The Pricing of Financial Derivatives under Transaction Costs

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Financial Derivatives and Transaction costs

Chapter 1

Introduction

Our aim is to develop a theory of arbitrage bounds for prices of contingent claims considering transaction costs, but regardless of other conceivable market frictions. However, we want to hold our assumptions on the market as general as convenient for the deduction of meaningful results that make good economic sense.

The main problems we have to deal with are how to

- model transaction costs?
- formulate a Fundamental Theorem of Asset Pricing under transaction costs?
- develop a dual characterization of arbitrage bounds under transaction costs ?

Of course we are not the first to adress these problems. Before we give a short survey of existing theory on arbitrage pricing and transaction costs, let us first explain the basic theoretical notions.

1.1 Basic Notions

A *financial derivative* is a *financial instrument,* i.e., a security or standardized financial contract, whose value and characteristics are derived in part from the value and characteristics of one or more other financial instruments or assets, the *underlyings.* In other words, a financial derivative is characterized by the fact that the claim resulting from an engagement in a financial derivative is *contingent* upon the prices or values of other, more basic financial instruments. Therefore, financial derivatives are also known as *contingent claims.*

Options and futures on stocks or currencies are familiar examples for financial derivatives. There is also a great diversity of more exotic products (see e.g. Hull, 1997). Possible underlyings are marketable commodities, securities, currencies or financial indices such as stock market indices, interest rates or any other standardized data, e.g. quoted temperatures. Moreover, marketable derivative instruments like options or future contracts can also serve as underlyings for more exotic derivatives such as options on futures or options on options.

The simplest example for a financial derivative according to our definition is a certificate on a stock market index or a certificate on any other basket of stocks. In fact, the value of such a security is derived in a straightforward way from the value of the underlying securities at any given time. Consequently, at any time one is able to calculate the current fair price of such a certificate as the weighted sum of the underlyings' current prices.

Unfortunately, typical financial derivatives such as futures and options lack this convenient feature. As these derivatives are essentially a bet on which way the value of the underlying instrument is going in the future, one has to await the issue of the bet at the so called *maturity date.* It is only then that the derivative's value can indeed be derived from the underlyings' prices in a definitely manner.

Thus the question arises, at which price a financial derivative should be traded before maturity. In other words, what is the *fair market value?* As *fair* market value we understand a price, at which an interested but not desperate seller could expect to find an interested but not desperate buyer or vice versa. So, the answer might depend upon the buyer's or seller's knowledge and assumptions on the underlyings' characteristics.

If one "knows" the possible values and the joint probability distribution of the underlyings, then one is able, at least theoretically, to derive the corresponding probability distribution for the payoffs from an investment in the derivative. Consequently, one can consider this investment as a fair game, if the expected payoff is greater or equal to the payoff from a secure investment until maturity.

But suppose that one party, say the buyer, can engage in a trading strategy in the underlyings that would yield a payoff pattern identical equal to that from the derivative investment. By a *trading strategy* we mean a sequence of financial transactions. Assume moreover that this strategy may be performed starting with a certain amount of *initial capital* and without adding or withdrawing any capital until maturity. Such a strategy is called a (replicating) *hedge* for the derivative. Then the buyer will compare the capital necessary to perform this hedge to the price to be paid for the derivative investment. If the initial capital for the hedge is less than the price of the corresponding derivative, then he will certainly not buy the derivative. This yields an upper bound for the price, that the buyer is willing to pay. It is remarkable that this bound is independent of the buyer's assumptions about the underlyings' probability distribution in as much as it does only depend on which prices are likely to occur. So this price bound is invariant to equivalent changes of measure.

A systematic treatment of price bounds induced by hedging strategies requires to allow for a larger class of trading strategies. If a strategy, starting with some inital capital, does not include the use of additional capital but provides for withdrawing capital and at all events yields a greater payoff than the derivative investment, then it is called a *super hedge* for the derivative. With these notions on hand we are now able to explain the concept of pricing by arbitrage.

1.2 Pricing by Arbitrage

Suppose that for a contingent claim there exists at least one super hedge. Provided that a buyer of the claim acts rationally, the maximum price at which he is willing to engage in the derivative investment will then equal the lowest amount of initial capital needed to perform himself a super hedge on this contingent claim. Under homogeneous trading conditions this maximum price will equal the lowest amount of intitial capital needed by the seller to perform a super hedge on the contingent claim. We call *homogenous trading conditions* the situation where every trader on the market acts under identical conditions, particularly with regard to trading and transaction costs. So in this situation, the maximum price that the buyer is willing to pay is equal to the lowest amount of capital that the seller of a claim needs in order to super-hedge his liability. Therefore this price is called the *seller's price.* In other words, if the seller achieved a higher price than the seller's price, then he would have the possibility to make a riskless profit, This is called an *arbitrage opportunity.* According to the above argumentation such an arbitrage opportunity should not exist on a market with homogeneous trading conditions. Therefore, the seller's price is an *upper arbitrage bound* for the fair price of a contingent claim on such a market. Because it is defined via super-hedging, the seller's price of a claim is also called its *super-hedging price.*

Analogously, a *lower arbitrage bound,* the so called *buyer's price,* can be found by the following argumentation. Suppose that at present date somebody has overdrawn his bank account by a certain amount of debt. Assume moreover that he has the opportunity to sell a certain contingent claim and reduce his debt with the proceeds. In other words, he is able to convert a part of this debt into the liability resulting from selling the claim. If he did so, then his liability as the seller of the claim would correspond to holding the negative claim. Suppose that alternatively the seller could also perform a super hedge on the negative of the claim. Let us assume that the initial "capital" for the super hedge (which would be negative in fact) was lower than the negative of the price for the claim. Then by performing the super hedge a greater part of debt could be turned into a liability that is not greater than that from selling the claim. Consequently he would decide not to sell the claim, but to perform the super hedge. This yields the following conclusion.

If the negative of a certain claim is super-hedgeable, then the minimum price that a seller could accept is given by the negative of the lowest amount of initial "capital" that is needed to super-hedge the negative of the claim. This means, that the minimum price equals the highest amount of debt satisfying the follwoing condition. Starting with the debt, one is able to perform a trading strategy resulting in a liability pattern, that is not worser than that resulting from selling the claim. Thus, under homogeneous trading conditions this minimum price corresponds to the highest amount of money that a buyer could lend against the claim as a secure garanty. This is true because by bearing a debt equal to the minimum price, holding the claim and performing the superhedge on the negative of the claim the buyer would end up without any debt at maturity. But if the price was lower than this price, then the buyer had an arbitrage opportunity. This reveals the lower arbitrage bound.

Pricing by arbitrage is only meaningful for financial derivatives whose underlyings are prices of marketable securities. Otherwise there is no way of super-hedging. Thus, before calculating arbitrage bounds, one has to think about a model of the underlying financial market. The first decision is, weather trading takes place in *discrete* or *continuous time,* i.e., at finitely many time points or continuously within a time interval. Moreover, one has to model the movement of prices and the transactions in the underlyings. The standard market models such as the Cox-Ross-

Rubinstein model in discrete time or the Black-Scholes model in continuous time are socalled *frictionless complete market models.* Henceforth we often simply speak of "markets" instead of a "market models". A market is called *frictionless,* if there are no such things as restrictions on trading, like prohibition of short selling, or different interest rates on borrowing and lending or transactions costs. A market is called *complete,* if every contingent claim is hedgeable (by a replicating hedge). For frictionless markets there exists a well developed theory of pricing by arbitrage. We refer to the textbooks of Karatzas (1997), Musiela, Rutkowksi(1997), Elliott, Kopp (1999), Karatzas, Shreve (1998) and Shiryaev (1999).

Frictionless complete markets are quite far away from reality. Due to the the following appealing property, however, they are very popular. In such markets, the upper and lower arbitrage bounds for the price of each contingent claim coincide, i.e., the seller's price equals the buyer's price. Consequently, in a frictionless complete financial market contingent claims are priceable by calculating the lowest initial amount of capital necessary in order to hedge the claim. These hedging strategies are constructed in such a way that the value of the portfolio resulting from the hedging strategy exactly equals the payoff of the contingent claim at maturity. Thus, the hedging strategy enables the writer of a contingent claim to eliminate the financial risk he would have to bear otherwise. In some not very complex discrete time models one can take advantage of this fact for a simultaneous calculation of price and strategy by a method called *backward induction.* This is possible in the Cox-Ross-Rubinstein model, where the price movement of a single underlying is modeled as a *binomial tree.* However the situation becomes more complex, if one has to model several underlyings or if one assumes continuous trading. Although in a discrete time model with several underlyings it is still possible to make use of backward induction, there is a much more efficient way of caculating prices. For this purpose, the initial optimization problem of finding the lowest intial capital for a hedge is transfromed into the dual problem which yields a *pricing formula* in terms of a *dual characterization.* The benefit of such a dual characterization of prices is that it theoretically allows to calculate prices using *martingale methods* because the *dual variables* are equivalent martingale measures. A probability measure is called a *martingale measures* for a (price) process, if the (price) process is a martingale. Given a financial market model for some underlyings, we call a probability measure simply *equivalent martingale measure,* if under this probability measure the price process of each underlying is a martingale and this measure is equivalent to the model's original probability measure. The famous Black-Scholes formula, for example, is such a dual characterization, although it has been derived in another way (c.f. Black, Scholes (1972,1973)). The set of admissible dual variables in the Black-Scholes formula consists of the unique equivalent martingale measure.

Dual characterizations may also be used to derive formulas for the sellers and the buyers prices in incomplete frictionless markets. A very elegant approach in this context is the concept of the *optional decomposition of supermartingales* (see Kramkov, 1996, Föllmer, Kabanov, 1998). It is also applicable in the case of certain restrictions on trading (see Föllmer, Kramkov, 1997). However, in every case, these pricing formulas rely on the *existence of an equivalent martingale measure* for the underlying asset price processes (or a probability measure with similar properties). But this existence assumption lacks an immediate economic justification. Hence the need arises to characterize the existence of equivalent martingale measures in terms of necessary and sufficient conditions that admit a direct economic interpretation. This is supplied by a certain type of theorems, that are called *Fundamental Theorems of Asset Pricing.* There is a multitude of Fundamental Theorems of Asset Pricing in different versions. Each of them states, that under certain premises the existence of an equivalent martingale measure is equivalent to the *absence* of some kind *of arbitrage opportunities.* These theorems mainly differ in assumptions on asset price processes, the time horizon of traders, admissible trading strategies and the definition of arbitrage opportunities.

The first theorems of this kind go back to Harisson, Kreps (1979), Kreps (1981) and Harrison, Pliska (1981) who introduced the notion of *"no free lunch"* with *"simple trading strategies"* in continuous time with finite horizon. Simple trading strategies are piecewise constant with a predetermined number of jumps. Their theory in continuous time heavily relies on postulations on consumer preferences. Without using such postulations, Harrison, Pliska (1981) already show the equivalence of the even weaker *"no arbitrage"* condition with the existence of a martingale measure for a finite state market in finite discrete time. Dalang et al. (1990), Kabanov, Kramkov (1994), Jacod, J., Shiryaev, A.N. (1998) and Shiryaev(1999) generalized this statement to arbitrary price processes in finite discrete time. Frittelli, Lakner (1995) give a Fundamental Theorem in a general continuous time financial market model. They characterize the existence of an equivalent martingale measure in terms of a "no free lunch" condition involving the topological closure of the *"set of achievable gains"* instead of taking limits of convergent sequences in this set. Delbaen (1992) and Schachermayer (1993) introduced the notion of *"no free lunch with bounded risk"* and stated their theorems in terms of simple trading strategies and limits of sequences. Their results refer to discrete, but infinite time sets (Schachermayer, 1993) and to continuous time when price processes are continuous and bounded (Delbaen, 1992). In order to achieve similar results for general semimartingale market models they introduced the notion of "no free lunch with vanishing risk" and applied it in a series of papers that culminated in Delbaen, Schachermayer (1998). In this latest paper they had to extend the class of equivalent martingale measures to the class of equivalent measures under which the price process is a so called "sigma-martingale" , a certain kind of martingale transform. With this definition they could show that if the price process is a semimartingale under a measure P, then the condition of "no free lunch with vanishing risk" is equivalent to the existence of an equivalent probability measure Q in that extended class. For the class of equivalent (local) martingale measures Delbaen, Schachermayer (1994) could prove this equivalence only in the case when the price process is a (locally) bounded semi-martingale. In summary, the Fundamental Theorems of Shiryaev(1999), Lakner (1995) and Delbaen, Schachermayer (1994, 1998) can be regarded as the current state of the art for frictionless markets.

1.3 Transaction Costs

The problem of transaction costs in the pricing of contingent claims is clearly motivated by practical considerations. Because of the lack of convenient pricing and hedging methods, however, transaction costs have been and still are neglected in the most part of pricing and hedging practice. The price of a European Call is often calculated in accordance with the concept of Black and Scholes, at times with some refinements concerning volatility, dividends, interest rates or alternative probability distributions of stock returns (c.f. Hull 1997), but regardless of transaction costs. Then one tries to approximate the theoretical replicating hedge by the following procedure. At the beginning one determines the number of underlyings in the hedge portfolio by calculating the Black-Scholes Delta-ratio, i.e. the first derivative of the Balck-Scholes price with respect to the underlying's price. Thereafter, as time passes and prices change, one recalculates the Black-Scholes Delta-ratio, but the portfolio is only adjusted if this ratio differs too much from the current proportion in the hedge portfolio. This *discrete adjustment of hedges* has already been suggested and analysed by Black, Scholes (1972) themselves in that paper in which they proposed their prominent pricing formula. Of course they also realized the important consequence, that if stock prices evolve according to their assumptions, then this hedging practice does not result in a replicating hedge and no super hedge either. Hence the final hedge return from a short position in an option and the hedge portfolio is not zero but becomes risky. However, Black and Scholes back up their method with the justified expectation, that in mean the hedging error turns to zero, at least if transaction costs are ignored. Some additional empirical research on the distribution of the returns of discretely adjusted option hedges has been done by Boyle, Emanuel (1980), who do not take account of transaction costs either. However, one of the practical reasons behind this strategy are obviously transaction costs such as commission fees and bid-ask spreads. Since at least for large transactions, things like minimum commission fees and fixed costs are neglegigible, the cost associated with a large transaction is approximately *proportional* to the traded volume, i.e., the quantity of money that is involved in the transaction. Thus it is quite self-suggesting to consider models with *proportional transaction costs.* The implementations of such models will be discussed in detail in the next chapter.

The first analysis of a discrete hedging strategy in a Black-Scholes world in account of transaction costs is by Leland (1985) and has been updated by Kabanov, Safarian (1997). They consider a stock market where the cost of a single transaction is a fixed fraction of its trading volume, i.e. number of shares times price for one share. Their objective is to calculate the limit hedging error, when the length of time intervals goes to zero. However such hedges always have to remain risky, since in the Black-Scholes model with proportional transaction costs it is impossible to replicate a European Call option. The reason is that necessary trading strategies, even if they were processes of bounded variation, are not of *uniformly* (in almost every ω) bounded variation and hence there is no almost sure bound for the resulting transaction costs. However, the writer of a European Call option could avoid any loss from selling the option, if he simply would sell the option for a price equal to the amount of money required for buying the underlying, then buy this underlying and hold it until maturity of the option. This trivial strategy is an example for a super hedge that is applicable to any market. It was first conjectured by Clark and Davis (1995) that this trivial strategy is the cheapest way of super-hedging the European Call option in the Black Scholes model with volume proportional transaction costs. In fact this conjecture has been confirmed in three different proofs (Soner et al., 1995, Leventhal, Skorohod, 1997, Cvitanić et al. 1999).

Leventhal, Skorohod (1997) actually work with a model where the price process of the underlying is a continuous semimartingale with a cad (continue à droite) filtration. Consequently the super-hedging price of the European Call option is higher than that of the underlying. Touzi (1999) has extended this result to a more general class of claims in a multidimensional Black-Scholes model. He considers claims whith payoffs that are a lower semicontinuous, bounded from below function of the price of one or several assets at a certain single date. For such claims the cheapest buy and hold super hedge is shown to be the optimal super hedge.

Although it is clear that these negative results do not apply for path-dependent options in general, it is obvious that, at least within the Black-Scholes model, super-hedging prices cannot be used for market making. However they are worth to be studied because they are fundamental for other more realistic hedging concepts such as *quantile-hedging* (see Föllmer and Leukert, 1999) or *utility maximization* (see Karatzas 1997, Kabanov, 1999). An application to quantile-hedging will be given in Chapter 9, where we also extend some results of Föllmer, Leukert (1999) to models with proportional transaction costs. Among others it will be shown, that quantile-hedging is about hedging path-dependent knock-out options. Consequently, the result of Touzi (1999) does not apply and one can expect to find better hedging schemes than buy and hold strategies. In Chapter 10 finally, we describe a class of security price processes in continuous time that we call log-Lipschitz processes. We show that log-Lipschitz price processes yield non-trivial super hedges in the presence of transaction costs. In particular, we prove the existence of non-trivial super hedges for the European Call option in continuous time markets with log-Lipschitz price processes.

Calculating the seller's price of a knock-out option in general is apparently not as simple as finding a buy and hold strategy. Therefore, despite of the above cited negative results, it seems useful to look for dual characterizations of arbitrage bounds in order to calculate them by martingale techniques. In the same view Cvitanić, Karatzas (1996) and Cvitanić (1997) derived a dual characterization of the seller's price of a claim in the Black Scholes model with proportional transaction costs, even though they already knew the above negative results on super-hedging. Kabanov (1999) then was the first who formulated a dual characterization of *super-hedging initial endowments* in a semimartingale model for a currency market with proportional transaction costs in continuous time. Super-hedging initial endowments can be understood as a generalization of super-hedging prices in that the initial amount of capital needed for a trading strategy can be held as a portfolio of assets traded in the market and needs not to be held as money only. His essential assumptions are the existence of dual variables, i.e. some kind of martingale measure, and the continuity of price processes together with some conditions on their variation over time. These assumptions are certainly satisfied for the multidimensional Black-Scholes model. Nevertheless it would be nice to have a dual characterization that is valid in a general semimartingal modell with much weaker conditions on the price processes. We are going to derive such a result in Chapter 8.

Moreover, one would like to have a Fundamental Theorem of Asset Pricing for general continuous time models in order to correlate the existence of the dual variables to the absence of arbitrage opportunities.

Up to now, such Fundamental Theorems have only been proven by Jouini (1996) and Jouini, Kallal (1995a). They deal with a security market model similar to that in Harrison, Kreps (1979), but with proportional transaction costs caused by bid-ask spreads. However their results are not completely satisfactory for the following reasons. First of all, their results are only valid for stock markets, where no direct exchange between assets is possible, but only buying and selling. Moreover they work in a general adapted price process framework and therefore can only allow for simple trading strategies. In addition they need unnatural integrability conditions on the prices process and on trading strategies. By the way, we do not know of any Fundamental theorem for a continuous time curreny market model that includes proportional transaction costs.

We intend to close this gap in chapter 7. There we develop two Fundamental Theorems, involving several notions of "free lunches", for a continuous time market where trading is restricted by differential cone constraints. These constraints include the case of self-financing with or without proportional transaction costs on stock or currency markets with a bid-ask spread or a cost structure like in Kabanov (1999). The notions of free lunch we introduce are defined in terms of convergent sequences in various topologies. Inspired from the papers of Jouini, Kallal (1995a) and Pham, Touzi (1999), who consider convergence in $L^2(P)$ resp. $L^1(P)$ for a given probability measure P, we introduce L^p *free lunches* for $1 \leq p < \infty$, but for a class $\mathcal{Q}$ of P-equivalent measures. The advantage of this approach in comparison to that of Jouini, Kallal and Pham, Touzi is, that our L^p free lunches do not depend on the arbitrary choice of a probability measure P. Moreover unlike Jouini, Kallal (1995a) or Jouini (1996) we do not need any integrability or topological conditions on the price process. This could also be achieved by further development of ideas we found in Clark (1993), that we also applied with another notion of arbitrage, that we call *L^∞-bounded free lunches.* These are free lunches in terms of convergence in probability, but with an additional feature, that relates them to convergence in the $\sigma(L^\infty, L^1)$- topology. In fact L^∞-bounded free lunches are comparable to "free lunches with bounded risk" or "free lunches with vanishing risk" (see above). Their consideration was inspired by a Theorem in Delbaen, Schachermayer (1994, Theorem 2) that we have reformulated in a more general version (see Theorem A.9). Since our results are also valid for the classic frictionless markets, it is worth noting, that our proofs remain rather short and comprehensible in comparison to those of Delbaen, Schachermayer (1994, 1998) although our assumptions on price processes are weaker. This could only be achieved by using a different notion of arbitrage. While Delbaen, Schachermayer (1994, 1998) work with "no free lunch with vanishing risk" we have to rely on the "no L^∞-bounded free lunch" condition and a finite time horizon. Moreover our L^∞-bounded free lunches are defined with respect to the units of the different assets held in the portfolio whereas Delbaen, Schachermayer (1994, 1998) just consider the value of a portfolio. Their "no arbitrage condition" as well as their "no free lunch with vanishing risk" condition only refers to terminal portfolios with values in L^∞. This has a severe consequence whenever price processes are not in L^∞. In fact in this very common case, the "no free lunch with vanishing risk" condition is not sufficient for ruling out arbitrage opportunities or free lunches that rely on simple buy and hold strategies. In constrast such free lunches are not possible if there is "no L^∞-bounded free lunch". All this led us to the supposition, that our L^∞-bounded free lunches are very well suited for the derivation of fundamental theorems, if one intends to hold assumptions on the price process as weak as possible. In fact, only for stochastic integration issues we have to impose that the price process is a semimartingale.

Another main advantage of our approach is that it covers stock markets as well as currency markets with or without arbitrary proportional transaction costs in continuous time. Besides the shortness and relative simplicity of proofs our main progress for frictionless markets is that we can characterize the existence of an equivalent martingale measure by "no free lunch"-conditions in terms of convergent sequences without any particular restrictions on the price process. This was achieved by some elementary techniques and a favorable application of the Halmos-Savage Theorem, similar to that in the proof of Theorem 7 in Clark (1993). Moreover the results remain valid, if we consider general processes in combination with simple trading strategies. In general, it will turn out that the choice of a suitable class of admissible trading strategies is related to the model for the price process. But this class will always include those simple trading strategies that satisfy a certain tame condition (see Lemma 6.10).

Surprisingly there is still a gap in theory for market models in discrete time, too. The theory is quite complete for security market models with a finite probability space (c.f. Naik, 1995). The Fundamental theorems of Jouini (1996) and Jouini, Kallal (1995a) also comprise the case of a discrete time security market with uncountable probability space. However they impose several technical assumptions such as integrability assumptions and other (see above) that we wish to relax. For a finite currency market Kabanov, Stricker (1999) show a Fundamental Theorem with proportional transaction costs. By the way, they do not model transaction costs by means of bid ask spreads but they use transaction cost factors that are applied to unique spot prices.

Pham, Touzi (1999) treat a stock market with uncountable probability space. They consider proportional transaction costs in addition to cone constraints on trading strategies, as for example the exclusion of short-sales. Note that their cone constraints are not differential cone constraints, as we are going to set up. They intend to complete the theory of Jouini (1996) and Jouini, Kallal (1995a) in that they want to prove a Fundamental Theorem using the "no arbitrage" condition instead of the stronger "no free lunch" condition. However, their technique of proof fails in multiperiod settings. So their result is only valid for a one period market. In Chapter 4 we are going to show a multiperiod version of this result in a more general security market with a very different technique of proof, but very similar non-degeneracy assumption on the price process. Before this, we develop Fundamental Theorems for a general security market with proportional transaction costs, including stocks and currency markets. First we state a condition sufficient in order to generalize the theorem of Dalang et. al.(1990) and Kabanov, Kramkov (1994) to markets with transaction costs. It will turn out that this condition is satisfied, only if transaction costs are small in comparison to maximal price changes. Besides, this will reveal that in a frictionless market the "no arbitrage" condition is equivalent to a "no certain loss" condition. And this equivalence is preserved under the condition of "small transaction costs". For the general case we prove Fundamental Theorems using several "no free lunch" conditions, similar to that in the continuous time case (see above).

In chapter 5 we derive a dual characterization of superhedging initial endowments in a general discrete time market model with infinite state space. Additionally we state a stronger dual characterization under the already mentioned non-degeneracy assumption. We did so in order to complement the numerous studies of superhedging in binomial and multinomial or other finite security markets. Especially for the Cox-Ross-Rubinstein model with proportional transaction costs super-hedging prices are not as unrealistic as in the Black Scholes model. In fact, most papers about super-hedging under proportional transaction costs deal with this model (see Bensaid et al., 1992, Boyle, Vorst, 1992, Ediringshe, 1993, Mercurio, Vorst, 1997). It is only if the set of trading dates tends to infinity, that the initial capital for the cheapest super hedge converges to the capital needed for the cheapest static super hedge. This has been shown by Koehl, Pham, Touzi (1999) and Touzi (1999).

As we have outlined, most attention in research has been devoted to markets with proportional transaction costs. One of the reasons therefore is simply that the theoretical results, that can be expected, will depend on how transaction costs are modelled. In practice, apart from time and effort, there are two kinds of costs that arise with trading securities. First there is a spread between ask and bid price, which results from the fact that everyone would like to buy at low price and sell at a high price. This spread in fact causes transaction costs that are proportional to the traded volume. Second, there are such things as commission fees etc., which typically consist of a fixed part, i.e. a minimum fee, and a variable part, that is somehow proportional to the traded volume. Hence it seems appropriate to model transaction costs by a function that is piecewise affine linear and concave in the traded volume, with a jump at zero.

But it is easy to see that with such a model, the arbitrage bounds for the price per unit of a certain derivative could depend on how many units are going to be traded. For example if one introduces such a transaction cost structure into the Black-Scholes model, then the seller's price of x units of a European Call option on a certain security would equal the price of x units of the stock plus transaction costs. Consequently the price of a unit of the Call, net of transaction costs, would depend on how many Calls one buys at one time. One might call it an irony, that the assumption of proportional transaction cost seems to deliver more suggestive results than a modeling that is nearer to reality. This is the main reason, why we consider solely proportional transaction costs, since moreover for sufficiently large transactions the associated costs are indeed almost proportional.

Chapter 2

Modelling Proportional Transaction Costs

Existing literature proposes several distinct models for different market situations, such as stock vs. currency markets or constant proportional transaction cost factors vs. bid-ask spreads. Similar are we going to model proportional transaction costs in different market situations such as stock or currency markets with or without transaction costs, that may be deterministic or random. However we will find that in each of these situations the set of feasible transactions is always characterized by certain cone constraints. Although each different situation will result in a different cone, these cones have a lot of similarities. Thus, by only using the common features of the different cones, that correspond to different market situations, we are able to study these different market situations at the same time. This is indeed a great advantage of our conception.

To provide a clear arrangement of statements and calculations we apply the following symbols and notations:

$$x \star y := (x_i y_i)_{i \in \{0,\dots,d\}}$$

for componentwise multiplication of vectors $x, y \in \mathbb{R}^{d+1}$,

$$xy := \sum_{i=0}^{d} x_i y_i$$

for the usual scalar product $x, y \in \mathbb{R}^{d+1}$,

$$\rho^+ := \max(\rho, 0) \ , \quad \rho^- := \max(-\rho, 0)$$

for the positive respectively negative part of a real number $\rho \in \mathbb{R}$.

2.1 General Framework

We consider a market where $d+1$ financial assets $i = 0, 1, \ldots, d$ are traded at time points $t \in \mathcal{T} \subset [0, T]$, $T \in \mathbb{R}_+$. Within this chapter, we need not to specify wether $\mathcal{T}$ is a discrete or continuous time set, because we just study transactions at a single arbitrary time point $t \in \mathcal{T}$. However in subsequent chapters, when analyzing trading strategies, we will have to distinguish between discrete and continuous time trading. Part II will deal with discrete time trading while Part III covers continuous time.

Asset 0 is always taken as a numeraire in the market. The other assets play the role of underlyings for contingent claims, and will be referred to as securities. These securities may be stocks, currencies or any other financial assets for which at time $t \in \mathcal{T}$ there is a quoted market price. As usual, uncertainty and information structure in this market are modelled as a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathbb{F} := (\mathcal{F}(t))_{t \in \mathcal{T}}$. For every asset $i = 1, \ldots, d$ there will be something like a spot price process $X_i(\omega, t)$ that quotes the price of asset i in units of the numeraire asset 0 at (ω, t). For convenience, we also define $\forall (\omega, t) : X_0(\omega, t) = 1$. The interpretation of $X_i(\omega, t)$ will result from its concrete definition and the characteristics of the specific market. Exemplary models are discussed in sections 2.2 to 2.6. Since it is in the nature of a spot price to be revealed not later than at time t, we postulate that for each $t \in \mathcal{T}$ the map $X_i := X_i(., t)$ is a $\mathcal{F}(t)$-measurable random variable on $(\Omega, \mathcal{F}, P)$. The definition

$$X(\omega, t) = (X_i(\omega, t))_{i \in \{0, \ldots, d\}} \quad , (\omega, t) \in \Omega \times \mathcal{T} \tag{2.1}$$

then yields the $\mathbb{R}_+^{d+1}$-valued $\mathbb{F}$-adapted price process $X = (X(t))_{t \in \mathcal{T}}$. Securities are always quoted at a positive price. Hence it is reasonable to assume

$$\forall (\omega, t) \in \Omega \times \mathcal{T} \forall i \in \{0, \ldots, d\} : X_i(\omega, t) > 0 \ . \tag{2.2}$$

We prefer the pointwise formulation (2.2) to an "almost sure" statement for the only reason, that we want to beware of fruitless elaborations on trivial items, such as avoidance of divisions by zero on null sets. In some market models, e.g., if asset 0 represents a bank account, a bond or any other kind of asset that is bearing interest, it may be necessary to allow for something like a price change in asset 0. In such a case, we propose to introduce the strictly positive $\mathbb{F}$-adapted process $S_0 = (S_0(t))_{t\in\mathcal{T}}$ in order to model this price change. So $S_0(\omega, t) > 0$ stands for the price of asset 0 at (ω, t) in units of "money". The price process $S = (S_i)_{i=0,\dots,d}$ is then defined by

$$S_i(t) := S_0(t)X_i(t) \quad i = 1, \dots, d,$$

where $S_i(t)$ denotes the price of the $i-$th asset $(i = 0, \dots d)$ in units of "money" at time t. The transformation of prices from units of money to units of asset 0 is a kind of discounting. Hence $X = \frac{1}{S_0}S$ is also called the *discounted price process,* because it quotes the values of assets $i = 0, \dots, d$ in units of asset 0. When using X instead of S we speak of the *discounted market.*

Henceforth we always consider the discounted market in order to simplify the representation of results and proofs. By discounting price processes, we transform the original market into a market with a riskless asset. Therefore, one must not forget this transformation, because it may change some features of financial instruments, if asset 0 is not riskless before the transformation (an example is given below). A collection of financial assets is called a *portfolio.* Translated from common speach into our market model, a portfolio is a $d+1$-tuple $h = (h_i)_{i\in\{0,\dots,d\}} \in \mathbb{R}^{d+1}$ where h_i represents the number of units of asset i that the trader holds. Of course, at a given time t it might be favorable to hold different portfolios at different "market states" $\omega \in \Omega$. A *random portfolio* at time t is a $\mathbb{R}^{d+1}$-valued $\mathcal{F}(t)$-measurable random variable $H(t)$. *Portfolio processes* will be introduced in subsequent chapters, because their definition is related to the specific set of trading dates $\mathcal{T}$. The notion of a *European contingent claim* maturing at time T, however, is independent of the specific $\mathcal{T}$. In fact, such a claim is nothing else than a random portfolio at time T, i.e., a $\mathbb{R}^{d+1}$-valued $\mathcal{F}(T)$-measurable random variable $C = (C_i)_{i\in\{0,1,\dots,d\}}$. Since we are not going to consider contingent claims other than European, we will simply call them *claims.* A component C_i , $i = 0, \dots, d$, is interpreted as the units of asset i, that the seller of the claim will supply to the buyer. There, supply of negative quantities means delivery from buyer to seller. For a European Call on security 1 with strike K and real delivery of the asset we have

$$C_0 = -K1_{\{S_1>K\}},\ C_1 = 1_{\{S_1>K\}} \text{ and } \quad \forall i \geq 2 : C_i = 0 \ ,$$

whereas with cash settlement we have

$$C_0 = (S_1(T) - K)^+ \quad \text{and} \quad \forall i \geq 1 :\ C_i = 0 \quad .$$

Let us see which features of a contingent claim may change under the discounting transformation. Consider once more the European Call option on security 1 with cash-settlement maturing at T with deterministic strike K. For simplicity of their structure such Options are labeled as "Vanilla" options in distinction to more "exotic" options. In the original market the payoff from the Call at maturity is $(S_1(T) - K)^+$. However, in the discounted market this payoff becomes

$$\frac{(S_1(T) - K)^+}{S_0(T)} = (X(T) - \frac{K}{S_0(T)})^+ \ .$$

Consequently, if asset 0 has not been riskless before discounting, then discounting changes the Vanilla Call in an exotic Call with stochastic strike. So we have to be careful about weather the riskiness of component 0 of a certain random portfolio will change by discounting.

2.2 Security Market without Transaction Costs

In this section we want to give a first example for how feasible transactions can be characterized by differential cone constraints. Since for the moment, we are just dealing with a simple frictionless security market model, many of the following explanations may seem to be needless at first glance. However, we do so in order to elucidate the frictionless market model as a special case of each of the transaction cost models in sections 2.3 to 2.6.

Given a security market with a numeraire, represented by asset 0, and d securities, represented by assets $i = 1, \ldots, d$, suppose that trading does not cause any transaction costs. This means, there are no bid-ask spreads or commission fees. Instead, every security may be bought or sold at it's spot price. Denote $X_i(\omega, t)$ the spot price of security $i \in \{1, \ldots, d\}$ at $(\omega, t) \in \Omega \times \mathcal{T}$ in units of asset 0. Let $X = (X_i)_{i \in \{0, \ldots, d\}}$, be the $d+1$-dimensional, $\mathbb{F}$-adapted, discounted spot price process defined according to (2.1) and satisfying (2.2).

For one unit of asset i given to the market at (ω, t), one can expect to receive $X_i(\omega, t)$ units of asset 0 and vice versa. Consequently, if one holds a portfolio $h = (h_0, h_1, \ldots, h_d) \in \mathbb{R}^{d+1}$, and decides to *liquidate* it at (ω, t), i.e., to sell h_i^+ and to buy h_i^- quantities of each asset i, he can expect to get

$$hX(\omega, t) = \sum_{i=0}^{d} h_i X_i(\omega, t)$$

units of asset 0. This is the same quantity, that would be necessary in order to build the portfolio h. Moreover given $hX(\omega,t)$ units of asset 0 we could build any other portfolio $\tilde{h} \in \mathbb{R}^{d+1}$ that satisfies the equation

$$hX(\omega,t) = \tilde{h}X(\omega,t) \ . \tag{2.3}$$

Instead of only liquidating or building a portfolio we could also be interested in rebalancing it by means of several transactions at (ω,t). By *rebalancing* we mean that the portfolio h is changed into another portfolio $\tilde{h}$ without adding or withdrawing any capital, i.e., the transactions within the *rebalancement* are *self-financing.* If we also allow for withdrawing capital, then we call this a *rebalancement with consumption.*

Given a portfolio h, which portfolios $\tilde{h}$ can we obtain then by rebalancements? It is easy to see that after liquidating portfolio h we are able to build any portfolio $\tilde{h}$ that satisfies (2.3). Since there are no transaction costs it makes no difference weather for example we sell h_i units of asset i and buy $\tilde{h}_i < h_i$ units of the same asset at the same time or simply sell $h_i - \tilde{h}_i$ units of asset i. Hence equation (2.3) characterizes the entire set of portfolios that are attainable by rebalancements (without consumption) given h. Moreover, if we allow for consumption, then we simply have to replace "=" by "$\geq$" in (2.3).

Of course, in the presence of transaction costs we would not always liquidate h before building $\tilde{h}$. Instead, we would calculate the difference $\tilde{h} - h$ and then perform the transactions necessary to build the *differential portfolio* $\tilde{h} - h$. Hence another way to characterize the set of feasible rebalancements is to say, that we need no capital in order to build the differential portfolio $\tilde{h} - h$. If we allow for consumption, this yields the condition

$$0 \geq (\tilde{h} - h)X(\omega,t) \ , \tag{2.4}$$

which of course is only a reformulation of (2.3) (with "=" replaced by "$\geq$"). Multiplying (2.4) by -1, we see that this is the same as saying that liquidation of portfolio $h - \tilde{h}$ yields a non-negativ amount of asset 0. Since this is a characterzation of feasible rebalancements, that will not change by taking account of transaction costs, we are going to focus on the set of portfolios with non-negative liquidation value. For every $(\omega,t) \in \Omega \times \mathcal{T}$ this set is defined by

$$K(\omega,t) = \{h \in \mathbb{R}^{d+1} : hX(\omega,t) \geq 0\} \ .$$

(2.2) implies that for P-almost every $(\omega,t) \in \Omega \times \mathcal{T}$ we have $\mathbb{R}^{d+1}_{+} \subset K(\omega,t)$. Given $K(\omega,t)$, one may ask, how much prices $X(\omega,t)$ can vary without changing $K(\omega,t)$.

We are going to consider variations of the form $r \star X(\omega, t)$ with $r \in \mathbb{R}^{d+1}$ constrained to $r_0 = 1$, because it is senseless to change the numeraire. This motivates us to consider the compact convex sets

$$K_0'(\omega, t) := \{r \in \{1\} \times \mathbb{R}^d : \inf_{h \in K(\omega,t)} h(r \star X(\omega, t)) \geq 0\} \quad , (\omega, t) \in \Omega \times \mathcal{T}.$$

Indeed, it is easy to see that for every $(\omega, t) \in \Omega \times \mathcal{T}$ we have

$$K_0'(\omega, t) := \{(1)^{d+1}\} \; .$$

Of course the situation will slightly change, when we introduce transaction costs. However $K_0'(\omega, t)$ will remain a compact convex set and prove useful in order to characterize $K(\omega, t)$, since, as we are going to see, we have

$$K(\omega, t) = \{h \in \mathbb{R}^{d+1} : \forall r \in K_0'(\omega, t) : h(r \star X(\omega, t)) \geq 0\} \; .$$

A very similar characterization is obtained if instead of $K_0'(\omega, t)$ for every $(\omega, t) \in \Omega \times \mathcal{T}$ we consider the convex cone

$$\begin{aligned} K'(\omega, t) &:= -\mathbb{R}_+ K_0'(\omega, t) \\ &= \{z = (z_0, \ldots, z_d) \in \mathbb{R}^{d+1} : \exists r \in K_0'(\omega, t) \exists z_0 \in \mathbb{R}_+ : z = -z_0 r\} \; . \end{aligned}$$

From the definition of $K_0'(\omega, t)$ it is evident that for all $(\omega, t) \in \Omega \times \mathcal{T}$ the cone $K'(\omega, t)$ satisfies

(2.5) $$K'(\omega, t) = \{z \in \mathbb{R}^{d+1} : \sup_{h \in K(\omega,t)} h(z \star X(\omega, t)) \leq 0\} \; .$$

For each $(\omega, t) \in \Omega \times \mathcal{T}$ equation (2.5) reveals $K'(\omega, t)$ as the *dual cone* of $K(\omega, t)$ with respect to the scalar product $< .,. > (\omega, t)$ defined by

(2.6) $$< h, z > (\omega, t) := h(z \star X(\omega, t)) = \sum_i h_i z_i X_i(\omega, t) \quad , \; h, z \in \mathbb{R}^{d+1} \; .$$

In Convex Analysis dual cones are sometimes referred to as *polar cones* (see Rockafellar, 1970, p.121). The set

$$-K'(\omega, t) = \mathbb{R}_+ K_0'(\omega, t) \subset \mathbb{R}_+^{d+1}$$

is the *dual positive cone* of $K(\omega, t)$. We also consider $-K'(\omega, t)$, because in some situations this convex cone is easier to tract than the compact convex set $K_0'(\omega, t)$.

2.3 Stock Market with Constant Transaction Cost Factors

Suppose the $\mathbb{R}_+^{d+1}$-valued $\mathbb{F}$-adapted process X defined according to (2.1) and satisfying (2.2) is the discounted spot price process of a stock market. However, there are constant, volume proportional transaction costs. This means, for every $i \in \{1, \ldots, d\}$ there are constant transaction cost factors λ_i, μ_i, that impinge on ask (buying) and bid (selling) prices in the following way. If at (ω, t) somebody wants to buy h_i units of asset i, then he has to pay $h_i(1 + \lambda_i)X_i$ units of asset 0 and if he sells h_i units of asset i, then he gets $h_i(1 - \mu_i)X_i$ units of asset 0.

Suppose that we hold a portfolio $h \in \mathbb{R}^{d+1}$ which we want to rebalance at (ω, t). Which portfolios are attainable by a rebalancement? As we have also explained in the case of no transaction costs, a portfolio $\tilde{h}$ is obviously attainable by a rebalancement (with consumption) if we need no capital in order to build the differential portfolio $\tilde{h} - h$. Let us look which transactions are necessary for this purpose. For every i we have to buy $(\tilde{h}_i - h_i)^+$ quantities of asset i and to sell $(\tilde{h}_i - h_i)^-$ quantities of asset i. Suppose we have given the negative of the differential portfolio, i.e., we hold $h - \tilde{h}$. Then the transactions necessary in order to liquidate $h - \tilde{h}$ would be exactly the same as that in order to build $\tilde{h} - h$. So we can conclude that a portfolio $\tilde{h}$ is attainable by a rebalancement of h (with consumption), if and only if the portfolio $h - \tilde{h}$ has non-negative liquidation value. Again we only need to focus on the set $K(\omega, t)$ of portfolios with non-negative liquidation value at (ω, t). By analyzing the transactions necessary in order to liquidate a portfolio we get

$$K(\omega, t) = \{h \in \mathbb{R}^{d+1} : h_0 + \sum_{j=1}^{d} h_j^+(1 - \mu_j)X_j(\omega, t) - \sum_{j=1}^{d} h_j^-(1 + \lambda_j)X_j(\omega, t) \geq 0\} \; .$$

In fact, given a portfolio $h \in K(\omega, t)$, one may sell h_j^+ units of asset j and buy h_j^- units of asset j for every $j \in \{1, \ldots, d\}$ which, together with the h_0 units of money that one already holds, will yield exactly

$$h_0 + \sum_{j=1}^{d} h_j^+(1 - \mu_j)X_j(\omega, t) - \sum_{j=1}^{d} h_j^-(1 + \lambda_j)X_j(\omega, t) \geq 0$$

units of asset 0. We see immediately that this yield does not decrease, if we replace $1 - \mu$ and $1 + \lambda$ by some

$$r \in \{1\} \times [1 - \mu_1, 1 + \lambda_1] \times \ldots \times [1 - \mu_d, 1 + \lambda_d] \; . \tag{2.7}$$

Moreover for any r not satisfying (2.7) it is easy to find $h \in K(\omega,t)$ such that $h(r \star X(\omega,t)) < 0$. Consequently for

$$K_0'(\omega,t) := \{r \in \{1\} \times \mathbb{R}^d : \inf_{h \in K(\omega,t)} h(r \star X(\omega,t)) \geq 0\} \quad , (\omega,t) \in \Omega \times \mathcal{T}$$

we have

$$K_0'(\omega,t) = \{r = (1, r_1, \ldots, r_d) \in \mathbb{R}^{d+1} : \forall j \in \{1,\ldots,d\} : 1 - \mu_j \leq r_j \leq 1 + \lambda_j\} \ .$$

Hence $K_0'(\omega,t)$ is compact and independent of $\omega \in \Omega$. Moreover we see that

$$\begin{aligned} K(\omega,t) &= \{h \in \mathbb{R}^{d+1} : h_0 + \sum_{j=1}^d h_j^+(1-\mu_j)X_j(\omega,t) - \sum_{j=1}^d h_j^-(1+\lambda_j)X_j(\omega,t) \geq 0\} \\ &= \{h \in \mathbb{R}^{d+1} : \forall r \in K_0'(\omega,t) : h(r \star X(\omega,t)) \geq 0\} \ . \end{aligned}$$

From the definition of $K_0'(\omega,t)$ it is evident, that for every $(\omega,t) \in \Omega \times \mathcal{T}$ the cone $-K'(\omega,t) := \mathbb{R}_+ K_0'(\omega,t)$ satisfies

$$-K'(\omega,t) = \{z \in \mathbb{R}^{d+1} : \inf_{h \in K(\omega,t)} h(z \star X(\omega,t) \geq 0\} \ .$$

For each $(\omega,t) \in \Omega \times \mathcal{T}$ this again reveals $-K'(\omega,t)$ as the dual positive cone of $K(\omega,t)$ with respect to the scalar product $< .,. > (\omega,t)$ defined like in (2.6). Note also, that (2.2) implies that for every $(\omega,t) \in \Omega \times \mathcal{T}$ we have $\mathbb{R}_+^{d+1} \subset K(\omega,t)$.

2.4 Stock Market with Bid-Ask Spread

In this section we model a stock market with a bid-ask spread, by assuming that there are two $\mathbb{R}_+^{d+1}$-valued $\mathbb{F}$-adapted processes $\underline{X} = (\underline{X}_i)_{i\in\{0,\ldots,d\}}$ and $\overline{X} = (\overline{X}_i)_{i\in\{0,\ldots,d\}}$, where $\underline{X}_i(\omega,t)$ is the bid price of asset i at (ω,t) and $\overline{X}_i(\omega,t)$ is the corresponding ask price. Securities are always bought and sold at a positive price. Moreover the ask price of a security cannot be lower than it's bid price. Hence it is reasonable to assume

$$\forall(\omega,t) \in \Omega \times \mathcal{T} \ \forall i \in \{0,\ldots,d\} : 0 < \underline{X}_i(\omega,t) \leq \overline{X}_i(\omega,t) \ . \tag{2.8}$$

As we have pointed out in the two previous sections, a portfolio h can be changed into a portfolio $\tilde{h}$ by a rebalancement with consumption at (ω,t), if and only if the portfolio $h - \tilde{h}$ has a non-negative liquidation value at (ω,t). The set of portfolios h that have non-negative liqidation value at (ω,t) is obviously given by

$$K(\omega,t) = \{h \in \mathbb{R}^{d+1} : h_0 + \sum_{j=1}^d h_j^+ \underline{X}_j(\omega,t) - \sum_{j=1}^d h_j^- \overline{X}_j(\omega,t) \geq 0\} \ .$$

From (2.8) it is clear, that for every $(\omega, t) \in \Omega \times \mathcal{T}$ we have $\mathbb{R}_+^{d+1} \subset K(\omega, t)$. Let us define an artificial spot price process by setting $X := \underline{X}$. Then like in section 2.3 it is easy to see that with the compact convex set

$$K'_0(\omega, t) := \{r = (1, r_1, \ldots, r_d) \in \mathbb{R}^{d+1} : \forall j \in \{1, \ldots, d\} : \underline{X}_j \le r_j X_j \le \overline{X}_j\}$$

and the convex cone $-K'(\omega, t) := \mathbb{R}_+ K'_0(\omega, t)$, we have

$$K(\omega, t) = \{h \in \mathbb{R}^{d+1} : \inf_{r \in K'_0(\omega,t)} h(r \star X(\omega, t)) \ge 0\}$$

$$-K'(\omega, t) = \{z \in \mathbb{R}^{d+1} : \inf_{h \in K(\omega,t)} h(z \star X(\omega, t) \ge 0\} \ .$$

The interpretation is anologous to that in the previous example. Note also that for every $r \in \mathbb{R}^{d+1}$ and every $t \in \mathcal{T}$ we obviously have $\{\omega \in \Omega : r \in K'_0(\omega, t)\} \in \mathcal{F}(t)$.

2.5 Currency Market with Constant Transaction Cost Factors

In this example we turn to a currency market with volume proportional transaction cost factors. Suppose there are $d + 1$ assets where asset 0 deserves as a numeraire, we may think of the domestic currency. Let X be a $\mathbb{R}_+^{d+1}$-valued $\mathbb{F}$-adapted process, defined according to (2.1) and satisfying (2.2), where $X_i(\omega, t)$ denotes the "spot" price of asset i in units of asset 0. Since on a currency market every currency is exchangeable with any other currency, there will be two matrices

$$\lambda = (\lambda_{ij})_{i,j \in \{0,\ldots,d\}} \in \mathbb{R}_+^{(d+1)\times(d+1)} \quad , \quad \mu = (\mu_{ij})_{i,j \in \{0,\ldots,d\}} \in \mathbb{R}_+^{(d+1)\times(d+1)}$$

of proportional transaction cost factors, that will be applied on spot prices in the following way. If at (ω, t) somebody wants to buy h_i units of asset i, then he has to pay

$$h_j = h_i(1 + \lambda_{ji}) \frac{X_i}{X_j}$$

units of asset j, which is the same as to say that if he sells h_j units of asset j, then he gets

$$h_i = h_j(1 - \mu_{ji}) \frac{X_j}{X_i}$$

units asset i. Consequently, in order to avoid contradictions, we have to assume

$$\forall i, j : (1 - \mu_{ij})(1 + \lambda_{ij}) = 1 \ .$$

It is self-suggesting to set

$$\forall i : \mu_{ii} = \lambda_{ii} = 0 \ .$$

Suppose again, that we have given a portfolio h of currencies, that we want to rebalance at (ω, t). In order to change portfolio h into a portfolio $\tilde{h}$ we have to build the differential portfolio $\tilde{h} - h$. If instead of $\tilde{h}$ we end up with a portfolio $\tilde{h} + \xi$ with $\xi \in \mathbb{R}_+^{d+1}$, then we may consume ξ. So, if we allow for consumption not only of currency 0, but also of any other currency, then we can say that a portfolio $\tilde{h}$ may be obtained by a rebalancement of portfolio h at (ω, t) with consumption, if and only if there exists a $\xi \in \mathbb{R}^{d+1}$, such that we could change portfolio $0 \in \mathbb{R}^{d+1}$ into portfolio $\tilde{h} + \xi - h$ by a rebalancement without consumption at (ω, t). What are the necessary transactions? This is not so easy to answer as in the case of a stock market, because now, all assets are directly interchangable. However it is clear that in order to build portfolio $\tilde{h} + \xi - h$ from starting with 0, we have to perform exactly the same transactions, as for the liquidation of portfolio $h - \tilde{h} - \xi$. Suppose that we perform this transactions, but starting with portfolio $h - \tilde{h} = (h - \tilde{h} - \xi) + \xi$. Then since we "ignore" ξ, we end up with portfolio ξ, a portfolio that is non-negative in every component.

Let us summarize the results of the above thoughts in one sentense: In order to change a given portfolio h into another portfolio $\tilde{h}$ by a rebalancement (with consumption), we have to perform exactly the same transactions as if we wanted to rebalance the portfolio $h - \tilde{h}$ in order to get a portfolio that is non-negative in every component. So we just have to focus on the closed convex cone

$$\begin{aligned} K(\omega, t) \ := \ & \{h \in \mathbb{R}^{d+1} : \exists \varepsilon \in \mathbb{R}_+^{d+1} \exists \mathbf{h} = (h_{ij}) \in \mathbb{R}^{(d+1)\times(d+1)} \\ & \forall i, j : h_i = \varepsilon_i + \sum_j h_{ij}^+ - \sum_j h_{ij}^-, \quad h_{ij}^+ = (1 + \lambda_{ij}) h_{ji}^- \frac{X_j(\omega, t)}{X_i(\omega, t)}\} \ . \end{aligned} \tag{2.9}$$

This is true, because by (2.9) we can understand $K(\omega, t)$ as the set of portfolios, that may be transformed by selffinancing transactions into a new portfolio ε without any short position. In fact, if we hold a portfolio $h \in \mathbb{R}^{d+1}$ satisfying

$$\forall i, j : h_i = \varepsilon_i + \sum_j h_{ij}^+ - \sum_j h_{ij}^-, \quad h_{ij}^+ = (1 + \lambda_{ij}) h_{ji}^- \frac{X_j(\omega, t)}{X_i(\omega, t)} \ . \tag{2.10}$$

with $\varepsilon \in \mathbb{R}_+^{d+1}$ and $\mathbf{h} = (h_{ij}) \in \mathbb{R}^{(d+1)\times(d+1)}$, then for every i, j we can sell h_{ij}^+ units of asset i in exchange for h_{ji}^- units of asset j and buy h_{ij}^- units of asset i in exchange for h_{ji}^+ units of asset j. This transactions will result in the new portfolio $\varepsilon \in \mathbb{R}_+^{d+1}$

without any short position. Again, (2.2) implies that for every $(\omega, t) \in \Omega \times \mathcal{T}$ we have $\mathbb{R}_+^{d+1} \subset K(\omega, t)$.

In order to find a simpler characterization of $K(\omega, t)$, we define the convex sets

$$\begin{aligned} \tilde{K}_0'(\omega, t) &:= \{r = (1, r_1, \ldots, r_d) \in \mathbb{R}^{d+1} : \forall i, j \in \{0, \ldots, d\} : r_j - (1 + \lambda_{ij}) r_i \leq 0\} \\ -\tilde{K}'(\omega, t) &:= \mathbb{R}\tilde{K}_0'(\omega, t) \\ \tilde{K}(\omega, t) &:= \{h \in \mathbb{R}^{d+1} : \forall r \in \tilde{K}_0'(\omega, t) : h(r \star X(\omega, t)) \geq 0\} \ . \end{aligned}$$

This notation is appropriate, because $-\tilde{K}'(\omega, t)$ is indeed the dual positive cone of $\tilde{K}(\omega, t)$ with respect to the scalar product defined like in (2.6). In fact by its definition, $\tilde{K}(\omega, t)$ is the dual positive cone of $-\tilde{K}'(\omega, t)$, and since $-\tilde{K}'(\omega, t)$ is certainly nonempty and closed we have $\tilde{K}(\omega, t) = -(-\tilde{K}')'(\omega, t)$ according to a theorem in Rockafellar (1970, Theorem 14.1). Note $\tilde{K}_0'(\omega, t)$ independent of $\omega \in \Omega$.

We are going to show

$$(2.11) \qquad \forall (\omega, t) \in \Omega \times \mathcal{T} : K(\omega, t) = \tilde{K}(\omega, t) \ .$$

In order to do this, we also consider

$$(2.12) \qquad K_0'(\omega, t) := \{r \in \{1\} \times \mathbb{R}^d : \forall h \in K(\omega, t) : h(r \star X(\omega, t)) \geq 0\}$$

and the dual positive cone of $K(\omega, t)$, i.e.,

$$-K'(\omega, t) = \mathbb{R}_+ K_0'(\omega, t) = \{z \in \mathbb{R}^{d+1} : \forall h \in K(\omega, t) : h(z \star X(\omega, t)) \geq 0\} \ .$$

Since $K(\omega, t)$ and $\tilde{K}(\omega, t)$ are closed convex cones, Theorem 14.1 in Rockafellar (1970) implies that (2.11) is equivalent with

$$(2.13) \qquad \forall (\omega, t) \in \Omega \times \mathcal{T} : -K'(\omega, t) = -\tilde{K}'(\omega, t) \ .$$

In order to show the inclusion "$\subseteq$" in (2.13), fix an arbitrary (ω, t) and consider a $z \in -K'(\omega, t)$. Then we have

$$h(z \star X(\omega, t)) \geq 0$$

for every $h \in K(\omega, t)$, i.e., for every $h \in \mathbb{R}^{d+1}$ for which there exist $\varepsilon \in \mathbb{R}_+^{d+1}$ and $\mathbf{h} = (h_{ij}) \in \mathbb{R}^{(d+1)\times(d+1)}$ satisfying (2.10). This means, that for every $\varepsilon \in \mathbb{R}_+^{d+1}$ and

$\mathbf{h} = (h_{ij}) \in \mathbb{R}^{(d+1)\times(d+1)}$ satisfying (2.10) we have

$$\begin{aligned}
0 &\le \sum_i z_i(\varepsilon_i + \sum_j h_{ij}^+ - \sum_j h_{ij}^-)X_i(\omega,t) \\
&= \sum_i z_i\varepsilon_i X_i(\omega,t) + \sum_{ij} z_i h_{ij}^+ X_i(\omega,t) - \sum_{ij} z_i h_{ij}^- X_i(\omega,t) \\
&= \sum_i z_i\varepsilon_i X_i(\omega,t) + \sum_{ij} z_i(1+\lambda_{ij})h_{ji}^- \frac{X_j(\omega,t)}{X_i(\omega,t)} X_i(\omega,t) - \sum_{ij} z_i h_{ij}^- X_i(\omega,t) \\
&= \sum_i z_i\varepsilon_i X_i(\omega,t) + \sum_{ij} z_i(1+\lambda_{ij})h_{ji}^- X_j(\omega,t) - \sum_{ji} z_j h_{ji}^- X_j(\omega,t) \\
&= \sum_i z_i\varepsilon_i X_i(\omega,t) + \sum_{ij}(z_i(1+\lambda_{ij}) - z_j)h_{ji}^- X_j(\omega,t) \ .
\end{aligned}$$

Hence it follows

$$\forall i,j \in \{0,\ldots,d\} : \quad z_i(1+\lambda_{ij}) - z_j \ge 0$$

which is equivalent with $z \in \mathbb{R}\tilde{K}_0'(\omega,t) = -\tilde{K}'(\omega,t)$. The above calculation may also be used to verify the converse inclusion.

Since(2.11) also implies

$$\forall(\omega,t) \in \Omega \times \mathcal{T} : -K_0'(\omega,t) = -\tilde{K}_0'(\omega,t) \ ,$$

we conclude that every $r \in -K_0'(\omega,t)$ satisfies the following inequlities for every $(\omega,t) \in \Omega \times \mathcal{T}$:

$$\forall i \in \{1,\ldots,d\} : \quad \frac{r_0}{1+\lambda_{i0}} \le r_i \le (1+\lambda_{j0})r_0 \ .$$

This shows that $-K_0'(\omega,t)$ is compact for every $(\omega,t) \in \Omega \times \mathcal{T}$.

2.6 Currency Market with Bid-Ask Spreads

This section deals with a currency market with bid-ask spreads. Suppose there are $d+1$ assets (currencies) where asset 0 deserves as a numeraire. Let $\underline{X} = (\underline{X}_{ij})_{i,j\in\{0,\ldots,d\}}$ and $\overline{X} = (\overline{X}_{ij})_{i,j\in\{0,\ldots,d\}}$ be $\mathbb{R}^{(d+1)\times(d+1)}$-valued $\mathbb{F}$-adapted processes, where $\underline{X}_{ij}(\omega,t)$ denotes the bid price and $\overline{X}_{ij}(\omega,t)$ denotes the ask price of asset i in units of asset j at (ω,t). This means that if at (ω,t) somebody wants to buy h_i units of asset i, then he has to pay $h_j = h_i\overline{X}_{ij}(\omega,t)$ units of asset j, which is the same as to say that if he sells h_j units of asset j, then he gets $h_i = h_j \underline{X}_{ji}(\omega, t)$ units of asset i. Thus in order to keep symmetry, we have to assume

$$\forall i, j \in \{0, \dots, d\} : \quad \underline{X}_{ji} \overline{X}_{ij} \equiv 1 \ .$$

It is self-suggesting to introduce the convention

$$\forall i \in \{0, \dots, d\} : \quad \underline{X}_{ii} = \overline{X}_{ii} \equiv 1 \ .$$

Currencies are always bought and sold at a positive exchange rate. Moreover the ask rate of a currency cannot be lower than it's bid rate. Hence, it is reasonable to assume

$$\forall (\omega, t) \in \Omega \times \mathcal{T} \ \forall i, i \in \{0, \dots, d\} : 0 < \underline{X}_{ij}(\omega, t) \leq \overline{X}_{ij}(\omega, t). \tag{2.14}$$

Let us define the spot price process $X = (X_0, X_1, \dots, X_d)$ by $X_0 \equiv 1$ and $X_i = \underline{X}_{i0}$ for $i = 1, \dots, d$. This definition is somehow arbitrary. In fact, any $\mathbb{F}$-adapted process X satisfying

$$\forall i \in \{0, \dots, d\} : \underline{X}_{i0} \leq X_i \leq \overline{X}_{i0}$$

could serve as spot price process without changing the quality of results. For the same reasons as in the case with constant transaction cost factors we are interested in the closed convex cone

$$\begin{aligned} K(\omega, t) &= \{h \in \mathbb{R}^{d+1} : \exists \varepsilon \in \mathbb{R}_+^{d+1} \exists \mathbf{h} = (h_{ij}) \in \mathbb{R}^{(d+1)\times(d+1)} \\ &\qquad \forall i, j : h_i = \varepsilon_i + \sum_j h_{ij}^+ - \sum_j h_{ij}^-, \quad h_{ij}^+ = h_{ji}^- \overline{X}_{ji}\} \ . \end{aligned} \tag{2.15}$$

of portfolios, that can be transformed by self-financing transactions into a new portfolio ε without any short position. In fact, if we hold a portfolio $h \in \mathbb{R}^{d+1}$ satisfying (2.15) at (ω, t) with $\varepsilon \in \mathbb{R}_+^{d+1}$ and $\mathbf{h} = (h_{ij}) \in \mathbb{R}^{(d+1)\times(d+1)}$, then for every i, j we can sell h_{ij}^+ units of asset i in exchange for h_{ji}^- units of asset j and buy h_{ij}^- units of asset i in exchange for h_{ji}^+ units of asset j. This transactions will result in the new portfolio $\varepsilon \in \mathbb{R}_+^{d+1}$ without any short position. From (2.14) it is clear, that for every $(\omega, t) \in \Omega \times \mathcal{T}$ we have $\mathbb{R}_+^{d+1} \subset K(\omega, t)$.

Like in the previous sections, we want to find a simpler characterization of $K(\omega, t)$. Therefore we define

$$\begin{aligned} \tilde{K}_0'(\omega, t) &:= \{r = (1, r_1, \dots, r_d) \in \mathbb{R}^{d+1} : \forall i, j \in \{0, \dots, d\} : \\ &\qquad X_j(\omega, t) r_j - \overline{X}_{ji}(\omega, t) X_i(\omega, t) r_i \leq 0\} \\ -\tilde{K}'(\omega, t) &:= \mathbb{R} \tilde{K}_0'(\omega, t) \\ \tilde{K}(\omega, t) &= \{h \in \mathbb{R}^{d+1} : \forall r \in \tilde{K}_0'(\omega, t) : h(r \star X(\omega, t)) \geq 0\} \ . \end{aligned}$$

This notation is appropriate, because $-\tilde{K}'(\omega,t)$ is indeed the dual positive cone of $\tilde{K}(\omega,t)$ with respect to the scalar product defined like in (2.6) (c.f. Section 2.5). Note also that for every $r \in \mathbb{R}^{d+1}$ and every $t \in \mathcal{T}$ we obviously have

$$\{\omega \in \Omega : r \in \tilde{K}'_0(\omega,t)\} \in \mathcal{F}(t) \ .$$

We are going to show

$$\forall(\omega,t) \in (\Omega,\mathcal{T}) : K(\omega,t) = \tilde{K}(\omega,t) \ . \tag{2.16}$$

Therefore, we consider

$$\begin{aligned} K'_0(\omega,t) &:= \{r \in \{1\} \times \mathbb{R}^d : \forall h \in K(\omega,t) : h(r \star X(\omega,t)) \geq 0\} \\ -K'(\omega,t) &= \mathbb{R}_+ K'_0(\omega,t) \ . \end{aligned}$$

Since $K(\omega,t)$ and $\tilde{K}(\omega,t)$ obviously are closed convex cones, Theorem 14.1 in Rockafellar (1970) implies that (2.16) holds, if and only if

$$\forall(\omega,t) \in (\Omega,\mathcal{T}) : -K'(\omega,t) = -\tilde{K}'(\omega,t) \ .$$

In order to show $K'(\omega,t) \subseteq \tilde{K}'(\omega,t)$ fix an arbitrary (ω,t) and consider a $z \in -K'(\omega,t)$. Then this z satisfies

$$\sum_i z_i h_i X_i(\omega,t) \geq 0$$

for every $h \in K(\omega,t)$, i.e. for every $h \in \mathbb{R}^{d+1}$, for that there exist $\varepsilon \in \mathbb{R}_+^{d+1}$ and $\mathbf{h} = (h_{ij}) \in \mathbb{R}^{(d+1)\times(d+1)}$ satisfying (2.15). Hence for every $\varepsilon \in \mathbb{R}_+^{d+1}$ and $\mathbf{h} = (h_{ij}) \in \mathbb{R}^{(d+1)\times(d+1)}$ satisfying (2.15) we have

$$\begin{aligned} 0 &\leq \sum_i z_i(\varepsilon_i + \sum_j h_{ij}^+ - \sum_j h_{ij}^-)X_i(\omega,t) \\ &= \sum_i z_i\varepsilon_i X_i(\omega,t) + \sum_{ij} z_i h_{ij}^+ X_i(\omega,t) - \sum_{ij} z_i h_{ij}^- X_i(\omega,t) \\ &= \sum_i z_i\varepsilon_i X_i(\omega,t) + \sum_{ij} z_i h_{ji}^- \overline{X}_{ji}(\omega,t) X_i(\omega,t) - \sum_{ij} z_i h_{ij}^- X_i(\omega,t) \\ &= \sum_i z_i\varepsilon_i X_i(\omega,t) + \sum_{ij} z_i h_{ji}^- \overline{X}_{ji}(\omega,t) X_i(\omega,t) - \sum_{ji} z_j h_{ji}^- X_j(\omega,t) \\ &= \sum_i z_i\varepsilon_i X_i(\omega,t) + \sum_{ij}(z_i \overline{X}_{ji}(\omega,t) X_i(\omega,t) - z_j X_j(\omega,t))h_{ji}^- \ . \end{aligned}$$

Hence it follows

$$\forall i,j \in \{0,\ldots,d\}: \quad z_i \overline{X}_{ji}(\omega,t) X_i(\omega,t) - z_j X_j(\omega,t) \geq 0$$

which is equivalent with $z \in \mathbb{R}\tilde{K}_0'(\omega,t) = -\tilde{K}'(\omega,t)$. The above calculation also may be used to verify the converse inclusion.

Since (2.16) also implies

$$\forall (\omega,t) \in \Omega \times \mathcal{T} : -K_0'(\omega,t) = -\tilde{K}_0'(\omega,t) \ ,$$

we conclude that every $r \in -K_0'(\omega,t)$ satisfies the following inequlities for every $(\omega,t) \in \Omega \times \mathcal{T}$:

$$\forall i \in \{1,\ldots,d\}: \quad \underline{X}_{i0}(\omega,t) = \frac{X_0(\omega,t) r_0}{\overline{X}_{0i}(\omega,t)} \leq X_i(\omega,t) r_i \leq \overline{X}_{i0}(\omega,t) X_0(\omega,t) r_0 = 1 \ .$$

This shows that $-K_0'(\omega,t)$ is compact for every $(\omega,t) \in \Omega \times \mathcal{T}$.

2.7 Differential Cone Constraints

In Sections 2.2 to 2.6 we introduced models for trading in stock and currency markets with different kinds of proportional transaction costs. The focus was layed on characterizing the set of feasible transactions at each single $(\omega,t) \in \Omega \times \mathcal{T}$. We supposed that at (ω,t) we had given a certain portfolio $h \in \mathbb{R}^{d+1}$ that we wanted to rebalance by means of self-financing transactions (with consumption). The question was, which portfolios $\tilde{h}$ we could achieve. We found, that at (ω,t) a portfolio $\tilde{h}$ is attainable by a rebalancement of a given portfolio h, if and only if the difference portfolio $h - \tilde{h}$ lies in a certain closed convex cone $K(\omega,t)$ with $\mathbb{R}_+^{d+1} \subset K(\omega,t)$. This cone describes the set of portfolios, that may be rebalanced at (ω,t) in such a way, that every component becomes non-negative. In each market model we characterized $K(\omega,t)$ by means of a compact convex set $K_0'(\omega,t)$ whose elements could be interpreted as modified prices. The closed convex cone $-K'(\omega,t) := \mathbb{R}_+ K_0'(\omega,t)$ turned out to be the dual positive cone of $K(\omega,t)$ with respect to a scalar product $< .,. > (\omega,t)$ defined like in (2.6). In fact, in every market model we got the following dual characterizations for every $(\omega,t) \in \Omega \times \mathcal{T}$:

$$\text{(2.17)} \qquad \begin{aligned} K(\omega,t) &= \{h \in \mathbb{R}^{d+1} : \forall r \in K_0'(\omega,t) : h(r \star X(\omega,t)) \geq 0\} \\ &= \{h \in \mathbb{R}^{d+1} : \forall z \in -K'(\omega,t) : h(z \star X(\omega,t)) \geq 0\} \end{aligned}$$

$$(2.18)\ \mathbb{R}_+K_0'(\omega,t) = -K'(\omega,t) = \{z \in \mathbb{R}^{d+1} : \forall h \in K(\omega,t) : h(z \star X(\omega,t)) \geq 0\}.$$

Moreover, we always had

$$\forall t \in \mathcal{T} \forall r \in \mathbb{R}^{d+1} : \{\omega \in \Omega : r \in K_0'(\omega,t)\} \in \mathcal{F}(t)\ . \tag{2.19}$$

$$\forall t \in \mathcal{T} \forall h \in \mathbb{R}^{d+1} : \{\omega \in \Omega : h \in K(\omega,t)\} \in \mathcal{F}(t)\ . \tag{2.20}$$

These similarities lead us to consider market models, where trading is restricted by abstract *differential cone constraints* within the general framework presented in Section 2.1. In order to define admissible portfolio strategies in subsequent chapters, we will have to deal with rebalancements of random portfolios. Therefore we want to state the general definition of admissible rebalencements in terms of random portfolios. The following definitions and assumptions are postulated to hold for the remainder of this section as well as throughout all subsequent chapters.

Definitions and Assumptions 2.1 *Assume the general market framework of section 2.1. For every $(\omega,t) \in \Omega \times \mathcal{T}$ let $K(\omega,t) \subset \mathbb{R}^{d+1}$ be a closed convex cone satisfying*

$$\mathbb{R}_+^{d+1} \subset K(\omega,t) \tag{2.21}$$

and $K_0'(\omega,t) \subset \{1\} \times \mathbb{R}^d$ a compact convex set satisfying the dual characterization (2.17). For every $(\omega,t) \in \Omega \times \mathcal{T}$ define the dual positive cone

$$\mathbb{R}_+K_0'(\omega,t) = -K'(\omega,t)$$

according to (2.18). Assume moreover that the measurability conditions (2.19) and (2.20) hold. Then we understand $K(\omega,t)$ as the the set of portfolios, that may be rebalanced at (ω,t) in such a way, that every component becomes non-negative. In accordance with this interpretation of $K(\omega,t)$, a random portfolio $H(t) \in (L^0(\mathcal{F}(t)))^{d+1}$ is attainable by a rebalancement of a given random portfolio $H(t-) \in (L^0(\mathcal{F}(t)))^{d+1}$ at time $t \in \mathcal{T}$, if and only if for P-almost every $\omega \in \Omega$ the differential portfolio $H(t-) - H(t)$ satisfies the differential cone constraint

$$H(\omega,t-) - H(\omega,t) \in K(\omega,t)\ . \tag{2.22}$$

Note that for the moment, the notation $H(t-)$ is only used in order to connote that portfolio $H(t-)$ may have been built some time before t, but this notation has no technical meaning so far.

Using the dual characterization (2.17) of $K(\omega,t)$, we can reformulate (2.22) as

$$\begin{aligned} 0 &\le \inf_{r\in K_0'(\omega,t)} \left(H(\omega,t-)-H(\omega,t)\right)\left(r\star X(\omega,t)\right) \\ &= \inf_{z\in -K'(\omega,t)} \left(H(\omega,t-)-H(\omega,t)\right)\left(z\star X(\omega,t)\right)\ . \end{aligned}$$

By introducing the sets of measurable selections

$$K(t) := \{U\in (L^0(\mathcal{F}(t)))^{d+1} : U(.)\in K(.,t)\ \ P-a.s.\}\ ,$$

$$K_0'(t) := \{V\in (L^0(\mathcal{F}(t)))^{d+1} : V(.)\in K_0'(.,t)\ \ P-a.s.\}\ ,$$

$$K'(t) := \{W\in (L^0(\mathcal{F}(t)))^{d+1} : W(.)\in K'(.,t)\ \ P-a.s.\}\ ,$$

the almost sure validity of the differential cone constraint (2.22) becomes equivalent to any of the following constraints:

$$H(t-)-H(t)\in K(t)$$

$$\forall R(t)\in K_0'(t) : (H(t-)-H(t))(R(t)\star X(t))\ge 0$$

$$\forall Z(t)\in K'(t) : (H(t-)-H(t))(R(t)\star X(t))\le 0\ .$$

Denote $\mathcal{R}$ the class of $\mathbb{R}^{d+1}$-valued processes $R=(R(t))_{t\in\mathcal{T}}$ such that there exists a $F\in\mathcal{F}$ with $P(F)=1$ and

$$\forall(\omega,t)\in F\times\mathcal{T}:\quad R(\omega,t)\in K_0'(\omega,t)\ . \tag{2.23}$$

$\diamondsuit$

Of course, the models introduced in Sections 2.2 to 2.6 may be refined in some directions. One may think of modelling interconnected stock and currency markets with proportional transactions costs. While the definitions of $K(\omega,t)$ and $K_0'(\omega,t)$ then will get a little more complex, the properties postulated in **Definitions and Assumptions (2.1)** will still be satisfied as long as there are no frictions on the market other than proportional transaction costs or restrictions on which assets are direct interchangeable.

Remark 2.2 According to (2.18) the condition (2.21) is equivalent to

$$\forall y\in\mathbb{R}_+^{d+1}\ \forall r\in K_0'(\omega,t) : yr\ge 0\ . \tag{2.24}$$

Moreover we always have $\mathbb{R}_+^{d+1}\cap -K(\omega,t)=\{0\}$.

$\diamondsuit$

Remark 2.3 According to Corollary 1D in Rockafellar (1976) condition (2.20) implies that for every $t \in \mathcal{T}$ the multifunction

$$K(.,t) : \omega \mapsto K(\omega, t)$$

is $\mathcal{F}(t)$-measurable. The same ist true for the multifunction $K_0'(.,t)$.

$\diamondsuit$

Proposition 2.4 *Suppose $(\omega, t) \in \Omega \times \mathcal{T}$, $h \in \mathbb{R}^{d+1}$. Then we have $h \in \partial K(\omega, t)$, if and only if $h \in K(\omega, t)$ and there exists a $r \in K_0'(\omega, t)$ satisfying*

$$h(r \star X(\omega, t)) = 0 \ . \tag{2.25}$$

Proof. Denote 1_{d+1} the $d+1$-dimensional vector with all components equal to 1.

i) "$\Rightarrow$": Suppose $h \in \partial K(\omega, t)$. Then there is a sequence $(h^n)_{n \in \mathbb{N}}$ in $\mathbb{R}^{d+1}$ that converges to h and satisfies $h^n \notin K(\omega, t)$ for every n. This implies that for every $n \in \mathbb{N}$ there exists $r^n \in K_0'(\omega, t)$ such that $h^n(r^n \star X(\omega, t)) < 0$. Since $K_0'(\omega, t)$ is compact by definition, there is a subsequence $(r^{k(n)})_{k \in \mathbb{N}}$ converging to some $r \in K_0'(\omega, t)$ and consequently we have

$$0 \leq h(r \star X(\omega, t)) = \lim_{k \to \infty} h^{k(n)}(r^{k(n)} \star X(\omega, t)) \leq 0$$

which implies (2.25).

ii) "$\Leftarrow$": Suppose $h \in K(\omega, t)$ and $r \in K_0'(\omega, t) \subset \{1\} \times \mathbb{R}_+^d$ satisfy (2.25). Remembering (2.2) then, we have

$$\forall \varepsilon > 0 : \quad (h - \varepsilon 1_{d+1})(r \star X(\omega, t)) = -\varepsilon 1_{d+1}(r \star X(\omega, t)) < 0$$

thus $(h - \varepsilon 1_{d+1}) \notin K(\omega, t)$ for any $\varepsilon > 0$. This shows $h \in \partial K(\omega, t)$.

$\diamondsuit$

Lemma 2.5 *Let $(\omega, t) \in \Omega \times \mathcal{T}$. Define the multifunctions $\Phi(., \omega, t)$, $\Upsilon(., \omega, t)$ from $\mathbb{R}^{d+1}$ to $\mathbb{R}^{d+1}$ by*

$$\Phi(h, \omega, t) := \partial K(\omega, t) \cap (h - \mathbb{R}_+^{d+1})$$

$$\Upsilon(h, \omega, t) := \{r \in K_0'(\omega, t) : h(r \star X(\omega, t)) = 0\}$$

Then these multifunctions satisfy

$$\text{(2.26)} \qquad \forall h \in K(\omega, t) : \quad \Phi(h, \omega, t) \neq \emptyset$$

$$\text{(2.27)} \qquad \forall h \in \partial K(\omega, t) : \quad \Upsilon(h, \omega, t) \neq \emptyset \ .$$

The multifunctions $\Phi(., \omega, t)$ and $\Upsilon(., \omega, t)$ are convex-valued, closed-valued and $\mathbb{B}^{d+1}$-measurable.

Proof. (2.27) is a direct implication of Proposition 2.4. For $h \in \partial K(\omega, t)$ the assertion (2.26) is trivial. Suppose $h \in K(\omega, t) \setminus \partial K(\omega, t)$ and

$$\partial K(\omega, t) \cap (h - \mathbb{R}_+^{d+1}) = \Phi(\omega, t) = \emptyset \ .$$

Then because of $h \in K(\omega, t)$, we had

$$-\mathbb{R}_+^{d+1} \subset h - \mathbb{R}_+^{d+1} \subseteq K(\omega, t)$$

which is a contradiction to Remark 2.2.

Since the sets $\partial K(\omega, t)$ and $h - \mathbb{R}_+^{d+1}$ are closed and convex, $\Phi(h, \omega, t)$ is also closed and convex. The set $\Upsilon(h, \omega, t)$ is obviously closed by definition.

In order to proof the measurability of $\Phi(., \omega, t)$ and $\Upsilon(., \omega, t)$ we use Corollary 1D in Rockafellar (1976,p. 164). According to this corollary, since $\Phi(., \omega, t)$ and $\Upsilon(., \omega, t)$ are convex- and closed-valued, it suffices to show, that

$$\Phi^{-1}(., \omega, t)(g) = \{h \in \mathbb{R}^{d+1} : g \in \Phi(h)\}$$

and $\Upsilon^{-1}1(., \omega, t)(g)$ are $\mathbb{B}^{d+1}$-measurable. For $g \notin \partial K(\omega, t)$ we have $\Phi^{-1}(., \omega, t)(g) = \emptyset$. For $g \in \partial K(\omega, t)$ we have

$$\Phi^{-1}(., \omega, t)(g) = \{h : g \in h - \mathbb{R}_+^{d+1}\} = g + \mathbb{R}_+^{d+1} \in \mathbb{B}^{d+1} \ .$$

Thus $\Phi(., \omega, t)$ is $\mathbb{B}^{d+1}$-measurable. For $r \notin K_0'(\omega, t)$ we have $\Upsilon^{-1}(., \omega, t)(r) = \emptyset$. For $r \in K_0'(\omega, t)$ the set

$$\Upsilon^{-1}(., \omega, t)(r) = \{h \in \mathbb{R}^{d+1} : h(r \star X(\omega, t)) = 0\}$$

is closed by definition and hence $\mathbb{B}^{d+1}$-measurable.

$\diamondsuit$

Lemma 2.6 *Let $t \in \mathcal{T}$ and $V \in K(t)$. For every omega $\in \Omega$ define the multifunction $\Phi(.,\omega,t)$ as in Lemma 2.5. Define also the multifunction $\Psi(.,t)$ from Ω to $\mathbb{R}^{d+1}$ by*

$$\Psi(\omega,t) := \Phi(V(\omega,t),\omega,t) \ .$$

Then $\Psi(.,t)$ is $\mathcal{F}(t)$-measurable. Moreover there exist a $\tilde{V} \in K(t)$ and a $W \in K_0'(t)$ satisfying

$$\forall i \in \{0,1,\ldots,d\} : \tilde{V}_i \leq V_i \tag{2.28}$$

$$\tilde{V}(W \star X(t)) = 0 \ . \tag{2.29}$$

Note that (2.29) is equivalent to

$$\tilde{V}(t) \in \partial K(t) := \{U \in K(t) : U \in \partial K(.,t) \text{ a.s.}\} \ .$$

Proof. As we know from Lemma 2.5, the multifunction $\Phi(.,\omega,t)$ is convex- and closed-valued. Hence, we can apply Corollary 1D of Rockafellar (1976,p. 164) and only have to show that for arbitrary $g \in \mathbb{R}^{d+1}$ we have $\Psi^{-1}(.,t)(g) \in \mathcal{F}(t)$. Since the random vector V and the multifunctions $K(.,t)$ are $\mathcal{F}(t)$-measurable (c.f. Remark 2.2) the calculation

$$\begin{aligned}\Psi^{-1}(.,t)(g) &= \{\omega \in \Omega : g \in \partial K(\omega,t) \cap (V(\omega,t) - \mathbb{R}_+^{d+1})\} \\ &= \{\omega \in \Omega : g \in \partial K(\omega,t)\} \cap \{\omega \in \Omega : g \in V(\omega,t) - \mathbb{R}_+^{d+1}\} \in \mathcal{F}(t)\end{aligned}$$

shows that $\Psi(.,t)$ is $\mathcal{F}(t)$-measurable. Moreover according to Lemma 2.5, for almost every $\omega \in \Omega$ we have

$$\Psi(\omega,t) = \Phi(V(\omega,t),\omega,t) \neq \emptyset$$

because $V(\omega,t) \in K(\omega,t)$.

By a measurable selection theorem (see Rockafellar, 1976, Theorem 1B, p.163) there exists an $\mathcal{F}(t)$-measurable $\tilde{V}$ such that for almost every $\omega \in \Omega$

$$\tilde{V}(\omega) \in \Psi(\omega,t) = \partial K(\omega,t) \cap (V(\omega,t) - \mathbb{R}_+^{d+1}) \ .$$

This implies $\tilde{V} \in K(t)$ and (2.28). For $\tilde{V}$ we define (almost everywhere) the multifunction Ξ from Ω to $\mathbb{R}^{d+1}$ by

$$\Xi(\omega,t) := \Upsilon(\tilde{V}(\omega,t),\omega,t) = \{r \in K_0'(\omega,t) : \tilde{V}(\omega,t)(r \star X(\omega,t)) = 0\}$$

where Υ is defined as in Lemma 2.5 . It is clear that $\Xi(.,t)$ is convex- and closed-valued. Hence again, from

$$\begin{aligned}\Xi^{-1}(.,t)(r) &= \{\omega \in \Omega : r \in K_0'(\omega,t), \tilde{V}(\omega,t)(r \star X(\omega,t)) = 0\} \\ &= \{\omega \in \Omega : r \in K_0'(\omega,t)\} \cap \{\omega \in \Omega : \tilde{V}(\omega)(r \star X(\omega,t)) = 0\}\end{aligned}$$

we see that $\Xi(.,t)$ is $\mathcal{F}(t)$-measurable, because $K_0'(.,t)$ and $\tilde{V}$ are $\mathcal{F}$-measurable. By the measurable selection theorem, already cited above, we can choose W as desired.

$\diamondsuit$

Part II

Discrete Time

Chapter 3

The Discrete Time Market

In this chapter we restrict the general framework of Chapter 2 to the case of discrete time trading. That means, we consider a financial market where $d+1$ primary financial assets $i = 0, 1, \ldots, d$ are traded at discrete time points $t \in \mathcal{T} = \{0, 1, \ldots, T\}$, $0 < T \in \mathbb{N}$. Let us shortly recall some technical assumptions. Uncertainty and information structure in this market are modelled as a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t \in \mathcal{T}}$. We assume that the prices of assets $i \in \{0, \ldots d\}$ in units of asset 0 are given by a $\mathbb{R}_+^{d+1}$-valued $\mathbb{F}$-adapted process X defined according to (2.1) and satisfying (2.2). Moreover we assume that

$$\forall \omega \in \Omega : X_i(\omega, 0) = x_i > 0 \ .$$

Note also, that $s_0 = 1$ and $X_0(t) = 1$ for every $t \in \mathcal{T}$.

In Chapter 2, particularly section 2.7, we have seen, that in the presence of transaction costs, self-financing rebalancements of portfolios are easily described by certain cone constraints. This coherence is formulated in **Definitions and Assumptions 2.1**, which we always take for granted. The concerning cone $K(\omega, t)$ is interpreted as the set of portfolios, that can be rebalanced at (ω, t) in such a way that every component is non-negative. It is this property that is connoted by the term "solvency cone".

Definition 3.1 *A portfolio process consists of an initial portfolio* $h = (h_i)_{i \in \{0,\ldots,d\}} \in \mathbb{R}^{d+1}$ *and a* $\mathbb{R}^{d+1}$*-valued* $\mathbb{F}$*-adapted process* $H = (H_i)_{i \in \{0,\ldots,d\}}$*. Denote* $\mathcal{H}$ *the* set of portfolio processes. *For every portfolio process* $(h, H) \in \mathcal{H}$ *we define the process*

$H(.-) = (H(t-))_{t\in\mathcal{T}}$ by

$$H(0-) := h$$
$$H(t-) := H(t-1) \quad , \; t > 0.$$

Note that $H(.-)$ is $\mathbb{F}$-predictable.

We define the process $\Delta H = (\Delta H_i)_{i\in\{0,\ldots,d\}}$ by

(3.1) $$\Delta H_i := H_i - H_i(.-) \quad , \; i = 0, \ldots, d,$$

and understand $H_i(t)$ as number of units of asset i held after all transactions at time t whereas $H_i(t-)$ represents the number of units of asset i before all transactions at time t. In particular $H_i(0-) = h$ is the initial holding in asset i, i.e., before any transaction is settled.

Definitions 3.2 *Let $G = (G_0, \ldots, G_d)$ and $Y = (Y_0, \ldots, Y_d)$ be two $\mathbb{R}^{d+1}$-valued, $\mathbb{F}$-adapted processes. Then the $\mathbb{F}$-adapted process*

$$G(.-) \bullet Y = (G(.-) \bullet Y(t))_{t\in\mathcal{T}}$$

is defined by

$$G(.-) \bullet Y(t) := \sum_j G_j(.-) \bullet Y_j(t) \quad t \in \mathcal{T}$$

with $G_j \bullet Y_j(t) := \sum_{s=1}^t G_j(s-)\Delta Y_j(s)$ and the convention $G(.-) \bullet Y(0) = 0$. The $\mathbb{F}$-adapted process

$$Y \bullet G = Y(.-) \bullet G + \Delta Y \bullet G$$

is defined in a similar way by

$$\Delta Y_j \bullet G_j(\omega, t) := \sum_{s=1}^{t} \Delta Y_j(s)\Delta G_j(s) \; .$$

The product GY given by

$$GY := G(0)Y(0) + G(.-) \bullet Y + Y \bullet G$$

is also $\mathbb{F}$-adapted.

$\diamondsuit$

For the $\mathbb{R}^{d+1}$-valued, $\mathbb{F}$-adapted process X and a portfolio process $(h, H) \in \mathcal{H}$, the processes $H_i(.-) \bullet X_i$ and

$$X_i \bullet H_i = X_i(.-) \bullet H_i + \Delta X_i \bullet H_i$$

satisfy

$$H_i X_i = H_i X_i(0) + H_i(.-) \bullet X_i + X_i \bullet H_i \ .$$

This yields the formula

$$HX = H(0)X(0) + H(.-) \bullet X + X \bullet H \ .$$

In this formula $H(.-) \bullet X$ represents the change of portfolio value due to price changes and $X \bullet H$ decribes the changes of portfolio value due to transactions.

In our introduction, we have already explained why we are interested in so called self-financing trading strategies. In fact, we are more concerned with the admissible portfolio processes resulting from such strategies. Hence, we define the set of admissible portfolio processes by only allowing for self-financing rebalancements (with consumption) instead of admitting arbitrary transactions. This is formalized as follows.

Definitions 3.3 *Let $\mathcal{A}$, the class of admissible portfolio processes, consist of all $(h, H) \in \mathcal{H}$ satisfying the differential cone constraints*

$$\forall t \in \mathcal{T} : -\Delta H(t) \in K(t) \ .$$

Using the class $\mathcal{R}$ introduced in **Definitions and Assumptions 2.1** *it is straight forward to verify that $(h, H) \in \mathcal{H}$, if and only if for every $R \in \mathcal{R}$ the process $(R \star X) \bullet H$ is decreasing and $(R \star X)\Delta H(0) \leq 0$.*

Denote

$$\mathcal{A}^\infty := \{(h, H) \in \mathcal{H} : (H(T))^- \in (L^\infty(\mathcal{F}(T), P))^{d+1}\}$$

the set of "tame" admissible portfolio processes. We have $(h, H) \in \mathcal{A}^\infty$, if and only if the short positions of $H(T)$ are P-almost surely bounded. This means that one is not allowed to borrow an "infinite" sum of money or to sell short "infinitely" many units of an asset. It is not very audacious to state, that this assumption does always apply in practice. Note also that our tame condition only refers to terminal positions, whereas the tame conditions usually used troughout literature refer to the set $\mathcal{T}$.

$\diamondsuit$

Remark 3.4 It is straight forward to show that for every $t \in \mathcal{T}$ the set $K(t)$ is a convex cone. Thus, it is easy to verify that $\mathcal{A}$ and $\mathcal{A}^\infty$ are convex cones in $\mathcal{H}$.

$\diamondsuit$

Theorem 3.5 *For every $(h, H) \in \mathcal{A}$ there exist a $(h, \tilde{H}) \in \mathcal{A}$ and a $R \in \mathcal{R}$ satisfying*

$$\forall t \in \mathcal{T}\ \forall i \in \{0, \dots, d\} : \tilde{H}_i(t) \geq H_i(t) \ ,$$

$$\forall t \in \mathcal{T} : (R(t) \star X(t))\Delta\tilde{H}(t) = 0 \ . \tag{3.2}$$

Note that (3.2) is equivalent with

$$\forall t \in \mathcal{T} : \ -\Delta\tilde{H}(t) \in \partial K(t) = \{U \in K(t) : U \in \partial K(., t)\text{a.s.}\} \ .$$

Proof. If $(h, H) \in \mathcal{A}$, then for every $t \in \mathcal{T}$ we have $-\Delta H(t) \in K(t)$. Hence according to Lemma 2.6, for every $t \in \mathcal{T}$ there exist a $\tilde{V}(t) \in K(t)$ and a $W(t) \in K_0'(t)$ satisfying

$$\tilde{V}(t)(W(t) \star X(t)) = 0$$

$$\forall i \in \{0, \dots, d\} : \quad \tilde{V}_i(t) \leq -\Delta H_i(t) \ .$$

If we define $R := (R(t))_{t \in \mathcal{T}}$ by $R(t) := W(t)$, $t \in \mathcal{T}$ and $\tilde{H} = (H(t))_{t \in \mathcal{T}}$ by $\tilde{H}(0) := h - \tilde{V}(0)$ and

$$\tilde{H}(t) := \tilde{H}(t-1) - \tilde{V}(t), \quad t = 1, \dots T$$

then it easy to verify that R and $\tilde{H}$ have the desired properties.

$\diamondsuit$

Definitions 3.6 *Denote $\mathcal{P}$ the class of all pairs (Q, R) satsifying the following conditions:*

(P1) Q is a probability equivalent to P on $\mathcal{F}(T)$,

(P2) $R \in \mathcal{R}$,

(P3) the process $R \star X$ is a Q-Martingale.

Define

$$\mathcal{A}^{\mathcal{P}} := \{(h,H) \in \mathcal{A} : \forall (Q,R) \in \mathcal{P} : H(R \star X) \text{ is a } Q-\text{supermartingal}\}$$

if $\mathcal{P} \neq \emptyset$ *and otherwise* $\mathcal{A}^{\mathcal{P}} := \emptyset$.

$\diamondsuit$

Lemma 3.9 will explain this definitions further. Chapter 4 will be contributed to necessary and sufficient conditions for $\mathcal{P} \neq \emptyset$.

Remark 3.7 If $\mathcal{P} \neq \emptyset$, then

$$\mathcal{A}^{\mathcal{P}} = \{(h,H) \in \mathcal{A} : \forall (Q,R) \in \mathcal{P}\ \forall t \in [0,T] : H(R \star X) \in L^1(Q)\} \ .$$

Hence, this is an alternative to the above definition of $\mathcal{A}^{\mathcal{P}}$.

$\diamondsuit$

Definitions 3.8 *Denote* $\mathcal{C} := (L^0(\mathcal{F}(T)))^{d+1}$ *the set of contingent claims. We call* $(h,H) \in \mathcal{A}$ *a super hedge for* $C \in \mathcal{C}$*, if and only if* $H(T) - C \in K(T)$*. We call a claim* C *super-hedgeable (by* (h,H)*), if there exists a super hedge* (h,H) *for* C*. Denote*

$$\mathcal{C}^h := \{C \in \mathcal{C} : \exists H : (h,H) \in \mathcal{A},\ H(T) - C \in K(T)\} \ ,$$

the set of claims that are super-hedgeable with initial portfolio $h \in \mathbb{R}^{d+1}$ *and define*

$$\mathcal{C}^+ := \{C \in K(T) : \exists A \in \mathcal{F}(T),\ P(A) > 0,\ \forall \omega \in A : C(\omega) \in intK(\omega,T)\} \ .$$

For a subclass $\mathcal{B} \subset \mathcal{A}$ *we write*

$$\mathcal{C}^h|\mathcal{B} := \{C \in \mathcal{C}^h : \exists H : (h,H) \in \mathcal{B},\ H(T) - C \in K(T)\} \ .$$

$\diamondsuit$

Lemma 3.9 *Let* $(Q,R) \in \mathcal{P} \neq \emptyset$ *and* $C \in \mathcal{C}$ *be a contingent claim satisfying* $E_Q[(C(R(T) \star X(T)))^-] < \infty$*. Suppose* $(h,H) \in \mathcal{A}$ *is a super hedge for* C*. Then the process* $H(R \star X)$ *is a* Q*-supermartingale and*

$$E_Q[H(T)(R(T) \star X(T))] \leq h(R(0) \star X(0)) \ .$$

Proof. Let $(Q,R) \in \mathcal{P} \neq \emptyset$. If $(h,H) \in \mathcal{A}$ is a super hedge for C with $E_Q[\min\{C(R(T) \star X(T)), 0\}] > -\infty$ then from $(H(T) - C)(R(T) \star X(T)) \geq 0$ we get

$$E_Q[\min\{H(T)(R(T) \star X(T)), 0\}] > -\infty \ .$$

Let us first show that $H(.-) \bullet (R \star X)$ is a Q-Martingale. Since we are dealing with a finite discrete time set $\mathcal{T}$ and $(R \star X)$ is a Q-martingale by assumption, the martingale transform $H(.-) \bullet (R \star X)$ is a local Q-Martingale. Because of $R \in \mathcal{R}$, the process $(R \star X) \bullet H$ is decreasing and $(R \star X)\Delta H(0) \leq 0$. Hence from

$$H(R\star X) = H(0-)(R(0)\star X(0)) + (R(0)\star X(0))\Delta H(0) + H(.-)\bullet(R\star X) + (R\star X)\bullet H$$

we see that

$$H(.-) \bullet (R \star X)(T) \geq H(R \star X)(T) - H(0-)(R(0) \star X(0))(T)$$

and hence

$$E_Q[\min\{H(.-) \bullet (R \star X)(T), 0\}] > -\infty$$

which implies that $H(.-) \bullet (R \star X)$ is a Q-Martingale. We still have to show, that $(R \star X) \bullet H$ is a Q-supermartingale and

$$(R(0) \star X(0))\Delta H(0) \in L^1(Q) \ , \quad E_Q[(R(0) \star X(0))\Delta H(0)] \leq 0 \ .$$

Since the process $(R \star X) \bullet H$ is decreasing we have

$$\forall t \in \mathcal{T} : \ 0 \geq (R \star X) \bullet H(t)$$

and thus $E_Q[(R \star X) \bullet H(t)] \leq 0$ is defined for every t. Analogously from

$$(R \star X)\Delta H(0) \leq 0$$

we conclude

$$E_Q[(R(0) \star X(0))\Delta H(0)] \leq 0 \ .$$

Because $H(.-) \bullet RX$ is a Q-supermartingale, the following calculation is valid for every $t \in \mathcal{T}$

$$\begin{aligned} h(R(0) \star X(0)) &\geq h(R(0) \star X(0)) + E_Q[(R(0) \star X(0))\Delta H(0)] \\ &\quad + E_Q[H(.-) \bullet (R \star X)(t)] + E_Q[(R \star X) \bullet H(t)] \\ &\geq h(R(0) \star X(0)) + E_Q[(R(0) \star X(0))\Delta H(0)] \\ &\quad + E_Q[H(.-) \bullet (R \star X)(T)] + E_Q[(R \star X) \bullet H(T)] \\ &= E_Q[H(T)(R(T) \star X(T))] \\ &\geq E_Q[\min\{H(T)(R(T) \star X(T)), 0\}] > -\infty \ . \end{aligned}$$

Hence $(RX) \bullet H$ is a Q-supermartingale.

$\diamondsuit$

Remark 3.10 Suppose $\mathcal{P} \neq \emptyset$. Then we have $\mathcal{A} \cap \mathcal{A}^{\infty} \subset \mathcal{A}^{\mathcal{P}}$. This is a direct implication of Lemma 3.9.

Chapter 4

Fundamental Theorems of Asset Pricing in Discrete Time

In this chapter we develop Fundamental Theorems of Asset Pricing in discrete time under varying assumptions on price processes. We are going to treat the cases of finite and infinite state space Ω separately, because the two cases involve different techniques of proofs. Since in every case, we have to rely on the same definition of "arbitrage", however, this will be discussed in advance.

4.1 Arbitrage

Definition 4.1 *A portfolio process* $(h, H) \in \mathcal{A}$ *is called an arbitrage, if* $-h \in K(0)$ *and* $H(T) \in \mathcal{C}^+$*.*

The following lemma states a sufficient condition for the absence of arbitrage in a general discrete time market. The converse implication does not hold in this generality, but in many cases, as will be revealed in the subsequent sections.

Lemma 4.2 *Suppose* $\mathcal{P} \neq \emptyset$*. Then there is no arbitrage in* $\mathcal{A}$*.*

Proof. Let $(Q, R) \in \mathcal{P} \neq \emptyset$ and $(h, H) \in \mathcal{A}$ with $-h \in K(0)$ and $H(T) \in K(T)$. We only need to show $H(T) \notin \mathcal{C}^+$. From $H(T) \in K(T)$ we have

$$H(T)(R(T) \star X(T)) \geq 0 \ .$$

Thus, if we define the claim $C := H(T)$ then we have

$$E_Q[C(R(T) \star X(T))] = E_Q[H(T)(R(T) \star X(T))] \geq 0 \ .$$

Since this clearly implies $E_Q[(C(R(T) \star X(T)))^-] < \infty$, we are in the situation of Lemma 3.9. According to this lemma then, the process $(R \star X) \bullet H$ is a Q-supermartingale and we get

$$0 \geq h(R(0) \star X(0)) \geq E_Q[H(T)(R(T) \star X(T))] \geq 0 \ .$$

Thus we have $E_Q[H(T)(R(T) \star X(T))] = 0$. Because of $H(T)(R(T) \star X(T)) \geq 0$ this implies

$$H(T)(R(T) \star X(T)) = 0 \ .$$

Applying Proposition 2.4 we conclude that for almost every $\omega \in \Omega$ we have

$$H(\omega, T) \in \partial K(\omega, T)$$

and thus $H(T) \notin \mathcal{C}^+$.

$\diamondsuit$

4.2 Finite Case

In this section we deal with the simple, but often treated case of finite Ω as it occurs for example with the Cox Ross Rubinstein model. For such markets Harrison, Pliska (1981) already show the equivalence of the "no arbitrage" condition and the "no free lunch" condition in a frictionless market. Kabanov, Stricker(1999) have developed a Fundamental Theorem for a currency market with proportional transaction costs, when Ω is finite. Below we give a proof, that adopts their ideas, but the notions and the techniques are modified in order to support the generalization to infinite Ω in the Section 4.4. Note also that the subsequent results for finite Ω also hold for finite $\mathcal{F}$.

Theorem 4.3 *Suppose $\Omega = \{\omega_1, \ldots, \omega_n\}$. If there is no arbitrage in $\mathcal{A}$, then $\mathcal{P} \neq \emptyset$.*

Proof. For $i = 0, \ldots, d$ we define the measures $X_i(T)P$ by

$$\frac{dX_i(T)P}{dP} = X_i(T)$$

and write

$$L_X^p := \times_{i=0,\dots,d} L^p(X_i P) \ , \quad p \in \{0\} \cup [1, \infty).$$

Since Ω is finite we have $L_X^0 = L_X^p$ for every $p \geq 1$. Define

$$\bar{\mathcal{C}}^+ := \{C \in \mathcal{C}^+ : E_P[CX] = 1\} \ .$$

The absence of arbitrage in $\mathcal{A}$ obviously implies

$$\bar{\mathcal{C}}^+ \cap \mathcal{C}^0 = \emptyset \ .$$

It is easy to see that

$$\mathcal{C}^0 = 0 + \sum_{t=0}^{T} K(t)$$

is a convex cone containing the claim $V = 0$. Using the finiteness of Ω and the closedness of every $K(\omega)$, $\omega \in \Omega$, one can show that $\mathcal{C}^0$ is $L_X^1(\mathcal{F}(T))$-closed. This is true, because the closedness of $K(\omega)$, $\omega \in \Omega$, implies the closedness of $0 + \sum_{t=0}^{T} K(\omega, t)$, $\omega \in \Omega$. The set $\bar{\mathcal{C}}^+$ is convex and $L_X^1(\mathcal{F}(T))$-closed. Thus according to a separating hyperplane theorem there exists a $Z(T) \in L_X^\infty(\mathcal{F}(T))$ such that

$$\sup_{C \in \mathcal{C}^0} E_P[C(Z(T) \star X(T))] = 0 < \inf_{C \in \mathcal{C}^+} E_P[C(Z(T) \star X(T))] \ . \tag{4.1}$$

From $-K(T) \subset \mathcal{C}^0$ we have $Z(T) \in -K'(T)$. Otherwise there would exist a $C \in K(T)$ with $P(C(Z(T) \star X(T)) < 0) > 0$. Defining $F := \{C(Z(T) \star X(T)) < 0\}$ then, we had $C1_F \in K(T)$ and $E_P[1_F C(Z(T) \star X)] < 0$, thus a contradiction to (4.1).

Moreover we have $Z_0(T) > 0$. In fact, for arbitrary $F \in \mathcal{F}(T)$ with $P(F) > 0$ the claim $C^F := \dfrac{1_F}{X(T)P(F)}(1, 0, \dots, 0)$ is an element of $\bar{\mathcal{C}}^+$ and consequently

$$0 < E_P[C^F(Z(T) \star X(T))] = E_P[\frac{1_F Z_0(T)}{P(F)}] = \frac{E_P[1_F Z_0(T)]}{P(F)}$$

for all $F \in \mathcal{F}(T)$ with $P(F) > 0$. Remembering $X_0(T) = 1$, we can renormalize $Z^C(T)$ in order to get $E_Q[Z_0^C(T)] = 1$.

Let us define the process $Z = (Z_i)_{i \in \{0,\dots,d\}}$ by

$$Z_i(t) := \frac{E_P[Z_i(T)X_i(T)|\mathcal{F}(t)]}{X_i(t)} \quad , \ t \in \mathcal{T} \ .$$

With this definition the process $Z \star X$ is a P-martingale. We want to show now that

$$\forall t \in \mathcal{T} : Z(t) \in -K'(t) \; . \tag{4.2}$$

We already know $Z(T) \in -K'(T)$. For a fixed $s \in \mathcal{T} \setminus \{T\}$ we take an arbitrary $G(s) \in K(s)$. Now we consider an arbitrary nonnegative $\xi \in L^0(P, \mathcal{F}(s))$ and the buy and hold strategy $(0, H)$ defined by

$$H(\omega, t) := -\xi(\omega) G(\omega, s) 1_{\{s,\dots,T\}}(\omega, t)$$

.

Since $-\Delta H(t) \in K(t)$ for every $t \in \mathcal{T}$, we have $(0, H) \in \mathcal{A}$ and moreover the strategy $(0, H)$ is a super hedge for the claim

$$C := H(T) = -\xi G(s)$$

and hence $C \in \mathcal{C}^0$. Consequently, the separating inequality (4.1) together with the tower properties of conditional expectations yield

$$\begin{aligned}
0 &\geq E_P[C(Z(T) \star X(T)] \geq -E_P[\xi G(s)(Z(T) \star X(T))] \\
&= -E_P[E_P[\xi G(s)(Z(T) \star X(T))|\mathcal{F}(s)]] \\
&= -E_P[\xi E_P[G(s)(Z(T) \star X(T))|\mathcal{F}(s)]] \\
&= -E_P[\xi G(s) E_P[(Z(T) \star X(T))|\mathcal{F}(s)]] \\
&= -E_P[\xi G(s)(Z(s) \star X(s))] \; .
\end{aligned}$$

Since ξ was arbitrary chosen, we conclude $G(s)Z(s) \geq 0$. For $Z(s) \in -K'(s)$, it is sufficent that for almost every ω we have

$$\forall h \in K(\omega, s) : hZ(\omega, s) \geq 0 \; .$$

But since $G(s) \in K(s)$ was arbitrary chosen and so was s, this is proven now. Hence we have (4.2). Since $\mathcal{T}$ is finite, this implies that for almost every ω we have

$$\forall t \in \mathcal{T} : Z(\omega, t) \in -K'(\omega, t) \; . \tag{4.3}$$

The measure Q defined by

$$\frac{dQ}{dP} = Z_0(T)$$

is equivalent to P on $\mathcal{F}(T)$ and because of (4.3) the process $R := \frac{1}{Z_0} Z$ is an element of $\mathcal{R}$. The process $Z_0(T)(R \star X)$ is a P-martingale. Thus $R \star X$ is a Q-martingale. So finally we have found a $(Q, R) \in \mathcal{P}$.

$\diamondsuit$

4.3 Small Transaction Costs

Dalang et al. (1989), Kabanov, Kramkov (1994), Irle (1998, Chapter 5) and Shiryaev (1999) prove Fundamental Theorems for frictionless markets with arbitrary Ω in finite discrete time. They make use of the equivalence of "no arbitrage" and "no local no arbitrage" that also holds in our general framework. "No local arbitrage" means that for every t and for every $V \in (L^0(\mathcal{F}(t-1)))^{d+1}$we have the implication

$$V\Delta X(t) \geq 0 \quad \Rightarrow \quad V\Delta X(t) = 0 \ .$$

In order to adopt the proofs of Dalang et alia (1989), Kabanov, Kramkov (1994), Irle (1998, Chapter 5) and Shiryaev (1999) to our general framework we would need the following Assumption 4.4.

Assumption 4.4 *There is a $R \in \mathcal{R}$ such that for every $t \in \{1, \ldots, T\}$ and for every $V \in (L^0(\mathcal{F}(t-1)))^{d+1}$ we have the implication*

$$V\Delta(R \star X(t)) \geq 0 \quad \Rightarrow \quad V\Delta(R \star X(t)) = 0 \ .$$

$\diamondsuit$

It is easy to see that with Assumption 4.4 the proofs of Dalang et alia (1989), Kabanov, Kramkov (1994), Irle (1998, Chapter 5) and Shiryaev (1999) can be applied to the process $R \star X$ instead of X in order to prove that $\mathcal{P} \neq \emptyset$. However, Assumption 4.4 will not be satisfied in general since the "no local arbitrage" condition in our framework corresponds to the following Assumption 4.5 that is in fact weaker.

Assumption 4.5 (No local Arbitrage) *For every $t \in \{1, \ldots, T\}$ and for every $V \in -K(t-1)$we have the implication*

$$V \in K(t) \quad \Rightarrow \quad V \in \partial K(t) \ .$$

An equivalent formulation involving $\mathcal{R}$ is:

For every $t \in \{1, \ldots, T\}$ and for every $V \in L^0(\mathcal{F}(t-1)))^{d+1}$ such that for every $R \in \mathcal{R}$ satisfying

$$V(R(t-1) \star X(t-1)) \leq 0 \quad , \quad V(R(t) \star X(t)) \geq 0$$

there is a $\bar{R} \in \mathcal{R}$ *such that*

$$V(\bar{R}(t) \star X(t)) = 0 \ .$$

$\diamondsuit$

Assumption 4.5 is weaker than Assumption 4.4, because $\bar{R}$ in Assumption 4.5 depends on $V \in L^0(\mathcal{F}(t-1)))^{d+1}$, whereas it is the crucial point of Assumption 4.4 that the R is the same for every $t \in \{1, \ldots, T\}$ and every $V \in L^0(\mathcal{F}(t-1)))^{d+1}$. Since the "no (local) arbitrage" condition in our general framework is in fact equivalent to Assumption 4.5, this leads us to the following conclusion:

Assumption 4.4 actually states, that there is no local arbitrage, if and only if there is no local arbitrage in a frictionless market with price process $R \star X$ instead of X.

We claim that Assumption 4.4 cannot hold, unless the transaction costs on the market are small in comparison to possible price changes. This is easy seen from a simple counter example. In fact, suppose at time $t-1$ you buy a certain security i at a price $X_i(t-1)$ with transaction costs $\lambda_i X_i(t-1)$, i.e., you pay $(1+\lambda_i)X_i(t-1)$. At time t you are going to sell the security at a price $X_i(t)$ under transaction costs $\mu_i X_i(t)$, i.e., you will get $(1-\mu_i)X_i(t)$. Your self-financing portfolio strategy $(0, H)$ with trading only in security i and at time points $t-1, t$ is clearly given by

$$H_0(t-1) = -(1+\lambda_i)X_i(t-1) \quad , \quad H_i(t-1) = 1 \ .$$

$$H_0(t) = (1-\mu_i)X_i(t) - (1+\lambda_i)X_i(t-1) \quad , \quad H_i(t) = 0 \ .$$

Now suppose that (P-almost surely) we have

$$(1-\mu_i)X_i(t) < (1+\lambda_i)X_i(t-1) \ .$$

Then you are going to make a certain loss, because even if the price of security i increases to the highest possible values, you lose money because of transaction costs. Moreover for any $R \in \mathcal{R}$ we have

$$\begin{aligned} -H(t-1)\Delta(R \star X(t)) &= H_i(t-1)(R_i(t-1)X_i(t-1) - R_i(t)X_i(t)) \\ &\geq (1-\mu_i)X_i(t-1) - (1+\lambda_i)X_i(t) > 0 \ . \end{aligned}$$

Thus, whatever $R \in \mathcal{R}$ we choose, Assumption 4.4 is not satisfied for $V = -H(t-1) = -1$. Consequently a necessary condition for Assumption 4.4 to hold is: There is no certain loss because of transaction costs, i.e., transaction costs are small in comparison to price changes. In other words, if Assumption 4.4 holds, then this implies that transaction costs are negligible in as much as arbitrage is concerned.

Of course one could ask, *how* small transaction costs have to be for Assumption 4.4 to hold. But we are not that much interested in sufficient conditions for Assumption 4.4, because the next section will treat the general case without additional assumptions.

4.4 General Case

In a general market situation, i.e., when Ω is arbitrary, it is convenient to introduce some notions of arbitrage opportunities weaker than that in Definition 4.1. One speaks of a free lunch, if there is an asymptotic arbitrage with respect to a suitable topology. The classical free lunches are defined with respect to the topology in probability P (see Definition 4.6).

Free lunches will also apply to continuous time trading (for a further discussion compare also for Chapter 7). Therefore, the content of this section is for the most part almost identique with that of Chapter 7. Nevertheless, in order to provide a closed representation for discrete as well as for continuous time, we are going to present results and proofs separately for both cases. Of course this inavoidably involves many redundancies which we deliberately accept, as they are not completely needless. In fact, Theorems 4.14 and 4.17 are slighthly stronger than their continuous time counterparts.

The notions of free lunches that we consider apart from classical free lunches can be ranged in two classes. The first class consists of so called L^∞-bounded free lunches in terms of uniformly bounded sequences converging in probability P. This convergence is invariant to an equivalent change of the probability measure. The second class consists of free lunches in terms of sequences converging in some $L^p(Q)$, $1 \leq p < \infty$. Since this convergence obviously depends on the choice of Q, it would be dissatisfactory to define free lunches only with respect to convergence in a single $L^p(Q)$ for some $Q \sim P$ or $Q = P$. Because then, the notion of arbitrage would depend on the choice of an arbitrary probability measure $Q \sim P$. In order to avoid this arbitrariness we consider the class

$$\mathcal{Q} := \{Q \sim P : \forall i \in \{0, \dots, d\} : X_i \in L^1(Q)\} .$$

From Lemma A.8 we know that this class is not empty.

For $Q \sim P$ we define the measures $X_i(T)Q$ by

$$\frac{dX_i(T)Q}{dQ} = X_i(T) \quad , \; i = 0, \ldots, d.$$

It is worth noting, that for $Q \in \mathcal{Q}$ the measures $X_i(T)Q$ are finite and we have $X_i(T)Q \sim P$ for every $i \in \{0, \ldots, d\}$. The latter is a consequence of condition (2.2). We denote

$$L^0 := (L^0(\mathcal{F}(T)))^{d+1}$$

the space of $\mathcal{F}(T)$-measurable random vectors,

$$L^\infty := (L^\infty(\mathcal{F}(T), P))^{d+1}$$

the space of P-almost surely bounded random vectors and

$$L^p_{XQ} := \times_{i=0,\ldots,d} L^p(\mathcal{F}(T), X_i Q) \quad , 1 \le p < \infty, \; Q \in \mathcal{Q}$$

the space of random vectors $V = (V_0, \ldots, V_d) \in L^0$ such that $V_i X_i \in L^p(Q, \mathcal{F}(T))$ for every $i \in \{0, \ldots, d\}$.

For $O \subset L^0$ we denote $\overline{O}^0$ the closure of O in with respect to the topology of (componentwise) convergence in probability $Q \sim P$, which is independent of Q because of $Q \sim P$.

For $O \subset L^p_{XQ}$, $1 \le p < \infty$, we write $\overline{O}^p_{XQ}$ for the closure of O in the L^p_{XQ}-norm topology of L^p_{XQ}.

Although L^∞ is independent of the choice of $Q \sim P$, we will deal with different weak* topologies on L^∞ depending on Q. For $Q \sim P$ let $\sigma(L^\infty, L^1_{XQ})$ denote the locally convex topology on L^∞ induced by the semi-norms

$$|.|_Z : L^\infty \to \mathbb{R}, \; C \mapsto \sum_{i=0}^{d} | \int_\Omega C_i Z_i X_i dQ| \quad , \; Z \in L^1_{XQ} \, .$$

Then we write $\overline{O}^\infty_{XQ}$ for the closure of O in the $\sigma(L^\infty, L^1_{XQ})$-topology of $L^\infty(\mathcal{F}(T))$.

Definitions 4.6 *A sequence $(h^n, H^n)_{n \in \mathbb{N}}$ in $\mathcal{A}$ is called a (classical) free lunch, if and only if for every $n \in \mathbb{N}$ we have $-h^n \in K(0)$ and there exists a $C \in \mathcal{C}^+$ such that $H^n(T)$ converges to C in probability. If in addition we have*

$$\forall n \in \mathbb{N} : \max_i |H^n_i(T)| \le \beta \quad P - a.s.$$

for some $\beta > 0$, then we call this a L^∞-bounded free lunch.

For $1 \leq p < \infty$ a sequence $(h^n, H^n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ is called a L^p-free lunch, if there exist a probability measure $Q \in \mathcal{Q}$ and a claim $C \in \mathcal{C}^+ \cap L^p_{XQ}$ such that

$$\forall n \in \mathbb{N} : H^n(T) \in L^p_{XQ}, \ -h^n \in K(0)$$

and $H^n(T)$ converges to C in L^p_{XQ}.

$\diamondsuit$

Proposition 4.7 *Suppose $\mathcal{B} \subset \mathcal{A}$. Then there is no arbitrage in $\mathcal{B}$, if and only if*

$$\forall h \in -K(0) : (\mathcal{C}^h|\mathcal{B}) \cap \mathcal{C}^+ = \emptyset \ .$$

Proof. Suppose there exists $h \in -K(0)$ such that $(\mathcal{C}^h|\mathcal{B}) \cap \mathcal{C}^+ \neq \emptyset$. Then there is a $(h, H) \in \mathcal{B}$ with $-h \in K(0)$ and $H(T) \in \mathcal{C}^+$, thus an arbitrage in $\mathcal{B}$.

Conversely, suppose there is an arbitrage $(h, H) \in \mathcal{B}$. Then we have $h \in -K(0)$ and

$$(\mathcal{C}^h|\mathcal{B}) \cap \mathcal{C}^+ \neq \emptyset \ .$$

$\diamondsuit$

Proposition 4.8 *Suppose $\mathcal{B} \subset \mathcal{A}$. Then there is no free lunch in $\mathcal{B}$, if and only if*

$$\forall h \in -K(0) : \overline{(\mathcal{C}^h|\mathcal{B})}^0 \cap \mathcal{C}^+ = \emptyset \ .$$

Proof. Suppose $h \in -K(0)$ and $\overline{(\mathcal{C}^h|\mathcal{B})}^0 \cap \mathcal{C}^+ \neq \emptyset$. Then there exist a sequence $(h^n, H^n)_{n\in\mathbb{N}}$ in $\mathcal{B}$ and a claim $C \in \mathcal{C}^+ \cap L^0$ such that $H^n(T)$ converges to C in probability P. Hence the sequence $(h, H^n)_{n\in\mathbb{N}}$ is a free lunch in $\mathcal{B}$.

Conversely, suppose there is a free lunch in $\mathcal{B}$. Then

$$\overline{(\mathcal{C}^0|\mathcal{B})}^0 \cap \mathcal{C}^+ \neq \emptyset \ .$$

$\diamondsuit$

Proposition 4.9 *Suppose $\mathcal{B} \subset \mathcal{A}$ and let $1 \leq p < \infty$. There is no L^p-free lunch in $\mathcal{B}$, if and only if*

$$\forall Q \in \mathcal{Q} \ \forall h \in -K(0) : \overline{((\mathcal{C}^h|\mathcal{B}) \cap L^p_{XQ})}^p_{XQ} \cap \mathcal{C}^+ = \emptyset \ .$$

Proof. Suppose $h \in -K(0)$, $Q \in \mathcal{Q}$ and

$$\overline{((\mathcal{C}^h|\mathcal{B}) \cap L^p_{XQ})}^p_{XQ} \cap \mathcal{C}^+ \neq \emptyset \; .$$

Then there exist a sequence $(h^n, H^n)_{n\in\mathbb{N}}$ in $\mathcal{B}$ and a claim $C \in \mathcal{C}^+ \cap L^p_{XQ}$ such that for every $n \in \mathbb{N}$ we have $H^n(T) \in L^p_{XQ}$ and $H^n(T)$ converges to C in L^p_{XQ}. Hence the sequence $(h, H^n)_{n\in\mathbb{N}}$ is a L^p-free lunch in $\mathcal{B}$.

Conversely, suppose there is a L^p-free lunch in $\mathcal{B}$. Then there exists a $Q \in \mathcal{Q}$ with

$$\overline{((\mathcal{C}^0|\mathcal{B}) \cap L^p_{XQ})}^p_{XQ} \cap \mathcal{C}^+ \neq \emptyset \; .$$

$\diamondsuit$

Remark 4.10 For a characterization of the absence of L^∞-bounded free lunches similar to Proposition 4.9 we will need to know that $L^1_{XQ} = L^1_{XQ}(\mathcal{F}(T))$ is separable for every $Q \in \mathcal{Q}$. This assumption is also necessary in so far as it is necessary for the weak* topology of the closed unit sphere of L^∞ to be metrizable (see Holmes, 1975, p. 72, Corollary 2). This property is needed to assure equivalence of weak* closedness and weak* sequential closedness. Although the separability assumption on L^1_{XQ} seems to be quite restrictive at first sight, it is in fact always satisfied in our discrete time model, if $\mathcal{F}(T)$ is generated by $\sigma(X_i(t) : i \in \{0, 1, \ldots, d\}, t \in \mathcal{T})$ and the P-null sets of $\mathcal{F}$. This is true, because for any probability space $(\Omega, \mathcal{F}, P)$ the space $L^1(P, \mathcal{F})$ is separable, if and only if there is a separable $\tilde{\mathcal{F}} \subset \mathcal{F}$ such that

$$\forall F \in \mathcal{F} \setminus \tilde{\mathcal{F}} : \quad P(F) \in \{0, 1\} \; .$$

There $\tilde{\mathcal{F}}$ is separable, if and only if it is generated by a countable subset $\mathcal{E} \subset \tilde{\mathcal{F}}$.

$\diamondsuit$

Proposition 4.11 *Suppose $\mathcal{F}(T)$ is separable (c.f. Remark 4.10) and let $\mathcal{B} \subset \mathcal{A}$. Then there is no L^∞-bounded free lunch in $\mathcal{B}$, if and only if*

$$\forall Q \in \mathcal{Q} \forall h \in -K(0) : \overline{((\mathcal{C}^h|\mathcal{B}) \cap L^\infty)}^\infty_{XQ} \cap \mathcal{C}^+ = \emptyset \; .$$

Proof.

i) Suppose there exists a $h \in -K(0)$ and a $Q \in \mathcal{Q}$ satisfying

$$\overline{((\mathcal{C}^h|\mathcal{B}) \cap L^\infty)}^\infty_{XQ} \cap \mathcal{C}^+ \neq \emptyset \; .$$

Then there is a $\lambda > 0$ such that the intersection of $\overline{((\mathcal{C}^h|\mathcal{B}) \cap L^\infty}^\infty_{XQ} \cap \mathcal{C}^+$ with

$$\lambda\mathcal{S}^\infty := \{C \in L^\infty : esssup \sum_i |C_i| \leq \lambda\}$$

is not empty. The intersection

$$\lambda\mathcal{S}^\infty \cap \overline{((\mathcal{C}^h|\mathcal{B}) \cap L^\infty)}^\infty_{XQ}$$

is closed with respect to the $\sigma(\mathcal{S}^\infty, L^1_{XQ})$-topology (c.f. the Krein-Smulian Theorem in Dunford, Schwartz, 1958, p.429, Theorem 7). Since L^1_{XQ} is separable, the $\sigma(\lambda\mathcal{S}^\infty, L^1_{XQ})$-topology is metrizable (c.f. Dunford, Schwartz, 1958, p.426, Theorem 1 or Holmes, 1975, p. 72, Corollary 2). Hence, for

$$C \in \lambda\mathcal{S}^\infty \cap \overline{((\mathcal{C}^h|\mathcal{B}) \cap L^\infty)}^\infty_{XQ} \cap \mathcal{C}^+$$

there exists a sequence $(H^n)_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$ we have $(h, H^n) \in \mathcal{B}$, $H^n(T) \in \lambda\mathcal{S}^\infty$ and $H^n(T)$ converges to C in the $\sigma(\lambda\mathcal{S}^\infty, L^1_{XQ})$-topology. Like in part ii) of the proof of Theorem A.9 we conclude that the sequence $(H^n(T))_{n\in\mathbb{N}}$ converges to C in probability P. Moreover, it is uniformly bounded by λ. Hence we have an L^∞-bounded free lunch.

ii) Conversely, suppose there is a L^∞-bounded free lunch, i.e., a sequence $(h^n, H^n)_{n\in\mathbb{N}}$, a claim $C \in \mathcal{C}^+ \cap L^\infty$ and a $\beta > 0$ with $(h^n, H^n) \in \mathcal{B}$, $h^n \in K(0)$,

$$\text{ess sup} \sum_i |H^n_i(T)| \leq \beta$$

and $H^n(T)$ converges to C in probability P. Note that from $h^n \in K(0)$ it is straight forward that $H^n(T) \in \mathcal{C}^0$. Suppose $Q \in \mathcal{Q}$ and for arbitrary $Z \in L^1_{XQ}$ define the measures Z_iX_iQ, $i = 0, \ldots, d$, in the usual way. Then it is clear that for every $i \in \{0, \ldots, d\}$ the sequence $(|H^n_i(T)|)_{n\in\mathbb{N}}$ is uniformly Z_iX_iQ-integrable. Hence $(|H^n_i(T)|)_{n\in\mathbb{N}}$ converges to C in $L^1(Z_iX_iQ)$ which implies

$$\lim_{n\to\infty} \sum_i \int_\Omega H^n_i(T) Z_i dX_iQ = \sum_i \int_\Omega C_i Z_i dX_iQ \ .$$

Since Z was chosen arbitrary in L^1_{XQ} , we conlcude that $H^n(T)$ converges to C in $\sigma(L^\infty, L^1_{XQ})$. Hence we have

$$\overline{((\mathcal{C}^0|\mathcal{B}) \cap L^\infty)}^\infty_{XQ} \cap \mathcal{C}^+ \neq \emptyset \ .$$

$\diamondsuit$

Remarks 4.12 Let $\mathcal{B} \subset \mathcal{A}$ and suppose there is no free Lunch in $\mathcal{B}$. Then there is no L^p-free lunch in $\mathcal{B}$. In fact, if $(h^n, H^n)_{n\in\mathbb{N}}$ and C constitute a L^p-free lunch, then, because L^p-convergence implies convergence in probability, there is a free lunch.

Proposition 4.8 implies that the postulation of no arbitrage in $\mathcal{B} \subset \mathcal{A}$ is equivalent with the postulation of no free lunch in $\mathcal{B}$, if and only if the set $(\mathcal{C}^h|\mathcal{B})$ is closed in the topology of (componentwise) convergence in probability P. Analogues statements result from Propositions 4.9 and 4.11.

$\diamondsuit$

Definition 4.13 *For $Q \in \mathcal{Q}$ and $1 \le q \le \infty$ define*

$$\mathcal{P}_Q^q := \{\tilde{Q} : \exists \tilde{R} \in \mathcal{R} : (\tilde{Q}, \tilde{R}) \in \mathcal{P}, \frac{d\tilde{Q}}{dQ}\tilde{R}(T) \in L_{XQ}^q\} \ .$$

Theorem 4.14 *Let $1 \le p < \infty$, $q = \frac{p}{p-1}$ for $1 < p$, and $q = \infty$ for $p = 1$. If*

$$\forall Q \in \mathcal{Q} : \mathcal{P}_Q^q \neq \emptyset \ ,$$

then there is no L^p-free lunch in $\mathcal{A}$.

Proof. Let $C \in K(T) \cap L_{XQ}^p$ be a contingent claim and $(h^n, H^n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{A}$ such that $-h^n \in K(0)$ and $H^n(T) \in L_{XQ}^p$ converges to C in L_{XQ}^p for some $Q \in \mathcal{Q}$. Choose an arbitrary $(\tilde{Q}, \tilde{R}) \in \mathcal{P}_Q^q$. Applying componentwise the Hölder inequality in account of $\tilde{R}(T)\frac{d\tilde{Q}}{dQ} \in L_{XQ}^q$ we get

$$\begin{aligned}
&\int_\Omega (|H^n(T) - C(T)| \star \tilde{R}(T)) \ X(T) d\tilde{Q} \\
&= \sum_i \int_\Omega |H_i^n(T) - C_i(T)| \tilde{R}_i(T) \ X_i(T) \frac{d\tilde{Q}}{dQ} dQ \\
&\le \sum_i \left(\int_\Omega |H_i^n(T) - C_i(T)|^p \ X_i(T) dQ \right)^{\frac{1}{p}} \left(\int_\Omega (\tilde{R}_i(T) \frac{d\tilde{Q}}{dQ})^q \ X_i(T) dQ \right)^{\frac{1}{q}} \ .
\end{aligned}$$

This inequality is also valid in the case $q = \infty$ with the convention

$$\int_\Omega (\tilde{R}_i(T) \frac{d\tilde{Q}}{dQ})^\infty X_i(T) dQ := \text{ess sup} \left(\tilde{R}_i(T) \frac{d\tilde{Q}}{dQ} \right) \in \mathbb{R}_+ \ .$$

The above inequality and the convergence of $H^n(T)$ to C in L_{XQ}^p imply that $(H^n(T) \star \tilde{R}(T))_{n\in\mathbb{N}}$ converges to C in $L_{X\tilde{Q}}^1$ (remember that $\tilde{R}(T)$ takes values in

$\{1\} \times \mathbb{R}^d_+$). Because for every $n \in \mathbb{N}$ the portfolio process $(h^n, H^n) \in \mathcal{A}$ is a super hedge for $H^n(T) \in L^p_{XQ}$, Lemma 3.9 implies, that $H^n(R \star X)$ is a Q-supermartingale. Consequently we have

$$0 \;\geq\; h(\tilde{R}(0) \star X(0)) \geq E_{\tilde{Q}}[H^n(T)(\tilde{R}(T) \star X(T))] \; .$$

Thus we have

$$0 \geq \lim_{n\to\infty} E_{\tilde{Q}}[(H^n(T) \star \tilde{R}(T))X(T))] = E_{\tilde{Q}}[(C \star \tilde{R}(T)) \star X(T)] \geq 0$$

which in account of $(C \star \tilde{R}(T)) \star X(T) \geq 0$ results in

$$C(R(T) \star X(T)) = 0 \; .$$

Hence, for almost every ω we have

$$C(\omega)(R(\omega, T) \star X(\omega, T) = 0 \; ,$$

which according to Proposition 2.4 implies that for almost every ω

$$C(\omega) \in \partial K(\omega, T)$$

and thus $C \notin \mathcal{C}^+$.

$\diamondsuit$

Theorem 4.15 *Let $1 \leq p < \infty$, $q = \frac{p}{p-1}$ for $1 < p$, and $q = \infty$ for $p = 1$. Suppose there is no L^p-free lunch in $\mathcal{A}$. Then*

$$\forall Q \in \mathcal{Q} : \mathcal{P}^q_Q \neq \emptyset \; .$$

Proof. Because $\mathcal{A}$ is convex in $\mathcal{H}$, it is clear that for any $Q \in \mathcal{Q}$ the set $\overline{(\mathcal{C}^0 \cap L^p_{XQ})}^p_{XQ}$ in $L^p_{XQ}(\mathcal{F}(T))$ is a convex cone containing 0. The set $\mathcal{C}^+ \cap L^p_{XQ}$ is also a convex cone in $L^p_{XQ}(\mathcal{F}(T))$. Fix an arbitrary $Q \in \mathcal{Q}$. According to Proposition 4.9 the absence of a L^p-free lunch in $\mathcal{A}$ implies

$$\overline{(\mathcal{C}^0 \cap L^p_{XQ})}^p_{XQ} \cap \mathcal{C}^+ = \emptyset \; .$$

Thus according to a separating hyperplane theorem, for every $C \in \mathcal{C}^+ \cap L^p_{XQ}$ there exists a $Z^C(T) \in L^q_{XQ}(\mathcal{F}(T))$ satisfying

$$\sup_{B \in \mathcal{C}^0 \cap L^p_{XQ}} E_Q[B(Z^C(T) \star X(T))] = 0 < E_Q[C(Z^C(T) \star X(T))] \; .$$

From

$$-K(T) \subset \mathcal{C}^0(T)$$

we have $Z^C(T) \in -K'(T)$. Otherwise there would exist a $B \in K(T)$ such that

$$Q(B(Z^C(T) \star X(T)) < 0) > 0 \ .$$

Defining

$$F := \{B(Z(T) \star X(T)) < 0\} \ ,$$

we had $-B1_F \in -K(T)$ and there would exist a $\mathcal{F}(T)$-measurable $\tilde{F} \subset F$ with $-B1_{\tilde{F}} \in K(T) \cap L^p_{XQ}$. Then we had $E_Q[-1_{\tilde{F}} B(Z(T) \star X(T))] > 0$, thus a contradiction to the separating inequality. Hence, we conclude $Z^C_0(T) \geq 0$.

Remembering $X_0(T) = 1$, we can renormalize $Z^C(T)$ in order to get $E_Q[Z^C_0(T)] = 1$ and define a Q-dominated probability measure Q^C by

$$\frac{dQ^C}{dQ} = Z^C_0(T) \ .$$

In summary, for every $C \in \mathcal{C}^+ \cap L^p_{XQ}$ there exists a $Z^C(T) \in -K'(T) \cap L^q_{XQ}$ and a P-dominated probability measure Q^C with $\dfrac{dQ}{dP} = Z^C_0(T)$.

Denote $\mathcal{M}$ the set of P-dominated probability measures Q^C, $C \in \mathcal{C}^+_0 \cap L^p_{XQ}$. For every $F \in \mathcal{F}(T)$ with $Q(F) > 0$ there exists a $C \in \mathcal{C}^+_0 \cap L^p_{XQ}$ with $Q^C(F) > 0$. In fact, the claim $C := 1_F(1, 0, \ldots, 0)$ is an element of $\mathcal{C}^+ \cap L^p_{XQ}$ satisfying

$$0 < E_Q[C(Z^C(T) \star X(T))] = E_Q[1_F Z^C_0(T)] = Q^C(F) \ .$$

Thus $(\Omega, \mathcal{F}(T), Q)$ and $\mathcal{M}$ meet the assumptions of the Halmos-Savage Theorem. According to this theorem, there is a countable subfamily $\mathcal{N} \subset \mathcal{M}$ that is equivalent to P. This means, there exist a sequence $(C^n)_{n\in\mathbb{N}}$ in $\mathcal{C}^+_0 \cap L^p_{XQ}$ and a sequence $(\lambda^n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $\forall n : \lambda_n > 0$ such that $\sum_n \lambda_n = 1$ and $Q = \sum_n \lambda^n Q^C_n$ is equivalent with P, i.e.,

$$\sum_n \lambda_n Z^{C_n}_0 > 0 \ . \tag{4.4}$$

Since for every $n \in \mathbb{N}$ we have $Z^{C_n} \in L^q_{XQ}$, there exists a sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that

$$\forall n \in \mathbb{N} \ \forall i \in \{0, \ldots, d\} : 0 < ||Z^{C_n}||_q \leq c_n \ .$$

There, $||.||_q$ denotes the norm in L^q_{XQ} defined by

$$||Y||_q := \sum_i (\int_\Omega |Y_i|^q X dQ)^{\frac{1}{q}} \ .$$

Define

$$\tilde{Z}(T) := \sum_n \frac{\lambda^n}{c_n} Z^n(T)$$

in terms of convergence in the Banach space L^q_{XQ}. Then we have $0 < ||\tilde{Z}(T)||_q \leq 1$ and $\tilde{Z}(T) \in \mathcal{F}(T)$. Moreover, since for every $\omega \in \Omega$ the cone $-K'(\omega, T)$ is closed, it follows by almost sure convergence of a subsequence of $(\sum_{k=1}^n \frac{\lambda^k}{c_k} Z^k(T))_{n\in\mathbb{N}}$, that $\tilde{Z}(T) \in -K'(T)$. In addition, because of (4.4) we have

$$0 < \sum_n \frac{\lambda^n}{c_n} Z^n_0(T)\tilde{Z}_0(T) \in L^q(Q) \ .$$

Thus, if we define

$$Z(T) := \frac{\tilde{Z}(T)}{E_Q[\tilde{Z}_0(T)]} \ , \qquad \frac{d\tilde{Q}}{dQ} := Z_0(T) \ ,$$

we obtain a P-equivalent probability measure $\tilde{Q}$. The definition of $Z(T)$ also yields

$$\sup_{B\in\mathcal{C}^0} E_Q[B(Z(T) \star X(T))] = 0 \ . \tag{4.5}$$

Moreover we have $Z(T) \in L^1_{XQ}$.

Let us define the process $Z = (Z_i)_{i\in\{0,\ldots,d\}}$ by

$$Z_i(t) := \frac{E_Q[Z_i(T)X_i(T)|\mathcal{F}(t)])}{X(t)} \ , \quad (t = 0, \ldots, T-1) \ .$$

With this definition the process $Z \star X$ is a Q-martingale .

We want to show now that

$$\forall t \in \mathcal{T} : Z(t) \in -K'(t) \ . \tag{4.6}$$

We already know that $Z(T) \in -K'(T)$. For a fixed $s \in \mathcal{T} \setminus \{T\}$ we take an arbitrary $G(s) \in K(s)$. Now we consider an arbitrary nonnegative $\xi \in L^\infty(Q, \mathcal{F}(s))$ and the sequence of buy and hold strategies $(0, H^n)_{n\in\mathbb{N}}$ defined by

$$H^n(\omega, t) := -\xi(\omega)G(\omega, s)1_{\{\max_i |G_i(s)|\leq n\}}(\omega, t)1_{\{s,\ldots,T\}}(\omega, t)$$

.

Since $-\Delta H^n(t) \in K(t)$ for all $t \in \mathcal{T}$, we have $(0, H^n) \in \mathcal{A}$. Moreover, the strategy $(0, H^n)$ is a super hedge for the claim

$$C^n := H^n(T) = -\xi G(s) 1_{\{\max_i |G_i(s)| \leq n\}}(\omega, t)$$

and hence $C^n \in \mathcal{C}^0$. It is clear that $C^n \in L^\infty_{XQ}(\mathcal{F}(T))$. Since XQ is a finite measure, (4.5) and the tower properties of conditional expectations yield the following calculation for every $n \in \mathbb{N}$:

$$\begin{aligned} 0 &\geq E_P[C^n(Z(T) \star X(T)] \geq -E_P[\xi G(s) 1_{\{\max_i |G_i(s)| \leq n\}}(Z(T) \star X(T))] \\ &= -E_P[E_P[\xi G(s) 1_{\{\max_i |G_i(s)| \leq n\}}(Z(T) \star X(T))|\mathcal{F}(s)]] \\ &= -E_P[\xi E_P[G(s) 1_{\{\max_i |G_i(s)| \leq n\}}(Z(T) \star X(T))|\mathcal{F}(s)]] \\ &= -E_P[\xi G(s) 1_{\{\max_i |G_i(s)| \leq n\}} E_P[(Z(T) \star X(T))|\mathcal{F}(s)]] \\ &= -E_P[\xi G(s) 1_{\{\max_i |G_i(s)| \leq n\}}(Z(s) \star X(s))] \ . \end{aligned}$$

Since ξ was arbitrary chosen, this implies $G(s) 1_{\{\max_i |G_i(s)| \leq n\}} Z(s) \geq 0$ for every $n \in \mathbb{N}$ and thus $G(s)Z(s) \geq 0$. In order to conclude $Z(s) \in -K'(s)$, it is sufficent to show that for almost every ω we have

$$\forall h \in K(\omega, s) : hZ(\omega, s) \geq 0 \ .$$

But since $G(s) \in K(s)$ was arbitrary chosen and so was s, this is proven now and we have (4.6). Because $\mathcal{T}$ is finite (4.6) implies that for almost every ω we have

$$\forall t \in \mathcal{T} : Z(\omega, t) \in -K'(\omega, t) \ . \tag{4.7}$$

As seen above, the measure $\tilde{Q}$ defined by $\frac{d\tilde{Q}}{dQ} = Z_0(T)$ is equivalent to $Q \sim P$ on $\mathcal{F}(T)$. Because of (4.7), the process $\tilde{R} := \dfrac{1}{Z_0} Z$ is an element of $\mathcal{R}$ satisfying

$$\frac{d\tilde{Q}}{dQ} R(T) = Z(T) \in L^q_{XQ} \ .$$

The process $Z_0(T)(\tilde{R} \star X)$ is a Q-martingale and thus $\tilde{R} \star X$ is a $\tilde{Q}$-martingale. So finally we have found a $(\tilde{Q}, \tilde{R}) \in \mathcal{P}^q_Q$. Since $Q \in \mathcal{Q}$ was arbitrary chosen, we conclude that for every $Q \in \mathcal{Q}$ we have $\mathcal{P}^q_Q \neq \emptyset$.

$\diamondsuit$

Theorem 4.16 *Let $1 \leq p < \infty$, $q = \frac{p}{p-1}$ for $1 < p$ and $q = \infty$ for $p = 1$. There is no L^p-free lunch in $\mathcal{A}$, if and only if*

$$\forall Q \in \mathcal{Q} : \mathcal{P}_Q^q \neq \emptyset \ .$$

Proof. Suppose there is no L^p free lunch in $\mathcal{A}$. Then according to Theorem 4.15 we have $\mathcal{P} \neq \emptyset$. For the other implication see Theorem 4.14.

$\diamondsuit$

Theorem 4.17 *If $\mathcal{P} \neq \emptyset$, then there is no L^∞-bounded free lunch in $\mathcal{A}$.*

Proof. Let $C \in K(T) \cap L^\infty$ be a contingent claim and (h^n, H^n) a sequence in $\mathcal{A}$ such that $-h^n \in K(0)$, $||H^n(T)||_\infty \leq \beta$ for some $\beta > 0$, and $H^n(T)$ converges to C in probability P. Choose an arbitrary $(Q, R) \in \mathcal{P}$. From $R(T) \in L^1_{XQ}$ we see that for every $i \in \{0, \ldots, d\}$ the measure $R_i X_i Q$ is finite. Hence, for every $i \in \{0, \ldots, d\}$ the uniformly boundedness of the sequence $(H_i^n(T))_{n \in \mathbb{N}}$ and the convergence of $H_i^n(T)$ in probability imply that $(H^n(T))_{n \in \mathbb{N}}$ converges to C in

$$L^1_{(R \star X)\tilde{Q}} := \times_{i=0,\ldots,d} L^1(R_i X_i Q) \ .$$

Since for every $n \in \mathbb{N}$ the portfolio process $(h^n, H^n) \in \mathcal{A}$ is a super hedge for $H^n(T) \in L^1_{X\tilde{Q}}$, Lemma 3.9 implies, that $H^n(R \star X)$ is a $\tilde{Q}$-supermartingale and we have

$$0 \ \geq \ h(R(0) \star X(0)) \geq E_Q[H^n(T)(R(T) \star X(T))] \ .$$

This implies

$$0 \geq \lim_{n \to \infty} E_{\tilde{Q}}[H^n(T)(\tilde{R}(T) \star X(T))] = E_{\tilde{Q}}[C(\tilde{R}(T) \star X(T))] \geq 0$$

which in account of $C(\tilde{R}(T) \star X(T)) \geq 0$ results in

$$C(R(T) \star X(T)) = 0 \ .$$

Hence for almost every ω we have

$$C(\omega)(R(\omega, T) \star X(\omega, T) = 0 \ .$$

According to Proposition 2.4 then for almost every ω we have

$$C(\omega) \in \partial K(\omega, T)$$

and thus $C \notin \mathcal{C}^+$.

$\diamondsuit$

Theorem 4.18 *Suppose $\mathcal{F}(T)$ is separable (c.f. Remark 4.10). If there is no L^∞-bounded free lunch in $\mathcal{A}$, then $\mathcal{P} \neq \emptyset$.*

Proof. Because $\mathcal{A}$ is convex in $\mathcal{H}$, for any $Q \in \mathcal{Q}$ the set $\overline{(\mathcal{C}^0 \cap L^\infty)}^\infty_{XQ}$ is obviously a convex cone in L^∞ containing 0. The set $\mathcal{C}^+ \cap L^\infty$ is also a convex cone in L^∞ . Fix an arbitrary $Q \in \mathcal{Q}$. According to proposition 4.11 the absence of L^∞-bounded free lunches in $\mathcal{A}$ implies

$$\overline{(\mathcal{C}^0 \cap L^\infty)}^\infty_{XQ} \cap \mathcal{C}^+ = \emptyset \ .$$

Thus according to a separating hyperplane theorem, for every $C \in \mathcal{C}^+ \cap L^\infty$ there exists a $Z^C(T) \in L^1_{XQ}(\mathcal{F}(T))$ satisfying

$$\sup_{B \in \mathcal{C}^0 \cap L^\infty} E_Q[B(Z^C(T) \star X(T))] = 0 < E_Q[C(Z^C(T) \star X(T))] \ . \tag{4.8}$$

From

$$-K(T) \subset \mathcal{C}^0$$

we have $Z^C(T) \in -K'(T)$. Otherwise there would exists a $B \in K(T)$ with $Q(B(Z^C(T) \star X(T)) < 0) > 0$. Defining

$$F := \{b \leq B(Z(T) \star X(T) < 0\}$$

then, we had $-B1_F \in -K(T)$ and there would exist a $\mathcal{F}(T)$-measurable $\tilde{F} \subset F$ with $-B1_{\tilde{F}} \in K(T) \cap L^\infty \subset L^1_{XQ}$. Then we had $E_Q[-1_{\tilde{F}} B(Z(T) \star X(T))] > 0$, thus a contradiction to the separating inequality (4.8). Hence we conclude $Z^C_0(T) \geq 0$.

Remembering $X_0(T) = 1$ we can renormalize $Z^C(T)$ in order to get $E_Q[Z^C_0(T)] = 1$ and define a Q-dominated probability measure Q^C by $\frac{dQ^C}{dQ} = Z^C_0(T)$.

In summary, for every $C \in \mathcal{C}^+ \cap L^\infty$ there exist an $Z^C(T) \in -K'(T) \cap L^1_{XQ}$ and a P-dominated probability measure Q^C with $\frac{dQ}{dP} = Z^C_0(T)$.

Define

$$\mathcal{C}^+_0 := \{C \in \mathcal{C}^+ : \forall i \geq 1 : C_i = 0\}$$

and deonte $\mathcal{M}$ the set of P-dominated probability measures Q^C, $C \in \mathcal{C}^+_0 \cap L^\infty$.

Then for every $F \in \mathcal{F}(T)$ with $Q(F) > 0$ there exists a $C \in \mathcal{C}^+_0 \cap L^\infty$ with $Q^C(F) > 0$. In fact, the claim $C := 1_F(1, 0, \dots, 0)$ is an element of $\mathcal{C}^+ \cap L^\infty$ satisfying

$$0 < E_Q[C(Z^C(T) \star X(T))] = E_Q[1_F Z^C_0(T)] = Q^C(F) \ .$$

Thus $(\Omega, \mathcal{F}(T), Q)$ and $\mathcal{M}$ satisfy the assumptions of the Halmos-Savage Theorem. According to this theorem there is a countable subfamily $\mathcal{N} \subset \mathcal{M}$ that is equivalent to P. Thus, there exist a sequence $(C^n)_{n\in\mathbb{N}}$ in $\mathcal{C}_0^+ \cap L^\infty$ and a sequence $(\lambda^n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $\lambda_n > 0$ for every $n \in \mathbb{N}$ such that $\sum_n \lambda_n = 1$ and $Q = \sum_n \lambda^n Q_n^C$ is equivalent with P, i.e.,

$$\sum_n \lambda_n Z_0^{C_n} > 0 \; . \tag{4.9}$$

Since for every $n \in \mathbb{N}$ we have $Z^{C_n} \in L^1_{XQ}$ there exists a sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that

$$\forall n \in \mathbb{N}: \quad 0 < ||Z^{C_n}||_1 \leq c_n$$

where $||.||_1$ denotes the norm in L^1_{XQ}.

Define then

$$\tilde{Z}(T) = \sum_n \frac{\lambda^n}{c_n} Z^n(T)$$

in terms of convergence in the Banach space L^1_{XQ}. Then we have

$$0 < ||\tilde{Z}(T)||_1 \leq 1 \; .$$

and $\tilde{Z}(T) \in \mathcal{F}(T)$. In addition, since for every $\omega \in \Omega$ the cone $-K'(\omega, T)$ is closed, it follows by almost sure convergence of a subsequence of $(\sum_{k=1}^{n} \frac{\lambda^k}{c_k} Z^k(T))_{n\in\mathbb{N}}$ that $\tilde{Z}(T) \in -K'(T)$. Moreover we have $0 < \tilde{Z}_0(T) \in L^1(Q)$, because of (4.9). Thus, if we define

$$Z(T) := \frac{\tilde{Z}(T)}{E_Q[\tilde{Z}_0(T)]} \; , \quad \frac{d\tilde{Q}}{dQ} := Z_0(T)$$

we get a P-equivalent probability measure $\tilde{Q}$. Moreover from the separating inequality (4.8) we have

$$\sup_{B\in\mathcal{C}^0\cap L^\infty} E_Q[B(Z(T) \star X(T))] = 0 \; . \tag{4.10}$$

Like in the proof of Theorem 4.15 we define a Q-martingale $Z \star X$ and the following assertions can be proved by copying the corresponding parts of the proof of Theorem 4.15.

The measure $\tilde{Q}$, defined by $\frac{d\tilde{Q}}{dQ} = Z_0(T)$, is equivalent to $Q \sim P$ on $\mathcal{F}(T)$ and since for almost every ω we have

$$\forall t \in \mathcal{T} : Z(\omega, t) \in -K'(\omega, t) \; ,$$

the process $\tilde{R} := \frac{1}{Z_0} Z$ is an element of $\mathcal{R}$ satisfying

$$R(T)\frac{d\tilde{Q}}{dQ} = Z(T) \in L^1_{XQ} .$$

The process $Z_0(T)(\tilde{R} \star X)$ is a Q-martingale and thus $\tilde{R} \star X$ is a $\tilde{Q}$-martingale. So finally we have found a $(\tilde{Q}, \tilde{R}) \in \mathcal{P}$.

$\diamondsuit$

Theorem 4.19 *There is no L^∞-bounded free lunch in $\mathcal{A}$, if and only if $\mathcal{P} \neq \emptyset$.*

Proof. Suppose there is no L^∞-bounded free lunch in $\mathcal{A}$. Then according to Theorem 4.18 we have $\mathcal{P} \neq \emptyset$. For the other implication see Theorem 4.17.

$\diamondsuit$

4.5 Non-Degeneracy Assumption

As we have already stated in Remarks 4.12, the condition of no arbitrage in $\mathcal{A}$ is equivalent to the condition of no free lunch in $\mathcal{B}$, if and only if the set $\mathcal{C}^h$ is closed in the topology of (componentwise) convergence in probability P. We show that the latter condition holds, if the price process X satisfies a certain non-degeneracy condition. For frictionless complete discrete time markets with finite Ω, this assumption on the price process will reduce to the no-arbitrage condition. Hence this non-degeneracy condition is not very restrictive. In fact we will show that it is satisfied for standard market models such as the Cox-Ross-Rubinstein or a discretized Black-Scholes model. Unfortunately, keeping the non-degeneracy condition as weak as possible requires an elaborate formulation of this condition that may seem a little obscure at first glance. But we will state a Lemma that helps to verify the non-degeneracy condition for standard market models.

Let us first introduce some usefull notation. For every $\omega \in \Omega$ and every $s \in \mathcal{T}$, like Koehl et al. (1999), we define the set

$$\ell(\omega, s) := \{\omega' \in \Omega : \forall Y = (Y(t))_{t=0,\ldots,T}, Y \text{ is } \mathbb{F}\text{-adapted} : Y^s(\omega') = Y^s(\omega)\} .$$

There the process $Y^s = (Y^s(t))_{t=0,\ldots,T}$ denotes the process Y stopped at time s, defined as

$$Y^s(\omega, t) = Y(\omega, \min\{s, t\}) \quad , \; (\omega, t) \in \Omega \times \mathcal{T} .$$

We denote $\mathbb{F}^X = (\mathcal{F}^X(t))_{t=0,\dots,N}$ the filtration generated by the process X. Very often throughout this section , we assume $\mathbb{F} = \mathbb{F}^X$, i.e.,

$$\forall t \in \mathcal{T} : \quad \mathcal{F}^X(t) = \mathcal{F}(t) \ . \tag{4.11}$$

Condition (4.11) poses actually no additional restrictions on trading, if at time $t \in \mathcal{T}$ traders' informations about future security prices are completely "mapped" in the conditional distribution $P^X(.|\mathcal{F}^X(t))$.

We also need an additional assumption on the differential cones. But this assumption will be seen to hold for the stock and currency markets featured in Sections 2.2 to 2.6.

Assumption 4.20 *For every $t \in \mathcal{T}$ the (according to Remark 2.3) $\mathcal{F}(t)$-measurable multifunction $K(.,t)$ is polyhedral the sense of Lemma B.8. Thus for every $t \in \mathcal{T}$ the number of extreme points and extreme directions of $K(\omega,t)$ is uniformly bounded in $\omega \in \Omega$. For definitions see Rockafellar(1970, Section 19).*

Observe that by Corollary 19.2.2 and Theorem 19.1 in Rockafellar (1970) this assumption is equivalent to the assumption that $(\omega,t) \in \Omega \times \mathcal{T}$ the dual cone multifunction $K'(.,t)$ is polyhedral and for every $t \in \mathcal{T}$ the number of extreme points and extreme directions of $K'(\omega,t)$ is uniformly bounded in $\omega \in \Omega$. (Note that in Rockafellar (1970) our dual cones are called polar cones.).

$\diamondsuit$

Remark 4.21 Assumption 4.20 is satisfied for the stock and currency markets featured in Sections 2.2 to 2.6. One only needs to look up the definitions of the corresponding $K_0'(\omega,t)$, $-K'(\omega,t)$ and/or $\tilde{K}_0'(\omega,t)$, $-\tilde{K}'(\omega,t)$ in order to verify this fact.

The following non-degeneracy assumption on the price process is a generalization of a no arbitrage condition known for finite frictionless complete markets in discrete time (c.f. Elliott, Kopp, 1999, Proposition 3.3.4).

Assumption 4.22 *There exist a set $F \in \mathcal{F}$ with $P(F) = 1$ and a $\mathbb{F}^X$-adapted process $R \in \mathcal{R}$, such that for every $G \in \mathcal{F}$ with $G \subset F$, $P(G) = 1$ and every $t \in \{0,1,\dots,T-1\}$ there exist random vectors*

$$X^p(t),\ R^p(t) : (\Omega, \mathcal{F}^X(t)) \to (\mathbb{R}^{d+1}, \mathbb{B}^{d+1}) \quad , \ p = 0,\dots,d$$

such that the following conditions are satisfied for every $\omega \in G$*:*

$$X^p(\omega,t) \in X(\ell^X(\omega,t) \cap G, t+1)$$

$$R^p(\omega,t) \in R(\ell^X(\omega,t) \cap G, t+1)$$

$$span(R^p(\omega,t) \star X^p(\omega,t) : p = 0,\dots,d) = \mathbb{R}^{d+1}$$

$$R(\omega,t) \star X(\omega,t) \in strconv(R^p(t) \star X^p(\omega,t) : p = 0,\dots,d) \ .$$

There, $strconv(R^p(t) \star X^p(\omega,t) \ : \ p = 0,\dots,d)$ *denotes the set of strict convex combinations*

$$\sum_{p=0}^{d} \lambda^p R^p(t) \star X^p(\omega,t) \ : \ \forall p \in \{0,\dots,p\} : \ \lambda^p > 0 \ .$$

$\diamondsuit$

In order to elucidate Assumption 4.22 let us look for an interpretation. For given $t \in \{0,1,\dots,T-1\}$ the random vectors $X^p(t)$ are nothing else, but $\mathcal{F}^X(t)$- measurable selections from the multifunction $G \ni \omega \mapsto X(\ell^X(\omega,t) \cap G, t+1) \subset \mathbb{R}^{d+1}$. The same is true for the random vectors $R^p(t)$. Assumption 4.22 then postulates that one may select $X^p(t)$ and $R^p(t)$, $p = 0,\dots,d$, in such a way that $R^p(t) \star X^p(t)$, $p = 0,\dots,d$, are linearly independent and $R(t) \star X(t)$ is a strict convex combination of them. This means, that starting from $R(t) \star X(t)$ at a certain time point t the modified price may move in one of $d+1$ directions to some $R^p(t) \star X^p(t)$. The span of these directions has dimension d. The dimension is not $d+1$, because X_0 does not change.

Now let us show that Assumption 4.22 is weaker in content than the non-degeneracy conditions used by Pham, Touzi (1999) and Pham (2000) although their results are not as strong as ours are going to be. For a stock market with proportional transaction cost factors, Pham, Touzi (1999, Lemma 3.2) prove that there is no local arbitrage, if and only if there is no local free lunch (c.f. Section 4.3). They require that the conditional covariance matrix $Var_P(X(t)|\mathcal{F}(t-1))$ exists and is invertible. Moreover their result is only locally valid. That means their result actually applies only to the two time points market $X(t-1), X(t)$. The reason for this is that "no local free lunch" is not equivalent to "no free lunch" in general. According to Touzi (2000, Remark 3.2), if $Var_P(X(t)|\mathcal{F}(t-1))$ exists and is invertible, then this yields the implication

(4.12) $\forall V \in (L^0(\mathcal{F}(t-1)))^{d+1} : \Big(V \Delta X(t) = 0 \ \Rightarrow \ \forall i \in \{1,\dots,d\} : \ V_i = 0 \Big) \ .$

By the way, note that although the title of Touzi (2000) contains the words "cone contraints" these constraints refer to constraints such as "no shortselling" and are not comparable with our differential cone constraints.

Let us now turn to Condition (4.12) that has a straightforward interpretation. It states that whenever one holds a portfolio h at time $t-1$ with non-zero positions in at least one security $i \in \{1, \dots, d\}$, then the spot value of this portfolio will change from time $t-1$ to time t with positive probability. Moreover if transaction costs are "small", then the absence of arbitrage implies that there is clearly a positive probability for an increase as well as for a decrease of the portfolio's value. Hence, the only way to stay on the safe side is to hold all the capital in the riskless asset 0. Obviously condition (4.12) agrees with what everybody experiences when investing in securities. Let us state this formally as

$$\forall V \in (L^0(\mathcal{F}(t-1)))^{d+1},\ \sum_{i=1}^{d} |V_i| \neq 0 :$$

$$P(V\Delta X(t) > 0 \mid \mathcal{F}(t-1))\ P(V\Delta X(t) < 0 \mid \mathcal{F}(t-1)) > 0\ . \tag{4.13}$$

The following argumentation is to show that in the absense of arbitrage, condition (4.13) is stronger "in content" than Assumption 4.22. Only because of measurable selection questions, we are not able to derive Assumption 4.22 from condition (4.13) in due form. But the argumetation will reveal, that this is only a matter of technique, that will vanish when considering concrete models.

Assume now, that there is no (local) arbitrage opportunity and condition (4.13) holds for two fixed time points $t-1, t$. Consider the class $\mathcal{V}$ of portfolios $V \in (L^0(\mathcal{F}(t-1)))^{d+1}$ such that for almost every $\omega \in \Omega$ we have

$$\forall i \in \{1, \dots\} :\ V_i(\omega) \in \{-1, 0, +1\}\ .$$

Then for every $V \in \mathcal{V}$ we have

$$V\Delta X(t) = \sum_{i=1}^{d} sign(V_i)\Delta X_i(t) = \sum_{i=1}^{d} 1_{\{V_i>0\}}\Delta X_i(t) - \sum_{i=1}^{d} 1_{\{V_i<0\}}\Delta X_i(t)\ .$$

According to condition (4.13) then, every $V \in \mathcal{V}$ satisfies

$$P\left(\sum_{i=1}^{d} 1_{\{V_i>0\}}\Delta X_i(t) > \sum_{i=1}^{d} 1_{\{V_i<0\}}\Delta X_i(t)\ \middle|\ \mathcal{F}(t-1)\right) > 0$$

as well as

$$P\left(\sum_{i=1}^{d} 1_{\{V_i>0\}}\Delta X_i(t) < \sum_{i=1}^{d} 1_{\{V_i<0\}}\Delta X_i(t) \;\Big|\; \mathcal{F}(t-1)\right) > 0 \ .$$

Thus we conclude that for every $B = \{0\}\times B_1\times\ldots\times B_d$ with $B_i \in \{(-\infty,0),(0,\infty)\}$, $i\in\{1,\ldots,d\}$, we have

$$P(\Delta X(t)\in B|\mathcal{F}(t-1)) > 0 \ .$$

From there it is clear that for almost every $\omega\in\Omega$ there exist $X^p(\omega,t)$, $p=0,\ldots,d$, such that

$$X^p(\omega,t) \in X(\ell^X(\omega,t),t+1)$$

$$span(X^p(\omega,t) : p=0,\ldots,d) = \mathrm{I\!R}^{d+1}$$

$$X(\omega,t)\in strconv(X^p(\omega,t) : p=0,\ldots,d) \ .$$

Defining then $R=(R(t))_{t\in\mathcal{T}}$ by $R(t):=1$, $t\in\mathcal{T}$, we nearly have reached Assumption 4.22 except for some technical matters. In order to see this define

$$\mathcal{B} := \{B = \{0\}\times B_1\times\ldots\times B_d : \forall i\in\{1,\ldots,d\} : B_i\in\{(-\infty,0),(0,\infty)\}\} \ .$$

Every $B\in\mathcal{B}$ is convex. Moreover, for every $B\in\mathcal{B}$ and for every $\omega\in\Omega$ there is a closed convex set $\tilde{B}(\omega)\subset B$ such that for almost every $\omega\in\Omega$ we have

$$P(\Delta X(t)\in\tilde{B}(\omega)|\mathcal{F}(t-1))(\omega) > 0 \ .$$

However, we do not know, wether we can choose $\tilde{B}(\omega)$, $\omega\in\Omega$ in such a way that the multifunction $\omega\mapsto B(\omega)$ is $\mathcal{F}^X(t-1)$-measurable. If we knew this, then we would be able to derive Assumption 4.22 in due form. This is stated by the following Lemma.

Lemma 4.23 *Suppose that for every $t\in\{0,\ldots,T-1\}$ there are $\mathcal{F}^X(t)$-measurable closed- and convex-valued multifunctions $B^p(t)$, $p=0,\ldots,d$, from Ω to $\mathrm{I\!R}^{d+1}$ such that the following conditions hold for almost every $\omega\in\Omega$:*

$$(4.14)\qquad \forall p\in\{0,\ldots,d\} : P(X(t+1)\in B^p(\omega,t)|\mathcal{F}^X(t))(\omega) > 0$$

$$\forall v=(v^0,\ldots,v^d)\in B^0(\omega,t)\times\ldots\times B^d(\omega,t) :\; X(\omega,t)\in strconv(v^p : p=0,\ldots,d) \ ,$$

$$\forall v=(v^0,\ldots,v^d)\in B^0(\omega,t)\times\ldots\times B^d(\omega,t) :\; span(v^p : p=0,\ldots,d) = \mathrm{I\!R}^{d+1} \ .$$

Then Assumption 4.22 holds.

Note, that $X(\tilde{\omega},t+1)\in B^p(\omega,t)$ is possible for $\tilde{\omega}\neq\omega$ and therefore (4.14) makes sense.

Proof. For every $t \in \{0, \ldots, T-1\}$ there is a set $F(t) \in \mathcal{F}(t)$ with $P(F) = 1$ such that the almost sure conditions hold for every $\omega \in F(t)$. Define $F = \bigcap_{t \in \{0,\ldots,T-1\}} F(t)$. Suppose $G \subset F$ with $P(G) = 1$. Then for $\omega \in G$ the conditions hold for every $t \in \{0, \ldots, T-1\}$. Since the multifunctions

$$\omega \mapsto B^p(\omega, t) \quad , \; p = 0, \ldots, d,$$

are $\mathcal{F}^X(t)$-measurable closed- and convexvalued, there exist $\mathcal{F}^X(t)$-measurable selections $X^p(t)$, $p = 0, \ldots, d$, such that

$$\forall \omega \in G : X^p(\omega, t) \in B^p(\omega, t) \; .$$

Choosing $R = (R(t))_{t \in \mathcal{T}}$ as $R(t) = 1$ and

$$\forall t \in \mathcal{T} : \; R_i^p(t) = 1 \quad , \; p = 0, \ldots, d, \; i = 0, \ldots, d \; .$$

The remainder is obvious.

◊

In order to illustrate Assumption 4.22 further we are going to verify it explicitely for two established standard market models. Our first example is very simple, but it has a little surprise in store.

Exampel 4.24 *(CRR-Model with transaction costs factors)* Consider a stock market as pictured in Section 2.3 with only two assets, i.e., $d = 1$. Let $T \in \mathbb{N}$ and the price processes X_0, X_1 starting with

$$X_0(0) = 1, \quad X_1(0) = x > 0$$

and then evolving according to

$$X_0(t) = 1 \quad , \; t \in \{1, \ldots, T\}$$

$$X_1(t) = X_1(t-1)\xi(t) \quad , \; t \in \{1, \ldots, T\}$$

There let $\xi(t), t \in \{1, \ldots, T\}$ be mutually independent random variables on $(\Omega, \mathcal{F}, P)$ with distribution

$$P(\xi(t) = u) = p = 1 - P(\xi(t) = d) \quad , \; 0 < d < u < \infty \; .$$

Let $\mathcal{F}(0) = \{\Omega, \emptyset\}$ and $\mathcal{F}(t) = \sigma(\xi(s) : s \leq t)$ for every $t \in \{t, \ldots, T\}$. Define $\mathbb{F} := (\mathcal{F}(t))_{t\in\{0,\ldots,T\}}$. Then the two-dimensional price process $X = (X_0, X_1)$ is $\mathbb{F}$-adapted. Let us look for necessary and sufficient conditions on transaction costs and price processes for Assumption 4.22 to be satisfied in this market. First of all it seems sensible to define for every t

$$X^0(t) = X(t)d$$

$$X^1(t) = X(t)u \ .$$

Let us define $K_0'(\omega, t)$ according to Section 2.3 and $\mathcal{R}$ according to **Definitions and Assumptions** 2.1. Then we have $R \in \mathcal{R}$ if and only if

$$\forall(\omega, t) \in \Omega \times \mathcal{T} : R_0(\omega, t) = 1 \ , \ 1 - \mu_1 \leq R_1(\omega, t) \leq 1 + \lambda_1 \ .$$

It is clear now that in the case $d < 1 < u$ we only need to choose $R_i^p(t) = 1$ for $p = 0, 1, \ i = 0, 1, \ t \in \mathcal{T}$, in order to satisfy Assumption 4.22. But if we have transaction costs, we do not really need to have $d < 1 < u$ for Assumption 4.22 to hold. In order to see this, observe first that

$$R^0(t) \star X^0(t) = (1, R_1^0(t)X_1(t)d)$$

$$R^1(t) \star X^0(t) = (1, R_1^1(t)X_1(t)u) \ .$$

Thus the condition

$$span\{R^0(t) \star X^0(t), R^1(t) \star X^1(t)\} = \mathbb{R}^2$$

is equivalent with

(4.15) $$R_1^0(t)X_1(t)d \neq R_1^1(t)X_1(t)u \ .$$

Moreover, the condition

$$R_1(t) \star X_1(t) \in strconv(R_1^0(t)X_1(t)d, R_1^1(t)X_1(t)u) \ ,$$

is equivalent with the existence of a $0 < \varepsilon < 1$ such that

(4.16) $$R_1(t) \star X_1(t) = \varepsilon R_1^0(t) \star X_1(t)d + (1 - \varepsilon)R_1^1(t) \star X_1(t)u \ .$$

From (4.15) and (4.16) we see that the inequalities

(4.17) $$\forall t \in \mathcal{T} : \ (1 + \lambda)X_1(t) > (1 - \mu)X_1(t)d$$

$$(4.18) \qquad \forall t \in \mathcal{T}: \; (1-\mu)X_1(t) < (1+\lambda)X_1(t)u \; .$$

are sufficient for Assumption 4.22 to hold for any choice $R \in \mathcal{R}$. Surprisingly, conditions (4.17) and (4.18) are also necessary for the absence of arbitrage. Indeed if one of these conditions is violated, then it is easy to construct an arbitrage strategy. For $\lambda = \mu = 0$ conditions (4.17) and (4.18) correspond exactly to the no-arbitrage condition $d < 1 < u$ known for the frictionless CRR-Model.

$\diamondsuit$

Exampel 4.25 *(Black-Scholes-Model with transaction costs, observed at equidistant time points)* Consider a stock market with two assets, i.e., $d = 1$. Let $T \in \mathbb{N}$ and the price processes X_0, X_1 starting with

$$X_0(0) = 1, \quad X_1(0) = x > 0$$

and evolving aacording to

$$X_0(t) = 1 \quad , \; t \in \{1, \ldots, T\}$$

$$X_1(t) = X_1(t-1)\xi(t) \quad , \; t \in \{1, \ldots, T\} \; .$$

There $\xi(t), t \in \{1, \ldots, T\}$ are identically distributed, mutually independent random variables on $(\Omega, \mathcal{F}, P)$ with logarhythmic normal distribution

$$ln\xi(t) \sim N(\mu - \frac{\sigma^2}{2}, \sigma) \; .$$

Let $\mathcal{F}(0) = \{\Omega, \emptyset\}$ and $\mathcal{F}(t) = \sigma(\xi(s) : s \leq t)$ for every $t \in \{1, \ldots, T\}$. Define $\mathbb{F} := (\mathcal{F}(t))_{t \in \{0, \ldots, T\}}$. Then the two-dimensional price process $X = (X_0, X_1)$ is $\mathbb{F}$-adapted. Choose an arbitrary $G \in \mathcal{F}$ with $P(G) = 1$. Since the T-dimensional random vector $\xi = (\xi(1), \ldots, \xi(T))$ has a T-dimensional log-normal distribution, there obviously exist some

$$0 < d_t < u_t < \infty \quad , \; t = 1, \ldots, T \; ,$$

such that the following conditions i) and ii) are satisfied:

i) For every $y = (y_1, \ldots, y_T) \in \{d_1, u_1\} \times \ldots \times \{d_T, u_T\}$ we have

$$y \in \xi^{-1}(G) \; .$$

ii) For every $t \in \{1, \dots, T\}$ we have

$$(1+\lambda)X_1(t) > (1-\mu)X_1(t)d_t$$

$$(1-\mu)X_1(t) < (1+\lambda)X_1(t)u_t$$

From the previous Example 4.24 it is clear then, that Assumption 4.22 is satisfied. $\diamondsuit$

The above discussion and illustration has shown that Assumption 4.22 is simply a quite natural non-degeneracy condition for the price process X in as much as it demands a certain minimum standard for modelling the uncertainty of asset price movements. Loosely spoken, the more "uncertain" prices evolve the more "likely" to hold is Assumption 4.22.

Our aim now is to proof, that under Assumption 4.22 the set $\mathcal{C}^0$ is L^0-closed in the topology of componentwise convergence in probability. This is done in Theorem 4.33. By Theorem 4.32 we are actually going to show that $\mathcal{C}^0$ is even closed with respect to almost sure convergence. In order to do this we first need to prove a lemma of convex analysis.

Lemma 4.26 *Suppose $B \subset \mathbb{R}^n$ is a bounded set and $K \subset \mathbb{R}^n$ is a convex cone containing 0 and with full dimension (this implies $\mathrm{int}K \neq \emptyset$). Then there exists $x \in \mathbb{R}^n$ such that $B \subset x + K$.*

Proof. As for given $x \in \mathbb{R}^n$ we have $B \subset x + K$, if and only if

$$x \in \bigcap_{b \in B} (b - K) \ ,$$

we only need to show

$$\bigcap_{b \in B} (b - K) \neq \emptyset \ .$$

Because B is bounded, we can find a cube Q with $B \subset Q$. Thus we actually have to prove

$$\bigcap_{q \in Q} (q - K) \neq \emptyset \ .$$

It is easy to verify that for $q_1, q_2 \in Q$ and $\lambda \in [0,1]$ we have

$$(q_1 - K) \cap (q_2 - K) \subset \lambda(q_1 - K) + (1-\lambda)(q_2 - K) = \lambda q_1 + (1-\lambda)q_2 - K.$$

In fact, $x = q_1 - k_1 = q_2 - k_2 \in (q_1 - K) \cap (q_2 - K)$ implies

$$x = \lambda(q_1 - k_1) + (1-\lambda)(q_2 - k_2) \in \lambda(q_1 - K) + (1-\lambda)(q_2 - K) \ .$$

Given the 2^n corners $\bar{q}_i$, $i = 1, \ldots, 2^n$, of the cube Q, we have the representation $Q = conv(\{\bar{q}_i : i = 1, \ldots, 2^n\})$ and thus

$$\bigcap_{i=1,\ldots,2^n} (\bar{q}_i - K) \subset \bigcap_{q \in Q} (q - K) \ .$$

Hence it suffices to show

$$\bigcap_{i=1,\ldots,2^n} (\bar{q}_i - K) \neq \emptyset \ . \tag{4.19}$$

Now observe that (4.19) holds if and only if there exist $\lambda_i \geq 0$ and $k_i \in K$, $i = 1, \ldots, 2^n$ such that

$$\forall i : \bar{q}_1 - \lambda_1 k_1 = \bar{q}_i - \lambda_i k_i \ .$$

This is equivalent to

$$\forall i : \lambda_1 k_1 - \lambda_i k_i = \bar{q}_1 - \bar{q}_i \ .$$

Hence (4.19) is true, if and only if there exist $\lambda_1 \geq 0$ and $k_1 \in K$ such that

$$\forall i : \bar{q}_1 - \bar{q}_i \in \lambda_1 k_1 - K \ .$$

But this is always true, because K is full dimensional and hence for every $k \in \text{int}K \neq \emptyset$ one has

$$\mathbb{R}^n = \{\lambda k : \lambda \geq 0\} - K \ .$$

$\diamondsuit$

Theorem 4.27 *Assume $\mathbb{F} = \mathbb{F}^X$. Suppose Assumption 4.20 is satisfied, Assumption 4.22 holds with $F \in \mathcal{F}$, $P(F) = 1$ and let $A \in \mathcal{F}$ such that $A \subset F$, $P(A) = 1$. Fix a $t \in \{0, \ldots, T-1\}$ and assume $V(t) \in (L^0(\mathcal{F}(t)))^{d+1}$ and $V(T) \in (L^0(\mathcal{F}(T)))^{d+1}$.*

Denote $\mathcal{H}^V(A)$ the subclass of admissible portfolio strategies $(h, H) \in \mathcal{A}$ satisfying

$$\forall \omega \in A : H(\omega, T) = V(\omega, T) \ ,$$

$$\forall \omega \in A : V(\omega, t) - H(\omega, t) \in K(\omega, t) \ ,$$

$$\forall (\omega, s) \in A \times \{t+1, \ldots, T\} : -\Delta H(\omega, s) \in K(\omega, s) \ . \tag{4.20}$$

Then the set

$$\mathcal{H}^V(A,t) := \{H(t) : (h,H) \in \mathcal{H}^V(A)\}$$

is $\mathcal{F}(t)$-measurable bounded on A, in the following sense: There exists a closed-valued, $\mathcal{F}(t)$-measurable multifunction $\hat{\Lambda}$ with bounded values $\hat{\Lambda}(\omega) \subset \mathbb{R}$ for $\omega \in A$ and satisfying

$$\forall \omega \in A : \{H(\omega,t) : H(t) \in \mathcal{H}^V(A,t)\} \subset \hat{\Lambda}(\omega) \ .$$

Moreover $\hat{\Lambda}$ is polyhedral in the sense of Lemma B.8 and hence the closed-valued multifunction $extr\hat{\Lambda}$ is $\mathcal{F}(t)$-measurable .

Proof. Since this proof is geometric in principle, we could not always achieve a pleasantly compact notation. Therefore we want to apologize in advance for the sometimes proliferating notation.

As a starting point for the proof, observe that (4.31) implies

$$\forall \omega \in A : \mathcal{H}^V(\omega,t) = (V(\omega,t) - K(\omega,t)) \cap \left(V(\omega,T) + \sum_{s=t+1}^{T} K(\omega,s) \right) \ .$$

If for $\omega \in A$ we define

$$\Lambda(\omega) := \{h : \forall \omega' \in \ell(\omega,t) : h \in (V(\omega',t) - K(\omega',t)) \cap \left(V(\omega',T) + \sum_{s=t+1}^{T} K(\omega',s) \right) \}$$

then since every $H(t) \in \mathcal{H}^V(A,t)$ is constant on the sets $\ell(\omega,t)$, we obtain

$$\forall \omega \in A : \{H(\omega,t) : H(t) \in \mathcal{H}^V(A,t)\} \subset \Lambda(\omega) \ .$$

Starting from Λ, we are going to construct a closed-valued multifunction $\hat{\Lambda}$ from Ω to $\mathbb{R}^{d+1}$ such that for arbitrary $\bar{\omega} \in A$ we have

$$\Lambda(\bar{\omega}) \subset \hat{\Lambda}(\bar{\omega}) \ ,$$

and $\hat{\Lambda}(\bar{\omega})$ is bounded. Moreover $\hat{\Lambda}$ will be $\mathcal{F}(k)$-measurable, i.e., for every closed set $B \subset \mathbb{R}^{d+1}$ we have

$$\hat{\Lambda}^{-1}(B) = \{\omega \in A : \hat{\Lambda}(\omega) \cap B \neq \emptyset\} \in \mathcal{F}(t) \ . \tag{4.21}$$

The assertions on $extr\hat{\Lambda}(\omega)$ are going to be revealed occasionally.

Construction of $\hat{\Lambda}$: Let $R \in \mathcal{R}$ be the process as supposed in Assumption 4.22. Then the following statements are true:

- For every $p \in \{0, \ldots, d\}$ there exist

$$X^p(t) : (\Omega, \mathcal{F}(t)) \to (\mathbb{R}^{d+1}, \mathbb{B}^{d+1}) \ ,$$

$$R^p(t) : (\Omega, \mathcal{F}(t)) \to (\mathbb{R}^{d+1}, \mathbb{B}^{d+1})$$

  such that

$$\forall \omega \in A : X^p(\omega, t) \in X(\ell^X(\omega, t) \cap A, t+1) \ ,$$

$$\forall \omega \in A : R^p(\omega, t) \in R(\ell^X(\omega, t) \cap A, t+1) \ .$$

- For all $\bar{\omega} \in A \subset F$ we have

$$span\,(R^p(\bar{\omega}, t) \star X^p(\bar{\omega}, t) : p = 0, \ldots, d) = \mathbb{R}^{d+1}$$

$$R(\bar{\omega}, t) \star X(\omega, t) \in strconv\,(R^p(\bar{\omega}, t) \star X^p(\bar{\omega}, t) : p = 0, \ldots, d) \ .$$

Henceforth, we are going to use the notation

$$(R \star X)(\omega, t) = R(\omega, t) \star X(\omega, t) \ .$$

Now, for every $p^{t+1} = 0, \ldots, d$ and every choice of $X^{p^{t+1}}$, $R^{p^{t+1}}$ we are able to choose a function $(p^{t+1}, \bar{\omega}) \mapsto \omega(p^{t+1}, \bar{\omega})$ form Ω to Ω such that for every $\bar{\omega} \in A$ we have

$$\omega(p^{t+1}, \bar{\omega}) \in \ell^X(\bar{\omega}, t) \cap A \ .$$

Then for these functions we have

$$X(\omega(p^{t+1}, \bar{\omega}), t+1) = X^{p^{t+1}}(\bar{\omega}, t) \ ,$$

$$R(\omega(p^{t+1}, \bar{\omega}), t+1) = R^{p^{t+1}}(\bar{\omega}, t)$$

and consequently

$$span((R \star X)(\omega(p^{t+1}, \bar{\omega}), t+1) : p^{t+1} = 0, \ldots, d) = \mathbb{R}^{d+1} \tag{4.22}$$

$$(R \star X)(\bar{\omega}, t) \in strconv\left((R \star X)(\omega(p^{t+1}, \bar{\omega}), t+1) : p^{t+1} = 0, \ldots, d\right) \ . \tag{4.23}$$

If we pursue this procedure inductively along the time index, then for every

$$\omega(p^{t+1}, \ldots, p^s), \bar{\omega}) \in \ell^X(\omega(p^{t+1}, \ldots, p^{s-1}, \bar{\omega}), s-1) \cap A$$

we obtain

$$\omega(p^{t+1}, \ldots, p^{s+1}, \bar{\omega}) \in \ell(\omega(p^{t+1}, \ldots, p^s, \bar{\omega}), s) \cap A$$

such that

$$(4.24)\quad span((R\star X)(\omega(p^{t+1},\dots,p^{s+1},\bar{\omega}),s+1):p^{s+1}=0,\dots,d)=\mathbb{R}^{d+1}$$

and

$$\begin{aligned}&(R\star X)(\omega(p^{t+1},\dots,p^{s},\bar{\omega}),s)\\ (4.25)\qquad &\in strconv\left((R\star X)(\omega(p^{t+1},\dots,p^{s+1},\bar{\omega}),s+1):p^{s+1}=0,\dots,d\right)\ .\end{aligned}$$

With this choice now, in account of

$$\{\omega(p^{t+1},\dots,p^{s},\bar{\omega}):p^{s}\in\{0,\dots,d\},s=t+1,\dots,T\}\subset\ell(\bar{\omega},k)$$

we define for every $\bar{\omega}\in A$

$$\begin{aligned}\hat{\Lambda}(\bar{\omega}):=\quad&(V(\bar{\omega},t)-K(\bar{\omega},t))\cap\\ \bigcap_{(p^{t+1},\dots,p^{T})\in\{0,\dots,d\}^{T-t-1}}&\left(V(\omega(p^{t+1},\dots,p^{T},\bar{\omega}),T)+\sum_{s=t+1}^{T}K(\omega(p^{t+1},\dots,p^{s},\bar{\omega}),s))\right)\ .\end{aligned}$$

From this construction it is clear that $\Lambda(\bar{\omega})\subset\hat{\Lambda}(\bar{\omega})$ for every $\bar{\omega}\in A$. Moreover in account of Assumption 4.20 we see that for every $\bar{\omega}\in A$ the set $\hat{\Lambda}(\bar{\omega})$ is polyhedral and the number of extreme points and extreme directions of $\hat{\Lambda}(\bar{\omega})$ is uniformly bounded in $\bar{\omega}\in A$. In particular, $\hat{\Lambda}$ is a polyhedral multifunction in the sense of Lemma B.8 (for the measurability conditions see below).

Boundedness of $\hat{\Lambda}$: For this purpose we need a notion of convex analysis:

If K is a nonempty convex **set**, then its **recession cone** is defined as the set (c.f. Rockafellar, 1970, Theorem 8.1)

$$0^{+}K:=\{y\in K:\forall k\in K:y+K\in K\}\ .$$

Obviously, if K is a convex cone containing 0, then it's recession cone coincides with K.

Observe that every

$$V(\omega(p^{t+1}),T)+\sum_{s=t+1}^{T}K(\omega(p^{t+1},\dots,p^{s}),s))$$

is a closed convex set with recession cone $\sum_{s=t+1}^{T}K(\omega(p^{t+1},\dots,p^{s}),s))$ whereas the closed convex set $V(\bar{\omega},k)-K(\bar{\omega},k)$ has the recession cone $-K(\bar{\omega},k)$. According to a corollary and a theorem in Rockafellar (1970, Corollary 8.3.3, Theorem 8.4) a nonempty intersection of closed convex sets is bounded, if and only if the intersection of all their recession cones consists of $\{0\}$. Hence we need only to prove

$$(4.26)\qquad (-K(\bar\omega,t)) \bigcap_{(p^{t+1},\ldots,p^T)\in\{0,\ldots,d\}^{T-t-1}} \sum_{s=t+1}^{T} K(\omega(p^{t+1},\ldots,p^s),s)) = \{0\}.$$

At this point it is usefull to observe, with the equality

$$K(\omega,s) = \{h\in \mathbb{R}^{d+1} : \inf_{r\in K_0'(\omega,s)} h(r\star X(\omega,s)) \geq 0\}$$

in mind, that for every $\bar r\in K_0'(\omega,s)$ we have

$$(4.27)\qquad K(\omega,s)\subset\{h\in\mathbb{R}^{d+1} : h(\bar r\star X(\omega,s))\geq 0\}\ .$$

We claim that the set

$$\bigcap_{(p^{t+1},\ldots,p^T)\in\{0,\ldots,d\}^{T-t-1}} \sum_{s=t+1}^{T} K(\omega(p^{t+1},\ldots,p^s,\bar\omega),s))$$

is a subset of

$$\{h\in\mathbb{R}^{d+1} : \forall p^{d+1}\in\{0,\ldots,d\} : h(R\star X)(\omega(p^{t+1},\bar\omega),t)\geq 0\}$$

In fact for given $\overline{(p^{t+1},\ldots,p^{T-1})}$, because of (4.27), (4.22) and (4.25), we have the inclusion

$$\bigcap_{(p^{t+1},\ldots,p^T):(p^{t+1},\ldots,p^{T-1})=\overline{(p^{t+1},\ldots,p^{T-1})}} K(\omega(p^{t+1},\ldots,p^T,\bar\omega),T))$$

$$\begin{aligned} = \{ \quad & h\in\mathbb{R}^{d+1} : \forall (p^{t+1},\ldots,p^T),(p^{t+1},\ldots,p^{T-1}) = \overline{(p^{t+1},\ldots,p^{T-1})} : \\ & h(R\star X)(\omega(p^{t+1},\ldots,p^T,\bar\omega),T)\geq 0\} \\ \subset \{ \quad & h\in\mathbb{R}^{d+1} : h(R\star X)(\omega\overline{(p^{t+1},\ldots,p^{T-1}}\bar\omega),T-1)\geq 0\}\ . \end{aligned}$$

Analogously, for given $\overline{(p^{t+1},\ldots,p^{T-2})}$ we have

$$\bigcap_{(p^{t+1},\ldots,p^T):(p^{t+1},\ldots,p^{T-2})=\overline{(p^{t+1},\ldots,p^{T-2})}} K(\omega(p^{t+1},\ldots,p^{T-1},\bar\omega),T-1))$$

$$\subset\{h\in\mathbb{R}^{d+1} : h(R\star X)(\omega\overline{(p^{t+1},\ldots,p^{T-2}},\bar\omega),T-2)\geq 0\}$$

and also

$$\begin{aligned}\{ \quad & h \in \mathbb{R}^{d+1} : \forall (p^{t+1}, \ldots, p^{T-1}), (p^{t+1}, \ldots, p^{T-2}) = \overline{(p^{t+1}, \ldots, p^{T-2})} : \\ & h(R \star X)(\omega(p^{t+1}, \ldots, p^{T-1}, \bar{\omega}), T-1) \geq 0\} \\ \subset \quad & \{h \in \mathbb{R}^{d+1} : h(R \star X)(\omega\overline{(p^{t+1}, \ldots, p^{T-2}}, \bar{\omega}), T-2) \geq 0\} \ .\end{aligned}$$

This implies

$$\begin{aligned}& \bigcap_{(p^{t+1},\ldots,p^{T}):(p^{t+1},\ldots,p^{T-2})=\overline{(p^{t+1},\ldots,p^{T-2})}} \quad \sum_{s=T-1}^{T} K(\omega(p^{t+1}, \ldots, p^{s}, \bar{\omega}), s)) \\ & \subset \{h \in \mathbb{R}^{d+1} : h(R \star X)(\omega\overline{(p^{t+1}, \ldots, p^{T-2}}, \bar{\omega}), T-2) \geq 0\} \ .\end{aligned}$$

Repeating this argumentation for $T-3, \ldots, t+1$ one can deduce the desired statement. Because of

$$-K(\bar{\omega}, k) \subset \{h \in \mathbb{R}^{d+1} : h(R \star X)(\bar{\omega}, k) \leq 0\}$$

we conclude that

$$(-K(\bar{\omega}, k)) \cap \bigcap_{(p^{k+1},\ldots,p^{T}) \in \{0,\ldots,d\}^{T-k-1}} \sum_{l=k+1}^{T} K(\omega(p^{k+1}, \ldots, p^{l}, \bar{\omega}), l))$$

is a subset of

$$\begin{aligned}\{0\} \quad = \quad & \{h \in \mathbb{R}^{d+1} : \forall p^{d+1} \in \{0, \ldots, d\} : h(R \star X)(\omega(p^{t+1}, \bar{\omega}), t) \geq 0 \\ & \qquad h(R \star X)(\bar{\omega}, k) \leq 0\} \ .\end{aligned}$$

There the equality to $\{0\}$ is a consequence of (4.22) and (4.23). In fact

$$h(R \star X)(\bar{\omega}, k) \leq 0$$

by (4.23) implies that there exist $\lambda_{p^{t+1}} > 0$, $p^{t+1} = 0, \ldots, d$, such that

$$\sum_{p^{t+1}=0}^{d} \lambda_{p^{t+1}} h(R \star X)(\omega(p^{t+1}, \bar{\omega}), t) \leq 0 \ .$$

Then from

$$\forall p^{d+1} \in \{0, \ldots, d\} : h(R \star X)(\omega(p^{t+1}, \bar{\omega}), t) \geq 0$$

we have

$$\sum_{p^{t+1}=0}^{d} \lambda_{p^{t+1}} h(R \star X)(\omega(p^{t+1}, \bar{\omega}), t) = 0 \ .$$

Consequently, (4.22) implies $h = 0$. This shows (4.26).

$\mathcal{F}(k)$-measurability of $\hat{\Lambda}$: We have to show (4.21). According to it's definition, the multifunction $\hat{\Lambda}$ is the intersection of the multifunction

$$\bar{\omega} \mapsto V(\bar{\omega}, t) - K(\bar{\omega}, t)$$

and the finitely many multifunctions

$$\bar{\omega} \mapsto V(\omega(p^{t+1}, \ldots, p^T, \bar{\omega}), T) + \sum_{s=t+1}^{T} K(\omega(p^{t+1}, \ldots, p^s, \bar{\omega}), s)$$

where $(p^{t+1}, \ldots, p^T) \in \{0, \ldots, d\}^{T-t-1}$. According to Assumption 4.20 each multifunction $K(., s)$ is $\mathcal{F}(s)$-measurable and polyhedral in the sense of Lemma B.8. This means every $K(., s)$ is the solution set of a system of finitely many inequalities with some $\mathcal{F}(s)$-measurable coefficients $Y^i(s)$, $i = 1, \ldots, m$.

In order to prove the $\mathcal{F}(t)$-measurability of $\hat{\Lambda}$ we proceed as follows. We show first that the functions

$$\bar{\omega} \mapsto X(\omega(p^{t+1}, \ldots, p^s, \bar{\omega}), s) \quad , \; s = t+1, \ldots, T$$

are $\mathcal{F}(t)$-measurable. Because of $\mathbb{F}^X = \mathbb{F}$ we know then that the other coefficient functions

$$\bar{\omega} \mapsto Y^i(\omega(p^{t+1}, \ldots, p^s, \bar{\omega}), s) \quad , \; s = t+1, \ldots, T$$

are also $\mathcal{F}(t)$-measurable and the same is true for the functions

$$\bar{\omega} \mapsto V(\omega(p^{t+1}, \ldots, p^T, \bar{\omega}), T) \; .$$

From Theorem 2J in Rockafellar (1976) then we know that the multifunctions $K(\omega(p^{t+1}, \ldots, p^s, .), s)$ are $\mathcal{F}(t)$-measurable. Using Corollary 1K, Proposition 1J and Theorem 1M in Rockafellar (1976) we can finally conclude that $\hat{\Lambda}$ is $\mathcal{F}(t)$-measurable.

Now we still have to show that the functions

$$\bar{\omega} \mapsto X(\omega(p^{k+1}, \ldots, p^l, \bar{\omega}), l)$$

are $\mathcal{F}(k)$-measurable. This is done by induction along the index l.

For $s = t+1$ and $B \in \mathbb{B}^{d+1}$ we have

$$(X(\omega(p^{t+1}, .), s))^{-1}(B) = \{\bar{\omega} \in A : X(\omega(p^{t+1}, \bar{\omega}), t+1) \in B\}$$

$$= \{\bar\omega \in A : X^{p^{t+1}}(\bar\omega, t) \in B\} = (X^{p^{t+1}}(., t))^{-1}(B) \in \mathcal{F}(t) \ .$$

Suppose the assertion is true for $t+1, \ldots, s$ and let $B \in I\!B^{d+1}$. Then for $s+1 \leq T$ and $B \in I\!B^{d+1}$ we have

$$\begin{aligned} &(X(\omega(p^{t+1}, \ldots, p^{s+1}, .), s+1))^{-1}(B) \\ &= \{\bar\omega \in A : X(\omega(p^{t+1}, \ldots, p^{s+1}, \bar\omega), t+1) \in B\} \\ &= \{\bar\omega \in A : X^{p^{s+1}}(\omega(p^{t+1}, \ldots, p^{s}, \bar\omega), s) \in B\} \\ &= \{\bar\omega \in A : \omega(p^{t+1}, \ldots, p^{s}, \bar\omega) \in (X^{p^{s+1}}(s))^{-1}(B)\} \ . \end{aligned}$$

Because of $(X^{p^{s+1}}(s))^{-1}(B) \in \mathcal{F}^X(s)$ there exists a $\mathbf{B} \in (I\!B^{d+1})^s$ such that

$$(X(0), \ldots, X(s))^{-1}(\mathbf{B}) = (X^{p^{s+1}}(s))^{-1}(B) \ .$$

Hence we have

$$\begin{aligned} &(X(\omega(p^{t+1}, \ldots, p^{s+1}, .), s+1))^{-1}(B) \\ &= \{\bar\omega \in A : \omega(p^{t+1}, \ldots, p^{s}, \bar\omega) \in (X^{p^{s+1}}(s))^{-1}(B)\} \\ &= \{\bar\omega \in A : \omega(p^{t+1}, \ldots, p^{s}, \bar\omega) \in (X(0), \ldots, X(s))^{-1}(\mathbf{B})\} \\ &= \{\bar\omega \in A : (X(\omega(p^{t+1}, \ldots, p^{s}, \bar\omega), 0), \ldots, X(\omega(p^{t+1}, \ldots, p^{s}, \bar\omega), s) \in \mathbf{B}\} \\ &= \{\bar\omega \in A : \Big(X(\bar\omega, 0), \ldots, X(\bar\omega, t), \\ &\qquad X(\omega(p^{t+1}, \bar\omega), t+1), \ldots X(\omega(p^{t+1}, \ldots, p^{s}, \bar\omega), s) \Big) \in \mathbf{B}\} \ . \end{aligned}$$

Since, by assumption, the functions

$$\bar\omega \mapsto X(\omega(p^{t+1}, \ldots, p^{t+i}, \bar\omega), t+i)$$

are $\mathcal{F}(t)$-measurable for $t+i \leq s$, we conclude that the set

$$\begin{aligned} &(X(\omega(p^{t+1}, \ldots, p^{s+1}, .), s+1))^{-1}(B) \\ &= \{\bar\omega \in A : \Big(X(\bar\omega, 0), \ldots, X(\bar\omega, t), \\ &\qquad X(\omega(p^{t+1}, \bar\omega), t+1), \ldots X(\omega(p^{t+1}, \ldots, p^{s}, \bar\omega), s) \Big) \in \mathbf{B}\} \end{aligned}$$

is an element of $\mathcal{F}(t)$, q.e.d..

$\diamondsuit$

Corollary 4.28 *Assume $\mathbb{F} = \mathbb{F}^X$. Suppose Assumption 4.20 is satisfied and Assumption 4.22 holds with F, $P(F) = 1$, and let $A \subset F \in \mathcal{F}$ with $P(A) = 1$. For $v \in \mathbb{R}^{d+1}$ and $V(T) \in L^0(\mathcal{F}(T))$ consider the class $\mathcal{H}^V(A)$ of portfolio processes $(h, H) \in \mathcal{A}$ satisfying*

$$h = H(0-) = v, \quad H(T) = V(T)$$

and

$$\forall(\omega, s) \in A \times \mathcal{T} : \quad -\Delta H(\omega, s) \in K(\omega, s) \ . \tag{4.28}$$

Then for every $t \in \{0, \dots, T-1\}$ the set

$$\mathcal{H}^V(A, t) := \{H(t) : (h, H) \in \mathcal{H}^V(A)\}$$

is $\mathcal{F}(t)$-measurable bounded on A, in the sense that there exists a closed-valued, $\mathcal{F}(t)$-measurable, multifunction $\hat{\Lambda}(., t)$ from Ω to $\mathbb{R}^{d+1}$ with bounded values $\hat{\Lambda}(\omega, t)$, $\omega \in A$ satisfying

$$\forall \omega \in A : \{H(\omega, t) : H(k) \in \mathcal{H}^V(A, t)\} \subset \hat{\Lambda}(\omega, k) \ . \tag{4.29}$$

Moreover $\hat{\Lambda}(., t)$ is polyderal in the sense of Lemma B.8. Hence $\sup_{\omega \in A} |extr\hat{\Lambda}(\omega, t)|$ is finite and the multifunction $extr\hat{\Lambda}(k) : \omega \mapsto extr\hat{\Lambda}(\omega, t)$ is also $\mathcal{F}(t)$-measurable.

Proof. The proof is done via induction over the time index. For $k = 0$ the assertion follows by Theorem 4.27.

Suppose the assertion is true for $s \leq t$, then $\mathcal{H}^V(A, t)$ is $\mathcal{F}(t)$-measurable bounded by $\hat{\Lambda}(\omega, t)$, i.e., for each $\omega \in A$ the set $\{H(\omega, t) : H(t) \in \mathcal{H}^V(A, t)\}$ is bounded by $\hat{\Lambda}(\omega, t)$ and $\hat{\Lambda}(., t)$ is a $\mathcal{F}(t)$-measurable, closed-valued, polyhedral multifunction. Moreover $\sup_{\omega \in A} |extr\hat{\Lambda}(\omega, t)|$ is finite and the multifunction $extr\hat{\Lambda}(., t) : \omega \mapsto extr\hat{\Lambda}(\omega, t)$ is also $\mathcal{F}(t)$-measurable.

According to Theorem 1B in Rockafellar (1976) and Lemma B.8 there are finitely many $W^i(., t) \in L^0(\mathcal{F}(t))^{d+1}$, $i = 1, \dots, m$, $m \in \mathbb{N}$ such that

$$\forall \omega \in A : extr\hat{\Lambda}(\omega, t) = \{W^i(\omega, t) : i = 1, \dots, m\} \ .$$

Because for ever $\omega \in A$ the cone $-K(\omega, t+1) \supset \mathbb{R}^{d+1}_+$ has full dimension and contains 0 and since $\hat{\Lambda}(\omega, t)$ is a bounded set, we can apply Lemma 4.26. According to this lemma we have

$$\forall \omega \in A \ \exists V(\omega, t+t) \in \mathbb{R}^{d+1} : \ \hat{\Lambda}(\omega, t) \subset V(\omega, t+1) - K(\omega, t+1) \ .$$

This means the multifunction

$$\omega \mapsto \Theta(\omega) := \{h \in \mathbb{R}^{d+1} : \hat{\Lambda}(\omega,t) \subset h - K(\omega,t+1)\}$$

has non-empty values on A. Observe now that we have

$$\Theta(\omega) = \bigcap_{g \in \hat{\Lambda}(\omega,t)} g + K(\omega,t+1) \; . \tag{4.30}$$

In fact $h \in \Theta(\omega)$ is equivalent to

$$\forall g \in \hat{\Lambda}(\omega,t) : \quad g \in h - K(\omega,t+1)$$

which in turn is equivalent to

$$\forall g \in \hat{\Lambda}(\omega,t) : \quad h \in g + K(\omega,t+1) \; .$$

Thus (4.30) is true. Consequently, because of

$$\hat{\Lambda}(\omega,k) = conv(extr(\hat{\Lambda})(\omega,t)) = conv(W^i(\omega,t), i = 1,\ldots,m)$$

we have

$$\Theta(\omega) = \bigcap_{i=1,\ldots,m} W^i(\omega,t)) + K(\omega,t+1) \; .$$

Because every $W^i(.,t))$, $i = 1,\ldots,m$, is $\mathcal{F}(t) \subset \mathcal{F}(t+1)$-measurable and $K(.,t+1)$ is $\mathcal{F}(t+1)$-measurable, Corollary 1K and Theorem 1M in Rockafellar (1976) tell us that the multifunction Θ is $\mathcal{F}(t+1)$-measurable. Hence according to Corollary 1C in Rockafellar (1976) we can select a $\mathcal{F}(t+1)$-measurable random vector $V(.,t+1)$ such that

$$\forall \omega \in A : \; V(\omega,t+1) \in \Theta(\omega) \; .$$

Denote $\mathcal{H}^{V(t+1),V(T)}(A)$ the subclass of admissible portfolio strategies $(h,H) \in \mathcal{A}$ satisfying

$$\forall \omega \in A : H(\omega,T) = V(\omega,T) \; ,$$

$$\forall \omega \in A : V(\omega,t+1) - H(\omega,t+1) \in K(\omega,t+1) \; ,$$

$$\forall (\omega,s) \in A \times \{t+2,\ldots,T\} : -\Delta H(\omega,s) \in K(\omega,s) \; . \tag{4.31}$$

Then according to Theorem 4.27 the set

$$\mathcal{H}^{V(t+1),V(T)}(A,t+1) := \{H(t+1) : (h,H) \in \mathcal{H}^{V(t+1),V(T)}(A)\}$$

is $\mathcal{F}(t+1)$-measurable bounded on A, in the following sense: There exists a closed-valued, $\mathcal{F}(t+1)$-measurable multifunction $\hat{\Lambda}(.,t+1)$ with bounded values $\hat{\Lambda}(\omega,t+1) \subset \mathbb{R}$ for $\omega \in A$ and satisfying

$$\forall \omega \in A : \{H(\omega,t+1) : H(t+1) \in \mathcal{H}^{V(t+1),V(T)}(A,t+1)\} \subset \hat{\Lambda}(\omega,t+1) \ .$$

Moreover $\hat{\Lambda}(.,t+1)$ is polyhedral in the sense of Lemma B.8 and hence the closed-valued multifunction $extr\hat{\Lambda}(.,t+1)$ is $\mathcal{F}(t+1)$-measurable .

Hence we only need to show that

$$\mathcal{H}^{V}(A,t+1) \subset \mathcal{H}^{V(t+1),V(T)}(A,t+1) \tag{4.32}$$

in order to conclude the desired assertions for $\mathcal{H}^{V}(A,t+1)$.

In order to verify (4.32) suppose $(h,H) \in \mathcal{H}^{V}(A)$. Then we have

$$\forall \omega \in A : \ H(\omega,t) - H(\omega,t+1) \in K(\omega,t+1)$$

but according to the definition of $V(.,t+1)$ we also have

$$\forall \omega \in A : \ V(\omega,k+1) - H(\omega,k) \in K(\omega,t+1) \ .$$

Because every $K(\omega,t)$ is a convex cone, this implies

$$\forall \omega \in A : \ V(\omega,k+1) - H(\omega,t+1) \in K(\omega,t+1) \ .$$

This shows (4.32) and so the proof is done.

$\diamondsuit$

Lemma 4.29 *Assume $\mathbb{F} = \mathbb{F}^X$. Suppose Assumption 4.20 and Assumption 4.22 are satisfied. Let $v \in \mathbb{R}^{d+1}$ and $V(T) = C \in \mathcal{F}(T)$ be a contingent claim. Suppose $(h^n,H^n)_{n\in\mathbb{N}}$ is a sequence of super hedges for C. Then there exists a set A that meets all the assumptions of Corollary* 4.28.

Proof. First, from the definition of super hedges and because we are working in a discrete time frame, it is easy to see that for every single (h^n,H^n) we can choose A^n so as to meet the required assumptions. But then it is clear that $A = \cap_{n\in\mathbb{N}} A_n$ also meets these assumptions.

$\diamondsuit$

Lemma 4.30 *Suppose Assumption 4.20 and Assumption 4.22 are satisfied. Let* $(C^n)_{n\in\mathbb{N}}$ *be a sequence of claims converging to* C *almost surely. Then there exists a claim* $\tilde{C}$ *such that almost surely* $C - \tilde{C} \in K(T)$ *and*

$$\forall n \in \mathbb{N} : C^n - \tilde{C} \in K(T) \; .$$

Proof. Define the multifunction

$$\Lambda(\omega) := \{h \in \mathbb{R}^{d+1} : \forall i \in \{0, \ldots, d\} : \|h_i\| \le \max(\max_{n\in\mathbb{N}} C_i^n, C_i)\} \; .$$

Then Λ is a $\mathcal{F}(T)$-measurable polyhedral multifunction and almost surely bounded. Thus Lemma B.8 applies to Λ. So there exist measurable selections $V^i(T)$, $i = 1, \ldots, m$, from $extr(\Lambda(.))$ such that

$$\omega \mapsto \bigcap_{i=1,\ldots,m} V_i(T) - K(\omega, T)$$

defines a $\mathcal{F}(T)$-measurable multifunction from that we can select a $\mathcal{F}(T)$-measurable random vector $\tilde{C}$ as required (c.f. the proof of Lemma 4.26).

◊

Lemmata 4.29 and 4.30 yield the following

Proposition 4.31 *Assume* $\mathbb{F} = \mathbb{F}^X$*. Let Assumption 4.20 and Assumption 4.22 be satisfied. Suppose* $(C^n)_{n\in\mathbb{N}}$ *is a sequence of claims converging almost surely to* C *and* $(h^n, H^n)_{n\in\mathbb{N}}$ *is a sequence in* $\mathcal{A}$*, such that for every* $n \in \mathbb{N}$ *the portfolio process* (h^n, H^n) *is a super hedge for* C^n*. Then there exist a claim* $\tilde{C} = V(T)$ *and a set* A *that meets all the assumptions of corollary* 4.28.

◊

Theorem 4.32 *Assume* $\mathbb{F} = \mathbb{F}^X$ *and suppose Assumption 4.20 and Assumption 4.22 are satisfied. Then the set* $\mathcal{C}^0$ *is closed with respect to almost sure convergence.*

Proof. Let $(C^n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}^0$ converging almost surely to C. We have to show $C \in \mathcal{C}^0$.

For every $n \in \mathbb{N}$ let $(0, H^n) \in \mathcal{A}$ be a super hedge for C^n. From Lemma 4.30 and Proposition 4.31 we know that we can find and a set A and a claim $V(T) = \tilde{C}$ such that $C - \tilde{C} \in K(T)$ and

$$\forall n \in \mathbb{N} : C^n - \tilde{C} \in K(T)$$

so as to meet all the assumptions of Corollary 4.28. According to this Corollary, for every $\omega \in A$ the sequence

$$y(\omega) = (0, H^n(\omega, 0), \ldots, H^n(\omega, T))_{n\in\mathbb{N}}$$

is bounded. Hence if we denote $\Xi(y)$ the set of limit points of a sequence $y = (y^n)$ with $y^n = (y^n_{it}) := (y^n_{it})_{i\in\{0,\ldots,d\},\ t\in\mathcal{T}} \in \mathbb{R}^{(d+1)\times(T+1)}$, then for all $\omega \in A$ we have

$$\Xi(y(\omega)) \neq \emptyset\ .$$

We want to choose now a (h, H) such that

$$\forall \omega \in A: \quad (h, H(\omega, 0), \ldots, H(\omega, T)) \in \Xi(y^\omega)$$

and H is $\mathbb{F}$-adapted. Then the proof is done.

In order to choose (h, H) as desired, it is convenient to focus on the limit points of every single component. For $(i, t) \in \{0, \ldots, d\} \times \mathcal{T}$, we denote $\Xi_{it}(y_{it})$ the set of limit points of a sequence $y_{it} = (y^n_{it})_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$.

We observe that

$$\Upsilon(y_{it}) = \begin{cases} \Xi_{it}(y_{it}) & \Xi(y_{it}) \neq \emptyset \\ \mathbb{R} & \text{otherwise} \end{cases}$$

defines a set-valued function from $(\mathbb{R})^{\mathbb{N}}$ to $\mathbb{R}$ in the sense of Luschgy (1989). Since $\mathbb{R}$ is metrizable and σ-compact we are in the situation of Proposition 1 in Luschgy (1985). According to this Proposition, for each $(i, t) \in \{0, \ldots, d\} \times \mathcal{T}$ there exists a measurable selection of Υ_{it}. This means for each $(i, t) \in \{0, \ldots, d\} \times \mathcal{T}$ there exists a $\left(\mathcal{B}^{\mathbb{N}}, \mathcal{B}\right)$ -measurable mapping $\gamma_{it} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ such that

$$\forall y_{it} \in \mathbb{R}^{\mathbb{N}} : \gamma_{it}(y_{it}) \in \Upsilon_{it}(y_{it})\ .$$

Consequently, the mapping $\gamma : \left(\mathbb{R}^{(d+1)\times(t+1)}\right)^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ defined by

$$\gamma(y) := (\gamma_{it}(y_{it}))_{i\in\{0,\ldots,d\},\ t\in\mathcal{T}}$$

is $\left(\left(\mathcal{B}^{\mathbb{N}}\right)^{(d+1)\times(T+1)}, \mathcal{B}^{\mathbb{N}}\right)$- measurable and because of

$$\left(\mathcal{B}^{\mathbb{N}}\right)^{(d+1)\times(T+1)} = \left(\mathcal{B}^{(d+1)\times(T+1)}\right)^{\mathbb{N}}$$

it is also $\left(\left(\mathcal{B}^{(d+1)\times(T+1)}\right)^{\mathbb{N}}, \mathcal{B}^{\mathbb{N}}\right)$- measurable. We define $H = (H_i)_{i=0,\ldots,d}$ by

$$H_i(\omega, t) = \gamma_{it}(y(\omega))\ .$$

We still have to show that H is $\mathbb{F}$-adapted and $(0, H) \in \mathcal{A}$.

Therefore, let $B \in \mathbb{B}$. Then we have $\gamma_{it}^{-1}(B) \in \mathbb{B}^{\mathbb{N}}$. The mapping $\omega \mapsto y_{it}(\omega)$ is $(\mathcal{F}(t), \mathbb{B}^{\mathbb{N}})$-measurable, because every $H_i^n(t)$ is $(\mathcal{F}(t), \mathbb{B})$-measurable. Thus we have

$$(H_i(t))^{-1}(B) = y_{it}^{-1}(\gamma_{it}^{-1}(B)) \in \mathcal{F}(t) \ .$$

Hence H is $\mathbb{F}$-adapted.

In account of $P(A) = 1$, $(0, H) \in \mathcal{A}$ will hold, if both of the following (sufficient) conditions (4.33) and (4.34) are satisfied:

$$\forall \omega \in A, \forall t \in \{0, \ldots, T\} : \quad -\Delta H(\omega, t) \in K(\omega, k) \tag{4.33}$$

$$\forall \omega \in A : \quad H(\omega, T) - C(\omega, T) \in K(\omega, T) \ . \tag{4.34}$$

In order to verify these conditions, let $\omega \in A$. Then for every $t \in \mathcal{T}$ and all $n \in \mathbb{N}$ we have

$$\Delta H^n(\omega, t) \in K(\omega, k)$$

$$H^n(\omega, T) - C^n(\omega, T) \in K(\omega, T)$$

and according to the construction of H there exists a subsequence $(H^{n(m)}(\omega))_{m \in \mathbb{N}}$ (dependent on ω) such that $H(\omega) = \lim_{m \to \infty} H^{n(m)}(\omega)$. Hence (4.33) and (4.34) follow from the closedness of the convex cones $K(\omega, t), t \in \{0, \ldots, T\}$.

$\diamondsuit$

Theorem 4.33 *Assume $\mathbb{F} = \mathbb{F}^X$ and suppose Assumption 4.20 and Assumption 4.22 are satisfied. Then the set $\mathcal{C}^0$ is $L^0(\mathcal{F}(T))$-closed. Hence there is no free lunch in $\mathcal{A}$ if and only if there is no arbitrage in $\mathcal{A}$.*

Proof. We are going to show that

$$\mathcal{C}^0 = \{C \in \mathcal{C} : \exists H : (0, H) \in \mathcal{A}, \ H(T) = C\}$$

is $L^0(\mathcal{F}(T))$-closed. Let $(C^n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{C}^0$ converging in $L^0(\mathcal{F}(T))$ to C with corresponding super hedges $(0, H^n)_{n \in \mathbb{N}}$ We have to show $C \in \mathcal{C}^0$.

Since convergence in L^0 means (componentwise) convergence in probability, there exists a subsequence $(C^{n(m)})_{m \in \mathbb{N}}$, such that $(C^{n(m)})_{m \in \mathbb{N}}$ converges almost surely to C. The strategy $(0, H^{n(m)})$ is a super hedge for $C^{n(m)}$. From Theorem 4.32 we conclude then, that $C \in \mathcal{C}^0$ and the proof is done.

$\diamondsuit$

Chapter 5

Dual Characterization of Super-Hedging Prices

Super-hedging a contingent claim is an extension of classical hedging in so far as the value of the terminal portfolio resulting from a superhedging strategy needs not to equal but only to dominate the payoff of the claim at maturity. The super-hedging price for a claim is the minimum initial amount of money needed to super-hedge the claim. In fricitionless markets it does not matter wether the initial capital of a trading strategy is held as a portfolio of assets or as money alone. Since this is different in the presence of transaction costs, it becomes usefull to consider super-hedging initial endowments. Unfortunately, the set of claims that are super-hedgeable with a given initial portfolio h is not L^0-closed in general. For this reason we introduce approximate super-hedging initial endowments (see Definitions 5.4).

We are going to derive a dual characterization of the set of approximate super-hedging initial endowments for a European claim C in a discrete time market with differential cone constraints. Then we show that under the non-degeneracy Assumption 4.22, the set of claims that are super-hedgeable without any initial capital is closed with respect to a suitable topology. This enables us to prove that the sets of approximate and exact super-hedging intial endowments are equal. As a consequence, the expectation representation formula for the approximate super-hedging price for C then is also valid for exact super-hedging. Moreover this implies, that under our assumptions the definition of super-hedging given in Jouini (1995) and the usual definition of exact super-hedging are equivalent. The latter result can be seen as a significant extension of a similar proposition that as been proven by Koehl et. al. (1999). In fact, they proofed the equality of superhedging-prices prices in the sense of Jouini (1995) and exact super-hedging prices only for claims that are classical hedgeable in a discret time market with two assets.

In Section 5.3 we shortly discuss the question wether there exists an analogue to the Optional Decomposition Theorem by Kramkov (1996). Indeed we can deduce an assertion that resembles somehow an Optional Decomposition Theorem, but unfortunately this resemblance brings only little returns.

5.1 General Case

Definitions 5.1 *A claim $C \in \mathcal{C} = L^0 = (L^0(\mathcal{F}(T)))^{d+1}$ is called marketable, if the claim C is super-hedgeable by some $(h^C, H^C) \in \mathcal{A}^P$ and the claim $-C$ is super-hedgeable by some $(h^{-C}, H^{-C}) \in \mathcal{A}^P$. Denote $\mathcal{C}^m$ the class of marketable claims.*

For $(Q, R) \in \mathcal{P}$ and $i \in \{0, \dots, d\}$ we define the measures $R_i(T)X_i(T)Q$ by

$$\frac{dR_i(T)X_i(T)Q}{dP} := R_i(T)X_i(T)\frac{dQ}{dP}$$

and write shortly

$$L^1_{(Q,R)} := \bigotimes_{i=0,\dots,d} L^1(R_i(T)X_i(T)Q, \mathcal{F}(T)) \ .$$

Remarks 5.2 If $C \in \bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$ and $(h, H) \in \mathcal{A}$ is a super-hedge for C, then from Theorem 3.9 we have $(h, H) \in \mathcal{A}^{\mathcal{P}}$, i.e., $H(R \star X)$ is a Q-supermartingale for every $(Q, R) \in \mathcal{P}$ and moreover

$$\sup_{(Q,R)\in\mathcal{P}} E_Q[C(R(T) \star X(T))] \leq h(R(0) \star X(0)) \ . \tag{5.1}$$

If $C \in L^0$ is super-hedgeable by some $(h, H) \in \mathcal{A}^{\mathcal{P}}$, then for all $(Q, R) \in \mathcal{P}$ we have

$$E_Q[C(R(T) \star X(T))] \leq h(R(0) \star X(0)) < \infty \ . \tag{5.2}$$

Thus from the definition of $\mathcal{C}^m$ we conclude that for all $(Q, R) \in \mathcal{P}$ and for all $C \in \mathcal{C}^m$:

$$-\infty < E_Q[C(R(T) \star X(T))] < \infty$$

which means that

$$\forall (Q,R) \in \mathcal{P}\ \forall C \in \mathcal{C}^m : C(R(T) \star X(T)) \in L^1(Q)\ .$$

However, we point out that this conclusion would be false, if we replaced $\mathcal{A}^{\mathcal{P}}$ by $\mathcal{A}$ in the definition of $\mathcal{C}^m$. This is seen from Example 5.3 were we consider a claim that is superhedgeable by a strategy $(h,H) \in \mathcal{A} \setminus \mathcal{A}^\infty$ (also c.f. Remark 3.10).

$\diamondsuit$

Exampel 5.3 Consider a market with two assets $i = 0,1$ that are traded at time points $t \in \mathcal{T} = \{0,1,2\}$. Let

$$K_0'(t) = \{(1,1)\} \quad , \; t = 0,1,2\ ,$$

which means, that there are no transaction costs and suppose $\mathcal{P} \neq \emptyset$. Then

$$\forall (Q,R) \in \mathcal{P}\ \forall t \in \mathcal{T} : \; R(t) = 1\ .$$

Denote $X_1(t)$ the price of asset 1 at time $t \in \mathcal{T}$ in units of asset 0. Suppose that there is a $(Q,R) \in \mathcal{P}$ such that

$$L^1(\Delta X_1(2) Q) \neq L^0(\mathcal{F}(1))\ .$$

Choose an arbitrary $G \in L^0(\mathcal{F}(1)) \setminus L^1(\Delta X_1(2)Q)$. Then the claim

$$C := (-G X_1(1), G)$$

is super-hedgeable by the portfolio processes $(h,H) \in \mathcal{A}$ with

$$0 = h = H(0) \quad , \quad C = H(1) = H(2)\ .$$

Moreover, the claim $-C$ is hedgeable by $(0,-H)$. It is obvious that $(0,H)$ and $(0,-H)$ are the cheapest super-hedges for C resp. $-C$. So, if we can show, that $(0,H),(0,-H) \notin \mathcal{A}^{\mathcal{P}}$, then we are sure that $C \notin \mathcal{C}^m$. In fact, because of $G \in L^0(\mathcal{F}(1)) \setminus L^1(\Delta X_1(2)Q)$ we have

$$CX(2) = H(2)X(2) = H_0(2) + H_1(2)X_1(2) = G\Delta X_1(2) \notin L^1(\Delta X_1(2)Q)$$

for some Q with $(Q,R) \in \mathcal{P}$. This implies $(0,H),(0,-H) \notin \mathcal{A}^{\mathcal{P}}$ and according to Remark 5.2 we have $C \notin \bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$.

$\diamondsuit$

Definitions 5.4 *Define the multifunction* Γ *from* L^0 *to* $\mathbb{R}^{d+1}$ *by*

$$\Gamma(C) := \{h \in \mathbb{R}^{d+1} : \exists (h, H) \in \mathcal{A} : H(T) - C \in K(., T) \text{ a.s.}\} \ ,$$

the set of super-hedging initial endowments for a claim $C \in L^0$.

For $(Q, R) \in \mathcal{P}$ *we define the multifunction* $\hat{\Gamma}_{(Q,R)}$ *from* $L^1_{(Q,R)}$ *to* $\mathbb{R}^{d+1}$ *by*

$$\hat{\Gamma}_{(Q,R)}(C) := \{h \in \mathbb{R}^{d+1} : \exists (C^n, h^n)_{n \in \mathbb{N}} : h^n \in \Gamma(C^n), C^n \in L^1_{(Q,R)}, C^n \stackrel{L^1_{(Q,R)}}{\longrightarrow} C, h^n \to h\} \ .$$

The multifunction $\hat{\Gamma}$ *from* $\bigcap_{(Q,R) \in \mathcal{P}} L^1_{(Q,R)}$ *to* $\mathbb{R}^{d+1}$ *is defined by*

$$\hat{\Gamma}(C) := \bigcap_{(Q,R) \in \mathcal{P}} \hat{\Gamma}_{(Q,R)}(C)$$

We understand $\hat{\Gamma}(C)$ *as the set of approximate super-hedging initial endowments for a claim* $C \in L^1_{(Q,R)}$. *In a smilar way we also define the multifunctions* $D_{(Q,R)}$ *from* $L^1_{(Q,R)}$ *to* $\mathbb{R}^{d+1}$ *by*

$$D_{(Q,R)}(C) = \{h \in \mathbb{R}^{d+1} : E_Q[C(R(T) \star X(T))] \leq h(R(0)X(0))\} \ ,$$

and D *from* $\bigcap_{(Q,R) \in \mathcal{P}} L^1_{(Q,R)}$ *to* $\mathbb{R}^{d+1}$ *by*

$$D(C) = \bigcap_{(Q,R) \in \mathcal{P}} D_{(Q,R)}(C) \ .$$

$\diamondsuit$

Remark 5.5 Note that $\Gamma(C) \neq \emptyset$, if and only if C is super-hedgeable. Moreover we have

$$\forall C \in \bigcap_{(Q,R) \in \mathcal{P}} L^1_{(Q,R)} : \quad \Gamma(C) \subseteq \hat{\Gamma}_{(Q,R)}(C) \ .$$

This is true, because for $h \in \Gamma(C)$ we could choose $(C^n, h^n) = (C, h)$ for all $n \in \mathbb{N}$, whereas in the case $\Gamma(C) = \emptyset$ the inclusion is trivial.

From the definitions of Γ and $\hat{\Gamma}$ it is also clear that both multifunctions coincide, if Ω or $\mathcal{F}$ has finitely many elements.

One reason for the consideration of $\hat{\Gamma}_{(Q,R)}(C)$ is, that in general markets the multifunction $\hat{\Gamma}_{(Q,R)}(C)$ is always closed (see the appendix on multifunctions for definitions), whereas Γ is neither closed-valued nor closed, in general. However, we will see that under the non-degeneracy Assumption 4.22 the multifunction Γ^{-1} is $L^1_{(Q,R)}$-closed for every $(Q,R) \in \mathcal{P}$. This property will enable us to prove $\Gamma(C) = D(C)$ for every $C \in L^1_{(Q,R)}$ in the subsequent section. Our aim in this section is to show

$$\forall C \in \bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)} : \quad \hat{\Gamma}(C) = D(C) \; . \tag{5.3}$$

We proceed as follows. Starting from the obvious inclusion

$$\forall (Q,R) \in \mathcal{P} \; \forall C \in L^1_{(Q,R)} : \quad \Gamma \subseteq D \subseteq D_{(Q,R)}$$

we show

$$\forall (Q,R) \in \mathcal{P} \; C \in L^1_{(Q,R)} : \hat{\Gamma}_{(Q,R)} \subseteq D_{(Q,R)}$$

by demonstrating that $\hat{\Gamma}_{(Q,R)}$ is the smallest $L^1_{(Q,R)}$-closed multifunction containing Γ. This then implies $\hat{\Gamma} \subseteq D$ on $\bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$. The inclusion $D \subseteq \hat{\Gamma}$ will result from

$$\forall (Q,R) \in \mathcal{P} \; \forall C \in L^1_{(Q,R)} : D(C) \subseteq \hat{\Gamma}_{(Q,R)}(C) \; .$$

This is obtained by applying for each $(Q,R) \in \mathcal{P}$ a separating hyperplane theorem to the disjoint sets $\hat{\Gamma}^{-1}_{(Q,R)}(0)$ and $\{C - h\}$ with $h \notin \hat{\Gamma}_{(Q,R)}(C)$.

Remark 5.6 For $(Q,R) \in \mathcal{P}$ we have $\Gamma \subseteq D \subseteq D_{(Q,R)}$ on $L^1_{(Q,R)}$.

In fact, suppose $C \in L^1_{(Q,R)}$ and $h \in \Gamma(C)$. Then there exists a portfolio process $(h,H) \in \mathcal{A}$ that super-hedges C. Then the assertion follows from Lemma 3.9. The case $\Gamma(C) = \emptyset$ is trivial.

$\diamondsuit$

Lemma 5.7 *For every $(Q,R) \in \mathcal{P}$ the multifunction $\hat{\Gamma}_{(Q,R)}$ is the smallest $L^1_{(Q,R)}$-closed multifunction containing Γ.*

Proof. i) The $L^1_{(Q,R)}$-closedness of $\hat{\Gamma}_{(Q,R)}$ is a consequence of Proposition B.2.

ii) For every $L^1_{(Q,R)}$-closed multifunction F containing Γ we have $\hat{\Gamma}_{(Q,R)} \subseteq F$: If we have $\hat{\Gamma}_{(Q,R)}(C) = \emptyset$ for some $C \in L^1_{(Q,R)}$, then according to Remark 5.5 this results in $\Gamma(C) = \emptyset$ and there is nothing to show for this C. In the case $\Gamma(C) \neq \emptyset$ we can choose $h \in \hat{\Gamma}_{(Q,R)}(C)$. Then there exists a sequence $(C^n, h^n)_{n\in\mathbb{N}}$ in $L^1_{(Q,R)}$ such that

$$\forall n \in: \quad h^n \in \Gamma(C^n) \subseteq F(C^n) \; ,$$

and

$$C^n \overset{L^1_{(Q,R)}}{\longrightarrow} C \quad , \quad h^n \to h \ .$$

From the closedness of F and Remark B.3 then, we conclude $h \in F(C)$.

$\diamondsuit$

Remark 5.8 According to Remark B.3, the $L^1_{(\bar{Q},\bar{R})}$-closedness of $\hat{\Gamma}_{(Q,R)}$ implies that $\hat{\Gamma}_{(Q,R)}$ and $\hat{\Gamma}^{-1}_{(Q,R)}$ are closed-valued.

$\diamondsuit$

Lemma 5.9 *We have $\hat{\Gamma} \subseteq D$ on $\bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$. In particular, we have*

$$\forall (Q,R) \in \mathcal{P} \forall C \in L^1_{(Q,R)} : \quad \hat{\Gamma}_{(Q,R)}(C) \subseteq D_{(Q,R)}(C)$$

Proof. According to Remark 5.6 we have $\Gamma \subseteq D_{(Q,R)}$ for every $(Q,R) \in \mathcal{P}$. It is straighforward to verify that every $D_{(Q,R)}$ is $L^1_{(Q,R)}$-closed. Thus according to Lemma 5.7 we have $\hat{\Gamma}_{(Q,R)} \subseteq D_{(Q,R)}$ for every $(Q,R) \in \mathcal{P}$. This finally implies

$$\hat{\Gamma} = \bigcap_{(Q,R)\in\mathcal{P}} \hat{\Gamma}_{(Q,R)} \subseteq \bigcap_{(Q,R)\in\mathcal{P}} D_{(Q,R)} = D \ .$$

$\diamondsuit$

Theorem 5.10 *We have $D \subseteq \hat{\Gamma}$ on* $\bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$.

Proof. This proof actually is a discrete time version of the proof of Theorem 8.10. The two proofs differ only in some technical matters.

Suppose $C \in \bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$.

i) First, we consider the case $\hat{\Gamma}(C) \neq \emptyset$ and prove that for every $(Q,R) \in \mathcal{P}$ we have $D(C) \subseteq \hat{\Gamma}_{(Q,R)}(C)$. For the case $\hat{\Gamma}(C) = \emptyset$ see ii).

For a fixed but arbitrary $(\bar{Q},\bar{R}) \in \mathcal{P}$ let us choose an arbitrary $h \notin \hat{\Gamma}_{(\bar{Q},\bar{R})}(C)$. We are going to show

$$h \notin D(C) = \bigcap_{(Q,R)\in\mathcal{P}} D_{(Q,R)}(C)$$

by specifying a $(\tilde{Q}, \tilde{R}) \in \mathcal{P}$ satisfying

$$E_{\tilde{Q}}[C(\tilde{R}(T) \star X(T))] > h(\tilde{R}(0) \star X(0)) \ .$$

From $h \notin \hat{\Gamma}_{(\bar{Q},\bar{R})}(C)$ we have $C \notin \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(h)$. This is equivalent to $C - h \notin \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)$. According to Remark 5.8, the convex set $\hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(x)$ is $L^1_{(\bar{Q},\bar{R})}$-closed for every $x \in \mathbb{R}^{d+1}$. Hence $\{(C-h)\}$ and $\hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)$ are strictly separated by some $\rho = (\rho_0, \ldots, \rho_d) \in L^\infty = (L^\infty(P, \mathcal{F}(T)))^{d+1}$. This means there exists a $\rho = (\rho_0, \ldots, \rho_d) \in L^\infty$ such that

$$\text{(5.4)} \qquad \sup_{V \in \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)} E_{\bar{Q}}[(V(\bar{R}(T) \star X(T) \star \rho)] < E_{\bar{Q}}[(C-h)(\bar{R}(T) \star X(T) \star \rho)] \ .$$

Since $V \in \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)$ is equivalent with $0 \in \hat{\Gamma}_{(\bar{Q},\bar{R})}(V)$ and because of $\hat{\Gamma}_{(\bar{Q},\bar{R})}(V) \subseteq D_{(\bar{Q},\bar{R})}(V)$ (see Lemma 5.9) we have

$$\sup_{V \in \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)} E_{\bar{Q}}[V(T)(\bar{R}(T) \star X(T) \star \rho(T))] \leq 0 \ .$$

As the claim $V \equiv 0$ obviously is an element of $\hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)$, we actually have equality. From

$$\forall \omega \in \Omega : \mathbb{R}^{d+1}_+ \subset K(\omega, T)$$

and $-K(T) \subset \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)$ we see that

$$\forall i \in \{0, \ldots, d\} : \rho_i \geq 0$$

is a necessary condition for (5.4) to hold.

If we define

$$Z(T) := R(T) \star \rho$$

then we have $Z(T) \star X(T) \in L^1_{\bar{Q}}$.

Let us define a $\mathbb{R}^{d+1}$-valued process $Z = (Z(t))_{t \in \mathcal{T}}$ by

$$Z_i(t) := \frac{E_Q[Z_i(T) X_i(T) | \mathcal{F}(t)])}{X_i(t)} \quad , \ t \in \mathcal{T}, \ i \in \{0, \ldots, d\} \ .$$

Then the separation inequality (5.4) can be written as

$$\text{(5.5)} \quad 0 = \sup_{V \in \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)} E_P[V(T)(Z(T) \star X(T))] < E_P[(C-h)(Z(T) \star X(T))] \ .$$

Note also that in account of

$$\sup_{V\in\hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(h)} E_{\bar{Q}}[(V-h)(\bar{R}(T)\star X(T)\star\rho)] = \sup_{V\in\hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)} E_{\bar{Q}}[(V(\bar{R}(T)\star X(T)\star\rho)]$$

the inequality (5.4) is equivalent to

$$\text{(5.6)} \qquad \sup_{V\in\hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(h)} E_P[V(T)(Z(T)\star X(T))] < E_P[C(Z(T)\star X(T))] \ .$$

We want to show

$$\text{(5.7)} \qquad \forall t\in\mathcal{T}: Z(t)\in -K'(t) \ .$$

For a fixed $s\in\mathcal{T}$ we take an arbitrary $G(s)\in K(s)$. Then for almost every $\omega\in\Omega$ we have $G(\omega,s)\in K(\omega,s)$. Now we consider an arbitrary nonnegative $\xi\in L^\infty(Q,\mathcal{F}(s))$ and the sequence of buy and hold strategies $(0,H^n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ defined by

$$H^n(\omega,t) := -\xi(\omega)G(\omega,s)1_{\{\max_i|G_i(s)|\le n\}}(\omega,t)1_{\{s,\dots,T\}}(\omega,t) \quad , \ t\in\mathcal{T}$$

in the case $s<T$ and

$$H^n(\omega,t) := 0 \quad , \ t\in\mathcal{T}$$

in the case $s=T$. Then we have

$$\forall i\in\{0,\dots,d\}: |H^n_i(T)|\le ||\xi||_\infty n \quad , \ n\in\mathbb{N}$$

and consequently $(0,H^n)\in\mathcal{A}^\infty\$ for every$n $\in\mathbb{N}$. Because of

$$\forall t\in\mathcal{T}: -\Delta H^n(t)\in K(t) \ ,$$

we conclude

$$(0,H^n)\in\mathcal{A}\cap\mathcal{A}^\infty\subset\mathcal{A}^{\mathcal{P}} \ .$$

Moreover, the strategy $(0,H^n)$ is a super hedge for the claim

$$V^n := -\xi G(s)1_{\{\max_i|G_i(s)|\le n\}}(\omega,T) \ .$$

This is true because in the case $s<T$ we have $V^n = H^n(T)$ and in the case $s=T$ we have

$$H^n(T)-V^n = \xi G(T)1_{\{\max_i|G_i(T)|\le n\}}\in K(T) \ .$$

This shows $0 \in \Gamma(V^n)$. Moreover, we have $V^n \in (L^\infty_{(Q,R)}(\mathcal{F}(T)))^{d+1}{}_{(Q,R)}$ and hence $V^n \in \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)$. Consequently, inequality (5.5) and the tower properties of conditional expectations admit the following calculation for every $n \in \mathbb{N}$

$$
\begin{aligned}
0 &\geq E_P[V^n(Z(T) \star X(T)] \geq -E_P[\xi G(s) 1_{\{\max_i |G_i(s)| \leq n\}}(Z(T) \star X(T))] \\
&= -E_P[E_P[\xi G(s) 1_{\{\max_i |G_i(s)| \leq n\}}(Z(T) \star X(T))|\mathcal{F}(s)]] \\
&= -E_P[\xi E_P[G(s) 1_{\{\max_i |G_i(s)| \leq n\}}(Z(T) \star X(T))|\mathcal{F}(s)]] \\
&= -E_P[\xi G(s) 1_{\{\max_i |G_i(s)| \leq n\}} E_P[(Z(T) \star X(T))|\mathcal{F}(s)]] \\
&= -E_P[\xi G(s) 1_{\{\max_i |G_i(s)| \leq n\}}(Z(s) \star X(s))] \ .
\end{aligned}
$$

Since ξ was arbitrary chosen, it follows $G(s) 1_{\{\max_i |G_i(s)| \leq n\}} Z(s) \geq 0$ for every $n \in \mathbb{N}$ and thus $G(s)Z(s) \geq 0$. In order to conclude that $Z(s) \in -K'(s)$, it suffices to show that for almost every ω we have

$$\forall h \in K(\omega, s) : hZ(\omega, s) \geq 0 \ .$$

But since $G(s) \in K(s)$ was arbitrary chosen and so was s, this is proven now. Thus (5.7) is true.

However, we do not know wether $Z_0(T) > 0$. For the construction of a pair $(\tilde{Q}, \tilde{R}) \in \mathcal{P}$ choose an arbitrary $(Q, R) \in \mathcal{P}$ and define the process Z^ε by

$$Z^\varepsilon(t) := (1-\varepsilon)Z(t) + \varepsilon E_P[\frac{dQ}{dP}|\mathcal{F}(t)]R(t) \quad , \ t \in \mathcal{T}.$$

Since for every $t \in \mathcal{T}$ $-K'(t)$ is a convex cone and $E_P[\frac{dQ}{dP}|\mathcal{F}(t)]R(t) \in K'(t)$, we conclude $\tilde{Z}(t) \in -K'(t)$ for every $t \in \mathcal{T}$. Moreover since $Q \sim P$ we have $Z_0^\varepsilon(T) > 0$. For $(Q^\varepsilon, R^\varepsilon)$ defined by

$$
\begin{aligned}
\frac{dQ^\varepsilon}{dP} &:= Z_0^\varepsilon(T) \\
R_i^\varepsilon(t) &:= \frac{Z_i^\varepsilon(t)}{Z_0^\varepsilon}(t) \quad , \ t \in \mathcal{T},
\end{aligned}
$$

we have $R_i^\varepsilon(t) \in K_0'(t)$ for every $t \in \mathcal{T}$. Moreover the process $R^\varepsilon = (R_i^\varepsilon(t))_{t \in \mathcal{T}}$ is $\mathbb{F}$-adapted. Since $\mathcal{T}$ has a finite number of elements this implies then $(Q^\varepsilon, R^\varepsilon) \in \mathcal{P}$ for every $\varepsilon > 0$. Because of

$$
\begin{aligned}
\sup_{V \in \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(h)} E_P[V(T)(Z^\varepsilon(T) \star X(T))] &\leq (1-\varepsilon) \sup_{V \in \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(h)} E_P[V(T)(Z(T) \star X(T))] \\
&\quad + \varepsilon \sup_{V \in \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(h)} E_P[V(T)(R(T) \star \frac{dQ}{dP} X(T))]
\end{aligned}
$$

there exists a $\varepsilon > 0$ such that the separating inequality (5.6) is satisfied with Z^ε instead of Z. We define $(\tilde{Q}, \tilde{R}) := (Q^\varepsilon, R^\varepsilon) \in \mathcal{P}$. Applying (5.6) to the claims C and $V := h \in \hat{\Gamma}^{-1}_{(\tilde{Q},\tilde{R})}(h)$, we finally get

$$h(\tilde{R}(0) \star X(0)) = E_{\tilde{Q}}[h(\tilde{R}(T) \star X(T))] < E_{\tilde{Q}}[C(\tilde{R}(T) \star X(T))] \ .$$

ii) For the case $\hat{\Gamma}(C) = \emptyset$ we show $D(C) = \emptyset$: Assume $D(C) \neq \emptyset$. Then for $h \in \mathbb{R}^{d+1} \setminus \hat{\Gamma}(C) = \mathbb{R}^{d+1}$ there exists a $(Q, R) \in \mathcal{P}$ such that $h \notin \hat{\Gamma}_{(Q,R)}(C)$. By copying part i) of this proof, with the only difference that maybe $\hat{\Gamma}_{(Q,R)}(C) = \emptyset$ which does not matter, one can verify that there exists a $(\tilde{Q}, \tilde{R}) \in \mathcal{P}$ such that

$$E_{\tilde{Q}}[C(\tilde{R}(T) \star X(T)] > h(\tilde{R}(0) \star X(0)) \ .$$

Since $\tilde{R}(0) \in K_0'(0) \subset (L^0(\mathcal{F}(0)))^{d+1}$ and $K_0'(\omega, t) \subset \{1\} \times \mathbb{R}_+^{d+1}$ (see **Definitions and Assumptions** 2.1) we almost surely have $R(0) = 1$. Since $h \in \mathbb{R}^{d+1}$ was arbitrary chosen, we can define the sequence $(h^n)_{h \in \mathbb{N}}$ by $h^n = (n, 0, \ldots, 0) \in \mathbb{R}^{d+1}$ and conclude that for every n there is a $(\tilde{Q}^n, \tilde{R}^n) \in \mathcal{P}$ such that almost surely

$$E_{\tilde{Q}^n}[C(\tilde{R}^n(T) \star X(T))] > h^n(\tilde{R}^n(0) \star X(0)) = n\tilde{R}^n_0(0) X_0(0) = n \ .$$

This obviously implies

$$\sup_{(Q,R) \in \mathcal{P}} E_Q[C(R(T) \star X(T))] = \infty \ .$$

and thus $D(C) = \emptyset$ which is a contradiction.

♢

From Lemma 5.9 and Theorem 5.10 we deduce

Theorem 5.11 *For all $C \in \bigcap_{(Q,R) \in \mathcal{P}} L^1_{(Q,R)}$ we have $\hat{\Gamma}(C) = D(C)$. Moreover $C \in \bigcap_{(Q,R) \in \mathcal{P}} L^1_{(Q,R)}$ is approximately super-hedgeable if and only if*

$$\sup_{(Q,R) \in \mathcal{P}} E_Q[C(R(T) \star X(T))] < \infty \ .$$

♢

From the equality of sets in Theorem 5.11 that gives a dual characterization of super-hedging intial endowments, one can derive a dual characterization of super-hedging prices. For $C \in \bigcap\limits_{(Q,R) \in \mathcal{P}} L^1_{(Q,R)}$ we define

$$\Pi(C, h_1, \ldots, h_d) := \inf\{h_0 : h = (h_0, h_1, \ldots, h_d) \in \hat{\Gamma}(C)\}$$

and understand $\Pi(C) := \Pi(C, 0, \ldots, 0)$ as the approximate super-hedging price for C inasmuch as in the case $-\infty < \Pi(C) < \infty$ it is equal to the minimum initial amount of money necessary to super hedge C approximately. In fact, if $-\infty < \Pi(C) < \infty$, then, because $\hat{\Gamma}(C)$ is closed, we have

$$\Pi(C, h_1, \ldots, h_d) := \min\{h_0 : h = (h_0, h_1, \ldots, h_d) \in \hat{\Gamma}(C)\} \ .$$

By contrast, the exact super-hedging price for C is given by $\tilde{\Pi}(C, 0, \ldots, 0)$ where

$$\tilde{\Pi}(C, h_1, \ldots, h_d) := \inf\{h_0 : h = (h_0, h_1, \ldots, h_d) \in \Gamma(C)\} \ .$$

It is clear that in the case $\hat{\Gamma}(C) = D(C) = \Gamma(C)$ exact and approximate superhedging prices are equal.

Remark 5.12 For a super-hedgeable claim C the price $\tilde{\Pi}(C, 0, \ldots, 0)$ is often called the **seller's price** of C and the **buyer's price** is given by $-\tilde{\Pi}(-C, 0, \ldots, 0)$ provided that $-C$ is superhedgable. As we explained in Chapter 1 Section 1.1, the buyers price $-\tilde{\Pi}(-C, 0, \ldots, 0)$ corresponds to the maximal amount of debt that an investor may borrow in order to buy the claim C and then perform a self-financing trading strategy, that enables him to compensate his debts at maturity of the claim.

$\diamondsuit$

From the definition of Π and remembering that for every $(Q, R) \in \mathcal{P}$ the process RX is a Q-martingale, we get

Theorem 5.13 *The following dual characterization for approximate super-hedging prices holds for all* $C \in \bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$*:*

$$\begin{aligned}
\Pi(C, h_1, \ldots, h_d) &= \inf\{h_0 : h = (h_0, \ldots, h_d) \in D(C) = \bigcap_{(Q,R)\in\mathcal{P}} D_{(Q,R)}(C)\} \\
&= \inf\{h_0 : h = (h_0, h_1, \ldots, h_d) \in \mathbb{R}^{d+1} : \\
&\qquad \sup_{(Q,R)\in\mathcal{P}} E_Q[C(R(T) \star X(T))] - \sum_{i=1}^{d} h_i R_i(0) X_i(0) \le h_0\} \\
&= \sup_{(Q,R)\in\mathcal{P}} E_Q[C_0 + \sum_{i=1}^{d} (C_i - h_i) R_i(T) X_i(T)]
\end{aligned}$$

$\diamondsuit$

The following proposition justifies Π as an approximate super-hedging price. In discrete time this price coincides with the free lunch price defined as in Jouini, Kallal (1995) and Jouini (1997).

Proposition 5.14 *For all* $C \in \bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$ *we have*

$$\begin{aligned}\Pi(C, h_1, \ldots, h_d) &= \inf\{x \in \mathbb{R} : \forall (Q,R) \in \mathcal{P} \exists (C^n, h^n) : h^n \in \Gamma(C^n), C^n \overset{L^1_{(Q,R)}}{\longrightarrow} C, \\ &\quad x = \liminf_n h_0^n, h_i = h_i^n\} \ .\end{aligned}$$

Proof. The proof is litterally identique to the Proof of Proposition 8.13 (see there).

$\diamondsuit$

5.2 Non-Degeneracy Assumption

Our aim in this section is to proof, that under Assumption 4.22 we have $\Gamma(C) = D(C)$ on $\bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$. This then implies the equality of approximate and exact super-hedging prices.

Theorem 5.15 *Assume* $\mathbb{F} = \mathbb{F}^X$ *and suppose Assumptions 4.20 and 4.22 are satisfied. Then for every* $(Q,R) \in \mathcal{P}$ *the set* $\Gamma^{-1}(0)$ *is* $L^1_{(Q,R)}$*-closed.*

Proof. Let $(Q,R) \in \mathcal{P}$ and suppose $(C^n)_{n\in\mathbb{N}}$ is a sequence in $\Gamma^{-1}(0)$ converging to C in $L^1_{(Q,R)}$. We have to show $C \in \Gamma^{-1}(0)$.

Observe first that according to Definitions 5.4 and Definition 3.8 we actually have

$$\Gamma^{-1}(0) = \mathcal{C}^0 \ .$$

Because of

$$R_i(T)X_i(T) > 0 \quad , \ i = 0, \ldots, d,$$

each of the measures $R_i(T)X_i(T)Q$, $i = 0, \ldots, d$ is equivalent to P. Thus there exists a subsequence $(C^{n(m)})_{m\in\mathbb{N}}$ in $\Gamma^{-1}(0) = \mathcal{C}^0$, such that $(C^{n(m)})_{m\in\mathbb{N}}$ converges to C almost surely. According to Theorem 4.32 this implies $C \in \mathcal{C}^0$.

$\diamondsuit$

Corollary 5.16 *Assume* $\mathbb{F} = \mathbb{F}^X$ *and suppose Assumptions 4.20 and 4.22 are satisfied. Then for every* $x \in \mathbb{R}^{d+1}$ *and every* $(Q,R) \in \mathcal{P}$ *the set* $\Gamma^{-1}(x)$ *is* $L^1_{(Q,R)}$*-closed.*

Proof. This follows from

$$C \in \Gamma^{-1}(x) \Leftrightarrow C - x \in \Gamma^{-1}(0)$$

and Theorem 5.15.

$\diamondsuit$

Theorem 5.17 *Assume* $\mathbb{F} = \mathbb{F}^X$ *and suppose Assumptions 4.20 and 4.22 are satisfied. Then the equality*

$$\Gamma(C) = D(C) = \hat{\Gamma}(C)$$

holds for all $C \in \bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$*. Moreover* $C \in \bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$ *is super-hedgeable, if and only if*

$$\sup_{(Q,R)\in\mathcal{P}} E_Q[C(R(T) \star X(T))] < \infty \ .$$

Proof. Suppose $C \in \bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$. Then according to Lemma 5.9 we have $\Gamma(C) \subseteq D(C)$. Thus we have only to prove $D(C) \subset \Gamma(C)$. Note first that we cannot simply apply Theorem 5.10, because from Corollary 5.16 we only know that the multifunction Γ^{-1} is close-valued. However in most parts, this proof is literally identique to that of Theorem 5.10. Hence for these parts we are going to refer to the proof of Theorem 5.10.

i) We consider first the case $\Gamma(C) \neq \emptyset$ and prove that we have $D(C) \subseteq \Gamma(C)$. For a fixed $(\bar{Q},\bar{R}) \in \mathcal{P}$ we choose a $h \notin \Gamma(C)$ and show

$$h \notin D(C) = \bigcap_{(Q,R)\in\mathcal{P}} D_{(Q,R)}(C)$$

by specfying some $(\tilde{Q},\tilde{R})$ with

$$E_{\tilde{Q}}[\sum_i C_i \tilde{R}_i(T) X_i(T)] > \sum_i h_i \tilde{R}_i(0) X_i(0) \ .$$

From $h \notin \Gamma(C)$ we have $C \notin \Gamma^{-1}(h) \cap L^1_{(Q,R)}$ which is equivalent to

$$C - h \notin \Gamma^{-1}(0) \cap L^1_{(Q,R)} \ .$$

Since for every $x \in \mathbb{R}^{d+1}$ the convex set $\Gamma^{-1}(x)$ is $L^1_{(\bar{Q},\bar{R})}$-closed (see Theorem 5.15 and Corollary 5.16), the sets $\{C - h\}$ and $\Gamma^{-1}(h) \cap L^1_{(Q,R)}$ are strictly separated by a $\rho = (\rho_0, \ldots, \rho_d)$ in $L^\infty = (L^\infty(P, \mathcal{F}(T)))^{d+1}$. This means there exists a $\rho = (\rho_0, \ldots, \rho_d) \in L^\infty(P)$ such that

$$\sup_{V \in \Gamma^{-1}(0) \cap L^1_{(Q,R)}} \sum_i E_{\bar{Q}}[\bar{R}_i(T) X_i(T) \rho_i V_i(T)] < \sum_i E_{\bar{Q}}[\bar{R}_i(T) X_i(T) \rho_i (C_i(T) - h_i)] \ .$$

The remainder of the proof is then identique to the proof of Theorem 5.10 with the only exception that we have to substitute $\hat{\Gamma}^{-1}$ by Γ^{-1}. The same applies for the case $\Gamma(C) = \emptyset$.

$\diamondsuit$

From Theorem 5.17 and Theorem 5.13 we get the following dual characterization of exact super hedging prices.

Theorem 5.18 *Suppose Assumptions 4.20 and 4.22 are satisfied. Then for every* $C \in \bigcap_{(Q,R) \in \mathcal{P}} L^1_{(Q,R)}$ *we have*

$$\tilde{\Pi}(C, h_1, \ldots, h_d) = \sup_{(Q,R) \in \mathcal{P}} E_Q[C_0 + \sum_{i=1}^{d} (C_i - h_i) R_i(T) X_i(T) \ ] \ .$$

Hence the approximate and exact super-hedging price coincide on $\bigcap_{(Q,R) \in \mathcal{P}} L^1_{(Q,R)}$.

$\diamondsuit$

According to Theorem 5.17 we have $\Gamma(C) = \hat{\Gamma}(C)$ for every $C \in \bigcap_{(Q,R) \in \mathcal{P}} L^1_{(Q,R)}$. Thus in the case $-\infty < \Pi(C, h_1, \ldots, h_d) < \infty$, the closedness of $\Gamma(C) = \hat{\Gamma}(C)$ implies

$$\tilde{\Pi}(C, h_1, \ldots, h_d) \quad = \quad \min\{h_0 \in \mathbb{R} : h = (h_0, h_1, \ldots, h_d) \in \Gamma(C)\} \ .$$

This means that $\tilde{\Pi}(Ch_1, \ldots, h_d)) =$ is not only the infimum but also the minimum over all h_0 such that C is super-hedgeable with initial capital h_0. Or in other words, if we are given the initial portfolio $h = (\Pi(C, h_1, \ldots, h_d), h_1, \ldots, h_d)$, then we can find a (h, H) that super-hedges C.

5.3 Optional Decomposition

In a frictionless market the class $\mathcal{P}$ only consists of elements (Q, R) such that

$$\forall t \in \{0, \dots, T\} \forall i \in \{0, \dots, d\} : R_i(t) = 1 \ .$$

Thus $\mathcal{P}$ is the class of all martingale measures for X. Let us extend $\mathcal{P}$ to the class of local martingale measures $\tilde{\mathcal{P}}$. A claim in a frictionless market like in Kramkov (1996) is simply a positive random variable f such that

$$\sup_{(Q) \in \tilde{\mathcal{P}}} E_Q[f] < \infty.$$

For such claims Kramkov (1996, Theorem 3.2) showed his Optional Decomposition Theorem. According to this theorem - adapted to our notation - there exist a portfolio process $(h, H) \in \mathcal{A}^{\mathcal{P}}$ and a non-decreasing consumption process G such that

$$H(t-)X(t) = hX(0) + \bar{H}(t-) \bullet X(t) - G(t) = \operatorname{ess} \sup_{Q \in \tilde{\mathcal{P}}} E_Q[f | \mathcal{F}(t)] \ .$$

For our transaction cost framework, we know sofar that for every claim $C \in \bigcap_{(Q,R) \in \mathcal{P}} L^1_{(Q,R)}$, satisfying

$$\sup_{(Q,R) \in \mathcal{P}} E_Q[C(R(T) \star X(T)] < \infty$$

there exists a super hedge $(h, H) \in \mathcal{A}^{\mathcal{P}}$ such that

$$h_0 = \sup_{(Q,R) \in \mathcal{P}} E_Q[C_0 + \sum_{i=1}^{d} (C_i - h_i) R_i(T) X_i(T)] \ .$$

It is possible to show, that the strategy (h, H) satisfies

$$H_0(t) - G(t) = \operatorname{ess} \sup_{(Q,R) \in \mathcal{P}} E_Q[C_0 + \sum_{i=1}^{d} (C_i - H_i(t)) R_i(T) X_i(T)] \ , \tag{5.8}$$

$$H_0(t) - G(t) = \operatorname{ess} \sup_{(Q,R) \in \mathcal{P}} E_Q[H_0(t+1) + \sum_{i=1}^{d} (H_i(t+1) - H_i(t)) R_i(t+1) X_i(t+1)] \ , \tag{5.9}$$

with a non-decreasing consumption process G. It might also be shown, that there exists a $R^* \in \{R : (Q, R) \in \mathcal{P}\}$ such that the process $\sum_{i=0}^{d} H_i X_i R_i^*$ is a Q-martingale for all Q with $(Q, R^*) \in \mathcal{P}$. But such propositions are of no general use for so long as we do not know wether

$$\sup_{(Q,R)\in\mathcal{P}} E_{(Q,R)}[\ldots] = \sup_{Q:(Q,R^*)\in\mathcal{P}} E_{(Q,R^*)}[\ldots] \ .$$

And this question does not seem to have a positive answer. However, in some special situations it might be administrable to apply equations (5.8) and (5.9) in order to calculate a super hedge by a kind of backward induction.

Part III

Continuous Time

Chapter 6

The Continuous Time Market

In this chapter the general framework of Chapter 2 is worked out for trading in continuous time. We consider a financial market where $d+1$ primary financial assets $i = 0, 1, \dots, d$ are traded continuously within a time intervall $\mathcal{T} = [0, T], T > 0$. Uncertainty and information structure in this market are modelled as a probability space $(\Omega, \mathcal{F}, P)$ with a right continuous filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$. We assume that the prices of assets $i \in \{0, \dots d\}$ in units of asset 0 are given by a $\mathbb{R}_+^{d+1}$-valued $\mathbb{F}$-semimartingale X defined according to (2.1) and satisfying (2.2). Moreover we assume that

$$\forall \omega \in \Omega : X_i(\omega, 0) = x_i > 0 \ .$$

Recall, that $x_0 = 1$ and $X_0(t) = 1$ for every $t \in \mathcal{T}$. Following the definitions of Elliott (1982) or Jacod and Shiryaev (1987) every semimartingale is assumed to be cadlag, i.e., almost every sample path is right continuous and admits left-hand limits.

The following Lemma 6.1 shows, that it is no real restriction to assume that the σ-algebra $\mathcal{F}(T)$ is separable. This assumption is needed later, in order to prove Theorem 7.15.

Lemma 6.1 *Suppose $Y = (Y(t))_{t\in[0,T]}$, $T > 0$, is a $\mathbb{R}^n$-valued (cadlag) process on $(\Omega, \mathcal{F}, P)$. Consider the filtration $\mathbb{F} = (\mathcal{F}(t))_{t\in[0,T]}$, defined by*

$$\mathcal{F}(t) := \bigcap_{\tilde{t}>t} \sigma(Y(s) : s \in [0, \tilde{t}]) \quad , \ t \in [0, T),$$

and $\mathcal{F}(T) = \sigma(Y(s) : s \in [0,T])$. *Then there is a (cadlag) modification* $\tilde{Y}$ *of* Y *satisfying* $\tilde{Y}(\Omega) \subset Y(\Omega)$ *and generating the filtration* $\mathbb{G} := (\mathcal{G}(t))_{t\in[0,T]}$, *defined by*

$$\mathcal{G}(t) := \bigcap_{\tilde{t}>t} \sigma(\tilde{Y}(s) : s \in [0,\tilde{t}]) \quad , \ t \in [0,T)$$

and $\mathcal{G}(T) = \sigma(\tilde{Y}(s) : s \in [0,T])$, *with the following properties.*

The σ*-algebra* $\mathcal{G}(T)$ *is separable and so is any* $L^p(\mathcal{G},P)$, $p \in \{0\} \cup [1,\infty]$. *For every* $t \in [0,T]$ *the* σ*-algebras* $\mathcal{F}(t)$ *and* $\mathcal{G}(t)$ *are equal up to sets of probability zero or one, i.e.,*

$$\forall t \in [0,T] \ \forall A \in (\mathcal{G}(t) \setminus \mathcal{F}(t)) \cup (\mathcal{F}(t) \setminus \mathcal{G}(t)) : \quad P(A)P(\Omega \setminus A) = 0 \ .$$

Consequently, $L^p(\mathcal{F},P)$ *is separable for any* $p \in \{0\} \cup [1,\infty]$. *Moreover, if* Y *is a* $\mathbb{F}$*-semimartingale, then* $\tilde{Y}$ *is a* $\mathbb{G}$*-semimartingale.*

Proof. The map $Y : \omega \mapsto Y(\omega,.)$ takes values in $(\mathbb{R}^n)^{[0,T]}$ and we obviously have

$$\mathcal{F}(T) = Y^{-1}((\mathcal{B}^n)^{[0,T]}) \ .$$

Since the process Y is cadlag, there is a $A \in \mathcal{F}(T)$ with $P(A) = 1$ such that $Y(\omega,.)$ is a cadlag function for every $\omega \in A$. The restriction $Y|A$ of Y to A takes values in the Skorohod space $D(\mathbb{R}^n)$,i.e., the space of cadlag functions on $[0,T]$. Choose an arbitrary $\bar{\omega} \in A$ and let us define the process $\tilde{Y}$ by

$$\tilde{Y}(\omega,t) := Y(\omega,t)1_A(\omega) + Y(\bar{\omega},t)1_{\Omega\setminus A}(\omega) \ .$$

Denote $\mathcal{D}(T)$ the Borel σ-algebra generated by the open sets of the Skorohod-Topology on $D(\mathbb{R}^n)$ (for definitions and properties on the Skorohod-Topoloy see Jacod, Shiryaev (1987)). $\mathcal{D}(T)$ is generated by all projections

$$\Pi(t) : f \in D(\mathbb{R}^n) \mapsto f(t) \in \mathbb{R}^n, t \in [0,T] \ .$$

Consequently, as the map $\tilde{Y} : \omega \mapsto \tilde{Y}(\omega,.)$ takes values in $D(\mathbb{R}^n)$, we have

$$G(T) = \tilde{Y}^{-1}(\mathcal{D}(T)) \ .$$

Since $\mathcal{D}(T)$ is the trace σ-algebra of $(\mathcal{B}^n)^{[0,T]}$ in $D(\mathbb{R}^n)$, it is easy to see that $(Y|A)^{-1}(\mathcal{D}(T))$ is equal to $A\mathcal{F}(T) = \{A \cap F : F \in \mathcal{F}(T)\}$, i.e., the trace of $\mathcal{F}(T)$ in $A \subset \Omega$. Since for $B \in \mathcal{D}(T)$ we have

$$\tilde{Y}^{-1}(B) = \begin{cases} Y^{-1}(B) \cup (\Omega \setminus A) & : \quad \bar{\omega} \in X^{-1}(B) \\ Y^{-1}(B) & : \quad \text{otherwise} \end{cases}$$

it follows

$$\mathcal{G}(T) = \sigma((Y|A)^{-1}(\mathcal{D}(T)), \{A\}) = \sigma(A\mathcal{F}(T), \{A\}) \ . \tag{6.1}$$

This shows that $\mathcal{G}(\mathcal{T})$ and $\mathcal{F}(T)$ only differ about sets of probability zero or one. Analogously, for $t \in [0, T)$ we have

$$\mathcal{G}(t) = \sigma(A\mathcal{F}(t), \{A\}) \ .$$

Since $\mathcal{D}(T)$ is separable, so is $\mathcal{G}(T)$ according to (6.1).

Assume that Y is a $\mathbb{F}$-semimartingale. In order to show that $\tilde{Y}$ is a $\mathbb{G}$-semimartingale we define the filtration $\mathbb{F}^A := (\mathcal{F}^A(t))_{t\in[0,T]}$ where

$$\mathcal{F}^A(t)) = \sigma(\mathcal{F}(t), \{A\}) \ .$$

Then according to Jacod (1979, p.297, Theoreme 9.36) Y is a $\mathbb{F}^A$-semimartingale. Since $\tilde{Y}$ is a cadlag modification of Y, the same is true for $\tilde{Y}$. As a cadlag $\mathbb{G}$-adapted process, $\tilde{Y}$ is $\mathbb{G}$-optional in particular. According to Jacod (1979, p.287, Theoreme 9.19)then, the process $\tilde{Y}$ is a $\mathbb{G}$-semimartingale

$\diamondsuit$

We have seen in Chapter 2, how self-financing rebalancements of portfolios are described by certain cone constraints. The solvency cone $K(\omega, t)$ is interpreted as the set of portfolios, that can be rebalanced at (ω, t) in such a way that every component is non-negative. This coherence formulated in **Definitions and Assumptions 2.1** is valid independent of wether $\mathcal{T}$ is a finite time set or a time intervall. However in continuous time, we have to impose some additional technical assumptions concerning $K_0'(\omega, t)$.

Assumption 6.2 *Assume that for every $(\omega, t) \in \Omega \times [0, T]$ the compact convex set $K_0'(\omega, t) \subset \{1\} \times \mathbb{R}_+^d$ has the following property: For every $\mathbb{R}^{d+1}$-valued $\mathbb{F}$-adapted cadlag process R the condition*

$$\forall t \in [0, T] : R(t) \in K_0'(t) \tag{6.2}$$

implies $R \in \mathcal{R}$.

Proposition 6.3 *Let Y be a $\mathcal{Y}$-valued $\mathbb{F}$-adapted cadlag process on $(\Omega, \mathcal{F}, P)$, $\mathcal{Y} \subset \mathbb{R}^N$, and $f = (f_1, \ldots, f_N) : \mathbb{R}^{d+1} \times \mathcal{Y} \to \mathbb{R}^n$ a continuous function. If for every $(\omega, t) \in \Omega \times \mathcal{T}$ we have*

$$K_0'(\omega, t) = \{r \in \{1\} \times \mathbb{R}^d : \forall i \in \{1, \ldots, N\} : \quad f_i(r, Y(\omega, t)) \le 0\} \ ,$$

then Assumption 6.2 is satisfied.

Proof. Suppose R is a $\mathbb{R}^{d+1}$-valued $\mathbb{F}$-adapted cadlag process satisfying (6.2). Then for every $t \in [0,T]$ there exists a set $A(t) \in \mathcal{F}(t)$ with $P(A) = 1$ and

$$\forall \omega \in A(t) \forall i \in \{1,\ldots,N\} : \; f_i(R(\omega,t), Y(\omega,t)) \leq 0 \; .$$

Define $S := \mathbb{Q} \cap [0,T] \cup \{T\}$ and

$$A := \bigcap_{t \in S} \{\omega \in A(t) \forall i \in \{1,\ldots,N\} : f_i(R(\omega,t), Y(\omega,t)) \leq 0\} \; .$$

Then we have $A \in \mathcal{F}$ and $P(A) = 1$. Moreover, there exists a set $B \in \mathcal{F}$ with $P(B) = 1$ such that $Y(\omega,.)$ and $R(\omega,.)$ are cadlag for every $\omega \in B$. Since $P(A \cap B) = 1$, we only need to show, that for all $\omega \in A \cap B$ we have

$$\forall t \in \mathcal{T} \forall i \in \{1,\ldots,N\} : \; f_i(R(\omega,t), Y(\omega,t)) \leq 0 \; .$$

In order to show this, choose $\omega \in A \cap B$ and $t \in \mathcal{T} \setminus S$ arbitrarily. Then there is a sequence $(t_n)_{n \in \mathbb{N}}$ in S with $t_n \searrow t$. For every $n \in \mathbb{N}$ we have

$$\forall i \in \{1,\ldots,N\} : \quad f_i(R(\omega,t_n), Y(\omega,t_n)) \leq 0 \; .$$

Since $Y(\omega,.)$ and $R(\omega,.)$ are cadlag and f is continuous, this implies

$$\forall i \in \{1,\ldots,N\} : \quad f_i(R(\omega,t), Y(\omega,t)) \leq 0 \; .$$

$\diamondsuit$

Remark 6.4 It is easy to see, that each of the compact convex multifunctions $(\omega,t) \mapsto K_0(\omega,t)$ corresponding to the exemplary models given in Sections 2.2 to 2.6 satisfies the assumptions on $K_0(\omega,t)$ given in Proposition 6.3. Thus, if the concerning ask, bid and spot price processes are assumed to be cadlag, then, according to Proposition 6.3, Assumption 6.2 is satisfied. Hence Assumption 6.2 is not really a restriction. $\diamondsuit$

Definitions 6.5 *A* portfolio process consists of an initial portfolio $h = (h_i)_{i \in \{0,\ldots,d\}} \in \mathbb{R}^{d+1}$ and a $\mathbb{R}^{d+1}$-valued $\mathbb{F}$-adapted cadlag process $H = (H_i)_{i \in \{0,\ldots,d\}}$ of finite variation. This means, for each $i \in \{0,\ldots,d\}$ the process H_i is (almost surely) of finite variation, i.e., almost every sample path of H_i is of finite variation on each compact subset of $[0,\infty[$. This implies, that H is a $\mathbb{F}$-semimartingale with a "deterministic martingale component".

The finite variation property is a natural assumption since otherwise we had to be alert to infinitely high transaction costs. Denote $\mathcal{H}$ the set of portfolio processes. For every portfolio process $(h,H) \in \mathcal{H}$ we define the process $H(.-) = (H(t-))_{t\in[0,T]}$ by $H(0-) := h$ and

$$H(t-) := \lim_{s \nearrow t} H(s) \quad , \; t > 0.$$

Note that the finite variation property already implies the existence of limits on the right and on the left for H_i (see Natanson, S. 245, Folgerung 2) , whereas from the continuity on the left it follows that $H_i(.-)$ is predictable.

We define the processes $\Delta H = (\Delta H_i)_{i\in\{0,\ldots,d\}}$ by

$$\Delta H_i := H_i - H(.-)_i \quad , \; i \in \{0,\ldots,d\},$$

and understand $H_i(t)$ as number of units of asset i held after all transactions at time t whereas $H_i(t-)$ represents the units of asset i before all transactions at time t . In particular $H_i(0-)$ is the initial holding in asset i, i.e., before any transaction is settled.

$\diamondsuit$

Definitions 6.6 *Let $G = (G_0,\ldots,G_d)$ and $Y = (Y_0,\ldots,Y_d)$ be two $\mathbb{R}^{d+1}$-valued (cadlag) semimartingales and assume that G is a process of bounded variation. Then any of the following integrals is well defined by the Ito formula (c.f. Elliott Theorem 12.21). The process $G(.-)\bullet Y = (G(.-)\bullet Y(t))_{t\in[0,T]}$ is defined by the stochastic integrals*

$$G(.-)\bullet Y(t) := \sum_j G_j(.-)\bullet Y_j(t)$$

with $G_j \bullet Y_j(t) := \int_0^t G_j(s-)dY_j(s)$. The process $Y \bullet G = Y(.-)\bullet H + \Delta Y \bullet H$ is defined in a similar way, but pathwise by the Stieltjes Integrals

$$Y_j(.-)\bullet G_j(\omega,t) := \int_0^t Y_j(.-)(\omega,s)dG_{j\bullet}(\omega,s)$$

and

$$\Delta Y_j \bullet G_j(\omega,t) := \sum_{0<s\leq t} \Delta Y_j(s)\Delta G_j(s) \; .$$

The product GY, given by

$$GY = G(0)Y(0) + G(.-) \bullet Y + Y \bullet G \ ,$$

is a semimartingale (c.f. Elliott, 1982, Corollaries 12.22 and 12.23).

$\diamondsuit$

For the $\mathbb{R}^{d+1}$-valued semimartingal X and a portfolio process $(h, H) \in \mathcal{H}$, where H is a $(d+1)$-dimensional semimartingale of bounded variation, the processes $H_i(.-) \bullet X_i$ and

$$X_i \bullet H_i = X_i(.-) \bullet H_i + \Delta X_i \bullet H_i$$

are well defined according to Definition 10. They also satisfy

$$H_i X_i = H_i X_i(0) + H_i(.-) \bullet X_i + X_i \bullet H_i \quad , \ i \in \{0, \dots, d\}.$$

This yields the partial integration formula

$$HX = H(0)X(0) + H(.-) \bullet X + X \bullet H \ .$$

In this partial integration formula $H(.-) \bullet X$ represents the change of portfolio value due to price changes and $X \bullet H$ decribes the changes of portfolio value due to transactions.

Similar as in the discrete time framework, we are interested in the admissible portfolio processes resulting from self-financing trading strategies. Again, we define the set of admissible portfolio processes by only allowing for self-financing rebalancements (with consumption) instead of admitting arbitrary transactions. However, in continuous time, we also have to take account of “continuous trading flows”. Such flows are possible in theory because we do not account for fixed costs in trading. Therefore, the involved differential cone constraints have to be interpreted as kind of “stochastic differential inclusions”. This is formalized as follows.

Definitions 6.7 *Let $\mathcal{A}^{\mathcal{R}}$ be the class of portfolio processes $(h, H) \in \mathcal{H}$ such that for every process $R \in \mathcal{R}$ we have $(R \star X)(0)\Delta H(0) \leq 0$ and the process $(R \star X) \bullet H$ is decreasing, i.e., for P-almost every $\omega \in \Omega$ we have*

$$\forall s, t \in [0, T], \ s \leq t : \quad (R \star X) \bullet H(\omega, s) \geq (R \star X) \bullet H(\omega, t) \ .$$

In a short notation this condition may be written as a stochastic differential inclusion of the form

$$\forall t \in [0,T] : -dH(t) \in K(t) \ ,$$

hence a differential cone constraint.

Denote $\mathcal{A}^s$ *the class of portfolio processes* (h,H) *such that* H *is piecewise constant on* $[0,T]$ *and has a finit number of jumps. This means that* $H \in \mathcal{A}^s$*, if and only if there is a time set* $\mathcal{T}(H) = \{t_0, t_1, \ldots, t_N\} \subset [0,T]$*,* $n \in \mathbb{N}$ *such that*

$$\forall k \in \{1,\ldots,N\} \ \forall t_{i-1} \le t < t_i \ \forall \omega \in \Omega : \quad H(\omega,t) = H(\omega,t_{i-1}) \ .$$

Processes in $\mathcal{A}^s$ *are called simple.*

Denote

$$\mathcal{A}^\infty := \{(h,H) \in \mathcal{H} : (H(T))^- \in (L^\infty(\mathcal{F}(T),P))^{d+1}\}$$

the set of "tame" portfolio processes. We have $(h,H) \in \mathcal{A}^\infty$*, if and only if the short positions of* $H(T)$ *are* P*-almost surely bounded. This means that one is not allowed to borrow an "infinite" sum of money or to sell short "infinitely" many units of an asset. This assumption actually always applies in practice. Note also that our tame condition only refers to terminal positions, whereas the tame conditions usually used troughout literature refer to the set* $\mathcal{T}$*.*

Let us denote $\mathcal{A} \subset \mathcal{H}$ *the class of admissible portfolio processes. The definition of this class should correspond somehow to the given model for the price process. Since we do not intend to confine to a particular asset price model, we avoid a more exact definition of* $\mathcal{A}$*. However we postulate that* $\mathcal{A}$ *is a convex cone and we will always assume*

$$\mathcal{A}^s \cap \mathcal{A}^{\mathcal{R}} \cap \mathcal{A}^\infty \subseteq \mathcal{A} \subseteq \mathcal{A}^{\mathcal{R}} \ . \tag{6.3}$$

$\diamondsuit$

Remark 6.8 According to it's definition the class $\mathcal{A}^s$ is obviously a convex cone in $\mathcal{H}$. Moreover it is straight forward to show that for every $t \in [0,T]$ the set $K(t)$ is a convex cone in $(L^0(\mathcal{F}(t)))^{d+1}$. Thus, it is easy to verify that $\mathcal{A}^{\mathcal{R}}$ and $\mathcal{A}^\infty$ are convex cones, too. So (6.3) and the assumption that $\mathcal{A}$ is a convex cone will not result in contradictions .

$\diamondsuit$

Definitions 6.9 *Let $\mathcal{P}$ be the class of all pairs (Q, R) satsifying the following conditions:*

(P1) Q is a probability measure equivalent to P on $\mathcal{F}(T)$,

(P2) $R \in \mathcal{R}$,

(P3) the process $R \star X$ is a (cadlag) Q-Martingale.

Define

$$\mathcal{A}^{\mathcal{P}} := \{(h, H) \in \mathcal{A}^{\mathcal{R}} : \forall (Q, R) \in \mathcal{P} : H(R \star X) \text{ is a } Q-\text{supermartingal}\}$$

if $\mathcal{P} \neq \emptyset$ and otherwise $\mathcal{A}^{\mathcal{P}} := \emptyset$.

$\diamondsuit$

Lemma 6.12 will explain this definition further. There the condition that $H(.-) \bullet (R \star X)$ should be a Q-supermartingale is used to rule out "doubling" strategies. The next chapter then will be contributed to necessary and sufficient conditions for $\mathcal{P} \neq \emptyset$.

Lemma 6.10 *If $\mathcal{P} \neq \emptyset$, then we have*

$$\mathcal{A}^s \cap \mathcal{A}^{\mathcal{R}} \cap \mathcal{A}^{\infty} \subseteq \mathcal{A}^{\mathcal{P}} \subseteq \mathcal{A}^{\mathcal{R}} .$$

Proof. Suppose $(h, H) \in \mathcal{A}^s \cap \mathcal{A}^{\mathcal{R}} \cap \mathcal{A}^{\infty}$ with corresponding $\mathcal{T}(H) := \{t_0, \ldots, t_N\}$. Let us define the filtration

$$\mathbb{F}(\mathcal{T}(H)) := (\mathcal{F}(t_k))_{k \in \{0, \ldots, N\}}$$

and consider the market with trading restricted to the discrete time set $\mathcal{T}(H)$. Since the sets $K(t)$, $K'(t)$ and $K'_0(t)$ are well defined for $t \in \mathcal{T}(H)$, we can define $\mathcal{R}(\mathcal{T}(H))$, $\mathcal{A}(\mathcal{T}(H))$, $\mathcal{A}^{\infty}(\mathcal{T}(H))$ and $\mathcal{A}^{\mathcal{P}}(\mathcal{T}(H))$ according to the definitions in discrete time (see Chapter 3). Then we have

$$(h, H) \in \mathcal{A}(\mathcal{T}(H)) \cap \mathcal{A}^{\infty}(\mathcal{T}(H))$$

and by Remark 3.10 this implies $(h, H) \in \mathcal{A}^{\mathcal{P}}(\mathcal{T}(H))$. Comparing the definitions of $\mathcal{A}^{\mathcal{P}}(\mathcal{T}(H))$ and $\mathcal{A}^{\mathcal{P}}$ we finally see that $(h, H) \in \mathcal{A}^{\mathcal{P}}$.

$\diamondsuit$

Definitions 6.11 *Denote* $\mathcal{C} := (L^0(\mathcal{F}(T)))^{d+1}$ *the set of contingent claims. We call* $(h,H) \in \mathcal{A}$ *a* super hedge *for a* $C \in \mathcal{C}$*, if and only if we have* $H(T) - C \in K(T)$*. We call a claim* $C \in \mathcal{C}$ *super-hedgeable (by* (h,H)*), if there exists a super hedge* (h,H) *for* C*. Denote*

$$\mathcal{C}^h := \{C \in \mathcal{C} : \exists H : (h,H) \in \mathcal{A},\ H(T) - C \in K(T)\}$$

the set of claims, that are super-hedgeable with initial portfolio $h \in \mathbb{R}^{d+1}$ *and define*

$$\mathcal{C}^+ := \{C \in K(T) : \exists A \in \mathcal{F}(T),\ P(A) > 0,\ \forall \omega \in A : C(\omega) \in intK(\omega, T)\}$$

where for $\omega \in \Omega$ *the set* $intK(\omega,T)$ *is the interior of* $K(\omega,t)$ *in the metric topology of* $\mathbb{R}^{d+1}$*.*

For a subclass $\mathcal{B} \subset \mathcal{A} \subset \mathcal{A}^{\mathcal{R}}$ *we write*

$$\mathcal{C}^h|\mathcal{B} := \{C \in \mathcal{C}^h : \exists H : (h,H) \in \mathcal{B},\ H(T) - C \in K(T)\}\ .$$

Lemma 6.12 *Suppose* $(Q,R) \in \mathcal{P} \neq \emptyset$ *and let* $C \in \mathcal{C}$ *be a contingent claim with*

$$E_Q[(C(R(T) \star X(T)))^-] < \infty\ .$$

Suppose $(h,H) \in \mathcal{A}^{\mathcal{R}}$ *is a super hedge for* C *and* $H(.-) \bullet (R \star X)$ *is a* Q*-supermartingale. Then the process* $H(R \star X)$ *is a* Q*-supermartingale and*

$$E_Q[C(R(T) \star X(T))] \leq E_Q[H(T)(R(T) \star X(T))] \leq h(R(0) \star X(0))\ .$$

Proof. Because of

$$H(R \star X) = H(0-)(R(0) \star X(0)) + (R(0) \star X(0))\Delta H(0) + H(.-) \bullet (R \star X) + (R \star X) \bullet H$$

there is only to show that $(R \star X) \bullet H$ is a Q-supermartingale and

$$(R(0) \star X(0))\Delta H(0) \in L^1(Q) \quad , \quad E_Q[(R(0) \star X(0))\Delta H(0)] \leq 0\ .$$

First of all note, that because of $R \in \mathcal{R}$, the process $(R \star X) \bullet H$ is decreasing and $(R \star X)\Delta H(0) \leq 0$. Consequently we have

$$\forall t \in [0,T] :\ 0 \geq (R \star X) \bullet H(t)$$

and thus $0 \geq E_Q[(R \star X) \bullet H(t)]$ is defined for every t and analogously

$$E_Q[(R(0) \star X(0))\Delta H(0)] \leq 0\ .$$

We still have to show that $E_Q[(R \star X) \bullet H(t)] > -\infty$ for every t and

$$E_Q[(R(0) \star X(0))\Delta H(0)] > -\infty \ .$$

Because $(h, H) \in \mathcal{H}$ is a super hedge for C we have

$$H(T)(R(T) \star X(T)) \geq C(R(T) \star X(T)) \ .$$

Taking into account that $H(.-) \bullet RX$ is a Q-supermartingale, the following calculation is feasible for every $t \in [0, T]$:

$$\begin{aligned} h(R(0) \star X(0)) \ &\geq \ h(R(0) \star X(0)) + E_Q[(R(0) \star X(0))\Delta H(0)] \\ &\quad + E_Q[H(.-) \bullet (R \star X)(t)] + E_Q[(R \star X) \bullet H(t)] \\ &\geq \ h(R(0) \star X(0)) + E_Q[(R(0) \star X(0))\Delta H(0)] \\ &\quad + E_Q[H(.-) \bullet (R \star X)(T)] + E_Q[(R \star X) \bullet H(T)] \\ &= \ E_Q[H(T)(R(T) \star X(T))] \geq E_Q[C(R(T) \star X(T)) \\ &\geq \ E_Q[\min\{C(R(T) \star X(T)), 0\}] > -\infty \ . \end{aligned}$$

Hence $(RX) \bullet H$ is a Q-supermartingale. $\diamondsuit$

Remark 6.13 If $(h, H) \in \mathcal{A}^{\mathcal{R}}$ and $(Q, R) \in \mathcal{P}$, then the process $H(.-) \bullet (R \star X)$ is always a local martingal. In fact, since H is a $\mathbb{F}$-adapted cadlag process, it is optional (see Elliott, 1982, Theorem 6.35). This implies that $H(.-)$ is locally bounded (see Elliott, 1982, Lemma 11.48) and $H(.-) \bullet (R \star X)$ is a local martingale.

$\diamondsuit$

Chapter 7

Fundamental Theorems of Asset Pricing in Continuous Time

In this Chapter we approach the question, how to characterize the condition $\mathcal{P} \neq \emptyset$ by the absense of arbitrage opportunities. The main results are stated in Fundamental Theorems 7.12 and 7.15.

We are going to introduce three different notions of arbitrage opportunities. The first consists of the classical free lunches in terms of sequences converging in probability P. The research of Lakner (1995) and Delbaen, Schachermayer (1994, 1998), who introduced different notions of free lunches, already reveals that even for frictionless markets the classical notion of free lunches is not well suited for deriving Fundamental Theorems. Instead of free lunches with vanishing risk considered by Delbaen, Schachermayer (1994, 1998) we introduce L^∞-bounded free lunches in terms of uniformly bounded sequences converging in probability P. This convergence is invariant to equivalent changes of probability measure. In addition, we introduce free lunches in terms of sequences converging in some $L^p(Q)$, $1 \leq p < \infty$. Since this convergence obviously depends on the choice of Q it is dissatisfactory in our opinion, to define free lunches only with respect to convergence in a single $L^p(Q)$ for some $Q \sim P$ or P itself like it was done by Jouini, Kallal (1995a) with $p = 2$ and Pham, Touzi (1999) in discrete time with $p = 1$. In fact, if one does so, the notion of arbitrage depends on the arbitrary choice of a probability measure. In order to avoid this arbitrariness we consider the class

$$\mathcal{Q} := \{Q \sim P : \forall i \in \{0, \dots, d\} : X_i \in L^1(Q)\} \ .$$

From Lemma A.8 we know that this class is not empty for any given probability P.

For $Q \sim P$ we define the measures $X_i(T)Q$ by

$$\frac{dX_i(T)Q}{dQ} = X_i(T) \quad , \; i = 0, \ldots, d.$$

It is worth noting, that for $Q \in \mathcal{Q}$ the measures $X_i(T)Q$ are finite and we have $X_i(T)Q \sim P$ for every $i \in \{0, \ldots, d\}$. The latter is a consequence of condition (2.2). We denote

$$L^0 := (L^0(\mathcal{F}(T))^{d+1}$$

the space of $\mathcal{F}(T)$-measurable random vectors,

$$L^\infty := (L^\infty(\mathcal{F}(T), P))^{d+1}$$

the space of P-almost surely bounded random vectors and

$$L^p_{XQ} := \times_{i=0,\ldots,d} L^p(\mathcal{F}(T), X_i Q) \quad , 1 \leq p < \infty, \; Q \in \mathcal{Q}$$

the space of random vectors $V = (V_0, \ldots, V_d) \in L^0$ such that $V_i X_i \in L^p(Q)$ for every $i \in \{0, \ldots, d\}$.

For $O \subset L^0(\mathcal{F}(T))$ we denote $\overline{O}^0$ the closure of O in with respect to the topology of (componentwise) convergence in probability $Q \sim P$, which is independent of Q because of $Q \sim P$.

For $O \subset L^p_{XQ}(\mathcal{F}(T))$, $1 \leq p < \infty$, we write $\overline{O}^p_{XQ}$ for the closure of O in the L^p_{XQ}-norm topology of $L^p_{XQ}(\mathcal{F}(T))$.

Although L^∞ is independent of the choice of $Q \in \mathcal{Q}$, we will deal with different weak* topologies on L^∞ depending on Q. For $Q \in \mathcal{Q}$ let $\sigma(L^\infty, L^1_{XQ})$ denote the locally convex topology on L^∞ induced by the semi-norms

$$|.|_Z : L^\infty \to \mathbb{R}, \; C \mapsto \sum_{i=0}^{d} | \int_\Omega C_i Z_i X_i dQ| \quad , \; Z \in L^1_{XQ} \; .$$

Then we write $\overline{O}^\infty_{XQ}$ for the closure of O in the $\sigma(L^\infty, L^1_{XQ})$-topology of L^∞.

Definitions 7.1 *A portfolio process $(h, H) \in \mathcal{A}$ is called an arbitrage, if $-h \in K(0)$ and $H(T) \in \mathcal{C}^+$.*

A sequence $(h^n, H^n)_{n\in\mathbb{N}}$ *in* $\mathcal{A}$ *is called a free lunch, if and only if for every* $n \in \mathbb{N}$ *we have* $-h^n \in K(0)$ *and there exists a* $C \in \mathcal{C}^+$ *such that* $H^n(T)$ *converges to* C *in probability* P*. If in addition we have*

$$\forall n \in \mathbb{N} : \max_i |H_i^n(T)| \leq c \quad P-a.s.$$

for some $c > 0$*, then we call this a* L^∞*-bounded free lunch.*

For $1 \leq p < \infty$ *a sequence* $(h^n, H^n)_{n\in\mathbb{N}}$ *in* $\mathcal{A}$ *is called a* L^p*-free lunch, if there exist a probability measure* $Q \in \mathcal{Q}$ *and a claim* $C \in \mathcal{C}^+ \cap L^p_{XQ}$ *such that*

$$\forall n \in \mathbb{N} : H^n(T) \in L^p_{XQ},\ -h^n \in K(0)$$

and $H^n(T)$ *converges to* C *in* L^p_{XQ}*.*

$\diamondsuit$

Proposition 7.2 *Suppose* $\mathcal{A} \subset \mathcal{A}^{\mathcal{P}}$*. Then there is no arbitrage in* $\mathcal{A}$*.*

Proof. If $\mathcal{P} = \emptyset$ then $\mathcal{A} \subset \mathcal{A}^{\mathcal{P}} = \emptyset$ and there is nothing to prove.

Suppose $\mathcal{P} \neq \emptyset$. Let $(h, H) \in \mathcal{A} \subset \mathcal{A}^{\mathcal{P}}$ such that $-h \in K(0)$ and $H(T) \in K(T)$. Then for any $(Q, R) \in \mathcal{P}$ we have

$$0 \;\; \geq \;\; h(\tilde{R}(0) \star X(0)) \geq E_{\tilde{Q}}[H(T)(\tilde{R}(T) \star X(T))] \; .$$

In account of $H(T)(\tilde{R}(T) \star X(T)) \geq 0$ this results in

$$H(T)(R(T) \star X(T)) = 0 \; .$$

Hence for almost every ω we have

$$H(\omega, T)(R(\omega, T) \star X(\omega, T)) = 0 \; .$$

According to Proposition 2.4 this implies that for almost every ω

$$H(\omega, T) \in \partial K(\omega, T)$$

and thus $H(T) \notin \mathcal{C}^+$.

$\diamondsuit$

Proposition 7.3 *Suppose $\mathcal{B} \subset \mathcal{A}$. Then there is no arbitrage in $\mathcal{B}$, if and only if*

$$\forall h \in -K(0): \ (\mathcal{C}^h|\mathcal{B}) \cap \mathcal{C}^+ = \emptyset \ .$$

Proof. Suppose there exists a $h \in -K(0)$ such that $(\mathcal{C}^h|\mathcal{B}) \cap \mathcal{C}^+ \neq \emptyset$. Then there is a $(h, H) \in \mathcal{B}$ satisfying $-h \in K(0)$ and $H(T) \in \mathcal{C}^+$, thus an arbitrage in $\mathcal{B}$.

Conversely, suppose there is an arbitrage $(h, H) \in \mathcal{B}$. Then we have $h \in -K(0)$ and $(\mathcal{C}^h|\mathcal{B}) \cap \mathcal{C}^+ \neq \emptyset$.

♢

Proposition 7.4 *Suppose $\mathcal{B} \subset \mathcal{A}$. Then there is no free lunch in $\mathcal{B}$, if and only if*

$$\forall h \in -K(0) : \overline{(\mathcal{C}^h|\mathcal{B})}^0 \cap \mathcal{C}^+ = \emptyset \ .$$

Proof. Suppose $h \in -K(0)$ and $\overline{(\mathcal{C}^h|\mathcal{B})}^0 \cap \mathcal{C}^+ \neq \emptyset$. Then there exist a sequence $(H^n)_{n\in\mathbb{N}}$ and a claim $C \in \mathcal{C}^+ \cap L^0$ such that $(h, H^n) \in \mathcal{B}$ for every $n \in \mathbb{N}$ and $H^n(T)$ converges to C in probability P. Thus, the sequence $(h, H^n)_{n\in\mathbb{N}}$ is a free lunch in $\mathcal{B}$.

Conversely, suppose there is a free lunch in $\mathcal{B}$. Then we have $\overline{(\mathcal{C}^0|\mathcal{B})}^0 \cap \mathcal{C}^+ \neq \emptyset$.

♢

Proposition 7.5 *Suppose $\mathcal{B} \subset \mathcal{A}$ and let $1 \leq p < \infty$. There is no L^p-free lunch in $\mathcal{B}$, if and only if*

$$\forall Q \in \mathcal{Q} \ \forall h \in -K(0) : \overline{((\mathcal{C}^h|\mathcal{B}) \cap L^p_{XQ})}^p_{XQ} \cap \mathcal{C}^+ = \emptyset \ .$$

Proof. Suppose $h \in -K(0)$, $Q \in \mathcal{Q}$ and

$$\overline{((\mathcal{C}^h|\mathcal{B}) \cap L^p_{XQ})}^p_{XQ} \cap \mathcal{C}^+ \neq \emptyset \ .$$

Then there exist a sequence $(H^n)_{n\in\mathbb{N}}$ and a claim $C \in \mathcal{C}^+ \cap L^p_{XQ}$ such that for every $n \in \mathbb{N}$ we have $(h, H^n) \in \mathcal{B}$, $H^n(T) \in L^p_{XQ}$ and $H^n(T)$ converges to C in L^p_{XQ}. Hence the sequence $(h, H^{n(k)})_{k\in\mathbb{N}}$ is a L^p-free lunch in $\mathcal{B}$.

Conversely, suppose there is a L^p-free lunch in $\mathcal{B}$. Then there exists a $Q \in \mathcal{Q}$ such that

$$\overline{((\mathcal{C}^0|\mathcal{B}) \cap L^p_{XQ})}^p_{XQ} \cap \mathcal{C}^+ \neq \emptyset \ .$$

♢

Remark 7.6 For a similar characterization of the absence of L^∞-bounded free lunches we will need the additional assumption that L^1_{XQ} is separable for every $Q \in \mathcal{Q}$. This assumption is also necessary in as much as it is necessary for the weak* topology of the closed unit sphere of L^∞ to be metrizable (see Holmes, 1975, p. 72, Corolarry 2). This property is needed to assure equivalence of weak* closedness and weak* sequential closedness. The separability assumption is in fact no real restriction for the cadlag price process X. This was shown in Lemma 6.1.

$\diamondsuit$

Proposition 7.7 *Let $\mathcal{B} \subset \mathcal{A}$ and suppose that for every $Q \in \mathcal{Q}$ the space L^1_{XQ} is separable (c.f. Remark 7.6). Then there is no L^∞-bounded free lunch in $\mathcal{B}$, if and only if*

$$\forall Q \in \mathcal{Q} \forall h \in -K(0) : \overline{((\mathcal{C}^h|\mathcal{B}) \cap L^\infty)}^{\infty}_{XQ} \cap \mathcal{C}^+ = \emptyset \ .$$

Proof. The Proof is literally the same as that of Proposition 4.11.

$\diamondsuit$

Remarks 7.8 Let $\mathcal{B} \subset \mathcal{A}$ and suppose there is no free Lunch in $\mathcal{B}$. Then there is no L^p-free lunch in $\mathcal{B}$. In fact, if $(h^n, H^n)_{n\in\mathbb{N}}$ and C constitute an L^p-free lunch then because L^p-convergence implies convergence in probability there is a free lunch.

Proposition 7.4 implies that the postulation of no arbitrage in $\mathcal{B} \subset \mathcal{C}$ is equivalent with the postulation of no free lunch in $\mathcal{B}$, if and only if the set $((\mathcal{C}^h|\mathcal{B}) \cap L^0)$ is closed in the topology of (componentwise) convergence in probability P. Analogues statements result from propositions 7.5 and 7.7.

$\diamondsuit$

Definition 7.9 *For $Q \in \mathcal{Q}$ and $1 \leq q \leq \infty$ define*

$$\mathcal{P}^q_Q := \{\tilde{Q} : \exists \tilde{R} \in \mathcal{R} : (\tilde{Q}, \tilde{R}) \in \mathcal{P}, \tilde{R}(T)\frac{d\tilde{Q}}{dQ} \in L^q_{XQ}\} \ .$$

Theorem 7.10 *Let $1 \leq p < \infty$, $q = \frac{p}{p-1}$ for $1 < p$, and $q = \infty$ for $p = 1$. Suppose $\mathcal{A} \subset \mathcal{A}^{\mathcal{P}}$. If*

$$\forall Q \in \mathcal{Q} : \mathcal{P}^q_Q \neq \emptyset \ ,$$

then there is no L^p-free lunch in $\mathcal{A}$.

Proof. Let $C \in K(T) \cap L^p_{XQ}$ be a contingent claim and $(h^n, H^n)_{n\in\mathbb{N}}$ a sequence in $\mathcal{A}$ such that $-h^n \in K(0)$ and $H^n(T) \in L^p_{XQ}$ converges to C in L^p_{XQ} for some $Q \in \mathcal{Q}$. Choose an arbitrary $(\tilde{Q}, \tilde{R}) \in \mathcal{P}^q_Q$. Applying componentwise the Hölder inequality in account of $\tilde{R}(T)\frac{d\tilde{Q}}{dQ} \in L^q_{XQ}$ we get

$$\begin{aligned}
&\int_\Omega (|H^n(T) - C(T)| \star \tilde{R}(T))\ X(T) d\tilde{Q} \\
&= \sum_i \int_\Omega |H^n_i(T) - C_i(T)| \tilde{R}_i(T)\ X_i(T) \frac{d\tilde{Q}}{dQ} dQ \\
&\leq \sum_i \left(\int_\Omega |H^n_i(T) - C_i(T)|^p\ X_i(T) dQ \right)^{\frac{1}{p}} \left(\int_\Omega (\tilde{R}_i(T) \frac{d\tilde{Q}}{dQ})^q\ X_i(T) dQ \right)^{\frac{1}{q}} .
\end{aligned}$$

This inequality is also valid in the case $q = \infty$ with the convention

$$\int_\Omega (\tilde{R}_i(T) \frac{d\tilde{Q}}{dQ})^\infty X_i(T) dQ := \text{ess sup} \left(\tilde{R}_i(T) \frac{d\tilde{Q}}{dQ} \right) \in \mathbb{R}_+ .$$

The above inequality and the convergence of $H^n(T)$ in L^p_{XQ} imply that $(H^n(T) \star \tilde{R}(T))_{n\in\mathbb{N}}$ converges to C in $L^1_{X\tilde{Q}}$. Because of $(h^n, H^n)_{n\in\mathbb{N}} \in \mathcal{A} \subset \mathcal{A}^{\mathcal{P}}$ we have

$$0 \geq h(\tilde{R}(0) \star X(0)) \geq E_{\tilde{Q}}[H^n(T)(\tilde{R}(T) \star X(T))]$$

and consequently

$$0 \geq \lim_{n\to\infty} E_{\tilde{Q}}[H^n(T)(\tilde{R}(T) \star X(T))] = E_{\tilde{Q}}[C(\tilde{R}(T) \star X(T))] \geq 0 .$$

In account of $C(\tilde{R}(T) \star X(T)) \geq 0$ this results in

$$C(R(T) \star X(T)) = 0 .$$

Hence for almost every ω we have

$$C(\omega)(R(\omega, T) \star X(\omega, T)) = 0 .$$

According to Proposition 2.4 this implies that for almost every ω

$$C(\omega) \in \partial K(\omega, T)$$

and thus $C \notin \mathcal{C}^+$.

◊

Theorem 7.11 *Let $1 \leq p < \infty$, $q = \frac{p}{p-1}$ for $1 < p$, and $q = \infty$ for $p = 1$. Suppose there is no L^p-free lunch in $\mathcal{A}^s \cap \mathcal{A}^{\mathcal{R}} \cap \mathcal{A}^\infty$. Then*

$$\forall Q \in \mathcal{Q} : \mathcal{P}_Q^q \neq \emptyset \ .$$

Proof. Because $\mathcal{A}^{s\mathcal{R}\infty} := \mathcal{A}^s \cap \mathcal{A}^{\mathcal{R}} \cap \mathcal{A}^\infty$ is convex, it is obvious that for any $Q \in \mathcal{Q}$ the set $\overline{((\mathcal{C}^0|\mathcal{A}^{s\mathcal{R}\infty}) \cap L^p_{XQ})}^p_{XQ}$ is a convex cone in $L^p_{XQ}(\mathcal{F}(T))$ containing 0. The set $\mathcal{C}^+ \cap L^p_{XQ}$ is also a convex cone in $L^p_{XQ}(\mathcal{F}(T))$. Fix an arbitrary $Q \in \mathcal{Q}$. According to Proposition 7.5, the absence of L^p-free lunches in $\mathcal{A}^{s\mathcal{R}\infty}$ implies

$$\overline{((\mathcal{C}^0|\mathcal{A}^{s\mathcal{R}\infty}) \cap L^p_{XQ})}^p_{XQ} \cap \mathcal{C}^+ = \emptyset \ .$$

Thus, according to a separating hyperplane theorem, for every $C \in \mathcal{C}^+ \cap L^p_{XQ}$ there exists a $Z^C(T) \in L^q_{XQ}(\mathcal{F}(T))$ satisfying

$$\sup_{B \in (\mathcal{C}^0|\mathcal{A}^{s\mathcal{R}\infty}) \cap L^p_{XQ}} E_Q[B(Z^C(T) \star X(T))] = 0 < E_Q[C(Z^C(T) \star X(T))] \ .$$

From

$$-K(T) \cap L^\infty \subset (\mathcal{C}^0(T)|\mathcal{A}^{s\mathcal{R}\infty})$$

we have $Z^C(T) \in -K'(T)$. In fact, suppose we had $Z^C(T) \notin -K'(T)$. Then in account of the definition of $-K(T)$ there would exist a $B \in K(T)$ such that

$$Q(B(Z^C(T) \star X(T)) < 0) > 0 \ .$$

Defining

$$F := \{B(Z(T) \star X(T)) < 0\} \ ,$$

we had $-B1_F \in -K(T)$ and there would exist a $\mathcal{F}(T)$-measurable $\tilde{F} \subset F$ such that

$$-B1_{\tilde{F}} \in -K(T) \cap L^\infty \subset -K(T) \cap L^p_{XQ} \ .$$

Because of $\tilde{F} \subset F$ this would result in

$$E_Q[-1_{\tilde{F}} B(Z(T) \star X(T))] > 0 \ ,$$

a contradiction to the separating inequality.

From $Z^C(T) \in -K'(T)$ we have $Z_0^C(T) \geq 0$. Moreover, remembering $X_0(T) = 1$ we can renormalize $Z^C(T)$ in order to get $E_Q[Z_0^C(T)] = 1$ and define a Q-dominated probability measure Q^C by

$$\frac{dQ^C}{dQ} = Z_0^C(T) \ .$$

In summary, for every $C \in \mathcal{C}^+ \cap L^p_{XQ}$ there exists a $Z^C(T) \in -K'(T) \cap L^q_{XQ}$ and a P-dominated probability measure Q^C with $\frac{dQ^C}{dP} = Z^C_0(T)$. Define

$$\mathcal{C}^+_0 := \{C \in \mathcal{C}^+ : \forall i \geq 1 : C_i = 0\}$$

and denote $\mathcal{M}$ the set of P-dominated probability measure Q^C, $C \in \mathcal{C}^+_0 \cap L^p_{XQ}$. For every $F \in \mathcal{F}(T)$ with $Q(F) > 0$ there exists a $C \in \mathcal{C}^+_0 \cap L^p_{XQ}$ with $Q^C(F) > 0$. In fact, the claim $C := 1_F(1, 0, \ldots, 0)$ is an element of $\mathcal{C}^+ \cap L^p_{XQ}$ satisfying

$$0 < E_Q[C(Z^C(T) \star X(T))] = E_Q[1_F Z^C_0(T)] = Q^C(F) \ .$$

Thus $(\Omega, \mathcal{F}(T), Q)$ and $\mathcal{M}$ meet the assumptions of the Halmos-Savage Theorem. According to this theorem, there is a countable subfamily $\mathcal{N} \subset \mathcal{M}$ that is equivalent to P. So there exist a sequence $(C^n)_{n \in \mathbb{N}}$ in $\mathcal{C}^+_0 \cap L^p_{XQ}$ and a sequence $(\lambda^n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ with $\forall n : \lambda_n > 0$ such that $\sum_n \lambda_n = 1$ and $Q = \sum_n \lambda^n Q^C_n$ is equivalent with P, i.e.,

$$\sum_n \lambda_n Z^{C_n}_0 > 0 \ .$$

Since for every $n \in \mathbb{N}$ we have $Z^{C_n} \in L^q_{XQ}$, there exists a sequence $(c_n)_{n \in \mathbb{N}}$ in $\mathbb{R}_+$ with

$$\forall n \in \mathbb{N} \forall i \in \{0, \ldots, d\} : 0 < ||Z^{C_n}||_q \leq c_n \ .$$

There, $||.||_q$ denotes the norm in L^q_{XQ} defined by

$$||Y||_q := \sum_i (\int_\Omega |Y_i|^q X dQ)^{\frac{1}{q}} \ .$$

Define

$$\tilde{Z}(T) := \sum_n \frac{\lambda^n}{c_n} Z^n(T)$$

in terms of convergence in the Banach space L^q_{XQ}. Then we have $0 < ||\tilde{Z}(T)||_q \leq 1$ and $\tilde{Z}(T) \in \mathcal{F}(T)$. Moreover, since for every $\omega \in \Omega$ the cone $-K'(\omega, T)$ is closed, it follows by almost sure convergence of a subsequence of $(\sum_{k=1}^n \frac{\lambda^k}{c_k} Z^k(T))_{n \in \mathbb{N}}$, that $\tilde{Z}(T) \in -K'(T)$. In addition, we have $0 < \tilde{Z}_0(T) \in L^q(Q)$ because of $X_0(T) = 1$. Thus, if we define

$$Z(T) := \frac{\tilde{Z}(T)}{E_Q[\tilde{Z}_0(T)]} \ , \qquad \frac{d\tilde{Q}}{dQ} := Z_0(T) \ ,$$

we obtain a P-equivalent probability measure $\tilde{Q}$. The definition of $Z(T)$ also yields

$$\sup_{B\in(\mathcal{C}^0|\mathcal{A}^s\cap\mathcal{A}^{\mathcal{R}}\cap\mathcal{A}^\infty)} E_Q[B(Z(T)\star X(T))] = 0 \ . \tag{7.1}$$

Moreover we have $Z(T)\in L^1_{XQ}$. Thus, according to Theorem 1.42 in Jacod, Shiryaev (1987, p. 11) there exists a uniformly integrable cadlag Q-martingale $Z\star X$ satisfying

$$\forall i\in\{0,\ldots,d\}: P(\forall t\in[0,T:(Z\star X)_i(t)=E_Q[Z_i(T)X_i(T)|\mathcal{F}(t)])=1 \ .$$

Let us define the cadlag process Z by

$$Z_i(t):=\frac{(Z\star X)_i(t)}{X_i(t)} \quad ,\ t\in[0,T),\ i\in\{0,\ldots,d\} \ .$$

The redefinition of $Z(T)$ is clearly irrelevant, because it only affects an evascent set. We want to show now that

$$\forall t\in[0,T]: Z(t)\in -K'(t) \ . \tag{7.2}$$

We already know $Z(T)\in -K'(T)$. For a fixed $s\in[0,T)$ we choose an arbitrary $G(s)\in K(s)$. Then for almost every $\omega\in\Omega$ we have $G(\omega,s)\in K(\omega,s)$. Now we consider an arbitrary nonnegative $\xi\in L^\infty(Q,\mathcal{F}(s))$ and the sequence of buy and hold strategies $(0,H^n)_{n\in\mathbb{N}}$ in $\mathcal{A}^s$ defined by

$$H^n(\omega,t):=-\xi(\omega)G(\omega,s)1_{\{\max_i|G_i(s)|\le n\}}(\omega,t)1_{[s,T]}(\omega,t) \ .$$

We have

$$\forall i\in\{0,\ldots,d\}: |H^n_i(T)|\le ||\xi||_\infty n \quad ,\ n\in\mathbb{N}$$

and consequently $(0,H^n)\in\mathcal{A}^\infty$ for every $n\in\mathbb{N}$. Because of $\forall t\in[0,T]: -dH^n(t)\in K(t)$, we conclude $(0,H^n)\in\mathcal{A}^{s\mathcal{R}\infty}$. Moreover, the strategy $(0,H^n)$ is a super hedge for the claim

$$C^n:=H^n(T)=-\xi G(s)1_{\{\max_i|G_i(s)|\le n\}}(\omega,t) \ .$$

This shows $C^n\in(\mathcal{C}^0|\mathcal{A}^{s\mathcal{R}\infty})$. It is clear that $C^n\in L^\infty_{XQ}(\mathcal{F}(T))$. Consequently, since the measures $X_i(T)Q$, $i=0,\ldots,d$ are finite , equation (7.1) and the tower properties of conditional expectations admit the following calculation for every $n\in\mathbb{N}$

$$\begin{aligned}
0 &\ge E_P[C^n(Z(T)\star X(T)]\ge -E_P[\xi G(s)1_{\{\max_i|G_i(s)|\le n\}}(Z(T)\star X(T))]\\
&= -E_P[E_P[\xi G(s)1_{\{\max_i|G_i(s)|\le n\}}(Z(T)\star X(T))|\mathcal{F}(s)]]\\
&= -E_P[\xi E_P[G(s)1_{\{\max_i|G_i(s)|\le n\}}(Z(T)\star X(T))|\mathcal{F}(s)]]\\
&= -E_P[\xi G(s)1_{\{\max_i|G_i(s)|\le n\}}E_P[(Z(T)\star X(T))|\mathcal{F}(s)]]\\
&= -E_P[\xi G(s)1_{\{\max_i|G_i(s)|\le n\}}(Z(s)\star X(s))] \ .
\end{aligned}$$

Since ξ was arbitrary chosen, it follows $G(s)1_{\{\max_i |G_i(s)| \leq n\}} Z(s) \geq 0$ for every $n \in \mathbb{N}$ and thus $G(s)Z(s) \geq 0$. In order to conclude that $Z(s) \in -K'(s)$, it suffices to show that for almost every ω we have

$$\forall h \in K(\omega, s) : hZ(\omega, s) \geq 0 \ .$$

But since $G(\omega, s) \in K(s)$ was arbitrary chosen and so was s, this is proven now. Thus we have (7.2).

As seen above, the measure $\tilde{Q}$ defined by $\dfrac{d\tilde{Q}}{dQ} = Z_0(T)$ is equivalent to $Q \sim P$ on $\mathcal{F}(T)$. In account of (7.2) the definition $\tilde{R} := \frac{1}{Z_0} Z$ yields a cadlag process satisfying

$$\forall t \in [0, T] : \tilde{R}(t) \in K_0'(t) \ .$$

Assumption (6.2) then implies $R \in \mathcal{R}$. Note also that

$$\frac{d\tilde{Q}}{dQ} R(T) = Z(T) \in L^q_{XQ} \ .$$

The process $Z_0(T)(\tilde{R} \star X)$ is a Q-martingale and thus $\tilde{R} \star X$ is a $\tilde{Q}$-martingale. So finally we have found a $(\tilde{Q}, \tilde{R}) \in \mathcal{P}^q_Q$. Since $Q \in \mathcal{Q}$ was arbitrary chosen, we conclude that for all $Q \in \mathcal{Q}$ we have $\mathcal{P}^q_Q \neq \emptyset$.

♢

Theorem 7.12 (Fundamental Theorem of Asset Pricing) *Let $1 \leq p < \infty$, $q = \frac{p}{p-1}$ for $1 < p$ and $q = \infty$ for $p = 1$. There is no L^p-free lunch in $\mathcal{A}^s \cap \mathcal{A}^{\mathcal{R}} \cap \mathcal{A}^\infty$, if and only if*

(7.3) $$\forall Q \in \mathcal{Q} : \mathcal{P}^q_Q \neq \emptyset \ .$$

Proof. Suppose there is no L^p-free lunch in $\mathcal{A}^s \cap \mathcal{A}^{\mathcal{R}} \cap \mathcal{A}^\infty$. Then according to Theorem 7.11 we have (7.3).

Conversely, assume (7.3) is satisfied. Then in account of Lemma 6.10 Theorem 7.10 imlpies that there is no L^p-free lunch in $\mathcal{A}^s \cap \mathcal{A}^{\mathcal{R}} \cap \mathcal{A}^\infty$.

♢

Theorem 7.13 *Suppose $\mathcal{A} \subset \mathcal{A}^{\mathcal{P}}$. If $\mathcal{P} \neq \emptyset$, then there is no L^∞-bounded free lunch in $\mathcal{A}$.*

Proof. Let $C \in K(T) \cap L^\infty$ be a contingent claim and (h^n, H^n) a sequence in $\mathcal{A}$ such that $-h^n \in K(0)$, $||H^n(T)||_\infty \leq \beta$ for some $\beta > 0$, and $H^n(T)$ converges to C in probability P. Choose an arbitrary $(Q, R) \in \mathcal{P}$. Because of $R(T) \in L^1_{XQ}$, the measure $R_i X_i Q$ is finite for every $i \in \{0, \ldots, d\}$. Hence for every i the uniformly boundedness of the sequence $(H_i^n(T))_{n\in\mathbb{N}}$ and the convergence of $H_i^n(T)$ in probability imply, that $(H^n(T))_{n\in\mathbb{N}}$ converges to C in

$$L^1_{(R\star X)Q} := \times_{i=0,\ldots,d} L^1(R_i(T)X_i(T)Q) \ .$$

Because of $(h^n, H^n) \in \mathcal{A} \subset \mathcal{A}^{\mathcal{P}}$ we have

$$0 \ \geq \ h^n(R(0) \star X(0)) \geq E_Q[H^n(T)(R(T) \star X(T))] \ .$$

This and $C \in K(T) \cap L^\infty$ imply

$$0 \geq \lim_{n\to\infty} E_Q[H^n(T)(R(T) \star X(T))] = E_Q[C(R(T) \star X(T))] \geq 0$$

which in account of $C(R(T) \star X(T)) \geq 0$ results in

$$C(R(T) \star X(T)) = 0 \ .$$

Hence for almost every ω we have

$$C(\omega)(R(\omega, T) \star X(\omega, T) = 0 \ ,$$

which according to Proposition 2.4 implies that for almost every ω

$$C(\omega) \in \partial K(\omega, T)$$

and thus $C \notin \mathcal{C}^+$.

$\diamondsuit$

Theorem 7.14 *Suppose there is no L^∞-bounded free lunch in $\mathcal{A}^s \cap \mathcal{A}^{\mathcal{R}} \cap \mathcal{A}^\infty$ and assume that for every $Q \in \mathcal{Q}$ the space L^1_{XQ} is separable. Then $\mathcal{P} \neq \emptyset$.*

Proof. Because $\mathcal{A}^{s\mathcal{R}\infty} := \mathcal{A}^s \cap \mathcal{A}^{\mathcal{R}} \cap \mathcal{A}^\infty$ is a convex cone in $\mathcal{H}$, it is obvious that for any $Q \in \mathcal{Q}$ the set $\overline{((\mathcal{C}^0|\mathcal{A}^{s\mathcal{R}\infty}) \cap L^\infty)}^\infty_{XQ}$ is a convex cone containing 0. The set $\mathcal{C}^+ \cap L^\infty$ is also a convex cone. Fix an arbitrary $Q \in \mathcal{Q}$. According to proposition 7.7 the absence of an L^∞-bounded free lunch in $\mathcal{A}^s$ implies

$$\overline{((\mathcal{C}^0|\mathcal{A}^{s\mathcal{R}\infty}) \cap L^\infty)}^\infty_{XQ} \cap \mathcal{C}^+ = \emptyset \ .$$

Thus according to a separating hyperplane theorem, for every $C \in \mathcal{C}^+ \cap L^\infty$ there exists a $Z^C(T) \in L^1_{XQ}(\mathcal{F}(T))$ satisfying

$$\sup_{B \in (\mathcal{C}^0 | \mathcal{A}^{s\mathcal{R}\infty}) \cap L^\infty} E_Q[B(Z^C(T) \star X(T))] = 0 < E_Q[C(Z^C(T) \star X(T))] \ .$$

From

$$-K(T) \cap L^\infty \subset (\mathcal{C}^0(T) | \mathcal{A}^{s\mathcal{R}\infty})$$

we have $Z^C(T) \in -K'(T)$. In fact, suppose we had $Z^C(T) \notin -K'(T)$. Then in account of the definition of $-K(T)$ there would exist a $B \in K(T)$ such that

$$Q(B(Z^C(T) \star X(T)) < 0) > 0 \ .$$

Defining

$$F := \{B(Z(T) \star X(T)) < 0\} \ ,$$

we had $-B1_F \in -K(T)$ and there would exist a $\mathcal{F}(T)$-measurable $\tilde{F} \subset F$ such that

$$-B1_{\tilde{F}} \in -K(T) \cap L^\infty \subset -K(T) \cap L^1_{XQ} \ .$$

Because of $\tilde{F} \subset F$ this would result in

$$E_Q[-1_{\tilde{F}} B(Z(T) \star X(T))] > 0 \ ,$$

a contradiction to the separating inequality.

Consequently, we have $Z_0^C(T) \geq 0$. Moreover, remembering $X_0(T) = 1$ we can renormalize $Z^C(T)$ in order to get $E_Q[Z_0^C(T)] = 1$ and define a Q-dominated probability measure Q^C by

$$\frac{dQ^C}{dQ} = Z_0^C(T) \ .$$

In summary, for every $C \in \mathcal{C}^+ \cap L^\infty$ there exists a $Z^C(T) \in -K'(T) \cap L^1_{XQ}$ and a P-dominated probability measure Q^C with $\dfrac{dQ}{dP} = Z_0^C(T)$.

Define

$$\mathcal{C}_0^+ := \{C \in \mathcal{C}^+ : \forall i \geq 1 : C_i = 0\}$$

and denote $\mathcal{M}$ the set of P-dominated probability measures Q^C, $C \in \mathcal{C}_0^+ \cap L^\infty$.

Then for every $F \in \mathcal{F}(T)$ with $Q(F) > 0$ there exists a $C \in \mathcal{C}_0^+ \cap L^\infty$ with $Q^C(F) > 0$. In fact, the claim $C := 1_F(1, 0, \dots, 0)$ is an element of $\mathcal{C}^+ \cap L^\infty$ satisfying

$$0 < E_Q[C(Z^C(T) \star X(T))] = E_Q[1_F Z_0^C(T)] = Q^C(F) \ .$$

Thus $(\Omega, \mathcal{F}(T), Q)$ and $\mathcal{M}$ satisfy the assumptions of the Halmos-Savage Theorem. According to this theorem, there is a countable subfamily $\mathcal{N} \subset \mathcal{M}$ that is equivalent to P. So there exist a sequence $(C^n)_{n\in\mathbb{N}}$ in $\mathcal{C}_0^+ \cap L^\infty$ and a sequence $(\lambda^n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with $\forall n : \lambda_n > 0$ such that $\sum_n \lambda_n = 1$ and $Q = \sum_n \lambda^n Q_n^C$ is equivalent with P, i.e.,

$$\sum_{n\in\mathbb{N}} \lambda_n Z_0^{C_n} > 0 \ .$$

Since for every n we have $Z^{C_n} \in L^1_{XQ}$ there exists a sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ such that

$$\forall n \in \mathbb{N} : \quad 0 < ||Z^{C_n}||_1 \leq c_n \ .$$

There $||.||_1$ denotes the norm in L^1_{XQ}. Define then

$$\tilde{Z}(T) = \sum_n \frac{\lambda^n}{c_n} Z^n(T)$$

in terms of convergence in the Banach space L^1_{XQ}. This yields $0 < ||\tilde{Z}(T)||_1 \leq 1$ and $\tilde{Z}(T) \in \mathcal{F}(T)$. Since for every $\omega \in \Omega$ the cone $-K'(\omega, T)$ is closed, it follows by almost sure convergence of a subsequence of $(\sum_{k=1}^n \frac{\lambda^k}{c_k} Z^k(T))_{n\in\mathbb{N}}$ that $\tilde{Z}(T) \in -K'(T)$. Moreover, we have $0 < \tilde{Z}_0(T) \in L^1(Q)$, because $X_0(T) = 1$. Thus, if we define

$$Z(T) := \frac{\tilde{Z}(T)}{E_Q[\tilde{Z}_0(T)]} \ , \quad \frac{d\tilde{Q}}{dQ} := Z_0(T)$$

we get a P-equivalent probability measure $\tilde{Q}$. Moreover we have

$$\sup_{B\in(\mathcal{C}^0(T)|\mathcal{A}^{s\mathcal{R}\infty})} E_Q[B(Z(T) \star X(T))] = 0 \ . \tag{7.4}$$

Like in the proof of Theorem 7.11 we define a Q-martingale $Z \star X$ and a cadlag process Z. Then the following assertions can be proven by copying the corresponding parts of the proof of Theorem 7.11:

The measure $\tilde{Q}$ is equivalent to $Q \sim P$ on $\mathcal{F}(T)$ and we have

$$\forall t \in [0, T] : Z(t) \in -K'(t) \ ,$$

the process $\tilde{R} := \frac{1}{Z_0} Z$ is an element of $\mathcal{R}$ satisfying

$$\frac{d\tilde{Q}}{dQ} R(T) = Z(T) \in L^1_{XQ} \ .$$

The process $Z_0(T)(\tilde{R} \star X)$ is a Q-martingale and thus $\tilde{R} \star X$ is a $\tilde{Q}$-martingale. So finally we have found a$(\tilde{Q}, \tilde{R}) \in \mathcal{P}$.

$\diamondsuit$

Theorem 7.15 (Fundamental Theorem of Asset Pricing) *Suppose that for every $Q \in \mathcal{Q}$ the space L^1_{XQ} is separable. Then there is no L^∞-bounded free lunch in $\mathcal{A}^s \cap \mathcal{A}^{\mathcal{R}} \cap \mathcal{A}^\infty$, if and only if*

$$\mathcal{P} \neq \emptyset \ .$$

Proof. Suppose there is no L^∞-bounded free lunch in $\mathcal{A}^s \cap \mathcal{A}^{\mathcal{R}} \cap \mathcal{A}^\infty$. Then according to Theorem 7.14 we have (7.3).

Conversely, assume (7.3) is satisfied. Then in account of Lemma 6.10 Theorem 7.13 imlpies that there is no L^∞-bounded free lunch in $\mathcal{A}^s \cap \mathcal{A}^{\mathcal{R}} \cap \mathcal{A}^\infty$.

$\diamondsuit$

The most important conclusions of this chapter are stated by the following

Remarks 7.16 Theorem 7.15 shows that the "no L^∞-bounded free lunch" condition is very well suited for characterizing $\mathcal{P} \neq \emptyset$. In fact, we see that with L^∞-bounded free lunches we only need to consider trading strategies in $\mathcal{A}^s \cap \mathcal{A}^{\mathcal{R}} \cap \mathcal{A}^\infty$. Note that this feature would not change, if we considered trading in $[0, \infty)$ instead of $[0, T]$. Thus, our Theorems 7.15 and 7.12 disprove the conventional wisdom, that the class of "simple processes" is too "thin", in order to state a Fundamental Theorem in continuous time with semimartingale price processes (c.f. Delbaen, Schachermayer, 1994, Shiryaev, 1999, pp. 648). However, this is indeed true, if one considers free lunches with bounded or vanishing risk like in Delbaen, Schachermayer (1994,1998). But in account of our Fundamental Theorems this fact shows that free lunches with bounded or vanishing risk are simply less suitable in order to characterize the existence of an equivalent martingale measure than our notions of free lunch are. This is also confirmed by the fact, that in the case of discontinuous semimartingale price processes Delbaen, Schachermayer (1998) had to introduce the greater class of equivalent σ-martingale measures.

Note finally, what is the reason why the counterexamples of Delbaen, Schachermayer (1994, Example 7.5, Example 7.7) do not apply for our free lunches. They construct a market, where their "no free lunch condition with bounded resp. vanishing risk" is trivially satisfied within the class of simple portfolio processes. The reason therefore is, that their boundedness and risk conditions refer to the value process $H \bullet X$ of a portfolio strategy H. Our boundedness conditions, in contrast, are not imposed on the value process $H \bullet X$ but only on the portfolio process H. This is, why within our framework, we only need to consider simple portfolio processes. It is obvious that this in turn is the very reason for the simplicity of our proofs in contrast to that of Delbaen, Schachermayer (1994,1998), although we even allow for transaction costs.

$\diamondsuit$

Chapter 8

Dual Characterization of Super-Hedging Prices

Assumption 8.1 *Throughout this chapter we always assume $\mathcal{P} \neq \emptyset$. This implies $\mathcal{A}^{\mathcal{P}} \neq \emptyset$. Lemma 6.10, Lemma 6.12, Remark 6.13, Proposition 7.2 and Theorem 7.13 lead us to postulate $\mathcal{A} = \mathcal{A}^{\mathcal{P}}$.*

$\diamondsuit$

Remarks 8.2 The postulation $\mathcal{A} = \mathcal{A}^{\mathcal{P}}$ in Assumption 8.1 does not substantially differ from Kabanov's (1999) assumption on admissible strategies. This assumption is needed, because the supermartingale property excludes so called "doubling strategies", that could yield arbitrage opportunities. Usually these strategies are excluded by "tame conditions" postulating that $H \bullet X$ is bounded from below. However, we find that these tame conditions are too restrictive, because they exclude simple "sell and hold" strategies. Clearly, the usual definitions of admissible trading strategies used troughout literature on (super-) hedging in continuous time always result in value processes, that are super-martingales. In our notation this means $\mathcal{A} \subset \mathcal{A}^{\mathcal{P}}$. Hence by choosing $\mathcal{A} = \mathcal{A}^{\mathcal{P}}$, we certainly impose no severe restrictions on trading.

$\diamondsuit$

Under Assumption 8.1 we are going to derive a dual characterization of the set of "approximate" super-hedging initial endowments for a European claim C. As a result we obtain an expectation representation formula for the approximate super-hedging price for C. This price is defined as the minimum of all $h_0 \in \mathbb{R}$ such that for every $(Q, R) \in \mathcal{P}$ there exists an arbitrary good $L^1_{(Q,R)}$-approximation of C that is super-hedgeable with initial capital h_0 (for the Definition of $L^1_{(Q,R)}$ see (8.4)). The family $\mathcal{P}$ plays the same role as the martingale measures in the frictionless market models. The consideration of approximate instead of exact super-hedging makes it possible to derive the results under the only two conditions that the price process is a semimartingale and that $\mathcal{P}$ is not empty. Moreover, the expectation representation formula is a direct generalization of the expectation representation formula known for frictionless incomplete markets. However, we could not get any analogue to the optional decomposition theorem.

Our aim is the same as that of Kabanov (1999) who derives a dual characterization of super-hedging initial endowments for contingent claims in a financial market with proportional transaction costs where prices are given by semimartingales. We also have drawn some inspirations from the papers of Jouini (1995), Cvitanić and Karatzas(1996).

Dual characterizations of convex optimization problems are usually obtained by the use of separating hyperplane theorems for two disjoint closed convex sets. In Kabanov (1999) the first set consists of the claim C to be priced and the second set consists of all claims that are super-hedgeable with a certain initial endowment not sufficient to super-hedge C. Since the second set will not be closed with respect to $L^1_{(Q,R)}$ in general, Kabanov looks for sufficient conditions for the market that assure the closedness of this set. This conditions cover some models with continuous price processes but do not extend to point process models for example.

Our strategy in contrast is to look for a "weaker" definition of super-hedging prices such that these prices still have the nice property to be invariant against equivalent changes of probabilities while they are still calculable with the use of a dual characterization. This all should be done in more general market situations than those in Kabanov (1999). From the paper of Jouini, Kallal (1995) we got the idea to introduce approximate super-hedging prices. We have extended this concept in such a way, that we can define an approximate super-hedging price for a larger class of claims than Jouini, Kallal (1995). This new price concept enables us to derive a dual characterization of super-hedging prices similar to that in Cvitanić, Karatzas (1996), but in more general situations.

In currency markets it is more natural to consider super-hedging bundles instead of super-hedging prices. Therefore, we introduce approximate super-hedging initial endowments and obtain a similar dual characterization result as Kabanov (1999).

However, our result comprises a greater class of claims and is valuable without additional conditions on price processes. The dual characterization for super-hedging prices is then obtained by applying a functional operation on the set of approximate super-hedging bundles.

Definition 8.3 *We call a claim $C \in L^0(\mathcal{F}(\mathcal{T}))^{d+1}$ marketable, if both the claim C and $-C$ are super-hedgeable and denote by $\mathcal{C}^m$ the class of marketable claims.*

$\diamondsuit$

From the dual characterization of $K(T)$ by $K_0'(T)$ we know, that if (h, H) is a super hedge for C, then for every $R(T) \in K_0'(T)$ we have

$$H(T)(R(T) \star X(T)) \geq C(R(T) \star X(T)) \ . \tag{8.1}$$

Thus, if C is super-hedgeable by some $(h, H) \in \mathcal{A}$, then according to the definition of $\mathcal{A} = \mathcal{A}^{\mathcal{P}}$, for every $(Q, R) \in \mathcal{P}$ we have

$$E_Q[C(R(T) \star X(T))] \leq E_Q[H(T)(R(T) \star X(T))] \leq h(R(0) \star X(0)) < \infty \ . \tag{8.2}$$

This implies

$$\forall C \in \mathcal{C}^m \ \forall (Q, R) \in \mathcal{P}: \quad -\infty < E_Q[\sum_i C_i R_i(T) X_i(T)] < \infty \ . \tag{8.3}$$

For $(Q, R) \in \mathcal{P}$ let us define the measures $R_i(T)X_i(T)Q$, $i = 0, \dots, d$, by

$$\frac{dR_i(T)X_i(T)Q}{dP} := R_i(T)X_i(T)\frac{dQ}{dP}$$

and write simply

$$L^1_{(Q,R)} := \times_{i=0,\dots,d} L^1(R_i(T)X_i(T)Q) \ . \tag{8.4}$$

Remark 8.4 By (8.3) every marketable cash settlement claim is an element of $\bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$.

$\diamondsuit$

Unlike Kabanov (1999) we do not focus on

$$\Gamma(C) := \{h \in \mathbb{R}^{d+1} : \exists (h, H) \in \mathcal{A} : H(T) - C \in K(T)\}$$

the set of super-hedging initial endowments for a claim C. Instead we examine approximate super-hedging initial endowments. For $(Q,R) \in \mathcal{P}$ and $C \in L^1_{(Q,R)}$ the set of $L^1_{(Q,R)}$-approximate super-hedging initial endowments is defined by

$$\hat{\Gamma}_{(Q,R)}(C) := \{h \in \mathbb{R}^{d+1} : \exists (C^n, h^n)_{n\in\mathbb{N}} : h^n \in \Gamma(C^n), C^n \in L^1_{(Q,R)}, C^n \stackrel{L^1_{(Q,R)}}{\longrightarrow} C, h^n \to h\} \,.$$

One reason for the consideration of $\hat{\Gamma}_{(Q,R)}(C)$ is that, as will be seen later, $\hat{\Gamma}_{(Q,R)} : C \mapsto \hat{\Gamma}_{(Q,R)}(C)$ is a $L^1_{(Q,R)}$-closed multifunction from $L^1_{(Q,R)}$ to $\mathbb{R}^{d+1}$ (see the appendix on multifunctions for definitions). The multifunction $\Gamma(C) : C \mapsto \Gamma(C)$ is neither closed-valued nor $L^1_{(Q,R)}$-closed, in general.

For each $(Q,R) \in \mathcal{P}$ we also define the multifunction $D_{(Q,R)} : C \mapsto$ from $L^1_{(Q,R)}$ to $\mathbb{R}^{d+1}$ by

$$D_{(Q,R)}(C) = \{h \in \mathbb{R}^{d+1} : E_Q[C(R(T) \star X(T))] \le h(R(0) \star X(0))\} \,.$$

Finally, the multifunctions $\hat{\Gamma} : C \mapsto \hat{\Gamma}(C)$ and $D : C \mapsto D(C)$ from $\bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$ to $\mathbb{R}^{d+1}$ are defined by

$$\hat{\Gamma}(C) := \bigcap_{(Q,R)\in\mathcal{P}} \hat{\Gamma}_{(Q,R)}(C)$$
$$D(C) = \bigcap_{(Q,R)\in\mathcal{P}} D_{(Q,R)}(C) \,.$$

Remarks 8.5 We have $\Gamma(C) \neq \emptyset$, if and only if C is super-hedgeable.

The inclusion $\Gamma(C) \subseteq \hat{\Gamma}_{(Q,R)}(C)$ holds for every $C \in L^1_{(Q,R)}$, $(Q,R) \in \mathcal{P}$. In fact, for $h \in \Gamma(C)$ we can define $(C^n, h^n) = (C, h)$, $n \in \mathbb{N}$, whereas for $\Gamma(C) = \emptyset$ the inclusion is trivial.

$\diamondsuit$

Our aim now is to show

$$\forall C \in \bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)} : \quad \hat{\Gamma}(C) = D(C) \,.$$

We will proceed as follows. Starting from the obvious inclusion $\Gamma \subseteq D \subseteq D_{(Q,R)}$, that holds for all $(Q,R) \in \mathcal{P}$, we show

$$\forall (Q,R) \in \mathcal{P} : \hat{\Gamma}_{(Q,R)} \subseteq D_{(Q,R)} \,.$$

This is done by demonstrating that $\hat{\Gamma}_{(Q,R)}$ is the smallest $L^1_{(Q,R)}$-closed multifunction containing Γ which implies then $\hat{\Gamma} \subseteq D$. The inclusion $D \subseteq \hat{\Gamma}$ is obtained by the inclusions $D \subseteq \hat{\Gamma}_{(Q,R)}$ which follow from a separating hyperplane theorem applied to the disjoint sets $\{C-h\}, h \notin \hat{\Gamma}_{(Q,R)}(C)$ and $\hat{\Gamma}^{-1}_{(Q,R)}(0)$.

Remark 8.6 For $(Q,R) \in \mathcal{P}$ we have $\Gamma \subseteq D \subseteq D_{(Q,R)}$ on $L^1_{(Q,R)}$. In fact, for $h \in \Gamma(C)$ there exists a super hedge (h,H) for C such that h is the corresponding initial endowment. Then everything follows from (8.2). The case $\Gamma(C) = \emptyset$ is trivial.

$\diamondsuit$

Lemma 8.7 *For every $(Q,R) \in \mathcal{P}$ the multifunction $\hat{\Gamma}_{(Q,R)}$ is the smallest $L^1_{(Q,R)}$-closed multifunction containing Γ.*

Proof. i) By Proposition B.2 the multifunction $\hat{\Gamma}_{(Q,R)}$ is the $L^1_{(Q,R)}$-closure of Γ and thus $L^1_{(Q,R)}$-closed.

ii) For every $L^1_{(Q,R)}$-closed multifunction F containing Γ we have $\hat{\Gamma}_{(Q,R)} \subseteq F$:

In fact, if for some $C \in$ we have $\hat{\Gamma}_{(Q,R)}(C) = \emptyset$, then according to Remark 8.5 this results in $\Gamma(C) = \emptyset$ and there is nothing to show for this C.

In the case $\Gamma(C) \neq \emptyset$, choose an arbitrary $h \in \hat{\Gamma}_{(Q,R)}(C)$. Then there exists a sequence $(C^n, h^n)_{n\in\mathbb{N}}$ such that

$$h^n \in \Gamma(C^n) \subseteq F(C^n) \ , \quad C^n \overset{L^1_{(Q,R)}}{\longrightarrow} C \ , \quad h^n \to h \ .$$

The closedness of F and Remark B.3 then result in $h \in F(C)$.

$\diamondsuit$

Remark 8.8 From the $L^1_{(\bar{Q},\bar{R})}$-closedness of $\hat{\Gamma}_{(Q,R)}$ and Remark B.3 we conclude that $\hat{\Gamma}_{(Q,R)}$ and $\hat{\Gamma}^{-1}_{(Q,R)}$ are closed-valued.

Lemma 8.9 *We have $\hat{\Gamma} \subseteq D$.*

Proof. It is easy to see that $D_{(Q,R)}$ is $L^1_{(Q,R)}$-closed. Moreover we have $\Gamma \subseteq D_{(Q,R)}$, as was seen in Remark 8.6 . Thus, by Lemma 8.7 we conclude

$$\forall (Q,R) \in \mathcal{P}: \quad \hat{\Gamma}_{(Q,R)} \subseteq D_{(Q,R)} \ .$$

In sum we have

$$\hat{\Gamma} = \bigcap_{(Q,R)\in\mathcal{P}} \hat{\Gamma}_{(Q,R)} \subseteq \bigcap_{(Q,R)\in\mathcal{P}} D_{(Q,R)} = D \ .$$

$\diamondsuit$

Theorem 8.10 *We have* $D \subseteq \hat{\Gamma}$ *on* $\bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$.

Proof. Throughout this proof suppose $C \in \bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$.

i) First, we consider the case $\hat{\Gamma}(C) \neq \emptyset$ and prove that for every $(Q,R) \in \mathcal{P}$ we have $D(C) \subseteq \hat{\Gamma}_{(Q,R)}(C)$. For the case $\hat{\Gamma}(C) = \emptyset$ see ii).

For a fixed but arbitrary $(\bar{Q},\bar{R}) \in \mathcal{P}$ let us choose an arbitrary $h \notin \hat{\Gamma}_{(\bar{Q},\bar{R})}(C)$. We are going to show

$$h \notin D(C) = \bigcap_{(Q,R)\in\mathcal{P}} D_{(Q,R)}(C)$$

by specifying a $(\tilde{Q},\tilde{R}) \in \mathcal{P}$ satisfying

$$E_{\tilde{Q}}[C(\tilde{R}(T) \star X(T))] > h(\tilde{R}(0) \star X(0)) \ .$$

Since $h \notin \hat{\Gamma}_{(\bar{Q},\bar{R})}(C)$ we have $C \notin \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(h)$. This is equivalent to $C - h \notin \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)$. According to Remark 8.8 the convex set $\hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(x)$ is $L^1_{(\bar{Q},\bar{R})}$-closed for every $x \in \mathbb{R}^{d+1}$. Hence $\{(C-h)\}$ and $\hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)$ are strictly separated by some $\rho = (\rho_0,\ldots,\rho_d) \in (L^\infty(P))^{d+1}$. This means there exists a $\rho = (\rho_0,\ldots,\rho_d) \in (L^\infty(P))^{d+1}$ such that

$$\text{(8.5)} \qquad \sup_{V\in\hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)} E_{\bar{Q}}[(V(\bar{R}(T) \star X(T) \star \rho)] < E_{\bar{Q}}[(C-h)(\bar{R}(T) \star X(T) \star \rho)] \ .$$

Since $V \in \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)$ is equivalent with $0 \in \hat{\Gamma}_{(\bar{Q},\bar{R})}(V)$, the inclusion $\hat{\Gamma}_{(\bar{Q},\bar{R})}(V) \subseteq D_{(\bar{Q},\bar{R})}(V)$ results in

$$\sup_{V\in\hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)} E_{\bar{Q}}[V(T)(\bar{R}(T) \star X(T) \star \rho(T))] \leq 0 \ .$$

Because the claim $V \equiv 0$ obviously is an element of $\hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)$, we actually have equality. From

$$\forall \omega \in \Omega : \mathbb{R}^{d+1}_+ \subset K(\omega,T)$$

and $-K(T) \subset \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)$ we see that

$$\forall i \in \{0,\ldots,d\} : \rho_i \geq 0$$

is a necessary condition for (8.5) to hold.

If we define

$$Z(T) := R(T) \star \rho$$

then we have $Z(T) \star X(T) \in L^1_{\bar{Q}}$. According to Theorem 1.42 in Jacod, Shiryaev (1987, p. 11) there exists a uniformly integrable cadlag $\bar{Q}$-martingale $Z \star X$ satisfying

$$\forall i \in \{0,\ldots,d\} : P(\forall t \in [0,T] : (Z \star X)_i(t) = E_Q[Z_i(T)X_i(T)|\mathcal{F}(t)]) = 1$$

Let us define the cadlag process Z by

$$Z_i(t) := \frac{(Z \star X)_i(t)}{X_i(t)} \quad , \ t \in [0,T], \ i \in \{0,\ldots,d\} \ .$$

The redefinition of $Z(T)$ is clearly irrelevant, because it only affects an evascent set. The separation inequality (8.5) now is equivalent to

$$(8.6) \quad 0 = \sup_{V \in \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)} E_P[V(T)(Z(T) \star X(T))] < E_P[(C-h)(Z(T) \star X(T))] \ .$$

Note also that in account of

$$\sup_{V \in \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(h)} E_{\bar{Q}}[(V-h)(\bar{R}(T) \star X(T) \star \rho)] = \sup_{V \in \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)} E_{\bar{Q}}[(V(\bar{R}(T) \star X(T) \star \rho)]$$

the inequality (8.5) is equivalent to

$$(8.7) \qquad \sup_{V \in \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(h)} E_P[V(T)(Z(T) \star X(T))] < E_P[C(Z(T) \star X(T))] \ .$$

We want to show

$$(8.8) \qquad \forall t \in [0,T] : Z(t) \in -K'(t) \ .$$

For a fixed $s \in [0,T]$ we take an arbitrary $G(s) \in K(s)$. Then for almost every $\omega \in \Omega$ we have $G(\omega,s) \in K(\omega,s)$. Now we consider an arbitrary nonnegative $\xi \in L^\infty(Q,\mathcal{F}(s))$ and the sequence of buy and hold strategies $(0,H^n)_{n\in\mathbb{N}}$ in $\mathcal{A}^s$ defined by

$$H^n(\omega,t) := -\xi(\omega)G(\omega,s)1_{\{\max_i |G_i(s)| \leq n\}}(\omega,t)1_{[s,T]}(\omega,t) \quad , \ t \in [0,T]$$

in the case $s < T$ and

$$H^n(\omega, t) := 0 \quad , \ t \in [0,T]$$

in the case $s = T$. We have

$$\forall i \in \{0,\dots,d\} : |H_i^n(T)| \leq ||\xi||_\infty n \quad , \ n \in \mathbb{N}$$

and consequently $(0, H^n) \in \mathcal{A}^\infty$ for every $n \in \mathbb{N}$. Because of

$$\forall t \in [0,T] : -dH^n(t) \in K(t) \ ,$$

we conclude

$$(0, H^n) \in \mathcal{A}^s \cap \mathcal{A}^\mathcal{R} \cap \mathcal{A}^\infty \subset \mathcal{A}^\mathcal{P} = \mathcal{A} \ .$$

Moreover, the strategy $(0, H^n)$ is a super hedge for the claim

$$V^n := -\xi G(s) 1_{\{\max_i |G_i(s)| \leq n\}}(\omega, T) \ .$$

This is true because in the case $s < T$ we have $C^n = H^n(T)$ and in the case $s = T$ we have

$$H^n(T) - V^n = \xi G(T) 1_{\{\max_i |G_i(T)| \leq n\}} \in K(T) \ .$$

This shows $0 \in \Gamma(V^n)$. Moreover, it is clear that $V^n \in L^\infty_{(Q,R)}(\mathcal{F}(T)) \subset L^1_{(Q,R)}(\mathcal{F}(T))$. So we have indeed $V^n \in \hat{\Gamma}^{-1}_{(\bar{Q},\bar{R})}(0)$. Consequently, inequality (8.6) and the tower properties of conditional expectations admit the following calculation for every $n \in \mathbb{N}$

$$\begin{aligned}
0 &\geq E_P[V^n(Z(T) \star X(T)] \geq -E_P[\xi G(s) 1_{\{\max_i |G_i(s)| \leq n\}}(Z(T) \star X(T))] \\
&= -E_P[E_P[\xi G(s) 1_{\{\max_i |G_i(s)| \leq n\}}(Z(T) \star X(T))|\mathcal{F}(s)]] \\
&= -E_P[\xi E_P[G(s) 1_{\{\max_i |G_i(s)| \leq n\}}(Z(T) \star X(T))|\mathcal{F}(s)]] \\
&= -E_P[\xi G(s) 1_{\{\max_i |G_i(s)| \leq n\}} E_P[(Z(T) \star X(T))|\mathcal{F}(s)]] \\
&= -E_P[\xi G(s) 1_{\{\max_i |G_i(s)| \leq n\}}(Z(s) \star X(s))] \ .
\end{aligned}$$

Since ξ was arbitrary chosen, it follows $G(s) 1_{\{\max_i |G_i(s)| \leq n\}} Z(s) \geq 0$ for every $n \in \mathbb{N}$ and thus $G(s)Z(s) \geq 0$. In order to conclude that $Z(s) \in -K'(s)$, it suffices to show that for almost every ω we have

$$\forall h \in K(\omega, s) : hZ(\omega, s) \geq 0 \ .$$

But since $G(s) \in K(s)$ was arbitrary chosen and so was s, this is proven now. Thus (8.8) is true.

The problem now is that we do not know, wether $Z_0(T) > 0$. For the construction of a pair $(\tilde{Q}, \tilde{R}) \in \mathcal{P}$ choose an arbitrary $(Q, R) \in \mathcal{P}$ and define the process Z^ε by

$$Z^\varepsilon(t) := (1-\varepsilon)Z(t) + \varepsilon E_P[\frac{dQ}{dP}|\mathcal{F}(t)]R(t) \quad , \ t \in [0,T].$$

There we take a cadlag version of the process $(E_P[\frac{dQ}{dP}|\mathcal{F}(t)])_{t\in[0,T]}$ in order to make Z^ε a cadlag process. Since for every $t \in \mathcal{T}$ $-K'(t)$ is a convex cone and because of $E_P[\frac{dQ}{dP}|\mathcal{F}(t)]R(t) \in -K'(t)$, we conclude $\tilde{Z}(t) \in -K'(t)$ for every $t \in \mathcal{T}$. Moreover from $Q \sim P$ we have $Z_0^\varepsilon(T) > 0$. For $(Q^\varepsilon, R^\varepsilon)$ defined by

$$\frac{dQ^\varepsilon}{dP} := Z_0^\varepsilon(T)$$

$$R_i^\varepsilon(t) := \frac{Z_i^\varepsilon(t)}{Z_0^\varepsilon}(t) \quad , \ t \in \mathcal{T},$$

we have $R_i^\varepsilon(t) \in K_0'(t)$ for every $t \in \mathcal{T}$. Of course, the process $R^\varepsilon = (R_i^\varepsilon(t))_{t\in\mathcal{T}}$ is also $\mathbb{F}$-adapted and cadlag. According to Assumption 6.2 this implies then $(Q^\varepsilon, R^\varepsilon) \in \mathcal{P}$ for every $\varepsilon > 0$. Because of

$$\begin{aligned} \sup_{V\in\hat{\Gamma}^{-1}_{(\tilde{Q},\tilde{R})}(h)} E_P[V(T)(Z^\varepsilon(T)\star X(T))] \ \leq \ & (1-\varepsilon) \sup_{V\in\hat{\Gamma}^{-1}_{(\tilde{Q},\tilde{R})}(h)} E_P[V(T)(Z(T)\star X(T))] \\ & +\varepsilon \sup_{V\in\hat{\Gamma}^{-1}_{(\tilde{Q},\tilde{R})}(h)} E_P[V(T)(R(T)\star \frac{dQ}{dP}X(T))] \end{aligned}$$

there exists a $\varepsilon > 0$ such that the separating inequality (8.7) is satisfied with Z^ε instead of Z. We define $(\tilde{Q}, \tilde{R}) := (Q^\varepsilon, R^\varepsilon) \in \mathcal{P}$. Applying (8.7) to the claims C and $V := h$ in account of $V \in \hat{\Gamma}^{-1}_{(\tilde{Q},\tilde{R})}(h)$, we finally get

$$h(\tilde{R}(0)\star X(0)) = E_{\tilde{Q}}[h(\tilde{R}(T)\star X(T))] < E_{\tilde{Q}}[C(\tilde{R}(T)\star X(T))] \ .$$

ii) For the case $\hat{\Gamma}(C) = \emptyset$ we prove $D(C) = \emptyset$: Assume $D(C) \neq \emptyset$. Then for $h \in \mathbb{R}^{d+1} \setminus \hat{\Gamma}(C) = \mathbb{R}^{d+1}$ there exists a $(Q, R) \in \mathcal{P}$ such that $h \notin \hat{\Gamma}_{(Q,R)}(C)$. By copying part i) of this proof, with the only difference that maybe $\hat{\Gamma}_{(Q,R)}(C) = \emptyset$ which does not matter, one can verify that there exists a $(\tilde{Q}, \tilde{R}) \in \mathcal{P}$ such that

$$E_{\tilde{Q}}[C(\tilde{R}(T)\star X(T)] > h(\tilde{R}(0)\star X(0)) \ .$$

Since $\tilde{R}(0) \in K_0'(0) \subset (L^0(\mathcal{F}(0)))^{d+1}$ and $K_0'(\omega, t) \subset \{1\} \times \mathbb{R}_+^{d+1}$ (see **Definitions and Assumptions** 2.1) we almost surely have $R(0) = 1$. Since $h \in \mathbb{R}^{d+1}$ was arbitrary chosen, we may define a sequence $(h^n)_{h\in\mathbb{N}}$ by $h^n = (n, 0, \ldots, 0) \in \mathbb{R}^{d+1}$ and conclude that for every n there is a $(\tilde{Q}^n, \tilde{R}^n) \in \mathcal{P}$ such that almost surely

$$E_{\tilde{Q}^n}[C(\tilde{R}^n(T) \star X(T)] > h^n(\tilde{R}^n(0) \star X(0)) = n\tilde{R}^n_0(0)X_0(0) = n \ .$$

This obviously implies

$$\sup_{(Q,R)\in\mathcal{P}} E_Q[C(R(T) \star X(T))] = \infty \ .$$

and thus $D(C) = \emptyset$ which is a contradiction.

◇

From Lemma 8.9 and Theorem 8.10 we deduce

Theorem 8.11 *Under Assumption 8.1 we have the following dual characterization of super-hedging initial endowments:*

$$\forall C \in \bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)} : \quad \hat{\Gamma}(C) = D(C) \ .$$

Moreover, a claim $C \in \bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$ *is approximately super-hedgeable, if and only if*

$$\sup_{(Q,R)\in\mathcal{P}} E_Q[C(R(T) \star X(T))] < \infty \ .$$

◇

From the equality of sets in Theorem 8.11 that gives a dual characterization of super-hedging initial endowments, it is possible to derive a dual characterization of super-hedging prices. For $C \in \bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$ we define

$$\Pi(C, h_1, \ldots, h_d) := \inf\{h_0 : h = (h_0, h_1, \ldots, h_d) \in \hat{\Gamma}(C)\}$$

and understand $\Pi(C) := \Pi(C, 0, \ldots, 0)$ as the approximate super-hedging price for C inasmuch as in the case $-\infty < \Pi(C) < \infty$ it is equal to the minimum initial amount of money needed for super-hedging C approximately. In fact, if $-\infty < \Pi(C) < \infty$, then, because $\hat{\Gamma}(C)$ is closed, we have

$$\Pi(C, h_1, \ldots, h_d) := \min\{h_0 : h = (h_0, h_1, \ldots, h_d) \in \hat{\Gamma}(C)\} \ .$$

By contrast, the exact super-hedging price for C is given by $\tilde{\Pi}(C, 0, \ldots, 0)$ where

$$\tilde{\Pi}(C, h_1, \ldots, h_d) := \inf\{h_0 : h = (h_0, h_1, \ldots, h_d) \in \Gamma(C)\} \; .$$

From the convexity of $\hat{\Gamma}(C)$ it follows by Theorem 5.3 in Rockafellar (1970) that Π is convex in $h_1, \ldots, h_d$. It can be shown that Π is also convex in C. From the definition of Π and remembering that for every $(Q, R) \in \mathcal{P}$ the process $R \star X$ is a Q-martingale, we get

Theorem 8.12 *The following dual characterization for approximate super-hedging prices holds for all* $C \in \bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$:

$$\begin{aligned}
\Pi(C, h_1, \ldots, h_d) &= \inf\{h_0 : h = (h_0, \ldots, h_d) \in D(C) = \bigcap_{(Q,R)\in\mathcal{P}} D_{(Q,R)}(C)\} \\
&= \inf\{h_0 : h = (h_0, h_1, \ldots, h_d) \in \mathbb{R}^{d+1} : \\
&\qquad \sup_{(Q,R)\in\mathcal{P}} E_Q[C(R(T) \star X(T))] - h(R(0) \star X(0)) \leq h_0\} \\
&= \sup_{(Q,R)\in\mathcal{P}} E_Q[C_0 + \sum_{i=1}^{d} (C_i - h_i) R_i(T) X_i(T)] \; .
\end{aligned}$$

$\diamondsuit$

The following proposition justifies Π as an approximate super-hedging price that is defined similar to the free lunch price in Jouini, Kallal (1995) and Jouini (1997). Note however, that Jouini does not work in a semimartingale framework. He only allows for "simple" trading strategies, that are piecewise constant.

Proposition 8.13 *For all* $C \in \bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$ *we have*

$$\begin{aligned}
\Pi(C, h_1, \ldots, h_d) &= \inf\{x \in \mathbb{R} : \forall (Q, R) \in \mathcal{P} \; \exists (C^n, h^n) : \; \forall i \in \{1, \ldots d\} : \; h_i = h_i^n \; , \\
&\qquad h^n \in \Gamma(C^n), \; C^n \overset{L^1_{(Q,R)}}{\longrightarrow} C, \; x = \liminf_n h_0^n, \} \; .
\end{aligned}$$

Proof. i) "$\geq$": The case $\Pi(C, h_1, \ldots, h_d) = \infty$ is trivial. Suppose $\Pi(C, h_1, \ldots, h_d) < \infty$. Then by definition of Π there exists a $h_0 \geq \Pi(C, h_1, \ldots, h_d)$ such that $h = (h_0, \ldots, h_d) \in \hat{\Gamma}(C)$. This is equivalent to

$$\forall (Q, R) \in \mathcal{P} \; \exists (\tilde{C}^n, h^n), h^n \in \Gamma(\tilde{C}^n) : \quad \tilde{C}^n \overset{L^1_{(Q,R)}}{\longrightarrow} C, h^n \to h \; .$$

Defining the sequence $(C^n)_{n\in\mathbb{N}}$ by $C^n := \tilde{C}^n + h - h^n$, we have $C^n \overset{L^1_{(Q,R)}}{\longrightarrow} C$ and $h \in \Gamma(C^n)$ for every $n \in \mathbb{N}$. This shows that h_0 belongs to the set on the right side of the equality in question. Since this holds for all $h_0 \in \mathbb{R}^{d+1}$ with $h_0 \geq \Pi(C, h_1, \ldots, h_d)$ we are done.

ii) "$\leq$": Again the case $\inf\{\ldots\} = \infty$ is trivial. For the other case, suppose there exists an element h of the set on the right side of the equality such that $\Pi(C, h_1, \ldots, h_d) > h_0$ and let $(h^n)_{n\in\mathbb{N}}$ the corresponding sequence. Then there exists a subsequence of $(h^n)_{n\in\mathbb{N}}$ converging to h. Thus we have $h \in \hat{\Gamma}(C)$, which is a contradiction.

$\diamondsuit$

Remark 8.14 The proof of Proposition 8.13 shows that in the case $-\infty < \Pi(C, h_1, \ldots, h_d) < \infty$ we can simplify Π to

$$\begin{aligned}\Pi(C, h_1, \ldots, h_d) \quad = \quad & \min\{h_0 \in \mathbb{R} : \forall (Q,R) \in \mathcal{P} \exists C^n : \\ & h = (h_0, h_1, \ldots, h_d) \in \Gamma(C^n), C^n \overset{L^1_{(Q,R)}}{\longrightarrow} C\} \ .\end{aligned}$$

This means the approximate super-hedging price $\Pi(C)$ for a claim C is the minimum over all h_0 such that for every $(Q,R) \in \mathcal{P}$ there exists an arbitrary good $L^1_{(Q,R)}$-approximation of C that is hedgeable with initial capital h_0. Or in other words, if we are given the initial portfolio $h = (\Pi(C, h_1, \ldots, h_d), h_1, \ldots, h_d)$, then for any $(Q,R) \in \mathcal{P}$ we can find a sequence of portfolio processes $(h, H^n)_{n\in\mathbb{N}}$ and a sequence of claims $(C^n)_{n\in\mathbb{N}}$ converging to C in $L^1_{(Q,R)}$ such that

$$\forall n \in \mathbb{N} : H^n(T) - C^n(T) \in K(T) \ .$$

$\diamondsuit$

Part IV

Application

Chapter 9

Quantile-Hedging

Quantile-hedging was introduced by Föllmer, Leukert (1999) who deal with an incomplete, but frictionless continuous time market with a riskless bond and a risky security. For given $0 < \alpha \leq 1$, the α-quantile-hedging price of a contingent claim C is the minimum amount of initial capital needed in order to super-hedge the claim with probability α. In particular, a 1-quantile hedge is nothing else than a super hedge. Föllmer, Leukert (1999) derive a dual characterization of α-quantile-hedging prices. They find that for every claim C with non-negative payoff at maturity and given $0 < \alpha \leq 1$ there exists a set $A \in \mathcal{F}(T)$ with $P(A) \geq \alpha$ such that the α-quantile-hedging price of C equals the super-hedging price of the knock-out claim $C1_A$. Thus the dual characterization of the α-quantile-hedging price is obtained from the dual characterization of the super-hedging price for $C1_A$. However, except for special cases one does not know how to choose the set A. Moreover, Föllmer, Leukert (1999) approach also the question, how to maximize the probability of a succesfull super hedge, if one has given a certain amount of initial capital.

Our aim in this chapter is to generalize the results on quantile-hedging of Föllmer, Leukert (1999) to our general multi-asset framework with differential cone constraints. We will focus on the dual characterization of quantile-hedging-prices for claims with non-negative cash value. This may give a hint, how to generalize the corresponding results on maximizing the probability of a succesfull super hedge.

Since we are going to apply our results on the dual characterization of super-hedging prices, we will work in the framwork of Chapter 8 including Assumption 8.1. Of course, analogues results will also hold for discrete time markets as considered in Chapter 5.

For similar reasons as Föllmer, Leukert (1999) we only deal with claims $C \in K(T)$. According to the interpretation of $K(T)$, such claims have a non-negative cash-value at maturity. For cash-settlement claims this is equivalent to returning a non-negative payoff at maturity. If a claim does not have a non-negative cash-value it might be possible to shift it in such a way, that the condition is satisfied for the shifted claim. As we often will be concerned with sets of the form

$$\{\omega \in \Omega : H(\omega, T) - C(\omega) \in K(\omega, T)\}$$

we start with a remark on the measurability of such sets.

Remark 9.1 Suppose $H(t)$ is a $\mathcal{F}(t)$-measurable $\mathbb{R}^{d+1}$-valued random vector for $t \in \mathcal{T}$. Then the set $\{\omega \in \Omega : H(\omega, t) \in K(\omega, t)\}$ is $\mathcal{F}(t)$-measurable.

In fact, according to **Definitions and Assumptions 2.1** and Remark 2.3, the multifunction $\omega \mapsto K(\omega, t)$ is closed-valued and $\mathcal{F}(t)$-measurable. Since the multifunction $\omega \mapsto \{H(\omega, t)\}$ is also closed-valued and $\mathcal{F}(t)$-measurable, Theorem 1M in Rockafellar (1976) implies that the intersection

$$\{H(., t)\} \cap K(., t) : \omega \mapsto \{H(\omega, t)\} \cap K(\omega, t)$$

is closed valued and $\mathcal{F}(t)$-measurable. Thus the set

$$\begin{aligned} \{\omega \in \Omega : H(\omega, t) \in K(\omega, t)\} &= \{\omega \in \Omega : (\{H(., t)\} \cap K(., t))(\omega) \neq \emptyset\} \\ &= (\{H(t)\} \cap K(t))^{-1}(\mathbb{R}^{d+1}) , \end{aligned}$$

is $\mathcal{F}(t)$-measurable.

♢

Definitions 9.2 *Given a contingent claim $C \in K(T)$ and a probability level $0 < \alpha \leq 1$, we call $(h, H) \in \mathcal{A}$ an α-quantile hedge for C, if (h, H) satisfies the following two conditions*

$$P(H(., T) - C(.) \in K(., T)) \geq \alpha \tag{9.1}$$

$$H(T) \in K(T) \; . \tag{9.2}$$

A claim $C \in K(T)$ is called α-quantile-hedgeable (by (h, H)), if there exists an α-quantile hedge (h, H) for C.

♢

Condition (9.1) states that (h, H) super-hedges the claim C with a probability greater or equal to α while (9.2) is a kind of tame or solvency condition. In fact, it garanties that a loss from the final portfolio $H(T) - C$ is only generated by C.

Condition (9.2) enables us to characterize the class of α-quantile hedges for a claim C by the class of super hedges for the knock-out claim $C1_A$ with $P(A) \geq \alpha$. This will be done in Lemma 9.3. Note also that in the frictionless case condition (9.2) reduces to

$$H(T)X(T) \geq 0$$

This condition is at least not stronger than that imposed by Föllmer, Leukert (1999) who actually impose

$$\forall t \in [0, T] : H(t)X(t) \geq 0 \ .$$

Lemma 9.3 *Suppose $C \in K(T)$ is a contingent claim, $0 < \alpha \leq 1$ a probability level and $(h, H) \in \mathcal{A}$ is a portfolio process. Then (h, H) is an α-quantile hedge for C, if and only if there exists an $A \in \mathcal{F}(T)$ with $P(A) \geq \alpha$ such that (h, H) super-hedges the claim $C1_A$.*

Proof. i) If (h, H) is an α-quantile-hedging for C, then, according to Condition (9.1) and Remark 9.1 there exists an $A \in \mathcal{F}(T)$ with $P(A) \geq \alpha$ such that $(H(T)-C)1_A \in K(T)$. But because of $H(T) \in K(T)$ we have indeed $H(T) - C1_A \in K(T)$. Thus (h, H) super-hedges the claim $C1_A$.

ii) Let $A \in \mathcal{F}(T)$ with $P(A) \geq \alpha$. If (h, H) super-hedges the claim $C1_A$, then we have $H(T) - C1_A \in K(T)$. Because of $0 \in K(\omega, T)$ for all $\omega \in \Omega$, this yields

$$(H(T) - C)1_A \in K(T)$$

which implies Condition (9.1). Moreover, since the set $K(T)$ is a convex cone, we have

$$H(T) = (H(T) - C1_A) + C1_A \in K(., T) \ .$$

Thus (h, H) also satisfies condition (9.2).

$\diamondsuit$

For claims $C \notin K(T)$ whose cash value at maturity is bounded from below one can easily prove the following

Corollary 9.4 *Let $l \in \mathbb{R}_+^{d+1}$. Suppose $C \in -l + K(T)$ is a contingent claim, $(h, H) \in \mathcal{A}$ is a portfolio process and $0 < \alpha \leq 1$ is a probability level. Then we have*

$$P(H(.,T) - C(.) \in K(.,T)) \geq \alpha \ , \tag{9.3}$$

$$H(T) \in -l + K(T) \ , \tag{9.4}$$

if and only if there exists an $A \in \mathcal{F}(T)$ such that (h, H) super-hedges the claim $C1_A - l1_{\Omega \setminus A} = (C + l)1_A - l$.

Proof. Apply Lemma 9.3 to the claim $\tilde{C} = C + l$. Then deduce the assertions for the claim $C = \tilde{C} - l$.

$\diamondsuit$

Our aim now is to find a dual characterization of the set of quantile hedging initial endowments

$$\Gamma^\alpha(C) := \{ \ h \in \mathbb{R}^{d+1} : \exists A \in \mathcal{F}(T), P(A) \geq \alpha, \exists (h, H) \in \mathcal{A} : H(T) - C1_A \in K(T)\} \ .$$

According to Lemma 9.3 we have

$$\Gamma^\alpha(C) = \bigcup_{A \in \mathcal{F}(T) : P(A) \geq \alpha} \Gamma(C1_A) \ .$$

For the same reasons as with super-hedging, we are going to consider the set of approximate quantile-hedging endowments, which we define as

$$\begin{aligned} \hat{\Gamma}^\alpha(C) \ &:= \bigcup_{A \in \mathcal{F}(T) : P(A) \geq \alpha} \hat{\Gamma}(C1_A) \\ &= \{h \in \mathbb{R}^{d+1} : \exists A \in \mathcal{F}(T), P(A) \geq \alpha : \quad \forall (Q, R) \in \mathcal{P} \exists (h^n, C^n)_{n \in \mathbb{N}} : \\ &\qquad h^n \in \Gamma(C^n), C^n \in L^1_{(Q,R)}, C^n \overset{L^1_{(Q,R)}}{\longrightarrow} C1_A, h^n \to h\} \ . \end{aligned}$$

From Theorem 8.11 we conclude

$$\hat{\Gamma}^\alpha(C) = \bigcup_{A \in \mathcal{F}(T) : P(A) \geq \alpha} D(C1_A) \ .$$

If we define

$$\Pi^\alpha(C, h_1, \ldots, h_d) := \inf\{h_0 \in \mathbb{R} : h = (h_0, h_1, \ldots, h_d) \in \hat{\Gamma}^\alpha(C)\} \ ,$$

then similar as in Theorem 8.12 we get

$$\begin{aligned}
\Pi^\alpha(C,h_1,\ldots,h_d) &= \inf\{h_0\in\mathbb{R} : \exists h=(h_0,\ldots,h_d)\in \bigcup_{A\in\mathcal{F}(T):P(A)\geq\alpha} D(C1_A)\} \\
&= \inf\{h_0\in\mathbb{R} : \exists A\in\mathcal{F}(T), P(A)\geq\alpha, \exists h=(h_0,\ldots,h_d)\in D(C1_A)\} \\
&= \inf\{h_0\in\mathbb{R} : \exists h=(h_0,h_1,\ldots,h_d)\in\mathbb{R}^{d+1}\exists A\in\mathcal{F}(T), P(A)\geq\alpha : \\
&\qquad \sup_{(Q,R)\in\mathcal{P}} E_Q[C1_A(R(T)\star X(T))] - \sum_{i=1}^d h_iR_i(0)X_i(0)\leq h_0\} \\
&= \inf_{A\in\mathcal{F}(T):P(A)\geq\alpha}\left(\sup_{(Q,R)\in\mathcal{P}} E_Q[C_01_A+\sum_{i=1}^d(C_i1_A-h_i)R_i(T)X_i(T)]\right)
\end{aligned}$$

We understand $\Pi^\alpha := \Pi^\alpha(C,0,\ldots,0)$ as the approximate α-quantile-hedging price for C. Not better than Föllmer, Leukert (1999), we do not know under which conditions the optimization problem

$$\inf_{A\in\mathcal{F}(T):P(A)\geq\alpha}\left(\sup_{(Q,R)\in\mathcal{P}} E_Q[C_01_A+\sum_{i=1}^d(C_i1_A-h_i)R_i(T)X_i(T)]\right) \tag{9.5}$$

admits an optimal solution $A\in\mathcal{F}(T):P(A)\geq\alpha$. Therefore, in the style of Föllmer, Leukert (1999), we relax the restrictions of (9.5) by extending the admissible controls from the class of "non-randomized tests" 1_A to the larger class of randomized tests φ. We will see that for this modification of (9.5) there exists an optimal solution φ in the class of randomized tests.

Thus the set $\{A\in\mathcal{F}(T):P(A)\geq\alpha\}$ is replaced by the set of randomized tests

$$\Phi^\alpha := \{\varphi\in L^0(\Omega,\mathcal{F}(T)) : 0\leq\varphi\leq 1, E_P\varphi\geq\alpha\}$$

and we study the price

$$\begin{aligned}
\tilde\Pi^\alpha(C) &:= \inf_{\varphi\in\Phi^\alpha}\left(\sup_{(Q,R)\in\mathcal{P}} E_Q[C_0\varphi+\sum_{i=1}^d(C_i\varphi-h_i)R_i(T)X_i(T)]\right) \\
&= \inf_{\varphi\in\Phi^\alpha}\left(\sup_{(Q,R)\in\mathcal{P}} E_Q[\varphi\,(C(R(T)\star X(T))) - \sum_{i=1}^d h_iR_i(T)X_i(T)]\right)\ . \qquad (9.6)
\end{aligned}$$

Remark 9.5 For every $(Q,R)\in\mathcal{P}$ the linear function

$$F_{(Q,R)} : L^\infty(\Omega,\mathcal{F}(T))\to\mathbb{R}\ ,$$

$$\varphi \mapsto E_Q[\varphi\,(C(R(T) \star X(T))) - \sum_{i=1}^{d} h_i R_i(T) X_i(T)]$$

is convex in particular. Because the supremum of convex functions is a convex function, so is consequently the (non-linear) function

$$\varphi \mapsto \sup_{(Q,R)\in\mathcal{P}} F_{(Q,R)}(\varphi) \ .$$

Moreover, because Φ^α is a convex set, the optimization problem (9.6) is in fact a convex program.

$\diamondsuit$

The convex program (9.6) corresponds to calculating (approximate) super-hedging prices for claims of the type φC. What does this mean? Denote

$$\Phi = \bigcup_{0\le\alpha\le 1} \Phi^\alpha \ .$$

the set of randomized tests. If we define the function $\tilde{\varphi} : \Phi \times \mathcal{A} \to \Phi$ by

$$\tilde{\varphi}(\varphi, H) := 1_{\{H(.,T)-C\in K(.,T)\}} + \varphi 1_{\{H(.,T)-C\notin K(.,T)\}}$$

then we have

Proposition 9.6 *Let $(h, H) \in \mathcal{A}$ and $\varphi \in \Phi$. Then $H(T) - \varphi C \in K(T)$, if and only if $H(T) - \tilde{\varphi}(\varphi, H)C \in K(T)$.*

Proof. If $H(T) - \varphi C \in K(.,T)$ then one can easily verifiy that we have $H(T) - \tilde{\varphi}(\varphi, H)C \in K(.,T)$.

The other inclusion is trivial because $0 \le \varphi \le 1$.

$\diamondsuit$

As a consequence of Proposition 9.6, every $\varphi \in \Phi$ satisfies

$$\Gamma(\varphi C) = \{h \in \mathbb{R}^{d+1} : \exists (h, H) \in \mathcal{A} : H(T) - \tilde{\varphi}(\varphi, H)C \in K(T)\} \ .$$

So, if we super-hedge φC by (h, H) for $\varphi\Phi^\alpha$, we super-hedge C only with probability $\tilde{\alpha} := P(H(T) - C \in K(T)) \le \alpha$. For the loss of probability $\alpha - \tilde{\alpha}$ however, we are rewarded with the partial superhedge of C on $\{H(.,T) - C \notin K(.,T)\}$, on which set our loss is bounded by the cash value of $(1 - \varphi)C$. In this way φ can be interpreted as a succes ratio of the hedge and the condition $E\varphi \ge \alpha$ is a restricition on the average succes ratio of a hedge.

Theorem 9.7 *For every claim $C \in \bigcap_{(Q,R)\in\mathcal{P}} L^1_{(Q,R)}$ and $0 \le \alpha \le 1$ there exists a randomized test $\varphi^* \in \Phi^\alpha$ satisfying*

$$\tilde{\Pi}^\alpha(C) = \Pi(\varphi^* C) \tag{9.7}$$

Proof. According to the definition of $\tilde{\Pi}^\alpha(C)$ there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\tilde{\Phi}^\alpha$ such that

$$\tilde{\Pi}^\alpha(C) = \lim_{n\to\infty} \Pi(\varphi_n C) = \sup_{(Q,R)\in\mathcal{P}} E_P[\varphi_n C(R(T) \star X(T))\frac{dQ}{dP}] \ .$$

Since Φ is weak*-compact in $L^1(P)$ (see Witting, 1985, p.205), there exist a subsequence $(\varphi_{k(n)})_{n\in\mathbb{N}}$ and a $\varphi^* \in \Phi$ such that

$$\forall f \in L^1(P) : \lim_{n\to\infty} E_P[\varphi_{k(n)} f] = E_P[\varphi^* f] \ . \tag{9.8}$$

This implies

$$E_P\varphi^* = \lim_{n\to\infty} E_P\varphi_{k(n)} \ge \alpha \ ,$$

thus $\varphi^* \in \Phi^\alpha$. Consequently we have

$$\tilde{\Pi}^\alpha(C) = \inf_{\varphi\in\Phi^\alpha} \Pi(\varphi C) \le \Pi(\varphi^* C) \ .$$

Moreover we can apply (9.8) to $C(R(T) \star X(T))\frac{dQ}{dP} \in L^1(P)$ wich yields

$$\begin{aligned} & E_P[\varphi^* C(R(T) \star X(T))\frac{dQ}{dP}] \\ = \ & \lim_{n\to\infty} E_P[\varphi_{k(n)} C(R(T) \star X(T))\frac{dQ}{dP}] \\ \le \ & \lim_{n\to\infty} \sup_{(Q,R)\in\mathcal{P}} E_P[\varphi_{k(n)} C(R(T) \star X(T))\frac{dQ}{dP}] \\ = \ & \lim_{n\to\infty} \Pi(\varphi_{k(n)} C) = \tilde{\Pi}^\alpha(C) \ . \end{aligned}$$

And in this way we also get

$$\Pi(\varphi^* C) = \sup_{(Q,R)\in\mathcal{P}} E_P[\varphi^* \sum_{i=0}^{d} C_i R_i(T) X_i(T) \frac{dQ}{dP}] \le \tilde{\Pi}^\alpha(C) \ .$$

◇

In order to see the relation between quantile-hedging and value at risk, imagine to face the following problem. Given a contingent claim C, that we have taken short, we are interested about the initial endowments necessary for a hedge such that our final portfolio has a given Value At Risk at time T. This means that for a given confidence level $0 < \alpha \leq 1$ and a given level of loss $l < 0$ we look for a portfolio process $(h, H) \in \mathcal{A}$ such that the cash value of the portfolio $H(T) - C$ at time T is greater than $-l$ with probability at least α, i.e.,

$$P(H(T) - C + le^0 \in K(., T)) \geq \alpha$$

where $e^0 = (1, 0, \ldots, 0) \in \mathbb{R}^{d+1}$. What we need then is nothing else than an α-quantile-hedging for the claim $C - le^0$. So indeed quantile hedging is a dynamic version of the Value-At-Risk concept.

As Value-At-Risk is often difficult to deal with, there is another concept widely used in practice called Maximum Loss. The difference to Value-At-Risk is that with Maximum Loss one predefines a "reasonable" trust region $A_T \in \mathcal{F}(T)$ with $P(A_T) \geq \alpha$ and tries to find a portfolio such that the loss at maturity does not exceed a given level l on A_T. The existing literature (c.f. Luethi, Studer (1996) and Studer, G. (1999)) mainly deals with static Maximum Loss, i.e., $A_T \in \sigma(X(T))$. In our hedging context however, we could consider a dynamic version, i.e., A_T should also depend on the path of X. For a given claim C, a loss level l and the predefined set A_T with $P(A_T) \geq \alpha$ of maximum-loss hedging is equivalent to super-hedging the claim $(C - le^0)1_{A_T}$. Hence this could yield another field of application for super-hedging.

Chapter 10

Nontrivial Super Hedges with Log-Lipschitz Processes

There are three different proofs (Soner et al., 1995, Leventhal, Skorohod, 1997, Cvitanić et al. 1999) confirming the conjecture of Clark and Davis (1995) that the trivial super hedge for a European Call option is optimal in the Black Scholes model with volume proportional transaction costs. Touzi (1999) has extended this result to a more general class of claims in a multidimensional Black-Scholes model.

From these negative results one can draw two alternative conclusions depending to which paradigm one has a stronger disposition: the Black Scholes model or Pricing by Arbitrage.

If one leans towards the Black Scholes model, it appears consequential to look for alternatives of super-hedging such as quantile-hedging or utility maximization.

If one is convinced that Pricing by Arbitrage is the right way to value contingent claims, then the conclusion is to model stock prices by stochastic processes that result in reasonable hedging schemes and arbitrage bounds.

Our main intension in this chapter is to show, that there are reasonable security price models in continuous time that yield non-trivial super hedges in the presence of transaction costs. The idea is to generalize discrete time multinomial models in a non-trivial way to continuous time.

We will need the componentwise quotient of vectors. Thus we introduce the notation

$$\frac{x}{y} := (\frac{x_k}{y_k})_{k=0,\dots,d}$$

for the componentwise quotient of $x = (x_0, \ldots, x_d) \in \mathbb{R}^{d+1}$ and $y = (y_0, \ldots, y_d) \in (0, \infty)^{d+1}$.

Definition 10.1 *Suppose $\mathcal{T} = [0,T] \subset \mathbb{R}$ or $\mathcal{T} = \{0, \ldots, T\} \subset \mathbb{N}$. Let $X = (X_i(t))_{t\in\mathcal{T}, i\in\{1,\ldots,d\}}$, be a $\mathbb{F}$-adapted, $\mathbb{R}_+^d$-valued stochastic process on $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Assume that*

$$\forall i \in \{1, \ldots, d\}\ \forall (\omega, t) \in \Omega \times \mathcal{T}: \quad X_i(\omega, t) > 0 \ .$$

If there exists a polytope

$$B = conv\{b^0, \ldots, b^d\} \subset (0, \infty)^d$$

with $b^i \in (0,\infty)^d$, $i = 1, \ldots, d$, such that almost surely

$$\forall s,t \in \mathcal{T},\ s < t: \quad \frac{X(t)}{X(s)} \in e^{|t-s| \ln B} = B^{|t-s|} \ , \tag{10.1}$$

then we call X a (uniformly) log-Lipschitz process. We choose this denomination, because (10.1) obviously implies that there is a $F \in \mathcal{F}$ with $P(F) = 1$ such that the paths of

$$\ln X(\omega, .) = (\ln X_i(\omega, .))_{i\in\{1,\ldots,d\}} \quad , \ \omega \in F,$$

are uniformly Lipschitz (continuous), i.e.,

$$\forall s,t \in \mathcal{T},\ s < t: \ \max_{j=1,\ldots,d} |\ln X_j(\omega,t) - \ln X_j(\omega,s)| < |t-s| \max_{i,j} \ln b_j^i \ . \tag{10.2}$$

Conversely, if almost every path $\ln X(\omega, .)$ is uniformly Lipschitz in the usual sense and the Lipschitz constant can be chosen indepent of ω, then one is certainly able to find a polytope B such as to satisfy condition (10.1) (c.f. the proof of Proposition 10.2). Despite this equivalence, we prefer condition (10.1), because it is (10.1) that we are going to rely on later.

$\diamondsuit$

Proposition 10.2 *Suppose $Y = (Y_i(t))_{t\in\mathcal{T}, i\in\{1,\ldots,d\}}$ is a progressive measurable, $\mathbb{R}_+^{d+1}$-valued stochastic process on $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Assume moreover that Y is almost surely bounded, i.e., there is a constant $\beta > 0$ such that almost surely*

$$\forall t \in \mathcal{T}: \quad sup_{i\in\{1,\ldots,d\}} |Y_i(t)| \leq \beta \ .$$

Then the process X defined by

$$X(0) := x \in (0,\infty)^{d+1} \ ,$$

$$X_i(t) = \exp(\int_0^t Y_i(s)ds) \quad , \ i=1,\dots,d, \ t \in \mathcal{T}\setminus\{0\}$$

is a log-Lipschitz process.

Proof. According to it's defintion, the process X satisfies

$$|\ln X_i(t) - \ln X_i(s)| = |\int_s^t Y_i(s)ds| \le \beta(t-s) \quad , \ s<t, \ i=1,\dots,d.$$

Defining $\underline{\beta} := e^{-\beta}$ and $\overline{\beta} := e^{\beta}$ then, we get

$$\underline{\beta}e^{t-s} \le \frac{X_i(t)}{X_i(s)} \le \overline{\beta}e^{t-s} \quad , \ s<t, \ i=1,\dots,d.$$

Let us define $b^0,\dots,b^d \in \mathbb{R}^{d+1}$ by

$$b_j^0 = \underline{\beta} \qquad\qquad , \ j=1,\dots,d,$$

$$b_j^i = \underline{\beta} + \delta_{ij}(1+d)\overline{\beta} \quad , \ j=1,\dots,d$$

and $B = conv(\{b^0,\dots,b^d\} \subset \mathbb{R}^d$. Then with these definitions, it is straightforward to verify condition (10.1).

♢

There is certainly much more to say about the construction of log-Lipschitz processes than is stated in Proposition 10.2. However, this would go beyond the scope of this chapter. Instead, we confine ourselfes to some examples for the case $d=1$. From these examples and Proposition 10.2 it is clear then, how to construct examples of $\mathbb{R}^d$-valued log-Lipschitz processes with $d>1$. Note however, that in Proposition 10.2 the polytope B was only chosen to serve the purpose of the proof. For other purposes one may try to choose the polytope B as "small as possible" (see below). In the case $d=1$ however, the polytope reduces to an intervall and thus the choice is straightforward.

Remark 10.3 Suppose $(Y(t))_{t\in[0,T]}$ is a progressive measurable, $\mathbb{R}$-valued process on $(\Omega,\mathcal{F},\mathbb{F}^Y,P)$. Let $B=[b^0,b^1] \subset (0,\infty)$ be an intervall and $f:\mathbb{R}\to B$ a Borel-measurable function. Then the process $\ln f(Y) = (\ln f(Y(t)))_{t\in[0,T]}$ is progressive measurable and takes values in $[\ln b^0, \ln b^1]$. Thus according to Proposition 10.2, the process X defined by

$$X(0) := x \in (0,\infty) \ ,$$

$$X(t) = \exp\left(\int_0^t \ln Y(s) ds\right) \quad , \ t \in (0,T]$$

is log-Lipschitz.

$\diamondsuit$

Exampel 10.4 Suppose $(W(t))_{t\in[0,T]}$ is a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}^W, P)$. Let $B = [b^0, b^1] \in \mathbb{R}$ be an intervall and $f : \mathbb{R} \to B$ a Borel-measurable function. Then according to Remark 10.3 the process X_1 defined by

$$X_1(0) := x \in (0,\infty) \ ,$$

$$X_1(t) = \exp\left(\int_0^t \ln f(W(s)) ds\right) \quad , \ t \in (0,T]$$

is log-Lipschitz.

Moreover if f is continuous, then almost every path $X_1(\omega, .)$ is differentiable and

$$X_1'(\omega, .) := \frac{\partial X_1(\omega,t)}{\partial t} = X_1(\omega,t) \ln f(W(t)) \ .$$

The process (X_1, W) is Markov. In fact, we have

$$\begin{aligned}
\ln X_1(t) - \ln X_1(s) &= \int_s^t \ln f(W(r)) dr \\
&= \int_s^t \ln f(W(s)) dr + \int_s^t \ln f(W(r)) - \ln f(W(s)) dr \\
&= (t-s) \ln f(W(s)) + \int_s^t \ln f(W(r)) - \ln f(W(s)) dr \ ,
\end{aligned}$$

which yields

$$\begin{aligned}
E_P[\ln X_1(t) | \mathcal{F}^W(s)] &= \ln X_1(s) + (t-s) \ln f(W(s)) \\
&\quad + E_P\left[\int_s^t \ln f(W(r)) - \ln f(W(s)) dr | \mathcal{F}^W(s)\right] \\
&= \ln X_1(s) + (t-s) \ln f(W(s)) \\
&\quad + E_P\left[\int_s^t \ln f(W(r)) - \ln f(W(s)) dr\right] \ .
\end{aligned}$$

Suppose now that f is continuous and $X_1(t)$ describes the spot price process of a security. If there are no transaction costs, it is easy to construct a simple arbitrage strategy whenever there is a $A \in \mathcal{F}(s)$ with $P(A) > 0$ such that for every at $\omega \in A$ we have

$$X_1'(\omega, .)(s) \neq 0$$

In fact, if there is a $A \in \mathcal{F}(s)$ with $P(A) > 0$ such that for every at $\omega \in A$ we have

$$X_1'(\omega, .)(s) > 0$$

then we can buy the security at $(\omega, s) \in (A, s)$ and hold it untill $\tau \wedge T = \min\{\tau, T\}$ where

$$\tau(\omega) := \inf\{T \leq t > s : X_1'(\omega, .)(t(\omega)) = 0\} \quad , \ \omega \in \Omega,$$

is a stopping time (with the usual convention $\inf \emptyset = \infty$). From the Intermediate Value Theorem we know that the following two conditions hold

$$\forall \omega \in A : \quad \tau(\omega) > s$$

$$\forall (\omega, t) \in A \times [s, \tau(\omega) \wedge T) : \quad X_1'(\omega, .)(t) > 0 \ .$$

Thus, if we sell the security at time $\tau \wedge T$, then we have certainly gained the difference

$$X_1(\omega, \tau) - X(\omega, s) = \int_s^{\tau \wedge T} X_1'(\omega, .)(t) dt > 0 \ .$$

This means we have realized an arbitrage opportunity.

If there are transaction cost factors λ for buying and μ for selling then the above arbitrage opportunity vanishes, provided there is a subset $\tilde{A} \subset A$, $\tilde{A} \notin \mathcal{F}(s)$ with $P(\tilde{A} > 0$ such that every $\omega \in \tilde{A}$ we have

$$(1 - \mu) X_1(\omega, \tau) - (1 + \lambda) X_1(\omega, s) < 0 \ .$$

Hence in this case it seems to be possible to choose a continuous f in such a way that there are no (simple) arbitrage opportunities with the consequence $\mathcal{P} \neq \emptyset$ (see Theorems 7.15 and 7.12).

$\diamondsuit$

Example 10.4 shows that differentiable processes are not that suitable for modelling security prices, because the derivative of the price gives a hint, how to make a riskless profit, at least if transaction costs are negligable.

However, suppose the price process X_1 of security 1 is only piecewise differentiable on stochastic intervalls (τ_k, τ_{k+1}),$k \in \mathbb{N}$. But in τ_k, $k \in \mathbb{N}$ the process may suddenly change it's "direction" in the sense that at τ_k, $k \in \mathbb{N}$, the algebraic sign of the derivative will change with positive probability and non of the τ_k, $k \in \mathbb{N}$, except for $\tau_0 = 0$ is predictable. Then one can never be sure wether the price will rise or fall within the next seconds. This feature is modelled in the simplest possible way in the following

Exampel 10.5 Suppose τ_j, $j \in \mathbb{N}$, are the jump times of a $\mathbb{F}$-adapted Poisson process J on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ (stopped at T). Define the process $Y = (Y(t))_{t\in[0,T]}$ by $Y(0) = Y(\tau_0) = 1$ and

$$Y(t) = (-1)^{J(t)} \ .$$

Let $b = [d, u] \subset (0, \infty)$ with $d \leq u$ and $f : \mathbb{R} \to \{d, u\}$ Borel-measurable and such that

$$f(y) = d1_{(-\infty,0)}(y) + u1_{[0,\infty)}(y) \ .$$

Define then the process $X_1 = (X_1(t))_{t\in[0,T]}$ by $X_1(0) = x > 0$ and

$$\begin{aligned} X(t) &= e^{\int_0^t \ln f(Y(s))ds} \qquad t \in (0, T] \\ &= exp\left(\sum_{k=0}^{\infty} \int_0^t 1_{[\tau_{2k},\tau_{2k+1}]}(s) \ln u ds + \sum_{k=0}^{\infty} \int_0^t 1_{[\tau_{2k+1},\tau_{2(k+1)}]}(s) \ln d ds\right) \\ &= u^{\sum_{k=0}^{\infty} \tau_{2k+1}\wedge t - \tau_{2k}\wedge t} d^{\sum_{k=0}^{\infty} \tau_{2(k+1)}\wedge t - \tau_{2k+1}\wedge t} \ . \end{aligned}$$

From this representation we see that X is generated by a "binomial tree" with random ramifications. Since multinomial trees as in the Cox-Ross-Rubinstein model have poved usefull for modelling stock price movements, the same can be expected for multinomial trees with random ramifications or other log-Lipschitz tree processes.

From now on we make us of the following

Definitions and Assumptions 10.6 *Suppose* $X = (X_i(t))_{t\in[0,T],\ i\in\{0,\dots,d\}}$ *is the* $\mathbb{R}^{d+1}$*-valued, cadlag spot price process on* $(\Omega, \mathcal{F}, \mathbb{F}, P)$ *within the framework of Chapter 6. Then the process* $(X_i(t))_{t\in[0,T],\ i\in\{1,\dots,d\}}$ *is assumed to be log-Lipschitz with respect to a polytope*

$$B = conv\{b^0, \dots, b^d\} \subset (0, \infty)^d$$

with $b^i \in (0,\infty)^d$, $i = 1,\dots,d$ *(remember that* $X_0(t) = 1$ *for every* $t \in [0,T]$*).*

Denote $\mathcal{TT}$ *the class of all* $\mathcal{T} = \{t_0,\dots,t_N\} \subset [0,T]$ *such that*

$$\forall i \in \{1,\dots,N\}: \quad t_i - t_{i-1} = \frac{T}{N} \ .$$

We assume that for each $\mathcal{T} = \{t_0,\dots,t_N\} \in \mathcal{TT}$ *and for every* $i \in \{1,\dots,N\}$ *there is a* $\mathcal{F}(t_i)$*-measurable* $\mathbb{R}^{d+1}$*-valued random vector*

$$Y(t_i) : \Omega \to \Big(\{1\} \times \{b^0,\dots,b^d\}\Big)^{\frac{T}{N}}$$

such that

$$\forall j \in \{0,\dots,d\}: \quad P(Y(t_i) = (1,b^j)^{\frac{T}{N}}) > 0$$

and $Y(t_i)$ *is independent of* $\mathcal{F}(t_{i-1})$*. Note that this assumption certainly holds, if the filtration* $\mathbb{F}$ *was generated by a d-dimensional Brownian motion or a d-dimensional Poisson process (c.f. Examples 10.4 and 10.5).*

Suppose $\mathcal{T} \in \mathcal{TT}$ *and* $Y(t_i)$, $i = 1,\dots,N$*, as above. Then for every* $n \in \{0,\dots,N-1\}$ *we define the process* $\tilde{X}^{t_n} = \Big(\tilde{X}^{t_n}(t)\Big)_{t\in[0,T]}$ *by*

$$\tilde{X}^{t_n} = \begin{cases} X(t) & : \quad t \le t_n \\ \tilde{X}(t_{i-1}) & : \quad t_n \le t_{i-1} < t < t_i \\ Y(t_i) \star \tilde{X}(t_{i-1}) & : \quad t_n < t = t_i \end{cases} \ . \tag{10.3}$$

Thus each $\tilde{X}^{t_n}$, $i = 1,\dots,N$*, is* $\mathbb{F}$*-adapted, cadlag and satisifies*

$$\forall t \in [0,T]: \quad \tilde{X}_0^{t_n}(t) = 1 \ .$$

Moreover the process $\tilde{X} := \tilde{X}^{t_0}$ *is* $\mathbb{F}^{\mathcal{T}}$*-Markov with* $\mathbb{F}^{\mathcal{T}} := (\mathcal{F}(t_i))_{i=0,\dots,N}$*. In fact,* $\tilde{X}$ *is a multinomial tree process. Of course the definitions of the processes* $\tilde{X}^{t_n}$, $i = 1,\dots,N$ *depend on* $\mathcal{T} \in \mathcal{TT}$*. While we do not include this dependence into notation, we should bear it in mind however.*

For simplicity of exposition assume

$$\forall (\omega,t) \in \Omega \times [0,T]: \quad K_0'(\omega,t) = K_0' \ , \tag{10.4}$$

like it is the case in stock or currency markets with constant transaction cost factors (see Section 2.3 resp. 2.5).

For $x \in (0,\infty)^{d+1}$ we define the closed convex cone $K(x)$ by

$$K(x) := \{h \in \mathbb{R}^{d+1} : \forall r \in K_0' : h(r \star x) \geq 0\} \; . \tag{10.5}$$

We need this definition, because we have to deal simultaneously with several price processes simultaneously. In particular, for every $\omega \in \Omega$ we have

$$K(X(\omega,t)) = \{h \in \mathbb{R}^{d+1} : \forall r \in K_0' : h(r \star X(\omega,t)) \geq 0\} = K(\omega,t)$$

whereas $K(\tilde{X}^{t_n}(\omega,t))$ is the corresponding cone for the price process $\tilde{X}^{t_n}$. Without condition (10.4) and the definition (10.5) we would have to introduce different cones $K^{\tilde{X}^{t_i}}$ for the several price processes. In these premises condition (10.4)and the definition (10.5) actually state that the transaction cost factors do not depend on the price process, which is a reasonable assumption.

Analoguously, for a random vector $W(t) \in (L^0(\mathcal{F}(t)))^{d+1}$ we define

$$K(W(t)) := \{V(t) \in (L^0(\mathcal{F}(t)))^{d+1} : V(\omega,t) \in K(W(\omega,t))\} \; .$$

In particular for $W(t) = X(t)$, because of $K(X(\omega,t)) = K(\omega,t)$, we have

$$K(X(t)) = \{V(t) \in (L^0(\mathcal{F}(t)))^{d+1} : V(\omega,t) \in K(X(\omega,t))\} = K(t) \; .$$

$\diamondsuit$

For given $\mathcal{T} \in \mathcal{T\!T}$ we will have to deal with claims $C(t_i) \in (L^0(\mathcal{F}(t)))^{d+1}$, $t_i \in \{t_0, \ldots, t_N\}$ of the followig type. For a convex function $f : \mathbb{R}^{d+1} \to \mathbb{R}$ the claim $C = C^f = (C_0, \ldots, C_d)$ is defined by

$$C_0(t_i) = f(X(t))$$

$$(C_1(t), \ldots, C_d(t)) \in (L^0(\mathcal{F}(t_{i-1})))^d \; .$$

We imagine that a claim $C(t)$ matures at time t.

For claims of this type we need to define the following (dynamic) superhedging endowments.

For $x \in (0,\infty)^{d+1}$, $(c_1, \ldots, c_d) \in \mathbb{R}^d$, $t_i \in \{t_0, \ldots, t_{N-1}\}$ and a convex function $f : \mathbb{R}^{d+1} \to \mathbb{R}$, we define the t_i-local super-hedging endowments

$$\begin{aligned}\Gamma(x, f, (c_i)_{i=1,\ldots,d}, t_i) \;\; = \;\; & \{v \in \mathbb{R}^{d+1} : \exists h \in \mathbb{R}^{d+1} : v - h \in K(x), \\ & h - (f(Y(t_{i+1}) \star x), c_1, \ldots, c_d) \in K(Y(t_{i+1}) \star x)\}.\end{aligned}$$

In addition for $(v_1, \ldots, v_d) \in real^d$, we define the t_i-local super-hedging price

$$\begin{aligned}&\Pi(x, f, (c_i)_{i=1,\ldots,d}, (v_i)_{i=1,\ldots,d}, t_i)\\ &\quad = \inf\{v_0 \in \mathbb{R} : \exists v = (v_i)_{i=0,\ldots,d} : \quad v \in \Gamma(x, f, (c_i)_{i=1,\ldots,d}, t_i)\} \ .\end{aligned}$$

Analogously, we need to define these notions in account of measurability conditions and the price process. For $t_i \in \{t_0, \ldots, t_{N-1}\}$, $(C_1, \ldots, C_d) \in (L^0(\mathcal{F}(t_i)))^d$, and a function $f : \mathbb{R}^{d+1} \to \mathbb{R}$, we define the local super-hedging endowments

$$\begin{aligned}\Gamma^X(f, (C_i)_{i=1,\ldots,d}, t_i) &= \{V \in (L^0(\mathcal{F}(t_i)))^{d+1} : \exists H(t_i) \in (L^0(\mathcal{F}(t_i)))^{d+1} :\\ &\quad V - H(t_i) \in K(X(t_i)), H(t_i) - (f(X(t_{i+1})), C_1, \ldots, C_d) \in K(X(t_{i+1})\}\\ \Gamma^{\tilde{X}^{t_i}}(f, (C_i)_{i=1,\ldots,d}, t_i) &= \{V \in (L^0(\mathcal{F}(t_i)))^{d+1} : \exists H(t_i) \in (L^0(\mathcal{F}(t_i)))^{d+1} :\\ &\quad V - H(t_i) \in K(X(t_i)), H(t_i) - (f(\tilde{X}^{t_i}(t_{i+1})), C_1, \ldots, C_d) \in K(\tilde{X}^{t_i}(t_{i+1}))\}\\ \Gamma^{\tilde{X}}(f, (C_i)_{i=1,\ldots,d}, t_i) &= \{V \in (L^0(\mathcal{F}(t_i)))^{d+1} : \exists H(t_i) \in (L^0(\mathcal{F}(t_i)))^{d+1} :\\ &\quad V - H(t_i) \in K(X(t_i)), H(t_i) - (f(\tilde{X}(t_{i+1}), C_1, \ldots, C_d) \in K(\tilde{X}(t_{i+1})\}\end{aligned}$$

In addition for $(V_i)_{i=1,\ldots,d} \in (L^0(\mathcal{F}(t_i)))^d$ we define the super-hedging price

$$\begin{aligned}&\Pi^X(f, (C_i)_{i=1,\ldots,d}, (V_i)_{i=1,\ldots,d}, t_i)\\ &\quad = \operatorname{ess\,inf}\{V_0 \in \mathbb{R} : \exists V = (V_i)_{i=0,\ldots,d}(L^0(\mathcal{F}(t_i)))^{d+1} : V \in \Gamma^X(f, (C_i)_{i=1,\ldots,d}, t_i)\} \ ,\end{aligned}$$

$$\begin{aligned}&\Pi^{\tilde{X}^{t_i}}(f, (C_i)_{i=1,\ldots,d}, (V_i)_{i=1,\ldots,d}, t_i)\\ &\quad = \operatorname{ess\,inf}\{V_0 \in \mathbb{R} : \exists V = (V_i)_{i=0,\ldots,d}(L^0(\mathcal{F}(t_i)))^{d+1} : V \in \Gamma^{\tilde{X}^{t_i}}(f, (C_i)_{i=1,\ldots,d}, t_i)\} \ ,\end{aligned}$$

$$\begin{aligned}&\Pi^{\tilde{X}}(f, (C_i)_{i=1,\ldots,d}, (V_i)_{i=1,\ldots,d}, t_i)\\ &\quad = \operatorname{ess\,inf}\{V_0 \in \mathbb{R} : \exists V = (V_i)_{i=0,\ldots,d}(L^0(\mathcal{F}(t_i)))^{d+1} : V \in \Gamma^{\tilde{X}}(f, (C_i)_{i=1,\ldots,d}, t_i)\} \ .\end{aligned}$$

Lemma 10.7 *Let $\mathcal{T} = \{t_0, \ldots, t_{N-1}\} \in \mathcal{T\!T}$, $t_i \in \{t_0, \ldots, t_{N-1}\}$, $(C_1, \ldots, C_d) \in (L^0(\mathcal{F}(t_i)))^d$. Suppose the function $f : \mathbb{R}^{d+1} \to \mathbb{R}$ is convex. Then we have*

$$\Gamma^{\tilde{X}^{t_i}}(f, (C_i)_{i=1,\ldots,d}, t_i) \subset \Gamma^X(f, (C_i)_{i=1,\ldots,d}, t_i) \ .$$

Consequently, for every $(V_i)_{i=1,\ldots,d} \in (L^0(\mathcal{F}(t_i)))^d$ we have

$$\Pi^X(f, (C_i)_{i=1,\ldots,d}, (V_i)_{i=1,\ldots,d}, t_i) \leq \Pi^{\tilde{X}^{t_i}}(f, (C_i)_{i=1,\ldots,d}, (V_i)_{i=1,\ldots,d}, t_i) \ .$$

Proof. Suppose $V(t_i) \in \Gamma^{\tilde{X}^{t_i}}(f,(C_i)_{i=1,\ldots,d},t_i)$. Then there exists a $H(t_i) \in (L^0(\mathcal{F}(t_i)))^{d+1}$ such that $V(t_i) - H(t_i) \in K(X(t_i))$ and

$$H(t_i) - (f(\tilde{X}^{t_i}(t_{i+1})), C_1, \ldots, C_d) \in K(\tilde{X}^{t_i}(t_{i+1})) = K(Y(t_{i+1}) \star X(t_i)) \ .$$

This implies that for every $r \in K_0' \subset \{1\} \times \mathbb{R}^d_+$ we have

$$H_0(t_i) - f(Y(t_{i+1}) \star X(t_i)) + \sum_{k=1}^{d} (H_k(t_i) - C_k(t_i)) r_k Y_k(t_{i+1}) X_k(t_i) \geq 0 \ .$$

Since $Y_k(t_{i+1})$ is independent of $\mathcal{F}(t_{i+1})$ we conclude

$$\forall j \in \{0,\ldots,d\} \forall r \in K_0' :$$

$$H_0(t_i) - f((1,b^j) \star X(t_i)) + \sum_{k=1}^{d} (H_k(t_i) - C_k(t_i)) r_k (b_k^j)^{\frac{T}{N}} X_k(t_i) \geq 0 \ . \tag{10.6}$$

Because we almost surely have

$$\frac{X(t_{i+1})}{X(t_i)} \in \left(\{1\} \times conv\{b^0,\ldots,b^d\}^{\frac{T}{N}}\right)$$

there exits a $\lambda(t_{i+1}) = (\lambda_j(t_{i+1}))_{j=0,\ldots,d} \in (L^0(\mathcal{F}(t_{i+1})))^{d+1}$ with

$$\forall j \in \{0,\ldots,d\} : 0 \leq \lambda_j(t_{i+1}) \leq 1 \quad , \ \sum_{j=0}^{d} \lambda_j(t_{i+1}) = 1$$

such that

$$X(t_{i+1}) = \left(\sum_{j=0}^{d} \lambda_j(t_{i+1})(1,b^j)^{\frac{T}{N}} \right) \star X(t_i) = \sum_{j=0}^{d} \lambda_j(t_{i+1}) \left((1,b^j)^{\frac{T}{N}} \star X(t_i) \right) \ .$$

Because the function $f : \mathbb{R}^{d+1} \to \mathbb{R}$ is convex, we get

$$f(X(t_{i+1})) = f\left(\sum_{j=0}^{d} \lambda_j(t_{i+1}) \left((1,b^j)^{\frac{T}{N}} \star X(t_i) \right) \right) \leq \sum_{j=0}^{d} \lambda_j(t_{i+1}) f\left((1,b^j)^{\frac{T}{N}} \star X(t_i) \right) \ .$$

In account of (10.6) this implies that for every $r \in K_0'$ we have

$$\begin{aligned}
&H_0(t_i) - f(X(t_{i+1})) + \sum_{k=1} d(H_k(t_i) - C_k(t_i)) r_k Y_k(t_{i+1}) X_k(t_i) \\
&\geq H_0(t_i) - \sum_{j=0}^{d} \lambda_j(t_{i+1}) f\left((1,b^j) \star X(t_i) \right) + \sum_{k=1} d(H_k(t_i) - C_k(t_i)) r_k Y_k(t_{i+1}) X_k(t_i) \\
&= \sum_{j=0}^{d} \lambda_j(t_{i+1}) \left(H_0(t_i) - f\left((1,b^j) \star X(t_i) \right) + \sum_{k=1} d(H_k(t_i) - C_k(t_i)) r_k Y_k(t_{i+1}) X_k(t_i) \right) \\
&\geq 0 \ .
\end{aligned}$$

thus we have

$$H(t_i) - (f(X(t_{i+1}), (C_i)_{i=1,\dots,d}) \in K(X(t_i))$$

Because of $V(t_i) - H(t_i) \in K(X(t_i))$ and the convexity of the cone $K(X(t_i))$ this yields the desired assertions in account of the involved definitions.

$\diamondsuit$

Lemma 10.8 *Let* $(c_1, \dots, c_d) \in \mathbb{R}^d$, $\mathcal{T} = \{t_0, \dots, t_{N-1}\} \in \mathcal{T\!T}$, $t_i \in \{t_0, \dots, t_{N-1}\}$, $(v_1, \dots, v_d) \in real^d$ *and* $f : \mathbb{R}^{d+1} \to \mathbb{R}$ *a convex function. Then there is a convex function* $g : (0, \infty)^{d+1} \to \mathbb{R}$ *such that*

$$\forall x \in (0, \infty)^{d+1} : \quad \Pi(x, f, (c_i)_{i=1,\dots,d}, (v_i)_{i=1,\dots,d}, t_i) = g(x) \ .$$

Proof. First, recall that

$$\begin{aligned}
&\Pi(x, f, (c_i)_{i=1,\dots,d}, (v_i)_{i=1,\dots,d}, t_i) \\
&\quad = \inf\{v_0 \in \mathbb{R} : \exists v = (v_i)_{i=0,\dots,d} : v \in \Gamma(x, f, (c_i)_{i=1,\dots,d}, t_i)\}
\end{aligned}$$

where

$$\begin{aligned}
\Gamma(x, f, (c_i)_{i=1,\dots,d}, t_i) = \quad \{ \quad & \tilde{v} \in real^{d+1} : \exists h \in \mathbb{R}^{d+1} : \tilde{v} - h \in K(x), \\
& h - (f(Y(t_{i+1}) \star x), c_1, \dots, c_d) \in K(Y(t_{i+1}) \star x)\}.
\end{aligned}$$

We are going to apply our results on super-hedging in discrete time to the two time point market with the "price process" $(x, Y(t_{i+1}) \star x)$.

Therefore, we define the class $\tilde{\mathcal{P}}$ of all pairs $(\tilde{Q}, \tilde{R})$ satsifying the following conditions:

(P1) $\tilde{Q}$ is a probability equivalent to P on $\sigma(Y(t_{i+1}))$,

(P2) $\tilde{R} = (\tilde{r}, \tilde{R}(t_{i+1})) \in K_0' \times K_0'$ (a.s.),

(P3) $E_{\tilde{Q}}[\tilde{R}(t_{i+1}) \star Y(t_{i+1}) \star x] = \tilde{r} \star x$.

Observe now that our two time points market is actually finite because $\sigma(Y(t_{i+1}))$ has finitely many elements. In account of Remark 5.5 we can apply Theorem 5.13

and get

$$
\begin{aligned}
g(x) &:= \Pi(x, f, (c_i)_{i=1,\ldots,d}, (v_i)_{i=1,\ldots,d}, t_i) \\
&= \sup_{(\tilde{Q},\tilde{R})\in\tilde{\mathcal{P}}} E_{\tilde{Q}}[f(Y(t_{i+1})\star x) + \sum_{i=1}^{d}(c_i - v_i)R_i(t_{i+1})Y_i(t_{i+1})x_i] \\
&= \sum_{i=1}^{d}(c_i - v_i)\tilde{r}_i x_i + \sup_{(\tilde{Q},\tilde{R})\in\tilde{\mathcal{P}}} \sum_{j=0}^{d} \tilde{Q}(Y(t_{i+1}) = (1, b^j)^{\frac{T}{N}}) f((1, b^j)^{\frac{T}{N}} \star x) \ .
\end{aligned}
$$

The convexity of f implies that each of the functions

$$
x \mapsto f((1, b^j)^{\frac{T}{N}} \star x) \quad , \ j = 0, \ldots, d,
$$

is convex in x. Thus $g : x \mapsto g(x)$ is convex, because it consists of a finite sum of linear functions plus the pointwise supremum of a familiy of convex functions (c.f. Rockafellar, 1970, Theorem 5.2, Theorem 5.5).

♢

Lemma 10.9 *Let $\mathcal{T} = \{t_0, \ldots, t_{N-1}\} \in \mathcal{TT}$. Then for every $t_i \in \{t_0, \ldots, t_{N-1}\}$ and for P-almost every $\omega \in \Omega$ we have*

$$
X(\omega, t_i) \in conv(\tilde{X}(\Omega, t_i)) \ .
$$

Proof. For simplicity of exposition and without loss of generality assume $T = N$.

For t_0 the assertion is trivial.

Supose the assertion is true for t_i. Then we have

$$
\begin{aligned}
X(\omega, t_{i+1}) &\in conv(Y(\Omega, t_{i+1}) \star X(\omega, t_i)) \\
&\subset conv\left(Y(\Omega, t_{i+1}) \star conv(\tilde{X}(\Omega, t_i))\right) \\
&= conv(\tilde{X}(\Omega, t_{i+1})) \ .
\end{aligned}
$$

There we have used the notation

$$
A \star \tilde{A} := \{a \star \tilde{a} : a \in A, \tilde{a} \in \tilde{A}\} \quad , \ A, \tilde{A} \subset \mathbb{R}^{d+1} \ .
$$

♢

Lemma 10.10 *Let $\mathcal{T} = \{t_0, \ldots, t_{N-1}\} \in \mathcal{T\!T}$, $t_i \in \{t_0, \ldots, t_{N-1}\}$, $(C_1, \ldots, C_d) \in (L^0(\mathcal{F}(t_{i-1})))^d$. Suppose the function $f : \mathbb{R}^{d+1} \to \mathbb{R}$ is convex. Then for every $(V_i)_{i=1,\ldots,d} \in (L^0(\mathcal{F}(t_{i-1})))^d$ we have*

$$\Pi^{\tilde{X}^{t_i}}(f, (C_i)_{i=1,\ldots,d}, (V_i)_{i=1,\ldots,d}, t_i) \leq conv(\Pi^{\tilde{X}}(f, (C_i)_{i=1,\ldots,d}, (V_i)_{i=1,\ldots,d}, t_i)(\Omega)) \ .$$

in the sense that for P-almost every $\omega \in \Omega$ and every

$$\gamma \in conv(\Pi^{\tilde{X}}(f, (C_i)_{i=1,\ldots,d}, (V_i)_{i=1,\ldots,d}, t_i)(\Omega))$$

we have

$$\Pi^{\tilde{X}^{t_i}}(f, (C_i)_{i=1,\ldots,d}, (V_i)_{i=1,\ldots,d}, t_i)(\omega) \leq \gamma \ .$$

Proof. Observe first that according to Lemma 10.9, for P-almost every $\omega \in \Omega$ we have

$$\tilde{X}^{t_i}(\omega, t_i) = X(\omega, t_i) \in conv(\tilde{X}(\Omega, t_i)) \ .$$

Since $\tilde{X}^{t_i}(t_{i+1}) = Y(t_{i+1}) \star X(t_i)$ and $Y(t_{i+1})$ is independent of $\mathcal{F}(t_i)$, it suffices to show the assertion for given $(c_i)_{i=1,\ldots,d}, (v_i)_{i=1,\ldots,d} \in \mathbb{R}^d$ "conditional on"

$$\{(C_i)_{i=1,\ldots,d} = (c_i)_{i=1,\ldots,d}, (V_i)_{i=1,\ldots,d} = (v_i)_{i=1,\ldots,d}\} \in \mathcal{F}(t_{i-1})$$

for varying $x \in \tilde{X}^{t_i}(\Omega, t_i) = X(\Omega, t_i)$. In this way, the assertion follows by combining Lemma 10.9 with Lemma 10.8.

$\diamondsuit$

Definition 10.11 *For a subset $\mathcal{T} = \{t_0, \ldots, t_N\} \in \mathcal{T\!T}$ we define the class $\mathcal{A}^{\mathcal{T},X}$ of portfolio processes $(h, H) \in \mathcal{H}$ such that for every $i \in \{1, \ldots, N\}$ the process $H(.)$ is constant on $[t_{i-1}, t_i[$ and*

$$\forall i \in \{0, \ldots, d\} : \quad -\Delta H(t_i) \in K(X(t_i)) \ .$$

Analogously we define the classes $\mathcal{A}^{\mathcal{T},\tilde{X}}$ and $\mathcal{A}^{\mathcal{T},\tilde{X}^{t_i}}$, $i = 1, \ldots, N$.

$\diamondsuit$

Theorem 10.12 *Suppose $\mathcal{T} = \{t_0, t_1, t_2\} \in \mathcal{T\!T}$, i.e. $t_0 = 0$, $t_1 = \frac{1}{2}T$, $t_2 = T$. Let $f : (0, \infty)^{d+1} \to \mathbb{R}$ be a convex function. Let $(\tilde{h}, \tilde{H}) \in \mathcal{A}^{\mathcal{T},\tilde{X}}$ be a super hedge for the claim $(f(\tilde{X}(t_2), 0, \ldots, 0)$ with respect to the price process $\tilde{X}$. Then there is a portfolio process $(h, H) \in \mathcal{A}^{\mathcal{T},X}$ with $h = \tilde{h}$ that super-hedges the claim $(f(X(t_2)), 0, \ldots, 0)$ with respect to the price process X.*

Proof. Suppose $(\tilde{h},\tilde{H}) \in \mathcal{A}^{\mathcal{T},\tilde{X}}$ is a super hedge for the claim $(f(\tilde{X}(t_2),0,\dots,0)$ with respect to the price process $\tilde{X}$. Define $h=\tilde{h}$ and $H(t_0)=\tilde{H}(t_0)$. Then because of $X(t_0)=\tilde{X}(t_0)$ and $(\tilde{h},\tilde{H}) \in \mathcal{A}^{\mathcal{T},\tilde{X}}$, we have

$$\tilde{h}-\tilde{H}(t_0)=h-H(t_0) \in K(X(t_0)) = K(\tilde{X}(t_0)\ .$$

Since $(\tilde{h},\tilde{H}) \in \mathcal{A}^{\mathcal{T},\tilde{X}}$ is a super hedge for the claim $(f(\tilde{X}(t_2),0,\dots,0)$ we have

$$\begin{aligned} H_0(t_0) &\geq \Pi^{\tilde{X}}(f,(0_i)_{i=1,\dots,d},(H_i(t_0))_{i=1,\dots,d},t_1) \\ &= \operatorname{ess\,inf}\{V_0 \in \mathbb{R} : \exists V=(H_i(t_0))_{i=0,\dots,d}(L^0(\mathcal{F}(t_i)))^{d+1} : V \in \Gamma^X(f,(0_i)_{i=1,\dots,d},t_1)\} \end{aligned}$$

where

$$\begin{aligned} \Gamma^X(f,(0_i)_{i=1,\dots,d},t_1) &= \{V \in (L^0(\mathcal{F}(t_i)))^{d+1} : \exists H(t_1) \in (L^0(\mathcal{F}(t_i)))^{d+1} : \\ &\qquad V-H(t_1) \in K(X(t_i)), H(t_1)-(f(X(t_2)),0,\dots,0) \in K(X(t_2)\}\ . \end{aligned}$$

In account of Lemma 10.10 and Lemma 10.7 this implies

$$H_0(t_0) \geq \Pi^{\tilde{X}^{t_1}}(f,(0,\dots,0),(H_i(t_0))_{i=1,\dots,d},t_1) \geq \Pi^X(f,(0_i)_{i=1,\dots,d},(H_i(t_0))_{i=1,\dots,d},t_1)\ .$$

According to the definition of Π^X this implies

$$H(t_0) \in \Gamma^X(f,(0_i)_{i=1,\dots,d},t_1)\}\ .$$

This in turn implies that there exists a $H(t_1) \in (L^0(\mathcal{F}(t_1)))^{d+1}$ satisfying

$$H(t_0)-H(t_1) \in K(X(t_1))\ ,$$

$$H(t_1)-(f(X(t_2)),0,\dots,0) \in K(X(t_2)\ .$$

Finally we can choose $H(t_2)=(f(X(t_2)),0,\dots,0)$. It is clear then that the strategy H super hedges $(f(X(t_2)),0,\dots,0)$ with respect to the price system X.

$\diamondsuit$

Corollary 10.13 *Suppose* $\mathcal{T}=\{t_0,t_1,t_2\} \in \mathbb{T}$*, i.e.* $t_0=0$*,* $t_1=\frac{1}{2}T$*,* $t_2=T$*. Let* $f:(0,\infty)^2 \to \mathbb{R}$*,* $f(x)=(x_1-k)^+$*, be the payoff function of a (cash settlement) European Call option on asset* 1 *with exercise price* $k>0$*. Let* $(\tilde{h},\tilde{H}) \in \mathcal{A}^{\mathcal{T},\tilde{X}}$ *be a super hedge for the claim*

$$((\tilde{X}_1(t_2)-k)^+,0)$$

with respect to $\tilde{X}$*. Then there is a portfolio process* $(h,H) \in \mathcal{A}^{\mathcal{T},X}$ *with* $h=\tilde{h}$ *that super hedges the claim* $((X_1(t_2)-k)^+,0)$ *with respect to* X*.*

Proof. Since the function $(x_0, x_1) \mapsto f(x) = (x_1 - k)^+$ is convex, the assertion is a direct consequence of Theorem 10.12.

$\diamondsuit$

It is possible to prove Theorem 10.12 and Corollary 10.13 for arbitrary $\mathcal{T} \in \mathcal{TT}$. Unfortunately, this proof is more delicate, because for this purpose one needs to show mutliperiod versions of Lemma 10.10 and Lemma 10.7. Such a theorem could be usefull in order to calculate good upper bounds for super-hedging prices. For this purpose the polytope B should be chosen as small as possible. There are many other interseting questions concerning log-Lipschitz price processes and super-hedging prices. However, our main intension within this chapter was to show the existence of non-trivial super hedges for the European Call option in continuous time markets with log-Lipschitz price processes. The results are statet in our final

Remarks 10.14 The price process $(\tilde{X}(t))_{t\in\mathcal{T}}$, $\mathcal{T} = \{t_0, t_1, t_2\} \in \mathcal{TT}$, in Theorem 10.12 is a multinomial tree process. In particular, the price process $(\tilde{X}(t))_{t\in\mathcal{T}}$ in Corollary 10.13 is nothing else than a two period (i.e. three time points) Cox-Ross-Rubinstein price process. In this market, a portfolio strategy is non-static, if it allows for trading at time t_1. It is well known, that in the Cox-Ross-Rubinstein model with proportional transaction costs, there are non-trivial, non-static super hedges (even replicative hedges) for the European Call option that are cheaper than the simple buy and hold strategy, at least if transaction costs are within reasonable bounds in comparison to stock price movements (c.f. e.g. Boyle, Vorst (1992), Edirisinghe et al. (1993), Kusuoka (1995), Mercurio, Vorst (1997)). In the context of Corollary 10.13 this means that there is a non-static super hedge $(\tilde{h}, \tilde{H}) \in \mathcal{A}^{\mathcal{T},\tilde{X}}$ with $h_1 = 0$ and

$$h_0 < (1 + \lambda)X_1(t_0) \ .$$

According to Corollary 10.13 then, h_0 is an upper bound for the super-hedging price of the same European Call option but with respect to the price process X. Since this super-hedging price lies strictly below the price that we would have to pay for buying asset 1 at time t_0, the trivial buy and hold strategy is not optimal with respect to the price process X either.

Moreover, as $(\tilde{h}, \tilde{H}) \in \mathcal{A}^{\mathcal{T},\tilde{X}}$ is non-static, it may prove true in many cases, that there is not only a non-trivial but also a non-static super hedge $(h, H) \in \mathcal{A}^{\mathcal{T},X}$ with respect to X. However, this conjecture will only hold subject to suitable implementations of the price process X.

At least for the price process of Example 10.5 one is able to verify this directly, because the binomial tree structure makes it possible to calculate the portfolio process $(h, H) \in \mathcal{A}^{\mathcal{T},X}$ similar as in the Cox-Ross-Rubinstein model. In addition, provided that Theorem 10.12 respectively Corollary 10.13 hold for arbitrary $\mathcal{T} \in \mathcal{TT}$, one can calculate an upper bound for the super-hedging price with respect to every $\mathcal{T} \in \mathcal{TT}$. Because of transaction costs then, one can expect that there is something like an optimal $\mathcal{T}$. In fact, if the number of trading dates tends to infinity, it will not be efficient to trade at every date, because the transaction costs become relatively high compared to maximal possible price changes within a time period. Observe that this feature is characteristic for log-Lipschitz models in contrast to the Black-Scholes model.

This also elucidates, why trivial super-hedging is optimal within the Black-Scholes model under proportional transaction costs. The reason is simply that within the Black-Scholes model it seems useless to trade at a time point t_1,$0 = t_0 < t_1 < t_2 = T$, because the maximal possible price change from t_0 to t_1 is as large as that from t_0 to t_2. It is clear then, that this conclusion devolves to all time sets $\mathcal{T} \in \mathcal{TT}$. The only way out could be given by super hedges with trading at infinitely many time points. But because of transaction costs such strategies seem to be more expensive than the trivial super hedge. Because of these absurd observations we were looking for a class of processes with more plausible features. This finally lead us to consider log-Lipschitz processes.

$\diamondsuit$

Zusammenfassung

Die Arbeit entwickelt eine Arbitrage-Preistheorie für Finanzderivate unter Berücksichtugung von Transaktionskosten. Andere mögliche Marktreibungen werden ausgeschlossen. Ansonsten werden nur solche Annahmen an den Markt gestellt, die nötig sind, um ökonomisch sinnvolle, aussagekräftige Ergebnisse herzuleiten. Speziellere Annahmen wollten wir weitgehend vermeiden.

Die Arbeit trägt unter anderem zu folgenden Themen bei:

- Modellierung von Transaktionskosten,
- Fundamentalsatz der Preistheorie (mit und ohne Transaktionskosten) in diskreter und stetiger Zeit,
- Duale Charakterisierung von Arbitrage-Preisen (Superhedging-Preisen) unter Transaktionskosten,
- Quantile-Hedging unter Transaktionskosten,
- Alternativen zum Black-Scholes Modell in stetiger Zeit (unter Transaktionskosten).

In der Einleitung werden zunächst die theoretischen Grundbegriffe für die Analyse von Finanzderivaten eingeführt. Danach werden das Prinzip der Arbitrage-freien Bewertung erläutert und ein Überblick über die bestehende Preistheorie für reibungslose Märkte gegeben. Anschließend diskutieren wir die bestehende Theorie für Märkte mit (proportionalen) Transaktionskosten unter Einbeziehung unserer Resultate. Sowohl theoretische als auch Praxis-bezogene Überlegungen führen uns zu dem Schluss, dass man Transaktionskosten Volumen-proportional modellieren sollte. Man unterstellt damit, dass die mit einer Order verbundenen Transaktionskosten proportional zu Menge × Stück-Preis sind. Diese Annahme ist für die Praxis - zumindest bei Transaktionen ab einer gewissen Mindestgröße - durchaus zutreffend.

In Kapitel 2 stellen wir unseren allgemeinen Modellrahmen vor. In diesen Rahmen lassen sich Aktien- und Währungsmärkte mit oder ohne Transaktionskosten einordnen. Die Kostenstruktur kann dabei deterministisch oder zufällig sein. Damit berücksichtigen wir sowohl im voraus berechenbare Transaktionsgebühren wie auch zufällige Differenzen zwischen Geld- und Briefkursen. Trotz dieser geringen Einschränkungen können wir die zulässigen Transaktionen einheitlich durch differentielle Kegelrestriktionen beschreiben. Zwar sind die auftretenden Kegel bei Währungsmärkten andere als bei Aktienmärkten und bei deterministischen Transaktionskosten andere als bei schwankenden Differenzen zwischen Geld- und Briefkursen, doch diese Kegel haben wesentliche Gemeinsamkeiten. Indem wir uns im Rest der Arbeit nur auf die gemeinsamen Eigenschaften dieser Kegel beziehen, können wir erstmals ohne zusätzliche Fallunterscheidungen eine Theorie für alle genannten Marktsituationen gleichzeitig entwickeln. Allerdings müssen wir hinsichtlich der möglichen Handelsstrategien eine Unterscheidung zwischen Märkten mit diskreten Handelszeitpunkten und solchen mit kontinuierlichem Handel treffen. Aus technischen Gründen können für den Handel in diskreter Zeit teilweise inhaltlich stärkere Aussagen abgeleitet werden als in stetiger Zeit.

In Kapitel 3 wird der in Kapitel 2 dargestellte allgemeine Rahmen auf den Handel zu diskreten Zeitpunkten eingeschränkt. Wir führen den Begriff eines Portfolio-Prozesses ein. Portfolio-Prozesse beschreiben, wieviele Einheiten der am Markt gehandelten Wertpapiere sich zu welchen Zeitpunkten im Portfolio befinden. Der *Marktwert eines Portfolios* kann sich sowohl durch Preisänderungen der Anlagen als auch durch Transaktionskosten verändern. Dagegen kann sich die Zahl einer bestimmten Wertpapierart im Portfolio nur durch Kauf oder Verkauf dieses Wertpapieres verändern. Diese Eigenschaft macht es möglich, die Portfolioprozesse, die aus selbstfinanzierenden Handelsstrategien resultieren, allein durch die in Kapitel 2 eingeführten differentiellen Kegelbedingungen zu beschreiben.

In Kapitel 4 führen wir mehrer Definitionen für Arbitragegelegenheiten ein. Dabei handelt es sich um assymptotische Arbitragemöglichkeiten, sogenannte *Free Lunches*. Wir unterscheiden zwischen L^p-und L^∞-*Free Lunches*. Dazu korrespondierend beweisen wir je einen Fundamentalsatz der Preistheorie ohne zusätzliche Vorussetzungen an das Marktmodell. Die Abwesenheit von L^p-oder L^∞-*Free*

Lunches impliziert die Abwesenheit von klassischen Arbitragemöglichkeiten. Die umgekehrte Implikation gilt auf Märkten mit Transaktionskosten im allgemeinen nicht. Wir können sie aber für den Fall nachweisen, dass der Preisprozess eine zusätzliche Nicht-Degeneriertheitsbedingung erfüllt. Bisher war ein vergleichbares Resultat nur für Märkte mit zwei Handelszeitpunkten bekannt (Pham, Touzi ,1999).

In Kapitel 5 leiten wir schlielich noch eine duale Charakterisierung für approximative Super-Hedging-Bündel und Super-Hedging-Preise her. Wir können zeigen, dass approximative und exakte Super-Hedging-Bündel und Super-Hedging-Preise identisch sind falls die Nicht-Degeneriertheitsbedingung gilt.

Zeitkontinuierliche Märkte werden in Kapitel 6 dargestellt. Die zulässigen Portfolioprozesse werden ähnlich wie in diskreter Zeit allein durch die in Kapitel 2 eingeführten differentiellen Kegelbedingungen beschrieben. Zusätzliche Aufmerksamkeit legen wir jedoch auf sogenannte einfache Portfolioprozesse (*Simple Portfolio Processes*) und solche, die eine gewisse Beschränktheitsbedingung (*Tame Condition* erfüllen.

Mit Hilfe dieser einfachen, beschränkten Portfolioprozesse gelingt es uns, in Kapitel 7 je einen Fundamentalsatz der Preistheorie für L^p-und L^∞-*Free Lunches* in stetiger Zeit ohne zusätzliche Annahmen an den Preisprozess zu beweisen. Da wir aufgrund geschickter Definitionen mit einfachen Portfolioprozessen auskommen, ist unser Beweis weit weniger aufwendig als die Beweise der vergleichbaren Fundamentalsätze für sogenannte *Free Lunches with Vanishing Risk* von Delbaen und Schachermayer (1994, 1998).

In Kapitel 8 leiten wir schließlich noch eine duale Charakterisierung für approximative Super-Hedging-Bündel und Super-Hedging-Preise her. Dieses Ergebnis stelt eine Verallgemeinerung der Resultate von Kabanov (1999) dar. Dieser beweist eine duale Charakterisierung für exakte Super-Hedging-Bündel und Super-Hedging-Preise, setzt dabei aber die Stetigkeit des Preisprozesses voraus. Diese Bedingung konnten wir durch den Übergang zu approximativen Super-Hedging-Bündeln und Super-Hedging-Preisen vermeiden.

Die duale Charakterisierung für approximative Super-Hedging-Preise wird in Kapitel 9 verwendet um eine duale Charakterisierung von Quantile-Hedging-Preisen herzuleiten. Damit übertragen wir das entsprechende Resultat für reibungslose Märkte von Föllmer und Leukert (1999) auf Märkte mit Transaktionskosten.

In Kapitel 10 kommen wir auf die bekannte Vermutung von Clark und Davis zu sprechen, die inzwischen dreifach bewiesen ist (Soner et al., 1995, Leventhal, Skorohod, 1997, Cvitanić et al., 1999). Diese (bewiesene) Vermutung besagt, dass unter den Annahmen von Black und Scholes der optimale *Super Hedge* für jede Europäische Call Option in folgender trivialen Kaufen- und Halten-Strategie besteht: Bei Emission einer Call-Option erwirbt man gleichzeitig das Underlying und hält es bis zum Ausübungstag. Folglich ist der Super-Hedging-Preis einer Call Option exakt so hoch wie der Preis des Underlyings - ein absurdes Resultat. Wir ziehen daraus den Schluss, dass sich das Black-Scholes Modell zumindest unter Berücksichtigung von Transaktionskosten nicht zur Arbitragefreien Bewertung von Optionen eignet. Als eine mögliche Alternative schlagen wir eine Klasse von Prozessen vor, deren logarhythmierte Pfade gleichmäßig Lipschitz-stetig sind. Solche Log-Lipschitz Prozesse stellen im wesentlichen eine nichttriviale zeitstetige Verallgemeinerung von zeitdiskreten Prozessen mit multinomialen logarhythmierten Zuwächsen dar. Insbesondere lassen sich die Pfade von log-Lipschitz Prozessen durch Pfade geeigneter zeitdiskreter Prozesse mit multinomialen logarhythmierten Zuwächsen “eingrenzen”. Diese Eigenschaft kann genutzt werden, um nicht-triviale obere Schranken für Super-Hedging-Preise von Finanzderivaten mit konvexer Auszahlungsfunktion herzuleiten, wie etwa einer Europäischen Call Option. In ähnlicher Weise lassen sich damit auch nicht-triviale *Super Hedges* berechnen.

Appendix A

Banach Spaces

Let $\mathcal{X}$ be a Banach-space with norm $||.||$ and $\mathcal{X}^*$ the dual space of $\mathcal{X}$, i.e. the space of $||.||$-continous linear functionals on $\mathcal{X}$ equipped with the norm

$$||f|| = \sup_{||x|| \leq 1} |f(x)| \ .$$

Then with this norm $\mathcal{X}^*$ is a Banach-space (see Dunford-Schwartz, 1958, p. 60, Corollary 9). The closed unit balls of $\mathcal{X}$ resp. $\mathcal{X}^*$ are the sets

$$\mathcal{S} := \{x \in \mathcal{X} : ||x|| \leq 1\}$$

$$\mathcal{S}^* := \{f \in \mathcal{X}^* : ||f|| \leq 1\} \ .$$

The weak topology $\sigma(\mathcal{X}, \mathcal{X}^*)$ is the locally convex topology on $\mathcal{X}$ induced by the family of semi-norms

$$|.|_f : \mathcal{X} \to \mathbb{R}, \ x \mapsto |f(x)|, \quad f \in \mathcal{X}^* \ .$$

The weak* topology $\sigma(\mathcal{X}^*, \mathcal{X})$ is the locally convex topology on $\mathcal{X}$ induced by the semi-norms

$$|.|_x : \mathcal{X}^* \to \mathbb{R}, \ f \mapsto |f(x)|, \quad x \in \mathcal{X} \ .$$

Theorem A.1 *(Krein-Smulian Theorem, see Dunford-Schwartz, 1958, p. 429, Theorem 7) A convex set in $\mathcal{X}^*$ is $\sigma(\mathcal{X}^*, \mathcal{X})$-closed, if and only if its intersection with every positive multiple of $\mathcal{S}^*$ is $\sigma(\mathcal{X}^*, \mathcal{X})$-closed.*

The following Lemma is an exercise in Dunford-Schwartz (1958).

Lemma A.2 *(see Dunford-Schwartz, 1958, p. 437, Exercise 16) If $\mathcal{X}$ is a separable Banach-space, a convex subset $A \subset \mathcal{X}^*$ is $\sigma(\mathcal{X}^*, \mathcal{X})$-closed, if and only if $\forall n : f_n \in A$ and $\forall x \in \mathcal{X} : \lim_{n\to\infty} f_n(x) = f(x)$ imply $f \in A$.*

A Theorem similar to Lemma A.2 can be proofed, if we admit $\mathcal{X}$ to be a weakly compactly generated Banach space, which is a weaker assumption than separability.

Definition A.3 *(see Diestel, 1975, p.143) A Banach space $\mathcal{X}$ is said to be weakly compactly generated whenever there exists a weakly compact subset $\mathcal{K} \subset \mathcal{X}$ such that the linear span of $\mathcal{K}$ is dense in $\mathcal{X}$.*

Theorem A.4 *(see Diestel, 1975, p.163, Theorem 4) If $\mathcal{X}$ is a weakly compactly generated Banach space and $\mathcal{Z}$ is a closed linear subspace o $\mathcal{X}$, then the closed unit ball of $\mathcal{Z}^*$ is weak* sequentially compact.*

Corollary A.5 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mu_0, \mu_1, \ldots, \mu_d)$ a $d+1$-tuple of P-equivalent σ-finite measures on $(\Omega, \mathcal{F})$. Write $L^1 := \times_{i=0,\ldots,d} L^1(\mu_i)$ and $L^\infty := \times_{i=0,\ldots,d} L^\infty(\mu_i)$ and let $\sigma(L^1, L^\infty)$ be the weak topology on L^1 and $\sigma(L^\infty, L^1)$ the weak* topology on L^∞. Then the closed unit ball*

$$\mathcal{S}^\infty := \{C \in L^\infty : esssup_{i=0,\ldots,d}|C_i| \leq 1\}$$

is $\sigma(L^\infty, L^1)$ sequentially compact.

Proof. According to an example in Diestel (1975, p.143) the space $L^1(\mu)$ is weakly compactly generated for any σ-finite measure μ. Thus in our case, since the weak topology on a product space coincides with the product of the weak topologies, L^1 is also weakly compactly generated. Hence the assertion follows by Theorem A.4.

$\diamondsuit$

Lemma A.6 *Suppose $(\Omega, \mathcal{F}, \mu)$ is a σ-finite measure space. Then there exists a probabiltiy measure $P \sim \mu$ with Radon-Nikodym densitiy $\frac{dP}{d\mu} \in L^p(\mu)$ for every $1 \leq p \leq \infty$.*

Proof. The proof is almost identic with that of Lemma 17.6 in Bauer (1992). Although the assertion there is slightly weaker, Bauer's proof in fact covers our stronger assertion.

$\diamondsuit$

Lemma A.7 *Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and X is a $\mathbb{R}$-valued random variable with $X > 0$ P-a.s.. Then there exists a probability measure $\tilde{P} \sim P$ with Radon-Nikodym density $\frac{\tilde{P}}{P} \in L^\infty(P)$ for every $1 \leq q \leq \infty$ and satisfying $X \in L^1(\tilde{P})$.*

Proof. Consider the measure $XP \sim P$ defined by $\frac{dXP}{dP} := X$. Then as a result from the Radon-Nykodym theorem, since X is $\mathbb{R}$-valued, the measure XP is σ-finite. Then according to Lemma A.6 there exists a probability measure $Q \sim XP$ with Radon Nikodym density $f = \frac{dQ}{dXP} \in L^\infty(XP$ for every $1 \leq q \leq \infty$. We define the probability measure $\tilde{P} \sim P$ by

$$\frac{d\tilde{P}}{dP} := \frac{f}{E_P[f]} \; .$$

Note that $\frac{d\tilde{P}}{dP} \in L^\infty(P)$. We calulate

$$\begin{aligned} \int_\Omega X d\tilde{P} &= \int_\Omega X \frac{f}{E_P[f]} dP \\ &= \frac{1}{E_P[f]} \int_\Omega \frac{dQ}{dXP} X dP \\ &= \frac{1}{E_P[f]} \int_\Omega \frac{dQ}{dXP} dXP = \frac{1}{E_P[f]} \end{aligned}$$

and see that $X \in L(\tilde{P})$.

$\diamondsuit$

Lemma A.8 *Suppose $(\Omega, \mathcal{F}, P)$ is a probability space and $X = (X_i)_{i=0,\dots,d}$ is a $\mathbb{R}^{d+1}$-valued random vector with $X_i > 0$ P-a.s. for every $i \in \{0, \dots, d\}$. Then there exists a probability measure $\tilde{P} \sim P$ with Radon-Nikodym density $\frac{\tilde{P}}{P} \in L^\infty(P)$ and $X_i \in L^1(\tilde{P})$ for every $i \in \{0, \dots, d\}$.*

Proof. According to Lemma A.7, for $i \in \{0, \dots, d\}$ there exist probability measures $\tilde{P}_i$ sucht that $X_i \in L^1(\tilde{P}_i)$ and $\frac{d\tilde{P}_i}{dP} \in L^\infty(P)$. Hence if we define a finite measure

$\mu \sim P$ with $\frac{d\mu}{dP} \in L^\infty(P)$ for every $1 \le q \le \infty$ by

$$\frac{d\mu}{dP} := \min_{i\in\{0,\ldots,d\}} \frac{d\tilde{P}_i}{dP}$$

and then define $\tilde{P} \sim P$ by

$$\frac{\tilde{P}}{P} := \frac{1}{\mu(\Omega)}\frac{d\mu}{dP}$$

we get a probabiltiy measure $\tilde{P} \sim P$ as desired.

$\diamondsuit$

Theorem A.9 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mu_0, \mu_1, \ldots, \mu_d)$ a $d+1$-tuple of P-equivalent σ-finite measures on $(\Omega, \mathcal{F})$. Write $L^1 := \times_{i=0,\ldots,d} L^1(\mu_i)$ and $L^\infty := \times_{i=0,\ldots,d} L^\infty(\mu_i)$ and denote $\sigma(L^1, L^\infty)$ the weak topology on L^1 and $\sigma(L^\infty, L^1)$ the weak* topology on L^∞. If $\mathcal{K}$ is a convex cone in L^∞, then $\mathcal{K}$ is closed in $\sigma(L^\infty, L^1)$ if and only if for every sequence $(C^n)_{n\in\mathbb{N}}$ in $\mathcal{K}$ that is uniformly bounded and with every component $(C_i^n)_{n\in\mathbb{N}}$, $i = 0, 1\ldots,$ converging in probability P to a $C_i \in L^\infty$, we have $C = (C_i)_{i=0,\ldots,d} \in \mathcal{K}$.*

Proof.

Suppose $\mathcal{K}$ is a convex cone in L^∞. According to the Krein-Smulian Theorem, $\mathcal{K}$ is $\sigma(L^\infty, L^1)$ closed if and only if it's intersection with every positive multiple of

$$\mathcal{S}^\infty := \{C \in L^\infty : \operatorname{ess}\sup_{i=0,\ldots,d} |C_i| \le 1\}$$

is $\sigma(L^\infty, L^1)$-closed. Define $A(\lambda) := \mathcal{K} \cap \lambda\mathcal{S}^\infty$ for $\lambda > 0$. According to Corollary A.5, the set $\mathcal{S}^\infty$ is $\sigma(L^\infty, L^1)$ sequentially compact. Consequently $\mathcal{K}$ is closed, if and only if $A(\lambda)$ is $\sigma(L^\infty, L^1)$ compact for every $\lambda > 0$. In fact if $\mathcal{K}$ is $\sigma(L^\infty, L^1)$ closed then, since $\lambda\mathcal{S}^\infty$ is $\sigma(L^\infty, L^1)$ sequentially compact, every sequence in $A(\lambda) \subset \lambda\mathcal{S}^\infty$ has a $\sigma(L^\infty, L^1)$ convergent subsequence with limit in $A(\lambda)$, because $A(\lambda)$ is $\sigma(L^\infty, L^1)$ closed. The converse assertion is a direct implication of the Krein-Smulian theorem.

i) " $\Rightarrow$ ". Suppose $\mathcal{K}$ is a convex $\sigma(L^\infty, L^1)$ closed cone and let $(C^n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}$ that is uniformly bounded and with every component $(C_i^n)_{n\in\mathbb{N}}$ converging in probability P to a $C_i \in L^\infty$, $i = 0, 1\ldots, d$. Since for every i we have $\mu_i \sim P$, every component $(C_i^n)_{n\in\mathbb{N}}$ also converges in measure μ_i to this $C_i \in L^\infty$. Let $Y = (Y_i)_{i=0,\ldots,d}$ be an arbitrary element of L^1. Then for every i the uniformly bounded sequence $(C_i^n)_{n\in\mathbb{N}}$ is uniformly integrabel with respect to the finite measures $Y_i^+\mu_i$ and $Y_i^-\mu_i$ defined by

$$\frac{dY_i^+\mu_i}{d\mu_i} := \max\{Y_i, 0\}$$
$$\frac{dY_i^+\mu_i}{d\mu_i} := -\min\{Y_i, 0\}$$

and hence converging to C_i in $L^1(Y_i^+\mu_i)$ and in $L^1(Y_i^-\mu_i)$. Consequently, for every $i \in \{0,\ldots,d\}$ the sequence $(\int_\Omega C_i^n Y_i d\mu_i)_{n\in\mathbb{N}}$ converges to $\int_\Omega C_i Y_i d\mu_i$. Since Y was arbitrary chosen, we conclude that $(C^n)_{n\in\mathbb{N}}$ converges to C in the $\sigma(L^\infty, L^1)$ topology. Thus $C \in \mathcal{K}$, because otherwise $\mathcal{K}$ would not be closed with respect to $\sigma(L^\infty, L^1)$.

ii) " $\Leftarrow$ ". Suppose $\mathcal{K}$ is a convex cone in L^∞. We have to show that $A(\lambda)$ is sequentially $\sigma(L^\infty, L^1)$ compact for every $\lambda > 0$. For arbitrary $\lambda > 0$ let $(C^n)_{n\in\mathbb{N}}$ be a sequence in $A(\lambda) \subset \lambda\mathcal{S}^\infty$. Since $\lambda\mathcal{S}^\infty$ is $\sigma(L^\infty, L^1)$ sequentially compact there is a $\sigma(L^\infty, L^1)$ convergent subsequence $(C^{k(n)})_{k\in\mathbb{N}}$ with limit $C \in L^\infty$, i.e.

$$\forall Y \in L^1 : \lim_{k\to\infty} \sum_{i=0}^{d} \int_\Omega C_i^{k(n)} Y \, d\mu_i = \sum_{i=0}^{d} \int_\Omega C_i Y \, d\mu_i$$

which implies

$$\forall i \in \{0,\ldots,d\} \forall Y_i \in L^1(\mu_i) : \lim_{k\to\infty} \int_\Omega C_i^{k(n)} Y \, d\mu_i = \int_\Omega C_i Y \, d\mu_i \ .$$

Let $i \in \{0,\ldots\}$ and $B \in \mathcal{F}$ with $\mu_i(B) < \infty$ be arbitrary but fixed. Then consider the sets

$$A^k := B \cap \{C_i^k(n) - C_i \geq 0\} \in \mathcal{F}$$

and define

$$\mathcal{I} := \{I = \bigcap_{k\in\mathbb{N}} I^k : \forall k \in \mathbb{N} : I^k \in \{A^k, B \setminus A^k\}\} \ .$$

Let $I \in \mathcal{I}$. Then for every $k \in \mathbb{N}$ we either have $(C_i^k - C_i)1_{I^k} \geq 0$ or $(C_i^k - C_i)1_{I^k} \leq 0$. Having this in mind, by eventually separating $(C^{k(n)})_{k\in\mathbb{N}}$ into two subsequences it is easy to verify that

$$\lim_{k\to\infty} \int_\Omega (C_i^{k(n)} - C_i) 1_I d\mu_i = 0$$

implies

$$\lim_{k\to\infty} \int_\Omega |C_i^{k(n)} - C_i| 1_I d\mu_i = 0 \ .$$

Since $\mathcal{I}$ is countable by construction and because of $B = \cup_{I\in\mathcal{I}} I$, the σ-finiteness of μ implies

$$\lim_{k\to\infty} \int_\Omega |C_i^{k(n)} - C_i| 1_B d\mu_i = 0 \ .$$

Since B and i were arbitrary chosen, we are able to conclude the following. For each $i \in \{0, \ldots, d\}$ and for every $B \in \mathcal{F}$ with $\mu_i(B) < \infty$ we have for every $\alpha > 0$:

$$\begin{aligned} \lim_{k\to\infty} \mu_i(\{|C_i^{k(n)} - C_i| \geq \alpha\} \cap B) &= \lim_{k\to\infty} \mu_i(|C_i^{k(n)} - C_i| 1_B \geq \alpha) \\ &\leq \lim_{k\to\infty} \frac{1}{\alpha} \int_\Omega |C_i^{k(n)} - C_i| 1_B d\mu_i = 0 \ . \end{aligned}$$

There we have used the Chebyshev-Markov inequality. Hence every component $(C_i^{k(n)})_{n\in\mathbb{N}}$, $i = 0, 1 \ldots$, of the uniformly bounded sequence $(C^{k(n)})_{n\in\mathbb{N}}$ in $\mathcal{K}$ converges in probability $P \sim \mu_i$ to a $C_i \in L^\infty$. Thus we have $C = (C_i)_{i=0,\ldots,d} \in \mathcal{K}$.

$\diamondsuit$

Appendix B

Multifunctions

Definition B.1 *Suppose $\mathcal{C}$ and $\mathcal{X}$ are two sets and let F be a multifunction from $\mathcal{C}$ to $\mathcal{X}$, that maps $C \in \mathcal{C}$ to a subset $F(C) \subset \mathcal{X}$. The inverse F^{-1} of F is defined by*

$$F^{-1}(x) := \{C \in \mathcal{C} : x \in F(C)\} \ , x \in \mathcal{X} \ .$$

In addition we define

$$F(\mathcal{B}) := \bigcup_{C \in \mathcal{B}} F(C) \quad , \mathcal{B} \subset \mathcal{C}$$

$$F^{-1}(\mathcal{Y}) := \bigcup_{x \in \mathcal{Y}} F^{-1}(x) = \{C \in \mathcal{C} : F(C) \cap \mathcal{Y} \neq \emptyset\} \quad , \mathcal{Y} \subset \mathcal{X} \ .$$

If $\mathcal{X}$ is a metric space, then a multifunction F is called closed-valued, if for all C the set $F(C)$ is closed (comp. Rockafellar 1976, p.159). *The graph of a multifunction is defined as*

$$graph(F) = \{(C, x) : x \in F(C)\} \ .$$

Given a metric space $(\mathcal{C}, \mathcal{T})$ we call a multifunction F from $(\mathcal{C}, \mathcal{T})$ to $\mathbb{R}^{d+1}$ closed, if it's graph is closed (comp. Rockafellar, 1970, p. 415). (Rockafellar (1970) only considers convex-valued multifunctions from $\mathbb{R}^n$ to $\mathbb{R}^m$ and calls them convex processes.) For two multifunctions F_1, F_2 we define as a short notation the partial ordering

$$F_1 \subseteq F_2 :\Leftrightarrow \forall C \in \mathcal{C} : F_1(C) \subseteq F_2(C)$$

Proposition B.2 *If F is a multifunction from a metric space $\mathcal{C}$ to $\mathbb{R}^{d+1}$ (or another metric space) then the multifunction $\hat{F}$ defined by*

$$\hat{F}(C) = \{h \in \mathbb{R}^{d+1} : \exists (C^n, h^n) : h^n \in F(C^n), h^n \to h, C^n \to C\}$$

is closed. $\hat{F}$ is called the closure of F.

Proof. Suppose $(C^n, h^n)_{n\in\mathbb{N}}$ is a sequence such that $h^n \in F(C^n)$ for every $nin\mathbb{N}$ and $h^n \to h$, $C^n \to C$. Then for every n there exists a sequence $(C^{mn}, h^{mn})_{m,nin\mathbb{N}}$ such that $h^{mn} \in F(C^{mn})$ for every $m, nin\mathbb{N}$ and $h^{mn} \to h^n$ and $C^{mn} \to C^n$ for every $nin\mathbb{N}$. It is then possible by diagonal extraction to choose a sequence $(C^{m(n)}, h^{m(n)})_{n\in\mathbb{N}}$ such that $h^{m(n)} \in F(C^{m(n)})$ for every $nin\mathbb{N}$ and $C^{m(n)} \to C$, $h^{m(n)} \to h$. This shows $h \in \hat{F}(C)$.

$\diamondsuit$

Remark B.3 If F is a closed multifunction, then

$$(2.1) \qquad F(C) = \{h \in \mathbb{R}^{d+1} : \exists (C^n, h^n) : h^n \in F(C^n), h^n \to h, C^n \to C\}.$$

In fact, if F is closed, then the inclusion $F(C) \subseteq \ldots$ is trivial and the other inclusion follows from the closedness of the graph.

(2.1) also implies that F and F^{-1} are closed-valued. To verify this for F take a sequence $h^n \in F(C)$ converging to some $h \in \mathbb{R}^{d+1}$. For every n there exists a sequence $(C^{m,n}, h^{m,n})_{m\in\mathbb{N}}$ such that $h^{m,n} \in F(C^{m,n})$ for every $m \in \mathbb{N}$ and $C^{m,n} \to C$,$h^{m,n} \to h$. The remainder again is done by diagonal extraction. For F^{-1} the argumentation is similar.

$\diamondsuit$

Definition B.4 *(c.f. Rockafellar, 1976, p.159 f.) A closed-valued multifunction F from a measurable space $(\Omega, \mathcal{F})$ into $\mathbb{R}^n$ is said to be $\mathcal{F}$-measurable, if for every closed set $B \subset \mathbb{R}^n$ we have*

$$F^{-1}(B) \in \mathcal{F} \ .$$

Proposition B.5 *(c.f. Rockafellar, 1976, Proposition 1A) For a closed-valued multifunction F from a measurable space $(\Omega, \mathcal{F})$ into $\mathbb{R}^n$ the following properties are equivalent:*

i) F is measurable;

ii) for each $z \in \mathbb{R}^n$ the function $\omega \mapsto dist(z, F(\omega))$ is $\mathcal{F}$-measurable.

Definition B.6 *Let $(\Omega, \mathcal{F})$ be a measurable space. A function $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ is called a* **Caratheodory mapping** *if for every $x \in \mathbb{R}^n$ the function $F(., x)$ is $(\mathcal{F}, \mathbb{B}^m)$-measurable and for every $\omega \in \Omega$ the function $F(\omega, .)$ is continuous.*

Theorem B.7 *(c.f. Rockafellar, 1976, Theorem 2J) Let be* $(\Omega, \mathcal{F})$ *is a measurable space and suppose* $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^m$ *is a Caratheodory mapping. For every* $\omega \in \Omega$ *consider the set convex set*

$$\Phi(\omega) := \{x \in real^n : f(\omega, x) \leq 0\} .$$

Then the multifunction Φ *is* $\mathcal{F}$*-measurable, and hence* Φ *has a measurable selection where it is nonempty-valued.*

The following Lemma is actually a Corollary of Theorem 2J in Rockafellar (1976), although it is not straightforward.

Lemma B.8 *Let* $(\Omega, \mathcal{M})$ *be a measurable space and* Ξ *a* $\mathcal{M}$*-measurable polyhedral multifunction with convex values* $\Xi(\omega) \subset \mathbb{R}^n$ *,*$\omega \in \Omega$*. This means there are measurable vectors*

$$W_i = (W_{ij})_{j=0,1,\ldots,n} : \Omega \to \mathbb{R}^{n+1} \quad , \; i = 1, \ldots, m, \quad m \in \mathbb{N}$$

such that

$$\Xi(\omega) = \{x \in \mathbb{R}^n : \sum_{j=1}^{n} x_i W_i(\omega) \leq W_0(\omega)\} . \tag{2.2}$$

(Note that Ξ *is measurable according to Theorem 2J in Rockafellar (1976).) Denote* $extr\Xi(\omega)$ *the set of extreme points of* $\Xi(\omega)$*. Then* $extr\Xi : \omega \mapsto extr\Xi(\omega)$ *is a* $\mathcal{M}$*-measurable multifunction.*

Proof. From (2.2) we see that for every $\omega \in \Omega$ we have $x \in extr\Xi(\omega)$, if and only if $x \in \Xi(\omega)$ and there is a suitable subset $I \subset \{1, \ldots, m\}$ with $|I| = n$ such that x is the unique solution of the system of linear equations

$$\sum_{j=1}^{n} x_i W_i(\omega) \leq W_0(\omega) \quad , i \in I .$$

For $W_i = (W_{ij})_{j=0,1,\ldots,n}$ let $\hat{W}_i := (W_{ij})_{j=1,\ldots,n}$. Denote $\mathcal{I}$ the set of subsets $I \subset \{1, \ldots, m\}$ with $|I| = n$. Observe that $\mathcal{I}$ has finitely many elements. For every $I \in \mathcal{I}$ the set

$$M(I) := \{\omega \in \Omega : det(\hat{W}_i : i \in I) = 0\}$$

is $\mathcal{M}$-measurable, because the determinant $det(.)$ is a $\mathbb{B}^{n\times n}$-measurable function. For every $I \in \mathcal{I}$ let us define the multifunction Υ^I from Ω to $\mathbb{R}^n$ by $\Upsilon^I(\omega) = \emptyset$ for $\omega \in M(I)$ and

$$\Upsilon^I(\omega) = \{x \in \mathbb{R}^n : \sum_{j=1}^{n} x_i W_i(\omega) \leq W_0(\omega) \quad , i \in I\} \ .$$

Then from Theorem 1B and Theorem 2J in Rockafellar (1976) we know that Υ^I is $\mathcal{M}$-measurable.

Now it is clearly evident that for every $\omega \in \Omega$ we have

$$extr\Xi(\omega) = \bigcup_{I\in\mathcal{I}} \Upsilon^I(\omega) \cap \Xi(\omega) \ .$$

Hence, according to Propostion 1L and Theorem 1M in Rockafellar (1976), the multifunction $extr\Xi$ is $\mathcal{M}$-measurable.

$\diamondsuit$

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List of Symbols

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