int_0^{(4-2\kappa)/6} 1 - \tfrac{t^2}{2\sigma_\kappa^2} + \tfrac{t}{n-k}\mathrm{d}t \\ &= \tfrac{4-2\kappa}{6} - \tfrac{(4-2\kappa)^2}{36\cdot\sigma_\kappa^2} \cdot \big(\tfrac{4-2\kappa}{36} - \tfrac{\kappa}{2}\big) = \tfrac{4-2\kappa}{6} - \tfrac{(4-2\kappa)^2(1-5\kappa)}{36\cdot 9\sigma_\kappa^2} \geq \tfrac{4-2\kappa}{6} - \tfrac{4(1-5\kappa)}{9\cdot 9\sigma_\kappa^2} \\ &\geq \tfrac{4-2\kappa}{6} - \tfrac{1-2\kappa}{72\sigma_\kappa^2} \end{aligned}$$

since $k\log(1+\frac{t}{k}) + (n-k-1)\log(1-\frac{t}{n-k}) \geq t - \frac{t^2}{2k} - t + \frac{t}{n-k} - \frac{t^2}{2(n-k} = -\frac{t^2}{2\sigma_\kappa^2} + \frac{t}{n-k}$ and $\mathrm{e}^x \geq 1 + x$ in the first and $\frac{1-2\kappa}{72} - \frac{4(1-5\kappa)}{81} = \frac{71\kappa}{324} - \frac{23}{648} \geq \frac{71}{324\cdot 6} - \frac{23}{648} = \frac{1}{972} > 0$ in the last inequality.

Thus we have

$$F_{n,p_1}(k) \quad \leq \quad \tfrac{1}{2}$$

and hence p_1 is an upper bound of $p_{n,k},$, since $F_{n,p}(k)$ is decreasing in p.

Let now $p_0 := \frac{k}{n} + \frac{1}{2n} + \frac{1-2(k+1)/n}{6n}$ and first $2 \leq k \leq \frac{n}{2}$ and hence $n-k \geq 2$ and $\sigma_\kappa^2 \geq 2(1-\kappa) \geq 1$. Beforehand if $t \leq \frac{2}{3}$ we have

$$\begin{aligned}(1+\tfrac{t}{k})^k \cdot (1-\tfrac{t}{n-k})^{n-k-1} &= \mathrm{e}^{k\log(1+\frac{t}{k})+(n-k-1)\log(1-\frac{t}{n-k})} \leq \mathrm{e}^{\frac{t}{n-k}-\frac{t^2}{2\sigma_\kappa^2}+\frac{t^3}{3k^2}+\frac{t^2}{2(n-k)^2}} \\ &\leq 1+\tfrac{t}{n-k}-\tfrac{t^2}{2\sigma_\kappa^2}+\tfrac{t^3}{3k^2}+\tfrac{t^2}{2(n-k)^2}+\sum_{r=2}^{\infty}\tfrac{1}{r!}\cdot\left|\tfrac{t}{n-k}-\tfrac{t^2}{2\sigma_\kappa^2}+\tfrac{t^3}{3k^2}+\tfrac{t^2}{2(n-k)^2}\right|^r \\ &\leq 1+\tfrac{t}{n-k}-\tfrac{t^2}{2\sigma_\kappa^2}+\tfrac{t^3}{3k^2}+\tfrac{t^2}{2(n-k)^2}+\tfrac{t^2}{2(n-k)^2}\cdot\sum_{r=0}^{\infty}(\tfrac{t}{3(n-k)})^r \\ &\leq 1+\tfrac{t}{n-k}-\tfrac{t^2}{2\sigma_\kappa^2}+\tfrac{t^3}{3k^2}+\tfrac{t^2}{2(n-k)^2}+\tfrac{5t^2}{8(n-k)^2}\end{aligned}$$

since $\frac{t}{n-k}-\frac{t^2}{2\sigma_\kappa^2}+\frac{t^3}{3k^2}+\frac{t^2}{2(n-k)^2} \leq \frac{t}{n-k}$ and $-(\frac{t}{n-k}-\frac{t^2}{2\sigma_\kappa^2}+\frac{t^3}{3k^2}+\frac{t^2}{2(n-k)^2}) \leq -\frac{t}{n-k}+\frac{t^2}{2\sigma_\kappa^2} = \frac{t}{n-k}-\frac{t}{\sigma_\kappa^2}\cdot(2\kappa-\frac{t}{2}) \leq \frac{t}{n-k}$ in the penultimate and $\frac{t}{3(n-k)} \leq \frac{1}{9} \leq \frac{1}{5}$ in the ultimate inequality.

Thus we have analogous to above, only estimated in the other direction with $n \geq 2k \geq 4$

$$\begin{aligned} n\int_\kappa^{p_0}\tfrac{f_{n-1,p}(k)}{f_{n,\kappa}(k))}\mathrm{d}p &= \int_0^{(4-2\kappa-2/n)/6}(1+\tfrac{t}{k})^k\cdot(1-\tfrac{t}{n-k})^{n-k-1}\mathrm{d}t \\ &\leq \int_0^{(4-2\kappa-2/n)/6} 1+\tfrac{t}{n-k}-\tfrac{t^2}{2\sigma_\kappa^2}+\tfrac{t^3}{3k^2}+\tfrac{t^2}{2(n-k)^2}+\tfrac{5t^2}{8(n-k)^2}\mathrm{d}t \\ &= \left[t+\tfrac{t^2\kappa}{2\sigma_\kappa^2}-\tfrac{t^3}{6\sigma_\kappa^2}+\tfrac{t^4(1-\kappa)^2}{12\sigma_\kappa^4}+\tfrac{3t^3\kappa^2}{8\sigma_\kappa^4}\right]_{t=0}^{(2-\kappa-1/n)/3} \\ &\leq \tfrac{2-\kappa}{3}-\tfrac{1}{3n}+\tfrac{(2-\kappa)^2\kappa}{18\sigma_\kappa^2}-\tfrac{(2-\kappa)\kappa}{9\sigma_\kappa^2 n}+\tfrac{\kappa}{18\sigma_\kappa^2 n^2}-\tfrac{(2-\kappa)^3}{162\sigma_\kappa^2}+\tfrac{(2-\kappa)^2}{54n\sigma_\kappa^2}-\tfrac{2-\kappa}{54\sigma_\kappa^2 n^2}+\tfrac{1}{162\cdot 16\sigma_\kappa n} \\ &\quad +\tfrac{(2-\kappa)^4(1-\kappa)^2}{12\cdot 3^4\sigma_\kappa^4}+\tfrac{(2-\kappa)^3\kappa^2}{72\sigma_\kappa^4} \\ &= \tfrac{2-\kappa}{3}-\tfrac{1-2\kappa}{18\sigma_\kappa^2}+\tfrac{1}{\sigma_\kappa^2}\cdot\left(-\tfrac{\kappa(1-\kappa)}{3}+\tfrac{(2-\kappa)^2\kappa}{18}-\tfrac{(2-\kappa)^3}{162}+\tfrac{1-2\kappa}{18}\right)+\tfrac{1}{n^2\sigma_\kappa^2}\cdot\left(\tfrac{\kappa}{18}-\tfrac{2-\kappa}{54}\right) \\ &\quad +\tfrac{1}{\sigma_\kappa^4}\cdot\left(-\tfrac{(2-\kappa)\kappa^2(1-\kappa)}{9}+\tfrac{(2-\kappa)^2\kappa(1-\kappa)}{54}+\tfrac{(2-\kappa)^4(1-\kappa)^2}{12\cdot 3^4}+\tfrac{(2-\kappa)^3\kappa^2}{72}+\tfrac{\kappa(1-\kappa)}{2592}\right) \\ &= \tfrac{2-\kappa}{3}-\tfrac{1-2\kappa}{18\sigma_\kappa^2}+\underbrace{\tfrac{1-24\kappa+12\kappa^2+10\kappa^3}{162\sigma_\kappa^2}+\tfrac{128+67\kappa-1187\kappa^2+1312\kappa^3-32\kappa^4-188\kappa^5+8\kappa^6}{7776\sigma_\kappa^4}}_{=:h(\kappa)}-\tfrac{1-2\kappa}{27n^2\sigma_\kappa^2}.\end{aligned}$$

If the σ_κ^{-4} term is negative it follows from $1-24\kappa+12\kappa^2+10\kappa^3 \leq 1-\frac{24}{6}+\frac{12}{36}+\frac{10}{216} = -\frac{283}{108}$ that $h(\kappa) \leq 0$. If this is not the case, we have with $\sigma_\kappa^2 \geq 1$

$$\begin{aligned} h(\kappa) &\leq \tfrac{1-24\kappa+12\kappa^2+10\kappa^3}{162\sigma_\kappa^2}+\tfrac{128+67\kappa-1187\kappa^2+1312\kappa^3-32\kappa^4-188\kappa^5+8\kappa^6}{7776\sigma_\kappa^2} \\ &= \tfrac{176-1085\kappa-611\kappa^2+1792\kappa^3-32\kappa^4-188\kappa^5+8\kappa^6}{7776\sigma_\kappa^2} =: i(\kappa).\end{aligned}$$

Because of

$$\begin{aligned} i'(\kappa) &= \tfrac{-1085-1222\kappa+5376\kappa^2-128\kappa^3-940\kappa^4+48\kappa^5}{7776\sigma_\kappa^2} \leq \tfrac{-1085-1222\kappa+2688\kappa}{7776\sigma_\kappa^2} \leq \tfrac{-1085+(2688-1222)/2}{7776\sigma_\kappa^2} \\ &= -\tfrac{352}{7776\sigma_\kappa^2} < 0\end{aligned}$$

we get $i(\kappa) \le i(\frac{1}{6}) = -\frac{39\,535}{22\,674\,816} < 0$ and hence in both cases $h(\kappa) \le 0$ for all $\frac{1}{6} \le \kappa \le \frac{1}{2}$.

Thereby we finally have with Lemma 11.9 analogous to the estimation at the beginning of this poof

$$\begin{aligned}
F_{n,p_0}(k) &= F_{n,\kappa}(k) + \int_{\kappa}^{p_0} -n \cdot f_{n-1,p}(k)\mathrm{d}p \\
&\ge \tfrac{1}{2} + f_{n,\kappa}(k) \cdot \big(\tfrac{4-2\kappa}{6} - \tfrac{1-2\kappa}{18\sigma_\kappa^2} - n \cdot \int_{\kappa}^{p_0} \tfrac{f_{n-1,p}(k)}{f_{n,\kappa}(k)}\mathrm{d}p\big) \\
&= \tfrac{1}{2} + f_{n,\kappa}(k) \cdot \big(\tfrac{4-2\kappa}{6} - \tfrac{1-2\kappa}{18\sigma_\kappa^2} - \int_0^{(4-2\kappa-2/n)/6} (1+\tfrac{t}{k})^k \cdot (1-\tfrac{t}{n-k})^{n-k-1}\mathrm{d}t\big) \\
&\ge \tfrac{1}{2} + f_{n,\kappa}(k) \cdot \big(\tfrac{4-2\kappa}{6} - \tfrac{1-2\kappa}{18\sigma_\kappa^2} - (\tfrac{2-\kappa}{3} - \tfrac{1-2\kappa}{18\sigma_\kappa^2})\big) \;=\; \tfrac{1}{2}.
\end{aligned}$$

Thus we have $F_{n,p_1}(k) \le \frac{1}{2} \le F_{n,p_0}(k)$ and hence $p_{n,k} \in [p_0, p_1]$ in the case $2 \le k \le \frac{n}{2}$.

The case $k = 0$ can not occur according to requirement and if $k = 1$ we have for the admissible $n = 2, 3, 4, 5, 6$ numerically

$$\begin{aligned}
F_{2,\frac{1}{2}+\frac{1}{4}+\frac{1-2/2}{12}}(1) &= 0.4375... \le \tfrac{1}{2} \le 0.5556... = F_{2,\frac{1}{2}+\frac{1}{4}+\frac{1-4/2}{12}}(1) \\
F_{3,\frac{1}{3}+\frac{1}{6}+\frac{1-2/3}{18}}(1) &= 0.4722... \le \tfrac{1}{2} \le 0.5278... = F_{3,\frac{1}{3}+\frac{1}{6}+\frac{1-4/3}{18}}(1) \\
F_{4,\frac{1}{4}+\frac{1}{8}+\frac{1-2/4}{24}}(1) &= 0.4824... \le \tfrac{1}{2} \le 0.5188... = F_{4,\frac{1}{4}+\frac{1}{8}+\frac{1-4/4}{24}}(1) \\
F_{5,\frac{1}{5}+\frac{1}{10}+\frac{1-2/5}{30}}(1) &= 0.4875... \le \tfrac{1}{2} \le 0.5145... = F_{5,\frac{1}{5}+\frac{1}{10}+\frac{1-4/5}{30}}(1) \\
F_{6,\frac{1}{6}+\frac{1}{12}+\frac{1-2/6}{36}}(1) &= 0.4906... \le \tfrac{1}{2} \le 0.5121... = F_{6,\frac{1}{6}+\frac{1}{12}+\frac{1-4/6}{36}}(1).
\end{aligned}$$

If now $\frac{n}{2} \le k \le \frac{5}{6}n - 1$, we have with the symmetries $F_{n,p}(k) = 1 - F_{n,1-p}(n-k-1)$

$$\begin{aligned}
F_{n,\frac{k}{n}+\frac{1}{2n}+\frac{1-2(k+1)/n}{6n}}(k) &= 1 - F_{n,1-\frac{k}{n}-\frac{1}{2n}-\frac{1-2(k+1)/n}{6n}}(n-k-1)) \\
&= 1 - F_{n,\frac{n-k-1}{n}+\frac{1}{2n}+\frac{1-2(n-k-1)/n}{6n}}(n-k-1) \\
&\ge 1 - \tfrac{1}{2} \;=\; \tfrac{1}{2}
\end{aligned}$$

since with $\frac{n}{6} \le n-k-1 \le \frac{n}{2}$ we can apply the previous case. Analogous to above we also have

$$\begin{aligned}
F_{n,\frac{k}{n}+\frac{1}{2n}+\frac{1-2k/n}{6n}}(k) &= 1 - F_{n,1-\frac{k}{n}-\frac{1}{2n}-\frac{1-2k/n}{6n}}(n-k-1) \\
&= 1 - F_{n,\frac{n-k-1}{n}+\frac{1}{2n}+\frac{1-2(n-k-1+1)/n}{6n}}(n-k-1) \;\le\; \tfrac{1}{2}.
\end{aligned}$$

□

The next lemma is not actually used in this work and may hence be skipped. It was used in an earlier proof of Lemma 11.14, and is kept here for possible use elsewhere.

Lemma 11.11. *Let $n \in \mathbb{N}$ and $p \in (0, \frac{1}{2}]$. Further let as usual $m = \lfloor np \rfloor$ and $\delta = m + 1 - np \in (0, 1]$ as $\lambda = \frac{m+1}{n} \leq \frac{n/2+1}{n}$. Then the function*

$$h(\delta) \;:=\; f_{n,p}(m) \cdot \sigma \;=\; f_{n,\frac{m+1-\delta}{n}}(m) \cdot \sqrt{\tfrac{(m+1-\delta)(n-m-1+\delta)}{n}}$$

becomes maximal at the point $\delta_0 = \frac{n}{n+1} \cdot (\frac{1}{2} + \lambda) \leq 1$.

Proof. 1.) $h'(\delta_0) = 0$: We have

$$\begin{aligned} h'(\delta) &= (f_{n-1,p}(m) - f_{n-1,p}(p)) \cdot \sigma - \tfrac{1-2\lambda+2\delta/n}{2\sigma} \cdot f_{n,p}(m) \\ &= \tfrac{f_{n,p}(m)}{\sigma} \cdot \big(\tfrac{f_{n-1,p}(m) - f_{n-1,p}(m))}{f_{n,p}(m)} \cdot \sigma^2 - \tfrac{1-2\lambda+2\delta/n}{2}\big) \\ &= \tfrac{f_{n,p}(m)}{\sigma} \cdot \big((\tfrac{n-m}{n(1-p)} - \tfrac{m}{np}) \cdot np(1-p) - \tfrac{1-2\lambda+2\delta/n}{2}\big) \\ &= \tfrac{f_{n,p}(m)}{\sigma} \cdot \big((n-m)p - m(1-p) - \tfrac{1-2\lambda+2\delta/n}{2}\big) \\ &= \tfrac{f_{n,p}(m)}{\sigma} \cdot \big((1-\delta) - \tfrac{1-2\lambda+2\delta/n}{2}\big) \\ &= \tfrac{f_{n,p}(m)}{\sigma} \cdot \big(\tfrac{1}{2} + \lambda - \tfrac{n+1}{n}\delta\big) \end{aligned}$$

and hence obviously $h'(\delta_0) = 0$.

2.) $h''(\delta_0) < 0$: We have

$$\begin{aligned} h''(\delta_0) &= (\tfrac{f_{n,p}(m)}{\sigma})' \cdot [\tfrac{1}{2} + \lambda - \tfrac{n+1}{n}\delta_0] - \tfrac{n+1}{n} \cdot \tfrac{f_{n,p}(m)}{\sigma} \\ &= -\tfrac{n+1}{n} \cdot \tfrac{f_{n,p}(m)}{\sigma} \;<\; 0. \end{aligned}$$

3.) The computation in 1.) yields that δ_0 is the only root of $h'(\delta)$ and hence there can be no local minimum, consequently δ_0 is global maximum. ☐

Lemma 11.12. *If $n \in \mathbb{N}$ and $p \in [\frac{1}{2}, 1)$, then we have*

$$\tfrac{f(k)}{g(k)} \text{ is decreasing on } \{k \in \mathbb{Z} : np \leq k \leq n\}.$$

Proof. The claim is equivalent to $\frac{f(k+1)}{f(k)} < \frac{g(k+1)}{g(k)}$ for $np \leq k \leq n-1$. We have due to Hipp/Mattner (2007, Lemma 2.1)

$$(1+x)\mathrm{e}^{-x} \quad \leq \quad (1+y)\mathrm{e}^{-y} \quad \text{if } x, y \in \mathbb{R},\ 0 \leq y \leq |x|.$$

If we now set $x := -\frac{k-np}{nq}$, $y := \frac{k-np}{np} \geq 0$, then we have

$$|x| - y = \tfrac{k-np}{nq} - \tfrac{k-np}{np} = \tfrac{k-np}{nq}(1 - \tfrac{q}{p}) = \tfrac{k-np}{nq} \cdot \tfrac{p-q}{p} \geq 0$$

and thus the condition above is satisfied and it follows

$$\begin{aligned} \tfrac{f(k+1)}{f(k)} &= \frac{\binom{n}{k+1}p^{k+1}q^{n-k-1}}{\binom{n}{k}p^k q^{n-k}} = \tfrac{n-k}{k+1} \cdot \tfrac{p}{q} < \tfrac{n-k}{k} \cdot \tfrac{p}{q} = (1 - \tfrac{k-np}{nq})/(1 + \tfrac{k-np}{np}) \\ &\leq \mathrm{e}^{-\frac{k-np}{nq} - \frac{k-np}{np}} = \mathrm{e}^{-\frac{k-np}{npq}}. \end{aligned}$$

Further Hipp/Mattner (2007, Lemma 2.2) yields if $x, y, z \in \mathbb{R}$ and if $x \geq 0$, $y < z$ and $x + y + z > 0$

$$\mathrm{e}^{-\frac{x}{2}\cdot(x+y+z)} \quad \leq \quad \tfrac{\Phi(z+x) - \Phi(y+x)}{\Phi(z) - \Phi(y)} \quad \leq \quad \mathrm{e}^{-\frac{x}{2}\cdot(x+2y)}.$$

This time we set $x := \frac{1}{\sqrt{npq}}$, $y := \frac{k-np-1}{\sqrt{npq}}$ and $z := \frac{k-np}{\sqrt{npq}}$ and since $k \geq np$ we get

$$x \geq 0 \quad , \quad y < z \quad , \quad x + y + z = 2z \geq 0$$

and thus it follows

$$\tfrac{g(k+1)}{g(k)} = \tfrac{G(k+1) - G(k)}{G(k) - G(k-1)} = \tfrac{\Phi(z+x) - \Phi(y+x)}{\Phi(z) - \Phi(y)} \geq \mathrm{e}^{-\frac{x}{2}(x+y+z)} = \mathrm{e}^{-xz} = \mathrm{e}^{-\frac{k-np}{npq}}$$

and hence the claim. ∎

Lemma 11.13. *Let $n \in \mathbb{N}$ odd with $n \geq 3$ and $p = \frac{1}{2} - \frac{y}{n}$ for an $y \in [0, \frac{1}{2}]$ and hence $m := \lfloor np \rfloor = \frac{n-1}{2}$. Then we have*

(i) $f_{n-1,\frac{1}{2}}(m) \cdot y \cdot (1 - \frac{2(n-1)y^3}{3n^2}) \;\leq\; F_{n,p}(m) - \frac{1}{2} \;\leq\; f_{n-1,\frac{1}{2}}(m) \cdot y$

(ii) $-\frac{1/2-y}{\sqrt{2\pi}\,\sigma} \;\leq\; G_{n,p}(m) - \frac{1}{2} \;\leq\; -\frac{1/2-y}{\sqrt{2\pi}\,\sigma} + \frac{(1-2y)^3}{12\sqrt{2\pi}\,\sigma(n-1)}$

(iii) $f_{n,p}(m) \;\leq\; f_{n-1,\frac{1}{2}}(m) \cdot (1 - \frac{17(n-1)y^2}{12n^2} - \frac{1-2y}{n+1})$

(iv) $f_{n,p}(m+1) \;\leq\; f_{n-1,\frac{1}{2}}(m) \cdot (1 - \frac{1+2y}{n+1})$.

Proof. on (i): Since $\frac{\mathrm{d}}{\mathrm{d}p} F_{n,p}(m) = -n \cdot f_{n-1,p}(m)$ and $n - m - 1 = m$ we have

$$
\begin{aligned}
F_{n,p}(m) - \tfrac{1}{2} &= F_{n,p}(m) - F_{n,\frac{1}{2}}(m)) \;=\; -n \cdot \int_{1/2}^{p} f_{n-1,t}(m))\mathrm{d}t \\
&= n \cdot \int_{1/2-y/n}^{\frac{1}{2}} f_{n-1,t}(m)\mathrm{d}t \;=\; \int_{0}^{y} f_{n-1,\frac{1}{2}-\frac{t}{n}}(m)\mathrm{d}t \\
&= f_{n-1,\frac{1}{2}}(m) \cdot \int_{0}^{y} \frac{f_{n-1,\frac{1}{2}-\frac{t}{n}}(m)}{f_{n-1,\frac{1}{2}}(m)}\mathrm{d}t \;=\; f_{n-1,\frac{1}{2}}(m) \cdot \int_{0}^{y} (1 - \tfrac{2t}{n})^m (1 + \tfrac{2t}{n})^m \mathrm{d}t \\
&= f_{n-1,\frac{1}{2}}(m) \cdot \int_{0}^{y} (1 - \tfrac{4t^2}{n^2})^m \mathrm{d}t.
\end{aligned}
$$

From $1 - \frac{4t^2}{n^2} \leq 1$ the estimation upwards follows directly. On the other side integrating after estimating $(1 - \frac{4t^2}{n^2})^m \geq 1 - m \cdot \frac{4t^2}{n^2} = 1 - \frac{2(n-1)t^2}{n^2}$ due to Bernoulli's inequality yields the estimation downwards.

on (ii): First we have

$$
G_{n,p}(m) - \tfrac{1}{2} \;=\; \Phi(\tfrac{(n-1)/2-np}{\sigma} - \tfrac{1}{2} \;=\; \Phi(\tfrac{y-1/2}{\sigma}) - \tfrac{1}{2} \;=\; \tfrac{1}{2} - \Phi(\tfrac{1/2-y}{\sigma})
$$

and hence Lemma 11.1 yields immediately the estimation downwards as well as

$$
\tfrac{1}{2} - \Phi(\tfrac{1/2-y}{\sigma}) \;\leq\; -\tfrac{1/2-y}{\sqrt{2\pi}\,\sigma} \cdot \mathrm{e}^{-\frac{(1/2-y)^2}{6\sigma^2}} \;\leq\; -\tfrac{1/2-y}{\sqrt{2\pi}\,\sigma} \cdot (1 - \tfrac{(1/2-y)^2}{6(n-1)/4}) \;=\; -\tfrac{1/2-y}{\sqrt{2\pi}\,\sigma} + \tfrac{(1-2y)^3}{12(n-1)}
$$

since $\sigma^2 \geq \frac{(n-1)(n+1)}{4n} \geq \frac{n-1}{4}$ and hence also the estimation upwards.

on (iii): We have

$$
\begin{aligned}
\frac{f_{n,p}(m)}{f_{n-1,\frac{1}{2}}(m))} &= \frac{n}{m+1}\cdot(2p)^m(2(1-p))^m(1-p)\\
&= \frac{2(1-p)n}{n+1}\cdot(4p(1-p))^m\\
&= \frac{n+2y}{n+1}\cdot(1-\frac{4y^2}{n^2})^m \;\le\; \frac{n+2y}{n+1}\cdot(1-\frac{4y^2k}{n^2}\cdot(1-\frac{4y^2k}{2n^2}))\\
&= (1-\frac{1-2y}{n+1})\cdot(1-\frac{2y^2(n-1)}{n^2}\cdot(1-\frac{y^2(n-1)}{n^2}))\\
&\le (1-\frac{1-2y}{n+1})\cdot(1-\frac{2y^2(n-1)}{n^2}\cdot(1-\frac{2}{4\cdot 9}))\\
&= (1-\frac{1-2y}{n+1})\cdot(1-\frac{17y^2(n-1)}{9n^2})\\
&= 1-\frac{1-2y}{n+1}-\frac{17y^2(n-1)}{9n^2}\cdot(1-\frac{1-2y}{n+1})\\
&\le 1-\frac{1-2y}{n+1}-\frac{17y^2(n-1)}{9n^2}\cdot\frac{3}{4}
\end{aligned}
$$

since $(1-x)^d = \mathrm{e}^{d\log(1-x)} \le \mathrm{e}^{-dx} \le 1-dx+(dx)^2/2$ if $x, d \ge 0$.

on (iv): Since $p(1-p) \le \frac{1}{4}$ we have

$$
\begin{aligned}
\frac{f_{n,p}(m+1)}{f_{n-1,\frac{1}{2}}(m))} &= \frac{n}{m+1}\cdot(2p)^m(2(1-p))^m\cdot p \;=\; \frac{2pn}{n+1}\cdot(4p(1-p))^m\\
&\le \frac{2pn}{n+1} \;=\; \frac{n-2y}{n+1} \;=\; 1-\frac{1+2y}{n+1}.
\end{aligned}
$$

□

In the following lemma, sufficient for this work as it stands, the somewhat unnatural assumption "$\lfloor np \rfloor \neq np$" was introduced to shorten the proof a bit.

Lemma 11.14. *Let* $n, m \in \mathbb{N},\ n \geq 6$ *and* $p \in [\frac{1}{6}, \frac{5}{6}]$ *with* $m := \lfloor np \rfloor \neq np$. *Then we have*

$$\begin{array}{ll} (i) & f(m) \ < \ \frac{2}{\sqrt{2\pi}} \cdot \frac{p^2+q^2}{\sigma} \\ (ii) & f(m+1) \ < \ \frac{2}{\sqrt{2\pi}} \cdot \frac{p^2+q^2}{\sigma} \\ (iii) & g(m+1) \ < \ \frac{2}{\sqrt{2\pi}} \cdot \frac{p^2+q^2}{\sigma} \\ (iv) & f(m) + f(m+1) - g(m+1) \ < \ \frac{2}{\sqrt{2\pi}} \cdot \frac{p^2+q^2}{\sigma}. \end{array}$$

Proof. We may assume $p \leq \frac{1}{2}$ in each of the four cases, since with $\lfloor np \rfloor = n - \lceil nq \rceil$ and $\lceil nq \rceil = \lfloor nq \rfloor + 1$ for $p > \frac{1}{2}$ and hence $q < \frac{1}{2}$ we have

$$\begin{aligned} f_{n,p}(\lfloor np \rfloor) &= f_{n,q}(\lceil nq \rceil) \ = \ f_{n,q}(\lfloor nq \rfloor + 1) \\ g_{n,p}(\lfloor np \rfloor + 1) &= \Phi(\tfrac{\lfloor np \rfloor + 1 - np}{\sigma}) - \Phi(\tfrac{\lfloor np \rfloor - np}{\sigma}) \ = \ \Phi(\tfrac{np - \lfloor np \rfloor}{\sigma}) - \Phi(\tfrac{\lfloor np \rfloor - np - 1}{\sigma}) \\ &= \Phi(\tfrac{\lfloor nq \rfloor + 1 - nq}{\sigma}) - \Phi(\tfrac{\lfloor nq \rfloor - nq}{\sigma}) \\ &= g_{n,q}(\lfloor nq \rfloor + 1). \end{aligned}$$

<u>on (i):</u> $f(m) < \frac{2}{\sqrt{2\pi}} \cdot \frac{p^2+q^2}{\sigma}$

Due to Lemma 11.4, $\sqrt{1-x} \leq 1 - \frac{x}{2} \leq \mathrm{e}^{-x/2}$ if $0 < x < 1$, and $\ell \log(1 - \frac{x}{\ell}) + (n - \ell) \log(1 + \frac{x}{n-\ell}) \leq -\frac{x^2}{2\ell} - \frac{x^2}{2(n-\ell)} - \frac{x^3}{3\ell^2} + \frac{x^3}{3(n-\ell)^2} - \frac{x^4}{4\ell^3} - \frac{x^4}{4(n-\ell)^3} = -\frac{x^2}{2\sigma_\lambda^2} - \frac{x^3(1-2\lambda)}{3\sigma_\lambda^4)} - \frac{x^4(\lambda^3+(1-\lambda)^3)}{4\sigma_\lambda^6}$ we have

$$\begin{aligned} f(m) \cdot \sqrt{2\pi}\,\sigma &= \sqrt{2\pi}\,\sigma_\lambda \cdot f_{n,\lambda}(\ell) \cdot \tfrac{f_{n,p}(m)}{f_{n,\lambda}(\ell)} \cdot \tfrac{\sigma}{\sigma_\lambda} \\ &= \sqrt{2\pi}\,\sigma_\lambda \cdot f_{n,\lambda}(\ell) \cdot \tfrac{\ell(1-p)}{(n-\ell+1)p} \cdot (\tfrac{np}{\ell})^\ell (\tfrac{n(1-p)}{n-\ell})^{n-\ell} \cdot \sqrt{\tfrac{np \cdot n(1-p)}{\ell(n-\ell)}} \\ &= \sqrt{2\pi}\,\sigma_\lambda \cdot f_{n,\lambda}(\ell) \cdot (1 + \tfrac{\delta(n+1)-\ell}{(\ell-\delta)(n-\ell+1)}) \cdot (1 - \tfrac{\delta}{\ell})^\ell (1 + \tfrac{\delta}{n-\ell})^{n-\ell} \cdot \sqrt{1 - \tfrac{(1-2\lambda)\delta + \delta^2/n}{\sigma_\lambda^2}} \\ &\leq \mathrm{e}^{-\frac{19}{320\sigma_\lambda^2}} \cdot \mathrm{e}^{\frac{\delta(n+1)-\ell}{(\ell-\delta)(n-\ell+1)}} \cdot \mathrm{e}^{-\frac{\delta^2}{2\sigma_\lambda^2} - \frac{\delta^3(1-2\lambda)}{3\sigma_\lambda^4} - \frac{\delta^4(\lambda^3+(1-\lambda)^3)}{2\sigma_\lambda^6}} \cdot \mathrm{e}^{-\frac{(1-2\lambda)\delta+\delta^2/n}{2\sigma_\lambda^2}}. \end{aligned}$$

If $\delta \leq \frac{\ell}{n+1}$, then obviously all exponents are negative and it follows $f(m) \cdot \sqrt{2\pi}\,\sigma \ \leq 1 \leq 2(p^2+q^2)$, so we may assume $\delta > \frac{\ell}{n+1}$ and then we have

$$
\begin{aligned}
\tfrac{\delta(n+1)-\ell}{(\ell-\delta)(n-\ell+1)} &= \tfrac{(n+1)\delta/n-\lambda}{\sigma_\lambda^2}\cdot\tfrac{\ell}{\ell-\delta}\cdot\tfrac{n-\ell}{n-\ell+1}\\
&= \tfrac{(n+1)\delta/n-\lambda}{\sigma_\lambda^2}\cdot(1+\tfrac{\delta}{\ell}+\tfrac{\delta^2}{\ell^2}\cdot\tfrac{\ell}{\ell-\delta})\cdot(1-\tfrac{1}{n-\ell}\cdot\tfrac{n-\ell}{n-\ell+1})\\
&\le \tfrac{(n+1)\delta/n-\lambda}{\sigma_\lambda^2}\cdot(1+\tfrac{\delta(1-\lambda)}{\sigma_\lambda^2}+\tfrac{\delta^2(1-\lambda)^2}{\sigma_\lambda^4}\cdot\tfrac{3}{2})\cdot(1-\tfrac{\lambda}{\sigma_\lambda^2}\cdot\tfrac{3}{4})\\
&\le (\tfrac{\delta-\lambda}{\sigma_\lambda^2}+\tfrac{\delta\lambda(1-\lambda)+(7\delta/6-\lambda)\delta(1-\lambda)}{\sigma_\lambda^4}+\tfrac{3(7\delta/6-\lambda)\delta^2(1-\lambda)^2}{2\sigma_\lambda^6})\cdot(1-\tfrac{\lambda}{\sigma_\lambda^2}\cdot\tfrac{3}{4})\\
&\le \tfrac{\delta-\lambda}{\sigma_\lambda^2}+\tfrac{\delta\lambda(1-\lambda)+(7\delta/6-\lambda)\delta(1-\lambda)-3(\delta-\lambda)\lambda/4}{\sigma_\lambda^4}\\
&\quad+\tfrac{3(7\delta/6-\lambda)\delta^2(1-\lambda)^2/2-3\lambda\cdot(\delta\lambda(1-\lambda)+(7\delta/6-\lambda)\delta(1-\lambda))/4}{\sigma_\lambda^6}\\
&= \tfrac{\delta-\lambda}{\sigma_\lambda^2}+\tfrac{7\delta^2(1-\lambda)/6-3(\delta-\lambda)\lambda/4}{\sigma_\lambda^4}+\tfrac{(7\delta/6-\lambda)\delta(1-\lambda)\cdot(3\delta(1-\lambda)/2-3\lambda/4)-3\delta\lambda^2(1-\lambda)/4}{\sigma_\lambda^6},
\end{aligned}
$$

where we used $n\ge 6$, $\ell-\delta\ge\ell-1\ge 2$ and $n-\ell\ge n/2\ge 3$ in the first inequality and $\frac{n+1}{n}\ge\frac{7}{6}$ in the second one. On the other hand we have with $0\le 1-4pq\le 1-4\cdot\frac{5}{36}=\frac{4}{9}$

$$
\begin{aligned}
\log(2(p^2+q^2)) &= \log(1+1-4pq)\ \ge\ 1-4pq-\tfrac{(1-4pq)^2}{2}\ =\ (1-8p^2q^2)\\
&= \tfrac{1}{2}-8\cdot\big(\lambda(1-\lambda)-\tfrac{\delta}{n}\cdot(1-2\lambda+\tfrac{\delta}{n})\big)^2\\
&= \tfrac{1}{2}-8\lambda^2(1-\lambda)^2+\tfrac{16\delta\lambda(1-\lambda)(1-2\lambda+\delta/n)}{n}-\tfrac{8\delta^2(1-2\lambda+\delta/n)^2}{n^2}\\
&= \tfrac{(1-2\lambda)^2(1+4\lambda(1-\lambda))}{2}+\tfrac{16\lambda^2(1-\lambda)^2(1-2\lambda)\delta}{\sigma_\lambda^2}+\tfrac{8\delta^2\lambda^2(1-\lambda)^2\cdot(2\lambda(1-\lambda)-(1-2\lambda)^2)}{\sigma_\lambda^4}\\
&\quad-\tfrac{16\delta^3\lambda^3(1-\lambda)^3(1-2\lambda)}{\sigma_\lambda^6}-\tfrac{8\delta^4\lambda^4(1-\lambda)^4}{\sigma_\lambda^8}\\
&\ge \tfrac{3(1-\lambda)(1-2\lambda)^2(1+4\lambda(1-\lambda))}{2\sigma_\lambda^2}+\tfrac{16\lambda^2(1-\lambda)^2(1-2\lambda)\delta}{\sigma_\lambda^2}+\tfrac{8\delta^2\lambda^2(1-\lambda)^2\cdot(2\lambda(1-\lambda)-(1-2\lambda)^2)}{\sigma_\lambda^4}\\
&\quad-\tfrac{16\delta^3\lambda^3(1-\lambda)^3(1-2\lambda)}{\sigma_\lambda^6}-\tfrac{8\delta^4\lambda^4(1-\lambda)^3}{3\sigma_\lambda^6}\\
&> \tfrac{3(1-\lambda)(1-2\lambda)^2(1+4\lambda(1-\lambda))}{2\sigma_\lambda^2}+\tfrac{40\lambda^2(1-\lambda)^2(1-2\lambda)\delta}{3\sigma_\lambda^2}+\tfrac{16\delta^2\lambda^3(1-\lambda)^3}{\sigma_\lambda^4}-\tfrac{8\delta^4\lambda^4(1-\lambda)^3}{3\sigma_\lambda^6}
\end{aligned}
$$

since $\sigma_\lambda^2\ge 3(1-\lambda)$ in the first and $16-\frac{8\delta(1-2\lambda)}{\sigma_\lambda^2}-\frac{16\delta^2\lambda(1-\lambda)}{\sigma_\lambda^4}>16-\frac{8(1-2\lambda)}{3(1-\lambda)}-\frac{16\lambda(1-\lambda)}{9(1-\lambda)^2}\cdot\frac{3}{2}=\frac{40}{3}$ in the second inequality.

Thus it follows together

$$
\begin{aligned}
\log(\tfrac{f(m)\sqrt{2\pi}\sigma}{2(p^2+q^2)}) &< -\tfrac{19}{320\sigma_\lambda^2}-\tfrac{\delta^2}{2\sigma_\lambda^2}-\tfrac{\delta^3(1-2\lambda)}{3\sigma_\lambda^4}-\tfrac{\delta^4(\lambda^3+(1-\lambda)^3)}{4\sigma_\lambda^6}-\tfrac{(1-2\lambda)\delta+\delta^2/n}{2\sigma_\lambda^2}\\
&\quad+\tfrac{\delta-\lambda}{\sigma_\lambda^2}+\tfrac{7\delta^2(1-\lambda)/6-3(\delta-\lambda)\lambda/4}{\sigma_\lambda^4}+\tfrac{(7\delta/6-\lambda)\delta(1-\lambda)\cdot(3\delta(1-\lambda)/2-3\lambda/4)-3\delta\lambda^2(1-\lambda)/4}{\sigma_\lambda^6}\\
&\quad-\tfrac{3(1-2\lambda)^2}{2\sigma_\lambda^2}-\tfrac{40\lambda^2(1-\lambda)^2(1-2\lambda)\delta}{3\sigma_\lambda^2}-\tfrac{16\delta^2\lambda^3(1-\lambda)^3}{\sigma_\lambda^4}+\tfrac{8\delta^4\lambda^4(1-\lambda)^3}{3\sigma_\lambda^6}\\
&= \tfrac{1}{\sigma_\lambda^2}\cdot\Big(-\tfrac{19}{320}+\delta-\lambda-\tfrac{\delta^2}{2}-\tfrac{(1-2\lambda)\delta}{2}-\tfrac{3(1-2\lambda)^2}{2}-\tfrac{40\delta\lambda^2(1-\lambda)^2(1-2\lambda)}{3}\Big)\\
&\quad+\tfrac{1}{\sigma_\lambda^4}\cdot\Big(-\tfrac{\delta^3(1-2\lambda)}{3}-\tfrac{\delta^2\lambda(1-\lambda)}{2}+\tfrac{7\delta^2(1-\lambda)}{6}-\tfrac{3(\delta-\lambda)\lambda}{4}-16\delta^2\lambda^3(1-\lambda)^3\Big)\\
&\quad-\tfrac{\delta^4(\lambda^3+(1-\lambda)^3)}{2\sigma_\lambda^6}+\tfrac{\delta(1-\lambda)}{\sigma_\lambda^6}\cdot(\tfrac{7\delta}{6}-\lambda)(\tfrac{3\delta(1-\lambda)}{2}-\tfrac{3\lambda}{4})-\tfrac{3\delta\lambda^2(1-\lambda)}{4\sigma_\lambda^6}+\tfrac{8\delta^4\lambda^4(1-\lambda)^3}{3\sigma_\lambda^6}.
\end{aligned}
$$

We further have

$$\begin{aligned}
& -\tfrac{\delta^4(\lambda^3+(1-\lambda)^3)}{4} + (\tfrac{7\delta}{6}-\lambda)\delta(1-\lambda)\cdot(\tfrac{3\delta(1-\lambda)}{2}-\tfrac{3\lambda}{4}) - \tfrac{3\delta\lambda^2(1-\lambda)}{4} + \tfrac{8\delta^4\lambda^4(1-\lambda)^3}{3} \\
= \; & \delta^3(1-\lambda)\cdot\Big(-\tfrac{\delta(\lambda^3+(1-\lambda)^3)}{4(1-\lambda)} + (\tfrac{7}{6}-\tfrac{\lambda}{\delta})(\tfrac{3(1-\lambda)}{2}-\tfrac{3\lambda}{4\delta}) - \tfrac{3\lambda^2}{4\delta^2} + \tfrac{8\delta\lambda^4(1-\lambda)^2}{3}\Big) \\
= \; & \delta^3(1-\lambda)\cdot\Big(-\tfrac{\delta\lambda^3}{4(1-\lambda)} - \tfrac{\delta(1-\lambda)^2}{4} + \tfrac{7(1-2\lambda)}{4} + \tfrac{7\lambda}{4} - \tfrac{7\lambda}{8\delta} - \tfrac{3\lambda(1-\lambda)}{2\delta} + \tfrac{8\delta\lambda^4(1-\lambda)^2}{3}\Big) \\
\le \; & \delta^3(1-\lambda)\cdot\Big(\tfrac{7(1-2\lambda)}{4} - \tfrac{\lambda^3}{4(1-\lambda)} - \tfrac{(1-\lambda)^2}{4} + \tfrac{7\lambda}{4} - \tfrac{7\lambda}{8} - \tfrac{3\lambda(1-\lambda)}{2} + \tfrac{8\lambda^4(1-\lambda)^2}{3}\Big) \\
\le \; & \delta^3(1-\lambda)\cdot\Big(\tfrac{7(1-2\lambda)}{4} - \tfrac{\lambda^3}{4} - \tfrac{\lambda^4}{4} - \tfrac{(1-\lambda)^2}{4} + \tfrac{7\lambda}{4} - \tfrac{7\lambda}{8} - \tfrac{3\lambda(1-\lambda)}{2} + \tfrac{8}{3\cdot 2^6}\Big) \\
= \; & \delta^3(1-\lambda)\cdot\Big(\tfrac{7(1-2\lambda)}{4} - \tfrac{5}{24} - \tfrac{\lambda}{8} + \tfrac{5\lambda^2}{4} - \tfrac{\lambda^3}{4} - \tfrac{\lambda^4}{4}\Big) \;\le\; \delta^3(1-\lambda)\cdot\tfrac{7(1-2\lambda)}{4}
\end{aligned}$$

since $-\frac{5}{24}-\frac{\lambda}{8}+\frac{5\lambda^2}{4}-\frac{\lambda^3}{4}-\frac{\lambda^4}{4} \le -\frac{5}{24}-\frac{1}{8\cdot 2}+\frac{5}{4\cdot 4}-\frac{1}{4\cdot 8}-\frac{1}{4\cdot 16} = -\frac{1}{192} < 0$ in the last inequality.

Additionally from

$$\begin{aligned}
& -\tfrac{19}{320} + \delta - \lambda - \tfrac{\delta^2}{2} - \tfrac{(1-2\lambda)\delta}{2} \;=\; -\tfrac{19}{320} + \tfrac{\delta}{2} - \lambda(1-\delta) - \tfrac{\delta^2}{2} \\
\le \; & -\tfrac{19}{320} + \tfrac{\delta}{2} - \tfrac{1-\delta}{6} - \tfrac{\delta^2}{2} \;=\; -\tfrac{19}{320} + \tfrac{2\delta}{3} - \tfrac{1}{6} - \tfrac{\delta^2}{2} \;\le\; -\tfrac{19}{320} - \tfrac{1}{6} + \tfrac{2^2/3^2}{4/2} \;=\; -\tfrac{11}{2880} \;<\; 0
\end{aligned}$$

and $\sigma_\lambda^2 \ge 3(1-\lambda)$ it follows

$$\begin{aligned}
\tfrac{\sigma_\lambda^4}{1-\lambda}\cdot\log(\tfrac{f(m)\sqrt{2\pi}\sigma}{2(p^2+q^2)}) \;<\; & 3\cdot\Big(-\tfrac{19}{320} + \delta - \lambda - \tfrac{\delta^2}{2} - \tfrac{(1-2\lambda)\delta}{2} - \tfrac{3(1-2\lambda)^2}{2} - \tfrac{40\delta\lambda^2(1-\lambda)^2(1-2\lambda)}{3}\Big) \\
& -\tfrac{\delta^3(1-2\lambda)}{3(1-\lambda)} - \tfrac{\delta^2\lambda}{2} + \tfrac{7\delta^2}{6} - \tfrac{3(\delta-\lambda)\lambda}{4(1-\lambda)} - 16\delta^2\lambda^3(1-\lambda)^2 + \tfrac{7\delta^3(1-2\lambda)}{12(1-\lambda)} \\
= \; & -\tfrac{57}{320} + 3\delta - 3\lambda - \tfrac{3\delta^2}{2} - \tfrac{\delta^2\lambda}{2} + \tfrac{7\delta^2}{6} - \tfrac{3(1-\lambda+\delta-1)\lambda}{4(1-\lambda)} - 16\delta^2\lambda^3(1-\lambda)^2 \\
& +(1-2\lambda)\cdot\Big(-\tfrac{3\delta}{2} - \tfrac{9(1-2\lambda)}{2} - 40\delta\lambda^2(1-\lambda)^2 + \tfrac{\delta^3}{4(1-\lambda)}\Big) \\
= \; & -\tfrac{57}{320} + 3\delta - 3\lambda - \tfrac{7\delta^2}{12} - \tfrac{3\lambda}{4} + \tfrac{3(1-\delta)}{4} - 16\delta^2\lambda^3(1-\lambda)^2 \\
& +(1-2\lambda)\delta\cdot\Big(-\tfrac{3\delta}{2} - \tfrac{9(1-2\lambda)}{2} - 40\delta\lambda^2(1-\lambda)^2 + \tfrac{\delta^3}{4(1-\lambda)} + \tfrac{\delta^2}{4} - \tfrac{3(1-\delta)}{4(1-\lambda)}\Big) \\
\le \; & -\tfrac{57}{320} + \tfrac{3}{4} + (3-\tfrac{3}{4})\delta - (3+\tfrac{3}{4})\lambda - \tfrac{7\delta^2}{12} - 16\delta^2\lambda^3(1-\lambda)^2 \\
& +(1-2\lambda)\cdot\Big(-\tfrac{3\delta}{2} - \tfrac{9(1-2\lambda)}{2} - \tfrac{40\cdot 25\delta}{36^2} + \tfrac{\delta^3}{2} + \tfrac{\delta^2}{4} - \tfrac{3\cdot 6(1-\delta)}{4\cdot 5}\Big) \\
= \; & \tfrac{183}{320} + \tfrac{9}{4}\delta - \tfrac{15}{4}\lambda - \tfrac{7\delta^2}{12} - 16\delta^2\lambda^3(1-\lambda)^2 \\
& -(1-2\lambda)\cdot\Big(\delta\cdot(\tfrac{3}{2} - \tfrac{\delta^2}{2} + \tfrac{40\cdot 25}{36^2} - \tfrac{\delta}{4} - \tfrac{9}{10}) - \tfrac{9(1-2\lambda)}{2} - \tfrac{9}{10}\Big) \\
\le \; & \tfrac{183}{320} + \tfrac{9}{4}\delta - \tfrac{15}{4}\lambda - \tfrac{7\delta^2}{12} - 16\delta^2\lambda^3(1-\lambda)^2 - (1-2\lambda)\cdot\Big(\tfrac{3\delta}{5} + \tfrac{9(1-2\lambda)}{2} + \tfrac{9}{10}\Big) \\
\le \; & \tfrac{183}{320} + \tfrac{9}{4} - \tfrac{15}{4}\lambda - \tfrac{7}{12} - 16\lambda^3(1-\lambda)^2 - (1-2\lambda)\cdot\Big(\tfrac{3}{5} + \tfrac{9(1-2\lambda)}{2} + \tfrac{9}{10}\Big) \\
= \; & -\tfrac{3611}{960} + \tfrac{69\lambda}{4} - 18\lambda^2 - 16\lambda^3(1-\lambda)^2 \\
\le \; & -\tfrac{3611}{960} + \tfrac{69\lambda}{4} - 18\lambda^2 - 16\lambda(-\tfrac{1}{432} + \tfrac{56\lambda}{432}) \;=\; -\tfrac{3611}{960} + \tfrac{1867\lambda}{108} - \tfrac{542\lambda^2}{27} \\
\le \; & -\tfrac{3611}{960} + \tfrac{1867^2\cdot 27}{4\cdot 108^2\cdot 542} \;=\; -\tfrac{186013}{4682880} \;<\; 0
\end{aligned}$$

since $\lambda^2(1-\lambda)^2+\frac{1}{432}-\frac{56\lambda}{432} = \frac{(1-2\lambda)(6\lambda-1)}{12}\cdot(-\frac{1}{36}+\frac{4\lambda}{3}-\lambda^2) \geq \frac{(1-2\lambda)(6\lambda-1)}{12}\cdot(-\frac{1}{36}+\frac{4}{3\cdot 6}-\frac{1}{36}) = \frac{(1-2\lambda)(6\lambda-1)}{72} \geq 0$ in the penultimate inequality.

Thus we have $f(m) \leq \frac{2(p^2+q^2)}{\sqrt{2\pi}\,\sigma}$ in the case $m \geq 2$ and $m \leq \frac{n}{2}-1$.

Is however $m \leq 1$, which means $m = 1$ since $n \geq 6$ and $p \geq \frac{1}{6}$, we have

$$\begin{aligned}
\log(f_{n,p}(1)\cdot\sigma) &= \log(npq^{n-1}\sqrt{npq}) \;=\; \tfrac{3}{2}\cdot(\log(n))+\log(p))+(n-\tfrac{1}{2})\cdot\log(1-p)\\
&\leq \tfrac{3}{2}\cdot(\log(n))+\log(\tfrac{3}{2(n+1)}))+(n-\tfrac{1}{2})\cdot\log(1-\tfrac{3}{2(n+1)})\\
&\leq \tfrac{3}{2}\cdot(\log(12))+\log(\tfrac{3}{2\cdot 13}))+(12-\tfrac{1}{2})\cdot\log(1-\tfrac{3}{2\cdot 13}) \;=\; -0.9218...\\
&< -0.9189... \;=\; -\tfrac{1}{2}\cdot\log(2\pi),
\end{aligned}$$

where we used in the first inequality $\frac{\mathrm{d}}{\mathrm{d}p}[\log(p))+(n-\frac{1}{2})\cdot\log(1-p)] = \frac{3}{2p}-\frac{n-1/2}{1-p} = 0 \Leftrightarrow p = \frac{3}{2n+1}$ and in the second one

$$\begin{aligned}
[\tfrac{3}{2}\cdot(\log(n))+\log(\tfrac{3}{2(n+1)}))+(n-\tfrac{1}{2})\cdot\log(1-\tfrac{3}{2(n+1)})]' &= \tfrac{3}{2n(n+1)}+\log(1-\tfrac{3}{2(n+1)})+\tfrac{3}{2(n+1)}\\
&\geq \tfrac{3}{2n(n+1)}-\tfrac{3^2}{2\cdot 2^2(n+1)^2}\cdot\tfrac{4}{3} \;=\; \tfrac{3}{2n(n+1)^2}
\end{aligned}$$

since $\sum_{r=0}^{\infty}(\frac{3}{2(n+1)})^r \leq \sum_{r=0}^{\infty}(\frac{1}{4})^r = \frac{4}{3}$.

The case $m = \frac{n-1}{2}$ is shown at the end of this proof for all four cases (i)-(iv).

<u>on (ii):</u> $f(m+1) \leq \frac{2}{\sqrt{2\pi}}\cdot\frac{p^2+q^2}{\sigma}$

This case is easy, since it follows if $m \leq \frac{n}{2}-1$ from the proof of (i) and $\sigma_\lambda \geq \sigma$

$$\begin{aligned}
f(m+1) &= f_{n,\lambda}(\ell)\cdot(\tfrac{np}{\ell})^{\ell}\cdot(\tfrac{n(1-p)}{n-\ell})^{n-\ell}\\
&\leq \tfrac{1}{\sqrt{2\pi}\,\sigma_\lambda}\cdot \mathrm{e}^{-\frac{19}{320\sigma_\lambda^2}-\frac{\delta^2}{2\sigma_\lambda^2}} \;<\; \tfrac{1}{\sqrt{2\pi}\,\sigma_\lambda} \;\leq\; \tfrac{1}{\sqrt{2\pi}\,\sigma} \;\leq\; \tfrac{2(p^2+q^2)}{\sqrt{2\pi}\,\sigma}.
\end{aligned}$$

The case $m = \frac{n-2}{2}$ is considered as already mentioned at the end of this proof.

<u>on (iii):</u> $g(m+1) < \frac{2}{\sqrt{2\pi}} \cdot \frac{p^2+q^2}{\sigma}$

This case is trivial, since for all $k \in \mathbb{Z}$ we have

$$g(k) \;=\; \Phi(\tfrac{k-np}{\sigma}) - \Phi(\tfrac{k-np-1}{\sigma}) \;<\; \tfrac{1}{\sigma} \cdot \varphi(0) \;=\; \tfrac{1}{\sqrt{2\pi}\,\sigma} \;\leq\; \tfrac{2(p^2+q^2)}{\sqrt{2\pi}\,\sigma}.$$

<u>on (iv):</u> $f(m) + f(m+1) - g(m+1) < \frac{2}{\sqrt{2\pi}} \cdot \frac{p^2+q^2}{\sigma}$

First of all we have $m \geq \lfloor n/6 \rfloor \geq 1$. Now let firstly $m \leq \frac{n}{2} - 1$ hence $\lambda := \frac{\ell}{n} := \frac{m+1}{n} \leq \frac{1}{2}$.

1.) We have analogous to (i)

$$\begin{aligned}
f(m+1) + f(m) \;&=\; f_{n,\lambda}(\ell) \cdot ((\tfrac{np}{\ell})^{\ell} \cdot (\tfrac{nq}{n-\ell})^{n-\ell} + \tfrac{m+1}{n-m} \cdot \tfrac{n-m-1+\delta}{m+1-\delta} \cdot (\tfrac{np}{\ell})^{\ell} \cdot \tfrac{nq}{n-\ell})^{n-\ell} \\
&=\; f_{n,\lambda}(\ell) \cdot (\tfrac{np}{\ell})^{\ell} \cdot (\tfrac{nq}{n-\ell})^{n-\ell} \cdot \big(1 + \tfrac{(m+1)(n-m-1+\delta)}{(m+1-\delta)(n-m)}\big) \\
&=\; f_{n,\lambda}(\ell) \cdot (\tfrac{np}{\ell})^{\ell} \cdot (\tfrac{nq}{n-\ell})^{n-\ell} \cdot \big(2 + \tfrac{(n+1)\delta-(m+1)}{(m+1-\delta)(n-m)}\big) \\
&=\; 2 \cdot f_{n,\lambda}(\ell) \cdot (1 - \tfrac{\delta}{\ell})^{\ell} \cdot (1 + \tfrac{\delta}{n-\ell})^{n-\ell} \cdot \big(1 + \tfrac{(n+1)\delta-(m+1)}{2(m+1-\delta)(n-m)}\big) \\
&\leq\; 2 \cdot f_{n,\lambda}(\ell) \cdot \mathrm{e}^{-\frac{\delta^2}{2\sigma_\lambda^2}} \cdot \big(1 + \tfrac{(n+1)\delta-(m+1)}{2(m+1-\delta)(n-m)}\big) \\
&\leq\; 2 \cdot f_{n,\lambda}(\ell) \cdot \mathrm{e}^{-\frac{\delta^2}{2\sigma_\lambda^2}} \cdot \mathrm{e}^{\frac{(n+1)\delta-(m+1)}{2(m+1-\delta)(n-m)}} \\
&\leq\; 2 \cdot \frac{1}{\sqrt{2\pi}\,\sigma_\lambda} \cdot \mathrm{e}^{-\frac{19}{320\sigma_\lambda^2}} \cdot \mathrm{e}^{-\frac{\delta^2}{2\sigma_\lambda^2}} \cdot \mathrm{e}^{\frac{(n+1)\delta-(m+1)}{2(m+1-\delta)(n-m)}}.
\end{aligned}$$

2.) Further we have due to $\sigma \leq \sigma_\lambda$ and Lemma 11.1

$$\begin{aligned}
g(m+1) \;&=\; \Phi(\tfrac{m+1-np}{\sigma}) - \Phi(\tfrac{m-np}{\sigma}) \;=\; \Phi(\tfrac{\delta}{\sigma}) - \Phi(\tfrac{\delta-1}{\sigma}) \;\geq\; \Phi(\tfrac{\delta}{\sigma_\lambda}) - \Phi(\tfrac{\delta-1}{\sigma_\lambda}) \\
&=\; \Phi(\tfrac{\delta-1/2}{\sigma_\lambda} + \tfrac{1}{2\sigma_\lambda}) - \Phi(\tfrac{\delta-1/2}{\sigma_\lambda} - \tfrac{1}{2\sigma_\lambda}) \\
&\geq\; \frac{1}{\sqrt{2\pi}\,\sigma_\lambda} \cdot \exp(-\tfrac{(\delta-1/2)^2}{2\sigma_\lambda^2} + \tfrac{(\delta-1/2)^2/\sigma_\lambda^2-1}{24\sigma_\lambda^2} - \tfrac{(\delta-\frac{1}{2})^4}{960\sigma_\lambda^8}) \\
&\geq\; \frac{1}{\sqrt{2\pi}\,\sigma_\lambda} \cdot \exp(-\tfrac{(\delta-1/2)^2}{2\sigma_\lambda^2} - \tfrac{1}{24\sigma_\lambda^2}) \\
&=\; \frac{1}{\sqrt{2\pi}\,\sigma_\lambda} \cdot \mathrm{e}^{\frac{1}{2\sigma_\lambda^2}\cdot(-\delta^2+\delta-\frac{1}{3})}.
\end{aligned}$$

3.) We also have with $1+t \le \mathrm{e}^t$ for all $t \in \mathbb{R}$

$$
\begin{aligned}
& \mathrm{e}^{-\frac{\delta^2}{2\sigma_\lambda^2}} \cdot \big(2e^{-\frac{19}{320\sigma_\lambda^2}+\frac{(n+1)\delta-(m+1)}{2(m+1-\delta)(n-m)}} - \mathrm{e}^{\frac{1}{2\sigma_\lambda^2}\cdot(\delta-\frac{1}{3})}\big) \\
= \ & \mathrm{e}^{-\frac{\delta^2}{2\sigma_\lambda^2}-\frac{19}{320\sigma_\lambda^2}+\frac{(n+1)\delta-(m+1)}{2(m+1-\delta)(n-m)}} \cdot \big(2-\mathrm{e}^{\frac{1}{2\sigma_\lambda^2}\cdot(\delta-\frac{1}{3})-(-\frac{19}{320\sigma_\lambda^2}+\frac{(n+1)\delta-(m+1)}{2(m+1-\delta)(n-m)})}\big) \\
\le \ & \mathrm{e}^{-\frac{\delta^2}{2\sigma_\lambda^2}-\frac{19}{320\sigma_\lambda^2}+\frac{(n+1)\delta-(m+1)}{2(m+1-\delta)(n-m)}} \cdot \big(2-\big(1+\frac{1}{2\sigma_\lambda^2}\cdot(\delta-\frac{1}{3})-(-\frac{19}{320\sigma_\lambda^2}+\frac{(n+1)\delta-(m+1)}{2(m+1-\delta)(n-m)})\big)\big) \\
= \ & \mathrm{e}^{-\frac{\delta^2}{2\sigma_\lambda^2}-\frac{19}{320\sigma_\lambda^2}+\frac{(n+1)\delta-(m+1)}{2(m+1-\delta)(n-m)}} \cdot \big(1-\frac{1}{2\sigma_\lambda^2}\cdot(\delta-\frac{1}{3})-\frac{19}{320\sigma_\lambda^2}+\frac{(n+1)\delta-(m+1)}{2(m+1-\delta)(n-m)}\big)\big) \\
\le \ & \mathrm{e}^{-\frac{\delta^2}{2\sigma_\lambda^2}-\frac{19}{320\sigma_\lambda^2}+\frac{(n+1)\delta-(m+1)}{2(m+1-\delta)(n-m)}} \cdot \mathrm{e}^{-\frac{1}{2\sigma_\lambda^2}\cdot(\delta-\frac{1}{3})-\frac{19}{320\sigma_\lambda^2}+\frac{(n+1)\delta-(m+1)}{2(m+1-\delta)(n-m)})} \\
= \ & \exp\Big(\frac{1}{\sigma_\lambda^2}\cdot(-\frac{\delta^2+\delta}{2}+\frac{23}{480}+\frac{(n+1)\delta-(m+1)}{(m+1-\delta)(n-m)}\cdot\frac{(m+1)(n-m-1)}{n})\Big) \\
\le \ & \exp\Big(\frac{1}{\sigma_\lambda^2}\cdot(-\frac{\delta^2+\delta}{2}+\frac{23}{480}+\frac{(n+1)\delta-(m+1)}{n}\cdot\frac{n-m-1}{n-m}\cdot(1+\frac{\delta}{k})))
\end{aligned}
$$

since $\frac{m+1}{m+1-\delta} = 1 + \frac{\delta}{m+1-\delta} \le 1 + \frac{\delta}{k}$. Thus we have together

$$
f(m)+f(m+1)-g(m+1) \quad \le \quad \frac{1}{\sqrt{2\pi}\,\sigma_\lambda}\exp\Big(\frac{1}{\sigma_\lambda^2}\big(\frac{23}{480}-\frac{\delta^2+\delta}{2}+\frac{(n+1)\delta-(m+1)}{n}\cdot\frac{n-m-1}{n-m}(1+\frac{\delta}{k})\big)\Big).
$$

4.) On the other hand we have with $0 \le 2(p^2+q^2)-1 \le 2\cdot\frac{26}{36}-1=\frac{4}{9}$

$$
\begin{aligned}
\frac{2}{\sqrt{2\pi}}\cdot\frac{p^2+q^2}{\sigma} \quad & \ge \quad \frac{1}{\sqrt{2\pi}\,\sigma_\lambda}\cdot\mathrm{e}^{\log(1+2(p^2+q^2)-1)} \quad \ge \quad \frac{1}{\sqrt{2\pi}\,\sigma_\lambda}\cdot\mathrm{e}^{2(p^2+q^2)-1-(2(p^2+q^2)-1)^2/2} \\
& \ge \quad \frac{1}{\sqrt{2\pi}\,\sigma_\lambda}\cdot\mathrm{e}^{(2(p^2+q^2)-1)\cdot(1-\frac{2}{9})} \\
& = \quad \frac{1}{\sqrt{2\pi}\,\sigma_\lambda}\cdot\exp\Big(\frac{7}{9}\cdot((1-2\lambda)^2+\frac{4(1-2\lambda)\delta}{n}+\frac{4\delta^2}{n^2})\Big).
\end{aligned}
$$

5.) Since the monotonicity of the exponential function it remains to prove the positivity of the following function

$$
h(\delta) \quad := \quad \frac{7}{9}\big((1-2\lambda)^2+\frac{4(1-2\lambda)\delta}{n}+\frac{4\delta^2}{n^2}\big)-\frac{1}{\sigma_\lambda^2}\big(-\frac{\delta^2+\delta}{2}+\frac{23}{480}+\frac{(n+1)\delta-(m+1)}{n}\cdot\frac{n-m-1}{n-m}(1+\frac{\delta}{k})\big).
$$

If first $m=1$ we have with $n \ge 6$

$$
\begin{aligned}
h''(\delta) \quad & = \quad \frac{7}{9}\cdot\frac{8}{n^2}-\frac{1}{\sigma_\lambda^2}\cdot(-1+\frac{2(n+1)}{n}\cdot\frac{(n-m-1)}{(n-m)}) \quad = \quad \frac{1}{\sigma_\lambda^2}\cdot(\frac{56\lambda(1-\lambda)}{9n}+1-2\cdot(1-\frac{m+1}{n(n-m)})) \\
& \le \quad \frac{1}{\sigma_\lambda^2}\cdot(\frac{56\lambda(1-\lambda)}{9\cdot 6}+1-2\cdot(1-\frac{\lambda}{5})) \quad \le \quad \frac{1}{\sigma_\lambda^2}\cdot(\frac{56}{9\cdot 6\cdot 4}+1-2\cdot(1-\frac{1}{10})) \\
& = \quad \frac{1}{\sigma_\lambda^2}\cdot(\frac{7}{27}-\frac{4}{5}) \quad < \quad 0,
\end{aligned}
$$

hence h is concave and $h(0) = \frac{7}{9} \cdot (1-2\lambda)^2 - \frac{1}{\sigma_\lambda^2} \cdot (\frac{23}{480} - \frac{(m+1)}{n} \cdot \frac{n-2}{n-1}) \geq \frac{1}{\sigma_\lambda^2} \cdot (-\frac{23}{480} + \frac{1}{6} \cdot \frac{4}{5}) > 0$ as well as

$$\begin{aligned}
h(1) &= \tfrac{7}{9} \cdot ((1-2\lambda)^2 + \tfrac{4(1-2\lambda)}{n} + \tfrac{4}{n^2}) - \tfrac{1}{2(1-\lambda)} \cdot (-1 + \tfrac{23}{480} + 2(1-\lambda)) \\
&= \tfrac{1}{1-\lambda} \cdot \big(\tfrac{7}{9} \cdot ((1-2\lambda)^2(1-\lambda) + 2(1-2\lambda)(1-\lambda)\lambda + \lambda^2(1-\lambda)) - \tfrac{1-2\lambda}{2} - \tfrac{23}{960}\big) \\
&= \tfrac{1}{1-\lambda} \cdot \big(\tfrac{7}{9} \cdot (1 - 3\lambda + 3\lambda^2 - \lambda^3) - \tfrac{1-2\lambda}{2} - \tfrac{23}{960}\big) \\
&= \tfrac{1}{1-\lambda} \cdot \big(\tfrac{5}{18} - \tfrac{4}{3}\lambda + \tfrac{7}{3}\lambda^2 - \tfrac{7}{9}\lambda^3 - \tfrac{23}{960}\big) \\
&\geq \tfrac{1}{1-\lambda} \cdot \big(\tfrac{5}{18} - \tfrac{4}{3}\lambda + (\tfrac{7}{3} - \tfrac{7}{18})\lambda^2 - \tfrac{23}{960}\big) \\
&\geq \tfrac{1}{1-\lambda} \cdot \big(\tfrac{5}{18} - (\tfrac{4}{3})^2/(\tfrac{7}{3} - \tfrac{7}{18}) \cdot \tfrac{1}{4} - \tfrac{23}{960}\big) \\
&= \tfrac{1}{1-\lambda} \cdot \big(\tfrac{5}{18} - \tfrac{8}{35} - \tfrac{23}{960}\big) \;=\; \tfrac{509}{20160} \;>\; 0.
\end{aligned}$$

Thus we have $h(\delta) > 0$.

We may now assume $m \geq 2$. Here we distinguish between two cases:

(a) $\delta \in [0, \frac{m+1}{n+1}]$. Then with $n - m - 1 \geq m + 1 \geq 3$, $n \geq 6$ we have

$$\begin{aligned}
\sigma_\lambda^2 \cdot f(\delta) &\geq \tfrac{\delta^2+\delta}{2} - \tfrac{23}{480} - (\tfrac{n+1}{n}\delta - \lambda) \cdot \tfrac{n-m-1}{n-m} \cdot (1 + \tfrac{\delta}{k}) \\
&\geq \tfrac{\delta^2+\delta}{2} - \tfrac{23}{480} - (\tfrac{7}{6}\delta - \lambda) \cdot \tfrac{3}{4} \;=\; \tfrac{\delta^2}{2} - \tfrac{3}{8}\delta + \tfrac{3}{4}\lambda - \tfrac{23}{480} \\
&\geq -\tfrac{3^2}{8^2 \cdot 2} + \tfrac{3}{4} \cdot \tfrac{1}{6} - \tfrac{23}{480} \;=\; \tfrac{7}{128} - \tfrac{23}{480} \;=\; \tfrac{13}{1920} \;>\; 0.
\end{aligned}$$

(b) $\delta \in (\frac{m+1}{n+1}, 1]$. Then it follows from $\frac{(n+1)(n-m-1)}{n(n-m)} \leq 1$ and $\frac{n}{n+1} \geq \frac{6}{7}$ as $\frac{35(1-2\lambda)}{81} + \frac{3\lambda}{7} \leq \frac{1}{2}$

$$\begin{aligned}
\sigma_\lambda^2 \cdot f(\delta) &\geq \tfrac{7}{9} \cdot \sigma_\lambda^2 \cdot ((1-2\lambda)^2 + \tfrac{4(1-2\lambda)\delta}{n}) + \tfrac{\delta^2+\delta}{2} - \tfrac{23}{480} - \tfrac{(n+1)\delta-(m+1)}{n} \cdot \tfrac{n-m-1}{n-m} \cdot (1 + \tfrac{\delta}{k})) \\
&= \tfrac{7\ell(1-2\lambda)^2(1-\lambda)}{9} + \tfrac{28(1-2\lambda)(1-\lambda)\lambda\delta}{9} + \tfrac{\delta^2+\delta}{2} - \tfrac{23}{480} - (\delta - \tfrac{n\lambda}{n+1}) \cdot \tfrac{(n+1)n-m-1}{n(n-m)} \cdot (1 + \tfrac{\delta}{k})) \\
&\geq \tfrac{7\cdot 3\cdot(1-2\lambda)^2}{9\cdot 2} + \tfrac{28\cdot(1-2\lambda)\cdot 5\cdot\delta}{9\cdot 36} + \tfrac{\delta^2+\delta}{2} - \tfrac{23}{480} - (\delta - \tfrac{6}{7}\lambda) \cdot (1 + \tfrac{\delta}{2})) \\
&= \tfrac{7\cdot(1-2\lambda)^2}{6} + \tfrac{35\cdot(1-2\lambda)\delta}{81} - \tfrac{\delta}{2} - \tfrac{23}{480} + \tfrac{6\lambda}{7} + \tfrac{3\lambda\delta}{7} \\
&\geq \tfrac{7\cdot(1-2\lambda)^2}{6} + \tfrac{35\cdot(1-2\lambda)}{81} - \tfrac{1}{2} - \tfrac{23}{480} + \tfrac{6\lambda}{7} + \tfrac{3\lambda}{7} \\
&= \tfrac{7}{6} + \tfrac{35}{81} - \tfrac{1}{2} - \tfrac{23}{480} - \lambda \cdot (\tfrac{14}{3} + \tfrac{70}{81} - \tfrac{9}{7}) + \lambda^2 \cdot \tfrac{14}{3} \;=\; \tfrac{89}{81} - \tfrac{23}{480} - \tfrac{2407}{567} \cdot \lambda + \tfrac{14}{3} \cdot \lambda^2 \\
&\geq \tfrac{89}{81} - \tfrac{23}{480} - \tfrac{2407^2}{567^2} \cdot \tfrac{3}{14\cdot 4} \;=\; 0.0854... \;>\; 0.
\end{aligned}$$

We now consider finally for (i)-(iv) the case $m = \frac{n-1}{2}$, hence $n \geq 7$ since n must be odd. Then $p = \frac{m+1-\delta}{n} = \frac{1}{2} + \frac{1-2\delta}{2n}$ for a $\delta \in [\frac{1}{2}, 1]$. Then we have with $p + q = 1$

$$\begin{aligned} f(m+1) + f(m) &= \frac{n!}{((n+1)/2)!((n-1)/2)!} \cdot p^{\frac{n+1}{2}} q^{\frac{n-1}{2}} + \frac{n!}{((n+1)/2)!((n-1)/2)!} \cdot p^{\frac{n-1}{2}} q^{\frac{n+1}{2}} \\ &= \frac{n!}{((n+1)/2)!((n-1)/2)!} \cdot (\tfrac{1}{2} + \tfrac{1-2\delta}{2n})^{\frac{n}{2}} \cdot (\tfrac{1}{2} - \tfrac{1-2\delta}{2n})^{\frac{n}{2}} \cdot (pq)^{-\frac{1}{2}} \\ &= \frac{n!}{((n+1)/2)!((n-1)/2)!} \cdot 2^{-n} \cdot (1 - \tfrac{(1-2\delta)^2}{n^2})^{\frac{n}{2}} \cdot (pq)^{-\frac{1}{2}}. \end{aligned}$$

It follows from the Stirling formula that $\mathrm{e}^{\frac{1}{12n+1}} \leq \frac{n!}{\sqrt{2\pi n}} \cdot (\frac{e}{n})^n \leq \mathrm{e}^{\frac{1}{12n}}$ and hence

$$\begin{aligned} \frac{n!}{(\frac{n-1}{2})!(\frac{n+1}{2})!} &\leq \sqrt{2\pi n} \cdot (\tfrac{n}{e})^n \cdot \mathrm{e}^{\frac{1}{12n}} \cdot \frac{1}{2\pi} \cdot \sqrt{\frac{4}{(n-1)(n+1)}} \cdot \left(\frac{2e}{n-1}\right)^{\frac{n-1}{2}} \cdot \left(\frac{2e}{n+1}\right)^{\frac{n+1}{2}} \cdot \mathrm{e}^{-\frac{1}{6(n-1)+1} - \frac{1}{6(n+1)+1}} \\ &= \frac{1}{\sqrt{2\pi n}} \cdot 2^{n+1} \cdot \sqrt{\frac{n^2}{(n-1)(n+1)}} \cdot \left(\frac{n}{n-1}\right)^{\frac{n-1}{2}} \cdot \left(\frac{n}{n+1}\right)^{\frac{n+1}{2}} \cdot \mathrm{e}^{\frac{1}{12n} - \frac{1}{6(n-1)+1} - \frac{1}{6(n+1)+1}} \\ &= \frac{1}{\sqrt{2\pi n}} \cdot 2^{n+1} \cdot \sqrt{1 + \frac{1}{n^2-1}} \cdot \sqrt{(1 + \tfrac{1}{n-1})^{n-1} \cdot (1 - \tfrac{1}{n+1})^{n+1}} \cdot \mathrm{e}^{\frac{1}{12n} - \frac{1}{3n} \cdot \frac{(n+1/6)n}{(n-5/6)(n+7/6)}} \\ &\leq \frac{1}{\sqrt{2\pi n}} \cdot 2^{n+1} \cdot \mathrm{e}^{\frac{1}{12n} - \frac{1}{3n} \cdot \frac{7}{8} + \frac{7}{96n} - \frac{47}{96n}} \\ &= \frac{1}{\sqrt{2\pi n}} \cdot 2^{n+1} \cdot \mathrm{e}^{-\frac{5}{8n}} \end{aligned}$$

since $\frac{(n+1/6)n}{(n-5/6)(n+7/6)} \geq \frac{(n+1/6)}{n+7/6} \geq \frac{n}{n+1} \geq \frac{7}{8}$ and $1 + \frac{1}{n^2-1} \leq \mathrm{e}^{\frac{1}{n^2-1}} = \mathrm{e}^{\frac{1}{n} \cdot \frac{n}{n^2-1}} \leq \mathrm{e}^{\frac{1}{n} \cdot \frac{7}{48}}$ and also

$$\begin{aligned} \log\left((1 + \tfrac{1}{n-1})^{n-1} \cdot (1 - \tfrac{1}{n+1})^{n+1}\right) &\leq -\frac{1}{2(n-1)} - \frac{1}{2(n+1)} + \frac{1}{3(n-1)^2} - \frac{1}{3(n+1)^2} \\ &= -\frac{n}{n^2-1} + \frac{4n}{3(n^2-1)^2} \\ &= -\frac{1}{n} \cdot (1 + \tfrac{1}{n^2-1} \cdot (1 - \tfrac{4}{3} \cdot \tfrac{n^2}{n^2-1})) \\ &\leq -\frac{1}{n} \cdot (1 + \tfrac{1}{48} \cdot (1 - \tfrac{4}{3} \cdot \tfrac{49}{48})) \\ &= -\frac{1}{n} \cdot (1 - \tfrac{13}{48 \cdot 36}) \leq -\frac{47}{48n}. \end{aligned}$$

Thus we finally have with $(1 + \frac{t}{n^2})^{n^2} = \mathrm{e}^{n^2 \log(1 + \frac{t}{n^2})} \leq \mathrm{e}^{n^2 \cdot \frac{t}{n^2}} = \mathrm{e}^t$ for $t \in (-n^2, n^2)$

$$\begin{aligned} f(m+1) + f(m) &\leq \frac{1}{\sqrt{2\pi n}} \cdot \mathrm{e}^{-\frac{5}{8n}} \cdot 2^{n+1} \cdot 2^{-n} \cdot (1 - \tfrac{(1-2\delta)^2}{n^2})^{\frac{n}{2}} \cdot (pq)^{-\frac{1}{2}} \\ &= \frac{2}{\sqrt{2\pi}\,\sigma} \cdot \mathrm{e}^{-\frac{5}{8n}} \cdot \left(1 - \frac{(1-2\delta)^2}{n^2}\right)^{n^2 \cdot \frac{1}{2n}} \\ &\leq \frac{2}{\sqrt{2\pi}\,\sigma} \cdot \mathrm{e}^{-\frac{5}{8n} - \frac{(1-2\delta)^2}{2n}}. \end{aligned}$$

Further 3.) in the previous part (iv) of the proof and $\sigma_\lambda^2 = n \cdot \frac{(n-1)(n+1)}{4n^2} = \frac{n}{4} \cdot \frac{n^2-1}{n^2} \geq \frac{n}{4} \cdot \frac{48}{49} = \frac{12}{49n}$ and $\sigma_\lambda \leq \sigma$ yield

$$g(m+1) = \Phi(\tfrac{\delta}{\sigma}) - \Phi(\tfrac{\delta-1}{\sigma}) \geq \frac{1}{\sqrt{2\pi}\,\sigma_\lambda} \cdot \mathrm{e}^{\frac{1}{2\sigma_\lambda^2}\cdot(-\delta^2+\delta-\frac{1}{3})} \geq \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \mathrm{e}^{\frac{49}{24n}\cdot(-\delta^2+\delta-\frac{1}{3})}.$$

Thus with $\frac{49}{24n} \cdot (-\delta^2 + \delta - \frac{1}{3}) = -\frac{49}{24n} \cdot (\frac{(1-2\delta)^2}{4} + \frac{1}{12}) = -\frac{(1-2\delta)^2}{2n} - \frac{(1-2\delta)^2}{96n} - \frac{49}{12\cdot 24n} \geq -\frac{(1-2\delta)^2}{2n} - \frac{5}{8n}$ we have

$$\begin{aligned} f(m+1) + f(m) - g(m+1) &\leq \frac{1}{\sqrt{2\pi}\,\sigma} \cdot (2 \cdot \mathrm{e}^{-\frac{5}{8n} - \frac{(1-2\delta)^2}{2n}} - \mathrm{e}^{\frac{49}{24n}\cdot(-\delta^2+\delta-\frac{1}{3})}) \\ &\leq \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \mathrm{e}^{-\frac{5}{8n} - \frac{(1-2\delta)^2}{2n}} \\ &< \frac{2(p^2+q^2)}{\sqrt{2\pi}\,\sigma}. \end{aligned}$$

From the computations above also follows

$$\begin{aligned} f(m+1) \leq f(m) &= \frac{n!}{((n+1)/2)!((n-1)/2)!} \cdot p^{\frac{n-1}{2}} q^{\frac{n+1}{2}} \\ &\leq \frac{1}{\sqrt{2\pi\sigma}} \cdot \mathrm{e}^{-\frac{5}{8n} - \frac{(1-2\delta)^2}{2n}} \cdot 2q = \frac{1}{\sqrt{2\pi\sigma}} \cdot \mathrm{e}^{-\frac{5}{8n} - \frac{(1-2\delta)^2}{2n}} \cdot (1 - \frac{1-2\delta}{n}) \\ &\leq \frac{1}{\sqrt{2\pi\sigma}} \cdot \mathrm{e}^{-\frac{5}{8n} - \frac{(1-2\delta)^2}{2n} + \frac{1-2\delta}{n}} \leq \frac{1}{\sqrt{2\pi\sigma}} \cdot \mathrm{e}^{-\frac{5}{8n} + \frac{1}{2n}} \\ &< \frac{1}{\sqrt{2\pi\sigma}}. \end{aligned}$$

Finally we mention that the case $m = \lfloor np \rfloor = \frac{n}{2}$ for $p \leq \frac{1}{2}$ can not occur, since else would be $np = \frac{n}{2} \in \mathbb{N}$, which was excluded in the requirement. $\square$

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