Extension Operators with Optimal Continuity Estimates

Dissertation

zur Erlangung des akademischen Grades eines Doktors der Naturwissenschaften (Dr. rer. nat.)

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vorgelegt von

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Trier, Juni 2018

Abstract

Given a compact set $K \subset \mathbb{R}^d$, the theory of extension operators examines the question, under which conditions on K, the linear and continuous restriction operators

$$r_n : \mathscr{E}^n(\mathbb{R}^d) \to \mathscr{E}^n(K), f \mapsto \left(\partial^\alpha f\big|_K\right)_{|\alpha|\leq n}, n \in \mathbb{N}_0 \text{ and}$$

$$r : \mathscr{E}(\mathbb{R}^d) \to \mathscr{E}(K), f \mapsto \left(\partial^\alpha f\big|_K\right)_{\alpha\in\mathbb{N}_0^d},$$

have a linear and continuous right inverse. This inverse is called extension operator and this problem is known as Whitney's extension problem, named after Hassler Whitney. In this context, $\mathscr{E}^n(K)$ respectively $\mathscr{E}(K)$ denote spaces of Whitney jets of order n respectively of infinite order. With $\mathscr{E}^n(\mathbb{R}^d)$ and $\mathscr{E}(\mathbb{R}^d)$, we denote the spaces of n-times respectively infinitely often continuously partially differentiable functions on $\mathbb{R}^d$. Whitney already solved the question for finite order completely in his papers [Whi34a], [Whi34b] and [Whi34c]. He showed that it is always possible to construct a linear and continuous right inverse E_n for r_n. This work is concerned with the question of how the existence of a linear and continuous right inverse of r, fulfilling certain continuity estimates, can be characterized by properties of K. On $\mathscr{E}(K)$, we introduce a full real scale of generalized Whitney seminorms $(\|\cdot\|_{s,K})_{s\geq 0}$, where $\|\cdot\|_{s,K}$ coincides with the classical Whitney seminorms for $s \in \mathbb{N}_0$. We equip also $\mathscr{E}(\mathbb{R}^d)$ with a family $(\|\cdot\|_{s,L})_{s\geq 0}$ of those seminorms, where L shall be a a compact set with $K \subset \mathring{L}$. This family of seminorms on $\mathscr{E}(\mathbb{R}^d)$ suffices to characterize the continuity properties of an extension operator E, since we can without loss of generality assume that $E(\mathscr{E}(K)) \subset \mathscr{D}^s(L)$.
In Chapter 2, we introduce basic concepts and summarize the classical results of Whitney and Stein.
In Chapter 3, we modify the classical construction of Whitney's operators E_n and show that $\|E_n(\cdot)\|_{s,L} \leq C\|\cdot\|_{s,K}$ for $s \in [n, n+1)$.
In Chapter 4, we generalize the results of Frerick, Jordá and Wengenroth published in [FJW16b] and show that LMI(1) for K implies the existence of an extension operator E without loss of derivatives, i.e. we have it fulfils $\|E(\cdot)\|_{s,L} \leq C\|\cdot\|_{s,K}$ for all $s \geq 0$. We show that a large class of self similar sets, which includes the Cantor set and the Sierpinski triangle, admits an extensions operator without loss of derivatives.
In Chapter 5 we generalize the results of Frerick, Jordá and Wengenroth in [FJW11] and

show that WLMI(r) for $r \geq 1$ implies the existence of a tame linear extension operator E having a homogeneous loss of derivatives, such that $\|E(\cdot)\|_{s,L} \leq C\| \cdot \|_{(r+\varepsilon)s,K}$ for all $s \geq 0$ and all $\varepsilon > 0$.
In the last chapter we characterize the existence of an extension operator having an arbitrary loss of derivatives by the existence of measures on K.

Zusammenfassung

Gegeben sei eine kompakte Menge $K \subset \mathbb{R}^d$. Die Theorie der Extensionsoperatoren beschäftigt sich mit der Frage, welche Eigenschaften K haben muss, damit die linearen und stetigen Einschränkungen

$$r_n : \mathscr{E}^n(\mathbb{R}^d) \to \mathscr{E}^n(K), f \mapsto \left(\partial^\alpha f\big|_K\right)_{|\alpha|\leq n}, n \in \mathbb{N}_0 \text{ und}$$

$$r : \mathscr{E}(\mathbb{R}^d) \to \mathscr{E}(K), f \mapsto \left(\partial^\alpha f\big|_K\right)_{\alpha\in\mathbb{N}_0^d},$$

eine lineare und stetige Rechtsinverse besitzen. Die Inverse wird als Extensionsoperator bezeichnet und dieses Problem ist bekannt als Whitneys Extensionsproblem, benannt nach Hassler Whitney. In diesem Zusammenhang bezeichnen $\mathscr{E}^n(K)$ beziehungsweise $\mathscr{E}(K)$ die Räume der Whitney-Funktionen auf K der Ordnung n beziehungsweise unendlicher Ordnung. Mit $\mathscr{E}^n(\mathbb{R}^d)$ beziehungsweise $\mathscr{E}(\mathbb{R}^d)$ bezeichnen wir die Räume n-mal beziehungsweise unendlich oft stetig partiell differenzierbarer Funktionen auf $\mathbb{R}^d$. Whitney löste dieses Problem für r_n bereits vollständig in seinen Veröffentlichungen [Whi34a], [Whi34b] und [Whi34c]. Er konnte zeigen, dass es immer, unabhängig von den Eigenschaften von K, möglich ist eine lineare und stetige Rechtsinverse E_n von r_n zu konstruieren. Diese Arbeit beschäftigt sich mit der Frage, wie die Existenz einer linearen und stetigen Rechtsinversen von r mit gewissen Stetigkeitsabschätzungen durch Eigenschaften von K charakterisiert werden kann. Auf $\mathscr{E}(K)$ führen wir eine reelle Skala von verallgemeinerten Whitney-Seminormen $(\|\cdot\|_{s,K})_{s\geq 0}$ ein, wobei $\|\cdot\|_{s,K}$ für $s \in \mathbb{N}_0$ mit den klassischen Whitney-Seminormen übereinstimmt. Auch $\mathscr{E}(\mathbb{R}^d)$ statten wir mit einer reellen Skala dieser Seminormen $(\|\cdot\|_{s,L})_{s\geq 0}$ aus, wobei $L \subset \mathbb{R}^d$ kompakt ist mit $\mathring{L} \supset K$. Diese Familie von Seminormen genügt um die Stetigkeitseigenschaften eines Extensionsoperators E zu untersuchen, da ohne Beschränkung der Allgemeinheit $E(\mathscr{E}(K)) \subset \mathscr{D}^s(L)$.
In Kapitel 2 führen wir grundlegende Begriffe ein und stelle auch die klassischen Ergebnisse von Whitney und Stein zusammengefasst dar.
In Kapitel 3 beweisen wir, dass der klassische Extensionsoperator E_n von Whitney $\|E_n(\cdot)\|_{s,L} \leq C\|\cdot\|_{s,K}$ für alle $s \in [n, n+1)$ erfüllt.
Aufbauend darauf verallgemeinern wir in Kapitel 4 die in [FJW16b] von Frerick, Jordá und Wengenroth veröffentlichten Ergebnisse und zeigen, dass die Eigenschaft LMI(1)

für K die Existenz eine Extensionsoperators E ohne Verlust impliziert. Das bedeutet, dass E die Ungleichung $||E(\cdot)||_{s,L} \leq C|| \cdot ||_{s,K}$ für alle $s \geq 0$ erfüllt. Wir zeigen, dass eine große Klasse selbstähnlicher Mengen, welche unter anderem die Cantor-Menge und das Sierpinski-Dreieck enthält, einen Extensionsoperator ohne Verlust zulässt.
In Kapitel 5 verallgemeinern wir die von Frerick, Jordá und Wengenroth in [FJW11] veröffentlichten Ergebnisse und zeigen, dass WLMI(r) für $r \geq 1$ die Existenz eines zahm-linearen Extensionsoperators mit homogenem Verlust impliziert, welcher für alle $s \geq 0$ und $\varepsilon > 0$ die Ungleichung $||E(\cdot)||_{s,L} \leq C|| \cdot ||_{(r+\varepsilon)s,K}$ erfüllt.
Im letzten Kapitel charakterisieren wir die Existenz eines Extensionsoperators mit beliebig vorgegebenem Verlust durch die Existenz von Maßen auf K.

Acknowledgements

It is my pleasure to express my gratitude to my supervisor Prof. Dr. Leonhard Frerick for his invaluable support and guidance during the development of this thesis and for the opportunity to learn from his great expertise. Furthermore, I want to thank Prof. Dr. Jochen Wengenroth and Prof. Dr. Enrique Jordá, both for their great support and for kindly accepting to be second and third supervisor to this thesis. Particularly, I am indepted to Prof. Dr. Enrique Jordá for spending much of his time during his visits in Trier on working with me and answering my numerous questions. I also want to express my gratitude to all my colleagues at the University of Trier for the pleasant working atmosphere and the best coffee. I would also like to thank Dr. Bernhard Dierolf for finding all the missing commas and typos in the manuscript. Last, but not least, I want to thank my family for their great support and motivation.

Contents

Chapter 1

Introduction

The main basis of this work are the three papers [Whi34a], [Whi34b] and [Whi34c] published by Hassler Whitney in 1934. They offered a new and fruitful approach to deal with the following extension problem. Consider a compact subset K of $\mathbb{R}^d$ and a continuous function $f : K \to \mathbb{R}$. The question arises how to decide if the domain of this function f can be extended to $\mathbb{R}^d$ or an open superset of K, such that the extended function is n-times or even infinite often continuously partially differentiable. Another way of stating this question is, to find a meaningful definition of differentiability for functions defined in compact or closed sets. We will use the symbols $\mathscr{E}^n(K)$ respectively $\mathscr{E}(K)$ for those spaces. Whitney found a way to achieve this by using the well known Taylor theorem as a definition instead of getting it as a conclusion from the definition of differentiability. In the classical setting, Taylor's theorem shows that a function f, which is n-times continuously partially differentiable at some point x_0, can be locally approximated by its Taylor polynomial $T^n_{x_0}$, which is a polynomial of degree $\leq n$. The coefficients of the polynomial depend on the derivatives of the function at this point. For the difference $f - T^n_{x_0}$ between the function and its Taylor polynomial, the remainder $R^n_{x_0}$, the asymptotic behaviour at x_0 can be described by $R^n_{x_0}(x) = o(|x - x_0|^n)$ as $x \to x_0$. Since the Taylor polynomial already contains the derivatives of the function, it is necessary to replace these derivatives by other functions if one wants to use this theorem as a definition for differentiability. Therefore the objects contained in the spaces $\mathscr{E}^n(K)$ respectively $\mathscr{E}(K)$ cannot just be single functions but have to be families of functions $\left(f^{(\alpha)}\right)_{|\alpha|\leq n}$ in $\mathscr{E}^n(K)$ and $\left(f^{(\alpha)}\right)_{\alpha\in\mathbb{N}_0^d}$ in $\mathscr{E}(K)$ which we will call Whitney jets of order n respectively of infinite order later on. For these jets, the 'formal' Taylor polynomial of order n centred at $y \in K$ can very naturally be defined as

$$T^n_y\left(\left(f^{(\alpha)}\right)_{|\alpha|\leq n}\right)(x) := \sum_{|\alpha|\leq n} \frac{f^{(\alpha)}(y)}{\alpha!}(x - y)^\alpha.$$

The corresponding remainder itself has to be a family of functions, where the β-th entry is defined as

$$R_y^n\left(\left(f^{(\alpha)}\right)_{|\alpha|\leq n}\right)^{(\beta)}(x) := f^{(\beta)}(x) - \partial^\beta T_y^n\left(\left(f^{(\alpha)}\right)_{|\alpha|\leq n}\right)(x).$$

With that, the space $\mathscr{E}^n(K)$ is defined as the space of all jets $\left(f^{(\alpha)}\right)_{|\alpha|\leq n} \in \prod_{|\alpha|\leq n}\mathscr{C}(K)$, for which the remainder has the following asymptotic behaviour

$$\lim_{t\to 0}\sup_{|\beta|\leq n}\left\{\left|R_y^n\left(\left(f^{(\alpha)}\right)_{|\alpha|\leq n}\right)^{(\beta)}(x)\right||x-y|^{|\beta|-n} : x,y\in K, 0<|x-y|\leq t\right\}. \tag{1.1}$$

This is indeed the same asymptotic property, which the classical remainder shows in Taylor's theorem. The space $\mathscr{E}^n(K)$ endowed with the norm

$$\begin{aligned}\left\|\left(f^{(\alpha)}\right)_{|\alpha|\leq n}\right\|_{n,K} := &\sup_{|\beta|\leq n, x\in K}\left|f^{(\beta)}(x)\right| \\ &+ \sup_{|\beta|\leq n, t>0}\left\{\left|R_y^n\left(\left(f^{(\alpha)}\right)_{|\alpha|\leq n}\right)^{(\beta)}(x)\right||x-y|^{|\beta|-n} : x,y\in K, 0<|x-y|\leq t\right\}\end{aligned} \tag{1.2}$$

is a Banach space. The space $\mathscr{E}(K)$ is constructed as the projective limit of the $\mathscr{E}^n(K)$. Also for $F\subset\mathbb{R}^d$ closed, the spaces $\mathscr{E}^n(F)$ and $\mathscr{E}(F)$ can be constructed as projective limits of the spaces $\mathscr{E}^n(K_l)$ respectively $\mathscr{E}(K_l)$ for a fundamental sequence $(K_l)_{l\in\mathbb{N}}$ of compact sets for F.

Whitney could show, that this is indeed a meaningful definition of differentiability in the sense, that the continuous and linear restriction operators

$$r_n : \mathscr{E}^n(\mathbb{R}^d)\to\mathscr{E}^n(K), f\mapsto\left(\partial^\alpha f\big|_K\right)_{|\alpha|\leq n}, \text{ and}$$

$$r : \mathscr{E}(\mathbb{R}^d)\to\mathscr{E}(K), f\mapsto\left(\partial^\alpha f\big|_K\right)_{|\alpha|\in\mathbb{N}_0^d}$$

are surjective. Furthermore he was able to prove in a constructive way, that the restriction operators r_n have a linear and continuous right inverse, called extension operator.

The research which build up on these findings was mainly dedicated to the question, in which circumstances also the restriction operator r has a continuous and linear right inverse. It is relatively easy to see that not all geometries of K allow such an extension operator. Counterexamples are for instance singletons and sets contained in a hyperplane. But there are also examples of sets which even coincide with the closure of their interior and still do not admit an extension operator. Tidten proved in [Tid79] that certain sequences of pairwise disjoint intervals and sets with exponential cusps do not admit such an operator. On the other hand many classes of sets could be characterized which admit an extension operator. For instance Seeley constructed in [See64] an extension operator for half spaces and Stein dealt with sets having a Lipschitz boundary in [Ste70]. Stein's result was then generalized by Bierstone in [Bie78] and Frerick in [Fre07b] later on.

This work is also dedicated to the question of the characterization for 'good' geometries of a compact sets, such that they admit an extension operator. It is mainly based on the ideas of Frerick, Jordá and Wengenroth which they exhibited in the papers [FJW16b] and [FJW11]. They approached the question again in a constructive way by constructing an operator which resembles the one Whitney constructed. The operators E_n Whitney constructed as inverse of r_n has the following structure

$$E_n\left(\left(f^{(\alpha)}\right)_{|\alpha|\le n}\right)(x) = \begin{cases} f^{(0)}(x), x \in K \\ \sum_{i\in\mathbb{N}} \varphi_i(x) \sum_{|\alpha|\le n} \frac{f^{(\alpha)}(x_i)}{\alpha!}(x-x_i)^\alpha, x \notin K \end{cases},$$

where $(\varphi_i)_{i\in\mathbb{N}}$ is a partition of K^c satisfying certain conditions, and the points $x_i \in K$ depend on the supports of the φ_i. The idea is to interpolate the entries $f^{(\alpha)}$ of the jets by measures $\mu_{\alpha,i}$, which results in an operator E of the following structure

$$E\left(\left(f^{(\alpha)}\right)_{\alpha\in\mathbb{N}_0^d}\right)(x) = \begin{cases} f^{(0)}(x), x \in K \\ \sum_{i\in\mathbb{N}} \varphi_i(x) \sum_{|\alpha|\le m(i)} \frac{\mu_{\alpha,i}(f^{(0)})}{\alpha!}(x-x_i)^\alpha, x \notin K \end{cases},$$

where the sequence $(m(i))_{i\in\mathbb{N}}$ has to be chosen in the course of the construction. The measures $\mu_{\alpha,i}$ only depend on the first entry of the jet $f^{(0)}$, which has the additional charme that the extension operator E also simultaneously extends all jets of finite order. The operator E can therefore be applied to a jet $\left(f^{(\alpha)}\right)_{|\alpha|\le n} \in \mathscr{E}^n(K)$ but it is not necessarily a continuous operator from $\mathscr{E}^n(K)$ to $\mathscr{E}^n(\mathbb{R}^d)$. The continuity propterties of the operator on the projective scale can be characterized by means of a 'loss of derivatives', which shall be a mapping $\sigma : \mathbb{N}_0 \to \mathbb{N}_0$ with $\sigma(n) \ge n$, such that $E : \mathscr{E}^{\sigma(n)}(K) \to \mathscr{E}^n(\mathbb{R}^d)$ is continuous. In [FJW16b], the authors characterized the existence of an extension operator having no loss of derivatives, which means $\sigma(n) = n$ for all $n \in \mathbb{N}_0$. In [FJW11] they characterized the existence of a tame linear extension operator, which means $\sigma(n) = cn$ for some $c \ge 1$. Since $\sigma(n)$ is then not always a natural number, the last case shows that it is also necessary to generalize the definition of the Whitney spaces $\mathscr{E}^n(K)$ and to 'fill the gaps' between $\mathscr{E}^n(K)$ and $\mathscr{E}^{n+1}(K)$, respectivelly to fill the gaps between the seminorms $||\cdot||_{n,K}$ and $||\cdot||_{n+1,K}$ on $\mathscr{E}(K)$. This is done in [FJW11] in a very natural way by replacing the term $|x-y|^{|\beta|-n}$ by $|x-y|^{s-n}$ for $n \le s < n+1$. In this work, we also generalize the right side of the operator, so that we have a full real scale of seminorms on $\mathscr{E}(K)$ and $\mathscr{E}(\mathbb{R}^d)$. In Chapter 3, we generalize the construction of Whitneys operators E_n, such that E_n also extends the spaces $\mathscr{E}^s(K)$ for $s \in [n, n+1)$. In the Chapters 4 and 5, we generalize the results of the papers [FJW16b] and [FJW11] to the full real scale of seminorms. In the last Chapter, we characterize the existence of an extension operator with an arbitrary loss by means of the existence of measures.

Chapter 2

Preliminaries

2.1 Some Basic Notations and Results

As a very basic notation in multidimensional analysis we want to introduce the multiindex notation. It allows a very short and compact notation which resembles the notation in dimension 1.

2.1 Definition. *We call any vector $\alpha \in \mathbb{N}_0^d$ a (d-dimensional) multiindex. For $x \in \mathbb{R}^d$ and $\beta \in \mathbb{N}_0^d$, we set*

1. $|\alpha| := \sum_{i=1}^{d} \alpha_i$,
2. $x^\alpha := \prod_{i=1}^{d} x_i^{\alpha_i}$.
3. *The addition of two multiindices of the same dimension as well as the multiplication of a multiindex with a scalar is defined as in the d-dimensional Euclidean space.*
4. $\binom{\alpha}{\beta} := \frac{\alpha!}{(\alpha-\beta)!\beta!}$.
5. *For $\beta \in \mathbb{N}_0^d$ we define the relation*

$$\beta \leq \alpha \Leftrightarrow \beta_i \leq \alpha_i \textit{ for all } i \in \{1, ..., d\}.$$

In the following remark we gather some easy consequences which will often be used throughout this work.

2.2 Remark. *1. For two multiindices α, β it is obviously true that $|\alpha + \beta| = |\alpha| + |\beta|$ and if $\beta \leq \alpha$ we also have $|\alpha - \beta| = |\alpha| - |\beta|$.*

2. *For an arbitrary multi-index $\alpha \in \mathbb{N}_0^d$, the following inequality holds for all $x \in \mathbb{R}^d$*

$$|x^\alpha| \leq |x|^{|\alpha|},$$

where $|x|$ denotes the Euclidean length of x. This follows easily by definition:

$$|x^\alpha| = \left| \prod_{1 \leq i \leq d} x_i^{\alpha_i} \right| \leq \prod_{1 \leq i \leq d} |x_i|^{\alpha_i} \leq \prod_{1 \leq i \leq d} |x|^{\alpha_i} = |x|^{|\alpha|}.$$

2.3 Definition. *Spaces of differentiable functions*
Let $\Omega \subset \mathbb{R}^d$ be an open set. Then we define for $n \in \mathbb{N}_0$

$$\mathscr{E}^n(\Omega) := \{f : \partial^\alpha f \text{ exists and is continuous in } \Omega \text{ for all } |\alpha| \leq n\},$$

$$\mathscr{E}(\Omega) := \{f : \partial^\alpha f \text{ exists and is continuous in } \Omega \text{ for all } \alpha \in \mathbb{N}_0^d\}.$$

To define the topology on those spaces, let $(K_l)_{l \in \mathbb{N}}$ be a fundamental sequence of compact sets for Ω. Then let the supremum seminorm be defined as

$$|\cdot|_{n,K_l} : \mathscr{E}^n(\Omega) \to [0, \infty), f \mapsto \sup\{|\partial^\alpha f(x)| : x \in K_l, |\alpha| \leq n\}.$$

The system of seminorms $(|\cdot|_{n,K_l})_{l \in \mathbb{N}}$ defines a Fréchet space topology on $\mathscr{E}^n(\Omega)$ and the system $(|\cdot|_{l,K_l})_{l \in \mathbb{N}_0}$ defines a Fréchet space topology on $\mathscr{E}(\Omega)$.

Now we turn our attention to Taylor's theorem, which is the key result leading us to the results of Whitney in the next section.

2.4 Theorem. *Taylor's theorem*
Let $f : \mathbb{R}^d \to \mathbb{R}$ be n-times continuously partially differentiable at $y \in \mathbb{R}^d$. Then for all $|\alpha| = k$ there exists a $g_\alpha : \mathbb{R}^d \to \mathbb{R}$ with $\lim_{x \to y} g_\alpha(x) = 0$ such that

$$\begin{aligned} f(x) &= \sum_{|\alpha| \leq n} \frac{\partial^\alpha f(y)}{\alpha!}(x-y)^\alpha + \sum_{|\alpha|=n} g_\alpha(x)(x-y)^\alpha \\ &= T_y^n(f)(x) + R_y^n(f)(x), \end{aligned}$$

with $R_y^n(x) = o(|x-y|^n)$. The polynomial $T_y^n(f)$ is called Taylor polynomial of order n centred at y and $R_y^n(f)$ is called the Taylor remainder.

2.5 Remark. 1. *For $f \in \mathscr{E}^{n+1}(\mathbb{R}^d)$ there exist a lot of representations of $R_y^n(f)$, which we only state here. Let $\theta \in [0, 1]$*

- *Lagrange representation: $R_y^n(f)(x) = \sum_{|\alpha|=n+1} \frac{\partial^\alpha f(y+\theta(x-y))}{\alpha!}(x-y)^\alpha$,*
- *Cauchy representation: $R_y^n(f)(x) = (n+1)(1-\theta)^n \sum_{|\alpha|=n+1} \frac{\partial^\alpha f(y+\theta(x-y))}{\alpha!}(x-y)^\alpha$.*

2. *It is also possible to formulate a converse Taylor theorem. If for a given $f : \mathbb{R}^d \to \mathbb{R}$ and all $|\alpha| \leq n$ there exist continuous functions $f^{(\alpha)} : \mathbb{R}^d \to \mathbb{R}$ and $g_\alpha : \mathbb{R}^d \to \mathbb{R}$ with $\lim_{x\to y} g_\alpha(x) = 0$ such that*

$$f(x) = \sum_{|\alpha|\leq n} \frac{f^{(\alpha)}(y)}{\alpha!}(x-y)^\alpha + \sum_{|\alpha|=n} g_\alpha(x)(x-y)^\alpha,$$

then f is n-times continuously partially differentiable at y and $\partial^\alpha f(y) = f^{(\alpha)}(y)$. A proof can be found in [Oli54].Thus, Taylor's theorem can also be used to define partial differentiability.

2.2 Differentiable Functions on Closed Sets in the Sense of Whitney

Taking up on Remark 2.5, we now go in the direction Whitney took in [Whi34a] in order to use Taylor's formula to define differentiability of functions on closed sets. We begin with the definition of the 'formal' Taylor polynomial and its corresponding remainder for a family of arbitrary functions on a compact set $K \subset \mathbb{R}^d$. In the following, the Greek letters α and β always represent a multiindex unless stated otherwise.

2.6 Definition. *Let $n \in \mathbb{N}_0$ and $F = \left(f^{(\alpha)}\right)_{|\alpha|\leq n}$ be a family of continuous functions defined on a compact set $K \subset \mathbb{R}^d$, hereafter referred to as jet (of order n). For each such jet we can define its 'formal' Taylor polynomial of order n centred at $y \in K$ evaluated at the point $x \in \mathbb{R}^d$ as*

$$T^n_y(F)(x) := \sum_{|\alpha|\leq n} \frac{f^{(\alpha)}(y)}{\alpha!}(x-y)^\alpha.$$

For the partial derivatives of the Taylor polynomial we get

$$\partial^\beta T^n_y(F)(x) = T^{n-|\beta|}_y\left(\left(f^{\alpha+\beta}\right)_{|\alpha|\leq n-|\beta|}\right)(x) = \sum_{|\alpha|\leq n-|\beta|} \frac{f^{(\alpha+\beta)}(y)}{\alpha!}(x-y)^\alpha.$$

We define the corresponding Taylor remainder $R^n_y(F)^{(\alpha)} \in \prod_{|\alpha|\leq n} \mathscr{C}(K)$ as

$$R^n_y(F)^{(\alpha)}(x) := f^{(\alpha)}(x) - \partial^\alpha T^n_y(F)(x).$$

We note that the Taylor polynomial can be defined on whole $\mathbb{R}^d$, whereas the remainder can only be defined on K and that the definition of both does not depend on any smoothness properties of the entries of F. In addition to jets of finite order, we also call a countably infinite family of continuous functions $\left(f^{(\alpha)}\right)_{\alpha\in\mathbb{N}_0^d}$ on K a jet (of infinite order).

Before giving the definition of the spaces of Whitney jets, we state the following lemma.

2.7 Lemma. *Let $K \subset \mathbb{R}^d$ be compact and let $F = \left(f^{(\alpha)}\right)_{|\alpha|\le n}$ be a jet of order n defined on K. Then for $x, y \in K$, $n \in \mathbb{N}_0, |\beta| \le n$ and each $z \in \mathbb{R}^d$ it is true that*

$$\partial^\beta T_x^n(F)(z) - \partial^\beta T_y^n(F)(z) = \sum_{|\alpha|\le n} \frac{(z-x)^\alpha}{\alpha!} R_y^n(F)^{(\alpha+\beta)}(x).$$

Proof. A proof can be found in [Ste70] page 177. □

In the following we will make extensive use of the following well known notations for a real number $s \ge 0$

- $\lfloor s \rfloor := \max\{n \in \mathbb{N}_0 : n \le s\}$, which denotes the integer value of s or floor of s,
- $\{s\} := s - \lfloor s \rfloor$ which denotes the fractional part of s.

The following definition of the Whitney spaces as the (in a certain sense) correct description for the space of differentiable functions on closed sets is rather technical, but it will be fully justified by Whitney's results, which we will present in the following section.

2.8 Definition. *For $K \subset \mathbb{R}^d$ compact, $s \ge 0$ and $F = \left(f^{(\alpha)}\right)_{|\alpha|\le\lfloor s\rfloor} \in \prod_{|\alpha|\le\lfloor s\rfloor} \mathscr{C}(K)$ we define for $t > 0$*

$$q_{s,K}(F,t) := \sup\left\{|R_y^{\lfloor s\rfloor}(F)^{(\alpha)}(x)||x-y|^{|\alpha|-s} : x, y \in K, 0 < |x-y| \le t, |\alpha| \le \lfloor s\rfloor\right\},$$

and with this, we set

$$\mathscr{E}^s(K) := \left\{F \in \prod_{|\alpha|\le\lfloor s\rfloor} \mathscr{C}(K) : \lim_{t\to 0} q_{s,K}(F,t) = 0\right\},$$

$$\mathscr{E}(K) := \left\{F \in \prod_{\alpha\in\mathbb{N}_0^d} \mathscr{C}(K) : \lim_{t\to 0} q_{s,K}(F,t) = 0 \quad \text{for all } s \ge 0\right\}.$$

We endow the space $\mathscr{E}^s(K)$ with the norm

$$||F||_{s,K} := |F|_{\lfloor s\rfloor,K} + \sup_{t>0} q_{s,K}(F,t),$$

where

$$|F|_{\lfloor s\rfloor,K} := \sup\left\{|f^{(\alpha)}(x)| : x \in K, |\alpha| \le \lfloor s\rfloor\right\}.$$

The space $\mathscr{E}(K)$ is equipped with the Fréchet space topology induced be the family of seminorms $(\|\cdot\|_{s,K})_{s\in\mathbb{N}_0}$. In the case of $s \in \mathbb{N}_0$, the definition of the spaces $\mathscr{E}^s(K)$ coincides with the definition of the classical Whitney spaces given in [Whi34a]. The completeness of the spaces $\mathscr{E}^s(K)$ for $s \notin \mathbb{N}_0$ is shown in the next remark. If M is a closed set, we choose a fundamental sequence of compact sets $(K_l)_{l\in\mathbb{N}}$ and define $\mathscr{E}^s(M)$ and $\mathscr{E}(M)$ again as projective limits. For an open set Ω and a fundamental sequence $(K_l)_{l\in\mathbb{N}}$ thereof, the classical Whitney norms $\|\cdot\|_{n,K_l}$ also induce the Fréchet space topology on the spaces $\mathscr{E}^n(\Omega)$ (by the canonical identification of f with the jet $(\partial^\alpha f)_{|\alpha|\le n}$), which we introduced in Definition 2.3. This definition can be generalized naturally to all $s \ge 0$ by setting

$$\mathscr{E}^s(\Omega) := \left\{ f \in \mathscr{E}^{\lfloor s \rfloor}(\Omega) : \left(\partial^\alpha f\Big|_K\right)_{|\alpha|\le\lfloor s\rfloor} \in \mathscr{E}^s(K) \text{ for all compact } K \subset \Omega \right\}.$$

Furthermore we define

$$\mathscr{D}^s(K) := \left\{ f \in \mathscr{E}^s(\mathbb{R}^d) : supp(f) \subset K \right\}.$$

Equipped with the norm $\|f\|_{s,K} := \left\|\left(\partial^\alpha f\Big|_K\right)_{|\alpha|\le\lfloor s\rfloor}\right\|_{s,K}$, the spaces $\mathscr{D}^s(K)$ are complete. In the following we will mostly not mention the set in the subscript of the norms if it is clear on which set they are defined.

2.9 Remark. *1. The reason why we define the generalized Whitney spaces $\mathscr{E}^s(K)$ for $s \notin \mathbb{N}_0$ as we did it, is that we want $\mathscr{E}(K)$ to be dense in $\mathscr{E}^s(K)$ for all s. If instead we would merely impose Lipschitz conditions as Stein did in [Ste70], this is not the case any more. To see this, let $f : [-1,1] \to \mathbb{R}$, $x \mapsto \sqrt{|x|}$. Then $f \in Lip^{\frac{1}{2}}([-1,1])$, but is not approximable with respect to the corresponding norm by a smooth function.*

2. For a compact set $K \subset \mathbb{R}^d$ and $s \ge 0$, $\mathscr{E}^s(K)$ is a Banach space. To see this, let $(F_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{E}^s(K)$. Since the convergence in the Whitney norm implies convergence in the supremum norm and since $\prod_{|\alpha|\le\lfloor s\rfloor}\mathscr{C}(K)$ equipped with the supremum norm is a Banach space, there exists an $F \in \prod_{|\alpha|\le\lfloor s\rfloor}\mathscr{C}(K)$ such that $\lim_{n\to\infty}|F-F_n|_{\lfloor s\rfloor} = 0$. To show that $F \in \mathscr{E}^s(K)$, let $\varepsilon > 0$. Depending on ε, we find $m \in \mathbb{N}$, such that $\|F_n - F_m\|_s < \frac{\varepsilon}{3}$ for all $n \ge m$. Depending on m we find a $t > 0$ so that

$$\frac{|R_y^{\lfloor s\rfloor}(F_m)^{(\beta)}(x)|}{|x-y|^{s-|\beta|}} < \frac{\varepsilon}{3}$$

for all $0 < |x-y| < t$. Since we have

$$|R_y^{\lfloor s\rfloor}(F)^{(\beta)}(x) - R_y^{\lfloor s\rfloor}(F_n)^{(\beta)}(x)||x-y|^{|\beta|-s}$$

$$\leq \left(|f^{(\beta)}(x) - f_n^{(\beta)}(y)| + \sum_{|\alpha| \leq \lfloor s \rfloor - |\beta|} \frac{|f^{(\alpha+\beta)}(y) - f_n^{(\alpha+\beta)}(y)|}{\alpha!} |x-y|^{|\alpha|} \right) |x-y|^{|\beta|-s}$$

$$\leq |F - F_n|_{\lfloor s \rfloor} \left(|x-y|^{|\beta|-s} + \sum_{|\alpha| \leq \lfloor s \rfloor - |\beta|} \frac{|x-y|^{|\alpha|+|\beta|-s}}{\alpha!} \right),$$

and because $(F_n)_{n \in \mathbb{N}}$ *converges uniformly to F we always find* $n = n(|x-y|) > m$ *such that*

$$\frac{|R_y^{\lfloor s \rfloor}(F)^{(\beta)}(x) - R_y^{\lfloor s \rfloor}(F_n)^{(\beta)}(x)|}{|x-y|^{s-|\beta|}} < \frac{\varepsilon}{3}.$$

We note that n just depends on the distance of x and y. So we get for an arbitrary pair $x, y \in K$ *with* $0 < |x-y| < t$

$$\frac{|R_y^{\lfloor s \rfloor}(F)^{(\beta)}(x)|}{|x-y|^{s-|\beta|}} \leq \frac{|R_y^{\lfloor s \rfloor}(F)^{(\beta)}(x) - R_y^{\lfloor s \rfloor}(F_n)^{(\beta)}(x)|}{|x-y|^{s-|\beta|}} + \frac{|R_y^{\lfloor s \rfloor}(F_n)^{(\beta)}(x) - R_y^{\lfloor s \rfloor}(F_m)^{(\beta)}(x)|}{|x-y|^{s-|\beta|}}$$
$$+ \frac{|R_y^{\lfloor s \rfloor}(F_m)^{(\beta)}(x)|}{|x-y|^{s-|\beta|}}$$
$$< \varepsilon.$$

The fact that n just depends on $|x-y|$ *shows that* $q_{s,K}(F,t)$ *exists and that it converges in t to 0.*

3. *For K convex and compact and* $s \in [0, \infty)$*, Lemma 2.10 shows that the conditions*

$$\lim_{t \to 0} q_{s,K}(F,t) = 0$$

and

$$\lim_{t \to 0} \sup \left\{ \frac{|f^{(\beta)}(x) - f^{(\beta)}(y)|}{|x-y|^{\{s\}}} : x, y \in K, 0 < |x-y| < t, |\beta| = \lfloor s \rfloor \right\} = 0$$

are equivalent. The proof of Lemma 2.10 also shows that the Whitney norm $\|\cdot\|_s$ *and the norm*

$$|||F|||_s := |F|_{\lfloor s \rfloor} + \sup_{t>0} \left\{ \frac{|f^{(\beta)}(x) - f^{(\beta)}(y)|}{|x-y|^{\{s\}}} : \quad x, y \in K, 0 < |x-y| \leq t, |\beta| = \lfloor s \rfloor \right\}.$$

are equivalent on $\mathscr{E}^s(K)$*. Therefore, for a compact but not necessarily convex set K it is always possible to equip the spaces* $\mathscr{D}^s(K)$ *with a norm* $|||\cdot|||_{s,L}$ *where* $L \supset K$ *compact and convex. Lemma 2.11 shows that the norm can be even simplified further by replacing the supremum norm summand by taking the supremum only over the highest degree derivatives.*

4. *Let s_1, s_2 be positive numbers such that $\lfloor s_1 \rfloor \leq s_1 \leq s_2 \leq \lfloor s_1 \rfloor + 1$. If K consists only of finitely many points, then $l := \min\{|x-y| : x, y \in K, x \neq y\} > 0$. Hence,*

$$\frac{|R_y^{\lfloor s_2 \rfloor}(F)^{(\beta)}(x)|}{|x-y|^{s_2-|\beta|}} = \frac{|R_y^{\lfloor s_1 \rfloor}(F)^{(\beta)}(x)|}{|x-y|^{s_1-|\beta|}}|x-y|^{s_1-s_2} \leq \frac{|R_y^{\lfloor s_1 \rfloor}(F)^{(\beta)}(x)|}{|x-y|^{s_1-|\beta|}} l^{s_1-s_2},$$

which gives

$$\begin{aligned} \|F\|_{s_2} &= |F|_{\lfloor s_2 \rfloor} + \sup_{t>0} q_{K,s_2}(F,t) \\ &\leq |F|_{\lfloor s_1 \rfloor} + l^{s_1-s_2} \sup_{t>0} q_{K,s_1}(F,t) \\ &\leq \max(1, l^{s_1-s_2}) \|F\|_{s_1}. \end{aligned}$$

Thus both norms are equivalent. Be that as it may, the case of finite sets K is not of great interest for the following work, since then the space $\mathscr{E}(K)$ cannot admit a continuous and linear extension operator as pointed out in [FJW11].

2.10 Lemma. *Let $K \subset \mathbb{R}^d$ be a convex and compact set, then the norms $\|\cdot\|_s$ and $|||\cdot|||_s$ are equivalent on $\mathscr{E}^s(K)$.*

Proof. It is obviously true that $|||F|||_s \leq \|F\|_s$ for all $F \in \mathscr{E}^s(K)$. So we only have to prove the existence of a constant $C > 0$ such that $\|F\|_s \leq C|||F|||_s$ for all $F \in \mathscr{E}^s(K)$. Following the classical theorem of Whitney, each entry $f^{(\beta)}$ of the jet F can be extended to an $\lfloor s \rfloor - |\beta|$-times continuously partially differentiable function on $\mathbb{R}^d$. Therefore we also can regard the entries of the Taylor remainder $R_x^{\lfloor s \rfloor}(F)^{(\beta)}$ as being $\lfloor s \rfloor - |\beta|$-times partially differentiable at any point $y \in K$ for each $x \in K$. This allows us to apply the mean value theorem. If $|\beta| = \lfloor s \rfloor - 1$, then we find a $z \in (x, y)$ such that

$$\frac{|R_x^{\lfloor s \rfloor}(F)^{(\beta)}(y)|}{|x-y|^{s-|\beta|}} = \frac{|R_x^{\lfloor s \rfloor}(F)^{(\beta)}(y) - R_x^{\lfloor s \rfloor}(F)^{(\beta)}(x)|}{|x-y|^{\{s\}+1}} = \frac{|\nabla(R_x^{\lfloor s \rfloor}(F)^{(\beta)})(z)(x-y)|}{|x-y|^{\{s\}+1}}.$$

Let $j := \max_{i \in \{1,\ldots,d\}} |R_x^{\lfloor s \rfloor} F^{(\beta+e_i)}(z)|$. Applying the Cauchy-Schwarz inequality and using the fact that $|x-y| > |x-z|$, we obtain

$$\frac{|R_x^{\lfloor s \rfloor}(F)^{(\beta)}(y)|}{|x-y|^{\{s\}+1}} \leq \frac{|\nabla(R_x^{\lfloor s \rfloor}(F)^{(\beta)})(z)|}{|x-y|^{\{s\}}} < \sqrt{d}\frac{|R_x^{\lfloor s \rfloor}(F)^{(\beta+e_j)}(z)|}{|x-z|^{\{s\}}} \leq \sqrt{d}|||F|||_s.$$

Proceeding inductively, we arrive at

$$\|F\|_s \leq \sqrt{d}^{\lfloor s \rfloor}|||F|||_s.$$

□

2.11 Lemma. *Let $K \subset \mathbb{R}^d$ be compact. Then for a compact and convex set $L \supset K$ the space $\mathscr{D}^s(K)$ can be equipped with the following equivalent norms: $\|\cdot\|_{s,K}$, $\|\cdot\|_{s,L}$, $|||\cdot|||_{s,L}$ and*

$$|||f|||'_{s,L} := \sup_{\substack{x\in L\\|\alpha|=\lfloor s\rfloor}} |\partial^\alpha f(x)| + \sup_{t>0}\left\{\frac{|\partial^\beta f(x)-\partial^\beta f(y)|}{|x-y|^{\{s\}}} : \; x,y\in K, 0<|x-y|\le t, |\beta|=\lfloor s\rfloor\right\}.$$

Proof. The equivalence of the first three norms is rather obvious using the preceding lemma. The only thing which is left to show is that there exists a positive constant C such that for each $f \in \mathscr{D}^s(K)$ the following inequality holds

$$|f|_{\lfloor s\rfloor} \le C \sup_{\substack{x\in L\\|\alpha|=\lfloor s\rfloor}} |\partial^\alpha f(x)|.$$

As in the proof of the preceding lemma, we can again apply the mean value theorem and proceed recursively. Let $|\alpha| = \lfloor s\rfloor - 1$ and choose $z \in L \setminus K$. Then we have for arbitrary $x \in K$

$$|\partial^\alpha f(x)| = |\partial^\alpha f(x) - \partial^\alpha f(z)| \le |\nabla(\partial^\alpha f)(y)||x-z| \le \lambda\sqrt{d} \sup_{\substack{x\in L\\|\beta|=\lfloor s\rfloor}} |\partial^\beta f(x)|,$$

where λ denotes the diameter of L. Proceeding recursively we can set $C := \lambda^{\lfloor s\rfloor}\sqrt{d}^{\lfloor s\rfloor}$. □

Now we reformulate the conditions on the Taylor remainder using a modulus of continuity. We do this to be able to formulate the results and proofs in the next chapter in an easier way than Whitney did, leaning on the notation which Malgrange used in [Mal67].

2.12 Definition. *An increasing, continuous and concave function $\mu : [0,\infty) \to [0,\infty)$ with $\mu(0) = 0$ is called a modulus of continuity.*

2.13 Remark. *Given two moduli of continuity μ_1 and μ_2, it is easy to show that also $\mu_1 \circ \mu_2$ is a modulus of continuity.*

The next theorem shows, that the asymptotical behaviour of the Taylor remainder can naturally be described with moduli of continuity. It is a generalisation of Theorem 2.2 in [Mal67].

2.14 Theorem. *The following statements are equivalent for each $F \in \mathscr{E}^s(K)$.*

1. $\lim_{t\to 0} q_{s,K}(F,t) = 0$.

2. *There exists a modulus of continuity μ such that*

$$\left|R_x^{\lfloor s\rfloor}(F)^{(\beta)}(y)\right| \leq |x-y|^{s-|\beta|}\mu(|x-y|) \tag{2.1}$$

for $x, y \in K$ and $|\beta| \leq \lfloor s\rfloor$. Moreover we can choose μ such that $\|F\|_s = |F|_{\lfloor s\rfloor} + \mu(diam(K))$.

3. *There exists a modulus of continuity μ_0 such that*

$$\left|\partial^\beta T_x^{\lfloor s\rfloor}(F)(z) - \partial^\beta T_y^{\lfloor s\rfloor}(F)(z)\right| \leq \mu_0(|x-y|)(|z-x|^{s-|\beta|} + |z-y|^{s-|\beta|}) \tag{2.2}$$

for $x, y \in K, z \in \mathbb{R}^d$ and $|\beta| \leq \lfloor s\rfloor$. The proof shows in fact that we can choose $\mu_0 = C\mu$, where C depends only on d and $\lfloor s\rfloor$.

Proof. To see that 1. is equivalent to 2. we only have to prove that 1. implies 2. We note that $q_{s,K}(F, t)$ is increasing in t and continuous at 0 which allows us to choose a modulus of continuity μ with $\mu(0) = 0, \mu(t) \geq q_{s,K}(F, t)$ for $t \in (0, \text{diam}(K))$ and $\mu(t) \equiv q_{s,K}(F, \text{diam}(K))$ for $t \geq \text{diam}(K)$.

No we show that 2. implies 3.
Using Lemma 2.7 and the assumptions, we obtain

$$\begin{aligned}
&\left|\partial^\beta T_x^{\lfloor s\rfloor}(F)(z) - \partial^\beta T_y^{\lfloor s\rfloor}(F)(z)\right| \\
= &\left|\sum_{|\alpha|\leq\lfloor s\rfloor-|\beta|} \frac{(z-x)^\alpha}{\alpha!} R_y^{\lfloor s\rfloor}(F)^{(\alpha+\beta)}(x)\right| \\
\leq &\sum_{|\alpha|\leq\lfloor s\rfloor-|\beta|} \frac{|z-x|^{|\alpha|}}{\alpha!} \left|R_y^{\lfloor s\rfloor}(F)^{(\alpha+\beta)}(x)\right| \\
\leq &\ \mu(|x-y|) \sum_{|\alpha|\leq\lfloor s\rfloor-|\beta|} \frac{|z-x|^{|\alpha|}|x-y|^{s-|\alpha|-|\beta|}}{\alpha!} \\
= &\ \mu(|x-y|)|x-y|^{\{s\}} \sum_{|\alpha|\leq\lfloor s\rfloor-|\beta|} \frac{|z-x|^{|\alpha|}|x-y|^{(\lfloor s\rfloor-|\beta|)-|\alpha|}}{\alpha!}
\end{aligned}$$

To be able to apply the binomial theorem on the sum, we group the multi-indices by their absolute value. To this end let $p(d, i) := \left|\{\alpha \in \mathbb{N}_0^d : |\alpha| = i\}\right| = \binom{d+i-1}{i}$. Remark that $p(d, i) \leq p(d, \lfloor s\rfloor - |\beta|)$ for all $i \leq \lfloor s\rfloor - |\beta|$. For the binomial theorem we have to replace $\frac{1}{\alpha!}$ by $\binom{\lfloor s\rfloor-|\beta|}{|\alpha|!}$. Therefore we have to make sure that there exists a constant C_0, only depending on $\lfloor s\rfloor$ and d, such that $\frac{1}{\alpha!} \leq C_0\binom{\lfloor s\rfloor-|\beta|}{|\alpha|!}$ for all $|\beta| \leq \lfloor s\rfloor$. So C_0 has to fulfil $C_0 \geq \frac{(\lfloor s\rfloor-|\beta|-|\alpha|)!|\alpha|!}{\alpha!(\lfloor s\rfloor-|\beta|)!}$. Because for fixed $\lfloor s\rfloor$ we have to consider only a finite number of

α and β, we can define $C_0 \coloneqq \max\left\{\frac{(\lfloor s\rfloor-|\beta|-|\alpha|)!|\alpha|!}{\alpha!(\lfloor s\rfloor-|\beta|)!} : |\beta| \leq \lfloor s\rfloor, |\alpha| \leq \lfloor s\rfloor - |\beta|\right\}$ and are done. Hence, we get

$$\begin{aligned}
&\left|\partial^\beta T_x^{\lfloor s\rfloor}(F)(z) - \partial^\beta T_y^{\lfloor s\rfloor}(F)(z)\right| \\
\leq\ & \mu(|x-y|)|x-y|^{\{s\}} \sum_{|\alpha|\leq\lfloor s\rfloor-|\beta|} \frac{|z-x|^{|\alpha|}|x-y|^{(\lfloor s\rfloor-|\beta|)-|\alpha|}}{\alpha!} \\
\leq\ & \mu(|x-y|)|x-y|^{\{s\}} \sum_{i\leq\lfloor s\rfloor-|\beta|, i\in\mathbb{N}} C_0 p(d,i)\binom{\lfloor s\rfloor-|\beta|}{i}|z-x|^i|x-y|^{\lfloor s\rfloor-|\beta|-i} \\
\leq\ & C_0 p(d,\lfloor s\rfloor)\mu(|x-y|)|x-y|^{\{s\}}(|z-x|+|x-y|)^{\lfloor s\rfloor-|\beta|} \\
\leq\ & C_0 p(d,\lfloor s\rfloor)\mu(|x-y|)|x-y|^{\{s\}}(|z-x|+|x-z|+|z-y|)^{\lfloor s\rfloor-|\beta|} \\
\leq\ & 2^{\lfloor s\rfloor-|\beta|}C_0 p(d,\lfloor s\rfloor)\mu(|x-y|)|x-y|^{\{s\}}(|z-x|+|z-y|)^{\lfloor s\rfloor-|\beta|}.
\end{aligned}$$

In both cases, $|z-x| \leq |z-y|$ and $|z-x| > |z-y|$, we get

$$\begin{aligned}
(|z-x|+|z-y|)^{\lfloor s\rfloor-|\beta|} =& |z-x|^{\lfloor s\rfloor-|\beta|} + |z-y|^{\lfloor s\rfloor-|\beta|} \\
&+ \sum_{1\leq i\leq\lfloor s\rfloor-|\beta|-1} \binom{\lfloor s\rfloor-|\beta|}{i}|z-x|^i|z-y|^{\lfloor s\rfloor-|\beta|-i} \\
\leq& \left(2 + \sum_{1\leq i\leq\lfloor s\rfloor-|\beta|-1}\binom{\lfloor s\rfloor-|\beta|}{i}\right)(|z-x|^{\lfloor s\rfloor-|\beta|} + |z-y|^{\lfloor s\rfloor-|\beta|}).
\end{aligned}$$

Setting

$$C_1(\lfloor s\rfloor, d) \coloneqq \max_{|\beta|\leq\lfloor s\rfloor}\left(2 + \sum_{1\leq i\leq\lfloor s\rfloor-|\beta|-1}\binom{\lfloor s\rfloor-|\beta|}{i}\right)2^{\lfloor s\rfloor}C_0 p(d,\lfloor s\rfloor),$$

we have so far

$$\left|\partial^\beta T_x^{\lfloor s\rfloor}(F)(z) - \partial^\beta T_y^{\lfloor s\rfloor}(F)(z)\right| \leq C_1(\lfloor s\rfloor, d)\mu(|x-y|)|x-y|^{\{s\}}(|z-x|^{\lfloor s\rfloor} + |z-y|^{\lfloor s\rfloor}).$$

Thus, it is left to show, that there exists a constant C_2, again depending only on d and $\lfloor s\rfloor$, such that:

$$|x-y|^{\{s\}}(|z-x|^{\lfloor s\rfloor} + |z-y|^{\lfloor s\rfloor}) \leq C_2(|z-x|^s + |z-y|^s).$$

For this we remark first that

$$|x-y|^{\{s\}} \leq |z-x|^{\{s\}} + |z-y|^{\{s\}}.$$

To see this, we may first assume that $|z-x| \geq |z-y|$. Then there is a $q \in [0,1]$ such that $q|z-x| = |z-y|$. So we get an equivalent formulation of our problem and we have to show that

$$f(q) \coloneqq (1+q)^{\{s\}} - q^{\{s\}} \leq 1 \quad \text{for all } q \in [0,1].$$

Since we have $f(0) = 1$, it is sufficient to show that the first derivative of f is negative on $(0, 1]$. We compute

$$f'(q) = \{s\}(1+q)^{\{s\}-1} - \{s\}q^{\{s\}-1},$$

and since $(1+q)^{\{s\}-1} < q^{\{s\}-1}$ for $q \in (0, 1]$, we have the desired result. Therefore, we get

$$\begin{aligned}|x-y|^{\{s\}}(|z-x|^{\lfloor s\rfloor} + |z-y|^{\lfloor s\rfloor}) &\leq (|z-x|^{\{s\}} + |z-y|^{\{s\}})(|z-x|^{\lfloor s\rfloor} + |z-y|^{\lfloor s\rfloor}) \\ &= |z-x|^s + |z-y|^s + |z-x|^{\{s\}}|z-y|^{\lfloor s\rfloor} + |z-x|^{\lfloor s\rfloor}|z-y|^{\{s\}} \\ &\leq 3(|z-x|^s + |z-y|^s),\end{aligned}$$

which shows the assertion.
The last implication, 3. implies 2., is again easy to prove.

$$\begin{aligned}\left|R_x^{\lfloor s\rfloor}(F)^{(\beta)}(y)\right| &= \left|f^{(\beta)}(y) - \partial^\beta T_x^{\lfloor s\rfloor}(F)(y)\right| \\ &= \left|\partial^\beta T_y^{\lfloor s\rfloor}(F)(y) - \partial^\beta T_x^{\lfloor s\rfloor}(F)(y)\right| \\ &\leq \mu_0(|x-y|)(|y-y|^{s-|\beta|} + |x-y|^{s-|\beta|}) \\ &= |x-y|^{s-|\beta|}\mu_0(|x-y|).\end{aligned}$$

Note that we can assume $s \notin \mathbb{N}_0$ and therefore $|y-y|^{s-|\beta|} = 0$ for each $\beta \leq \lfloor s\rfloor$. □

Before the end of this section, we prove two propositions which will be very helpful in the next chapters. To explain the purpose of these propositions, let $0 \leq s_0 \leq s$,

$$F = \left(f^{(\alpha)}\right)_{|\alpha|\leq\lfloor s\rfloor} \in \mathscr{E}^s(K),$$

and let $E : \mathscr{E}^s(K) \to \mathscr{E}^{s_0}(\mathbb{R}^d)$ denote the constructed extension operator. By construction we always know that $E(F)\big|_{K^c} \in \mathscr{C}^\infty(K^c)$ and that $\partial^\alpha E(F)(x) = f^{(\alpha)}(x)$ for all $x \in K$ and all $|\alpha| \leq s_0$. Thus, to prove that $E(F) \in \mathscr{E}^{s_0}(\mathbb{R}^d)$, we have to show that $E(F)$ admits continuous partial derivatives up to order s_0 on the boundary of K, which is done with Proposition 2.15, and that for a compact set L with $\mathring{L} \supset \operatorname{supp}(E(F))$:

$$\lim_{t\to 0} q_{s_0,L}\left(\left(\partial^\alpha E(F)\big|_L\right)_{|\alpha|\leq\lfloor s_0\rfloor}, t\right) = 0,$$

which is done with Proposition 2.16. Since the product of E with a test function φ with $\varphi \equiv 1$ on K is again an extension operator with the same continuity properties, we can without loss of generality assume that such a compact set L exists and is convex. The convexity of L saves us some computational effort by Remark 2.9. For the proof of Proposition 2.16, we follow Malgrange's proof of Complement 3.6 in [Mal67].

2.15 Proposition. *Let $K \subset \mathbb{R}^d$ be compact, $s \geq 0$, $f : \mathbb{R}^d \to \mathbb{R}$ such that $f \in \mathscr{C}^\infty(K^c)$, $f \equiv 0$ on K and such that for all $\beta \in \mathbb{N}_0^d$ with $|\beta| \leq s$ and $\varepsilon > 0$ there exists a $\delta > 0$ with $\left|\frac{\partial^\beta f(x)}{dist(x,K)^{s-|\beta|}}\right| < \varepsilon$ for all $x \in \{y \in K^c : dist(y, K) < \delta\}$, or shorter*

$$|\partial^\beta f(x)| = o(dist(x, K)^{s-|\beta|}) \text{ for all } \beta \in \mathbb{N}_0^d \text{ with } |\beta| \leq s \text{ and } x \to \partial K. \tag{2.3}$$

Then f admits partial derivatives up to order $\lfloor s \rfloor$ in $\mathbb{R}^d$ and $\partial^\beta f \equiv 0$ on K for each $|\beta| \leq \lfloor s \rfloor$.

Proof. We will prove the existence of the partial derivatives only for those of first order, i.e. we show that

$$\lim_{h \to 0} \frac{|f(x_0 + he) - f(x_0)|}{h} = 0 \tag{2.4}$$

for an arbitrary $x_0 \in \partial K$ and some unit vector e. The same argument can be used for all the other partial derivatives. First we remark that f is continuous in $\mathbb{R}^d \backslash \partial K$ by assumption and the continuity in ∂K follows immediately from (2.3). In order to show (2.4), let $\varepsilon > 0$. With (2.3) we find a $\delta > 0$ such that

$$\left|\frac{\partial f}{\partial x_1}(x)\right| / |x - x_0|^{s-1} < \varepsilon \text{ for all } x \in U_\delta(x_0) \cap K^c. \tag{2.5}$$

We choose $h \in U_\delta(0)$ such that $x_0 + he \in K^c \cap U_\delta(x_0)$ (if no such h exists we have $\frac{\partial f}{\partial x_1}(x) = 0$). Because K is compact we find $\tilde{x} \in [x_0 + he, x_0] \cap K$ which minimizes the distance of $x_0 + he$ to K. According to $\tilde{x}$ we find a $0 < |\tilde{h}| < |h|$ which fulfils $x_0 + he = \tilde{x} + \tilde{h}e$ and we obtain

$$\begin{aligned} \frac{|f(x_0 + he) - f(x_0)|}{h^s} &= \frac{|f(x_0 + he) - f(\tilde{x})|}{h^s} \\ &= \frac{|f(\tilde{x} + \tilde{h}e) - f(\tilde{x})|}{\tilde{h}^s} \frac{\tilde{h}^s}{h^s} \\ &\leq \frac{|f(\tilde{x} + \tilde{h}e) - f(\tilde{x})|}{\tilde{h}^s}. \end{aligned} \tag{2.6}$$

Since $[x_0 + he, \tilde{x}) \subset K^c$, f is continuous on $[x_0 + he, \tilde{x}]$ and $\frac{\partial f}{\partial x_1}$ exists on $(x_0 + he, \tilde{x})$, we can apply the mean value theorem to get a $\hat{x} \in (x_0 + he, \tilde{x})$ such that

$$\frac{\partial f}{\partial x_1}(\hat{x}) = \frac{f(\tilde{x} + \tilde{h}e) - f(\tilde{x})}{\tilde{h}}.$$

Together with (2.5) and (2.6) we have for $h \leq 1$

$$\frac{|f(x_0 + he) - f(x_0)|}{h} \leq \frac{|f(x_0 + he) - f(x_0)|}{h^s} \leq \left|\frac{\partial f}{\partial x_1}(\hat{x})\right| / |\hat{x} - x_0|^{s-1} \leq \varepsilon,$$

which gives the desired result. Since $f \equiv 0$ on K we already know that $\partial^\beta f \equiv 0$ on $\mathring{K}$ for all $\beta \leq \lfloor s \rfloor$. The calculation above shows that this is true on the whole set K. □

2.16 Proposition. *Let $K \subset \mathbb{R}^d$ be compact, $s \geq 0$ and $\in \mathscr{E}^{\lfloor s \rfloor}(\mathbb{R}^d)$ such that $F := \left(\partial^\alpha f\big|_K\right)_{|\alpha|\leq\lfloor s\rfloor} \in \mathscr{E}^s(K)$ and $f \in \mathscr{E}^{\lfloor s\rfloor+1}(K^c)$. Then $f \in \mathscr{E}^s(\mathbb{R}^d)$ provided that the following conditions hold:*

1. *There is a modulus of continuity μ such that for all $|\beta| \leq \lfloor s \rfloor$ and $x \in \mathbb{R}^d$*

$$|\partial^\beta f(x) - \partial^\beta T_a^{\lfloor s\rfloor}(F)(x)| \leq \mu(dist\,(x,K)) \cdot dist\,(x,K)^{s-|\beta|}, \tag{2.7}$$

where a denotes a point in K with $|x-a| = dist\,(x,K)$.

2. *For $|\beta| = \lfloor s \rfloor + 1$ there is a modulus of continuity μ such that for all $x \in K^c$*

$$|\partial^\beta f(x)| \leq \mu(dist\,(x,K)) \cdot dist\,(x,K)^{s-|\beta|}. \tag{2.8}$$

Proof. In order to show that $f \in \mathscr{E}^s(\mathbb{R}^d)$, we prove that $\left(\partial^\alpha f\big|_L\right)_{|\alpha|\leq\lfloor s\rfloor} \in \mathscr{E}^s(L)$ for each convex and compact $L \supset K$, where $\mathscr{E}^s(L)$ is equipped with the norm $|||\cdot|||_{s,L}$.

So we have to show that

$$\lim_{t\to 0}\sup\left\{\frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x-y|^{\{s\}}} : 0 < |x-y| < t, x, y \in L, |\beta| = \lfloor s \rfloor\right\} = 0. \tag{2.9}$$

We split the proof in three different cases. The first case is that both points, x and y, belong to K. We apply inequality (2.2) of Theorem 2.14 to prove this case. The second case is that only one of the points lies in K. To show this case we apply inequality (2.7). The last case is that both points are located in $L \setminus K$ for which we use inequality (2.8).

1. Let $x, y \in K$. Since $F \in \mathscr{E}^s(K)$ it is clear that

$$\frac{|\partial^\beta f(x) - \partial^\beta f(y)|}{|x-y|^{\{s\}}} \leq \mu_0(|x-y|),$$

where μ_0 shall be the modulus of continuity belonging to the jet F, which exists by Theorem 2.14.

2. Let $x \in L \setminus K$ and $y \in K$. First we choose a point $a \in K$, such that $|x-a| = \operatorname{dist}(x,K)$. Then inequality (2.7) shows that

$$\begin{aligned}|\partial^\beta f(x) - \partial^\beta f(y)| &\leq |\partial^\beta f(x) - \partial^\beta f(a)| + |\partial^\beta f(a) - \partial^\beta f(y)| \\ &\leq |\partial^\beta f(x) - \partial^\beta T_a^{\lfloor s\rfloor} F(x)| + \mu_0(|a-y|)|a-y|^{\{s\}} \\ &\leq \mu(|x-a|)|x-a|^{\{s\}} + \mu_0(2|x-y|)2^{\{s\}}|x-y|^{\{s\}} \\ &\leq (\mu(|x-y|) + 2^{\{s\}}\mu_0(2|x-y|))|x-y|^{\{s\}},\end{aligned}$$

which proves this case because

$$\mu(|x-y|) + 2^{\{s\}}\mu_0(2|x-y|) \to 0 \quad \text{for } |x-y| \to 0.$$

3. For the last case we assume $x, y \in L \setminus K$. We split this case in two subcases. First we assume that $\operatorname{dist}(x, K) \geq 2|x-y|$. This condition ensures that the line segment $[x, y]$ is contained in $L \setminus K$. Applying the mean value theorem, we get a $z \in (x, y)$ such that
$$\partial^\beta f(x) - \partial^\beta f(y) = \nabla(\partial^\beta f(z))(x-y).$$
Now we can apply inequality (2.8) on the right side to get
$$\begin{aligned} |\partial^\beta f(x) - \partial^\beta f(y)| &\leq |\nabla(\partial^\beta f)(z)||x-y| \\ &\leq \sqrt{d}\mu(\operatorname{dist}(z, K)) \cdot \operatorname{dist}(z, K)^{s-|\beta|-1}\,|x-y| \\ &= \sqrt{d}\mu(\operatorname{dist}(z, K)) \cdot \operatorname{dist}(z, K)^{\{s\}-1}\,|x-y|. \end{aligned} \tag{2.10}$$
Since $\operatorname{dist}(x, K) \geq 2|x-y|$, we have $\operatorname{dist}(z, K) \geq |x-y|$ and therefore $\frac{|x-y|^{1-\{s\}}}{\operatorname{dist}(z,K)^{1-\{s\}}} < 1$. Furthermore, μ being concave yields that for all $0 < a < 1$ and all $x \geq 0$ the following inequality holds
$$a\mu(x) = a\mu(x) + (1-a)\mu(0) \leq \mu(ax + (1-a)0) = \mu(ax).$$
So we get
$$\frac{\mu(d(z, K))}{\operatorname{dist}(z, K)^{1-\{s\}}}|x-y| \leq \mu\left(\operatorname{dist}(z, K)^{\{s\}}\,|x-y|^{1-\{s\}}\right)|x-y|^{\{s\}} \leq \mu\left(\lambda^{\{s\}}|x-y|^{1-\{s\}}\right)|x-y|^{\{s\}},$$
which proves our assertion, since
$$\lim_{|x-y|\to 0} \mu\left(\lambda^{\{s\}}|x-y|^{1-\{s\}}\right) = 0.$$

The second case is that $\operatorname{dist}(x, K) < 2|x-y|$. For this we choose $a, b \in K$ such that
$$|x-a| = \operatorname{dist}(x, K)\,, \quad |y-b| = \operatorname{dist}(y, K)\,,$$
which leads to
$$\begin{aligned} &|y-b| \leq |y-a| \leq |x-y| + |x-a| \leq 3|x-y|, \\ &|a-b| \leq |x-a| + |x-y| + |y-b| \leq 2|x-y| + |x-y| + 3|x-y| = 6|x-y|. \end{aligned}$$
And therefore we can expand
$$\partial^\beta f(x) - \partial^\beta f(y) = \partial^\beta f(x) - \partial^\beta f(a) + \partial^\beta f(a) - \partial^\beta f(b) + \partial^\beta f(b) - \partial^\beta f(y),$$
which permits us to use the derived inequalities from the first two cases. Using inequality (2.7) results in
$$|\partial^\beta f(x) - \partial^\beta f(y)| \leq |\partial^\beta f(x) - \partial^\beta f(a)| + |\partial^\beta f(a) - \partial^\beta f(b)| + |\partial^\beta f(b) - \partial^\beta f(y)|$$

$$\leq \mu(|x-a|)|x-a|^{\{s\}} + \mu_0(|a-b|)|a-b|^{\{s\}} + \mu(|b-y|)|b-y|^{\{s\}}$$
$$\leq \mu(2|x-y|)2^{\{s\}}|x-y|^{\{s\}} + \mu_0(6|x-y|)6^{\{s\}}|x-y|^{\{s\}} + \mu(3|x-y|)3^{\{s\}}|x-y|^{\{s\}}$$
$$\leq (2^{\{s\}} + 6^{\{s\}} + 3^{\{s\}})\tilde{\mu}(6|x-y|)|x-y|^{\{s\}}, \tag{2.11}$$

where $\tilde{\mu}$ is a modulus of continuity satisfying $\tilde{\mu} \geq \max\{\mu, \mu_0\}$. Since the last term converges to 0 if $|x-y| \to 0$, the proof of (2.9) is complete.

□

2.3 The Results of Whitney and Stein

In this section we provide an overview of the classical results of Whitney and Stein. We chose those two because Whitney's work was seminal for the whole theory of extension operators and Stein's result on the extension problem for Sobolev spaces also yields a result for the central problem (2.12).

In 1934, Hassler Whitney published the three groundbraking articles [Whi34a], [Whi34b] and [Whi34c] on the question of how to describe the space of functions defined on a closed set M in $\mathbb{R}^d$ which can be extended to a function defined on the whole $\mathbb{R}^d$ having a certain order of differentiability. The first article [Whi34a] already solved the problem for finite orders of differentiability completely, and in addition he proved that this extension can be achieved by a continuous and linear operator E_n, which is the right inverse of the restriction operator

$$r_n : \mathscr{E}^n(\mathbb{R}^d) \to \mathscr{E}^n(M), f \mapsto \left(\partial^\alpha f\big|_M\right)_{|\alpha|\leq n}.$$

To put it short, for all $n \in \mathbb{N}$ the following short sequence is exact and splits in the category of Fréchet spaces

$$0 \to \mathscr{I}^n_F(\mathbb{R}^d) \to \mathscr{E}^n(\mathbb{R}^d) \xrightarrow{r_n} \mathscr{E}^n(M) \to 0,$$

where $\mathscr{I}^n_M(\mathbb{R}^d) := \left\{f \in \mathscr{E}^n : \partial^\alpha f\big|_M \equiv 0 \text{ for all } |\alpha| \leq n\right\}$. The fact that $r_n \circ E_n = id$ means that $\mathscr{E}^n(M)$ is exactly the space of restrictions of n-times continuously partially differentiable functions on $\mathbb{R}^d$ to M. The extension problem, which he solved for the finite order case, is much more complicated in the infinite order case and even until today subject of current research. In the same paper he published the result, that the restriction

$$r : \mathscr{E}(\mathbb{R}^d) \to \mathscr{E}(M), f \mapsto \left(\partial^\alpha f\big|_M\right)_{|\alpha|\in\mathbb{N}^d_0}$$

is surjective, or equivalently the following sequence is exact

$$0 \to \mathscr{I}_F(\mathbb{R}^d) \to \mathscr{E}(\mathbb{R}^d) \xrightarrow{r} \mathscr{E}(M) \to 0, \tag{2.12}$$

where $\mathscr{I}_M(\mathbb{R}^d) := \left\{ f \in \mathscr{E} : \partial^\alpha f\big|_M \equiv 0 \text{ for all } \alpha \in \mathbb{N}_0^d \right\}$. Since then, most of the research in this field has been concerned with the problem to characterize those sets, for which the latter sequence splits or equivalently when this extension can also be achieved with a continuous and linear extension operator. Iconic counterexamples for closed sets which do not admit such an extension operators are singletons or the examples given by Tidten in [Tid79], e.g. the exponential cusp:

$$M_{\exp} := \left\{ (x,y) \in \mathbb{R}^2 : |y| \leq \exp\left(-x^{-1}\right), 0 \leq x \leq 1 \right\}.$$

In [Whi34c], Whitney examines another, perhaps more intuitive, way of defining differentiability on closed sets, i.e. by simply defining for an open and bounded set O:

$$\mathscr{C}^n(\bar{O}) := \left\{ f \in \mathscr{C}(\bar{O}) : f \in \mathscr{C}^n(O) \text{ and } \partial^\alpha f \text{ is uniformly cont. on } O \text{ for all } |\alpha| \leq n \right\}.$$

Equipped with the supremum norm $|\cdot|_{n,O}$, this space is a Banach space. Since all the partial derivatives are uniformly continuous, it is possible to uniquely extend them to the boundary of O. Whitney introduced a special regularity condition for sets and could show that if O fulfills this condition, the spaces $\mathscr{C}^n(\bar{O})$ and $\mathscr{E}^n(\bar{O})$ are isomorphic, and therefore each $f \in \mathscr{C}^n(\bar{O})$ together with all its partial derivatives can be extended to an n-times continuously partially differentiable function on $\mathbb{R}^d$. According to Whitney, this result also holds for unbounded O. A set satisfies this regularity condition if there exists a constant $C > 0$ such that for each $x, y \in O$ there exists a rectifiable arc Γ in O which connects x and y and so that the length $|\Gamma|$ fulfills the inequality $|\Gamma| \leq C|x - y|$. This way of defining smooth functions on closed sets has its downsides when it comes to the infinite order case. As usual, the space $\mathscr{C}^\infty(\bar{O})$ is constructed as projective limit of the spaces $\mathscr{C}^n(\bar{O})$ and is therefore a Fréchet space. Unfortunately it turned out that for special geometries of $\bar{O}$, the Whitney space $\mathscr{E}(\bar{O})$ is a proper and dense subspace of $\mathscr{C}^\infty(\bar{O})$. Since the restriction operator r, as mentioned above, is surjective, there exist functions $f \in \mathscr{C}^\infty(\bar{O})$, such that there is no extension $\tilde{f} \in \mathscr{E}(\mathbb{R}^d)$ with $\partial^\alpha \tilde{f}\big|_{\bar{O}} = \partial^\alpha f$ for all $\alpha \in \mathbb{N}_0^d$, which is of course totally counter-intuitive. One example of such a set in $\mathbb{R}^2$ is $[-1,1]^2 \setminus M_{\exp}$. On the other hand there are geometries as the sets O with Lipschitz boundary treated i.a. in [Ste70], [Bie78] and [Fre07b] for which $\mathscr{C}^\infty(\bar{O}) = \mathscr{E}(\bar{O})$ but not necessarily $\mathscr{C}^n(\bar{O}) = \mathscr{E}^n(\bar{O})$.

The elements of the Whitney spaces are families of functions called Whitney jets. The aim of the paper [Whi34b] is to investigate wether it is possible to reduce those jets to single functions, thus to find conditions which involve only $f^{(0)}$. Whitney could solve the problem in the one dimensional case. He showed that a continuous function f defined on a closed set M is in $\mathscr{E}^n(M)$ (in the sense that f is the restriction of an n-times continuously partially differentiable function on $\mathbb{R}^d$ or equivalently can be completed to a full Whitney jet with $f^{(0)} = f$) if and only if the n-th difference quotients show a certain convergence behaviour. The question of how to find a characterization in higher dimensions is also a very current field of research with important contributions by Glaeser [Gla58], Bierstone, Milman and Pawlucki in [BMP03], [BMP06], Frerick, Jordá and Wengenroth in [FJW16a] and last but not least Fefferman i.a. in [Fef06].

In his book [Ste70], Stein published two results concerning extension operators. The first one is a variation of Whitney's result on the finite order extension operators E_n. He showed that $E_n : \mathrm{Lip}(\gamma, F) \to \mathrm{Lip}(\gamma, \mathbb{R}^d)$ is continuous for each $n < \gamma \leq n+1$, where the Lipschitz spaces on closed sets M are defined as those $\left(f^{(\alpha)}\right)_{|\alpha|\leq n} \in \prod_{|\alpha|\leq n} \mathscr{C}(M)$ for which there exists a constant $L \geq 0$ such that for each $|\beta| \leq n$ and all $x, y \in M$ we have

$$|f^{(\beta)}(x)| \leq L \quad \text{and} \quad \left| R_y^k \left(\left(f^{(\alpha)}\right)_{|\alpha|\leq n} \right)^{(\beta)} (x) \right| \leq L|x-y|^{\gamma-|\beta|}.$$

The main reason for citing his work in this context is his second result about the extension problem for Sobolev spaces. For $k \in \mathbb{N}_0$ and $1 \leq p \leq \infty$ the Sobolev space $W^{k,p}(\mathbb{R}^d)$ denotes the space of all those functions from $L^p(\mathbb{R}^d)$ for which all partial derivates exist up to order k in the week sense and are also in $L^p(\mathbb{R}^d)$. Already in [Cal61], Calderón constructed extension operators for those spaces, but those operators depend on the degree of differentiability k and are not valid for the cases $p = 1$ and $p = \infty$. The big advantage of Stein's result is that he constructed one operator doing the extension for all integers k and all $1 \leq p \leq \infty$, so $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$ continuous. In order for this to work, Ω has to be locally the graph of a Lipschitz continuous function. As mentioned above, the spaces $\mathscr{C}^\infty(\bar{\Omega})$ and $\mathscr{E}(\bar{\Omega})$ then coincide. Stein shows in his proof that for $f \in \mathscr{C}^k(\bar{\Omega})(\subset W^{k,\infty}(\Omega))$ we have that $E(f) \in \mathscr{C}^k(\mathbb{R}^d)$. So Stein's operator also offers a solution for (2.12). Since the norm on $W^{k,\infty}(\Omega)$, as defined in [Ste70] on page 122, equals the supremum norm on $\mathscr{C}^k(\bar{\Omega})$, it is evident that Stein's operator fulfils the norm inequalities

$$|E(f)|_{k,L} \leq C|f|_{k,\bar{\Omega}},$$

for all $f \in \mathscr{C}^\infty(\bar{\Omega})$ and some compact set $\mathring{L} \supset \bar{\Omega}$. This is the best possible continuity estimate and we say that this operator has no loss of derivatives. We will prove a more general result on this kind of operators in Chapter 4.

The research on extension operators for Sobolev spaces continued i.a. with the works of Jones who generalized the results of Stein in [Jon81] to locally uniform domains with the downside that his operators were again depending on the order of differentiability. Rogers could connect the results of Stein and Jones in [Rog04] to construct an extension operator $E : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^d)$ working for all $k \in \mathbb{N}_0$ and all $1 \leq p \leq \infty$ and Ω being locally uniform.

Chapter 3

Generalized Extension Theorem of Whitney

In this chapter we generalize Whitney's result about finite order extension operators as formulated in Section 1.3. Whitney constructed his extension operators E_n between the spaces $\mathscr{E}^n(K)$ and $\mathscr{E}^n$ or likewise $\mathscr{D}^n(L)$ for some compact and convex L with $\mathring{L} \supset K$. Our aim is to expand Whitney's construction to obtain operators $E_s : \mathscr{E}^s(K) \to \mathscr{D}^s(L)$ for all $s \geq 0$, i.e. operators 'on the real scale' instead of just 'on the natural scale'. This extension is necessary for the next chapters, where we construct operators which are not only a solution to (2.12) but which also map simultaneously all the spaces $\mathscr{E}^s(K)$ into $\mathscr{E}^s(\mathbb{R}^d)$.

3.1 Statement of the Main Result

3.1 Theorem. *Whitney's extension theorem*
For each $s \geq 0$ and each compact set L with $\mathring{L} \supset K$, there is a linear and continuous mapping $E_s : \mathscr{E}^s(K) \to \mathscr{D}^s(L)$ such that for every jet $F = \left(f^{(\alpha)}\right)_{|\alpha| \leq \lfloor s \rfloor} \in \mathscr{E}^s(K)$ and every $x \in K$, we have $\partial^\beta E(F)(x) = f^\beta(x)$ for $|\beta| \leq \lfloor s \rfloor$.

The proof of this theorem shows that for all $n \in \mathbb{N}_0$ the operator is defined in the same way for all $s \in [n, n+1)$. Therefore the statement of the following lemma makes sense. The proof of this lemma is contained in the proof of the main theorem.

3.2 Lemma. *Let $n \in \mathbb{N}_0$ be arbitrary and $L \subset \mathbb{R}^d$ be a cube with $K \subset \mathring{L}$. Since $E : \mathscr{E}^s(K) \to \mathscr{D}^s(L)$ is continuous for all $s \in [n, n+1)$ there exist constants C_s such that $|||E(F)|||_{s,L} \leq C_s ||F||_{s,K}$. These constants can be chosen such that the mapping $s \mapsto C_s$ is continuous and bounded on $[n, n+1)$.*

3.2 Proof of the Main Result

Now we can prove the generalized version of Whitney's extension theorem. In the following the family of functions $(\varphi_i)_{i\in I}$ shall denote a certain partition of unity which is for instance given in Lemma 3.1, [Mal67]. We will list the important properties in the next lemma.

3.3 Lemma. *Whitney's partition of unity*
Let $K \subset \mathbb{R}^d$ be compact and $\Omega \supset K$ open. Then there exists a countable family of positive testfunctions $\varphi_i \in \mathscr{D}(\Omega \setminus K)$ with the following properties.

1. *$\sum_{i=1}^{\infty} \varphi_i(x) = 1$ for all $x \in \Omega \setminus K$ and each point belongs to at most N supports $supp(\varphi_i)$ for some constant $N \in \mathbb{N}$.*

2. *$supp(\varphi_i) \to K$ for $i \to \infty$, that is, for each $\varepsilon > 0$ there is $k \in \mathbb{N}$ such that $supp(\varphi_i) \subset \{x \in \mathbb{R}^d : dist\,(x, K, <)\, \varepsilon\}$ for all $i \geq k$.*

3. *$diam(supp(\varphi_i)) \leq 2dist\,(supp(\varphi_i), K)$.*

4. *There are constants c_β such that $|\partial^\beta \varphi_i(x)| \leq c_\beta dist\,(x, K)^{-|\beta|}$ for all $i \in \mathbb{N}, \beta \in \mathbb{N}_0^d$, and $x \in \mathbb{R}^d$.*

With the aid of this decomposition we can prove now the main result of this chapter.

Proof. For $s \in \mathbb{N}$ this is the classical extensions theorem of Whitney. So we may assume that $s \in (0, \infty) \setminus \mathbb{N}$. In the following let $L \subset \mathbb{R}^d$ be a compact and convex set, such that $K \subset \mathring{L}$. We will apply the partition of unity $(\varphi_i)_{i\in I}$ only on $L \setminus K$. As in the original proof of Whitney we take the following operator as our candidate for the desired extension:

$$E_s(F)(x) = \begin{cases} f^{(0)}(x), x \in K \\ \sum_{i\in\mathbb{N}} \varphi_i(x) T_{a_i}^{\lfloor s \rfloor}(F)(x), x \notin K \end{cases},$$

where a_i denotes a point in K such that $\operatorname{dist}(\operatorname{supp}(\varphi_i), K) = \operatorname{dist}(\operatorname{supp}(\varphi_i), a_i)$, and for the ease of short notation we set in the following $\tilde{f} := E_s(F)$. We note directly that obviously $E_s(F) \in \mathscr{E}(\mathbb{R}^d \setminus K)$. In the course of this proof μ shall denote a modulus of continuity for the jet F, so it fulfils (2.1) of Theorem 2.14. Furthermore let λ denote the diameter of K, so $\lambda := \sup_{x\in L} \operatorname{dist}(x, K)$.
We structure the following proof into four parts.

1. We prove the existence of a constant C depending only on $\lfloor s \rfloor, d$ and λ such that for every $|\beta| \leq \lfloor s \rfloor$, $a \in K$ and $x \in L$, we have:

$$|\partial^\beta \tilde{f}(x) - \partial^\beta T_a^{\lfloor s \rfloor}(F)(x)| \leq C\mu(|x - a|)|x - a|^{s-|\beta|}. \tag{3.1}$$

2. We show that for each $|\beta| > \lfloor s \rfloor$ we can find a constant C depending only on $\beta, \lfloor s \rfloor$ and λ, such that
$$|\partial^\beta \tilde{f}(x)| \leq C\mu(\mathrm{dist}\,(x, K))\mathrm{dist}\,(x, K)^{s-|\beta|}\,. \tag{3.2}$$

3. We show that $\tilde{f} \in \mathscr{D}^s(L)$.

4. We show that $E_s : \mathscr{E}^s(K) \to \mathscr{D}^s(L)$ is continuous. Furthermore we show that the continuity constants C_s, for which the inequality $|||E_s(F)|||_{s,L} \leq C_s \|F\|_{s,K}$ is true for every $F \in \mathscr{E}^s(K)$, can be chosen such that the mapping $s \mapsto C_s$ depends continuously on s on each interval $[n, n+1)$ for each $n \in \mathbb{N}_0$ and is bounded on these intervals.

1. So we begin with the first part and note that we have for every $x \in L \setminus K$,

$$\tilde{f}(x) - T_a^{\lfloor s \rfloor}(F)(x) = \sum_{i \in \mathbb{N}} \varphi_i(x) \left(T_{a_i}^{\lfloor s \rfloor}(F)(x) - T_a^{\lfloor s \rfloor}(F)(x) \right).$$

Hence applying Leibniz's formula, we arrive at

$$\partial^\beta \tilde{f}(x) - \partial^\beta T_a^{\lfloor s \rfloor}(F)(x) = \sum_{i \in \mathbb{N}} \sum_{l \leq \beta} \binom{\beta}{l} \partial^l \varphi_i(x) \partial^{\beta - l} \left(T_{a_i}^{\lfloor s \rfloor}(F)(x) - T_a^{\lfloor s \rfloor}(F)(x) \right).$$

First, we consider those terms for which $l = 0$. For $x \in \mathrm{supp}(\varphi_i)$ we have obviously $\mathrm{dist}\,(x, K) \leq |x - a|$ and $\mathrm{dist}\,(x, K) \leq |x - a_i| \leq 3\mathrm{dist}\,(x, K)$, and hence

$$\mu(|a - a_i|) \leq \mu(|x - a| + |x - a_i|) \leq \mu(4|x - a|) \leq 4\mu(|x - a|).$$

Now we use (2.1) of Theorem 2.14 and get a constant $C_0 = C_0(\lfloor s \rfloor, d)$ such that

$$\begin{aligned} & |\varphi_i(x)| \left| \partial^\beta T_{a_i}^{\lfloor s \rfloor}(F)(x) - \partial^\beta T_a^{\lfloor s \rfloor}(F)(x) \right| \\ \leq\; & C_0 \mu(|a - a_i|) \left(|x - a_i|^{s - |\beta|} + |x - a|^{s - |\beta|} \right) \\ \leq\; & 16 C_0 \mu(|x - a|) |x - a|^{s - |\beta|}, \end{aligned}$$

which is an inequality of the form needed for (3.1). Now we treat the terms for $l \neq 0$. First, we note that for all $x \in L \setminus K$ we have

$$\sum_{i \in \mathbb{N}} \partial^l \varphi_i(x) = 0.$$

Thus, we obtain

$$\sum_{i \in \mathbb{N}} \partial^l \varphi_i(x) \partial^{\beta - l} \left(T_{a_i}^{\lfloor s \rfloor}(F)(x) - T_a^{\lfloor s \rfloor}(F)(x) \right) = \sum_{i \in \mathbb{N}} \partial^l \varphi_i(x) \partial^{\beta - l} \left(T_{a_i}^{\lfloor s \rfloor}(F)(x) - T_b^{\lfloor s \rfloor}(F)(x) \right),$$

for an arbitrary $b \in K$. If we choose b such that $|x-b| = \operatorname{dist}(x,K)$, we have (as in the first case) by (2.1) of Theorem 2.14 the constant C_0 and by Lemma 3.3 a constant $C_1(l)$ such that

$$\begin{aligned}
&\left|\sum_{i\in\mathbb{N}} \partial^l\varphi_i(x)\partial^{\beta-l}\left(T^{\lfloor s\rfloor}_{a_i}(F)(x) - T^{\lfloor s\rfloor}_{a}(F)(x)\right)\right| \\
\leq\ & \sum_{i\in\mathbb{N}} \left|\partial^l\varphi_i(x)\right| \left|\partial^{\beta-l}\left(T^{\lfloor s\rfloor}_{a_i}(F)(x) - T^{\lfloor s\rfloor}_{b}(F)(x)\right)\right| \\
\leq\ & \sum_{i\in\mathbb{N}} C_1(l)\operatorname{dist}(x,K)^{-|l|}\, C_0\mu(|a_i-b|)(|x-a_i|^{s-|\beta|+|l|} + |x-b|^{s-|\beta|+|l|}) \\
\leq\ & 16NC_1(l)C_0\mu(|x-b|)|x-b|^{s-|\beta|}.
\end{aligned}$$

Setting

$$C := \max\{16C_0, 16NC_0 \max_{|l|\leq\lfloor s\rfloor} C(l)\},$$

we have proved the assertion.

2. Now, we turn our attention to the proof of inequality (3.2). For this we basically proceed in the same way as for the proof of inequality (3.1). Let $a \in K$ such that $\operatorname{dist}(x,K) = |x-a|$. Since $T^{\lfloor s\rfloor}_a(F)$ is a polynomial of order $\lfloor s\rfloor$ we have that $\partial^\beta T^{\lfloor s\rfloor}_a(F) \equiv 0$, which leads to

$$\begin{aligned}
|\partial^\beta \tilde{f}(x)| &= |\partial^\beta \tilde{f}(x) - \partial^\beta T^{\lfloor s\rfloor}_a(F)(x)| \\
&= \sum_{i\in\mathbb{N}}\sum_{l\leq\beta}\binom{\beta}{l}\partial^l\varphi_i(x)\partial^{\beta-l}\left(T^{\lfloor s\rfloor}_{a_i}(F)(x) - T^{\lfloor s\rfloor}_{a}(F)(x)\right).
\end{aligned}$$

Again we have $\partial^{\beta-l}\left(T^{\lfloor s\rfloor}_{a_i}(F)(x) - T^{\lfloor s\rfloor}_{a}(F)(x)\right) = 0$ if $|\beta-l| > \lfloor s\rfloor$ and therefore we just consider $|l| > 0$ and can argue exactly in the same way as in the first part of the proof.

3. Now that (3.1) and (3.2) are established, we can prove that $\tilde{f} \in \mathscr{D}^s(L)$. Because $\operatorname{supp}(\varphi_i) \subset L$ for each $i \in I$ it is clear that $\operatorname{supp}(\tilde{f}) \subset L$.

We first conclude that $\tilde{f}$ admits continuous partial derivatives up to order $\lfloor s\rfloor$ in $\mathbb{R}^d$. Since the existence is clear in K^c and $\mathring{K}$, we only have to prove it on ∂K. But since $F \in \mathscr{E}^{\lfloor s\rfloor}(K)$ and the operator E_s is the classical Whitney operator, this follows directly from the results in [Whi34a] or [Mal67]. The fact that $\tilde{f} \in \mathscr{D}^s(L)$ follows then directly from Proposition 2.16.

4. To show that the operator E_s is continuous, we equip $\mathscr{D}^s(L)$ with the norm $|||\cdot|||_{s,L}$ and we prove the existence of a constant $C = C(s,d,\lambda)$ such that $|||\tilde{f}|||_{s,L} \leq C\|F\|_{s,K}$. By definition of the norm we have

$$|||\tilde{f}|||_{s,L} = |\tilde{f}|_{L,\infty} + \sup_{t>0}\left\{\frac{|\partial^\beta \tilde{f}(x) - \partial^\beta \tilde{f}(y)|}{|x-y|^{\{s\}}} : x,y \in L, 0 < |x-y| < t, |\beta| = \lfloor s\rfloor\right\}.$$

In the following we will estimate both summands. Beginning with the supremum norm part we have for $|\beta| \leq \lfloor s \rfloor$ and arbitrary $x \in L, a \in K$

$$|\partial^\beta \tilde{f}(x)| \leq \left|\partial^\beta \tilde{f}(x) - \partial^\beta T_a^{\lfloor s \rfloor}(F)(x)\right| + \left|\partial^\beta T_a^{\lfloor s \rfloor}(F)(x)\right|.$$

Applying inequality (3.1) and Theorem 2.14 (note that $\mu(t) \leq \mu(\mathrm{diam}(K)) \leq \|F\|_s$ for all $t > 0$) we get a constant $C_0(\lfloor s \rfloor, d, \lambda)$ such that

$$\begin{aligned}\left|\tilde{f}^{(\beta)}(x) - \partial^\beta T_a^{\lfloor s \rfloor}(F)(x)\right| &\leq C_0 \mu(|x-a|)|x-a|^{s-|\beta|} \\ &\leq C_0 \mu(\mathrm{diam}(K)) \max(1, \lambda^s) \\ &\leq C_1(s, d, \lambda)\|F\|_{s,K}.\end{aligned}$$

For the second summand we have

$$\begin{aligned}|\partial^\beta T_\alpha^{\lfloor s \rfloor}(F)(x)| &= \left|\sum_{|\alpha| \leq \lfloor s \rfloor - |\beta|} \frac{f^{(\alpha+\beta)}(a)}{\alpha!}(x-a)^\alpha\right| \\ &\leq \sum_{|\alpha| \leq \lfloor s \rfloor - |\beta|} \frac{1}{\alpha!}\|F\|_{s,K}\lambda^{|\alpha|},\end{aligned}$$

and therefore we find a constant $C_2(\lfloor s \rfloor, d, \lambda)$ such that

$$|\partial^\beta T_\alpha^{\lfloor s \rfloor}(F)(x)| \leq C_2(\lfloor s \rfloor, d, \lambda),$$

which gives in the sum

$$|\partial^\beta \tilde{f}(x)| \leq (C_1 + C_2)\|F\|_{s,K}.$$

Furthermore we note that C_1 depends continuously on $\tilde{s}$ and for all $s \in [n, n+1)$ it is true that

$$C_0(n, d, \lambda) \leq C_1(s, d, \lambda) \leq C_0(n, d, \lambda) \max(1, \lambda^{n+1}).$$

It is left to show that we also find a constant $C = C(s, d, \lambda)$ with comparable properties for all $x, y \in L$ and all $|\beta| = \lfloor s \rfloor$ such that

$$\frac{|\partial^\beta \tilde{f}(x) - \partial^\beta \tilde{f}(y)|}{|x-y|^{\{s\}}} \leq C\|F\|_{s,K}. \tag{3.3}$$

To achieve this we can use the results from part three of this proof. If $x, y \in K$ we simply have

$$\frac{|\partial^\beta \tilde{f}(x) - \partial^\beta \tilde{f}(y)|}{|x-y|^{\{s\}}} \leq \mu(|x, y|) \leq \|F\|_{s,K}.$$

For $x \in L \setminus K$ and $y \in K$ we get by (2.9)

$$\frac{|\partial^\beta \tilde{f}(x) - \partial^\beta \tilde{f}(y)|}{|x-y|^{\{s\}}} \leq C\mu(|x-y|) \leq C\|F\|_{s,K},$$

where the constant C depends only on $\lfloor s \rfloor$ and not on the fractional part of s.

For $x, y \in L \setminus K$ we get taking the inequalities (2.10) and (2.11) together a constant which depends continuously on $\{s\}$ and which is bounded on each interval $[n, n+1)$ for each $n \in \mathbb{N}_0$. □

Chapter 4

Extension Operators without Loss of Derivatives

The main result of this chapter is a characterization of all compact sets $K \subset \mathbb{R}^d$ which admit an extension operator $E : \mathscr{E}(K) \to \mathscr{E}(\mathbb{R}^d)$ fulfilling the best possible continuity estimates, i.e.

$$|||E(F)|||_{s,L} \leq C_s ||F||_{s,K} \tag{4.1}$$

for all Whitney jets $F \in \mathscr{E}(K)$ and any compact and convex set L with $K \subset \mathring{L}$. The inequalities (4.1) are best possible because, since E is an extension operator, it doesn't change the jet itself, so the smoothness properties can only be preserved in the best case. In this case we say that E has no loss of derivatives or shortly no loss. First results in the construction of an extension operator without loss are due to Seeley [See64] for the case of closed half spaces, Stein [Ste70] for the case of compact sets with Lip_1-boundary. The main contribution in this chapter is a generalization of a result of Frerick, Jordá and Wengenroth given in [FJW16b].

The existence of an extension operator without loss is characterized by the geometry of the underlying compact set K. On the one hand we show the equivalence to the existence of certain measures supported in K, and on the other hand we show the equivalence to the validity of Markov inequalities for polynomials on K. Different types of this Markov type inequalities have already been considered by various authors in the attempt to characterize 'good' geometries of K such that $\mathscr{E}(K)$ admits an extension operator. To give a short overview of the use of Markov inequalities in the context of extension operators, we refer to the work of Pawłucki and Pleśniak in [PP86], [PP88], [PP89], [Ple90], Bos and Milman in [BM95], Frerick in [Fre07a] and Frerick, Jordá and Wengenroth in [FJW11].

Pleśniak proved in [Ple90] Theorem 3.3 that $\mathscr{E}(K)$, endowed with a weaker topology than generated by the Whitney seminorms, admits an extension operator if and only if there exist positive constants C and r such that for all polynomials $p \in \mathbb{C}[x_1, ..., x_d]$ the following inequality holds for each $\alpha \in \mathbb{N}_0^d$

$$|\partial^\alpha p|_{0,K} \leq C \deg(p)^{r|\alpha|} |p|_{0,K}.$$

Goncharov proved in [Gon96] that this global version of a Markov inequality is not necessary for the existence of an extension operator if $\mathscr{E}(K)$ is equipped with the Fréchet space topology generated by the Whitney semi norms. The authors Bos and Milman showed then in [BM95] that this global Markov inequality is equivalent to the validity of the following local version. K fulfils this local version if and only if there exist $r \geq 1$, $\varepsilon_0 > 0$ and $C_k > 1$ such that for all polynomials $p \in \mathbb{C}[x_1, ..., x_d]$ with $\deg(p) \leq k$, each $x_0 \in K$ and $0 < \varepsilon < \varepsilon_0$ the following inequality holds

$$|\nabla p(x_0)| \leq C_k \varepsilon^{-r} |p|_{0,B(x_0,\varepsilon)\cap K},$$

or equivalently for all $\alpha \in \mathbb{N}_0^d$

$$|\partial^\alpha p(x_0)| \leq C_k \varepsilon^{-r|\alpha|} |p|_{0,B(x_0,\varepsilon)\cap K}.$$

If a set K fulfils those inequalities for an exponent r we write in the following that it has LMI(r). More importantly they could show in Theorem E that both versions are again equivalent to the existence of an extension operator having a homogeneous loss of derivatives.

Using a weaker form of LMI, or short WLMI, this result was improved in [FJW11] and the loss of derivatives could be calculated depending on the exponent of the Markov inequality. In detail, K fulfils this weaker form of the Markov inequality LMI(r) if and only if K fulfils LMI(s) for all $s > r$. The resulting extension operator fulfils then $|E(F)|_n \leq C_{n,\varepsilon} ||F||_{(r+\varepsilon)n,K}$ for all $F \in \mathscr{E}(K), n \in \mathbb{N}_0$ and arbitrary $\varepsilon > 0$.

In Theorem 4.6 of [Fre07a], Frerick characterized the existence of an extension operator on a compact set K using different types of global and local Markov inequalities. Suitable for K let L be compact with $K \subset \mathring{L}$. Then K fulfils the global form of the inequality if and only if for all $\theta \in (0, 1)$ there are $r \geq 1$ and $C \geq 1$ such that for all polynomials $p \in \mathbb{C}[x_1, ..., x_d]$

$$\sup_{x\in K} |\nabla p(x)| \leq C \deg(p)^r \sup_{x\in L} |p(x)|^\theta \sup_{x\in K} |p(x)|^{1-\theta},$$

and K fulfils the local form if and only if for all $\theta \in (0, 1)$ there are $r \geq 1$ and $\varepsilon_0 > 0$ such that for all $k \in \mathbb{N}$ there is $C \geq 1$ such that for all polynomials $p \in \mathbb{C}[x_1, ..., x_d]$, all $x_0 \in K$ and all $\varepsilon \in (0, \varepsilon_0)$

$$|\nabla p(x_0)| \leq \frac{C}{\varepsilon^r} \sup_{|x-x_0|\leq\varepsilon} |p(x)|^\theta \sup_{|x-x_0|\leq\varepsilon, x\in K} |p(x)|^{1-\theta}.$$

We close this chapter by showing that a rather general family of self similar fractals, including well known sets as the Cantor set, the Sierpinski triangle and Koch curve, admits an extension operator without loss. For this we use an equivalent condition to LMI(1) and results on so called d-sets given in the paper [JSW84] of Jonsson, Sjögren and Wallin and as well results of Triebel in [Tri11] and Falconer in [Fal14].

4.1 Statement of the Main Result

In Theorem 4.1 we generalize the results of [FJW16b] in which the authors prove the equivalence of LMI(1) to the existence of an extension operator having no loss of derivatives on the natural scale. This means that E fulfils the norm inequalities $|E(F)|_n \leq C_n \|F\|_{n,K}$ for each $F \in \mathscr{E}(K)$. This operator is constructed in a way that it is even defined for all $f \in \mathscr{E}^0(K)$ and therefore simultaneously is an extension operator for all the spaces $\mathscr{E}^n(K)$. We will show in Theorem 4.1 that LMI(1) is also equivalent to the existence of an extension operator E having no loss of derivatives on the real scale, meaning that E fulfils $|||E(F)|||_{s,L} \leq C_s \|F\|_{s,K}$ for arbitrary compact and convex set L with $K \subset \mathring{L}$.

4.1 Theorem. *Extension operator without loss of derivatives*
For $K \subset \mathbb{R}^d$ compact, the following statements are equivalent

1. *K satisfies the LMI(1) condition.*
2. *For all $\alpha \in \mathbb{N}_0^d$, $x \in \partial K$ and $\varepsilon > 0$ there exist measures $\nu_{\alpha,x,\varepsilon}$ on K such that for each $s \geq 0$ and each jet $\left(f^{(\alpha)}\right)_{|\alpha| \leq \lfloor s \rfloor} \in \mathscr{E}^s(K)$,*

   (a) $\lim_{\varepsilon \to 0} \sup_{|\alpha| \leq \lfloor s \rfloor, x \in \partial K} \frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)}) - \varepsilon^{|\alpha|} f^{(\alpha)}(x)|}{\varepsilon^s} = 0,$

   (b) $\lim_{\varepsilon \to 0} \sup_{|\alpha| > \lfloor s \rfloor, x \in \partial K} \frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)})|}{\varepsilon^s} = 0.$
3. *K admits an extension operator without loss of derivatives which fulfils the norm inequalities on the real scale.*

In fact we show in the following two sections only that 1. $\Rightarrow$ 2. $\Rightarrow$ 3. The equivalence follows from the results in [FJW16b]. There it is shown that the LMI(1) condition is equivalent to the existence of an extension operator $E : \mathscr{E}(K) \to \mathscr{E}(\mathbb{R}^d)$ without loss of derivatives on the natural scale. Since the operator constructed in 3. has of course this property, the equivalence is established.

Thus taking all characterizations of the LMI(1) condition into account we yield the following corollary.

4.2 Corollary. *For $K \subset \mathbb{R}^d$ the following statements are equivalent*

1. *K satisfies the LMI(1) condition.*
2. *For all $\alpha \in \mathbb{N}_0^d$, $x \in \partial K$ and $\varepsilon > 0$ there exist measures $\nu_{\alpha,x,\varepsilon}$ on K such that for each $s \geq 0$ and $\left(f^{(\alpha)}\right)_{|\alpha| \leq \lfloor s \rfloor} \in \mathscr{E}^s(K)$,*

(a) $\lim_{\varepsilon\to 0} \sup_{|\alpha|\le\lfloor s\rfloor, x\in\partial K} \frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)}) - \varepsilon^{|\alpha|} f^{(\alpha)}(x)|}{\varepsilon^s} = 0,$

(b) $\lim_{\varepsilon\to 0} \sup_{|\alpha|>\lfloor s\rfloor, x\in\partial K} \frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)})|}{\varepsilon^s} = 0.$

3. *K admits an extension operator without loss of derivatives (on the real scale).*

4. *For all* $\alpha \in \mathbb{N}_0^d$, $x \in \partial K$ *and* $\varepsilon > 0$ *there exist measures* $\nu_{\alpha,x,\varepsilon}$ *on K such that for each* $n \in \mathbb{N}_0$ *and* $\left(f^{(\alpha)}\right)_{|\alpha|\le n} \in \mathscr{E}^n(K)$,

   (a) $\lim_{\varepsilon\to 0} \sup_{|\alpha|\le n, x\in\partial K} \frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)}) - \varepsilon^{|\alpha|} f^{(\alpha)}(x)|}{\varepsilon^n} = 0,$

   (b) $\lim_{\varepsilon\to 0} \sup_{|\alpha|>n, x\in\partial K} \frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)})|}{\varepsilon^n} = 0.$

5. *K admits an extension operator without loss of derivatives (on the natural scale).*

6. *There is* $\varrho \in (0,1)$*, such that for each* $x_0 \in K$ *and* $\varepsilon \in (0,1)$ *the set* $K \cap B(x_0,\varepsilon)$ *is not contained in any band of the form* $\{x \in \mathbb{R}^d : |\langle b, x - x_0\rangle| \le \varrho\varepsilon\}$ *where* $b \in \mathbb{R}^d$ *with* $|b| = 1$ *arbitrary.*

7. *It exists* $\varrho \in (0,1)$*, such that for each* $x_0 \in K$ *and* $\varepsilon \in (0,1)$ *there are d pairwise different points* $x_1, ..., x_d \in K \cap B(x_0,\varepsilon)$ *such that for all* $n \in \{0, ..., d-1\}$

$$dist\,(x_{n+1}, affhull\{x_0, ..., x_n\}) \ge \varrho\varepsilon.$$

   Here $affhull\{x_0, ..., x_n\}$ *shall denote the affine hull of the points.*

Especially the properties 6 and 7 have a purely geometric character which allows direct application to examples. The equivalence of property 6 to LMI(1) has been shown in [JSW84] in Theorem 1.3. The equivalent property 7 was proved in [BM95] in Theorem D. We will make extensive use of the last characterization in order to prove the existence of extension operators for our examples at the end of this chapter.

4.2 Construction of the Measures

We start with showing the sufficiency of LMI(1) for the existence of the measures. To achieve this, we first cite Proposition 5 of [FJW16b] which we will then use to prove a modified version of Proposition 4 in the same paper. Ongoing we make use of the following notation for 'blow-ups' of the underlying compact set K. For $\varepsilon > 0$ and $x \in K$ we write

$$A_{x,\varepsilon} := \varepsilon^{-1}(K - x) \cup \{y \in \mathbb{R}^d : |y| \ge \varepsilon^{-1}\}.$$

4.3 Proposition. *Let $K \subset \mathbb{R}^d$ be a compact subset satisfying LMI(1). Then there exists a continuous and radial function $\varrho : \mathbb{R}^d \to (0, \infty)$ with $|y|^n = o(\varrho(y))$ for $|y| \to \infty$ and all $n \in \mathbb{N}$ such that for each $x \in \partial K, \varepsilon \in (0,1)$ and $\alpha \in \mathbb{N}_0^d$ there exists a finite regular Borel measure $\mu := \mu_{\alpha,x,\varepsilon}$ on $A_{x,\varepsilon}$ with total variation $|\mu|(A_{x,\varepsilon}) \leq 1$ such that*

$$\int_{A_{x,\varepsilon}} y^\beta \frac{1}{\varrho(y)} d\mu(y) = \begin{cases} \alpha!, & \beta = \alpha \\ 0, & else \end{cases} .$$

Proof. A proof can be found in [FJW16b]. □

4.4 Proposition. *Let $K \subset \mathbb{R}^d$ be a compact subset satisfying LMI(1). Then for all $\alpha \in \mathbb{N}_0^d$, $x \in \partial K$ and $\varepsilon \in (0,1)$, there is a measure $\nu_{\alpha,x,\varepsilon}$ on K such that for each $s \geq 0$ and each $F = \left(f^{(\alpha)}\right)_{|\alpha| \leq \lfloor s \rfloor} \in \mathscr{E}^s(K)$,*

1.
$$\lim_{\varepsilon \to 0} \sup_{|\alpha| \leq \lfloor s \rfloor, x \in \partial K} \frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)}) - \varepsilon^{|\alpha|} f^{(\alpha)}(x)|}{\varepsilon^s} = 0,$$

2.
$$\lim_{\varepsilon \to 0} \sup_{|\alpha| > \lfloor s \rfloor, x \in \partial K} \frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)})|}{\varepsilon^s} = 0.$$

Proof. We assume without loss of generality that $K \subset B(0, \frac{1}{4})$. For $x \in \partial K$, $\varepsilon \in (0,1)$, $\alpha \in \mathbb{N}_0^d$ and $f \in \mathscr{C}(K)$ we define with the measure $\mu_{\alpha,x,\varepsilon}$ from Proposition 4.3

$$\nu_{\alpha,x,\varepsilon}(f) := \int_{\varepsilon^{-1}(K-x)} \frac{f(\varepsilon y + x)}{\varrho(y)} d\mu_{\alpha,x,\varepsilon}(y). \tag{4.2}$$

For each $f \in \mathscr{C}(\mathbb{R}^d)$ with support in $B(0, \frac{3}{4})$ we then have

$$\nu_{\alpha,x,\varepsilon}(f\big|_K) = \int_{A_{x,\varepsilon}} \frac{f(\varepsilon y + x)}{\varrho(y)} d\mu_{\alpha,x,\varepsilon}(y)$$

since $|x + \varepsilon y| > \frac{3}{4}$ whenever $|y| > \frac{1}{\varepsilon}$.

Multiplying Whitney's extension operator $E_s : \mathscr{E}^s(K) \to \mathscr{E}^s(\mathbb{R}^d)$ from Theorem 3.1 with a cut-off function, we may assume that $E_s(F) \in \mathscr{D}^s(B(0, \frac{3}{4}))$ for each $F \in \mathscr{E}^s(K)$. To shorten the notation, we will denote by $\tilde{f}$ the extension $E_s(F)$. Applying Taylor's theorem we find $\xi = \xi(x, \varepsilon, y, \tilde{f}) \in (x, x + \varepsilon y)$ such that

$$|\nu_{\alpha,x,\varepsilon}(f^{(0)}) - \varepsilon^{|\alpha|} f^{(\alpha)}(x)| = \left| \int_{A_{x,\varepsilon}} \frac{\tilde{f}(\varepsilon y + x)}{\varrho(y)} d\mu_{\alpha,x,\varepsilon}(y) - \varepsilon^{|\alpha|} \partial^\alpha \tilde{f}(x) \right|$$
$$= \left| \int_{A_{x,\varepsilon}} \left(\sum_{\gamma < \lfloor s \rfloor} \frac{\partial^\gamma \tilde{f}(x)}{\gamma!} \varepsilon^{|\gamma|} y^\gamma + \varepsilon^{\lfloor s \rfloor} \sum_{|\gamma| = \lfloor s \rfloor} \frac{\partial^\gamma \tilde{f}(\xi)}{\gamma!} y^\gamma \right) \frac{1}{\varrho(y)} d\mu_{\alpha,x,\varepsilon}(y) - \varepsilon^{|\alpha|} \partial^\alpha \tilde{f}(x) \right|.$$

Since

$$\sum_{|\gamma|\leq\lfloor s\rfloor,\gamma\neq\alpha}\left(\frac{\partial^\gamma \tilde f(x)}{\gamma!}\varepsilon^{|\gamma|}\int_{A_{x,\varepsilon}}\frac{y^\gamma}{\varrho(y)}d\mu_{\alpha,x,\varepsilon}(y)\right)=0,$$

we can subtract this term from the above and obtain

$$|\nu_{\alpha,x,\varepsilon}(f^{(0)})-\varepsilon^{|\alpha|}f^{(\alpha)}(x)|=\varepsilon^{\lfloor s\rfloor}\left|\int_{A_{x,\varepsilon}}\sum_{|\gamma|=\lfloor s\rfloor}\left(\frac{\partial^\gamma \tilde f(\xi)}{\gamma!}-\frac{\partial^\gamma \tilde f(x)}{\gamma!}\right)\frac{y^\gamma}{\varrho(y)}d\mu_{\alpha,x,\varepsilon}(y)\right|,$$

and therefore we have using $|\xi-x|\leq\varepsilon|y|$

$$\begin{aligned}\frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)})-\varepsilon^{|\alpha|}f^{(\alpha)}(x)|}{\varepsilon^s}&=\left|\int_{A_{x,\varepsilon}}\sum_{|\gamma|=\lfloor s\rfloor}\frac{1}{\gamma!}\left(\frac{\partial^\gamma \tilde f(\xi)-\partial^\gamma \tilde f(x)}{\varepsilon^{\{s\}}}\right)\frac{y^\gamma}{\varrho(y)}d\mu_{\alpha,x,\varepsilon}(y)\right|\\&\leq\sum_{|\gamma|=\lfloor s\rfloor}\frac{1}{\gamma!}\int_{A_{x,\varepsilon}}\frac{|\partial^\gamma \tilde f(\xi)-\partial^\gamma \tilde f(x)|}{\varepsilon^{\{s\}}}\left|\frac{y^\gamma}{\varrho(y)}\right|d|\mu_{\alpha,x,\varepsilon}|(y)\\&\leq\sum_{|\gamma|=\lfloor s\rfloor}\frac{1}{\gamma!}\int_{A_{x,\varepsilon}}\frac{|\partial^\gamma \tilde f(\xi)-\partial^\gamma \tilde f(x)|}{\varepsilon^{\{s\}}}\frac{|y|^{|\gamma|}}{|\varrho(y)|}d|\mu_{\alpha,x,\varepsilon}|(y)\\&\leq\sum_{|\gamma|=\lfloor s\rfloor}\frac{1}{\gamma!}\int_{A_{x,\varepsilon}}\frac{|\partial^\gamma \tilde f(\xi)-\partial^\gamma \tilde f(x)|}{|\xi-x|^{\{s\}}}\frac{|y|^{|\gamma|+\{s\}}}{|\varrho(y)|}d|\mu_{\alpha,x,\varepsilon}|(y).\end{aligned}$$

To show that the last integral converges uniformly in x to zero for $\varepsilon\to 0$, we split it in the integrals over the sets $A_{x,\varepsilon}\cap\{|y|>r\}$ and $A_{x,\varepsilon}\cap\{|y|\leq r\}$. For the first integral we use the facts

$$\frac{|\partial^\gamma \tilde f(\xi)-\partial^\gamma \tilde f(x)|}{|\xi-x|^{\{s\}}}\leq\left\|\tilde f\right\|_{s,B(0,\frac34)},$$

$|\mu_{\alpha,x,\varepsilon}|(A_{x,\varepsilon})\leq 1$ and $|y|^n/\varrho(y)\to 0$ for $|y|\to\infty$ for all $n\in\mathbb{N}$. Thus we get

$$\begin{aligned}&\int_{A_{x,\varepsilon}\cap\{|y|>r\}}\frac{|\partial^\gamma \tilde f(\xi)-\partial^\gamma \tilde f(x)|}{|\xi-x|^{\{s\}}}\frac{|y|^{|\gamma|+\{s\}}}{|\varrho(y)|}d|\mu_{\alpha,x,\varepsilon}|(y)\\&\leq\left\|\tilde f\right\|_{s,B(0,\frac34)}\sup_{A_{x,\varepsilon}\cap\{|y|>r\}}\frac{|y|^{|\gamma|+\{s\}}}{\varrho(y)}\\&\leq\left\|\tilde f\right\|_{s,B(0,\frac34)}\sup_{\{|y|>r\}}\frac{|y|^{|\gamma|+\{s\}}}{\varrho(y)}.\end{aligned}$$

The last term converges to zero for $r\to\infty$ and is independent of ε. For the integral over the second set we first observe that $|\xi-x|\leq\varepsilon|y|\leq r\varepsilon$, which implies since $\tilde f\in\mathscr{D}^s(B(0,\frac34))$ that uniformly

$$\frac{|\partial^\gamma \tilde f(\xi)-\partial^\gamma \tilde f(x)|}{|\xi-x|^{\{s\}}}\to 0\quad\text{for }\varepsilon\to 0.$$

Thus we get

$$\int_{A_{x,\varepsilon}\cap\{|y|\le r\}} \frac{|\partial^\gamma \tilde{f}(\xi) - \partial^\gamma \tilde{f}(x)|}{|\xi - x|^{\{s\}}} \frac{|y|^{|\gamma|+\{s\}}}{|\varrho(y)|} d|\mu_{\alpha,x,\varepsilon}|(y)$$
$$\le q_{s,B(0,\frac{3}{4})}(\tilde{f}, r\varepsilon) \sup_{y\in\mathbb{R}^d} \frac{|y|^{|\gamma|+\{s\}}}{|\varrho(y)|},$$

which converges for a fixed r to zero for $\varepsilon \to 0$. All together this proves the first property of the measures.

For the second property we proceed now analogously. With Taylor's theorem we find again a $\xi \in (x, x + \varepsilon y)$ such that

$$|\nu_{\alpha,x,\varepsilon}(f^{(0)})| = \left| \int_{A_{x,\varepsilon}} \frac{\tilde{f}(x+\varepsilon y}{\varrho(y)} d\mu_{\alpha,x,\varepsilon}(y) \right|$$
$$= \left| \int_{A_{x,\varepsilon}} \left(\sum_{|\gamma|<\lfloor s\rfloor} \frac{\partial^\gamma \tilde{f}(x)}{\gamma!} \varepsilon^{|\gamma|} y^\gamma + \varepsilon^{\lfloor s\rfloor} \sum_{|\gamma|=\lfloor s\rfloor} \frac{\partial^\gamma \tilde{f}(\xi)}{\gamma!} y^\gamma \right) \frac{1}{\varrho(y)} d\mu_{\alpha,x,\varepsilon}(y) \right|.$$

Since $|\alpha| > \lfloor s\rfloor$, we have that

$$\int_{A_{x,\varepsilon}} \frac{y^\gamma}{\varrho(y)} d\mu_{\alpha,x,\varepsilon}(y) = 0$$

for all $|\gamma| \le \lfloor s\rfloor$, and therefore

$$\int_{A_{x,\varepsilon}} \sum_{|\gamma|\le\lfloor s\rfloor} \frac{\partial^\gamma \tilde{f}(x)}{\gamma!} \varepsilon^{|\gamma|} \frac{y^\gamma}{\varrho(y)} d\mu_{\alpha,x,\varepsilon}(y) = 0.$$

Subtracting this, we get as for the first property that

$$\frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)})|}{\varepsilon^s} \le \sum_{|\gamma|=\lfloor s\rfloor} \frac{1}{\gamma!} \int_{A_{x,\varepsilon}} \frac{|\partial^\gamma \tilde{f}(\xi) - \partial^\gamma \tilde{f}(x)|}{|\xi - x|^{\{s\}}} \frac{|y|^s}{|\varrho(y)|} d|\mu_{\alpha,x,\varepsilon}|(y),$$

which allows the same reasoning as for the first property. □

4.3 Construction of the Extension Operator

In this section we prove that in Theorem 4.1 the existence of the measures on K implies the existence of an extension operator on $\mathscr{E}(K)$ without loss. The operator will be constructed with the help of the measures derived in the former section.

In the next lemma we list some additional properties of Whitney's partition of unity.

4.5 Lemma. *For $K \subset \mathbb{R}^d$ compact, let $(\varphi_i)_{i\in I}$ be a partion of unity of $\mathbb{R}^d \backslash K$ as it is defined in Lemma 3.1 of [Mal67]. Then this partition has the following three properties, where we denote by γ_i the distance of K to supp(φ_i).*

1. *For all $n \in \mathbb{N}$, $|\alpha| \leq n$ and $\beta \in \mathbb{N}_0^d$ we obtain positive constants $C_{\beta,n}$ independent of i such that*
$$\sup_{x\in supp(\varphi_i)} \left|\partial^\beta((x-x_i)^\alpha \varphi_i(x))\right| \leq C_{\beta,n}\gamma_i^{|\alpha|-|\beta|}.$$

2. *For all $n \in \mathbb{N}$, $|\alpha| > n$ and $\beta \in \mathbb{N}_0^d$ we obtain positive constants $C_{\beta,n}$ independent of i such that*
$$\sup_{x\in supp(\varphi_i)} \left|\partial^\beta((x-x_i)^\alpha \varphi_i(x))\right| \leq C_{\beta,n} 3^{|\alpha|} \sup_{\gamma\leq\alpha,\beta} \frac{\alpha!}{(\alpha-\gamma)!}\gamma_i^{|\alpha|-|\beta|}.$$

3. *For all $n \in \mathbb{N}$ and $\beta \in \mathbb{N}_0^d$ we have*
$$\sum_{|\alpha|>n} \sup_{\gamma\leq\alpha,\beta} \frac{1}{(\alpha-\gamma)!} 3^{|\alpha|} \leq 3^{|\beta|} e^{3d}(|\beta|+1)^d.$$

Proof. For the first inequality we apply Leibniz's formula and get
$$\left|\partial^\beta((x-x_i)^\alpha \varphi_i(x))\right| \leq \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} |\partial^\gamma((x-x_i)^\alpha)| \left|\partial^{\beta-\gamma}\varphi_i(x)\right|.$$

We note that $\partial^\gamma((x-x_i)^\alpha) = 0$ if $\gamma_j > \alpha_j$ for at least one $j \in \{1, ..., d\}$, thus we can assume that $|\alpha| - |\gamma| \geq 0$ in the following, or also that $\alpha - \gamma \in \mathbb{N}_0^d$ which allows the application of the inequality from Remark 2.2 part 2. Using part 3. of Lemma 3.3 we get for all $x \in \text{supp}(\varphi_i)$
$$|x - x_i| \leq \text{diam}(\text{supp}(\varphi_i)) + \gamma_i \leq 3\gamma_i,$$
and of course it is true that $\text{dist}(x, K) \geq \gamma_i$, which gives together with part 4. of the same lemma
$$\begin{aligned}
\left|\partial^\beta((x-x_i)^\alpha \varphi_i(x))\right| &\leq \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \frac{\alpha!}{(\alpha-\gamma)!} |x-x_i|^{|\alpha|-|\gamma|} c_\beta \text{dist}(x,K)^{|\gamma|-|\beta|} \\
&\leq \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \frac{\alpha!}{(\alpha-\gamma)!} 3^{|\alpha|-|\gamma|}\gamma_i^{|\alpha|-|\gamma|} c_\beta \gamma_i^{|\gamma|-|\beta|} \\
&= \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \frac{\alpha!}{(\alpha-\gamma)!} c_\beta 3^{|\alpha|-|\gamma|}\gamma_i^{|\alpha|-|\beta|}.
\end{aligned}$$

We can then set

$$C_{\beta,n} = \sup_{|\alpha|\leq n} \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} \frac{\alpha!}{(\alpha-\gamma)!} c_\beta 3^{|\alpha|-|\gamma|}.$$

For the second inequality we calculate in the same way but we set

$$C_{\beta,n} = \sum_{\gamma\leq\beta} \binom{\beta}{\gamma} c_\beta 3^{-|\gamma|}.$$

To get the third inequality we use Fubini's theorem to get

$$\begin{aligned}
\sum_{|\alpha|>n} \sup_{\gamma\leq\alpha,\beta} \frac{1}{(\alpha-\gamma)!} 3^{|\alpha|} &\leq 3^{|\beta|} \sum_{\gamma\leq\beta} \sum_{\substack{|\alpha|>n \\ \alpha\geq\gamma}} \frac{1}{(\alpha-\gamma)!} 3^{|\alpha-\gamma|} \\
&\leq 3^{|\beta|} \sum_{\gamma\leq\beta} \sum_{\alpha\in\mathbb{N}_0^d} \frac{1}{\alpha!} 3^{|\alpha|} \\
&= 3^{|\beta|} \sum_{\gamma\leq\beta} \left(\sum_{\alpha_1\in\mathbb{N}_0} \cdots \sum_{\alpha_d\in\mathbb{N}_0} \frac{1}{\alpha_1!} \cdots \frac{1}{\alpha_d!} 3^{\alpha_1} \cdots 3^{\alpha_d} \right) \\
&= 3^{|\beta|} \sum_{\gamma\leq\beta} \left(\sum_{j\in\mathbb{N}_0} \frac{1}{j!} 3^j \right)^d \\
&= 3^{|\beta|} e^{3d} \prod_{i=1}^{d} (\beta_i + 1) \\
&\leq 3^{|\beta|} e^{3d} (|\beta| + 1)^d
\end{aligned}$$

□

4.6 Theorem. *Construction of an extension operator without loss of derivatives*
Let $K \subset \mathbb{R}^d$ be compact. If for all $\alpha \in \mathbb{N}_0^d$, $x \in \partial K$ and $\varepsilon \in (0,1)$, there is a measure $\nu_{\alpha,x,\varepsilon}$ satisfying the conditions of Theorem 4.1, then there exists a linear and continuous extension operator $E : \mathscr{E}(K) \to \mathscr{E}(\mathbb{R}^d)$ without loss of derivatives.

Proof. In the following we denote with E_s the Whitney operator with the properties shown in Theorem 3.1. Then we set with the measures $\nu_{\alpha,x,\varepsilon}$ of Proposition 4.4

$$\mu_{\alpha,i} := \nu_{\alpha,x_i,\gamma_i} / \gamma_i^{|\alpha|}.$$

For $F = \left(f^{(\alpha)}\right)_{\alpha\in\mathbb{N}_0^d} \in \mathscr{E}(K)$, we define the sought-after extension operator E as follows

$$E(F)(x) := \begin{cases} f^{(0)}(x), x \in K \\ \sum_{i\in\mathbb{N}} \varphi_i(x) \sum_{|\alpha|\leq i} \frac{1}{\alpha!} \mu_{\alpha,i}(f^{(0)})(x-x_i)^\alpha, x \notin K. \end{cases}$$

We structure the proof in the following steps. The continuity of the operator is proved in the next section.

1. For all $s \geq 0, \beta \in \mathbb{N}_0^d$ and $F \in \mathscr{E}^s(K)$ we have
$$\left|\partial^\beta E(F)(x) - \partial^\beta E_s(F)(x)\right| = o(\operatorname{dist}(x, K)^{s-|\beta|}) \quad \text{for } x \to \partial K, \tag{4.3}$$
or equivalently:
For all $s \geq 0, \beta \in \mathbb{N}_0^d$ and $F \in \mathscr{E}^s(K)$ there exists a modulus of continuity μ_β such that
$$\left|\partial^\beta E(F)(x) - \partial^\beta E_s(F)(x)\right| \leq \mu_\beta(\operatorname{dist}(x, K))\operatorname{dist}(x, K)^{s-|\beta|}. \tag{4.4}$$

2. Let L be a compact and convex set with $\mathring{L} \supset K$, then $\left(\partial^\alpha \tilde{f}\right)_{|\alpha| \leq \lfloor s \rfloor} \in \mathscr{E}^s(L)$.

We start with the proof of inequality (4.3). For $x \in \mathbb{R}^d \setminus K$ we define $i(x) := \inf\{i \in I : x \in \operatorname{supp}(\varphi_i)\}$ and as we are only interested in the behaviour near the boundary of K, we can limit ourselves here to those $x \in L \setminus K$ with $i(x) > \lfloor s \rfloor$. Because of property 2 of Lemma 3.3 we then have that $i(x) \to \infty$ is equivalent to $x \to \partial K$. For $|\beta| \leq \lfloor s \rfloor, F \in \mathscr{E}^s(K)$ and $i(x) > \lfloor s \rfloor$ we have

$$\begin{aligned}
\partial^\beta E(F)(x) - \partial^\beta E_s(F)(x) = &\sum_{i \geq i(x)} \sum_{|\alpha| \leq \lfloor s \rfloor} \frac{1}{\alpha!}(\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i))\partial^\beta((x - x_i)^\alpha \varphi_i(x)) \\
&+ \sum_{i \geq i(x)} \sum_{\lfloor s \rfloor < |\alpha| \leq i} \frac{1}{\alpha!}\mu_{\alpha,i}(f^{(0)})\partial^\beta((x - x_i)^\alpha \varphi_i(x)).
\end{aligned}$$

We will estimate both terms. Using the hypotheses on the measures we get for $|\alpha| \leq \lfloor s \rfloor$

$$|\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i)| = o(\gamma_i^{s-|\alpha|}) \quad \text{as } i \to \infty.$$

Lemma 3.3 states that each $x \in \mathbb{R}^d \setminus K$ is contained in a finite number of supports which allows us to treat the series as finite sums. Thus we obtain together with Lemma 4.5.1 for the first term that

$$\begin{aligned}
&\left|\sum_{|\alpha| \leq \lfloor s \rfloor} \frac{1}{\alpha!}(\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i))\partial^\beta((x - x_i)^\alpha \varphi_i(x))\right| \\
\leq &\sum_{|\alpha| \leq \lfloor s \rfloor} \frac{1}{\alpha!}\left|\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i)\right| \left|\partial^\beta\left((x - x_i)^\alpha \varphi_i(x)\right)\right| \\
\leq &\sum_{|\alpha| \leq \lfloor s \rfloor} \frac{1}{\alpha!} \frac{\left|\nu_{\alpha,x_i,\gamma_i}(f^{(0)}) - \gamma_i^{|\alpha|} f^{(\alpha)}(x_i)\right|}{\gamma_i^{|\alpha|}} C_{\beta,\lfloor s \rfloor}\gamma_i^{|\alpha|-|\beta|} \\
= &\ o(\gamma_i^{s-|\beta|}) \quad \text{as } i \to \infty.
\end{aligned}$$

Since the conditions dist $(x, K) \to 0$, $i(x) \to \infty$ and $\gamma_{i(x)} \to 0$ are equivalent, it follows that

$$\lim_{x\to\partial K} \sum_{i\geq i(x)} \left| \sum_{|\alpha|\leq\lfloor s\rfloor} \frac{1}{\alpha!}(\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i))\partial^\beta((x-x_i)^\alpha\varphi_i(x)) \right| \operatorname{dist}(x,K)^{|\beta|-s} = 0.$$

For the second summand Lemma 4.5.2 and .3 imply that

$$\begin{aligned} &\left| \sum_{|\alpha|>\lfloor s\rfloor} \frac{1}{\alpha!}\mu_{\alpha,i}(f^{(0)})\partial^\beta((x-x_i)^\alpha\varphi_i(x)) \right| \\ \leq\ & \sum_{|\alpha|>\lfloor s\rfloor} \frac{1}{\alpha!}\gamma_i^{-|\alpha|} o(\gamma_i^s) C_{\beta,\lfloor s\rfloor} \sup_{\gamma\leq\alpha,\beta} \frac{\alpha!}{(\alpha-\gamma)!} 3^{|\alpha|}\gamma_i^{|\alpha|-|\beta|} \\ \leq\ & o(\gamma_i^s) C_{\beta,\lfloor s\rfloor} e^{3d}(|\beta|+1)^d 3^{|\beta|}\gamma_i^{-|\beta|} \\ =\ & o(\gamma_i^{s-|\beta|}) \quad \text{as } i\to\infty. \end{aligned}$$

As for the first summand we conclude

$$\lim_{x\to\partial K} \sum_{i\geq i(x)} \left| \sum_{\lfloor s\rfloor<|\alpha|\leq i} \frac{1}{\alpha!}\mu_{\alpha,i}(f^{(0)})\partial^\beta((x-x_i)^\alpha\varphi_i(x)) \right| \operatorname{dist}(x,K)^{|\beta|-s} = 0,$$

which proves (4.3).

For the proof that $E(F) \in \mathscr{E}^s(\mathbb{R}^d)$, we first note that (4.3) together with Proposition 2.15 shows that the function $E(F) - E_s(F)$ is $\lfloor s\rfloor$-times continuously partially differentiable on $\mathbb{R}^d$. Since we already know that $E_s(F) \in \mathscr{E}^s(\mathbb{R}^d)$ this gives $E(F) = E(F) - E_s(F) + E_s(F) \in \mathscr{E}^{\lfloor s\rfloor}(\mathbb{R}^d)$. Then inequality (4.4) allows the application of Proposition 2.16. For this we note that $\partial^\alpha E(F)(x) - \partial^\alpha E_s(F)(x) = 0$ for all $|\alpha| \leq \lfloor s\rfloor$ and $x \in K$, and therefore

$$\partial^\beta T_a^{\lfloor s\rfloor}\left((\partial^\alpha E(F) - \partial^\alpha E_s(F))_{|\alpha|\leq\lfloor s\rfloor}\right) = 0,$$

for all $|\beta| \leq \lfloor s\rfloor$ and $a \in K$. □

In the next section we will calculate the constants occurring in the continuity estimates to study their behaviour, which of course proves the continuity of E.

4.4 A Closer Look at the Continuity Estimates

In this section we prove the following result on the continuity constants.

4.7 Proposition. *Let $E : \mathscr{E}(K) \to \mathscr{E}(\mathbb{R}^d)$ be the extension operator constructed in Theorem 4.6. Then for each $s \geq 0$ there is a constant $C_s > 0$ such that the inequality $|||E(F)|||_{s,L} \leq C_s\|F\|_{s,K}$ holds for each $F \in \mathscr{E}^s(K)$ and each compact and convex set L with $\mathring{L} \supset K$. Furthermore these constants can be chosen such that the mapping $s \mapsto C_s$ is continuous and bounded on each interval $[n, n+1)$ for every $n \in \mathbb{N}_0$.*

Proof. Let $n \in \mathbb{N}_0$ arbitrary. In the course of this proof we will call a family of constants $(C_s)_{s\in[n,n+1)}$ admissible if they just depend on s, d and λ and if the mapping $s \mapsto C_s$ is continuous and bounded on each interval $[n, n+1)$ for each $n \in \mathbb{N}$. Let $E_s : \mathscr{E}^s(K) \to \mathscr{E}^s(\mathbb{R}^d)$ be again the generalized Whitney operator constructed in Theorem 3.1. Then we have by the triangle inequality

$$|||E(F)|||_{s,L} \leq |||(E - E_s)(F)|||_{s,L} + |||E_s(F)|||_{s,L}. \tag{4.5}$$

For the second summand with Lemma 3.2 we can find a family $\left(C_s^{(0)}\right)_{s\geq 0}$ of admissible constants. So we only have to check the continuity estimate for the first summand. The proof is structured in two steps.

1. We show that there is an admissible family of constants $\left(C_s^{(1)}\right)_{s\geq 0}$ such that

$$|(E - E_s)(F)|_{s,L} \leq C_s^{(1)} |||E_s(F)|||_{s,L}. \tag{4.6}$$

2. We show that there is an admissible family of constants $\left(C_s^{(2)}\right)_{s\geq 0}$ such that for all $x_1, x_2 \in L$ and all $|\beta| = \lfloor s \rfloor$

$$\frac{|\partial^\beta(E - E_s)(F)(x_1) - \partial^\beta(E - E_s)(F)(x_2)|}{|x_1 - x_2|^{\{s\}}} \leq C_s^{(2)} |||E_s(F)|||_{s,L}. \tag{4.7}$$

From those two inequalities it follows then directly that

$$|||(E - E_s)(F)|||_{s,L} \leq (C_s^{(1)} + C_s^{(2)})C_s^{(0)} \|F\|_{s,K},$$

where the family $\left((C_s^{(1)} + C_s^{(2)})C_s^{(0)}\right)_{s\geq 0}$ is of course admissible.

For the proof of inequality (4.6) it suffices to show the inequality for $|\beta| = \lfloor s \rfloor$ as Lemma 2.11 shows. So let $x \in L \setminus K$, for $x \in K$ this is trivial.

$$\begin{aligned}
\partial^\beta(E - E_s)(F)(x) = &\sum_{i\in\mathbb{N}} \sum_{|\alpha|\leq \min(i,\lfloor s \rfloor)} \frac{1}{\alpha!}(\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i))\partial^\beta(\varphi_i(x)(x - x_i)^\alpha) \\
&+ \sum_{i>\lfloor s \rfloor} \sum_{\lfloor s \rfloor<|\alpha|\leq i} \frac{1}{\alpha!}\mu_{\alpha,i}(f^{(0)})\partial^\beta(\varphi_i(x)(x - x_i)^\alpha) \\
&- \sum_{i\leq\lfloor s \rfloor} \sum_{i<|\alpha|\leq\lfloor s \rfloor} \frac{1}{\alpha!} f^{(\alpha)}(x_i)\partial^\beta(\varphi_i(x)(x - x_i)^\alpha).
\end{aligned}$$

In order to prove (4.6) we have to show the three following inequalities

1. For $i \in \mathbb{N}$ and $|\alpha| \leq \min(i, \lfloor s \rfloor)$ we show that

$$|\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i)||\partial^\beta(\varphi_i(x)(x - x_i)^\alpha| \leq C(s, d, \lambda, \varrho)|||E_s(F)|||_{s,L}. \tag{4.8}$$

2. For $\lfloor s \rfloor < |\alpha| \leq i$ we need to show that

$$\sum_{|\alpha|>\lfloor s \rfloor} \frac{1}{\alpha!} |\mu_{\alpha,i}(f^{(0)})||\partial^\beta(\varphi_i(x)(x-x_i)^\alpha)| \leq C(s,d,\lambda,\varrho)|||E_s(F)|||_{s,L}. \tag{4.9}$$

3. For $i \leq \lfloor s \rfloor$ and $i < |\alpha| \leq \lfloor s \rfloor$ we show that

$$|f^{(\alpha)}(x_i)||\partial^\beta(\varphi_i(x)(x-x_i)^\alpha)| \leq C(s,d,\lambda,\varrho)|||E_s(F)|||_{s,L}. \tag{4.10}$$

For the first case we get, using Lemma 4.5 a constant $C_{\beta,\lfloor s \rfloor}$

$$|\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i)||\partial^\beta(\varphi_i(x)(x-x_i)^\alpha| \leq C_{\beta,\lfloor s \rfloor} \frac{|\nu_{\alpha,x_i,\gamma_i}(f^{(0)}) - \gamma_i^{|\alpha|} f^{(\alpha)}|}{|\gamma_i^{\lfloor s \rfloor}|}.$$

Using now the technique of the proof of Proposition 4.4 we get

$$\begin{aligned}
\frac{|\nu_{\alpha,x_i,\gamma_i}(f^{(0)}) - \gamma_i^{|\alpha|} f^{(\alpha)}(x_i)|}{\gamma_i^{\lfloor s \rfloor}} &= \gamma_i^{\{s\}} \frac{|\nu_{\alpha,x_i,\gamma_i}(f^{(0)}) - \gamma_i^{|\alpha|} f^{(\alpha)}(x_i)|}{\gamma_i^s} \\
&= \gamma_i^{\{s\}} \left| \int_{A_{x_i,\gamma_i}} \sum_{|\gamma|=\lfloor s \rfloor} \frac{1}{\gamma!} \frac{\partial^\gamma E_s(F)(\xi) - \partial^\gamma E_s(F)(x_i)}{\gamma_i^{\{s\}}} \frac{y^\gamma}{\varrho(y)} d\mu_{\alpha,x_i,\gamma_i}(y) \right| \\
&\leq \lambda^{\{s\}} \int_{A_{x_i,\gamma_i}} \sum_{|\gamma|=\lfloor s \rfloor} \frac{1}{\gamma!} \frac{|\partial^\gamma E_s(F)(\xi) - \partial^\gamma E_s(F)(x_i)|}{\gamma_i^{\{s\}}} \frac{|y|^{|\gamma|}}{\varrho(y)} d|\mu_{\alpha,x_i,\gamma_i}|(y) \\
&\leq \lambda^{\{s\}} \int_{A_{x_i,\gamma_i}} \sum_{|\gamma|=\lfloor s \rfloor} \frac{1}{\gamma!} \frac{|\partial^\gamma E_s(F)(\xi) - \partial^\gamma E_s(F)(x_i)|}{|\xi - x|^{\{s\}}} \frac{|y|^s}{\varrho(y)} d|\mu_{\alpha,x_i,\gamma_i}|(y) \\
&\leq \lambda^{\{s\}} \sup_{y\in\mathbb{R}^d} \left(\frac{|y|^s}{\varrho(y)} \right) \left(\sum_{|\gamma|=\lfloor s \rfloor} \frac{1}{\gamma!} \right) |\mu_{\alpha,x_i,\gamma_i}|(A_{x_i,\gamma_i}) |||E_s(F)|||_{s,L}.
\end{aligned}$$

Setting

$$C(s,d,\lambda,\varrho) := C_{\beta,\lfloor s \rfloor} \lambda^{\{s\}} \sup_{y\in\mathbb{R}^d} \left(\frac{|y|^s}{\varrho(y)} \right) \left(\sum_{|\gamma|=\lfloor s \rfloor} \frac{1}{\gamma!} \right)$$

establishes (4.8) because $|\mu_{\alpha,x_i,\gamma_i}|(A_{x_i,\gamma_i}) \leq 1$ and the function ϱ depends only on K itself. It is clear that the family $(C(s,d,\lambda,\varrho))_{s\geq 0}$ is admissible.

For the proof of (4.9) we use the same approach and get with Lemma 4.5 a constant $C_{\beta,\lfloor s \rfloor}$ such that

$$\sum_{|\alpha|>\lfloor s \rfloor} \frac{1}{\alpha!} |\mu_{\alpha,i}(f^{(0)})||\partial^\beta(\varphi_i(x)(x-x_i)^\alpha)| = C_{\beta,\lfloor s \rfloor} \sum_{|\alpha|>\lfloor s \rfloor} \frac{|\nu_{\alpha,x_i,\gamma_i}(f^{(0)})|}{\gamma_i^{\lfloor s \rfloor}} 3^{|\alpha|} \sup_{\gamma\leq\alpha,\beta} \frac{1}{(\alpha-\gamma)!}.$$

Furthermore we have here also

$$\begin{aligned}\frac{|\nu_{\alpha,x_i,\gamma_i}(f^{(0)})|}{\gamma_i^s} &= \left|\int_{A_{x_i,\gamma_i}} \sum_{|\gamma|=\lfloor s\rfloor} \frac{1}{\gamma!} \frac{\partial^\gamma E_s(F)(\xi) - \partial^\gamma E_s(F)(x_i)}{\gamma_i^{\{s\}}} \frac{y^\gamma}{\varrho(x)} d\mu_{\alpha,x_i,\gamma_i}(y)\right| \\ &\le \int_{A_{x_i,\gamma_i}} \sum_{|\gamma|=\lfloor s\rfloor} \frac{1}{\gamma!} \frac{|\partial^\gamma E_s(F)(\xi) - \partial^\gamma E_s(F)(x_i)|}{\gamma_i^{\{s\}}} \frac{|y|^{|\gamma|}}{\varrho(x)} d|\mu_{\alpha,x_i,\gamma_i}|(y) \\ &\le \int_{A_{x_i,\gamma_i}} \sum_{|\gamma|=\lfloor s\rfloor} \frac{1}{\gamma!} \frac{|\partial^\gamma E_s(F)(\xi) - \partial^\gamma E_s(F)(x_i)|}{|\xi - x|^{\{s\}}} \frac{|y|^s}{\varrho(x)} d|\mu_{\alpha,x_i,\gamma_i}|(y) \\ &\le \sup_{y\in\mathbb{R}^d}\left(\frac{|y|^s}{\varrho(y)}\right)\left(\sum_{|\gamma|=\lfloor s\rfloor}\frac{1}{\gamma!}\right)|\mu_{\alpha,x_i,\gamma_i}|(A_{x_i,\gamma_i})|||E_s(F)|||_{s,L},\end{aligned}$$

which leads, using again $|\mu_{\alpha,x_i,\gamma_i}|(A_{x_i,\gamma_i}) \le 1$, to

$$\begin{aligned}&\sum_{|\alpha|>\lfloor s\rfloor} \frac{1}{\alpha!}|\mu_{\alpha,i}(f^{(0)})||\partial^\beta(\varphi_i(x)(x-x_i)^\alpha)| \\ \le\ & C_{\beta,\lfloor s\rfloor} \sum_{|\alpha|>\lfloor s\rfloor} \frac{|\nu_{\alpha,x_i,\gamma_i}(f^{(0)})|}{\gamma_i^s}\gamma_i^{\{s\}} 3^{|\alpha|} \sup_{\gamma\le\alpha,\beta} \frac{1}{(\alpha-\gamma)!} \\ \le\ & C_{\beta,\lfloor s\rfloor} \sup_{y\in\mathbb{R}^d}\left(\frac{|y|^s}{\varrho(y)}\right)\lambda^{\{s\}}\left(\sum_{|\gamma|=\lfloor s\rfloor}\frac{1}{\gamma!}\right)\sum_{|\alpha|>\lfloor s\rfloor} 3^{|\alpha|}\sup_{\gamma\le\alpha,\beta}\frac{1}{(\alpha-\gamma)!}|||E_s(F)|||_{s,L} \\ \le\ & C_{\beta,\lfloor s\rfloor} \sup_{y\in\mathbb{R}^d}\left(\frac{|y|^s}{\varrho(y)}\right)\lambda^{\{s\}}e^{3d}(\lfloor s\rfloor+1)^d 3^{\lfloor s\rfloor}|||E_s(F)|||_{s,L},\end{aligned}$$

which establishes (4.9) by setting

$$C(s,d,\lambda,\varrho) := C_{\beta,\lfloor s\rfloor}\sup_{y\in\mathbb{R}^d}\left(\frac{|y|^s}{\varrho(y)}\right)\lambda^{\tilde{s}}e^{3d}(\lfloor s\rfloor+1)^d 3^{\lfloor s\rfloor}.$$

For the proof of (4.10) let $i \le \lfloor s\rfloor$ and $i < |\alpha| \le \lfloor s\rfloor$. Since we have here again for a given s only a finite combination of i and α we can compute easily

$$|f^{(\alpha)}(x_i)||\partial^\beta(\varphi_i(x)(x-x_i)^\alpha)| \le |||E_s(F)|||_{s,L}C_{\beta,\lfloor s\rfloor}\max_{i\le\lfloor s\rfloor}\max_{i<|\alpha|\le\lfloor s\rfloor}\gamma_i^{|\alpha|-\lfloor s\rfloor}.$$

Thus the proof of (4.6) is complete.

Now we prove inequality (4.7) and compute for $x_1, x_2 \in L \setminus K$

$$
\begin{aligned}
&\partial^\beta(E-E_s)(F)(x_1) - \partial^\beta(E-E_s)(F)(x_2)\\
=&\partial^\beta\left(\sum_{i\in\mathbb{N}}\varphi_i(\cdot)\sum_{|\alpha|\le i}\frac{1}{\alpha!}\mu_{\alpha,i}(f^{(0)})(\cdot - x_i)^\alpha\right)(x_1) - \partial^\beta\left(\sum_{i\in\mathbb{N}}\varphi_i(\cdot)\sum_{|\alpha|\le\lfloor s\rfloor}\frac{1}{\alpha!}f^{(\alpha)}(x_i)(\cdot - x_i)^\alpha\right)(x_1)\\
&-\partial^\beta\left(\sum_{i\in\mathbb{N}}\varphi_i(\cdot)\sum_{|\alpha|\le i}\frac{1}{\alpha!}\mu_{\alpha,i}(f^{(0)})(\cdot - x_i)^\alpha\right)(x_2) + \partial^\beta\left(\sum_{i\in\mathbb{N}}\varphi_i(\cdot)\sum_{|\alpha|\le\lfloor s\rfloor}\frac{1}{\alpha!}f^{(\alpha)}(x_i)(\cdot - x_i)^\alpha\right)(x_2)\\
=&\sum_{i\in\mathbb{N}}\sum_{|\alpha|\le\min(i,\lfloor s\rfloor)}\frac{1}{\alpha!}(\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i))\left(\partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_1) - \partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_2)\right)\\
&+\sum_{i>\lfloor s\rfloor}\sum_{\lfloor s\rfloor<|\alpha|\le i}\frac{1}{\alpha!}\mu_{\alpha,i}(f^{(0)})\left(\partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_1) - \partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_2)\right)\\
&-\sum_{i\le\lfloor s\rfloor}\sum_{i<|\alpha|\le\lfloor s\rfloor}\frac{1}{\alpha!}f^{(\alpha)}(x_i)\left(\partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_1) - \partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_2)\right)
\end{aligned}
$$

and for $x_1 \in L \setminus K, x_2 \in K$ we compute similarly:

$$
\begin{aligned}
&\partial^\beta(E-E_s)(F)(x_1) - \partial^\beta(E-E_s)(F)(x_2)\\
=&\partial^\beta\left(\sum_{i\in\mathbb{N}}\varphi_i(\cdot)\sum_{|\alpha|\le i}\frac{1}{\alpha!}\mu_{\alpha,i}(f^{(0)})(\cdot - x_i)^\alpha\right)(x_1) - \partial^\beta\left(\sum_{i\in\mathbb{N}}\varphi_i(\cdot)\sum_{|\alpha|\le\lfloor s\rfloor}\frac{1}{\alpha!}f^{(\alpha)}(x_i)(\cdot - x_i)^\alpha\right)(x_1)\\
&-f^{(\beta)}(x_2) + f^{(\beta)}(x_2)\\
=&\sum_{i\in\mathbb{N}}\sum_{|\alpha|\le\min(i,\lfloor s\rfloor)}\frac{1}{\alpha!}(\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i))\partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_1)\\
&-\sum_{i\le\lfloor s\rfloor}\sum_{i<|\alpha|\le\lfloor s\rfloor}\frac{1}{\alpha!}f^{(\alpha)}(x_i)\partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_1)\\
&+\sum_{i>\lfloor s\rfloor}\sum_{\lfloor s\rfloor<|\alpha|\le i}\frac{1}{\alpha!}\mu_{\alpha,i}(f^{(0)})\partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_1).
\end{aligned}
$$

The case that both points belong to K is trivial.

We start with the case that both points belong to $L \setminus K$. Here we get three different terms which we have to estimate.

1. We show that there are admissible constants $\left(C_s^{(\alpha)}\right)_{s\ge 0}$ such that for each $s \ge 0$ and each $|\alpha| \le \lfloor s\rfloor$:

$$|\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i)||\partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_1) - \partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_2)||x_1 - x_2|^{-\{s\}} \le C_s^{(\alpha)}\|F\|_{s.K} \tag{4.11}$$

2. We show that there are admissible constants $\left(C_s^{(b)}\right)_{s\geq 0}$ such that for each $s \geq 0$ and each $|\alpha| \leq \lfloor s \rfloor$:

$$|f^{(\alpha)}(x_i)||\partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_1) - \partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_2)||x_1 - x_2|^{-\{s\}} \leq C_s^{(b)}\|F\|_{s,K} \quad (4.12)$$

3. We show that there are admissible constants $\left(C_s^{(c)}\right)_{s\geq 0}$ such that for each $s \geq 0$ and each $|\alpha| > \lfloor s \rfloor$:

$$|\mu_{\alpha,i}(f^{(0)})||\partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_1) - \partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_2)||x_1 - x_2|^{-\{s\}} \leq C_s^{(c)}\|F\|_{s,K} \quad (4.13)$$

Starting with the proof of (4.11) we distinguish here between the two subcases, $|x_1 - x_2| \geq \gamma_i$ and $|x_1 - x_2| < \gamma_i$ for the estimation of the second factor in the left side of (4.11). To make the notation easier we set $C_{\lfloor s \rfloor} := \max_{|\beta| \leq \lfloor s \rfloor + 1} C_{\beta,\lfloor s \rfloor}$ where the $C_{\beta,\lfloor s \rfloor}$ are the constants derived in Lemma 4.5. If we have $|x_1 - x_2| \geq \gamma_i$ we compute with Lemma 4.5

$$|\partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_1) - \partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_2)| \leq 2C_{\lfloor s \rfloor}\gamma_i^{|\alpha|-|\beta|}, \quad (4.14)$$

and for the second case we apply the mean value theorem and again Lemma 4.5 to get

$$|\partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_1) - \partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha))(x_2)| \leq \sqrt{d}C_{\lfloor s \rfloor}\gamma_i^{|\alpha|-|\beta|-1}|x_1 - x_2|. \quad (4.15)$$

For the case $|x_1 - x_2| \geq \gamma_i$ we get therefore that $|x_1 - x_2|^{-\{s\}} \leq \gamma_i^{-\{s\}}$ and using (4.14) we compute

$$\begin{aligned}
&|\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i)||\partial^\beta(\varphi_i(x_1)(x_1 - x_i)^\alpha - \varphi_i(x_2)(x_2 - x_i)^\alpha)||x_1 - x_2|^{-\{s\}} \\
\leq &2C_{\lfloor s \rfloor}\frac{|\nu_{\alpha,x_i,\gamma_i} - \gamma_i^{|\alpha|} f^{(\alpha)}(x_i)|}{\gamma_i^s}.
\end{aligned}$$

For the case $|x_1 - x_2| < \gamma_i$ we compute with (4.15)

$$\begin{aligned}
&|\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i)||\partial^\beta(\varphi_i(x_1)(x_1 - x_i)^\alpha - \varphi_i(x_2)(x_2 - x_i)^\alpha)||x_1 - x_2|^{-\{s\}} \\
\leq &\sqrt{d}C_{\lfloor s \rfloor}\frac{|\nu_{\alpha,x_i,\gamma_i} - \gamma_i^{|\alpha|} f^{(\alpha)}(x_i)|}{\gamma_i^{\lfloor s \rfloor + 1}}|x_1 - x_2|^{1-\{s\}} \\
\leq &\sqrt{d}C_{\lfloor s \rfloor}\frac{|\nu_{\alpha,x_i,\gamma_i} - \gamma_i^{|\alpha|} f^{(\alpha)}(x_i)|}{\gamma_i^s}.
\end{aligned}$$

In both cases we can proceed now as in the proof of (4.8) and (4.11) is established. For the proof of (4.13) we proceed basically in the same way and for (4.12) we have in the case that $|x_1 - x_2| \geq \gamma_i$

$$|f^{(\alpha)}(x_i)||\partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_1) - \partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_2)||x_1 - x_2|^{-\{s\}}$$

$$\leq |||E_s(F)|||_{s,L} 2C_{\lfloor s\rfloor} \max_{i\leq\lfloor s\rfloor} \max_{i<|\alpha|\leq\lfloor s\rfloor} \gamma_i^{|\alpha|-s}$$

and in the case that $|x_1 - x_2| < \gamma_i$ we get

$$\begin{aligned}
&|f^{(\alpha)}(x_i)||\partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_1) - \partial^\beta(\varphi_i(\cdot)(\cdot - x_i)^\alpha)(x_2)||x_1 - x_2|^{-\{s\}} \\
\leq &|||E_s(F)|||_{s,L} C_{\lfloor s\rfloor}\gamma_i^{|\alpha|-|\beta|-1}|x_1 - x_2|^{1-\{s\}} \\
\leq &|||E_s(F)|||_{s,L} \sqrt{d} C_{\lfloor s\rfloor} \max_{i\leq\lfloor s\rfloor} \max_{i<|\alpha|\leq\lfloor s\rfloor} \gamma_i^{|\alpha|-s}.
\end{aligned}$$

It remains to handle the case $x_1 \in L \setminus K$ and $x_2 \in K$. But as seen above this case is even easier and can be proved in the same manner. □

4.5 Examples

In the following examples we focus on some well known fractal sets and we will use Corollary 4.2 to prove that they admit an extension operator without loss of derivatives.

4.8 Example. *1. Cantor set*

The Cantor set is the set of all points which is iteratively defined by the following process. We start with the interval $C_0 = [0, 1]$ and remove from it the middle third to get $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. In the next step we remove again from both remaining intervals the middle third to get $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ and so on. Continuing like this we get for each $n \in \mathbb{N}$ that $C_n = \frac{C_{n-1}}{3} \cup \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right)$. The Cantor set C is then defined as $C := \bigcap_{n\in\mathbb{N}_0} C_n$. To show that C admits an extension operator without loss of derivatives we prove that C fulfils property 7 of Corollary 4.2. Let $x_0 \in C$ and $\varepsilon > 0$. Then we have to find $x_1 \in C$ with $|x_0 - x_1| > \varrho\varepsilon$ for some $\varrho > 0$ which does not depend on x_0 or ε. Since C_n is always a union of disjoint intervals, there is for each $n \in \mathbb{N}_0$ an interval $I_n \subset C_n$ which contains x_0. We choose now n so large that $I_{n-1} \cap B(x_0, \varepsilon)^c \neq \emptyset$ and $I_n \cap B(x_0, \varepsilon)^c = \emptyset$. The situation is depicted in Figure 4.1.

We choose x_1 now as the boundary point of I_n which has the greatest distance to x_0. Since x_0 cannot belong to the middle third of I_n, x_1 is uniquely determined. Then we have

$$|x_0 - x_1| \geq \frac{2}{3}|I_n| = \frac{2}{9}|I_{n-1}| \geq \frac{2}{9}\varepsilon.$$

So we can choose $\varrho = \frac{2}{9}$.

2. *Generalized versions of the Cantor set*

There are many ways to generalize the definition of the Cantor set. For instance it

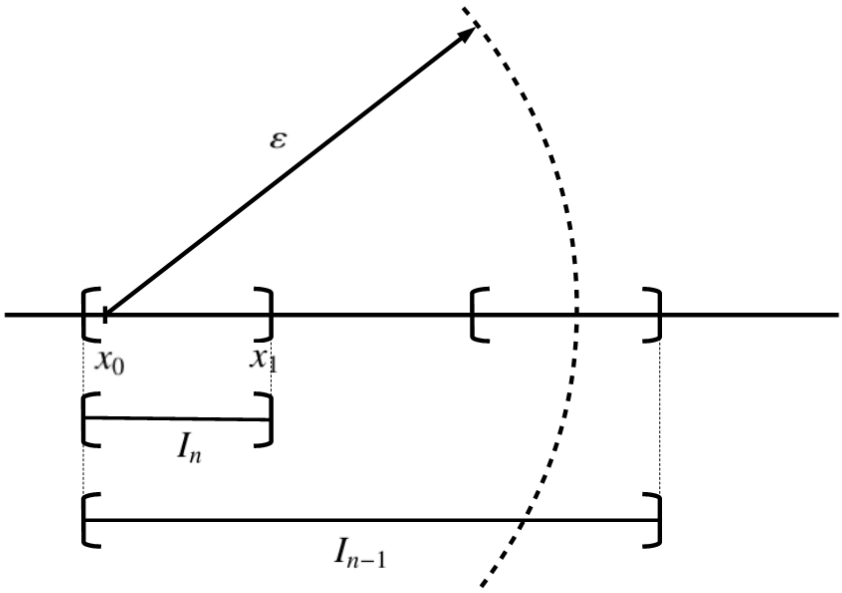

Figure 4.1: Choice of x_1 for the Cantor set

is possible to modify the length of middle interval which is taken out in each step. Instead of taking out one third of the original length, one could choose $l \in (0, 1)$ and take out from each Interval I a piece of length $l|I|$. The above proof remains valid also in this case and repeating the same steps yields $\varrho = \frac{1-l^2}{4}$.
A way to further generalize this setting would be to allow that l depends on the iteration step n, so you have a sequence $(l_n)_{n\in\mathbb{N}_0} \in (0, 1)^{\mathbb{N}_0}$. To be able to find a fixed ϱ in this case, 1 must not be an accumulation point of this sequence. This covers for instance the case of the Smith-Volterra-Cantor set where $l(n) = \frac{1}{4^{n+1}}$.

3. *Sierpinski triangle*
   The Sierpinski triangle is two dimensional analogon of the Cantor set which is constructed with triangles instead of intervals. For the iterative procedure we start with an equilateral triangle T_0. In the next step, T_0 is divided into 4 equilateral triangles of the same size and the middle one (without its boundary) is removed. The three remaining triangles form T_1. The same procedure is then applied to the three triangles to arrive at T_2 and so on. The first five iteration steps are depicted in Figure 4.2.
   The Sierpinski triangle is then again defined as $T := \bigcap_{n\in\mathbb{N}_0} T_n$. In order to prove that T admits an extension operator without loss of derivatives, we check again property 7 of Corollary 4.2. For that, we choose $x_0 \in T$ and $\varepsilon > 0$. We have to find $x_1, x_2 \in T$ and $\varrho > 0$ independent of x_0 and ε, such that

$$|x_0 - x_1| \geq \varrho\varepsilon, \quad dist\,(x_2, aff\{x_0, x_1\}) \geq \varrho\varepsilon.$$

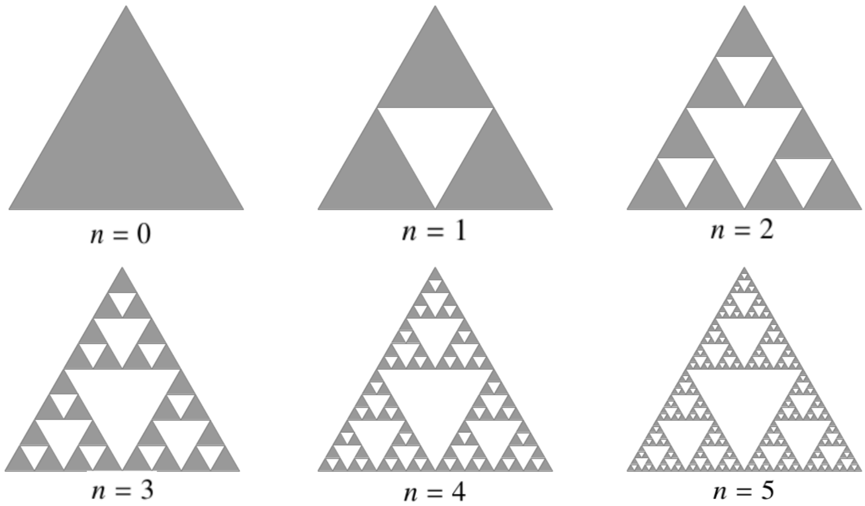

Figure 4.2: First steps in the construction of the Sierpinski triangle

Basically we follow the same approach as for the Cantor set. For x_0 we find a uniquely determined sequence of equilateral triangles $(\Delta_n)_{n\in\mathbb{N}_0}$ with $\Delta_0 = T_0$, such that $\Delta_n \supsetneq \Delta_{n+1}$ and $x_0 \in \Delta_n$ for each $n \in \mathbb{N}_0$. We choose $n \in \mathbb{N}_0$ with $\Delta_{n-1} \cap B(x_0, \varepsilon) \neq \emptyset$ and $\Delta_n \cap B(x_0, \varepsilon) = \emptyset$. Then we choose x_1 and x_2 to be the vertices of Δ_n which maximize the distance to x_0. Since x_0 cannot belong to the center of the circumscribed circle of Δ_n, both points are uniquely determined. The choice is visualized in Figure 4.3.
If we denote with $l(\Delta_n)$ the side length of Δ_n we have per construction $l(\Delta_n) = \frac{1}{2}l(\Delta_{n-1})$, and therefore

$$\begin{aligned} |x_0 - x_1| &\geq \frac{1}{2}l(\Delta_n) = \frac{1}{4}l(\Delta_n) > \frac{1}{4}\varepsilon, \\ dist\,(x_2, aff\{x_0, x_1\}) &\geq \frac{1}{2}l(\Delta_n) = \frac{1}{4}l(\Delta_n) > \frac{1}{4}\varepsilon. \end{aligned}$$

This shows that T fulfils property 7 of Corollary 4.2 with $\varrho = \frac{1}{4}$.

4. *Sierpinski tetrahedron*
The Sierpinski tetrahedron is the analogon of the Sierpinski triangle in three dimensions. It is constructed in the same way by starting with an equilateral tetrahedron and in each step the side lengths of the tetrahedra are divided by two. The first two iteration steps are shown in Figure 4.4. Since each side face of the tetrahedron is in each iteration step a Sierpinski triangle, an analogous choice of the points x_1, x_2, x_3 yields that the Sierpinski tetrahedron fulfils property 7 of Corollary 4.2 also with $\varrho = \frac{1}{4}$.

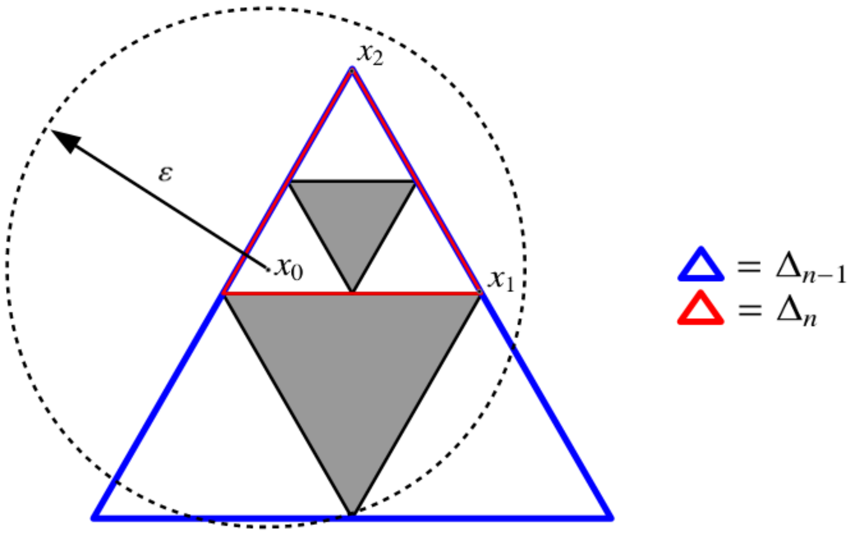

Figure 4.3: Choice of x_1 and x_2 for the Sierpinski triangle

All the above examples of fractal sets for which we rather manually verified the property 7 of Corollary 4.2 can also be treated with a more general approach to show that they have LMI(1). To describe this approach, we first have to introduce some basic concepts which play a central role in the theory of fractals. The first is the idea of a finite set of functions which is used to define a fractal set, but which also allows to iteratively approximate the set it defines.

4.9 Definition. *Let $D \subset \mathbb{R}^n$ be a closed set. Then a mapping $S : D \to D$ is called a contraction (contracting similarity) on D if there exists a $0 < r < 1$ such that for each $x, y \in D$ the following inequality holds*

$$|S(x) - S(y)| \leq (=) r|x - y|.$$

A finite set of contractions is called an iterated function system of short IFS. A set $F \subset \mathbb{R}^n$ is called attractor of an IFS $\{S_1, ..., S_m\}$ if

$$F = \bigcup_{i=1}^{m} S_i(D).$$

A fundamental property of an IFS is that it uniquely determines a non-empty and compact attractor, see for instance Theorem 9.1 in [Fal14] or Theorem 4.2 in [Tri11]. All the fractal sets we had in our last example can be defined via such an IFS where each

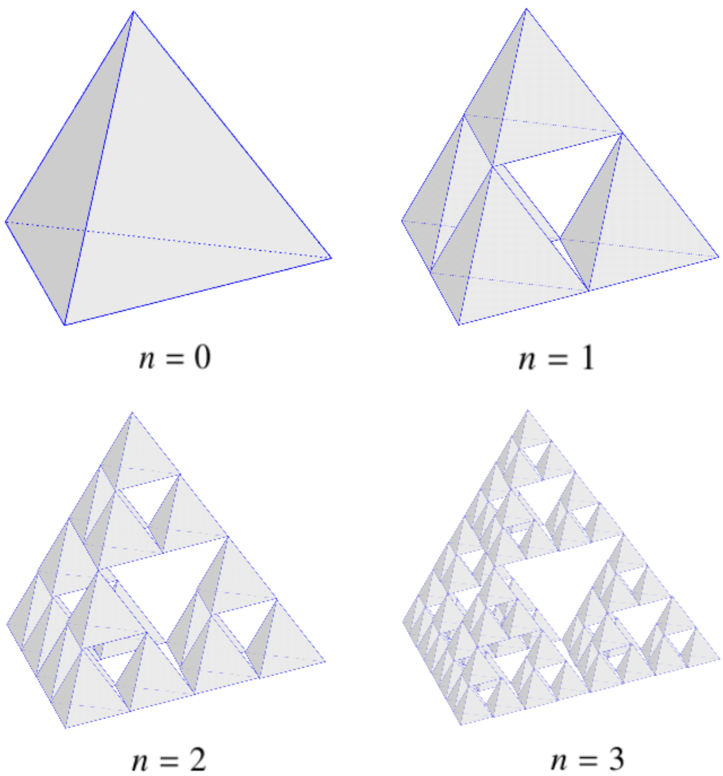

Figure 4.4: First steps in the construction of the Sierpinski tetrahedron

mapping is even a contracting similarity. For instance the Cantor set is the attractor of the IFS $\{S_1, S_2\}$, where $S_1 : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto \frac{1}{3}x$ and $S_2 : \mathbb{R}^n \to \mathbb{R}^n, x \mapsto \frac{1}{3}x + \frac{2}{3}$.

A very interesting property of such an IFS is, that it is not only possible to define a fractal set as attractor, but it is also possible to approximate the attractor by iteratively applying the mappings on a starting set. The starting set can indeed by any non-empty compact subset K of $\mathbb{R}^n$. This construction can be formalized in the following way for an IFS $\{S_1, ..., S_m\}$ and K compact:

$$S^0(K) := K, S^1(K) := \bigcup_{i=1}^{m} S_i(K), ..., S^k(K) := S^1(S^{k-1})(K).$$

Then it is true that $S^\infty(K) := \lim_{k\to\infty} S^k(K)$ exists in the metric space of all non-empty compact subsets of $\mathbb{R}^d$ equipped with the Hausdorff metric, and furthermore $S^\infty(K) = F$. For our purposes we need the following additional property of an IFS and its attractor.

4.10 Definition. *Let $\{S_1, ..., S_m\}$ be an IFS with attractor F. Then the IFS fulfils the* *open set condition if there exists a non-empty open set $O \subset \mathbb{R}^n$ such that*

$$\bigcup_{i=1}^{m} S_i(O) \subset O \text{ and } S_i(O) \cap S_j(O) = \emptyset \text{ for all } i \neq j.$$

The open set condition ist not as restrictive as it may seem. Again all our previously mentioned examples fulfill this condition. For the Cantor set we can choose the open interval $(0, 1)$. For the Sierpinski triangle (tetrahedron) one can take the interior of the triangle (tetrahedron) with which we started the iterative construction. Even the Koch curve fulfills this requirement as can be seen in Example 9.5 in [Fal14]

4.11 Definition. *Given an IFS of contraction similarities $\{S_m, ..., S_m\}$ with ratios $r_1, ..., r_m$ and attractor F, the unique solution d of the equation*

$$\sum_{i=1}^{m} r_i^d = 1$$

denotes the similarity dimension of F.

According to Theorem 9.3 in [Fal14], the similarity dimension d fulfils

$$d = \dim_H(F) = \dim_B(F),$$

where $\dim_H(F)$ denotes the Hausdorff dimension of F and $\dim_B(F)$ denotes the Box dimension of F. The last concept we have to introduce before we can apply the result of [JSW84] is the concept of d-sets.

4.12 Definition. *Let $M \subset \mathbb{R}^n$ and $0 \leq d \leq n$. Then M is called a d-set if there exists a Borel measure μ in $\mathbb{R}^n$ having the following properties:*

1. *$supp(\mu) = M$,*
2. *There exists constants $c_1, c_2 > 0$ such that for all $x \in M$ and all $0 < r < 1$*

$$c_1 r^d \leq \mu(B(x, r) \cap M) \leq c_2 r^d.$$

For those d-sets Jonsson, Sjögren and Walling published the following result in [JSW84].

4.13 Theorem. *If $F \subset \mathbb{R}^n$ is a d-set with $d > n - 1$, then F fulfils LMI(1).*

On the other hand, there is the following result which can be found in Theorem 4.7 in [Tri11].

4.14 Theorem. *Let $\{S_1, ..., S_n\}$ be an IFS of contracting similarities with attractor F that fulfils the open set condition and has similarity dimension d. Then F is self-similar and a compact d-set.*

Putting both theorems together, we get that each compact set $F \subset \mathbb{R}^n$, which is the attractor of an IFS of contracting similarities, which fulfils the open set condition and has similarity dimension $d > n - 1$, admits an extension operator without loss of derivatives. For instance the Koch curve in $\mathbb{R}^2$ fulfils all these requirements with $d = \frac{\log 4}{\log 3} \approx 1.26 > 1$ and therefore admits an extension operator without loss of derivatives.

Chapter 5

Tame Linear Extension Operators

In this chapter we show that the tame linear extension operator constructed in [FJW11] is also continuous on the real scale. Furthermore, in the last section of this chapter we present a modified construction of Whitney's finite order extension operators E_s. Frerick, Jordá and Wengenroth prove in [FJW11] that a compact set K fulfils WLMI(r) for some $r \geq 1$ if and only if it admits a tame linear extension operator. We introduced the weak local Markov inequality, or for short WLMI, already at the beginning of the last chapter. Tame linear means in this context, that the extension operator E fulfils the following continuity estimates for each $F \in \mathscr{E}(K), \varepsilon > 0$ and $m \in \mathbb{N}_0$

$$|E(F)|_m \leq C_{m,\varepsilon} \|F\|_{(r+\varepsilon)m}. \tag{5.1}$$

For a general definition of tame linear operators between Fréchet spaces see for instance [Vog87].

5.1 Statement of the Main Result

Our main result in this chapter is the transfer of the main result in [FJW11] from the natural to the real scale and reads as follows.

5.1 Theorem. *For a compact set $K \subset \mathbb{R}^d$, the following statements are equivalent.*

1. *K fulfils the WLMI(r) for some $r \geq 1$.*
2. *K admits a tame linear extension operator $E : \mathscr{E}(K) \to \mathscr{E}(\mathbb{R}^d)$ which extends simultaneously all $\mathscr{E}^s(K)$, and which fulfils the following continuity estimates for all convex and compact set $L \subset \mathbb{R}^d$ with $\mathring{L} \supset K$, each $F \in \mathscr{E}(K), \varepsilon > 0$ and $s \geq 0$*

$$|||E(F)|||_{s,L} \leq C_{s,\varepsilon} \|F\|_{(r+\varepsilon)s,K}. \tag{5.2}$$

5.2 Proof of the Main Result

For the proof we first note, that we only have to construct the operator given K fulfils WLMI(r). To achieve that we show that for $\mathring{L} \supset K$ compact and convex the following inequalities hold for each $F \in \mathscr{E}(K), \varepsilon > 0$ and $s \in [0, \infty)$:

$$|||E(F) - E_{\lfloor rs \rfloor}(F)|||_{s,L} \leq C_{s,\varepsilon} \|F\|_{(r+\varepsilon)s,K}, \tag{5.3}$$

where again $E_{\lfloor rs \rfloor} : \mathscr{E}^{\lfloor rs \rfloor}(K) \to \mathscr{D}^{\lfloor rs \rfloor}(L)$ denotes the Whitney operator. Since

$$\begin{aligned} & |||E(F) - E_{\lfloor rs \rfloor}(F)|||_{s,L} \\ = \; & |E(F) - E_{\lfloor rs \rfloor}(F)|_{\lfloor s \rfloor,L} \\ + \; & \sup_{t>0} \left\{ \frac{|\partial^\beta (E(F) - E_{\lfloor rs \rfloor}(F))(x_1) - \partial^\beta (E(F) - E_{\lfloor rs \rfloor}(F))(x_2)|}{|x_1 - x_2|^{\{s\}}} : \right. \\ & \left. x_1, x_2 \in L, 0 < |x_1 - x_2| \leq t, |\beta| = \lfloor s \rfloor \right\}, \end{aligned}$$

we only have to prove inequality (5.3) for the second summand. In the following calculation we will only focus on the case that $x_1, x_2 \in L \setminus K$, the cases that one of the points or even both belong to K are easier respectively trivial. In order to achieve a shorter notation we set in the following

$$\varphi_{i,\alpha}(x) := \varphi_i(x)(x - x_i)^\alpha.$$

We first prove the following estimation which holds for all families $(m(i))_{i\in\mathbb{N}}$, all $\beta \in \mathbb{N}_0^d$ and all families of complex numbers $(a_{\alpha,i})_{\alpha\in\mathbb{N}_0^d, i\in\mathbb{N}}$

$$\sum_{i=1}^{\infty} \sum_{0<|\alpha|\leq m(i)} |a_{\alpha,i}| \left|\partial^\beta \varphi_{i,\alpha}(x_1) - \partial^\beta \varphi_{i,\alpha}(x_2)\right| |x_1 - x_2|^{|\beta|-s} \leq C_\beta^{(3)} \sup_{i\in\mathbb{N}} C_{m(i)}^{(2)} \sup_{0<|\alpha|\leq m(i)} |a_{\alpha,i}| \gamma_i^{|\alpha|-s}. \tag{5.4}$$

To show this, we distinguish the cases $|x_1 - x_2| \geq \gamma_i$ and $|x_1 - x_2| < \gamma_i$. For the first case we get with (4.14) a positive constant $C_{\lfloor s \rfloor}$ with

$$\begin{aligned} \left|\partial^\beta \varphi_{i,\alpha}(x_1) - \partial^\beta \varphi_{i,\alpha}(x_2)\right| |x_1 - x_2|^{|\beta|-s} &\leq C_\beta \gamma_i^{|\alpha|-|\beta|} |x_1 - x_2|^{|\beta|-s} \\ &\leq C_\beta \gamma_i^{|\alpha|-s}. \end{aligned}$$

For the second case we compute with (4.15)

$$\begin{aligned} \left|\partial^\beta \varphi_{i,\alpha}(x_1) - \partial^\beta \varphi_{i,\alpha}(x_2)\right| |x_1 - x_2|^{|\beta|-s} &\leq C_\beta \gamma_i^{|\alpha|-|\beta|-1} |x_1 - x_2|^{1+|\beta|-s} \\ &\leq C_\beta \gamma_i^{|\alpha|-|\beta|-1} \gamma_i^{1+|\beta|-s} \\ &= C_\beta \gamma_i^{|\alpha|-s}. \end{aligned}$$

Thus, we get

$$\sum_{i=1}^{\infty}\sum_{0<|\alpha|\le m(i)} |a_{\alpha,i}|\left|\partial^{\beta}\varphi_{i,\alpha}(x_1)-\partial^{\beta}\varphi_{i,\alpha}(x_2)\right||x_1-x_2|^{|\beta|-s}$$
$$\le\ 2NC_{\beta}\sup_{i\in\mathbb{N}}\binom{d+m(i)-1}{m(i)}\sup_{0<|\alpha|\le m(i)}|a_{\alpha,i}|\gamma_i^{|\alpha|-s},$$

where N denotes as usual the maximum number of supports of the φ_i which can contain either x_1 or x_2, and $\binom{d+m(i)-1}{m(i)}$ is the number of multiindices with norm smaller or equal to $m(i)$. By setting $C^{(2)}_{m(i)} := \binom{d+m(i)-1}{m(i)}$ and $C^{(3)}_{\beta} := 2NC_{\beta}$ we have proved (5.4). From (5.4) we can also conclude that

$$\limsup_{x\to\partial K}\sup_{|\beta|\le\lfloor s\rfloor+1}\sum_{i=1}^{\infty}\sum_{0<|\alpha|\le m(i)}|a_{\alpha,i}|\left|\partial^{\beta}\varphi_{i,\alpha}(x_1)-\partial^{\beta}\varphi_{i,\alpha}(x_2)\right||x_1-x_2|^{|\beta|-s} \tag{5.5}$$
$$\le\ C^{(3)}_{\lfloor s\rfloor+1}\limsup_{i\to\infty}C^{(2)}_{m(i)}\sup_{0<|\alpha|\le m(i)}|a_{\alpha,i}|\gamma_i^{|\alpha|-s}.$$

According to the proof in [FJW11], the families $(\delta_i)_{i\in\mathbb{N}}$ and $(m(i))_{i\in\mathbb{N}}$ are chosen in a way that

$$\lim_{i\to\infty}C^{(1)}_{r+\frac{\delta_i}{2},m(i)}C^{(2)}_{m(i)}\gamma_i^{\frac{1}{2}\frac{\delta_i}{r+\delta_i}}=0.$$

Since $\lim_{i\to\infty}\frac{\delta_i}{r+\delta_i}=0$ we get for arbitrary $\varepsilon>0$ that $\frac{\delta_i}{r+\delta_i}<\frac{\varepsilon s}{r}$ for i big enough, and therefore we have

$$\sup_{i\in\mathbb{N}}C^{(1)}_{r+\frac{\delta_i}{2},m(i)}C^{(2)}_{m(i)}\gamma_i^{\frac{\varepsilon s}{2r}}<\infty. \tag{5.6}$$

Following Lemma 2 in [FJW11] we can choose measures $\mu_{\alpha,i}=\mu_{\alpha,x_0,\varepsilon(\alpha,i),m(i)}$ according to $\varepsilon(\alpha,i):=\gamma_i^{\frac{1}{r+\delta_i}}$ such that $\partial^{\alpha}P(x_0)=\mu_{\alpha,i}(P)$ for all polynomials P with $deg(P)\le m(i)$ and total variation $|\mu_{\alpha,i}|\le C^{(1)}_{r+\frac{\delta_i}{2},m(i)}\gamma_i^{-|\alpha|\frac{r+\frac{\delta_i}{2}}{r+\delta_i}}$. For those measures we get the following two inequalities. We assume that $\lfloor(r+\varepsilon)s\rfloor=\lfloor rs\rfloor+\varepsilon s$ so that the order of the Taylor polynomial fits to the order of the Whitney norm. The first one is true for $0<|\alpha|\le\lfloor rs\rfloor\le m(i)$:

$$|\mu_{\alpha,i}(f^{(0)})-f^{(\alpha)}(x_i)|=|\mu_{\alpha,i}(f^{(0)}-T^{\lfloor rs\rfloor}_{x_i}(F))|$$
$$\le\ C^{(1)}_{r+\frac{\delta_i}{2},m(i)}\gamma_i^{-|\alpha|\frac{r+\frac{\delta_i}{2}}{r+\delta_i}}\sup\left\{|f^0(x)-T^{\lfloor rs\rfloor}_{x_i}(F)(x)|:x\in B(x_i,\gamma_i^{\frac{1}{r+\delta_i}})\cap K\right\}$$
$$\le\ C^{(1)}_{r+\frac{\delta_i}{2},m(i)}\gamma_i^{-|\alpha|\frac{r+\frac{\delta_i}{2}}{r+\delta_i}}\sup\left\{\frac{|f^0(x)-T^{\lfloor rs\rfloor}_{x_i}(F)(x)|}{|x-x_i|^{(r+\varepsilon)s}}|x-x_i|^{(r+\varepsilon)s}:x\in B(x_i,\gamma_i^{\frac{1}{r+\delta_i}})\cap K\right\}$$

$$
\begin{aligned}
&\leq\; C^{(1)}_{r+\frac{\delta_i}{2},m(i)}\gamma_i^{-|\alpha|\frac{r+\frac{\delta_i}{2}}{r+\delta_i}}\gamma_i^{\frac{1}{r+\delta_i}(r+\varepsilon)s}\sup\left\{\frac{|f^0(x)-T^{\lfloor rs\rfloor}_{x_i}(F)(x)|}{|x-x_i|^{(r+\varepsilon)s}} : x\in B(x_i,\gamma_i^{\frac{1}{r+\delta_i}})\cap K\right\}\\
&\leq\; C^{(1)}_{r+\frac{\delta_i}{2},m(i)}\gamma_i^{-|\alpha|}\gamma_i^{\frac{rs+\varepsilon s}{r+\delta_i}}\|F\|_{(r+\varepsilon)s}.
\end{aligned}
\tag{5.7}
$$

The second inequality is valid for $\lfloor rs\rfloor < |\alpha| \leq m(i)$ and it can be proven in the same way as the first one using that for $|\alpha| > \lfloor rs\rfloor$ we have $\mu_{\alpha,i}(T^{\lfloor rs\rfloor}_{x_i}(F)(x)) = \partial^\alpha T^{\lfloor rs\rfloor}_{x_i}(F)(x) = 0$:

$$
|\mu_{\alpha,i}(f^{(0)})| = |\mu_{\alpha,i}(f^{(0)} - T^{\lfloor rs\rfloor}_{x_i}(F)(x))| \leq C^{(1)}_{r+\frac{\delta_i}{2},m(i)}\gamma_i^{-|\alpha|}\gamma_i^{\frac{rs+\varepsilon s}{r+\delta_i}}\|F\|_{(r+\varepsilon)s}. \tag{5.8}
$$

In this setting the operator E is defined at $x \in L \setminus K$ as

$$
E(F)(x) := \sum_{i\in\mathbb{N}}\sum_{|\alpha|\leq m(i)}\frac{1}{\alpha!}\mu_{\alpha,i}(f^{(0)})\varphi_{i,\alpha}(x).
$$

Hence we get as difference

$$
E(F)(x) - E_{\lfloor rs\rfloor}(F)(x) = T_{1,s}(F)(x) + T_{2,s}(F)(x),
$$

where

$$
\begin{aligned}
T_{1,s}(F)(x) := \sum_{i\geq j(s)}\Bigg(&\sum_{0<|\alpha|\leq\lfloor rs\rfloor}\frac{1}{\alpha!}\left(\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i)\right)\varphi_{i,\alpha}(x)\\
&+\sum_{\lfloor rs\rfloor<\alpha|\leq m(i)}\frac{1}{\alpha!}\mu_{\alpha,i}(f^{(0)})\varphi_{i,\alpha}(x)\Bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
T_{2,s}(F)(x) := \sum_{i=1}^{j(s)-1}\Bigg(&\sum_{0<|\alpha|\leq\lfloor rs\rfloor}\frac{1}{\alpha!}\left(\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i)\right)\varphi_{i,\alpha}(x)\\
&+\sum_{m(i)<\alpha|\leq\lfloor rs\rfloor}\frac{-f^{(\alpha)}(x_i)}{\alpha!}\varphi_{i,\alpha}(x)\Bigg).
\end{aligned}
$$

The index $j(s)$ is defined as the smallest index such that $m(i) \geq \lfloor rs\rfloor$ for $i \geq j(s)$. So we get for $x_1, x_2 \in L \setminus K$

$$
\begin{aligned}
&\left|\partial^\beta E(F)(x_1) - \partial^\beta E_{\lfloor rs\rfloor}(F)(x_1) - \left(\partial^\beta E(F)(x_2) - \partial^\beta E_{\lfloor rs\rfloor}(F)(x_2)\right)\right|\\
&\leq \left|\partial^\beta T_{1,s}(F)(x_1) - \partial^\beta T_{1,s}(F)(x_2)\right| + \left|\partial^\beta T_{2,s}(F)(x_1) - \partial^\beta T_{2,s}(F)(x_2)\right|.
\end{aligned}
$$

Starting with the first summand we have, using (5.4), (5.6), (5.7) and (5.8) for $|\beta| = \lfloor s \rfloor$:

$$\begin{aligned}
&\left|\partial^\beta T_{1,s}(F)(x_1) - \partial^\beta T_{1,s}(F)(x_2)\right| |x_1 - x_2|^{|\beta|-s} \\
\leq\ & \sum_{i=j(s)}^{\infty} \Bigg(\sum_{0<|\alpha|\leq\lfloor rs\rfloor} \frac{1}{\alpha!} \left|\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i)\right| \left|\partial^\beta \varphi_{i,\alpha}(x_1) - \partial^\beta \varphi_{i,\alpha}(x_2)\right| \\
&+ \sum_{\lfloor rs\rfloor<|\alpha|\leq m(i)} \frac{1}{\alpha!} |\mu_{\alpha,i}(f^{(0)})| \left|\partial^\beta \varphi_{i,\alpha}(x_1) - \partial^\beta \varphi_{i,\alpha}(x_2)\right| \Bigg) |x_1 - x_2|^{|\beta|-s} \\
\leq\ & \sum_{i=j(s)}^{\infty} \Bigg(\sum_{0<|\alpha|\leq\lfloor rs\rfloor} \frac{1}{\alpha!} C^{(1)}_{r+\frac{\delta_i}{2},m(i)} \gamma_i^{-|\alpha|} \gamma_i^{\frac{rs+\varepsilon s}{r+\delta_i}} \|F\|_{(r+\varepsilon)s} \left|\partial^\beta \varphi_{i,\alpha}(x_1) - \partial^\beta \varphi_{i,\alpha}(x_2)\right| |x_1 - x_2|^{|\beta|-s} \\
&+ \sum_{\lfloor rs\rfloor<|\alpha|\leq m(i)} \frac{1}{\alpha!} C^{(1)}_{r+\frac{\delta_i}{2},m(i)} \gamma_i^{-|\alpha|} \gamma_i^{\frac{rs+\varepsilon s}{r+\delta_i}} \|F\|_{(r+\varepsilon)s} \left|\partial^\beta \varphi_{i,\alpha}(x_1) - \partial^\beta \varphi_{i,\alpha}(x_2)\right| |x_1 - x_2|^{|\beta|-s} \Bigg) \\
\leq\ & 2 \sum_{i\in\mathbb{N}} \Bigg(\sum_{0<|\alpha|\leq m(i)} \frac{1}{\alpha!} C^{(1)}_{r+\frac{\delta_i}{2},m(i)} \gamma_i^{-|\alpha|} \gamma_i^{\frac{rs+\varepsilon s}{r+\delta_i}} \|F\|_{(r+\varepsilon)s} \left|\partial^\beta \varphi_{i,\alpha}(x_1) - \partial^\beta \varphi_{i,\alpha}(x_2)\right| |x_1 - x_2|^{|\beta|-s} \\
\leq\ & 2C^{(3)}_\beta \|F\|_{(r+\varepsilon)s} \sup_{i\in\mathbb{N}} C^{(1)}_{r+\frac{\delta_i}{2},m(i)} \gamma_i^{\frac{rs+\varepsilon s}{r+\delta_i}} C^{(2)}_{m(i)} \sup_{0<|\alpha|\leq m(i)} \gamma_i^{-|\alpha|} \gamma_i^{|\alpha|-s} \\
=\ & 2C^{(3)}_\beta \|F\|_{(r+\varepsilon)s} \sup_{i\in\mathbb{N}} C^{(1)}_{r+\frac{\delta_i}{2},m(i)} C^{(2)}_{m(i)} \gamma_i^{\frac{\varepsilon s-\delta_i s}{r+\delta_i}}.
\end{aligned}$$

The last supremum is finite because $\gamma_i^{\frac{\varepsilon s}{2r}} < \gamma_i^{\frac{\varepsilon s-\delta_i s}{r+\delta_i}} < \gamma_i^{\frac{\varepsilon s}{r}}$ for i large enough. The second summand can be treated in the same manor, and both together prove for $|\beta| = \lfloor s \rfloor$ the desired continuity estimate (5.3). Proceeding analogously as above, but using (5.5) yields for all $\beta \in \mathbb{N}_0^d$:

$$\begin{aligned}
\lim_{x\to\partial K} \left|\partial^\beta E(F)(x) - \partial^\beta E_{\lfloor rs\rfloor}(F)(x)\right| \operatorname{dist}(x,K)^{|\beta|-s} &\leq 2C^{(3)}_\beta \|F\|_{(r+\varepsilon)s} \lim_{i\to\infty} C^{(1)}_{r+\frac{\delta_i}{2},m(i)} C^{(2)}_{m(i)} \gamma_i^{\frac{\varepsilon s-\delta_i s}{r+\delta_i}} \\
&= 0.
\end{aligned}$$

With Proposition 2.15 and 2.16 we can finally conclude that $E(F) \in \mathscr{E}^s(\mathbb{R}^d)$.

5.3 A Modified Construction of Whitney's Operators

In this section we present a modified construction of Whitney's finite order extension operators $E_s : \mathscr{E}^s(K) \to \mathscr{E}^s(\mathbb{R}^d)$. The motivation for this construction mainly is the ε in our Theorem 5.1 and the aim to get rid of it. In order to get better continuity estimates, it could be beneficial to have more degrees of freedom in the construction. Theorem 4.9 in [Fre07a] offers such a possibility.

5.2 Theorem. *Let $K \subset \mathbb{R}^d$ be compact and assume that there are $\varepsilon_0 > 0, r \geq 1$ such that for every $k \in \mathbb{N}$ there is $C \geq 1$ such that for all $\varepsilon_0 > \varepsilon > 0$ and all $z \in \partial K$ there is $x \in K$ with $|x - z| < \varepsilon$ and*

$$|\partial^\alpha p(x)| \leq \frac{C}{\varepsilon^{r|\alpha|}} \sup_{|y-z|\leq\varepsilon, y\in K} |p(y)|$$

for all $p \in \mathbb{C}[x_1, ..., x_n], deg(p) \leq k, |\alpha| \leq k$. Then $\mathscr{E}(K)$ has (DN).

By Theorem 3.3 in the same paper, the condition (DN) for K is in fact equivalent to the existence of an extension operator $E : \mathscr{E}(K) \to \mathscr{E}(\mathbb{R}^d)$. But it is unknown which loss of continuity this operator has. Our original intent to prove a version of Theorem 5.1 for $\varepsilon = 0$ applying the same techniques failed. But we are able to present a modified construction of the operators E_s in the hope that it can be helpful for future results. In the classical construction, first the Whitney decomposition is constructed which gives a family $(\varphi_i)_{i\in\mathbb{N}}$ and then for each $i \in \mathbb{N}$ points $x_i \in \partial K$ are chosen, such that $\text{dist}(\text{supp}(\varphi_i), x) = \text{dist}(\text{supp}(\varphi_i), K)$. At these points we center the Taylor polynomials. In the new construction we will not center the Taylor polynomials at the points x_i but instead we will use the condition from Theorem 5.2 to choose for each of the points x_i another $\tilde{x}_i \in K$ with $|x_i - \tilde{x}_i| < \gamma_i^{1/r}$. If we check the proofs of the Theorems 4.1, 5.1 and also 6.2, the desired properties for the constructed extension operators E are always shown for the difference $E - E_s$. To get an easy expression for those differences, it is very convenient if both operators are 'centred' at the same points of K. For $s \geq 0$ and $F \in \mathscr{E}^s(K)$ the modified Whitney operator is then defined as

$$E(F)(x) = \begin{cases} f^{(0)}(x), x \in K \\ \sum_{i\in I} \varphi_i(x) T_{\tilde{x}_i}^{\lfloor s \rfloor}(F)(x), x \notin K \end{cases}.$$

In the following theorem we show that this so constructed operator maps $\mathscr{E}^{rs}(K)$ continuously into $\mathscr{E}^s(K)$.

5.3 Theorem. *Let $K \subset \mathbb{R}^d$ be compact. If K fulfils the conditions of Theorem 5.2 and we adjust the construction of the Whitney operator as described above, then this operator maps $\mathscr{E}^{rs}(K)$ continuously into $\mathscr{E}^s(\mathbb{R}^d)$ for each $s \geq 0$.*

Proof. In the following let $L \subset \mathbb{R}^d$ be an open cube such that $K \subset \mathring{L}$. We will apply the partition of unity $(\varphi_i)_{i\in I}$ only on $L \setminus K$. We structure the following proof into four parts.

1. We prove the existence of a constant C depending only on $\lfloor s \rfloor, d$ and λ and a modulus of continuity μ such that for every $|\beta| \leq \lfloor s \rfloor$, for $a \in K$ with $|x - a| = \text{dist}(x, K)$, $x \in L$, one has:

$$|\partial^\beta E(F)(x) - \partial^\beta T_a^{\lfloor s \rfloor}(F)(x)| \leq C\mu(|x - a|)|x - a|^{s-|\beta|}. \tag{5.9}$$

2. We show that for each $|\beta| = \lfloor s \rfloor + 1$ there is a constant C depending only on $\lfloor s \rfloor, \lambda$ and a modulus of continuity μ such that

$$|\partial^\beta E(F)(x)| \leq C\mu(\operatorname{dist}(x, K))\operatorname{dist}(x, K)^{s-|\beta|}. \tag{5.10}$$

3. We show that $E(F) \in \mathscr{D}^s(L)$.

4. We show that $E : \mathscr{E}^{rs}(K) \to \mathscr{D}^s(L)$ is continuous. Furthermore, we show that the continuity constants C_s, for which the inequality $|||E(F)|||_{s,L} \leq C_s \|F\|_{rs,K}$ is true for every $F \in \mathscr{E}^{rs}(K)$, can be chosen such that the mapping $s \mapsto C_s$ depends continuously on s on each interval $[n, n+1)$ for each $n \in \mathbb{N}_0$ and is bounded on these intervals.

1. Starting with the proof of (5.9), we first note that without loss of generality we can limit our calculations to those $x \in K^c$ with $\operatorname{dist}(x, K) \leq 1$. An application of Leibniz's formula shows that for each $x \in L \setminus K$ we have

$$\partial^\beta E(F)(x) - \partial^\beta T_a^{\lfloor s \rfloor}(F)(x) = \sum_{i \in \mathbb{N}} \sum_{l \leq \beta} \binom{\beta}{l} \partial^{\beta - l}\varphi_i(x)\partial^l \left(T_{\tilde{x}_i}^{\lfloor rs \rfloor}(F)(x) - T_a^{\lfloor s \rfloor}(F)(x)\right).$$

We observe that

$$T_a^{\lfloor s \rfloor}(F)(x) = T_a^{\lfloor rs \rfloor}(F)(x) - \sum_{\lfloor s \rfloor + 1 \leq |\alpha| \leq \lfloor rs \rfloor} \frac{f^{(\alpha)}(a)}{\alpha!}(x-a)^\alpha,$$

and thus we have for $|\beta| \leq \lfloor s \rfloor + 1$

$$\partial^\beta T_a^{\lfloor s \rfloor}(F)(x) = \partial^\beta T_a^{\lfloor rs \rfloor}(F)(x) - \sum_{\lfloor s \rfloor + 1 - |\beta| \leq |\alpha| \leq \lfloor rs \rfloor - |\beta|} \frac{f^{(\alpha+\beta)}(a)}{\alpha!}(x-a)^\alpha.$$

For $|\beta \leq \lfloor s \rfloor$ we then obtain

$$\begin{aligned}
&\left|\partial^{\beta - l}\varphi_i(x)\partial^l \left(T_{\tilde{x}_i}^{\lfloor rs \rfloor}(F)(x) - T_a^{\lfloor s \rfloor}(F)(x)\right)\right| \\
\leq\ &\operatorname{dist}(x, K)^{|l|-|\beta|} \left|\partial^l T_{\tilde{x}_i}^{\lfloor rs \rfloor}(F)(x) - \partial^l T_a^{\lfloor rs \rfloor}(F)(x)\right| \\
&+ \operatorname{dist}(x, K)^{|l|-|\beta|} \left| \sum_{\lfloor s \rfloor + 1 - |l| \leq |\alpha| \leq \lfloor rs \rfloor - |l|} \frac{f^{(\alpha)}(a)}{\alpha!}(x-a)^\alpha \right|.
\end{aligned}$$

From $|x_i - \tilde{x}_i| \leq \gamma_i^{\frac{1}{r}} \leq \operatorname{dist}(x, K)^{\frac{1}{r}}$ and $\operatorname{dist}(x, K) = |x - a| \leq 1$ it follows that

$$|x - \tilde{x}_i| \leq |x - x_i| + |x_i - \tilde{x}_i| \leq 3\operatorname{dist}(x, K) + \operatorname{dist}(x, K)^{\frac{1}{r}} \leq 4\operatorname{dist}(x, K)^{\frac{1}{r}} = 4|x - a|^{\frac{1}{r}},$$

and

$$|a - \tilde{x}_i| \leq |a - x| + |x - \tilde{x}_i| \leq 5|a - x|^{\frac{1}{r}}.$$

For the first of the two summands we get by an application of Theorem 2.14.3 that there exists a modulus of continuity μ_1 such that

$$\begin{aligned}
&\operatorname{dist}(x, K)^{|l|-|\beta|} \left|\partial^l T_{\tilde{x}_i}^{\lfloor rs \rfloor}(F)(x) - \partial^l T_a^{\lfloor rs \rfloor}(F)(x)\right| \\
\leq\ &\operatorname{dist}(x, K)^{|l|-|\beta|} \mu_1(a - \tilde{x}_i)(|x - \tilde{x}_i|^{rs-|l|} + |a - x|^{rs-|l|}) \\
\leq\ &\operatorname{dist}(x, K)^{|l|-|\beta|} \mu_1(5|a - x|^{\frac{1}{r}})(4^{rs-|l|}|a - x|^{\frac{1}{r}rs-\frac{1}{r}|l|} + |a - x|^{rs-|l|}) \\
\leq\ &\operatorname{dist}(x, K)^{|l|-|\beta|} \mu_1(5|a - x|^{\frac{1}{r}})(4^{rs}|a - x|^{s-|l|} + |a - x|^{s-|l|}) \\
\leq\ &4^{rs}\mu_1(5|a - x|^{\frac{1}{r}})|a - x|^{s-|\beta|},
\end{aligned}$$

which proves (5.9) for the first summand. For the second summand using that $|x-a| \leq 1$ and $|l| \leq \lfloor s \rfloor$, we calculate

$$\begin{aligned}
&\operatorname{dist}(x, K)^{|l|-|\beta|} \left| \sum_{\lfloor s \rfloor+1-|l| \leq |\alpha| \leq \lfloor rs \rfloor - |l|} \frac{f^{(\alpha)}(a)}{\alpha!}(x - a)^\alpha \right| \\
\leq\ &\operatorname{dist}(x, K)^{|l|-|\beta|} C|F|_{\lfloor rs \rfloor, K}|a - x|^{\lfloor s \rfloor+1-|l|} \\
=\ &C|F|_{\lfloor rs \rfloor, K}\operatorname{dist}(x, K)^{|l|-|\beta|} |a - x|^{s-|l|}|a - x|^{\lfloor s \rfloor+1-s} \\
=\ &C|F|_{\lfloor rs \rfloor, K}|a - x|^{s-|\beta|}|a - x|^{\lfloor s \rfloor+1-s}.
\end{aligned}$$

As constant C we can simply choose $\sum_{|\alpha| \leq \lfloor rs \rfloor} \frac{1}{\alpha!}$ which only depends on $\lfloor rs \rfloor$ and d and since $\lfloor s \rfloor + 1 - s > 0$ we can directly take $\mu_2 : [0, \infty) \to \mathbb{R}, x \mapsto C|F|_{\lfloor rs \rfloor, K}x^{\lfloor s \rfloor+1-s}$ as modulus of continuity in this case. Then choosing a modulus of continuity μ which fulfils

$$\mu(x) \geq \max(4^{rs}\mu_1(5|a - x|^{\frac{1}{r}}), \mu_2(x)),$$

finally proves (5.9).

2. In order to prove (5.10) we use the fact that $\partial^\beta T_a^{\lfloor s \rfloor}(F)(x) = 0$ for $|\beta| > \lfloor s \rfloor$. In the following we will just treat the case that $|\beta| \leq \lfloor rs \rfloor$ because otherwise we can use $\partial^\beta T_a^{\lfloor rs \rfloor}(F)(x) = 0$ and the below calculations just get easier. So for $\lfloor s \rfloor + 1 \leq |\beta| \leq \lfloor rs \rfloor$ we have

$$\begin{aligned}
&|\partial^\beta E(F)(x)| \\
=\ &|\partial^\beta E(F)(x) - \partial^\beta T_a^{\lfloor s \rfloor}(F)(x)| \\
=\ &\left|\partial^\beta E(F)(x) - \partial^\beta \left(\sum_{i \in \mathbb{N}} \varphi_i(x) T_a^{\lfloor s \rfloor}(F)(x) \right)\right| \\
\leq\ &\sum_{i \in \mathbb{N}} \sum_{l \leq \beta} \binom{\beta}{l} |\partial^{\beta-l}\varphi_i(x)||\partial^l T_{\tilde{x}_i}^{\lfloor rs \rfloor}(F)(x) - \partial^l T_a^{\lfloor s \rfloor}(F)(x)|
\end{aligned}$$

$$
\begin{aligned}
\le\ & \sum_{i\in\mathbb{N}}\sum_{l\le\beta}\binom{\beta}{l}|\partial^{\beta-l}\varphi_i(x)||\partial^l T^{\lfloor rs\rfloor}_{\tilde{x}_i}(F)(x)-\partial^l T^{\lfloor rs\rfloor}_a(F)(x)| \\
&+\sum_{i\in\mathbb{N}}\sum_{l\le\beta,|l|\le\lfloor s\rfloor}\binom{\beta}{l}|\partial^{\beta-l}\varphi_i(x)|\left|\sum_{\lfloor s\rfloor+1-|l|\le|\alpha|\le\lfloor rs\rfloor-|l|}\frac{f^{(\alpha+l)}(a)}{\alpha!}(x-a)^\alpha\right| \\
&+\sum_{i\in\mathbb{N}}\sum_{l\le\beta,|l|>\lfloor s\rfloor}\binom{\beta}{l}|\partial^{\beta-l}\varphi_i(x)|\left|\sum_{|\alpha|\le\lfloor rs\rfloor-|l|}\frac{f^{(\alpha+l)}(a)}{\alpha!}(x-a)^\alpha\right|.
\end{aligned}
$$

The first and second summand can obviously be treated in the same way as in the proof of (5.9). For the last summand we have

$$
\begin{aligned}
|\partial^{\beta-l}\varphi_i(x)|\left|\sum_{|\alpha|\le\lfloor rs\rfloor-|l|}\frac{f^{(\alpha+l)}(a)}{\alpha!}(x-a)^\alpha\right| &\le C|F|_{\lfloor rs\rfloor,K}\mathrm{dist}\,(x,K)^{|l|-|\beta|} \\
&= C|F|_{\lfloor rs\rfloor,K}\mathrm{dist}\,(x,K)^{|l|-|\beta|}\,|a-x|^{s-|l|}|a-x|^{|l|-s} \\
&= C|F|_{\lfloor rs\rfloor,K}\mathrm{dist}\,(x,K)^{s-|\beta|}\,|a-x|^{|l|-s}.
\end{aligned}
$$

Since $|l|>s$, here we can also choose a modulus of continuity μ satisfying

$$|a-x|^{|l|-s}\le|a-x|^{\lfloor s\rfloor+1-s}\le\mu|a-x|.$$

3. Now that (5.9) and (5.10) are established we can prove that $E(F)\in\mathscr{D}^s(L)$. Because $\mathrm{supp}(\varphi_i)\subset L$ for each $i\in I$ it is clear that $\mathrm{supp}(E(F))\subset L$. First we conclude that $E(F)$ admits continuous partial derivatives up to the order $\lfloor s\rfloor$ in $\mathbb{R}^d$. Since the existence is clear in K^c and $\mathring{K}$, we only have to prove it on ∂K. This can be done using (5.9) and proceeding as in the proof of Theorem 3.2 in [Mal67]. From Proposition 2.16 we obtain that $E(F)\in\mathscr{D}^s(L)$.

4. Here we can basically proceed in the same way as in part 4 of the proof of Theorem 3.1. □

Chapter 6

Extension Operators with an Arbitrary Loss of Derivatives

In this chapter we consider the question which we have already formulated in Section 2.3, to characterize the geometrical properties of a compact set K such that this set admits and extension operator $E : \mathscr{E}(K) \to \mathscr{E}(\mathbb{R}^d)$ or equivalently, when does the short exact sequence (2.12) split. The main result of this chapter is a characterization of all compact sets $K \subset \mathbb{R}^d$ which admit an extension operator with a prescribed loss of derivatives. In this context we describe the continuity properties of an extension operator E via a function $\sigma : [0, \infty) \to [0, \infty)$ having the properties defined in Definition 6.1. An extension operator $E : \mathscr{E}(K) \to \mathscr{E}(\mathbb{R}^d)$ is then said to have a loss of derivatives σ, if and only if it satisfies the inequality

$$|||E(F)|||_{s,L} \leq C_s ||F||_{\sigma(s),K},$$

for all $s \geq 0$, $F \in \mathscr{E}(K)$ and $\mathring{L} \supset K$ compact and convex. We construct the operator in the same fashion as in the previous chapter, so it is again an operator which extends all the spaces $\mathscr{E}^{\sigma(s)}(K)$ simultaneously to $\mathscr{E}^s(\mathbb{R}^d)$ and which resembles in its form the classical Whitney operator.

The characterization of the existence of an extension operator in terms of the existence of certain measures on K already turned out to be very fruitful in the previous chapters and it also offers a new approach for a question originally risen by Mityagin in [Mit61]. It is the very natural question of a geometric characterization of those compact sets $K \subset \mathbb{R}^d$ such that $\mathscr{E}(K)$ admits an extension operator. However, this result cannot be regarded as a final solution to this problem because it is far away from a nice geometric condition as for instance the inequality of Jonsson, Sjögren and Wallin offers in the case of operators with no loss.

6.1 Statement of the Main Result

In generalization of operators having no or a homogeneous loss of derivatives which we dealt with in the last chapters, we now want to allow for operators having an arbitrary continuity behaviour on the spectrum of Whitney spaces.

6.1 Definition. *Let $K \subset \mathbb{R}^d$ be compact and $E : \mathscr{E}(K) \to \mathscr{E}(\mathbb{R}^d)$ be an extension operator. For a surjective and monotonically increasing map $\sigma : [0,\infty) \to [0,\infty)$ we say that E has loss of derivatives σ if and only if $E : \mathscr{E}^{\sigma(s)}(K) \to \mathscr{E}^s(\mathbb{R}^d)$ is continuous for all $s \geq 0$, or equivalently if it fulfils the following norm inequality for all $s \geq 0$*

$$|||E(F)|||_{s,L} \leq C_s ||F||_{\sigma(s),K}$$

for all $F \in \mathscr{E}^{\sigma(s)}(K)$ and compact and convex sets $L \subset \mathbb{R}^d$ with $\mathring{L} \supset K$. In this terms, an operator E with no loss of derivatives has $\sigma = id$.

We have to be careful with this term, because if defined like this, an operator does not have a unique loss of derivatives. If σ is a loss of derivatives for E, then each surjective and monotonically increasing map $\tau : [0,\infty) \to [0,\infty)$ with $\tau \geq \sigma$ is also a loss of derivatives for E. And also defining 'the' loss of derivatives as the infimum over all such functions would not be appropriate as the results in [FJW11] indicate.

Our main result in this chapter is the following characterization.

6.2 Theorem. *For $K \subset \mathbb{R}^d$ compact and $\sigma : [0,\infty) \to [0,\infty)$ strictly monotonically increasing with $\sigma(0) = 0$, the following statements are equivalent:*

1. *$\mathscr{E}(K)$ admits an extension operator E with loss of derivatives σ.*
2. *For all $\alpha \in \mathbb{N}_0^d$, $x \in \partial K$ and $\varepsilon > 0$ there are measures $\nu_{\alpha,x,\varepsilon}$ on K such that for each $F \in \mathscr{E}^{\sigma(s)}$ and $n \in \mathbb{N}_0$*

   (a) $$\lim_{\varepsilon\to 0} \sup_{|\alpha|\leq\lfloor s\rfloor, x\in\partial K} \frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)}) - \varepsilon^{|\alpha|} f^{(\alpha)}(x)|}{\varepsilon^s} = 0,$$

   (b) $$\lim_{\varepsilon\to 0} \sup_{|\alpha|>\lfloor s\rfloor, x\in\partial K} \frac{|\nu_{\alpha,x,\varepsilon}(f^{(0)})|}{\varepsilon^s} = 0.$$

As an interesting consequence of this theorem we have the following corollary.

6.3 Corollary. *If a compact set $K \subset \mathbb{R}^d$ admits a continuous extension operator $E : \mathscr{E}(K) \to \mathscr{E}(\mathbb{R}^d)$, then we can also construct a 'Whitney like' extension operator with the same loss of derivatives.*

6.2 Construction of the Measures

In this section we prove the first part of the main theorem, i.e. the existence of the operator implies the existence of the measures.

A main tool in the construction of the measures is Eidelheit's theorem characterizing the solvability of an infinite system of equations.

6.4 Theorem. *Eidelheit's theorem*
Let E be a Fréchet space, $(U_k)_{k\in\mathbb{N}}$ be a fundamental system of zero neighbourhoods in E and let $(T_j)_{j\in\mathbb{N}}$ be linearly independent, continuous linear forms on E. Then the infinite system of equations

$$T_j x = y_j \quad \text{for all } j \in \mathbb{N}$$

is solvable for each sequence $y \in \omega$ if, and only if, the following holds:

$$dim((E')_{U_k^\circ} \cap span\{T_j : j \in \mathbb{N}\}) < \infty \quad \text{for all } k \in \mathbb{N}.$$

Proof. A proof can be found in [MV97], Theorem 26.27. □

The symbol $(E')_{U_k^\circ}$ is defined as the span of U_k°, so $(E')_{U_k^\circ} := \text{span} U_k^\circ = \cup_{t>0} t U_k^\circ$. Endowed with the Minkowski functional of U_k°, $(E')_{U_k^\circ}$ is a Banach space.

In the following we deal with sequence spaces. We will now shorty recall relevant definitions. In contrast to the 'classical' definitions of sequence spaces, we will define them over the index set $\mathbb{N}_0^d$, but basically, that won't change anything of relevance.

6.5 Definition. *For a given dimension $d \in \mathbb{N}$ we define the sequence spaces of sequences over the index set $\mathbb{N}_0^d$:*

- $\omega(\mathbb{N}_0^d) := \mathbb{C}^{\mathbb{N}_0^d}$.
- $\varphi(\mathbb{N}_0^d) := \left\{(x_\alpha)_{\alpha\in\mathbb{N}_0^d} \in \omega(\mathbb{N}_0^d) : \sharp\left\{\alpha \in \mathbb{N}_0^d : x_\alpha \neq 0\right\} < \infty\right\}$.
- $s(\mathbb{N}_0^d) := \left\{(x_\alpha)_{\alpha\in\mathbb{N}_0^d} \in \omega(\mathbb{N}_0^d) : \lim_{|\alpha|\to\infty} |x_\alpha||\alpha|^k = 0 \text{ for all } k \in \mathbb{N}\right\}$. *Since this space is nuclear by Example 29.4 in [MV97], the topology is generated by the fundamental system of seminorms $p_k((x_\alpha)_{\alpha\in\mathbb{N}_0^d}) = \sum_{\alpha\in\mathbb{N}_0^d} |x_\alpha||\alpha|^k$ as well as by the system $\tilde{p}_k((x_\alpha)_{\alpha\in\mathbb{N}_0^d}) = \sup_{\alpha\in\mathbb{N}_0^d} |x_\alpha||\alpha|^k$.*
- $s'(\mathbb{N}_0^d) := \left\{(x_\alpha)_{\alpha\in\mathbb{N}_0^d} \in \omega(\mathbb{N}_0^d) : p_k^*((x_\alpha)_{\alpha\in\mathbb{N}_0^d}) < \infty \text{ for one } k \in \mathbb{N}\right\}$ *shall denote the dual space of $s(\mathbb{N}_0^d)$. The Minkowski functional p_k^* of $U_k^\circ = \{y \in s' : |\sum_{\alpha\in\mathbb{N}_0^d} y_\alpha x_\alpha| \leq 1 \text{ for all } x \in s \text{ with } p_k(x) < 1\}$ is given by $p_k^*((x_\alpha)_{\alpha\in\mathbb{N}_0^d}) = \sup_{\alpha\in\mathbb{N}_0^d} |x_\alpha||\alpha|^{-k}$ (see [MV97] Lemma 27.12).*

With the help of Eidelheit's theorem, we now can prove the following representation lemma.

6.6 Lemma. *The mapping* $T : s(\mathbb{N}_0^d) \to \omega(\mathbb{N}_0^d), (\lambda_\beta)_{\beta\in\mathbb{N}_0^d} \mapsto \left(\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta \beta^\gamma\right)_{\gamma\in\mathbb{N}_0^d}$ *is surjective.*

Proof. The mapping T can be decomposed in countably infinite many linear forms T_γ, where for each $\gamma \in \mathbb{N}_0^d$ we have

$$T_\gamma\left((\lambda_\beta)_{\beta\in\mathbb{N}_0^d}\right) = \sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta \beta^\gamma,$$

and all these linear forms are obviously elements of $s'(\mathbb{N}_0^d)$. For the span $s'(\mathbb{N}_0^d)_{U_k^\circ}$ of U_k° we have

$$s'(\mathbb{N}_0^d)_{U_k^\circ} = \left\{(x_\alpha)_{\alpha\in\mathbb{N}_0^d} \in s'(\mathbb{N}_0^d) : p_k^*((x_\alpha)_{\alpha\in\mathbb{N}_0^d}) < \infty\right\}.$$

Thus, to be able to apply Eidelheit's theorem, we have to check for each $k \in \mathbb{N}$ that

$$\dim\left\{\left(\varrho_\gamma\right)_{\gamma\in\mathbb{N}_0^d} \in \varphi(\mathbb{N}_0^d) : \sup_{\beta\in\mathbb{N}_0^d}\left|\sum_{\gamma\in\mathbb{N}_0^d} \varrho_\gamma \beta^\gamma\right| |\beta|^{-k} < \infty\right\} < \infty.$$

We show by contradiction that for each $(\varrho_\gamma)_{\gamma\in\mathbb{N}_0^d} \in \varphi(\mathbb{N}_0^d)$ with

$$\sup_{\beta\in\mathbb{N}_0^d}\left|\sum_{\gamma\in\mathbb{N}_0^d} \varrho_\gamma \beta^\gamma\right| |\beta|^{-k} < \infty \tag{6.1}$$

it is true that $\varrho_\gamma = 0$ for $|\gamma| > k$. So we assume that $m := \max\{|\gamma| : \varrho_\gamma \neq 0\} > k$ and let $P_m(x) = \sum_{|\gamma|=m} \varrho_\gamma x^\gamma$. Following the assumption, there is $x = (x_1, ..., x_d) \in \mathbb{R}^d$ with $x_1, ..., x_d > 0$ such that $P_m(x) \neq 0$. Since the set $\{t\beta : \beta \in \mathbb{N}_0^d, t > 0\}$ is dense in $\{x \in \mathbb{R}^d : x \geq 0\}$, there is $\beta_0 \in \mathbb{N}_0^d$ with $P_m(\beta_0) \neq 0$. It is easy to see that for all $\beta_l := l\beta_0$ it is true for any $\gamma \in \mathbb{N}_0^d$ that $\beta_l^\gamma = l^{|\gamma|}\beta_0^\gamma$ and thus we get

$$\left|\sum_{|\gamma|=m} \varrho_\gamma \beta_l^\gamma\right| = |P_m(\beta_0)| l^m$$

and

$$\left|\sum_{|\gamma|<m} \varrho_\gamma \beta_l^\gamma\right| \leq l^{m-1} \sum_{|\gamma|<m} |\varrho_\gamma||\beta_0^\gamma|.$$

Since $m > k$ we have

$$\lim_{l\to\infty}\left|\sum_{|\gamma|\leq m} \varrho_\gamma \beta_l^\gamma\right| |\beta_l|^{-k} = \infty,$$

which is a contradiction to (6.1). □

The following lemma gives the solution to a certain moment problem. It can be seen as the analogon to Proposition 4.3.

6.7 Lemma. *Let $(k_\alpha)_{\alpha\in\mathbb{N}_0} \in \omega(\mathbb{N}_0^d)$ be arbitrary. Then there is $(\varrho_\beta)_{\beta\in\mathbb{N}_0^d} \in s(\mathbb{N}_0^d)$ such that for all $\alpha \in \mathbb{N}_0^d$ there is a sequence $(\lambda_\beta^{(\alpha)}) \in s(\mathbb{N}_0^d)$ with $|\lambda_\beta^{(\alpha)}| \leq \frac{\varrho_\beta}{2}$ for all $\beta \in \mathbb{N}_0^d$, satisfying*

$$\left(\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)}\beta^\gamma\right)_{\gamma\in\mathbb{N}_0^d} = (k_\alpha\delta_{\alpha\gamma})_{\gamma\in\mathbb{N}_0^d}.$$

Proof. We first observe that the set $\{(k_\alpha\delta_{\alpha,\gamma})_{\gamma\in\mathbb{N}_0^d} : \alpha \in \mathbb{N}_0^d\}$ is compact in the Fréchet space of all sequences $\omega(\mathbb{N}_0^d)$. This follows directly by Tychonov's theorem or the characterizations of compactness in metric spaces given in Proposition 4.8 in [MV97]. By Corollary 26.22 in [MV97] surjective maps between Fréchet spaces lift compact sets. This gives a compact set $K \subset s(\mathbb{N}_0^d)$ such that with the surjective mapping T of the previous Lemma we have

$$T(K) \supset \{(k_\alpha\delta_{\alpha,\gamma})_{\gamma\in\mathbb{N}_0^d} : \alpha \in \mathbb{N}_0^d\}.$$

Thus, we can choose a sequence $(\varrho_\beta)_{\beta\in\mathbb{N}_0^d} \in s(\mathbb{N}_0^d)$ satisfying $|\lambda_\beta| \leq \frac{\varrho_\beta}{2}$ for all $\beta \in \mathbb{N}_0^d$ and all $(\lambda_\beta)_{\lambda\in\mathbb{N}_0^d} \in K$. □

In the following theorem we construct measures approximating the derivatives of functions in $\mathscr{E}$ on the boundary of some compact set K.

6.8 Theorem. *Let $s \geq 0$ and $K \subset \mathbb{R}^d$ be compact. Then for all $\alpha \in \mathbb{N}_0^d$, $x \in \partial K$ and $\varepsilon > 0$ there are measures $\mu_{\alpha,x,\varepsilon}$ supported on a ball $\mathcal{B}$ with $\mathring{\mathcal{B}} \supset K$ such that for all $f \in \mathscr{E}^s(\mathbb{R}^d)$ we have*

1. $\lim_{\varepsilon\to 0} \sup_{|\alpha|\leq\lfloor s\rfloor, x\in\partial K} \dfrac{|\mu_{\alpha,x,\varepsilon}(f) - \varepsilon^{|\alpha|}\partial^\alpha f(x)|}{\varepsilon^s} = 0,$
2. $\lim_{\varepsilon\to 0} \sup_{|\alpha|>\lfloor s\rfloor, x\in\partial K} \dfrac{|\mu_{\alpha,x,\varepsilon}(f)|}{\varepsilon^s} = 0.$

Proof. By cutting off, it is enough to show the assertion for $f \in \mathscr{D}^s(\mathcal{B}) = \{g \in \mathscr{E}^s(\mathbb{R}^d) : \mathrm{supp}(g) \subset \mathcal{B}\}$ where $\mathcal{B}$ is a fixed ball containing K in its interior. We apply the previous lemma choosing $k_\alpha = \alpha!$ for each $\alpha \in \mathbb{N}_0^d$. Then there are sequences $(\lambda_\beta^{(\alpha)})_{\beta\in\mathbb{N}_0^d}$ and $(\varrho_\beta)_{\beta\in\mathbb{N}_0^d}$, both in $s(\mathbb{N}_0^d)$, such that for all $\alpha \in \mathbb{N}_0^d$

$$\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)}\beta^\gamma = \begin{cases} \alpha! & \text{for } \gamma = \alpha, \\ 0 & \text{else}, \end{cases} \tag{6.2}$$

and such that $|\lambda_\beta^{(\alpha)}| < \varrho_\beta$ for all $\alpha, \beta \in \mathbb{N}_0^d$. With this we define for each $g \in \mathscr{D}^0(\mathcal{B})$ and $x \in \partial K$

$$\mu_{\alpha,x,\varepsilon}(g) \coloneqq \sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} g(x+\varepsilon\beta).$$

We start with the proof of 1., hence let $|\alpha| \leq \lfloor s \rfloor$. With Taylor's theorem we then find a $\xi \in [x, x+\varepsilon\beta]$ such that

$$\left|\mu_{\alpha,x,\varepsilon}(f) - \varepsilon^{|\alpha|}\partial^\alpha f(x)\right| = \left|\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} f(x+\varepsilon\beta) - \varepsilon^{|\alpha|}\partial^\alpha f(x)\right|$$

$$= \left|\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \left(\sum_{|\gamma|<\lfloor s\rfloor} \frac{\partial^\gamma f(x)}{\gamma!}\varepsilon^{|\gamma|}\beta^\gamma + \varepsilon^n \sum_{|\gamma|=\lfloor s\rfloor} \frac{\partial^\gamma f(\xi)}{\gamma!}\beta^\gamma \right) - \varepsilon^{|\alpha|}\partial^\alpha f(x)\right|.$$

Using (6.2) we get

$$\varepsilon^{|\alpha|}\partial^\alpha f(x) = \alpha!\varepsilon^{|\alpha|}\frac{\partial^\alpha f(x)}{\alpha!} = \sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)}\beta^\alpha\varepsilon^{|\alpha|}\frac{\partial^\alpha f(x)}{\alpha!},$$

and

$$0 = \sum_{|\gamma|\leq\lfloor s\rfloor, \gamma\neq\alpha} \left(\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)}\beta^\gamma \right) \varepsilon^{|\gamma|}\frac{\partial^\gamma f(x)}{\gamma!} = \sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \sum_{|\gamma|\leq\lfloor s\rfloor, \gamma\neq\alpha} \beta^\gamma\varepsilon^{|\gamma|}\frac{\partial^\gamma f(x)}{\gamma!},$$

which results in

$$\varepsilon^{|\alpha|}\partial^\alpha f(x) = \sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \sum_{\gamma\leq\lfloor s\rfloor} \beta^\gamma\varepsilon^{|\gamma|}\frac{\partial^\gamma f(x)}{\gamma!}.$$

Therefore, we get using that $\varepsilon \leq |\xi - x||\beta|^{-1}$

$$\begin{aligned}
\left|\mu_{\alpha,x,\varepsilon}(f) - \varepsilon^{|\alpha|}\partial^\alpha f(x)\right| &= \varepsilon^{\lfloor s\rfloor}\left|\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \sum_{|\gamma|=\lfloor s\rfloor} \beta^\gamma\frac{1}{\gamma!}|\partial^\gamma f(\xi) - \partial^\gamma f(x)|\right| \\
&= \varepsilon^s\left|\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \sum_{|\gamma|=\lfloor s\rfloor} \beta^\gamma\frac{1}{\gamma!}\frac{|\partial^\gamma f(\xi) - \partial^\gamma f(x)|}{\varepsilon^{\{s\}}}\right| \\
&\leq \varepsilon^s \sum_{|\gamma|=\lfloor s\rfloor}\sum_{\beta\in\mathbb{N}_0^d} \varrho_\beta|\beta|^s\frac{|\partial^\gamma f(\xi) - \partial^\gamma f(x)|}{|\xi - x|^{\{s\}}} \\
&\leq \varepsilon^s \sum_{|\gamma|=\lfloor s\rfloor}\sum_{\beta\in\mathbb{N}_0^d} \varrho_\beta|\beta|^s \sup_{y\in(x,x+\varepsilon\beta]}\frac{|\partial^\gamma f(y) - \partial^\gamma f(x)|}{|y - x|^{\{s\}}},
\end{aligned}$$

where the last term is independent of α. We now show that the supremum over all boundary points of K of the last sum converges to 0 for each $|\gamma| = \lfloor s \rfloor$. To do this, we split the sum for some index $m \in \mathbb{N}$ into

$$\sup_{x\in\partial K}\sum_{\beta\in\mathbb{N}_0^d}\varrho_\beta|\beta|^s\sup_{y\in(x,x+\varepsilon\beta]}\frac{|\partial^\gamma f(y)-\partial^\gamma f(x)|}{|y-x|^{\{s\}}} \leq \sup_{x\in\partial K}\sum_{|\beta|\leq m}\varrho_\beta|\beta|^s\sup_{y\in(x,x+\varepsilon\beta]}\frac{|\partial^\gamma f(y)-\partial^\gamma f(x)|}{|y-x|^{\{s\}}}$$
$$+\sup_{x\in\partial K}\sum_{|\beta|>m}\varrho_\beta|\beta|^s\sup_{y\in(x,x+\varepsilon\beta]}\frac{|\partial^\gamma f(y)-\partial^\gamma f(x)|}{|y-x|^{\{s\}}}.$$

Since $f \in \mathscr{D}^s(B)$ and therefore has a compact support, there exists a constant $C > 0$ such that $\sup_{x,y\in\mathbb{R}^d}\frac{|\partial^\gamma f(x)-\partial^\gamma f(y)|}{|x-y|^{\{s\}}} < C$ for all $|\gamma| = \lfloor s \rfloor$. And since $(\varrho_\beta)_{\beta\in\mathbb{N}_0^d} \in s(\mathbb{N}_0^d)$ with $\varrho_\beta \geq 0$ for all β it follows that

$$\lim_{m\to\infty}\sum_{|\beta|>m}\varrho_\beta|\beta|^s = 0.$$

Thus, the second sum converges to 0 for $m \to \infty$ and is independent of the choice of ε. It is left to show that the first sum also converges to 0 but for $\varepsilon \to 0$. For this we use again the fact that since $f \in \mathscr{D}^s(B)$, all the $\partial^\gamma f$ are uniformly continuous, so we have that $\sup_{y\in(x,x+\varepsilon\beta]}\frac{|\partial^\gamma f(y)-\partial^\gamma f(x)|}{|y-x|^{\{s\}}} \to 0$ uniformly for $\varepsilon \to 0$. Then using

$$\sum_{\beta\leq m}\varrho_\beta|\beta|^s \leq \sum_{\beta\in\mathbb{N}_0^d}\varrho_\beta|\beta|^s,$$

for all $m \in \mathbb{N}$, finishes the proof for the case $|\alpha| \leq \lfloor s \rfloor$.
For the case $|\alpha| > \lfloor s \rfloor$ we argue similarly. With Taylor's theorem we again find a $\xi \in [x, x + \varepsilon\beta]$ such that

$$\left|\mu_{\alpha,x,\varepsilon}(f)\right| = \left|\sum_{\beta\in\mathbb{N}_0^d}\lambda_\beta^{(\alpha)}f(x+\varepsilon\beta)\right| = \left|\sum_{\beta\in\mathbb{N}_0^d}\lambda_\beta^{(\alpha)}\left(\sum_{|\gamma|<\lfloor s\rfloor}\frac{\partial^\gamma f(x)}{\gamma!}\varepsilon^{|\gamma|}\beta^\gamma + \varepsilon^{\lfloor s\rfloor}\sum_{|\gamma|=\lfloor s\rfloor}\frac{\partial^\gamma f(\xi)}{\gamma!}\beta^\gamma\right)\right|.$$

Since $|\alpha| > \lfloor s \rfloor$, we have that

$$\sum_{\beta\in\mathbb{N}_0^d}\lambda_\beta^{(\alpha)}\beta^\gamma = 0$$

for all $|\gamma| \leq \lfloor s \rfloor$, and therefore we get

$$\left|\sum_{\beta\in\mathbb{N}_0^d}\lambda_\beta^{(\alpha)}\sum_{|\gamma|<\lfloor s\rfloor}\frac{\partial^\gamma f(x)}{\gamma!}\varepsilon^{|\gamma|}\beta^\gamma\right| = 0 = \left|\sum_{\beta\in\mathbb{N}_0^d}\lambda_\beta^{(\alpha)}\sum_{|\gamma|=\lfloor s\rfloor}\frac{\partial^\gamma f(x)}{\gamma!}\varepsilon^{|\gamma|}\beta^\gamma\right|.$$

This results in

$$
\begin{aligned}
\left|\mu_{\alpha,x,\varepsilon}(f)\right| &= \left|\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \left(\sum_{|\gamma|<\lfloor s\rfloor} \frac{\partial^\gamma f(x)}{\gamma!}\varepsilon^{|\gamma|}\beta^\gamma + \varepsilon^{\lfloor s\rfloor} \sum_{|\gamma|=n} \frac{\partial^\gamma f(\xi)}{\gamma!}\beta^\gamma \right)\right| \\
&= \left|\sum_{\beta\in\mathbb{N}_0^d} \lambda_\beta^{(\alpha)} \left(-\sum_{|\gamma|=\lfloor s\rfloor} \frac{\partial^\gamma f(x)}{\gamma!}\varepsilon^{|\gamma|}\beta^\gamma + \varepsilon^{\lfloor s\rfloor} \sum_{|\gamma|=\lfloor s\rfloor} \frac{\partial^\gamma f(\xi)}{\gamma!}\beta^\gamma \right)\right| \\
&\leq \varepsilon^s \sum_{|\gamma|=n} \sum_{\beta\in\mathbb{N}_0^d} \varrho_\beta |\beta|^s \frac{|\partial^\gamma f(\xi) - \partial^\gamma f(x)|}{|\xi - x|^{\{s\}}} \\
&\leq \varepsilon^s \sum_{|\gamma|=n} \sum_{\beta\in\mathbb{N}_0^d} \varrho_\beta |\beta|^s \sup_{y\in(x,x+\varepsilon\beta]} \frac{|\partial^\gamma f(y) - \partial^\gamma f(x)|}{|y - x|^{\{s\}}},
\end{aligned}
$$

where again the last term is independent of α which ensures that the supremum over all $\alpha > \lfloor s\rfloor$ exists. The same argument as in the first case also shows the desired convergence property in this case. □

We prove now that 1. implies 2. in our main Theorem 6.2. We use the extension operator to project the measures constructed in the previous corollary from the dual space of $\mathscr{E}(\mathbb{R}^d)$ on the dual space of $\mathscr{E}(K)$.

6.9 Theorem. *Let $K \subset \mathbb{R}^d$ such that it admits an extension operator with loss σ. Then for all $\alpha \in \mathbb{N}_0^d$, $x \in \partial K$ and $\varepsilon > 0$ there are measures $\nu_{\alpha,x,\varepsilon}$ supported on K such that for each $F \in \mathscr{E}^{\sigma(s)}$ and $s \geq 0$*

1. $\lim_{\varepsilon\to 0} \sup_{|\alpha|\leq\lfloor s\rfloor, x\in\partial K} \dfrac{|\nu_{\alpha,x,\varepsilon}(f^{(0)}) - \varepsilon^{|\alpha|} f^{(\alpha)}(x)|}{\varepsilon^s} = 0$

2. $\lim_{\varepsilon\to 0} \sup_{|\alpha|>\lfloor s\rfloor, x\in\partial K} \dfrac{|\nu_{\alpha,x,\varepsilon}(f^{(0)})|}{\varepsilon^s} = 0.$

Proof. Let E denote the extension operator with loss σ. Then it is clear that for all $F = \left(f^{(\alpha)}\right)_{|\alpha|\leq\lfloor\sigma(s)\rfloor} \in \mathscr{E}^{\sigma(s)}(K)$ and all $x \in K$ we have

$$f^{(\alpha)}(x) = \partial^\alpha E(F)(x).$$

Since $E(F) \in \mathscr{E}^s(K)$, by taking the measures $\mu_{\alpha,x,\varepsilon}$ from the previous corollary we can define

$$\nu_{\alpha,x,\varepsilon}(f^{(0)}) := \mu_{\alpha,x,\varepsilon}(E(F)).$$

The proof is complete by applying the properties of the measures $\mu_{\alpha,x,\varepsilon}$. □

6.3 Construction of the Extension Operator

In this section we will construct the extension operator with the given measures. As already indicated, the main idea is to construct the operator as in Chapter 4. Thus, given the measures in Theorem 6.2 we set for $\mu_{\alpha,i} := \nu_{\alpha,x_i,\gamma_i}/\gamma_i^{|\alpha|}$:

$$E(F)(x) = \begin{cases} f^{(0)}(x), x \in K \\ \sum_{i\in\mathbb{N}} \varphi_i(x) \sum_{|\alpha|\leq i} \frac{1}{\alpha!}\mu_{\alpha,i}(f^{(0)})(x-x_i)^\alpha, x \notin K. \end{cases}$$

Again we split the proof in two parts

1. For $\beta \in \mathbb{N}_0^d$ and $x \in K^c$ we show that for each $F \in \mathscr{E}^{\sigma(s)}(K)$ we have

$$\left|\partial^\beta E(F)(x) - \partial^\beta E_s(F)(x)\right| = o(\operatorname{dist}(x,\partial K)^{s-|\beta|}) \quad \text{for } x \to \partial K. \tag{6.3}$$

   Using Proposition 2.15 and 2.16, this implies that $E(F) \in \mathscr{E}^s(L)$.

2. We show that E is continuous and has loss of derivatives σ.

In order to show (6.3) we can limit us again to those $x \in L \setminus K$ with $i(x) = \min\{i \in I : x \in \operatorname{supp}(\varphi_i)\} > \lfloor s\rfloor$. This allows us to express the difference $E - E_s$ very simple by

$$\begin{aligned}\partial^\beta E(F)(x) - \partial^\beta E_s(F)(x) = &\sum_{i\geq i(x)}\sum_{|\alpha|\leq\lfloor s\rfloor} \frac{1}{\alpha!}(\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i))\partial^\beta(\varphi_i(x)(x-x_i)^\alpha) \\ &+ \sum_{i\geq i(x)}\sum_{\lfloor s\rfloor<|\alpha|\leq i} \frac{1}{\alpha!}\mu_{\alpha,i}(f^{(0)})\partial^\beta(\varphi_i(x)(x-x_i)^\alpha).\end{aligned}$$

Starting with the first summand, we have by assumption

$$\lim_{\varepsilon\to 0}\sup_{|\alpha|\leq\lfloor s\rfloor, x\in\partial K} \frac{|\nu_{\alpha,x,\varepsilon}(f^0) - \varepsilon^{|\alpha|}f^{(\alpha)}(x)|}{\varepsilon^s} = 0.$$

Since $i(x) \to \infty$ is equivalent to $x \to \partial K$, we get for all $|\alpha| \leq \lfloor s\rfloor$ using the definition of $\mu_{\alpha,i}$:

$$\lim_{i\to\infty} \frac{|\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i)|}{\gamma_i^{s-|\alpha|}} = 0.$$

Thus, we can conclude with Lemma 4.5 for a given $i \geq i(x)$:

$$\begin{aligned}&\sup_{x\in\mathbb{R}^d}\left|\sum_{|\alpha|\leq\lfloor s\rfloor}\frac{1}{\alpha!}(\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i))\partial^\beta(\varphi_i(x)(x-x_i)^\alpha)\right|\gamma_i^{|\beta|-\lfloor s\rfloor} \\ = &\sup_{x\in\operatorname{supp}(\varphi_i)}\left|\sum_{|\alpha|\leq\lfloor s\rfloor}\frac{1}{\alpha!}(\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i))\partial^\beta(\varphi_i(x)(x-x_i)^\alpha)\right|\gamma_i^{|\beta|-|\alpha|}\gamma_i^{|\alpha|-\lfloor s\rfloor}\end{aligned}$$

$$
\begin{aligned}
&\leq \sup_{x\in\operatorname{supp}(\varphi_i)} \sum_{|\alpha|\leq\lfloor s\rfloor} \frac{1}{\alpha!} \frac{|\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i)|}{\gamma_i^{s-|\alpha|}} \frac{|\partial^\beta(\varphi_i(x)(x-x_i)^\alpha)|}{\gamma_i^{|\alpha|-|\beta|}} \\
&\leq C_{\lfloor s\rfloor} \sum_{|\alpha|\leq\lfloor s\rfloor} \frac{|\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i)|}{\gamma_i^{s-|\alpha|}}.
\end{aligned}
$$

Now let $\varepsilon > 0$ be arbitrary. Since the last term converges to 0 for $i \to \infty$, we can find $j \in \mathbb{N}$ such that for all $i \geq j$

$$
\sup_{x\in\mathbb{R}^d} \left| \sum_{|\alpha|\leq\lfloor s\rfloor} \frac{1}{\alpha!}(\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i))\partial^\beta(\varphi_i(x)(x-x_i)^\alpha) \right| \gamma_i^{|\beta|-\lfloor s\rfloor} < \frac{\varepsilon}{N},
$$

meaning that we have for all points $x \in L \setminus K$ for which $i(x) > j$

$$
\begin{aligned}
&\sum_{i\geq i(x)} \frac{1}{\operatorname{dist}(x,K)^{s-|\beta|}} \left| \sum_{|\alpha|\leq\lfloor s\rfloor} \frac{1}{\alpha!}(\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i))\partial^\beta(\varphi_i(x)(x-x_i)^\alpha) \right| \\
&\leq \sum_{i\geq i(x)} \frac{1}{\gamma_i^{s-|\beta|}} \left| \sum_{|\alpha|\leq\lfloor s\rfloor} \frac{1}{\alpha!}(\mu_{\alpha,i}(f^{(0)}) - f^{(\alpha)}(x_i))\partial^\beta(\varphi_i(x)(x-x_i)^\alpha) \right| \\
&\leq \varepsilon.
\end{aligned}
$$

Now we estimate the second summand. By assumption we have for all $|\alpha| > \lfloor s\rfloor$:

$$
\lim_{i\to\infty} \frac{|\mu_{\alpha,i}(f^{(0)})|}{\gamma_i^{s-|\alpha|}} = 0.
$$

Thus, Lemma 4.5.2 and .3 imply that

$$
\begin{aligned}
&\left| \sum_{|\alpha|>\lfloor s\rfloor} \frac{1}{\alpha!}\mu_{\alpha,i}(f^{(0)})\partial^\beta((x-x_i)^\alpha\varphi_i(x)) \right| \\
\leq\ & \sum_{|\alpha|>\lfloor s\rfloor} \frac{1}{\alpha!}\gamma_i^{-|\alpha|} o(\gamma_i^s) C_{\beta,\lfloor s\rfloor} \sup_{\gamma\leq\alpha,\beta} \frac{\alpha!}{(\alpha-\gamma)!} 3^{|\alpha|}\gamma_i^{|\alpha|-|\beta|} \\
\leq\ & o(\gamma_i^s) C_{\beta,\lfloor s\rfloor} e^{3d}(|\beta|+1)^d 3^{|\beta|}\gamma_i^{-|\beta|} \\
=\ & o(\gamma_i^{s-|\beta|}) \quad \text{as } i\to\infty.
\end{aligned}
$$

As for the first summand we conclude

$$
\lim_{x\to\partial K} \sum_{i\geq i(x)} \left| \sum_{\lfloor s\rfloor<|\alpha|\leq i} \frac{1}{\alpha!}\mu_{\alpha,i}(f^{(0)})\partial^\beta((x-x_i)^\alpha\varphi_i(x)) \right| \operatorname{dist}(x,K)^{|\beta|-s} = 0,
$$

which proves completes the proof of (6.3). The application of Proposition 2.15 and 2.16 then implies $E(F) \in \mathscr{E}^s(L)$.

To show the continuity of $E : \mathscr{E}^{\sigma(s)}(K) \to \mathscr{E}^s(\mathbb{R}^d)$, we first note that the operator $E - E_s$ takes its values in $\mathscr{I}^s(K) = \{g \in \mathscr{E}^s(\mathbb{R}^d) : \partial^\alpha g(x) = 0 \text{ for all } x \in K, |\alpha| \leq \lfloor s \rfloor\}$. This operator is continuous if we equip $\mathscr{I}^s(K)$ with the topology of pointwise convergence, which follows directly from the locally finite property of Whitney's partition of unity. Hence, using the closed graph theorem, we obtain the continuity of $E - E_s : \mathscr{E}^{\sigma(s)} \to \mathscr{I}^s(K)$ with respect to the Fréchet space topology on $\mathscr{I}^s(K)$. The continuity of E_s : $\mathscr{E}^{\sigma(s)} \to \mathscr{D}^s(L)$ permits to get positive constants C_s such that $|||E(F)|||_{s,L} \leq C_s \|F\|_{\sigma(s),K}$, which completes the proof.

6.4 Examples

In the following we list examples of compact sets admitting an extension operator having a loss of derivatives $\sigma \neq \mathrm{id}$.

6.10 Example. *1. For $r \geq 1$ we consider the following two dimensional cuspidal set*

$$K_r := \left\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \frac{1}{2}x^r \leq y \leq 2x^r\right\}.$$

In the paper [FJW11] of Frerick, Jordá and Wengenroth, the authors show in Example 5 that this set fulfills $LMI(r)$ and therefore also the weaker form $WLMI(r)$ which is per definition fulfilled if and only if $LMI(s)$ is fulfilled for each $s > r$. The main result of this paper is that a compact set $K \subset \mathbb{R}^d$ fulfills this weaker form of the local Markov inequality for some $r \geq 1$ if and only if K admits an extension operator with a tame linear loss of derivative (for a general definition of tame linear operators between Fréchet spaces see for instance [Vog87]), i.e. for all $\varepsilon > 0$ and $n \in \mathbb{N}_0$ there exists a positive constant $C_{n,\varepsilon} > 0$ such that for all $F \in \mathscr{E}(K)$ the following norm inequality holds

$$|E(F)|_n \leq C_{n,\varepsilon} \|F\|_{(r+\varepsilon)n,K}.$$

This result is generalized in appendix A to hold also for the real scale, i.e. the following inequality is satisfied for each $s \geq 0$ and each convex and compact $\mathring{L} \supset K$

$$|||E(F)|||_{s,L} \leq C_{s,\varepsilon} \|F\|_{(r+\varepsilon)s,K}.$$

Therefore the above cuspidal set also admits an extension operator which is continuous on the real scale.

2. *In Section 2.3 we already mentioned the results of Stein. He proved that if an open set Ω locally is the graph of a Lipschitz function, it is possible to construct an extension operator E which maps simultaneously all the Sobolev spaces $W^{k,p}(\Omega)$*

into $W^{k,p}(\mathbb{R}^d)$. *In [Bie78] Bierstone could prove a generalization of this result to a broader class of sets which he called Lipschitz domains. In order to briefly introduce this class, we say that a function* $\phi : \mathbb{R}^{d-1} \to \mathbb{R}$ *satisfies a Lipschitz condition of order* γ *with* $0 < \gamma \leq 1$ *if and only if there exists a constant* $C > 0$ *such that for all* $x, y \in \mathbb{R}^{d-1}$ *the inequality* $|\phi(x) - \phi(y)| \leq C|x - y|^\gamma$ *holds. Then the open set* $\{(x, y) \in \mathbb{R}^d : y > \phi(x)\}$ *and rotations thereof are called Lipschitz domains of class Lip* γ. *The class Lip* γ *also contains all open sets which are locally a graph of a Lip* γ *function in the above sense. A set* Ω *is locally the graph of a Lip* γ *function if for all* $a \in \partial\Omega$ *the exists an open neighbourhood of* U_a *of a such that* $\Omega \cap U_a \in$ *Lip* γ. *The result of Bierstone then reads as follows:*

If X is the closure of some open set $\Omega \in$ *Lip* γ, *then there exists an extension operator* $E : \mathscr{E}(X) \to \mathscr{C}^\infty(\mathbb{R}^d)$. *Furthermore if* $\frac{1}{\gamma} \in \mathbb{N}$, *then E can be constructed such that for each* $L \subset \mathbb{R}^d$ *compact there exists a* $K \subset X$ *compact so that for each* $m \in \mathbb{N}$ *there is a* $C > 0$ *such that the following inequality holds for each* $F \in \mathscr{E}(X)$

$$|E(F)|_{m,L} \leq C|F|_{\frac{m}{\gamma},K}.$$

Frerick proved in [Fre07b] a generalization of Bierstone's result and could get around the restriction that $\frac{1}{\gamma} \in \mathbb{N}$. *His extension operator fulfils the continuity estimates*

$$|E(F)|_{m,L} \leq C|F|_{\lceil \frac{m}{\gamma} \rceil,K}, \tag{6.4}$$

where $\lceil \frac{m}{\gamma} \rceil$ *denotes the smallest integer which is greater or equal than* $\frac{m}{\gamma}$. *Using this result, he could show that the cuspidal sets of the form*

$$K_\gamma := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, -x^{\frac{1}{\gamma}} \leq y \leq x^{\frac{1}{\gamma}}\},$$

of which the interior is indeed $\in$ *Lip* γ, *admit such an extension operator E and furthermore, the continuity estimates* (6.4) *are the best possible in the sense that there cannot exist* $\kappa \in (\gamma, 1]$ *such that E fulfilled* (6.4) *with* κ *instead of* γ.

3. *A prominent example for a set admitting an extension operator not having a tame linear loss of derivatives, was given by Goncharov in [Gon96]. The disjoint union*

$$K := \{0\} \cup \bigcup_{n \in \mathbb{N}} \left[b_n - \frac{1}{2} b_n, b_n + \frac{1}{2} b_n \right]$$

of shrinking intervals with $b_n := exp(-M^n)$ *and* $M \geq 3$. *Goncharov showed in his paper that this set does not fulfill LMI of any exponent but* $\mathscr{E}(K)$ *admits an extension operator. By the main result of [FJW11], this operator cannot be tame linear.*

Appendix A

Open Problems

In this appendix we gather some questions which could not be solved in this work and therefore offer an interesting outlook for future research on the topic of extension operators for spaces of Whitney jets.

1. We achieved in Chapter 6 a characterization of the existence of an extension operator on $\mathscr{E}(K)$ which has prescribed loss of derivatives. We found that the existence of an extension operator can be characterized by the existence of measures on K which locally approximate the entries of the Whitney jets on the boundary of K. The quality of the approximation depends on the loss of derivatives. In the Chapters 4 and 5, we gave characterizations of the existence for the special cases of operators having no or a homogeneous loss of derivatives. Apart from the existence of measures on K, it is possible in these cases to find characterizations of a more geometrical character, which are easier to verify for a given compact set. The question arises, if such a property can found which gives a characterization of the existence of extension operators having an arbitrary loss of derivatives.

2. Is there a 'universal' loss of derivatives $\sigma_0 : [0, \infty) \to [0, \infty)$ such that for all compact sets $K \subset \mathbb{R}^d$ which admit an extension operator, there is $\lambda > 0$ and an extension operator $E : \mathscr{E}(K) \to \mathscr{E}(\mathbb{R}^d)$ with loss of derivatives $\lambda\sigma_0$.

3. According to Example 1. in Section 6.4, the cuspidal set

$$K_r := \left\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, \frac{1}{2}x^r \leq y \leq 2x^r\right\}.$$

   for $r \geq 1$ admits a tame linear extension operator with a loss of $(r + \varepsilon)s \mapsto s$. All efforts to show that it is possible to get rid of the ε for $r > 1$, i.e. to show that a loss of only $rs \mapsto s$ can be achieved, failed so far. For the set

$$K := \left\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1, 0 \leq y \leq x^r\right\},$$

Bierstone already showed in [Bie78] for $r \in \mathbb{N}$, that it admits and extension operator having a loss of $\sigma(n) = rn$.

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