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Approximation of energy forms on finitely ramified fractals by discrete graphs and related metric measure spaces

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Contents

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Abstract

Our goal is to approximate energy forms on suitable fractals by discrete graph energies and certain metric measure spaces, using the notion of quasi-unitary equivalence. Quasi-unitary equivalence generalises the two concepts of unitary equivalence and norm resolvent convergence to the case of operators and energy forms defined in varying Hilbert spaces.

More precisely, we prove that the canonical sequence of discrete graph energies (associated with the fractal energy form) converges to the energy form (induced by a resistance form) on a finitely ramified fractal in the sense of quasi-unitary equivalence. Moreover, we allow a perturbation by magnetic potentials and we specify the corresponding errors.

This aforementioned approach is an approximation of the fractal from within (by an increasing sequence of finitely many points). The natural step that follows this realisation is the question whether one can also approximate fractals from outside, i.e., by a suitable sequence of shrinking supersets. We partly answer this question by restricting ourselves to a very specific structure of the approximating sets, namely so-called graph-like manifolds that respect the structure of the fractals resp. the underlying discrete graphs. Again, we show that the canonical (properly rescaled) energy forms on such a sequence of graph-like manifolds converge to the fractal energy form (in the sense of quasi-unitary equivalence).

From the quasi-unitary equivalence of energy forms, we conclude the convergence of the associated linear operators, convergence of the spectra and convergence of functions of the operators – thus essentially the same as in the case of the usual norm resolvent convergence.

Zusammenfassung

Das Ziel dieser Arbeit ist es, Energieformen auf geeigneten Fraktalen durch Energieformen, definiert auf diskreten Graphen und anderen graphartigen metrischen Maßräumen, zu approximieren. Dazu wenden wir das Konzept der *Quasi-unitären Äquivalenz* an. Quasi-unitäre Äquivalenz verallgemeinert die beiden bekannten Konzepte *unitäre Äquivalenz* und *Normresolventenkonvergenz* und erlaubt uns, Energieformen (und deren zugehörige Operatoren), die in verschiedenen Hilberträumen definiert sind, zu vergleichen.

Präziser gesagt, werden wir folgendes beweisen: Wir zeigen, dass die kanonische Folge von diskreten Graphen-Energieformen, induziert durch eine *Resistance-Form* auf einem *endlich-verzweigten Fraktal* (finitely ramified fractal), im Sinne der oben genannten verallgemeinerten Normresolventenkonvergenz gegen die Resistance-Form konvergiert. Wir zeigen außerdem, dass dies ebenfalls wahr ist, falls die Resistance-Form durch ein magnetisches Potenzial gestört ist.

Der oben beschriebene Prozess ist eine Approximation des Fraktals von innen (durch eine wachsende Folge diskreter Punkte). Die natürliche Frage, die sich nun stellt, ist es, ob wir Fraktale auch von außen approximieren können. Genauer: Können wir Fraktale durch eine Folge von schrumpfenden offenen Obermengen beliebig genau annähern? Wir können diese Frage teilweise beantworten: Wir zeigen, dass wir ein sog. PCF-Fraktal (post-critically finite self-similar fractals), eine Unterklasse der endlich-verzweigten Fraktale, durch eine geeignete Folge von graphartigen Mannigfaltigkeiten (die die Struktur der kanonischen Graphapproximationen genügen) approximieren können – wieder im Sinne der verallgemeinerten Normresolventenkonvergenz.

Die Folgerungen der Quasi-unitären Äquivalenz sind im Prinzip die gleichen, wie die der gewöhnlichen Normresolventenkonvergenz: Konvergenz der zugehörigen linearen Operatoren, Konvergenz der Spektra und Konvergenz von Funktionen der Operatoren.

CHAPTER 1

Introduction

Our goal is to apply the abstract setting of convergence for self-adjoint linear operators and their corresponding energy forms (i.e., symmetric closed quadratic forms) defined in different Hilbert spaces, developed in [**Pos06**] to the case of finitely ramified fractals that support an energy form in the sense of [**Kig03**] and [**Tep08**]. More precisely, a finitely ramified fractal can be approximated by a monotonously increasing sequence of finite discrete graphs and we prove that an appropriately scaled sequence of graph energies converges to the energy form on the fractal in the sense of [**Pos06**]. Moreover, from the sequence of discrete graphs, we can construct a sequence of graph-like manifolds. We show that – given reasonable assumptions – the corresponding energy forms also converge.

This convergence goes by the name *generalised norm resolvent convergence* and it is derived from the notion of *quasi-unitary equivalence* described in [**Pos06, Pos12**]. It generalises the two well-know concepts of unitary equivalence and norm resolvent convergence to the case of varying Hilbert spaces. The consequences are essentially the same as for the usual norm resolvent convergence, i.e., convergence of eigenvalues (w.r.t. multiplicity), convergence of eigenfunctions (not just in the Hilbert space norm but also in energy norm) and convergence of functions of the operator. In the following, we will illustrate our approach and the *plan of action* using the concrete example of the Sierpiński gasket, which is the most common example for a *post-critically finite self-similar fractal* (in the sense of [**Kig01**], see Definition 4.4.1), a subclass of finitely ramified fractals. Other common examples are the unit interval, the Koch curve or the pentagasket (see Figure 1).

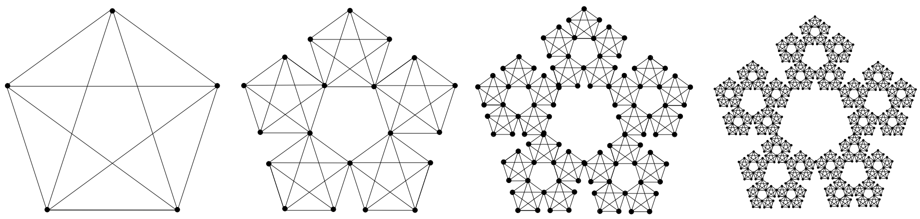

FIGURE 1. [**PS18a**, Fig. 1] The pentagasket's approximating graphs G_m for the generations $m = 0, \dots, 3$.

1.1. The Sierpiński gasket and its energy form

Let us briefly introduce the most common example of a post-critically finite self-similar fractal in the sense of [**Kig01**]. For a mathematical rigid and detailed construction of the following, we refer the reader to [**Str06**] or [**Kig01**].

1.1.1. The Sierpiński gasket. Let $V_0 := \{p_1, p_2, p_3\}$ be the vertices of an equilateral triangle in $\mathbb{R}^2$. We define a so-called *iterated function system (IFS)*

$F = \{F_j\}_{j=1,2,3}$ by

$$F_j \colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \qquad F_j(x) = \frac{1}{2} \cdot (x - p_j) + p_j, \qquad j = 1, 2, 3.$$

Then there exists a unique non-empty compact subset $K \subset \mathbb{R}^2$ such that

$$K = F(K) \coloneqq F_1(K) \cup F_2(K) \cup F_3(K)$$

and we call K the *Sierpiński gasket*. Moreover, we call V_0 the *boundary* of K. Of course, V_0 is not the topological boundary of K (as subset of $\mathbb{R}^2$) since the interior of K is empty.

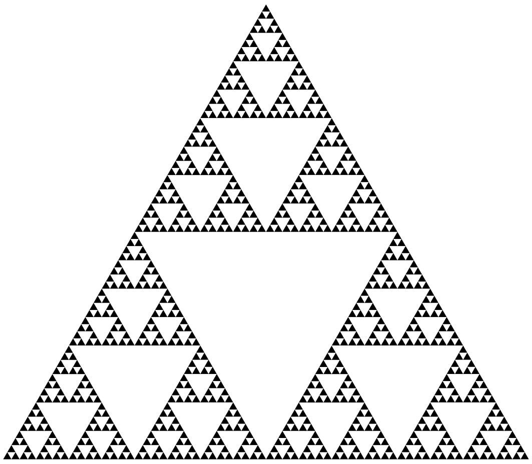

FIGURE 2. An approximation of the Sierpiński gasket.

The IFS does not only describe the fractal itself but also determines a cell structure on K in the following sense: Let $W_m \coloneqq \{1, 2, 3\}^m$ denote the collection of all words of length $m \geq 0$ over the alphabet $\{1, 2, 3\}$. We define a map via

$$W_m \ni w = w_1 \dots w_m \longmapsto F_w(K) \coloneqq F_{w_1} \circ \dots \circ F_{w_m}(K)$$

and for all $w \in W_m$, we call $F_w(K)$ an m-cell (see Figure 3). Note that we have $F_{wj}(K) \subset F_w(K)$ for all $w \in W_m$ and $j = 1, 2, 3$, since $F_j(K) \subset K$. Hence, a sequence of cells of K related to a word $w \coloneqq w^{(m)} = w_1 w_2 \dots w_m$ is decreasing in the *length* $|w| \coloneqq m$ and one can show that there is a unique point $z \in K$ such that $\lim_{m \to \infty} F_{w^{(m)}}(K) = \{z\}$.

We can also apply the IFS to the boundary V_0, i.e., for each $m \geq 0$, we define

$$V_m \coloneqq \bigcup_{w \in W_m} F_w(V_0).$$

The sequence $\{V_m\}_{m \in \mathbb{N}_0}$ is increasing, i.e., $V_m \subset V_{m+1}$ for all $m \geq 0$ and the limit $V_\star \coloneqq \lim_{m \to \infty} V_m$ is a dense subset of K (w.r.t. the relative topology). Moreover, it

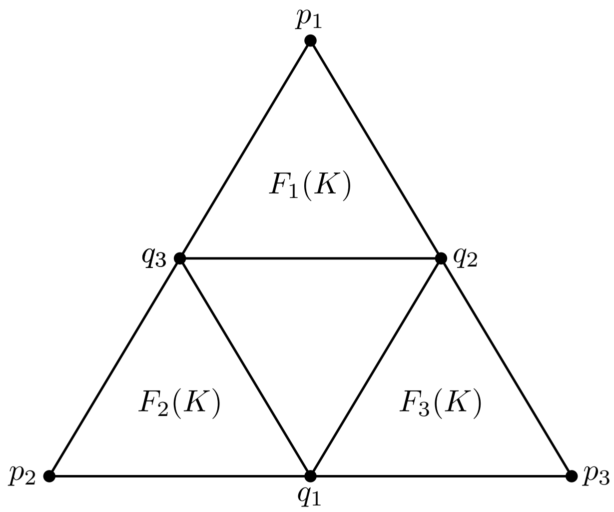

FIGURE 3. [**Kig01**, Fig. 1.2] The cell structure of the Sierpiński gasket K, described by the IFS $F = \{F_1, F_2, F_3\}$ with boundary $V_0 = \{p_1, p_2, p_3\}$. Here, $q_1 = F_2(p_3) = F_3(p_2)$, $q_2 = F_1(p_3) = F_3(p_1)$ and $q_3 = F_1(p_2) = F_2(p_1)$ are the three junction points where distinct 1-cells can intersect.

is a key property of fractals like the Sierpiński gasket that two distinct cells of the same generation m can only intersect in *finitely* many points, namely, we have

$$F_w(K) \cap F_{w'}(K) = F_w(V_0) \cap F_{w'}(V_0),$$

for all $w, w' \in W_m$ and $m \in \mathbb{N}$. This precisely means that any two distinct cells of generation m can only intersect in points of $V_m \setminus V_0$ and one sometimes refers to elements of $V_\star \setminus V_0$ as *junction points*.

1.1.2. Approximating sequence of discrete graphs. Next, we approximate the Sierpiński gasket from the inside by a monotonously increasing sequence of finite discrete graphs. Let G_0 be the complete graph with vertex set V_0 and for $m \in \mathbb{N}$, we define $G_m \coloneqq (V_m, E_m)$, with edges defined by

$$E_m \coloneqq \{ e \,|\, e = (p, q) \text{ such that } p \sim_m q \},$$

where $p \sim_m q$ if and only if $p, q \in V_m$, $p \neq q$ and there exists a word w of length $|w| = m$ such that $p, q \in F_w(K)$. Then there is a canonical energy form on G_m, given by

$$\mathcal{E}_m(f) \coloneqq \sum_{(p,q) \in E_m} \gamma_m \cdot |f(p) - f(q)|^2,$$

for all $f\colon V_m \longrightarrow \mathbb{C}$, where $\gamma_m \geq 1$ is a constant which we specify now. Note that the energy forms $\mathcal{E}_m$ are not necessarily related to each other in any sense. We would like to have the following connection: Let $\varrho\colon V_m \longrightarrow \mathbb{C}$. Then we want to specify γ_m such that the following minimisation problem,

$$\mathcal{E}_m(\varrho) = \min\{\, \mathcal{E}_{m+1}(f) \,\big|\, f\colon V_{m+1} \longrightarrow \mathbb{C} \text{ and } f\restriction_{V_m} = \varrho \,\}$$

has a solution for any given *boundary values* ϱ. More precisely, this means that we require the existence of a function $g\colon V_{m+1} \longrightarrow \mathbb{C}$ such that $\mathcal{E}_m(\varrho) = \mathcal{E}_{m+1}(g)$. In the case of the Sierpiński gasket the existence is guaranteed if we choose

$$\gamma_m = \left(\frac{5}{3}\right)^m.$$

The solution of the above defined minimisation problem g is uniquely determined and we call g the *harmonic extension (to V_{m+1}) with boundary values ϱ.*

The number $5/3$ is called *renormalisation factor* (and the particular value is of course specific to the Sierpiński gasket). For a general post-critically finite self-similar fractal, it is not clear, that a solution to the minimisation problem described above exists. However, if such a number exists, then we call $\{(G_m, \mathcal{E}_m)\}_{m\geq 0}$ a *compatible sequence.* In particular, if the sequence $\{(G_m, \mathcal{E}_m)\}_{m\geq 0}$ is compatible, then the graph energies fulfil the following relation: Let $f\colon V_m \longrightarrow \mathbb{C}$. We have

$$\mathcal{E}_m(f) = \sum_{j=1}^{3} \frac{5}{3} \cdot \mathcal{E}_{m-1}(f \circ F_j) = \sum_{w\in W_m} \left(\frac{5}{3}\right)^m \mathcal{E}_0(f \circ F_w).$$

1.1.3. Energy form on the Sierpiński gasket. The compatibility of the above defined sequence $\{(G_m, \mathcal{E}_m)\}_{m\geq 0}$ also ensures, that for each $u\colon V_\star \longrightarrow \mathbb{C}$, the limit

$$\mathcal{E}_\star(u) := \lim_{m\to\infty} \mathcal{E}_m(u\restriction_{V_m}),$$

exists in in $[0, \infty]$. This is an immediate consequence of the compatibility, since $u\restriction_{V_{m+1}}$ is just *any* extension of $u\restriction_{V_m}$ and not necessarily the harmonic extension, i.e., the sequence $\{\mathcal{E}_m(u\restriction_{V_m})\}_m$ is monotonously increasing. Moreover, since $V_\star$ is a dense subset of K, we can define an energy form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ on the Sierpiński gasket by extending $\mathcal{E}_\star$, where the domain $\operatorname{dom}\mathcal{E}$ consists of all continuous functions $u\colon K \longrightarrow \mathbb{C}$ such that the above limit $\mathcal{E}_\star(u\restriction_{V_\star})$ is finite, i.e.,

$$\operatorname{dom}\mathcal{E} = \{\, u \,\big|\, u \in \mathsf{C}(K) \text{ and } \lim_{m\to\infty} \mathcal{E}_m(u\restriction_{V_m}) < \infty \,\}.$$

A direct consequence of the compatibility is the following: The energy form $\mathcal{E}$ is *self-similar*, i.e., for all $u \in \operatorname{dom}\mathcal{E}$, we have

$$\mathcal{E}(u) = \sum_{j=1}^{3} \frac{5}{3} \cdot \mathcal{E}(u \circ F_j) = \sum_{w\in W_m} \left(\frac{5}{3}\right)^m \mathcal{E}(u \circ F_w).$$

Of course, in order to justify the term *energy form*, we need a suitable Hilbert space (or L_2-space), i.e., we need to introduce a measure. Let μ be the homogeneous and self-similar Hausdorff measure, i.e., μ is a Borel regular probability measure on K associating the same weight $\mu(F_w(K)) = 1/3^m$ to each m-cell. It can be shown that $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is indeed an energy form in $\mathsf{L}_2(K, \mu)$ (which is usually called *Dirichlet form* in this context).

On each discrete graph G_m, an edge weight $\gamma_m = \{\gamma_{e,m}\}_{e \in E_m}$ is fixed by our choice of the conductance $\gamma_{m,e} = (5/3)^m$ $(e \in E_m)$. Moreover, a suitable vertex weight $\mu_m = \{\mu_m(v)\}_{v \in V_m}$ is also already fixed by the choice of the measure μ on the fractal. It turns out that the following definition is the most natural one:

$$\mu_m(p) := \int_K \psi_{p,m}(x)\, \mathrm{d}\mu(x) = \begin{cases} \dfrac{2}{3} \cdot \Big(\dfrac{1}{3}\Big)^m & p \in V_m \setminus V_0, \\ \dfrac{1}{3} \cdot \Big(\dfrac{1}{3}\Big)^m & p \in V_0. \end{cases}$$

Here $\psi_{p,m} \colon K \longrightarrow [0,1]$ denotes the m-harmonic function with boundary values $\varrho = \mathbb{1}_{\{p\}}$ on V_m. The concrete numbers in the latter equality follow from the symmetry of the fractal. We will discuss this in more details later on. A simple calculation now shows that the symmetric linear operator Δ_{μ_m} associated with the discrete energy form $\mathcal{E}_m$ (and depending on the vertex measure μ_m) is given by

$$\Delta_{\mu_m} f(p) = \frac{3}{2} \sum_{q \sim_m p} 5^m \big(f(p) - f(q)\big),$$

for any function $f \colon V_m \longrightarrow \mathbb{C}$ and point $p \in V_m \setminus V_0$.

1.2. Main results

1.2.1. Quasi-unitary equivalence of energy forms. The notion of quasi-unitary equivalence was first introduced in [**Pos06**] and most of the mentioned consequences and results from Chapter 2 were already stated therein. However, we give a more structured introduction here and improve some of the results (published in [**PS19b**]). In particular, we give a simplified proof to Proposition 2.2.3 which states the transitivity of the notion.

What do we mean by transitivity? Let $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ be three energy forms defined in the Hilbert spaces $\mathscr{H}_1$, $\mathscr{H}_2$ and $\mathscr{H}_3$ respectively. Moreover, let $\mathcal{E}_1$ and $\mathcal{E}_2$ be quasi-unitarily equivalent and also $\mathcal{E}_2$ and $\mathcal{E}_3$. Then $\mathcal{E}_1$ and $\mathcal{E}_3$ are quasi-unitarily equivalent as well.

The transitivity is a very important consequence for our applications and one reason that makes most of our proofs quite nice: It allows us to combine several small steps instead of having to make one big one. More precisely, that means that we can utilise the *closeness* (i.e., the quasi-unitary equivalence of a sequence of energy forms with decreasing errors; the meaning of *close* in this context will become clear later in the introduction, when we introduce our approach) of the fractals to a sequence of discrete graphs and the closeness of graph-like manifolds to the same discrete graphs and then use the transitivity to conclude that such fractals and graph-like manifolds are also close. Moreover, in the special situation when two of the three energy forms are defined in the *same* Hilbert space (i.e., $\mathscr{H}_2 = \mathscr{H}_3$), we can apply Proposition 2.2.4, which gives us a better error estimate. We use the latter version when we introduce a magnetic potential for the energy form on a finitely ramified fractal. Here, our first step is to restrict ourselves to *simple* magnetic potentials (in the sense that they are constant on cells of a fixed generation) and in a second step, we prove that we can also allow general magnetic potential. In this particular situation, the energy form and the Hilbert space are fixed and we just change the 1-form.

Though the statement is not surprising, Theorem 2.3.8 (cf. [**PS18a**]) is a new result and states the convergence of eigenfunctions of the corresponding operators not just in norm but also in energy norm. Note that the main purpose of [**PS19b**] and Chapter 2 is to give a brief and easily accessible overview of the notion, to briefly explain the consequences and to improve some of the older results (with respect to applicability and error estimates) and simplify their proofs.

1.2.2. Approximating fractals by discrete graphs. In the second chapter, we apply the concept of quasi-unitary equivalence, mentioned above, to the case of magnetic energy forms on finitely ramified fractals generalising earlier results from [**PS18a**]. In this article, the authors showed that the canonical energy form on a post-critically finite self-similar fractal (see Definition 4.4.1) is quasi-unitarily equivalent with the approximating sequence of energy forms defined in the associated sequence of discrete graphs (approximating the fractal from the inside). In Chapter 3, we generalise these results in two ways. First, we allow a more general class of spaces: Our approach does not depend on the self-similarity of the fractal and we also do not need that at each junction point only finitely many cells may intersect. However, we do require the space to support a cellular structure and thus the possibility to approximate it by a sequence of discrete graphs (in the sense of [**Kig03**], which we also briefly explain in Section 3.1 and Section 3.2). Second, we generalise the result by allowing a magnetic potential. These potentials have lately become of more interest, which we comment in greater details in Chapter 3. The notion of quasi-unitary equivalence has already been applied to the concrete example of a magnetically perturbed energy form on the Sierpiński gasket in [**PS18c**].

Instead of just mentioning the results here, let us understand the basic idea behind our approach. Therefore, we focus on the concrete example where the fractal is the Sierpiński gasket with its standard energy form (as described above). This brief introduction follows the content of [**PS18a**] and [**PS18c**].

In the situation of Section 1.1 in this introduction, we have two energy forms. First, we have $\mathcal{E}_m$ defined in the Hilbert space $\ell_2(V_m, \mu_m)$ and we have $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ defined in $\mathsf{L}_2(K,\mu)$. The definition of quasi-unitary equivalence (see Definition 2.2.1) needs two pairs of so-called identification operators J, J' and J^1, J'^1, linking the Hilbert spaces and the form domains, respectively. Finding these operators is usually the difficult part. However, we can make use of the compatibility of $\{(G_m, \mathcal{E}_m)\}_{m\in\mathbb{N}_0}$.

Let us start by mapping a function, defined in $\ell_2(V_m, \mu_m)$, to an L_2-function on the Sierpiński gasket. We define $J := J_m \colon \ell_2(V_m,\mu_m) \longrightarrow \mathsf{L}_2(K,\mu)$ by

$$Jf(x) = \sum_{p\in V_m} f(p)\psi_{p,m}(x),$$

where $x \in K$ and where $\psi_{p,m} \colon K \longrightarrow [0,1]$ is the m-harmonic function with boundary values $\varrho := \mathbb{1}_{\{p\}}$ for $p \in V_m$. The choice of the operator J – in particular the multiplication by $\psi_{x,m}$ and not just a sort of indicator function – will become clear later, when we define J^1 and J'^1. For the other direction, we choose $J' \colon \mathsf{L}_2(K,\mu) \longrightarrow$

$\ell_2(V_m, \mu_m)$ to be the adjoint $J' \coloneqq J'_m = J^\star$, given by

$$J'u(p) = \frac{1}{\mu_m(p)} \langle u, \psi_{p,m} \rangle_{\mathsf{L}_2(K,\mu)},$$

for all $p \in V_m$. We need to verify that both operators satisfy part one of Definition 2.2.1, i.e., we need to verify the following inequalities:

$$\|Jf\| \le (1+\delta)\|f\| \qquad \bigl|\langle Jf, u\rangle - \langle f, J'u\rangle\bigr| \le \delta \|f\|\,\|u\|$$

for all $f \in \ell_2(V_m, \mu_m)$ and $u \in \mathsf{L}_2(K, \mu)$ and

$$\|f - J'Jf\| \le \delta \|f\|_{\mathcal{E}_m} \qquad \|u - JJ'u\| \le \delta \|u\|_{\mathcal{E}}$$

for all $f \in \ell_2(V_m, \mu_m)$ and $u \in \operatorname{dom} \mathcal{E}$, where $\|.\|$ denotes the corresponding Hilbert space norm and where $\|.\|_{\mathcal{E}}^2 \coloneqq \|.\|^2 + \mathcal{E}(.)$ denotes the energy norm.

The first inequality holds with $\delta = 0$, i.e., $\|Jf\| \le \|f\|$, which follows directly from the Cauchy-Young inequality and the fact that $\{\psi_{p,m}\}_{p \in V_m}$ is a partition of unity. Also, the second one is trivially fulfilled with $\delta = 0$ by the choice $J' = J^\star$. We comment on the missing two inequalities later.

Furthermore, we need two additional operators J^1 and J'^1 in order to compare the form domains. The simple choice $J^1 \colon \ell_2(V_m, \mu_m) \longrightarrow \operatorname{dom} \mathcal{E}$, $J^1 f = Jf$ in the case of the first operator makes sense, because the m-harmonic functions are all contained in $\operatorname{dom} \mathcal{E}$. Moreover, we define $J'^1 \colon \operatorname{dom} \mathcal{E} \longrightarrow \ell_2(V_m, \mu_m)$ to be the evaluation map $J'^1 u = u\restriction_{V_m}$. This choice is justified because functions in the domain of $\mathcal{E}$ are continuous. Now we need to prove that the following inequality

$$\bigl|\mathcal{E}(J^1 f, u) - \mathcal{E}_m(f, J'^1 u)\bigr| \le \delta \|f\|_{\mathcal{E}} \|u\|_{\widetilde{\mathcal{E}}},$$

holds for all $f \in \ell_2(V_m, \mu_m)$ and $u \in \operatorname{dom} \mathcal{E}$. Note that by polarisation and a simple argument (essentially using the definition of m-harmonic functions), we deduce the following formula,

$$\mathcal{E}(u, h) = \mathcal{E}_m(u\restriction_{V_m}, h\restriction_{V_m}),$$

whenever $u \in \operatorname{dom} \mathcal{E}$ and h is m-harmonic. This is true, since m-harmonic functions minimise the energy by the compatibility of the sequence of approximating graph energies. Hence, if we plug in $h = \psi_{p,m}$ we see that the above inequality holds with $\delta = 0$.

Remark 1.2.1. *We would like to stress that this property of the energy forms, i.e.*

$$\mathcal{E}(J^1 f, u) = \mathcal{E}_m(f, J'^1 u)$$

for all $f \in \ell_2(V_m, \mu_m)$ and $u \in \operatorname{dom} \mathcal{E}$, makes the comparison of the graph energies and the energy form on the Sierpiński gasket very pleasant. In our second application, where we compare a fractal with a graph-like manifold, we need to put in much more effort to prove that the energies are close. The reason is that points in a graph-like manifold do not have positive capacity in general and hence, we cannot define an evaluation map J'^1 that easily but we need to average over a suitable neighbourhood of the vertex.

Finally, we need to verify that the identification operators on the Hilbert space level and the ones on the form domains are also close, i.e., we need

$$\|J^1 f - Jf\| \le \delta \|f\|_{\mathcal{E}_m} \qquad\qquad \|J'^1 u - J'u\| \le \delta \|u\|_{\mathcal{E}}$$

for all $f \in \ell_2(V_m, \mu_m)$ and $u \in \operatorname{dom} \mathcal{E}$. Since $J = J^1$, the first inequality again holds with $\delta = 0$. By the choice of our identification operators, we could already meet four of seven inequalities. So, we just need to verify the following three inequalities:

$$\|f - J'Jf\| \le \delta \|f\|_{\mathcal{E}_m}, \qquad \|u - JJ'u\| \le \delta \|u\|_{\mathcal{E}} \qquad \text{and} \qquad \|J'^1 u - J'u\| \le \delta \|u\|_{\mathcal{E}},$$

where $f \in \ell_2(V_m, \mu_m)$ and $u \in \operatorname{dom} \mathcal{E}$ is arbitrary. These missing three inequalities need some more details about our fractals and the energy forms. Let us discuss them from left to right.

Let $f \colon V_m \longrightarrow \mathbb{R}$ and $p \in V_m$. Since $\mathbb{1}_K = \sum_{q \in V_m} \psi_{q,m}$, we have

$$\mu_m(p) = \int_K \psi_{p,m}(x) \,\mathrm{d}\mu(x) = \sum_{q \in V_m} \langle \psi_{p,m}, \psi_{q,m} \rangle_{\mathsf{L}_2(K,\mu)},$$

for each $p \in V_m$, and thus, for any vertex $q \in V_m$,

$$f(q) - J'Jf(q) = \frac{1}{\mu_m(q)} \sum_{p \in V_m} \big(f(q) - f(p)\big) \langle \psi_{p,m}, \psi_{q,m} \rangle_{\mathsf{L}_2(K,\mu)}.$$

Hence, estimating in norm using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\|f - J'Jf\|^2_{\ell_2(V_m,\mu_m)} \\
&\quad= \sum_{q \in V_m} \frac{1}{\mu_m(q)} \Big| \sum_{p \in V_m} \big(f(q) - f(p)\big) \langle \psi_{p,m}, \psi_{q,m} \rangle_{\mathsf{L}_2(K,\mu)} \Big|^2 \\
&\quad\le \sum_{q \in V_m} \frac{1}{\mu_m(q)} \sum_{p \in V_m} \Big(\frac{3}{5}\Big)^m \langle \psi_{p,m}, \psi_{q,m} \rangle^2_{\mathsf{L}_2(K,\mu)} \sum_{p \sim_m q} \Big(\frac{5}{3}\Big)^m \big|f(q) - f(p)\big|^2 \\
&\quad\le \Big(\frac{3}{5}\Big)^m \Big(\max_{q \in V_m} \frac{1}{\mu_m(q)} \sum_{p \in V_m} \langle \psi_{p,m}, \psi_{q,m} \rangle^2_{\mathsf{L}_2(K,\mu)} \Big) \mathcal{E}_m(f).
\end{aligned}$$

We can further estimate the maximum by

$$\begin{aligned}
\max_{q \in V_m} \frac{1}{\mu_m(q)} \sum_{p \in V_m} \langle \psi_{p,m}, \psi_{q,m} \rangle^2_{\mathsf{L}_2(K,\mu)} &\le \max_{q \in V_m} \sum_{p \in V_m} \langle \psi_{p,m}, \psi_{q,m} \rangle_{\mathsf{L}_2(K,\mu)} \\
&\le \max_{q \in V_m} \mu_m(q) \le \mu_{+,m},
\end{aligned}$$

where we used that $\langle \psi_{p,m}, \psi_{q,m} \rangle \le \langle \mathbb{1}, \psi_{q,m} \rangle = \mu_m(q)$ for the first inequality, the partition of unity property for the second one and where $\mu_{+,m}$ is defined by

$$\mu_{+,m} := \max\Big\{ \max_{w \in W_m} \mu(F_w(K)), \max_{q \in V_m} \mu_m(q) \Big\}.$$

This particular choice $\mu_{+,m} > 0$ will become clear after we have discussed the next inequality. Combining the above, we see that we can choose

$$\delta_m := \Big(\frac{3}{5}\Big)^{m/2} \sqrt{\mu_{+,m}}$$

and that $\delta_m \to 0$ exponentially as $m \to \infty$.

Instead of proving the second one, it is sufficient to verify that the inequality holds with J' replaced by J'^1 using Lemma 2.2.5 (this simplification does indeed impact the error making it slightly worse, but the difference is just a constant factor). In our situation, this is much easier to deal with since J'^1 is just the evaluation map. Moreover, we do not need to apply a Poincaré type estimate and thus our δ_m does not depend on an eigenvalue estimate. By the above mentioned lemma, we need to verify the following inequality,

$$\|u - JJ'^1 u\| \le \delta' \|u\|_{\mathcal{E}}.$$

Hence, the notion provides the user with some flexibility and may be adopted to the specific needs. For any function $u \in \operatorname{dom} \mathcal{E}$, we get

$$u - JJ'^1 u = \sum_{p \in V_m} u\psi_{p,m} - \sum_{p \in V_m} u(p)\psi_{p,m} = \sum_{p \in V_m} \big(u - u(p)\big)\psi_{p,m},$$

using again that $\{\psi_{p,m}\}_{p \in V_m}$ is a partition of unity and that we can evaluate functions in the domain of $\mathcal{E}$ by continuity. Moreover, we can further simplify the above expression by localisation to the m-cells of the Sierpiński gasket. More precisely, we have

$$u - JJ'^1 u = \sum_{w \in W_m} \sum_{p \in F_w(V_0)} \big(u - u(p)\big)\psi_{p,m}\restriction_{F_w(K)}$$

and, of course, this is also true for the L_2-norm $\|.\|_{\mathsf{L}_2(K,\mu)}$, since two different m-cells can only intersect in finitely many points (i.e., a set of measure zero). Thus,

$$\|u\|_{\mathsf{L}_2(K,\mu)} = \sum_{w \in W_m} \big\|u\restriction_{F_w(K)}\big\|_{\mathsf{L}_2(F_w(K),\mu)},$$

for each $u \in \mathsf{L}_2(K,\mu)$. From the self-similarity of the energy form $\mathcal{E}$, we can even deduce a very simple decomposition formula for the fractal energy form $\mathcal{E}$: Let $w \in W_m$ and $u \in \operatorname{dom} \mathcal{E}$. Then, we define the localised energy form

$$\mathcal{E}_{F_w(u)}(u\restriction_{F_w(K)}) := \Big(\frac{5}{3}\Big)^m \mathcal{E}(u \circ F_w)$$

and hence, by the self-similarity of the energy form, we get

$$\mathcal{E}(u) = \sum_{w \in W_m} \mathcal{E}_{F_w(u)}(u\restriction_{F_w(K)}).$$

The estimate in norm now follow from standard arguments and the fact that functions in the domain of the fractal energy form are actually $1/2$-Hölder continuous (w.r.t. the so-called *resistance metric* R, described naturally by the energy form; see Definition 3.1.2), i.e., we have

$$|u(x) - u(y)|^2 \le R(x,y)\mathcal{E}_{F_w(u)}(u\restriction_{F_w(K)}) \le \Big(\frac{3}{5}\Big)^m \mathcal{E}_{F_w(u)}(u\restriction_{F_w(K)}),$$

for any $u \in \operatorname{dom}\mathcal{E}$ and $x, y \in F_w(K)$ for any word w of length m. Now, we are prepared to estimate in norm. We have,

$$\begin{aligned}
\|u - JJ'^1 u\|^2_{\mathsf{L}_2(K,\mu)} &= \sum_{w \in W_m} \|u - JJ'^1 u\|^2_{\mathsf{L}_2(F_w(K),\mu)} \\
&\le \sum_{w \in W_m} \sum_{p,q \in F_w(V_0)} \langle \psi_{p,m}, \psi_{q,m} \rangle_{\mathsf{L}_2(F_w(K),\mu)} \cdot \\
&\qquad \cdot \max_{x,z \in F_w(K)} |u(x) - u(z)|^2 \cdot \max_{y,z \in F_w(K)} |u(y) - u(z)|^2 \\
&\le \Big(\frac{3}{5}\Big)^m \sum_{w \in W_m} \sum_{p,q \in F_w(V_0)} \langle \psi_{p,m}, \psi_{q,m} \rangle_{\mathsf{L}_2(F_w(K),\mu)} \mathcal{E}_{F_w(K)}(u\restriction_{F_w(K)}) \\
&= \Big(\frac{3}{5}\Big)^m \sum_{w \in W_m} \mu(F_w(K)) \mathcal{E}_{F_w(K)}(u\restriction_{F_w(K)}) \le \underbrace{\Big(\frac{3}{5}\Big)^m \mu_{+,m}}_{={\delta'_m}^2} \mathcal{E}(u).
\end{aligned}$$

And we conclude, that the above equation is fulfilled with the same error δ'_m. However, we need to be careful here: As we exchanged one of the inequalities from Definition 2.2.1, this leads to a modified error δ_m overall. We keep this in mind and fix the error later.

Finally, the missing inequality is basically using the same arguments, i.e., localisation, the $1/2$-Hölder continuity and standard arguments from Hilbert space theory. Let $u \in \operatorname{dom}\mathcal{E}$ and $p \in V_m$. Then

$$\begin{aligned}
J'u(p) - J'^1 u(p) &= \frac{1}{\mu_m(p)} \langle u - u(p)\mathbb{1}, \psi_{p,m} \rangle_{\mathsf{L}_2(K,\mu)} \\
&= \sum_{w \in W_{p,m}} \frac{1}{\mu_m(p)} \langle u - u(p)\mathbb{1}, \psi_{p,m} \rangle_{\mathsf{L}_2(F_w(K),\mu)},
\end{aligned}$$

where $W_{p,m} := \{\, w \mid w \in W_m \text{ such that } F_w(K) \ni p \,\}$ denotes the collection of words of length m whose associated cells contain p. We can estimate the above as follows:

$$\begin{aligned}
\big|J'u(p) - J'^1 u(p)\big|^2 &\le \Big(\sum_{w \in W_{p,m}} \frac{1}{\mu_m(p)} \int_{F_w(K)} |u - u(p)| \psi_{p,m}(x) \,\mathrm{d}\mu(x) \Big)^2 \\
&\le \Big(\frac{3}{5}\Big)^m \max_{w \in W_{p,m}} \mathcal{E}_{F_w(K)}(u\restriction_{F_w(K)}) \Big(\frac{1}{\mu_m(p)} \underbrace{\sum_{w \in W_{p,m}} \int_{F_w(K)} \psi_{p,m}(x) \,\mathrm{d}\mu(x)}_{=\mu_m(p)} \Big)^2 \\
&= \Big(\frac{3}{5}\Big)^m \max_{w \in W_{p,m}} \mathcal{E}_{F_w(K)}(u\restriction_{F_w(K)}) \le \Big(\frac{3}{5}\Big)^m \sum_{w \in W_{p,m}} \mathcal{E}_{F_w(K)}(u\restriction_{F_w(K)}),
\end{aligned}$$

where we used the Hölder continuity in the second inequality. Hence, in norm, we get

$$\begin{aligned}\|J'u - J'^1u\|^2_{\ell_2(V_m,\mu_m)} &\leq \Big(\frac{3}{5}\Big)^m \sum_{p\in V_m}\sum_{w\in W_{p,m}} \mathcal{E}_{F_w(K)}(u\restriction_{F_w(K)})\mu_m(p)\\ &= \Big(\frac{3}{5}\Big)^m \sum_{w\in W_m} \mathcal{E}_{F_w(K)}(u\restriction_{F_w(K)}) \sum_{p\in F_w(V_0)} \mu_m(p)\\ &\leq 3\mu_{+,m}\Big(\frac{3}{5}\Big)^m \sum_{w\in W_m} \mathcal{E}_{F_w(K)}(u\restriction_{F_w(K)}) = 3\mu_{+,m}\Big(\frac{3}{5}\Big)^m \mathcal{E}(u),\end{aligned}$$

where we used $|V_0| = 3$ to estimate $\sum_{p\in F_w(V_0)} \mu_m(p) \leq 3\mu_{+,m}$ in the last line.

Collecting all the individual error term δ_m (and δ'_m) and applying Lemma 2.2.5, we can calculate our desired error: The first two inequalities of Definition 2.2.1 hold with $\delta_a = 0$, we specified δ'_m from Lemma 2.2.5 by $\delta'_m = (3/5)^{m/2}\sqrt{\mu_{+,m}}$ and the last error is $\delta_c = 3(3/5)^{m/2}\sqrt{\mu_{+,m}}$. Hence, we get

$$\delta_m = (1+\sqrt{3})\Big(\frac{3}{5}\Big)^{m/2}\sqrt{\mu_{+,m}}.$$

We have proven that the discrete energy form $\mathcal{E}_m$ in $\ell_2(V_m,\mu_m)$ and the fractal energy form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ in $\mathsf{L}_2(K,\mu)$ are *δ_m-quasi unitarily equivalent*. Moreover, since this is true for all $m \in \mathbb{N}_0$ and, in addition, since the sequence $\{\delta_m\}_{m\in\mathbb{N}_0}$ converges to zero (as $m \to \infty$), we say that the sequence $\{\mathcal{E}_m\}_{m\in\mathbb{N}_0}$ of discrete graph energies converges to $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ in *generalised norm resolvent sense*. Let us formulate the above as our first main theorem:

Theorem 1.2.2 ([**PS18a**, Theorem 1.1]). *Let μ be a Borel regular probability measure of full support. The discrete graph energy form $\mathcal{E}_m$ in the Hilbert space $\ell_2(V_m,\mu_m)$ and the fractal energy form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ in $\mathsf{L}_2(K,\mu)$ are δ_m-quasi-unitarily equivalent with*

$$\delta_m = (1+\sqrt{3})\Big(\frac{3}{5}\Big)^{m/2}\sqrt{\mu_{+,m}}.$$

Moreover, since $\delta_m \to 0$ as $m \to \infty$, the sequence $\{\mathcal{E}_m\}_m$ converges to $\mathcal{E}$ in generalised norm resolvent sense.

Note, that in the calculations made above, we did not need the concrete representation of the measure, i.e., the theorem indeed holds for any measure μ as stated. If μ is the homogeneous self-similar measure with weights $1/3$, then $\mu_{+,m} = 1/3^m$ (which is the measure of an arbitrary m-cell) and the error is given by

$$\delta_m = \frac{1+\sqrt{3}}{5^{m/2}}.$$

As mentioned earlier, we will discuss this in greater details in Chapter 3. Moreover, we generalise the above theorem to the case of suitable finitely ramified fractals (specified there) and we allow a perturbation by magnetic potentials. Let us briefly present the idea for magnetic potentials here (following the introduction from [**HKM⁺17**]):

Let $\mathcal{F} := \operatorname{dom}\mathcal{E}$. We define the Hilbert module $\mathcal{H}$ of 1-forms on the Sierpiński gasket associated with the energy form $\mathcal{E}$ as the completion of $\mathcal{F}\otimes\mathcal{F}$ with respect to $\|.\|^2_{\mathcal{H}} = \langle .,.\rangle_{\mathcal{H}}$, with non-negative bilinear form

$$\langle u_1\otimes u_2, v_1\otimes v_2\rangle_{\mathcal{H}} := \frac{1}{2}\big(\mathcal{E}(u_1u_2v_2, v_1) + \mathcal{E}(v_1u_2v_2, u_1) - \mathcal{E}(u_1v_1, u_2v_2)\big),$$

modulo the space of elements with norm zero. Then there is a derivation

$$\partial\colon \mathcal{F}\longrightarrow\mathcal{H}, \qquad \partial u = u\otimes\mathbb{1},$$

i.e., a linear operator, satisfying the Leibniz rule $\partial(uv) = u\partial v + v\partial u$ for all $u,v\in\mathcal{F}$ and such that

$$\|\partial u\|^2_{\mathcal{H}} = \mathcal{E}(u),$$

for all $u\in\mathcal{F}$. Let $a\in\mathcal{H}$ be a real valued 1-form. Then, we can define the magnetic energy form on the Sierpiński gasket with magnetic potential a by

$$\mathcal{E}^a(u) := \big\|(\partial + ia)u\big\|^2_{\mathcal{H}},$$

where $u\in\operatorname{dom}\mathcal{E}^a = \mathcal{F} = \operatorname{dom}\mathcal{E}$.

Next, we need to find a suitable discrete approximation for the potential: Since $\mathcal{E}$ is a resistance form, the module structure of $\mathcal{H}$ extends to multiplication with more general functions. In particular, we can multiply by $\mathbb{1}_{K_w}$. Thus, we define

$$\mathcal{H}_m := \operatorname{span}\big\{\, h\otimes\mathbb{1}_{K_w} \,\big|\, h \text{ is an } m\text{-harmonic function and } w\in W_m \,\big\}.$$

These spaces are closed (since they are finite dimensional) and we have $\mathcal{H}_m\subset\mathcal{H}_{m+1}$. Moreover, $\bigcup_m \mathcal{H}_m$ is a dense subset of $\mathcal{H}$. Let $P_m\colon\mathcal{H}\longrightarrow\mathcal{H}_m$ be the orthogonal projection and denote $a_m := P_m a$ for any $a\in\mathcal{H}$. Then, one can show that the sequence $\{a_m\}_m$ is compatible, i.e., $P_m a_{m+1} = a_m$, and that $\|a_m\|_{\mathcal{H}} \le \|a_{m+1}\|_{\mathcal{H}}$. Moreover, we have

$$\|a_m - a\|_{\mathcal{H}} \to 0 \qquad \text{as} \qquad m\to\infty.$$

Conversely, for any bounded sequence $\{a_m\}_m$ with $a_m\in\mathcal{H}_m$, there exists an $a\in\mathcal{H}$ such that $\lim_{m\to\infty} a_m = a$ in $\mathcal{H}$.

Now we are prepared to define magnetic energy forms on G_m. Let us fix $m\in\mathbb{N}$. Let $a_m\in\mathcal{H}_m$. Then there exist m-harmonic functions $A_w\in\operatorname{dom}\mathcal{E}$ such that

$$a_m = \sum_{w\in W_m}\partial A_w\mathbb{1}_{K_w} = \sum_{w\in W_m} A_w\otimes\mathbb{1}_{K_w}$$

and this finally allows us to define the discrete magnetic energy form on G_m associated with a magnetic potential a_m by

$$\mathcal{E}^{a_m}_m(f) = \sum_{w\in W_m}\sum_{p,q\in F_w(V_0)}\left(\frac{5}{3}\right)^m\left|f(p)\mathrm{e}^{\mathrm{i}A_w(p)} - f(q)\mathrm{e}^{\mathrm{i}A_w(q)}\right|^2.$$

To prove the convergence $\mathcal{E}^{a_m}_m\to\mathcal{E}^{a_m}$ in generalised norm resolvent sense for magnetic potentials $a_m\in\mathcal{H}_m$, we use literally the same arguments as in the non-perturbed case but we need to modify our partition of unity: We define

$$\psi^{a_m}_{p,m} := \sum_{w\in W_{p,m}} \mathrm{e}^{\mathrm{i}A_w(p)-\mathrm{i}A_w}\psi_{p,m}\restriction_{\operatorname{int}(K_w)}$$

and extend continuously to the boundary. We denote the extension also by $\psi_{p,m}^{a_m}$. Note that we have $\psi_{p,m}^{a_m}\restriction_{V_m} = \mathbb{1}_{\{p\}}$ and that $\psi_{p,m}^{a_m} \in \operatorname{dom} \mathcal{E}$. Moreover, $\psi_{p,m}^{a_m}$ is m-harmonic with respect to $\mathcal{E}^{a_m}$. In the case of a simple potential (i.e., if $a_m \in \mathcal{H}_m$), we get the same result as in Theorem 1.2.2 with the same error δ_m.

If $a \in \mathcal{H}$ is not contained in $\mathcal{H}_m$, then we can just define $a_m := P_m a \in \mathcal{H}_m$ and we need to prove that $\mathcal{E}_m^{a_m} \to \mathcal{E}^a$ in generalised norm resolvent sense. This is again an application of the transitivity of the notion of quasi-unitary equivalence (see Proposition 2.2.4). It turns out that this approach works for suitable potentials but with slightly more complicated error δ_m. In Subsection 3.3.2 we discuss all the details.

1.2.3. Approximating fractals by metric graphs. The content of this section is already published [**PS18c**, Section 5.1].

A metric graph M_m is given by a discrete (and oriented) graph $G_m = (V_m, E_m)$ together with a function $\ell_m \colon E_m \longrightarrow (0, \infty)$ that assigns a length $\ell_{m,e} > 0$ to each edge $e \in E_m$. Let us fix the length scaling $\ell_{m,e} = 1/2^m$, which is motivated by the geometry of the Sierpiński gasket (the IFS scales also with contraction ratio $1/2$). Then M_m is the topological space

$$M_m := \bigcup_{e \in E_m} M_{m,e}/\omega,$$

where we identify the endpoints of $M_{m,e} := [0, \ell_{m,e}]$ via the function ω that maps $0 \in M_{m,e}$ onto the initial and $\ell_{m,e} \in M_{m,e}$ onto the terminal vertex according to the graph structure. As metric on M_m we choose the shortest path between points x and y. The (canonical) measure ν_m on M_m is the sum of the Lebesgue measures $\mathrm{d}x_e$ on $M_{m,e}$. We can actually identify M_m with the m^{th} iteration of the IFS applied to an equilateral triangle M_0 of side length 1, as we have chosen the same length scaling $\ell_{m,e} = 1/2^m$ as for the IFS.

As Hilbert space on M_m we have $\mathsf{L}_2(M_m, \nu_m)$ with norm (and hence inner product)

$$\|u\|^2_{\mathsf{L}_2(M_m,\nu_m)} = \sum_{e \in E_m} \int_0^{\ell_{m,e}} |u_e(x)|^2 \,\mathrm{d}x_e$$

where we identify functions u with the family $\{u_e\}_{e \in E_m}$, $u_e \colon [0, \ell_{m,e}] \longrightarrow \mathbb{C}$. The (rescaled) energy form on the metric graphs is then given by

$$\mathcal{E}_{M_m}(u) := 3 \cdot \left(\frac{5}{4}\right)^m \|u'\|^2_{\mathsf{L}_2(M_m,\nu_m)} \qquad \text{with} \qquad \operatorname{dom} \mathcal{E}_{M_m} = \mathsf{H}^1(M_m),$$

where $\mathsf{H}^1(M_m)$ consists of functions $u_e \in \mathsf{H}^1(M_{m,e})$ that are continuous on M_m. The associated operator is the Kirchhoff Laplacian. Note here, that we need to multiply the energy form on the metric graph with the factor $3 \cdot (5/4)^m$ in order to take into account the different nature of metric graphs and the Sierpiński gasket.

Let μ be the homogeneous self-similar measure on K – this assumption is for simplicity only. For the identification operators, we define

$$\widetilde{J}_m \colon \ell_2(G_m, \mu_m) \longrightarrow \mathsf{L}_2(M_m, \nu_m), \quad \widetilde{J}_m f := \frac{1}{\sqrt{3}}\Big(\frac{2}{3}\Big)^{m/2} \sum_{p \in V_m} f(p)\widetilde{\psi}_{p,m}$$

where $\widetilde{\psi}_{p,m}$ denotes the affine linear function on the edges of M_m with vertex values $\mathbb{1}_{\{p\}}$. It is again a key feature of the family $\{\widetilde{\psi}_{p,m}\}_{x \in V_m}$ that it forms a partition of unity. For $\widetilde{J}'_m$ we choose the adjoint of J_m, i.e.,

$$\widetilde{J}'_m \colon \mathsf{L}_2(M_m, \nu_m) \longrightarrow \ell_2(G_m, \mu_m), \quad \widetilde{J}'_m u(p) = \sqrt{3} \cdot \Big(\frac{3}{2}\Big)^{m/2} \frac{1}{\nu_m(p)} \int_{M_m} u\widetilde{\psi}_{p,m} \, \mathrm{d}\nu_m,$$

where $\nu_m(p) = \int \widetilde{\psi}_{p,m} \, \mathrm{d}\nu_m$. As before, we choose $\widetilde{J}^1_m = \widetilde{J}_m$ and $\widetilde{J}'^1_m$ as the evaluation (but also normalised by the inverse of $c_m^2 := {}^1\!/\!_3 \cdot ({}^2\!/\!_3)^m$). Our main result here is the following:

Theorem 1.2.3 ([**PS18b**, Thm. 5.4]). *The discrete energy form $\mathcal{E}_m$ on $G_m = (V_m, E_m)$ and the rescaled energy form $\mathcal{E}_{M_m}$ on the associated metric graph M_m are δ'_m-quasi-unitarily equivalent with $\delta'_m = \mathrm{O}({}^1\!/\!_{5^{m/2}})$.*

From this result, together with Theorem 1.2.2 and the transitivity of the notion of quasi-unitary equivalence (see Proposition 2.2.3), we conclude:

Corollary 1.2.4 ([**PS18b**, Cor. 5.6]). *The energy form $\mathcal{E}$ on the Sierpiński gasket and the corresponding energy $\mathcal{E}_{M_m}$ on the associated metric graphs are δ'_m-quasi-unitary equivalent, where $\delta'_m = \mathrm{O}({}^1\!/\!_{5^{m/2}})$.*

It follows from Definition 2.1.2 and Proposition 2.2.2 that the rescaled metric graph (Kirchhoff) Laplacians Δ_{M_m} converge to the Laplacian Δ on the Sierpiński gasket (with homogeneous self-similar probability measure) in generalised norm resolvent sense with error of order $\mathrm{O}({}^1\!/\!_{5^{m/2}})$.

Note that the error is of the same order as before when we compared fractals and discrete graphs. This is not surprising as we will see in Subsection 3.4.1, that a metric graph is just a finitely ramified fractal.

1.2.4. Approximating fractals by graph-like manifolds. The content of this section is already published [**PS18c**, Section 5.2].

Roughly speaking, a *graph-like manifold* X_m with underlying discrete graph $G_m = (V_m, E_m)$ is a Riemannian manifold of dimension $d \geq 2$ glued together from vertex neighbourhoods $\check{X}_{m,p}$ and edge neighbourhoods $X_{m,e}$ respecting the structure of the discrete graph G_m, i.e.,

$$X_m = \bigcup_{p \in V_m} \check{X}_{m,p} \cup \bigcup_{e \in E_m} X_{m,e},$$

where $\partial_e \check{X}_{m,p} := \check{X}_{m,p} \cap X_{m,e} \neq \emptyset$ if $e \in E_{m,p}$ (which denotes the subset of all edges with initial or terminal vertex p) and all other pairs of $\check{X}_{m,p}$ and $X_{m,e}$ are disjoint. Moreover, $\partial_e \check{X}_{m,p}$ is isometric with a $(d-1)$-dimensional Riemannian manifold Y_m

and $X_{m,e}$ is isometric with $[0, \ell_{m,e}] \times Y_m$ for some $\ell_{m,e} > 0$ (see Definition 4.1.1). We assume that the transversal manifold Y_m and the vertex manifolds $\check{X}_{m,p}$ scale as

$$Y_m = \mathrm{E}^m Y_0 \qquad \text{and} \qquad \check{X}_{m,v} = \mathrm{E}^m \check{X}_p,$$

for $1/10 < \mathrm{E} < 1/2$, respectively, with $\mathrm{vol}_{d-1} Y_0 = 1$. Here, εX is a Riemannian manifold $(X, \varepsilon^2 g)$ constructed from (X, g) by length scaling with the factor ε. We provide more details and precise definitions later in Chapter 4 (see also [**Pos12**] or [**PS18b**, Section 4]).

A simple example is the $\mathrm{E}^m/2$-neighbourhood of the metric graph M_m embedded in $\mathbb{R}^2$ (obtained e.g. as the m^{th} application of the IFS to an equilateral triangle of side length 1). In this case, $Y_m = [-\mathrm{E}^m/2, \mathrm{E}^m/2]$. In order to make some space for the vertex neighbourhoods $\check{X}_{m,p}$, we need to slightly shorten the length $\ell_{m,e}$ in $[0, \ell_{m,e}] \times Y_m$ compared to the one of the metric graph, which is $1/2^m$. But this shortened length will not affect the error (see the discussion in [**PS18b**, Ex. 4.2]).

Let ν_m denote the Riemannian measure on X_m. Associated with the graph-like manifold and the measure, there is a canonical Hilbert space $\mathsf{L}_2(X_m, \nu_m)$ with norm

$$\|u\|^2_{\mathsf{L}_2(X_m,\nu_m)} = \int_{X_m} |u(x)|^2 \,\mathrm{d}\nu_m(x)$$

and a natural (rescaled) energy form on X_m defined by

$$\mathcal{E}_{X_m}(u) = 3 \cdot \Big(\frac{5}{4}\Big)^m \int_{X_m} |\nabla u(x)|^2_x \,\mathrm{d}\nu_m(x),$$

where ∇ is the gradient and $|\cdot|_x$ denotes the norm induced by the Riemannian metric tensor at $x \in X_m$. Moreover, $\mathrm{dom}\,\mathcal{E}_{X_m} = \mathsf{H}^1(X_m)$ is the closure of compactly supported Lipschitz continuous functions on X_m with respect to the energy norm $\|\cdot\|_{\mathsf{H}^1(X_m)}$.

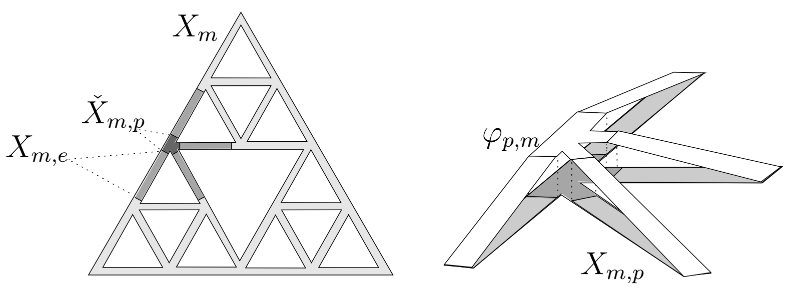

FIGURE 4. [**PS18c**, Fig. 1] Left: A graph-like manifold X_m according to the discrete graph G_m ($m = 2$) with boundary and transversal manifold Y_m being an interval. Right: A function $\varphi_{p,m}$ used to embed a discrete graph value into $X_{m,p}$.

We choose $\mathrm{E} = 1/2\sqrt{5}$ in the next theorem, as this choice gives the optimal error. Moreover, we assume that the length scales as $\ell_{m,e} = 1/2^m(1 + \mathrm{o}(1))$.

Theorem 1.2.5 ([**PS18b**, Thm. 5,7]). *The discrete energy form* $\mathcal{E}_m$ *on* $G_m = (V_m, E_m)$ *and the rescaled energy* $\mathcal{E}_{X_m}$ *on the approximating sequence of graph-like manifolds are* δ_m*-quasi-unitary equivalent where* $\delta_m = \mathrm{O}(1/5^{m/4})$.

The identification operators are chosen similarly as before (see e.g. the comparison of the Sierpiński gasket with its associated sequence of discrete graphs), but with factor c_m resp. $1/c_m$ with $c_m = (2/3)^m \cdot (3E^{(d-1)m})^{-1}$. Here, we use a piecewise affine linear function $\varphi_{p,m}$ which is 1 on $\check{X}_{m,p}$ and 0 on $\check{X}_{m,p'}$ for all $p' \neq v$. It is again crucial that the family $\{\varphi_{p,m}\}_{p \in V_m}$ is a partition of unity. Again by transitivity, we conclude:

Corollary 1.2.6 ([**PS18b**, Cor. 5.12]). *The rescaled energy form* $\mathcal{E}_{X_m}$ *on the manifold and the energy form on the Sierpiński gasket* $\mathcal{E}$ *are* δ'_m*-quasi-unitarily equivalent with* $\delta'_m = \mathrm{O}(1/5^{m/3})$.

We conclude again from Definition 2.1.2 and Proposition 2.2.2 that the rescaled (Neumann) Laplacians Δ_{X_m} converge to the Laplacian Δ_μ on the Sierpiński gasket (with homogeneous self-similar probability measure μ) in generalised norm resolvent sense with error of order $\mathrm{O}(1/5^{m/3})$.

Note that we cannot treat the case when the transversal length scale E equals the longitudinal length scale $1/2$ ($\mathrm{E} = 1/2$), i.e., when X_m is obtained by applying the IFS to X_0, as we require $1/10 < \mathrm{E} < 1/2$ in a more general version of our theorem. However, we can choose E as close to $1/2$ as we want but at the cost of a worse error term. For a more in depth discussion of the scaling, we refer to Remark 4.4.6 (see also [**PS18b**, Rem. 5.8]).

1.3. Previous and related works

In [**Pos06, Pos12**] the author introduced the notion of *quasi-unitary equivalence*, which generalises the concept of *norm* resolvent convergence to the case of energy forms resp. their associated symmetric operators defined in different Hilbert spaces. The consequences (such as convergence of spectra, of operator functions etc.) are basically the same as in the case of the usual norm resolvent convergence. It turns out in many applications (see e.g. [**KhP18, PS18a, AP18, PS18d**]) that the setting is tailor-made for these kind of linear approximation problems on varying spaces and might even be easier to apply than the weaker notion of strong (or Γ-)convergence which we briefly comment on now.

Variational convergence (such as Γ- or *Mosco convergence*) of discrete energy forms to suitable energy forms on metric measure spaces is quite popular recently. In [**Mos94**], Mosco introduced a notion of convergence of energy forms (defined in a fixed Hilbert space), nowadays called “Mosco convergence”. It is equivalent to some sort of strong resolvent convergence, hence our results are stronger as they provide convergence in operator norm. Nevertheless, we believe that the conditions of quasi-unitary equivalence are often easier to check than the ones for Mosco convergence. The notion of Mosco convergence was extended by Kuwae and Shioya to the case of varying Hilbert spaces in [**KS03**, Section 2], which goes by the name Γ-convergence. Kasue [**Kas10**] considers sequences of compact metric spaces with resistance metric and energy forms, e.g. finite resistance networks (weighted graphs with trivial vertex weights $\mu = 1$ and variable edge weights γ) and Γ-convergence of such sequences, e.g., to infinite graphs. Hinz and Teplyaev [**HT15**, Theorem 1.2] consider approximations of a bounded Dirichlet form by a sequence of finite weighted graphs in the sense of Mosco. The finite weighted graphs appear as Dirichlet forms on a finitely generated measure space. The partition of unity (which we e.g. introduced before when we discussed the concrete example: Sierpiński gasket) is just the corresponding finite family of indicator functions; as the limit Dirichlet form is bounded, these indicator functions are in its domain. Hinz and Teplyaev then generalise the result to unbounded Dirichlet forms as any Dirichlet form can be approximated by a bounded Dirichlet form.

In [**Kig01, Kig03, Kig12**], Kigami introduced the notion of *resistance forms*, defined in suitable metric spaces and defines the associated objects like harmonic functions, Green's function and Green's operator. These forms are characterised by the existence of an approximating and increasing sequence of discrete graph energies. A metric space together with a resistance form is sometimes called *resistance space* in the literature. We want to emphasise that the definition of a resistance form itself does not require a measure. On the one hand, this makes the notion quite flexible and we make use of this flexibility in the sense that we do not really need to worry about the measure defined on the metric measure space. And on the other hand, it motivates the question, whether a resistance form induces an energy form (called *Dirichlet form* in this context) or not. We comment on this in greater details in Chapter 3, see in particular Subsection 3.1.1 and Subsection 3.1.2. One of the main example classes for resistance spaces is the collection of post-critically finite self-similar fractals that support an energy form in the sense of [**Kig01**] or [**Str06**]. The most prominent example for such a fractal is the Sierpiński gasket (which we briefly introduced earlier). The case of post-critically finite self-similar fractals (with arbitrary Borel regular probability measure of full support) has already been treated in Theorem 1.1 of [**PS18a**], where the authors showed that an energy form on such a fractal and the energy forms on the associated sequence of approximating discrete graphs are quasi-unitary equivalent. The reason, we emphasise the measure is, that our approach does not rely on a specific choice of a measure (like e.g. the self-similar Hausdorff measure or the energy measure).

The Green's function on a post-critically finite fractal is defined in [**Kig01**, Sec. 3.5] and [**Str06**, Theorem 2.6.1 and Section 4.4] as a monotone limit of approximations: In our notation, the approximation is stated as the uniform convergence of the integral kernel of $JR_m(0)J^\star$ to the integral kernel of $R_\infty(0)$, where $R_m(z) \coloneqq (\Delta_m - z)^{-1}$ is the resolvent of the Laplacian Δ_m with Dirichlet boundary conditions (so that $z = 0 \notin \sigma(\Delta_m)$). From a uniform convergence of the integral kernels one can also conclude the operator convergence $\|R_\infty(0) - JR_m(0)J^\star\| \to 0$ provided the underlying space has finite measure. Our approach using quasi-unitary equivalence also provides the aforementioned operator convergence, but it can easily be extended to non-compact spaces such as fractafolds (see e.g. [**Str03**]) and it specifies the convergence speed explicitly.

In [**Tep08**], Teplyaev introduced the notion of *finitely ramified fractals* and showed that they also support a resistance form (and when this resistance form induces a Dirichlet form) – given some reasonable assumptions on the structure of the space. Moreover, the author gives sufficient conditions under which the Dirichlet form can be represented as the (squared) L_2-norm of a derivation, called weak gradient there. Thus, the results from Teplyaev generalise the ones from Kigami not just by dropping the assumption of self-similarity of the space but also by allowing a more general cell structure.

In [**IRT12**], Ionescu, Rogers and Teplyaev study derivations and Fredholm modules associated with local regular Dirichlet forms on finitely ramified fractals. Their starting point are results from [**CS03**], where Cipriani and Sauvageot proved the existence of a derivation, acting similarly as the gradient in the usual case, for sufficiently well-behaving Dirichlet forms using the abstract framework of $\mathrm{C}^\star$-algebras. It turns out that the *nice* structure of finitely ramified fractals allows a quite concrete representation of such derivations. In particular, they prove that the Hilbert space module of 1-forms (associated with a local regular Dirichlet form) respects the structure of the space in the sense that there exists a direct sum decomposition of the Hilbert space module with respect to the cell structure. From this observation, they then deduce an analogue of the Hodge decomposition. We briefly explain the idea of this construction and present some of their results in Subsection 3.2.3. Moreover, this derivation is then used in [**HR16**], where Hinz and Rogers introduce a magnetic resistance space (i.e., a metric measure space together with a regular, strongly local resistance form). In addition, the authors provide sufficient conditions on the measure for closability of the magnetic form and thus the existence of a self-adjoint magnetic Laplacian and gauge invariance. In [**HT13**], Hinz and Teplyaev study the special case of Schrödinger operators on post-critically self-similar fractals under the assumption that the energy measures are absolutely continuous with respect to the given reference measure, also known under the name energy dominance.

The idea to approximate the energy form on the fractal by the sequence of discrete graph energy forms (defined on the corresponding sequence of approximating discrete graphs) is not new. The usual method to solve this approximation problem for certain fractals is the so-called method of *spectral decimation*, first introduced in [**FS92**]. The idea behind this concept is to calculate the eigenvalues of generation m from the previous generation $m-1$ by the pre-image of some rational function. This sequence of eigenvalues converge to the eigenvalues of the fractal (cf. [**Shi96**, Section 3.1]). However, this method only works for a limited class of finite self-similar fractals like e.g. the Sierpński gasket and it requires the use of a specific Hausdorff measure, the so-called self-similar measure. Moreover, one also gets only convergence of eigenvalues and not operator convergence. Our approach presented here does not rely on self-similarity of the space or a particular choice of the measure. A generalised version of the spectral decimation method was recently used to prove that a magnetic energy form on the Sierpiński gasket can be approximated by its magnetic graph energies (cf. [**HKM$^+$17**]). A similar result was shown in [**PS18d**], where the authors also treated the Sierpiński gasket and proved that the canonical magnetic energy form and the magnetic graph energies are quasi-unitary equivalent (extending results from the aforementioned article [**PS18a**]). In [**BCH$^+$17**], the authors applied the spectral decimation method to the Diamond lattice fractal which is an example of a self-similar finitely ramified fractal (which is not post-critically finite in the sense of [**Kig01**]).

In the recent article [**HM17**], Hinz and Meinert use a metric graph approximation to approximate solutions of some non-linear differential equations on the Sierpiński gasket. The choice to approximate by metric graphs is quite natural because the canonical energy form on such graphs is also induced by a resistance form (see e.g. [**Tep08**]).

The above mentioned methods have one thing in common: They all rely on an approximation of the fractal or the space from within, i.e., by finding an increasing sequence of isolated (and in each generation finitely many) points *inside* the space that converges in the limit (in the sense that the union of all these sets is a dense subset of the space). The natural question that arises from this discovery is, whether one can also approximate a fractal from *outside* by a sequence of shrinking open supersets. In [**BHS09**], Berry, Heilmann and Strichartz study sequences of such sets that are generated by the iterated function system associated with a self-similar fractal. They provide numerical results for this approach which give evidence that the Neumann eigenvalues of these open sets converge to the ones of the fractal. In [**BSU08**], the authors present better numerical results by allowing the sequence of open sets be independent from the iterated function system. Our idea, presented in Chapter 4 is a little different, since we technically do not really approximate the fractal from outside. Our approximating sequence is not strictly constructed by applying the iterated function system, since we do need more flexibility in the proper scaling of the building blocks. Also, our sequence is not nested. But still, our approach is more similar to the latter one: We give a mathematically rigorous proof to show that the canonical energy form on a sequence of suitable graph-like manifolds converge to the one on a symmetric post-critically finite self-similar fractal by applying the aforementioned concept of quasi-unitary equivalence. Hence, we can conclude operator convergence and the convergence of the eigenvalues. Moreover, our analysis confirms the proper rescaling factor $\tau_m = (5/4)^m$ found in [**BHS09**] for the Sierpiński gasket.

The idea to use a sequence of discrete graphs to construct graph-like manifolds approximating a fractal is not new. In [**KuZh98**] and [**BCG01**] such an approach is used to show some untypical behaviour of the wave equation or the heat kernel. In [**BaBK06**] the authors show certain parabolic Harnack inequalities and their stability under rough isometries. In particular, once, such an inequality holds on a weighted graph, then it follows that it also holds on an associated graph-like manifold.

In some applications, graph-like manifolds can be seen as a more realistic model than their one dimensional counterpart metric graphs. Hence, such manifolds have been used to approximate metric graphs, see e.g. [**EP05, Pos06, MV06**] and also the extensive list of references in [**Pos12**, Ch. 1].

In [**MV07, MV09, MV15**], Mosco and Vivaldi construct approximations, where the fractal energy form is part of the limit energy form (as some sort of delta-potential supported on a singular set). More precisely, the authors construct a sequence of *weighted* energy forms on open domains that *Mosco-converge* to an energy form on this open domain plus the natural energy form on the underlying nested fractal. The weights of the energy form become large on "ε_m-neighbourhoods" of the approximating (embedded) graph at generation m and $\varepsilon_m \to 0$ as $m \to \infty$. Actually, this "ε_m-neighbourhood" is only a true neighbourhood away from the vertices at generation m, and reduces to a point in transversal direction at a vertex; moreover, the weight becomes singular at a vertex. In particular, in [**MV07**], this method is applied to the Koch curve and in [**MV09**] to the Sierpiński gasket.

In [**EL13**], the generation m is fixed, but the transversal parameter ε of the domain, where the weight becomes large, tends to 0. The limit energy form is the energy form on the original domain plus a term supported on the approximating (metric) graph (being a hypersurface with singularties at the vertices).

There are quite some articles about the relation between (mostly infinite or classes of finite) graphs and (non-compact or classes of compact) Riemannian manifolds under the name discretisation of a manifold; most authors are interested in questions whether certain properties are invariant under so-called rough isometries, using a related property on a discrete graph. We mention here only the works of [**DP76, Kan86a, Kan86b, Cou92, Man05, CGR16**] and references therein; and the monograph [**Cha01**] where Chavel defines similar maps as our J and J', called *smoothing* and *discretisation* there. The interest in all these works is to have uniform control of classes of manifolds and discrete graphs, e.g. that $f \longmapsto J'Jf$ is bounded by a constant times the energy norm of f, but the constant is not supposed to be small as in our case. One exception is [**BIK14**] where Burago, Ivanov and Kurylev compare eigenvalues of compact Riemannian manifolds with eigenvalues of a discrete weighted graph and show that they are close to each other.

CHAPTER 2

Quasi-unitary equivalence: A generalised norm-resolvent convergence

In this chapter, we introduce the abstract notion of *quasi-unitary equivalence*. This notion generalises the two concepts *norm resolvent convergence* and *unitary equivalence*. It is formulated for energy forms (i.e., closed, quadratic forms) defined in a Hilbert space and their corresponding self-adjoint, symmetric operators. Quasi-unitary equivalence allows us to compare energy forms respectively operators defined in different Hilbert spaces. The concept was first introduced in [**Pos06**] and later presented in greater details in the monograph [**Pos12**]. This chapter is published in [**PS19b**]. The purpose of this article is to give an easy to follow and clear overview of the notion and make some of the statements from the above mentioned papers more precise and improve or simplify them.

2.1. Quasi-unitary equivalence for operators

First we define a "distance" between two non-negative, self-adjoint operators Δ and $\widetilde{\Delta}$ acting in separable Hilbert spaces $\mathscr{H}$ and $\widetilde{\mathscr{H}}$ and we allow the Hilbert spaces to be *different*. The distance is expressed in terms of a parameter $\delta \geq 0$, and appears in the concept of δ*-quasi-unitary equivalence*, which we introduce now.

The operator Δ induces a natural *scale of Hilbert spaces* $\mathscr{H}^k := \operatorname{dom} \Delta^{k/2}$ endowed with the norm $\|f\|_k := \|(\Delta+1)^{k/2} f\|_{\mathscr{H}}$ for $k \geq 0$. For negative powers, we let $\mathscr{H}^{-k}$ be the completion of $\mathscr{H}$ under the norm $\|f\|_{-k} := \|(\Delta+1)^{-k/2} f\|_{\mathscr{H}}$. Moreover the inner product $\langle .,. \rangle_{\mathscr{H}}$ extends continuously onto the dual pairing $\mathscr{H}^{-k} \times \mathscr{H}^k$. Similarly, we have a scale of Hilbert spaces $\widetilde{\mathscr{H}}^k$ associated with $\widetilde{\Delta}$.

Definition 2.1.1 (Quasi-unitary equivalence for operators). *Let $\delta \geq 0$. Moreover, let $J\colon \mathscr{H} \longrightarrow \widetilde{\mathscr{H}}$ and $J'\colon \widetilde{\mathscr{H}} \longrightarrow \mathscr{H}$ be linear operators on the Hilbert spaces.*

(i) The operator J is called δ-quasi-unitary *with* δ-quasi-adjoint *J' (for the operators Δ and $\widetilde{\Delta}$) if*

$$\|Jf\| \leq (1+\delta)\|f\| \qquad \big|\langle Jf, u\rangle - \langle f, J'u\rangle\big| \leq \delta \|f\|\|u\| \tag{2.1.1}$$

for all $f \in \mathscr{H}$ and $u \in \widetilde{\mathscr{H}}$ and if

$$\|f - J'Jf\| \leq \delta\|f\|_2 \qquad \|u - JJ'u\| \leq \delta\|u\|_2 \tag{2.1.2}$$

for all $f \in \mathscr{H}^2, u \in \widetilde{\mathscr{H}}^2$. We call J and J' identification operators.

(ii) We say that the operators Δ and $\widetilde{\Delta}$ are δ-close *if*

$$\big|\langle Jf, \widetilde{\Delta} u\rangle_{\widetilde{\mathscr{H}}} - \langle J\Delta f, u\rangle_{\mathscr{H}}\big| \leq \delta\|f\|_2\|u\|_2 \tag{2.1.3}$$

for all $f \in \mathscr{H}^2$ and $u \in \widetilde{\mathscr{H}}^2$.

(iii) We say that Δ and $\widetilde{\Delta}$ are δ-(operator-)quasi-unitarily equivalent*, if* (2.1.1)–(2.1.3) *are fulfilled, i.e., we have the following equivalent operator norm*

estimates

$$\|J\| \le 1+\delta \qquad\qquad \|J^* - J'\| \le \delta \tag{2.1.1'}$$

$$\|(\mathrm{id}_{\mathscr{H}} - J'J)R\| \le \delta \qquad\qquad \|(\mathrm{id}_{\widetilde{\mathscr{H}}} - JJ')\widetilde{R}\| \le \delta \tag{2.1.2'}$$

$$\|\widetilde{R}J - JR\| \le \delta \tag{2.1.3'}$$

where $R := (\Delta+1)^{-1}$ *and* $\widetilde{R} := (\widetilde{\Delta}+1)^{-1}$ *denotes the resolvents in* -1.

By the above assumptions it follows that J' is also bounded. More precisely, we have

$$\|J'\| \le \|J' - J^*\| + \|J^*\| \le 1 + 2\delta, \tag{2.1.4}$$

using $\|J^*\| = \|J\|$ and (2.1.1').

The concept of quasi-unitary equivalence *generalises unitary equivalence*: If $\delta = 0$ in the above definition then J is unitary with inverse $J^* = J'$ by (2.1.1') and (2.1.2'). Thus the corresponding operators Δ and $\widetilde{\Delta}$ are *unitarily equivalent* by (2.1.3').

Moreover, the above definition allows us to define a generalised notion for norm resolvent convergence:

Definition 2.1.2 (Generalised norm resolvent convergence). *Let* Δ_m *be a self-adjoint and non-negative operator acting in* $\mathscr{H}_m$ *for* $m \in \bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$. *We say that the sequence* $\{\Delta_m\}_{m\in\mathbb{N}}$ converges in generalised norm resolvent sense (with error estimate $\{\delta_m\}_{m\in\mathbb{N}}$) *to* Δ_∞, *if* Δ_m *and* Δ_∞ *are* δ_m*-quasi-unitarily equivalent with* $\delta_m \to 0$.

The notion of generalised norm resolvent convergence indeed *generalises* the concept of *norm resolvent convergence*: Assume that the operators all act in the same Hilbert space $\mathscr{H}$, i.e., that $\mathscr{H} = \mathscr{H}_m$ for all $m \in \mathbb{N} \cup \{\infty\}$. Then we are in the situation of the classical definition for norm resolvent convergence and hence, the sequence $\{\Delta_m\}_{m\in\mathbb{N}}$ converges in *norm resolvent sense* to an operator Δ_∞ if and only if

$$\left\|(\Delta_m+1)^{-1} - (\Delta_\infty+1)^{-1}\right\| \to 0 \qquad \text{as} \quad m \to \infty \tag{2.1.5}$$

(see e.g. [**RS80**, Sec. VIII.7]). If we choose the identification operators (for each operator Δ_m) J_m and J'_m to be the identity operator on $\mathscr{H}$, then (2.1.1') and (2.1.2') are fulfilled with $\delta_m = 0$, and (2.1.3') with $\delta_m \to 0$ is equivalent with (2.1.5). Moreover, a sequence of operators also converges in *generalised* norm resolvent sense if there is a sequence of unitary operators $J_m \colon \mathscr{H}_m \longrightarrow \mathscr{H}_\infty$ such that

$$\|J_m(\Delta_m+1)^{-1}J_m^* - (\Delta_\infty+1)^{-1}\| \to 0.$$

Moreover, the notion of operator-quasi-unitary equivalence is transitive in the following sense (the proof is similar to the one of Proposition 2.2.4 and we slightly improved the error term $\hat{\delta}$ compared to the one given in [**Pos12**, Proposition 4.2.5]):

Proposition 2.1.3. *Assume that* $\delta, \widetilde{\delta} \in [0,1]$. *Assume in addition that* Δ *and* $\widetilde{\Delta}$ *are* δ*-quasi-unitarily equivalent with identification operators* J *and* J', *and that* $\widetilde{\Delta}$ *and* $\widehat{\Delta}$ *are* $\widetilde{\delta}$*-quasi-unitarily equivalent with identification operators* $\widetilde{J}$ *and* $\widetilde{J}'$. *Then*

Δ and $\widehat{\Delta}$ are $\hat{\delta}$-quasi-unitarily equivalent with identification operators $\hat{J} = \widetilde{J}J$ and $\hat{J}' = J'\widetilde{J}'$, where $\hat{\delta} = 5\delta + 5\widetilde{\delta}$.

2.2. Quasi-unitary equivalence for energy forms

It is actually more convenient to start with the quadratic forms $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ associated with the non-negative, self-adjoint operators Δ and $\widetilde{\Delta}$, and develop a slightly more elaborated version of quasi-unitary equivalence. This approach avoids dealing with the sometimes complicated operator domains and graph norms. Nevertheless, in applications, it turns out that these conditions are easily verified as we will see later.

Again, let $\mathscr{H}$ and $\widetilde{\mathscr{H}}$ be two arbitrary separable Hilbert spaces over the field $\mathbb{C}$. We say that $\mathcal{E}$ is an *energy form in $\mathscr{H}$* if $\mathcal{E}$ is a closed, non-negative quadratic form in $\mathscr{H}$, i.e., if $\mathcal{E}(f) := \mathcal{E}(f,f)$ for some sesquilinear form $\mathcal{E}\colon \mathscr{H}^1 \times \mathscr{H}^1 \longrightarrow \mathbb{C}$, denoted by the same symbol, if $\mathcal{E}(f) \geq 0$ and if $\mathscr{H}^1 := \operatorname{dom}\mathcal{E}$, endowed with the energy norm $\|.\|_{\mathcal{E}}$ defined by

$$\|f\|_{\mathcal{E}}^2 := \|f\|_{\mathscr{H}}^2 + \mathcal{E}(f), \tag{2.2.1}$$

for all $f \in \operatorname{dom}\mathcal{E}$, is itself a Hilbert space and dense (as a set) in $\mathscr{H}$. We call the corresponding non-negative, self adjoint operator Δ (see e.g. [**Kat66**, Sec. VI.2]) the *Laplacian* associated with $\mathcal{E}$. Similarly, let $\widetilde{\mathcal{E}}$ be an energy form in $\widetilde{\mathscr{H}}$ with Laplacian $\widetilde{\Delta}$. Note that $\|f\|_1 = \|f\|_{\mathcal{E}}$ and that $\|f\|_{\mathcal{E}} \leq \|f\|_2$ in the terminology of Section 2.1 of this chapter.

In addition to the identification operators from Definition 2.1.1, now we also need identification operators J^1 and J'^1 acting on the form domains $\mathscr{H}^1$ and $\widetilde{\mathscr{H}}^1$.

Definition 2.2.1 (Quasi-unitary equivalence for energy forms). *Let $\delta \geq 0$. Moreover, let $J\colon \mathscr{H} \longrightarrow \widetilde{\mathscr{H}}$ and $J'\colon \widetilde{\mathscr{H}} \longrightarrow \mathscr{H}$, resp. $J^1\colon \mathscr{H}^1 \longrightarrow \widetilde{\mathscr{H}}^1$ and $J'^1\colon \widetilde{\mathscr{H}}^1 \longrightarrow \mathscr{H}^1$ be linear operators on the Hilbert spaces and energy form domains.*

(i) We say that J is δ-quasi-unitary *with* δ-quasi-adjoint *J' (for the energy forms $\mathcal{E}$ and $\widetilde{\mathcal{E}}$) if*

$$\|Jf\| \leq (1+\delta)\|f\| \qquad \big|\langle Jf, u\rangle - \langle f, J'u\rangle\big| \leq \delta\|f\|\|u\| \tag{2.2.2a}$$

for all $f \in \mathscr{H}$ and $u \in \widetilde{\mathscr{H}}$ and if

$$\|f - J'Jf\| \leq \delta\|f\|_{\mathcal{E}} \qquad \|u - JJ'u\| \leq \delta\|u\|_{\widetilde{\mathcal{E}}} \tag{2.2.2b}$$

for all $f \in \mathscr{H}^1$ and $u \in \widetilde{\mathscr{H}}^1$.

(ii) We say that the operators J^1 and J'^1 are δ-compatible *(with the identification operators J and J') if*

$$\|J^1 f - Jf\| \leq \delta\|f\|_{\mathcal{E}} \qquad \|J'^1 u - J'u\| \leq \delta\|u\|_{\widetilde{\mathcal{E}}} \tag{2.2.2c}$$

for all $f \in \mathscr{H}^1$ and $u \in \widetilde{\mathscr{H}}^1$.

(iii) We say that the energy forms $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are δ-close *if*

$$\big|\widetilde{\mathcal{E}}(J^1 f, u) - \mathcal{E}(f, J'^1 u)\big| \leq \delta\|f\|_{\mathcal{E}}\|u\|_{\widetilde{\mathcal{E}}} \tag{2.2.2d}$$

for all $f \in \mathscr{H}^1$ *and* $u \in \widetilde{\mathscr{H}}^1$.

(iv) *We say that* $\mathcal{E}$ *and* $\widetilde{\mathcal{E}}$ *are* δ-quasi-unitarily equivalent, *if* (2.2.2a)–(2.2.2d) *are fulfilled.*

We have the following relation between quasi-unitary equivalence for quadratic forms and operators; the last conclusion has already been shown in [**Pos12**, Proposition 4.4.15]:

PROPOSITION 2.2.2. *If the forms* $\mathcal{E}$ *and* $\widetilde{\mathcal{E}}$ *are* δ*-quasi-unitarily equivalent then we have*

$$\|\widetilde{R}(z)J - JR(z)\| \le C(z)\delta, \tag{2.2.3}$$

where $R(z) := (\Delta - z)^{-1}$ *and* $\widetilde{R}(z) := (\widetilde{\Delta} - z)^{-1}$ *denotes the corresponding resolvents in* $z \in \mathbb{C} \setminus (\sigma(\Delta) \cup \sigma(\widetilde{\Delta}))$ *and where*

$$C(z) := 4\Big(1 + \frac{|z+1|}{d(z, \sigma(\Delta) \cup \sigma(\widetilde{\Delta}))}\Big)^2. \tag{2.2.4}$$

In particular, the associated operators Δ *and* $\widetilde{\Delta}$ *are* 4δ*-quasi-unitarily equivalent.*

PROOF. For $g \in \mathscr{H}$ and $v \in \widetilde{\mathscr{H}}$, we estimate

$$\begin{aligned}
\big|\langle(\widetilde{R}(z)J - JR(z))g, v\rangle\big| &= \big|\langle g, J^*\widetilde{R}(\overline{z})v\rangle - \langle JR(z)g, v\rangle\big| \\
&= \big|\langle \Delta f, J^*u\rangle - \langle Jf, \widetilde{\Delta}u\rangle\big| \\
&\le \big|\big\langle \Delta f, \big((J^* - J') + (J' - J'^1)\big)u\big\rangle\big| + \big|\mathcal{E}(f, J'^1u) - \widetilde{\mathcal{E}}(J^1f, u)\big| \\
&\quad + \big|\langle (J^1 - J)f, \widetilde{\Delta}u\rangle\big| \\
&\le 2\delta\|f\|_2\|u\| + \delta\|f\|_1\|u\|_1 + \delta\|f\|\,\|u\|_2 \le 4\delta\|f\|_2\|u\|_2,
\end{aligned}$$

where $f = R(z)g$ and $u = \widetilde{R}(\overline{z})v$. Then

$$\|f\|_2 = \|(\Delta+1)R(z)g\| \le \|(\Delta+1)R(z)\|\,\|g\|$$

and

$$\|(\Delta+1)R(z)\| = \sup_{\lambda\in\sigma(\Delta)} \frac{\lambda+1}{|\lambda - z|} \le 1 + \sup_{\lambda\in\sigma(\Delta)} \frac{|z+1|}{|\lambda - z|} = 1 + \frac{|z+1|}{d(z, \sigma(\Delta))}$$

using the spectral theorem. A similar estimate holds for $\|u\|_2$. In particular, the resolvent estimate follows. For the second statement, note that for $z = -1$ we have $C(-1) = 4$, hence (2.1.3) holds with 4δ. The remaining estimates (2.1.1)–(2.1.2) follow from the quasi-unitary equivalence of the forms and the fact that $\|f\|_1 \le \|f\|_2$ and similarly for u. ☐

In particular, if we choose the rough estimate $\sigma(\Delta) \cup \sigma(\widetilde{\Delta}) \subset [0, \infty)$, then

$$C(z) \le 1 + \frac{|z+1|}{d(z, [0,\infty))}. \tag{2.2.5}$$

For $\operatorname{Re} z \geq 0$, the rough estimate (2.2.5) equals $1 + {|z+1|}/{|\operatorname{Im} z|}$ and for $\operatorname{Re} z < 0$ the it equals $1 + {|z+1|}/{|z|}$. Hence, we have

$$C(z) \leq 1 + \frac{|z+1|}{|\operatorname{Im} z|} \quad \text{resp.} \quad C(z) \leq 1 + \frac{|z+1|}{|z|}$$

for $\operatorname{Re} z \geq 0$ resp. $\operatorname{Re} z < 0$.

Let us again discuss the following special case here: If $\delta = 0$ in (2.2.2a)–(2.2.2c), then J is a unitary operator with $J' = J^*$, $J^1 = J\restriction_{\mathscr{H}^1}$ and $J'^1 = J^*\restriction_{\widetilde{\mathscr{H}}^1}$; hence — in this situation — we can assume without loss of generality that $\mathscr{H} = \widetilde{\mathscr{H}}$, $J = J' = \operatorname{id}_{\mathscr{H}}$ and $\operatorname{dom} \mathcal{E} = \operatorname{dom} \widetilde{\mathcal{E}}$. In particular, $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are δ-quasi-unitarily-equivalent if and only if

$$|\widetilde{\mathcal{E}}(f,u) - \mathcal{E}(f,u)| \leq \delta \|f\|_{\mathcal{E}} \|u\|_{\widetilde{\mathcal{E}}} \tag{2.2.6a}$$

for all $f, u \in \mathscr{H}^1 := \operatorname{dom} \mathcal{E} = \operatorname{dom} \widetilde{\mathcal{E}}$. Using the fact that $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are symmetric, it is sufficient if (2.2.6a) only holds for $f = u$, i.e., (2.2.6a) is equivalent with

$$|\widetilde{\mathcal{E}}(f) - \mathcal{E}(f)| \leq \hat{\delta} \|f\|^2_{\mathcal{E}} \tag{2.2.6b}$$

for all $f \in \mathscr{H}^1$. For the implication (2.2.6a)⇒(2.2.6b) one can use $\hat{\delta} = \delta\sqrt{{2+\delta}/{2-\delta}}$ (provided $\delta < 2$) and for (2.2.6b)⇒(2.2.6a) one can use $\delta = {\hat{\delta}}/{\sqrt{1-\hat{\delta}}}$ (provided $\hat{\delta} < 1$). This situation has also been studied in [**BrF17**]; basically, their Theorem 2 is the implication (2.2.6b)⇒(2.2.6a) together with Proposition 2.2.2 (with $z = -1$ and 4δ replaced by δ, as $\delta = 0$ in (2.2.2a)–(2.2.2c)).

In particular, if $\{\mathcal{E}_m\}_{m\in\mathbb{N}}$ is a sequence of energy forms acting in the *same* Hilbert space as $\mathcal{E}_\infty$, i.e., $\mathscr{H}_m = \mathscr{H}_\infty$ with the *same* domain $\operatorname{dom} \mathcal{E}_m = \operatorname{dom} \mathcal{E}_\infty$ for all $m \in \mathbb{N}$, then (with all identification operators being the corresponding identity operators) (2.2.2a)–(2.2.2c) are trivially fulfilled with $\delta = 0$. Moreover, (2.2.2d) with $\delta = \delta_m \to 0$ is equivalent with

$$\big|\mathcal{E}_\infty(f) - \mathcal{E}_m(f)\big| \leq \hat{\delta}_m \|f\|^2_{\mathcal{E}_\infty},$$

for all $f \in \operatorname{dom} \mathcal{E}_\infty$ where δ_m and $\hat{\delta}_m$ are related as above. This is the classical situation of Kato [**Kat66**, Theorem VI.3.6] or [**RS80**, Theorem VIII.25(c)], and we conclude (using Proposition 2.2.2) that the operators Δ_m associated with $\mathcal{E}_m$ *converge* to Δ_∞ in *norm resolvent sense*, see (2.1.5). Note that both classical results do not state the convergence speed of the norm of the resolvent difference.

Another useful implication is the transitivity of quasi-unitary equivalence for energy forms. It was originally proved in [**Pos12**, Proposition 4.4.16]; we give a simpler proof here and improve the error slightly.

Proposition 2.2.3. *Let $\delta, \widetilde{\delta} \in [0,1]$. Assume that $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are δ-quasi-unitarily equivalent with identification operators J, J^1, J' and J'^1. Moreover, assume that $\widetilde{\mathcal{E}}$ and $\widehat{\mathcal{E}}$ are $\widetilde{\delta}$-quasi-unitarily equivalent with identification operators $\widetilde{J}$, $\widetilde{J}^1$, $\widetilde{J}'$ and $\widetilde{J}'^1$. Assume in addition that, for all $f \in \mathscr{H}^1$ and $w \in \widehat{\mathscr{H}}^1$,*

$$\|J^1 f\|_{\widetilde{\mathcal{E}}} \leq (1+\delta)\|f\|_{\mathcal{E}} \quad \textit{and} \quad \|\widetilde{J}'^1 w\|_{\widetilde{\mathcal{E}}} \leq (1+\widetilde{\delta})\|w\|_{\widehat{\mathcal{E}}}.$$

Then $\mathcal{E}$ and $\widehat{\mathcal{E}}$ are $\hat{\delta}$-quasi-unitarily equivalent with $\hat{\delta} = 14(\delta + \widetilde{\delta})$.

Proof. We define the identification operators by $\widehat{J} := \widetilde{J}J$, $\widehat{J}^1 := \widetilde{J}^1 J^1$, $\widehat{J}' := J'\widetilde{J}'$ and $\widehat{J}'^1 := J'^1\widetilde{J}'^1$ and we set $R := (\Delta + 1)^{-1}$, $\widetilde{R} := (\widetilde{\Delta} + 1)^{-1}$ and $\widehat{R} := (\widehat{\Delta} + 1)^{-1}$. Then $\widehat{J}$ is bounded, because

$$\|\widehat{J}\| = \|\widetilde{J}J\| \leq (1 + \widetilde{\delta})(1 + \delta) \leq 1 + \frac{3}{2}(\delta + \widetilde{\delta}).$$

The second inequality in (2.2.2a) follows from

$$\|\widehat{J}^* - \widehat{J}'\| \leq \|J^*(\widetilde{J}^* - \widetilde{J}')\| + \|(J^* - J')\widetilde{J}'\| \leq (1 + \delta)\widetilde{\delta} + \delta(1 + 2\widetilde{\delta}) \leq \frac{5}{2}\delta + \frac{5}{2}\widetilde{\delta}$$

as $\|\widetilde{J}'\| \leq 1 + 2\delta$ by (2.1.4). The first inequality in (2.2.2b) is also satisfied because

$$\begin{aligned}\|f - \widehat{J}'\widehat{J}f\| &\leq \|f - J'Jf\| + \|J'(J - J^1)f\| + \|J'(\mathrm{id}_{\widetilde{\mathscr{H}}} - \widetilde{J}'\widetilde{J})J^1\| + \|J'\widetilde{J}'\widetilde{J}(J^1 - J)f\| \\ &\leq \Big(\delta + (1 + 2\delta)\big(\delta + \widetilde{\delta}(1 + \delta) + (1 + 2\widetilde{\delta})(1 + \widetilde{\delta})\delta\big)\Big)\|f\|_{\mathcal{E}} \leq 14(\delta + \widetilde{\delta})\|f\|_{\mathcal{E}}\end{aligned}$$

and the second one follows by similar arguments. Next we prove that the two inequalities in (2.2.2c) also hold. We estimate

$$\begin{aligned}\|(\widehat{J}^1 - \widehat{J})f\| &\leq \|(\widetilde{J}^1 - \widetilde{J})J^1 f\| + \|\widetilde{J}(J^1 - J)f\| \\ &\leq \big(\widetilde{\delta}(1 + \delta) + (1 + \widetilde{\delta})\delta\big)\|f\|_{\mathcal{E}} \leq 2(\delta + \widetilde{\delta})\|f\|_{\mathcal{E}}\end{aligned}$$

and

$$\begin{aligned}\|(\widehat{J}'^1 - \widehat{J}')w\| &\leq \|(J'^1 - J')\widetilde{J}'^1 w\| + \|J'(\widetilde{J}'^1 - \widetilde{J}')w\| \\ &\leq \big(\delta(1 + \widetilde{\delta}) + (1 + 2\delta)\widetilde{\delta}\big)\|w\|_{\widehat{\mathcal{E}}} \leq \frac{5}{2}(\delta + \widetilde{\delta})\|w\|_{\widehat{\mathcal{E}}}.\end{aligned}$$

For inequality (2.2.2d) we estimate

$$\begin{aligned}&\big|\widehat{\mathcal{E}}(\widehat{J}^1 f, w) - \mathcal{E}(f, \widehat{J}'^1 w)\big| \\ &\quad\leq \big|\widehat{\mathcal{E}}(\widetilde{J}^1 J^1 f, w) - \widetilde{\mathcal{E}}(J^1 f, \widetilde{J}'^1 w)\big| + \big|\widetilde{\mathcal{E}}(\widetilde{J}^1 f, \widetilde{J}'^1 w) - \mathcal{E}(f, J'^1\widetilde{J}'^1 w)\big| \\ &\quad\leq \widetilde{\delta}\|J^1 f\|_{\widetilde{\mathcal{E}}}\|w\|_{\widehat{\mathcal{E}}} + \delta\|f\|_{\mathcal{E}}\|\widetilde{J}'^1 w\|_{\widetilde{\mathcal{E}}} \\ &\quad\leq \big(\widetilde{\delta}(1 + \delta) + \delta(1 + \widetilde{\delta})\big)\|f\|_{\mathcal{E}}\|w\|_{\widehat{\mathcal{E}}} \leq 2(\delta + \widetilde{\delta})\|f\|_{\mathcal{E}}\|w\|_{\widehat{\mathcal{E}}}.\end{aligned}$$

□

In our application to magnetic energy forms on finitely ramified fractals in Chapter 3 (see in particular Subsection 3.3.2), we only need a simplified version because the limit forms are both defined in the same Hilbert space and only the potential is varying.

Proposition 2.2.4. *Let $\delta \geq 0$ and $\widetilde{\delta} \in [0, 1)$. Assume that the energy forms $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are δ-quasi-unitarily equivalent with identification operators J, J^1 and J'^1. Moreover, assume that $\widetilde{\mathcal{E}}$ and $\hat{\mathcal{E}}$ are energy forms in the same Hilbert space $\widetilde{\mathscr{H}}$ with common domain $\widetilde{\mathscr{H}}^1$ such that*

$$\big|\widetilde{\mathcal{E}}(u) - \hat{\mathcal{E}}(u)\big| \leq \widetilde{\delta}\|u\|^2_{\hat{\mathcal{E}}}, \tag{2.2.8a}$$

for all $u \in \widetilde{\mathscr{H}}^1$. *Finally, we assume that*

$$\|J^1 f\|_{\widetilde{\mathcal{E}}} \leq (1+\delta)\|f\|_{\mathcal{E}}. \tag{2.2.8b}$$

Then $\mathcal{E}$ *and* $\hat{\mathcal{E}}$ *are* $\hat{\delta}$*-quasi-unitarily equivalent with error*

$$\hat{\delta} = \widetilde{\delta} \cdot \frac{1+\delta}{1-\widetilde{\delta}} + \delta(1+\widetilde{\delta}).$$

Proof. Again, we need to verify the conditions from Definition 2.2.1 for the forms $\mathcal{E}$ and $\hat{\mathcal{E}}$ with the identification operators J, J^1 and J'^1.

As the boundedness of J in the above mentioned definition involves no form, it is clear that it is also fulfilled with the same error δ.

From the quasi-unitary equivalence of $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ it follows that (2.2.2b)–(2.2.2c) is fulfilled with error δ for the energy norm $\|u\|_{\widetilde{\mathcal{E}}}$. But from (2.2.8a) we conclude that

$$(1-\widetilde{\delta})^2\|u\|^2_{\hat{\mathcal{E}}} \leq (1-\widetilde{\delta})\|u\|^2_{\hat{\mathcal{E}}} \leq \|u\|^2_{\widetilde{\mathcal{E}}} \leq (1+\widetilde{\delta})\|u\|^2_{\hat{\mathcal{E}}} \leq (1+\widetilde{\delta})^2\|u\|^2_{\hat{\mathcal{E}}}.$$

In particular, (2.2.2b) and (2.2.2c) are fulfilled for $\mathcal{E}$ and $\hat{\mathcal{E}}$ with δ replaced by $\delta(1+\widetilde{\delta})$. For the remaining estimate (2.2.2d) note first that we have

$$\widetilde{\mathcal{E}}(u,v) - \hat{\mathcal{E}}(u,v) = \frac{1}{4}\big(\widetilde{\mathcal{E}}(u+v) - \hat{\mathcal{E}}(u-v)\big)$$

by the symmetry of $\widetilde{\mathcal{E}} - \hat{\mathcal{E}}$. It follows that

$$\big|\widetilde{\mathcal{E}}(u,v) - \hat{\mathcal{E}}(u,v)\big| \leq \frac{1}{4}\widetilde{\delta}\big(\|u+v\|^2_{\hat{\mathcal{E}}} + \|u-v\|^2_{\hat{\mathcal{E}}}\big) = \frac{1}{2}\widetilde{\delta}\big(\|u\|^2_{\hat{\mathcal{E}}} + \|v\|^2_{\hat{\mathcal{E}}}\big)$$

using (2.2.8a) and the parallelogram equality. Hence, we have

$$\sup_{\|u\|_{\hat{\mathcal{E}}}, \|v\|_{\hat{\mathcal{E}}} \leq 1} \big|\widetilde{\mathcal{E}}(u,v) - \hat{\mathcal{E}}(u,v)\big| \leq \frac{1}{2}(\widetilde{\delta} + \widetilde{\delta}) = \widetilde{\delta}.$$

Thus, we get the following estimate

$$\big|\widetilde{\mathcal{E}}(u,v) - \hat{\mathcal{E}}(u,v)\big| \leq \widetilde{\delta}\|u\|_{\hat{\mathcal{E}}}\|v\|_{\hat{\mathcal{E}}}.$$

Finally, we see that we can estimate the distance between the energy forms as follows,

$$\begin{aligned}
\big|\hat{\mathcal{E}}(J^1 f, u) - \mathcal{E}(f, J'^1 u)\big| &\leq \big|\hat{\mathcal{E}}(J^1 f, u) - \widetilde{\mathcal{E}}(J^1 f, u)\big| + \big|\widetilde{\mathcal{E}}(J^1 f, u) - \mathcal{E}(f, J'^1 u)\big| \\
&\leq \widetilde{\delta}\|J^1 f\|_{\hat{\mathcal{E}}}\|u\|_{\hat{\mathcal{E}}} + \delta\|f\|_{\mathcal{E}}\|u\|_{\widetilde{\mathcal{E}}} \\
&\leq \frac{\widetilde{\delta}}{1-\widetilde{\delta}}\|J^1 f\|_{\widetilde{\mathcal{E}}}\|u\|_{\hat{\mathcal{E}}} + \delta(1+\widetilde{\delta})\|f\|_{\mathcal{E}}\|u\|_{\hat{\mathcal{E}}} \\
&\leq \Big(\widetilde{\delta} \cdot \frac{1+\delta}{1-\widetilde{\delta}} + \delta(1+\widetilde{\delta})\Big)\|f\|_{\mathcal{E}}\|u\|_{\hat{\mathcal{E}}}.
\end{aligned}$$

Collecting all the individual error terms, the claim follows as stated. □

We end this section with the following remark: The conditions, more precisely the inequalities in Definition 2.2.1 are not written in stone. It is a useful feature of the definition that it provides us with some flexibility in terms of the inequalities. The following lemma is one example. This flexibility will be helpful in Chapter 3 (in particular in Subsection 3.3.1) where the modified inequality helps to avoid a Poincaré-type estimate, i.e., to bypass an estimate of the first non-zero eigenvalue. Note that this trick works when we estimate a finitely ramified fractal and its associated discrete graph – however, it is not a universal tool to solve any situation: In our second application to graph-like manifolds, it turns out that this approach is not helpful at all.

LEMMA 2.2.5 ([**PS18a**, Lem. 2.4]). *Assume that* (2.2.2a) *is fulfilled with* $\delta_{\mathrm{a}} > 0$ *and* (2.2.2c) *with* $\delta_{\mathrm{c}} > 0$. *If for all* $u \in \widetilde{\mathscr{H}}^1$,

$$\|u - JJ'^1 u\| \le \delta' \|u\|_{\widetilde{\mathcal{E}}}, \tag{2.2.2b'}$$

holds, then the second inequality in (2.2.2b) *is fulfilled with* $\delta = \delta' + (1 + \delta_{\mathrm{a}})\delta_{\mathrm{c}}$.

In particular, if all conditions (2.2.2) *are fulfilled for some* $\delta > 0$, *except for the second one in* (2.2.2b) *which is replaced by* (2.2.2b'), *then* $\mathcal{E}$ *and* $\widetilde{\mathcal{E}}$ *are* $\widetilde{\delta}$-*quasi-unitarily equivalent with* $\widetilde{\delta} = \delta' + (1 + \delta)\delta$.

PROOF. Let $u \in \widetilde{\mathscr{H}}^1$. By our assumptions, we have

$$\begin{aligned} \|u - JJ'u\| &\le \|u - JJ'^1 u\| + \|J(J'^1 - J')u\| \\ &\le \|u - JJ'^1 u\| + \|J\| \|(J'^1 - J')u\| \le \big(\delta' + (1 + \delta_{\mathrm{a}})\delta_{\mathrm{c}}\big)\|u\|_{\widetilde{\mathcal{E}}} \end{aligned}$$

and if $\delta', \delta_{\mathrm{a}}, \delta_{\mathrm{c}} \le \delta$, then the error estimate is greater or equal to $2\delta + \delta^2$, as claimed. □

2.3. Consequences of the notion of quasi-unitary equivalence

Let us now state some consequences of the notion of generalised unitary equivalence energy forms respectively non-negative symmetric associated operators. Roughly speaking, the consequences are nearly the same as for the classical norm resolvent convergence. For more details we again refer the interested reader to the monologue [**Pos12**] (in particular Chapter 4), [**Pos06**] and references therein.

Assume that Δ is a non-negative, self-adjoint operator in a (complex) Hilbert space $\mathscr{H}$ and denote by $R(z) := (\Delta - z)^{-1}$ its resolvent. Let $U \subset \mathbb{C}$ be an open neighbourhood of $\sigma(\Delta) \subset \mathbb{C}$ such that ∂U is locally the graph of a Lipschitz continuous function and such that $\partial U \cap \sigma(\Delta) = \emptyset$. Moreover, let $\eta \colon U \longrightarrow \mathbb{C}$ be a holomorphic function. Then the integral

$$\eta(\Delta) := -\frac{1}{2\pi \mathrm{i}} \int_{\partial U} \eta(z) R(z) \, \mathrm{d}z \tag{2.3.1}$$

is defined in the operator norm topology provided

$$C_{\eta,\sigma} := \frac{1}{2\pi} \int_{\partial U} \frac{|\eta(z)|}{d(z,\sigma)} \, \mathrm{d}|z| < \infty$$

for $\sigma := \sigma(\Delta)$.

One example for η are spectral projections: If U encloses a compact subset K of $\sigma(\Delta)$, then $\mathbb{1}_U(\Delta)$ (defined with $\eta = \mathbb{1}_U$ in (2.3.1)) is the spectral projection onto K.

Theorem 2.3.1. *Assume that the forms $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ corresponding to the operators Δ and $\widetilde{\Delta}$ are δ-quasi-unitarily equivalent (or that* (2.2.3) *holds), and that U is an open subset such that ∂U is locally Lipschitz and such that $\partial U \cap (\sigma(\Delta) \cup \sigma(\widetilde{\Delta})) = \emptyset$. Then*

$$\|\eta(\widetilde{\Delta})J - J\eta(\Delta)\| \leq C_\eta \delta, \tag{2.3.2}$$

where C_η is defined in (2.3.3).

Proof. Since the integrals for $\eta(\Delta)$ and $\eta(\widetilde{\Delta})$ exist in operator norm, we have

$$\eta(\widetilde{\Delta})J - J\eta(\Delta) = -\frac{1}{2\pi \mathrm{i}} \int_{\partial U} \eta(z)\big(\widetilde{R}(z)J - JR(z)\big)\,\mathrm{d}z.$$

Taking the operator norm on both sides and using (2.2.3) and (2.2.4), we obtain

$$\|\eta(\widetilde{\Delta})J - J\eta(\Delta)\| \leq \delta \cdot \underbrace{\frac{2}{\pi} \int_{\partial U} |\eta(z)| \Big(1 + \frac{|z+1|}{d(z, \sigma(\Delta) \cup \sigma(\widetilde{\Delta}))}\Big)^2 \mathrm{d}|z|}_{=:C_\eta}. \tag{2.3.3}$$

In particular, $C_\eta < \infty$ implies that $C_{\eta,\sigma(\Delta)} < \infty$ and $C_{\eta,\sigma(\widetilde{\Delta})} < \infty$. □

Remark 2.3.2. *Of course, there is also a functional calculus for measurable functions being continuous in a neighbourhood of $\sigma(\Delta)$. In that case, the error $C_\eta\delta$ has to be replaced by a function $\Phi_\eta(\delta)$ with the property that $\Phi_\eta(\delta) \to 0$ as $\delta \to 0$. In particular, we lose the information about the convergence speed (see* [**Pos06**, Theorem A.8] *for details). Nevertheless, the following result remains true (with a modified error term, see* [**Pos06**, Theorem A.10]*).*

Proposition 2.3.3 ([**Pos12**, Lem. 4.2.13]). *Let $\delta \in [0,1]$. Assume that the energy forms $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ (with corresponding operators Δ and $\widetilde{\Delta}$, respectively) are δ-quasi-unitarily equivalent with identification operators J and J'. Assume in addition that* (2.3.2) *holds. Then*

$$\|\eta(\widetilde{\Delta}) - J\eta(\Delta)J'\| \leq C'_\eta\delta \quad \text{and} \quad \|\eta(\Delta) - J'\eta(\widetilde{\Delta})J\| \leq C'_\eta\delta,$$

with

$$C'_\eta \coloneqq 5 \sup_{\lambda \in [0,\infty)\cap U} |\eta(\lambda)(\lambda+1)^{1/2}| + 3C_\eta.$$

Let us calculate explicitly the constants C_η and C'_η for two examples of the function η:

Example 2.3.4.

(i) ***Spectral projections:*** *Let $I \coloneqq (a,b)$ such that $-1 < a < b$ and $a, b \notin \sigma(\Delta) \cup \sigma(\widetilde{\Delta}) =: S$ with $d(\{a,b\}, S) \geq \varepsilon$ for some $\varepsilon > 0$.*[1] *We want to compare the spectral projections $\mathbb{1}_I(\Delta)$ and $\mathbb{1}_I(\widetilde{\Delta})$, defined via the functional*

---

[1] If we aim in operator convergence of spectral projections, it is a standard assumption that ∂I is in the resolvent set of at least one of the operators; if δ is small enough, it can then be shown that ∂I is also in the resolvent set of the other operator, see e.g. [**RS80**, Theorem VIII.23 (b)]. For *strong* convergence, the assumption can be weakened to exclude that ∂I are eigenvalues.

calculus for self-adjoint operators. Let $U := I \times \mathrm{i}(-\varepsilon, \varepsilon) \subset \mathbb{C}$ be a rectangle enclosing I. Note that we have $\mathbb{1}_I(\Delta) = \eta(\Delta)$ with $\eta = \mathbb{1}_U$ where the latter operator function is defined via the holomorphic functional calculus (2.3.1); a similar statement holds for $\widetilde{\Delta}$. A straightforward estimate shows that

$$C_\eta = \frac{4}{\pi}(b - a + \varepsilon)\Big(1 + \sqrt{1 + \Big(\frac{b+1}{\varepsilon}\Big)^2}\Big)^2 = \mathrm{O}(b).$$

Moreover, $C'_\eta = 5\sqrt{b+1} + 3C_\eta = \mathrm{O}(b)$.

(ii) ***Heat operator:*** *For the heat operator, we have $\eta_t(\lambda) = \mathrm{e}^{-t\lambda}$ for $t \geq 0$. In this example we let the open neighbourhood U of the spectrum be the open sector with half-angle $\theta \in (0, \pi/2)$ and vertex at -1 and moreover, we assume that U is symmetric with respect to the real axis. Then we have*

$$C_{\eta_t} \leq \frac{4}{\pi}\int_0^\infty \mathrm{e}^{-tr\cos\theta}\Big(1 + \frac{1}{\sin\theta}\Big)^2 \mathrm{d}r = \frac{4}{\pi\cos\theta}\Big(1 + \frac{1}{\sin\theta}\Big)^2 \cdot \frac{1}{t},$$

since $d(z, \sigma(\Delta) \cup \sigma(\widetilde{\Delta})) \geq |z+1|\sin\theta$. The minimum in $\theta \in (0, \pi/2)$ of the right hand side is achieved when $\theta = \pi/4$. Hence the minimal value is

$$\frac{4}{\pi\cos\frac{\pi}{4}}\Big(1 + \frac{1}{\sin\frac{\pi}{4}}\Big)^2 \cdot \frac{1}{t} = \frac{12\sqrt{2} + 16}{\pi} \cdot \frac{1}{t} \leq \frac{11}{t}.$$

Moreover, as

$$\sup_{\lambda \in [0,\infty)} |\mathrm{e}^{-t\lambda}(\lambda+1)^{1/2}| \leq \begin{cases} 1 & t \geq 1/2, \\ 1/\sqrt{2t} & t \in [0, 1/2], \end{cases}$$

we conclude that a rough estimate is

$$C'_{\eta_t} = \frac{33}{t} + 5. \tag{2.3.4}$$

In particular, we conclude the following convergence result for the solution of the heat equation:

Corollary 2.3.5. *Let $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ be two energy forms with associated operators Δ and $\widetilde{\Delta}$. Assume that $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ are δ-quasi-unitarily equivalent. Let f_t resp. u_t be the solution of the heat equations*

$$\partial_t f_t + \Delta f_t = 0 \qquad \textit{resp.} \qquad \partial_t u_t + \widetilde{\Delta} u_t = 0$$

for $t > 0$. If $f_0 = J'u_0$, then for any $T > 0$ we have

$$\|u_t - Jf_t\|_{\widetilde{\mathscr{H}}} \leq C_{\eta_T}\delta\|u_0\|_{\widetilde{\mathscr{H}}}$$

for all $t \in [T, \infty)$ with $C'_{\eta_T} = \mathrm{O}(1/T)$ as $T \to 0$.

Proof. Note that we have $u_t = \mathrm{e}^{-t\widetilde{\Delta}}u_0$ and $f_t = \mathrm{e}^{-t\Delta}J'u_0$. Then

$$\|u_t - Jf_t\|_{\widetilde{\mathscr{H}}} = \|(\mathrm{e}^{-t\widetilde{\Delta}} - J\mathrm{e}^{-t\Delta}J')u_0\|_{\widetilde{\mathscr{H}}} \leq C'_{\eta_t}\delta\|u_0\|_{\widetilde{\mathscr{H}}}.$$

We apply Proposition 2.3.3 and the concrete estimate for C'_{η_t} to conclude the claim. ∎

As in the case of usual norm convergence the operator norm convergence of spectral projections implies the *convergence of spectra* (also called *spectral exactness*):

COROLLARY 2.3.6 ([**Pos12**, Theorem 4.3.3]). *If a sequence* $\{\Delta_m\}_{m\in\mathbb{N}}$ *converges in generalised norm resolvent sense to* Δ_∞*, then*

$$\bar{d}(\sigma(\Delta_m), \sigma(\Delta_\infty)) \to 0$$

as $m \to \infty$*, where*

$$\bar{d}(A, B) := \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\}$$

defines a weighted Hausdorff metric between two closed sets $A, B \subset [0, \infty)$*. Here,* $d(a, b) := |(a + 1)^{-1} - (b + 1)^{-1}|$ *is a weighted metric on* $[0, \infty)$*.*

If the operators have purely discrete spectrum, we can specify the error estimate:

COROLLARY 2.3.7. *Let* $\lambda_k(\Delta_m)$ *resp.* $\lambda_k(\Delta_\infty)$ *denote the* k^{th} *eigenvalue of* Δ_m *resp.* Δ_∞ *(in increasing order and repeated according to their multiplicity). Then*

$$\left|\lambda_k(\Delta_m) - \lambda_k(\Delta_\infty)\right| \le C_k\delta_m$$

for all $m \in \mathbb{N}$ *such that* $\dim \mathscr{H}_m \ge k$*, where* C_k *depends only on* $\lambda_k(\Delta_\infty)$*.*

In the case of purely discrete spectrum (or isolated eigenvalues) we can approximate an eigenfunction also in energy norm:

THEOREM 2.3.8 ([**PS18a**, Proposition 2.6]). *Let* $\mathcal{E}$ *and* $\widetilde{\mathcal{E}}$ *be two* δ*-quasi-unitarily equivalent energy forms with associated operators* Δ *and* $\widetilde{\Delta}$*. Assume that* $\widetilde{\Phi}$ *is an eigenvector of* $\widetilde{\Delta}$*, such that its eigenvalue* $\widetilde{\lambda}$ *is discrete in* $\sigma(\widetilde{\Delta})$*, i.e., there is an open disc* D *in* $\mathbb{C}$ *such that* $\sigma(\widetilde{\Delta}) \cap D = \{\widetilde{\lambda}\}$*. Then there exists a normalised eigenvector* Φ *of* Δ *with* $\Phi \in \operatorname{ran} \mathbb{1}_D(\Delta)$ *and a universal constant* C *depending only on* $\widetilde{\lambda}$ *(and the radius of* D*) such that*

$$\|J^1\Phi - \widetilde{\Phi}\|_{\widetilde{\mathcal{E}}} \le C\delta.$$

Note that the eigenvalue $\widetilde{\lambda}$ does not necessarily need to have finite multiplicity.

Before we prove the above theorem, we need to state some additional consequences of the notion of quasi-unitary equivalence.

We first provide the notation $\|A\|_{-1\to1} = \|(\widetilde{\Delta}+1)^{1/2}A(\Delta+1)^{1/2}\|$ for an operator $A\colon \mathscr{H}^{-1} \longrightarrow \widetilde{\mathscr{H}}^1$. Moreover, $(J'^1)^*\colon \mathscr{H}^{-1} \longrightarrow \widetilde{\mathscr{H}}^{-1}$ denotes the dual map of $J'^1\colon \widetilde{\mathscr{H}}^1 \longrightarrow \mathscr{H}^1$ with respect to the dual pairing $\mathscr{H}^1 \times \mathscr{H}^{-1}$ induced by the inner product on $\mathscr{H}$ and similarly on $\widetilde{\mathscr{H}}$.

PROPOSITION 2.3.9 (see [**Pos12**, Ch. 4]). *Let* $\eta\colon [0, \infty) \longrightarrow \mathbb{C}$ *be a function continuous in a neighbourhood* U *of* $\sigma(\widetilde{\Delta})$ *such that* $\lim_{\lambda\to\infty}(\lambda + 1)^{1/2}\eta(\lambda)$ *exists. Then there is a function* $\delta \longmapsto \varepsilon(\delta)$ *such that* $\varepsilon(\delta) \to 0$ *if* $\delta \to 0$ *and*

$$\|J'\eta(\widetilde{\Delta}) - \eta(\Delta)J'\| \le \varepsilon(\delta), \tag{2.3.5a}$$

$$\big|\|J'u\| - \|u\|\big| \le 3\delta\|u\|_1 \tag{2.3.5b}$$

$$\|J^1\eta(\Delta) - \eta(\widetilde{\Delta})(J'^1)^*\|_{-1\to1} \le \varepsilon(\delta) \tag{2.3.5c}$$

for any pair of δ-quasi-unitarily equivalent energy forms $\mathcal{E}$ and $\widetilde{\mathcal{E}}$ with associated operators Δ and $\widetilde{\Delta}$, where $\varepsilon = \varepsilon_\eta$ depends only on η and U. If $\eta(\lambda) = \mathrm{e}^{-t\lambda}$ (for $t > 0$) or $\eta = \mathbb{1}_D$ (if $\partial D \cap \sigma(\widetilde{\Delta}) = \emptyset$ for an open set $D \subset \mathbb{C}$ with smooth boundary) we can choose $\varepsilon(\delta) = C\delta$ for some constant $C > 0$.

Proof of Theorem 2.3.8. Set $P := \mathbb{1}_D(\Delta)$ and $\widetilde{P} := \mathbb{1}_D(\widetilde{\Delta})$. Note that $\operatorname{ran} P$ may consist of the linear combination of several eigenvectors if $\widetilde{\lambda}$ is not a simple eigenvalue. We have

$$\begin{aligned}\|PJ'\widetilde{\Phi}\| &\geq \|J'\widetilde{P}\widetilde{\Phi}\| - \|(PJ' - J'\widetilde{P})\widetilde{\Phi}\| \\ &\geq \|J'\widetilde{\Phi}\| - C_\eta\delta\|\widetilde{\Phi}\| \geq 1 - \underbrace{\big(3(\widetilde{\lambda}+1)^{1/2} + C_\eta\big)}_{=:C_1}\delta\end{aligned}$$

by (2.3.5a) with $\eta = \mathbb{1}_D$ and (2.3.5b) and similarly,

$$\|PJ'\widetilde{\Phi}\| \leq \|J'\widetilde{P}\widetilde{\Phi}\| + \|(PJ' - J'\widetilde{P})\widetilde{\Phi}\| \leq \|J'\widetilde{\Phi}\| + C_\eta\delta\|\widetilde{\Phi}\| \leq 1 + C_1\delta,$$

and therefore

$$\big|\|PJ'\widetilde{\Phi}\| - 1\big| \leq C_1\delta. \tag{2.3.6}$$

In particular, $\|PJ'\widetilde{\Phi}\| > 0$ for δ small enough. Let

$$\Phi := \frac{1}{\|PJ'\widetilde{\Phi}\|} PJ'\widetilde{\Phi},$$

then $\Phi \in \mathscr{H}^1$ and for $\delta < {}^1/_{C_1}$ we have

$$\begin{aligned}\|J^1\Phi - \widetilde{\Phi}\|_1 &= \Big\|\frac{1}{\|PJ'\widetilde{\Phi}\|} J^1PJ'\widetilde{\Phi} - \widetilde{\Phi}\Big\|_1 \\ &\leq \frac{1}{\|PJ'\widetilde{\Phi}\|}\Big(\big\|(J^1P - \widetilde{P}(J'^1)^*)J'\widetilde{\Phi}\big\|_1 + \big\|\widetilde{P}((J'^1)^*J'\widetilde{\Phi} - \widetilde{\Phi})\big\|_1 \\ &\qquad\qquad + \big|1 - \|PJ'\widetilde{\Phi}\|\big|\|\widetilde{\Phi}\|_1\Big) \\ &\leq \frac{1}{1 - C_1\delta}\Big(C_\eta\delta\|\widetilde{\Phi}\|_{-1} + (\widetilde{\lambda}+1)\big\|(J'^1)^*J' - \mathrm{id}\big\|_{0\to-1}\|\widetilde{\Phi}\|_1 \\ &\qquad\qquad + C_1\delta\|\widetilde{\Phi}\|_1\Big) \\ &\leq \frac{1}{1 - C_1\delta}\big(C_\eta + (\widetilde{\lambda}+1)^{3/2}C'' + C_1(\widetilde{\lambda}+1)^{1/2}\big)\delta\end{aligned}$$

using (2.3.5c), $\|\widetilde{P}\|_{-1\to1} = 1 + \widetilde{\lambda}$ and (2.3.6) for the second inequality, and

$$\begin{aligned}\big\|(J'^1)^*J' - \mathrm{id}\big\|_{0\to-1} &= \big\|(J')^*J'^1 - \mathrm{id}\big\|_{1\to0} \\ &\leq \|(J')^* - J\|\|J'^1\|_{1\to0} + \|JJ'^1 - \mathrm{id}\|_{1\to0} \\ &\leq \delta(1 + 3\delta) + \delta' \leq (1 + 3/C_1 + C')\delta =: C''\delta\end{aligned}$$

using also

$$\|J'^1\|_{1\to0} \leq \|J'^1 - J'\|_{1\to0} + \|J' - J^*\|_{0\to0} + \|J^*\|_{0\to0} \leq 1 + 3\delta \leq 1 + 3/C_1$$

as $\|A\|_{1\to 0} \leq \|A\|_{0\to 0}$ and $\delta < {}^1\!/_{C_1}$, and using $\delta' \leq C'\delta$ for some constant $C' > 0$; the latter estimate can be seen similarly as in Lemma 2.2.5. $\square$

CHAPTER 3

Convergence of finite dimensional approximations of magnetic Laplacians on finitely ramified fractals

The aim of this chapter is to apply the abstract notion of quasi-unitary equivalence introduced in Chapter 2 to the case of energy forms (induced by resistance forms) on finitely ramified fractals and approximate them by discrete energy forms defined on the canonical sequence of approximating discrete graphs.

Therefore, we first introduce the abstract theory of resistance forms in Section 3.1 and discuss when a resistance form induces an energy form (called Dirichlet form in this context; see Definition 3.1.8). In Section 3.2, we first introduce the notion of finitely ramified fractals. Then we define a resistance form on a suitable dense subset as the limit of the graph energies. At the end of the second subsection, we formulate our main assumption for this chapter in Assumption 3.2.11. This assumption allows us to extend the resistance form, defined on the dense subset to the finitely ramified fractal. Moreover, it will simplify the notation and also allows us to define magnetic potentials, i.e., introducing 1-forms associated with the energy.

In Section 3.3, we prove that — under Assumption 3.2.11 — an energy form (induced by a resistance form) and perturbed by a magnetic potential on a finitely ramified fractal and its canonical sequence of graph energies are quasi-unitarily equivalent and that the error converges to zero exponentially.

The quintessence of this chapter are the results formulated in the preprint [**PS18d**].

3.1. A brief introduction of resistance forms and resistance metrics

We begin by introducing the notion of resistance forms in the first subsection and in the second one, we comment on the question whether a resistance form induces a Dirichlet form. Note that the theory of resistance forms is more general and not necessarily related to a fractal. However, the class of post-critically finite self-similar fractals — a subclass of finitely ramified fractals — is probably the most common example. For more details and proofs of the claimed statements, we refer to the monograph [**Kig01**] for a detailed overview; see also [**Kig03, Kig12**] for some additional information.

3.1.1. Resistance forms and resistance metric. Let us first define, what we mean by a resistance form.

Definition 3.1.1 (Resistance Form). *Let X be a set. A non-negative, quadratic form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ in $\ell(X) := \{\, u \,|\, u\colon X \longrightarrow \mathbb{C} \,\}$ is called a* resistance form *if it has the following properties.*

- *[RF1] The domain $\operatorname{dom}\mathcal{E}$ is a linear subspace of $\ell(X)$ containing the constants and $\mathcal{E}(u) = 0$ if and only if u is constant on X.*
- *[RF2] $\big(\operatorname{dom}\mathcal{E}/\mathbb{C}, \mathcal{E}\big)$ is a Hilbert space.*
- *[RF3] For any finite subset $V \subset X$ and $f\colon V \longrightarrow \mathbb{C}$ there exists a function $u \in \operatorname{dom}\mathcal{E}$ such that $u\restriction_V = f$.*

[RF4] For two arbitrary points $x, y \in X$ the following supremum is finite

$$\sup\Big\{ \frac{|u(x)-u(y)|^2}{\mathcal{E}(u)} \,\Big|\, u \in \operatorname{dom}\mathcal{E} \text{ and } \mathcal{E}(u) > 0 \Big\}.$$

[RF5] (Markov property) For any real-valued function $u \in \operatorname{dom}\mathcal{E}$, we have

$$u_\star := \min\{\max\{u,0\},1\} \in \operatorname{dom}\mathcal{E} \qquad \text{and} \qquad \mathcal{E}(u) \geq \mathcal{E}(u_\star).$$

Note that the definition of a resistance form on X is independent of a measure. A resistance form on a measure space (X, μ) may induce a Dirichlet form (see Definition 3.1.8) on the space $\mathsf{L}_2(X,\mu)$ of square-integrable functions on X with respect to the measure μ. We will comment on this in more detail in Subsection 3.1.2.

Note also that the supremum in [RF4] is actually a maximum and it defines a distance on X (cf. [**Kig03**, Prop. 2.10]). Thus, the following definition is justified.

Definition 3.1.2. *We define the* resistance metric $R_{(\mathcal{E},\operatorname{dom}\mathcal{E})}(.,.)$ *on X associated with the resistance form $(\mathcal{E},\operatorname{dom}\mathcal{E})$ by*

$$R_{(\mathcal{E},\operatorname{dom}\mathcal{E})}(x,y) := \sup\Big\{ \frac{|u(x)-u(y)|^2}{\mathcal{E}(u)} \,\Big|\, u \in \operatorname{dom}\mathcal{E} \text{ and } \mathcal{E}(u) > 0 \Big\}$$

for all $x, y \in X$. We call $R_{(\mathcal{E},\operatorname{dom}\mathcal{E})}(x,y)$ the effective resistance *between the points x and y (with respect to the resistance form $(\mathcal{E},\operatorname{dom}\mathcal{E})$).*

In what follows, we simply write $R(.,.)$ instead of the more precise $R_{(\mathcal{E},\operatorname{dom}\mathcal{E})}(.,.)$ if the underlying resistance form is clear from the context.

For any function $u \in \operatorname{dom}\mathcal{E}$ and points x and y in X we deduce from [RF4] that

$$|u(x)-u(y)|^2 \leq \mathcal{E}(u)R(x,y), \tag{3.1.1}$$

i.e., functions in the domain of $\mathcal{E}$ are uniformly $1/2$-Hölder continuous with respect to the resistance metric R. Hence, these functions naturally extend to continuous functions on the completion Ω of the set X with respect to the resistance metric R and $(\mathcal{E},\operatorname{dom}\mathcal{E})$ becomes a resistance form on (Ω, R). The resistance metric associated with $(\mathcal{E},\operatorname{dom}\mathcal{E})$ on Ω is the canonical extension of the resistance metric on X (cf. [**Kig03**, Thm. 2.12]). By the above we may always assume without loss of generality that (X, R) is a complete metric space.

Let us take a closer look at the special case of a resistance form $\mathcal{E}$ on a finite set X. For more details we refer the interested reader to [**Kig01**, Section 2.1].

Example 3.1.3 (Resistance forms on finite sets). *Let V be a finite set. A symmetric bilinear form $\mathcal{E}$ on $\ell(V) = \{\, f \mid f \colon V \longrightarrow \mathbb{C} \,\}$ is a resistance form if*

[D1] $\mathcal{E}(f) \geq 0$ for any $f \in \ell(V)$,
[D2] $\mathcal{E}(f) = 0$ if and only if $f \in \ell(V)$ is constant on V,
[D3] $\mathcal{E}$ has the Markov property.

We define a measure μ on V as the uniform distribution i.e., the sum of all the Dirac measures on V. Then there is a canonical inner product and norm given by

$$\langle f, g\rangle = \sum_{p\in V} f(p)\overline{g(p)} \qquad \text{and} \qquad \|f\|^2 = \sum_{p \in V} |f(p)|^2$$

and thus, $\mathcal{E}$ defines a (closed) quadratic form in $\ell_2(V,\mu)$, where

$$\ell_2(V,\mu) := \{\, f \mid f\colon V \longrightarrow \mathbb{C}, \|f\| < \infty \,\}.$$

It is well known that there is a one-to-one correspondence between symmetric bilinear forms and symmetric linear operators on $\ell(V)$ by the map

$$T \longmapsto \big(f \mapsto \mathcal{E}_T(f) := \langle Tf, f\rangle\big).$$

Working this out, we see that for any resistance form $\mathcal{E}$ on V there is a unique symmetric linear operator $H\colon \ell(V) \longrightarrow \ell(V)$ with the following properties

[H1] $H \geq 0$ is positive semi-definite,
[H2] $Hu = 0$ if and only if u is constant on V,
[H3] $H_{pq} := H\mathbb{1}_{\{q\}}(p) \leq 0$ for all $p, q \in V$, $p \neq q$,

where $\mathbb{1}_{\{q\}}\colon \ell(V) \longrightarrow \mathbb{R}$, $\mathbb{1}_{\{q\}}(p) = 1$ if $p = q$ and 0 otherwise. We call H the Laplace operator *associated with the resistance form $\mathcal{E} := \mathcal{E}_H$ on V. Note that, for each $f \in \ell(V)$,*

$$\mathcal{E}_H(f) := \langle f, Hf\rangle = \frac{1}{2}\sum_{p,q\in V} H_{pq}|f(p) - f(q)|^2.$$

Hence, any symmetric linear operator H, satisfying [H1], [H2] and [H3] also defines a resistance form on V. Moreover, for any Laplace operator H on V and points p and q in V, the effective resistance between p and q associated with H) *is defined by*

$$R_H(p,q) = \big(\min\{\, \mathcal{E}_H(u) \,\big|\, u \in \ell(V), u(p) = 1, u(q) = 0 \,\}\big)^{-1}. \tag{3.1.2}$$

As stated before, R_H defines a metric on V. The names are motivated by the link to electrical networks on V (cf. [**Kig01**, Chapter 2]*; in particular Lemma 2.1.15).*

Let $V \subset X$ be a finite set. Then, we define the *trace* $\mathcal{E}_V$ of $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ on V by

$$\mathcal{E}_V(\varrho) := \inf\{\, \mathcal{E}(u) \,\big|\, u \in \operatorname{dom}\mathcal{E} \text{ and } u\restriction_V = \varrho \,\} \tag{3.1.3}$$

for $\varrho \in \ell(V)$. This infimum exists, i.e., it is actually a minimum and it is attained by a unique function $h \in \operatorname{dom}\mathcal{E}$ (cf. [**Kig03**, Prop. 2.15]). Note that the notion of a trace of a resistance form can be generalised to arbitrary subsets $Y \subset X$. We refer the interested reader to [**Kig12**, Ch. 8].

Definition 3.1.4 (Harmonic functions). *The function $h \in \operatorname{dom}\mathcal{E}$, solving the minimisation problem* (3.1.3)*, is called* V-harmonic function (with boundary values ϱ on V)*. If ϱ is the characteristic function $\mathbb{1}_{\{p\}}$ of the set $\{p\} \subset V$, we denote the associated V-harmonic function by ψ_p^V.*

Note that the space of V-harmonic functions is a finite subspace of $\operatorname{dom}\mathcal{E}$ and a basis is given by $\{\psi_p^V\}_{p\in V}$, i.e., $h = \sum_{p\in V} h(p)\psi_p^V$ for any V-harmonic $h \in \operatorname{dom}\mathcal{E}$. The evaluation of $h \in \operatorname{dom}\mathcal{E}$ in points p of X is justified by (3.1.1).

By polarisation and a simple argument (cf. [**Kig03**, Lem. 2.20(2)]), we see that

$$\mathcal{E}(u,h) = \mathcal{E}_V(u\restriction_V, h\restriction_V) \tag{3.1.4}$$

for any $u \in \operatorname{dom}\mathcal{E}$ and V-harmonic functions h. Moreover, V-harmonic functions satisfy the following *maximum principle.*

Proposition 3.1.5 (Maximum principle, [**Kig03**, Prop. 2.18]). *A V-harmonic function u on X attains its maximum and minimum on V, i.e.,*

$$\min_{p\in V} u(p) = \min_{x\in X} u(x) \leq \max_{x\in X} = \max_{p\in V} u(p).$$

Let $\{V_m\}_{m\in\mathbb{N}_0}$ be a sequence of finite subsets of X and $\mathcal{E}_{V_m}$ be a *finite dimensional resistance form* on V_m for each m as defined in Example 3.1.3 (see also [**Kig03**, Def. 2.1]). We call $\{(V_m, \mathcal{E}_m)\}_{m\geq 0}$ a *compatible sequence* if

$$V_m \subset V_{m+1} \quad \text{and} \quad \mathcal{E}_{V_m}(\varrho) = \inf\big\{\, \mathcal{E}_{m+1}(f) \,\big|\, f\colon V_{m+1} \longrightarrow \mathbb{C} \text{ and } f{\restriction_{V_m}} = \varrho \,\big\} \tag{3.1.5}$$

for all $\varrho\colon V_m \longrightarrow \mathbb{C}$ and $m \geq 0$.

Proposition 3.1.6 ([**Kig03**, Thm. 2.13, Thm. 2.14]). *Let $\{(V_m, \mathcal{E}_{V_m})\}_{m\in\mathbb{N}_0}$ be a compatible sequence and $V_\infty \coloneqq \bigcup_{m\geq 0} V_m$. Then $(\mathcal{E}_\infty, \operatorname{dom}\mathcal{E}_\infty)$ is a resistance form on V_∞ where*

$$\mathcal{E}_\infty(u) = \lim_{m\to\infty} \mathcal{E}_{V_m}(u{\restriction_{V_m}}) \quad \text{and} \quad \operatorname{dom}\mathcal{E}_\infty = \big\{\, u \,\big|\, u \in \ell(V_\infty), \lim_{m\to\infty} \mathcal{E}_{V_m}(u{\restriction_{V_m}}) < \infty \,\big\}.$$

Moreover, if (Ω, R) denotes the completion of V_∞ with respect to $R_{(\mathcal{E}_\infty, \operatorname{dom}\mathcal{E}_\infty)}$, then R is the extension of the resistance metric associated with $(\mathcal{E}_\infty, \operatorname{dom}\mathcal{E}_\infty)$ on Ω.

Conversely, let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be a resistance form on a set X and R its associated resistance metric. Assume that (X, R) is separable. If $\{V_m\}_{m\in\mathbb{N}_0}$ is an increasing sequence of finite subsets V_m of X such that $V_\infty = \bigcup_{m\geq 0} V_m$ is dense in (X, R) then $\{(V_m, \mathcal{E}_{V_m})\}_{m\in\mathbb{N}_0}$ is a compatible sequence and $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ equals $(\mathcal{E}_\infty, \operatorname{dom}\mathcal{E}_\infty)$.

We end this section by introducing the notion of *regularity (of a resistance form).* Regularity in this context means that the domain of the resistance form contains sufficiently many functions to approximate continuous functions uniformly by a function with finite energy (cf. [**Kig12**, Ch. 6]).

Let $\mathrm{C}_0(X)$ be the Banach space of compactly supported R-continuous functions mapping X into $\mathbb{C}$, endowed with the supremum norm $\|.\|_\infty$, defined by

$$\|u\|_\infty \coloneqq \sup_{x\in X} |u(x)| \in [0, \infty],$$

for any function $u \in \ell(X)$.

Definition 3.1.7 (Regular resistance form). *Let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be a resistance form on X. We say that $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is* regular *if $\operatorname{dom}\mathcal{E} \cap \mathrm{C}_0(X)$ is dense in $(\mathrm{C}_0(X), \|.\|_\infty)$.*

If (X, R) is a compact metric space then $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is regular. However, this is not true in general, e.g., it is not enough if (X, R) is locally compact (see [**Kig12**, Cor. 6.4 and Ex. 5.5]).

3.1.2. Resistance forms as Dirichlet forms. In this subsection we discuss how to obtain a *(regular) Dirichlet form* (see Definition 3.1.8) from a (regular) resistance form. For more details on Dirichlet forms we refer to [**FOT94, Kig01, Kig12**] and references therein.

In this section, we assume that $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is a resistance form on X with associated resistance metric R. Moreover, we assume, that (X, R) is a separable, complete and locally compact metric space.

Let μ be a Borel regular measure on (X, R) such that $0 < \mu(B_R(x, r)) < \infty$ for all $x \in X$ and $r > 0$, where $B_R(x, r)$ denotes the open ball with center point x and radius r in (X, R), i.e.,

$$B_R(x, r) := \{\, y \mid y \in X, R(x, y) < r \,\}.$$

By the assumption on the measure μ, it follows that $\mathrm{C}_0(X)$ is a dense subset of $\mathsf{L}_2(X, \mu)$.

DEFINITION 3.1.8 (Dirichlet form). *A densely defined closed quadratic form* $\mathfrak{h}$ *with domain* $\operatorname{dom}\mathfrak{h}$ *in* $\mathsf{L}_2(X, \mu)$ *is called* Dirichlet form on X *if it has the Markov property, i.e., if* $u_\star := \min\{\max\{u, 0\}, 1\} \in \operatorname{dom}\mathfrak{h}$ *and* $\mathfrak{h}(u) \geq \mathfrak{h}(u_\star)$ *for any real-valued function* $u \in \operatorname{dom}\mathfrak{h}$. *The Dirichlet form is said to be* regular *if* $\operatorname{dom}\mathfrak{h} \cap \mathrm{C}_0(X)$ *is dense in* $(\mathrm{C}_0(X), \|.\|_\infty)$ *and in* $(\operatorname{dom}\mathfrak{h}, \|.\|_\mathfrak{h})$, *where* $\|.\|_\mathfrak{h}^2 := \|.\|_{\mathsf{L}_2(X,\mu)}^2 + \mathfrak{h}(.)$.

By our assumptions $(\mathcal{E}, \operatorname{dom}\mathcal{E} \cap \mathsf{L}_2(X, \mu))$ is a closed quadratic form in $\mathsf{L}_2(X, \mu)$ (cf. [**Kig01**, Thm. 2.4.1]), i.e. $(\operatorname{dom}\mathcal{E} \cap \mathsf{L}_2(X, \mu), \|.\|_\mathcal{E})$ is a Hilbert space. Let $\mathcal{D}$ denote the closure of $\operatorname{dom}\mathcal{E} \cap \mathrm{C}_0(X)$ in $\operatorname{dom}\mathcal{E} \cap \mathsf{L}_2(X, \mu)$ with respect to the norm $\|.\|_\mathcal{E}$. Note that $\mathcal{D}$ is a subset of $\mathsf{L}_2(X, \mu)$ because $\operatorname{dom}\mathcal{E} \cap \mathrm{C}_0(X) \subset \operatorname{dom}\mathcal{E} \cap \mathsf{L}_2(X, \mu)$. Moreover, $\mathcal{D} = \operatorname{dom}\mathcal{E}$ if (X, R) is compact.

PROPOSITION 3.1.9 ([**Kig12**, Thm. 9.4]). *Let* $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ *be a regular resistance form on* X. *Then* $(\mathcal{E}, \mathcal{D})$ *is a regular Dirichlet form in* $\mathsf{L}_2(X, \mu)$.

If it is clear from the context, we will simply write $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ for the resistance form on X and also for the associated Dirichlet form in $\mathsf{L}_2(X, \mu)$. It is understood that in the latter case we identify its domain with $\mathcal{D} \subset \mathsf{L}_2(X, \mu)$ defined above.

It is an interesting property of (regular) Dirichlet forms induced by resistance forms, that single points in X have positive capacity. Let us recall the definition of capacity of a set (with respect to a Dirichlet form).

DEFINITION 3.1.10 (Capacity with respect to $\mathcal{E}$). *Let* $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ *be a regular Dirichlet form in* $\mathsf{L}_2(X, \mu)$ *induced by a regular resistance form. Let* U *be an open subset of* X. *We define the* capacity of U (with respect to $\mathcal{E}$) *by*

$$\operatorname{Cap}(U) := \inf\big\{\, \|u\|_\mathcal{E}^2 \,\big|\, u \in \mathcal{D} \text{ and } u(x) \geq 1 \text{ for all } x \in U \,\big\}$$

with the convention that $\inf \emptyset := \infty$. *For an arbitrary set* $A \subset X$ *we set*

$$\operatorname{Cap}(A) := \inf\big\{\, \operatorname{Cap}(U) \,\big|\, U \subset X \text{ open and } U \supset A \,\big\}.$$

PROPOSITION 3.1.11 ([**Kig12**, Thm. 9.9]). *Any point* x *of* X *has positive capacity, i.e.,* $0 < \operatorname{Cap}(\{x\}) < \infty$. *Moreover,* $0 < \inf_{x \in K} \operatorname{Cap}(\{x\})$ *for any compact subset* K *of* X.

3.2. Resistance forms on finitely ramified fractals

In this section we briefly explain the notion of finitely ramified fractals and introduce a canonical resistance form defined as a limit of finite-dimensional resistance forms. Moreover, at the end of the second subsection, we will make our main assumption (see Assumption 3.2.11) which ensures, that the fractal supports a well behaving energy form. For more details, in particular topological properties, and proofs of the claimed statements, we refer to [**Tep08**] and references therein.

3.2.1. Finitely ramified fractals. In this subsection, we follow the introduction of finitely ramified fractals given in [**Tep08**].

Definition 3.2.1 (Finitely ramified fractal). *A compact metric space* (K, d) *is called* finitely ramified fractal *if there is a* cell structure $\{K_\alpha\}_{\alpha\in\mathcal{A}}$ *and a* boundary (vertex) structure $\{V_\alpha\}_{\alpha\in\mathcal{A}}$ *such that*

(i) $\mathcal{A}$ *is a countable index set with* $0 \in \mathcal{A}$*;*
(ii) each K_α *is a distinct compact connected subset of* K*;*
(iii) each V_α *is a finite subset of* K_α *with at least two elements;*
(iv) if $K_\alpha = \cup_{j=1}^k K_{\alpha_j}$ *then* $V_\alpha = \cup_{j=1}^k V_{\alpha_j}$*;*
(v) there exists a filtration $\{\mathcal{A}_m\}_{m\in\mathbb{N}_0}$ *such that*
- $\mathcal{A}_m$ *are finite subsets of* $\mathcal{A}$*,* $\mathcal{A}_0 = \{0\}$ *and* $K_0 = K$*;*
- $\mathcal{A}_m \bigcap \mathcal{A}_n = \emptyset$ *if* $n \neq m$*;*
- *for any* $\alpha \in \mathcal{A}_m$ *there exist* $\alpha_1, \ldots, \alpha_k \in \mathcal{A}_{m+1}$ *such that*

$$K_\alpha = \bigcup_{\ell=1}^{k} K_{\alpha_\ell};$$

(vi) $K_\alpha \cap K_\beta = V_\alpha \cap V_\beta$ *for any* $\alpha, \beta \in \mathcal{A}_m$ *with* $\alpha \neq \beta$*;*
(vii) for any strictly decreasing infinite cell sequence $K_{\alpha_1} \supsetneq K_{\alpha_2} \supsetneq \ldots$ *there exists* $x \in K$ *such that* $\bigcap_{m\geq 1} K_{\alpha_m} = \{x\}$*.*

If the above conditions are satisfied, then $(K, \{K_\alpha\}_{\alpha\in\mathcal{A}}, \{V_\alpha\}_{\alpha\in\mathcal{A}})$ *is called a* finitely ramified cell structure.

Notation 3.2.2. *If* $\alpha \in \mathcal{A}_m$*, we call each* K_α *an* m*-cell. Moreover, we define the collection of all* m*-cell boundary points by* $V_m := \bigcup_{\alpha\in\mathcal{A}_m} V_\alpha$ *and we set* $V_\infty := \bigcup_{m\geq 0} V_m$*.*

From the definition, it follows that $V_m \subset V_{m+1}$ for all $m \geq 0$ and hence, the sequence $\{V_m\}_{m\in\mathbb{N}}$ is monotonously increasing. Since V_α is finite for each $\alpha \in \mathcal{A}_m$ the union of m-cell boundary vertices V_m has finite cardinality and the limit V_∞ is a countable subset of K.

By (v) and (vii) any cell is the union of at least two smaller cells. But each cell contains at least two points and is compact and connected by (ii) and (iii). Thus each cell K_α is uncountable and so is K.

Let us briefly introduce an important subclass of the class of finitely ramified fractals, the so-called post-critically finite self-similar fractals (see also Definition 4.4.1).

Example 3.2.3 (Post-critically finite self-similar fractals [**Kig01, Str06**]). *Let* (K, d) *be a compact metric space. We call* K *a* self-similar set *(or simply* self-similar*) if* K *is non-empty and if there exists a family* $F = \{F_j\}_{j=1,\ldots,N}$ *of contractive similarities* $F_j\colon K \longrightarrow K$ *such that*

$$K = F(K) := F_1(K) \cup \cdots \cup F_N(K). \tag{3.2.1}$$

The family F is called an iterated function system (IFS)*. Conversely, if we start with an IFS $F = \{F_j\}_{j=1,\dots,N}$ in a compact metric space (X, d) then there exists a unique non-empty compact subset $K \subset X$ such that* (3.2.1) *holds. This is a consequence of Banach's fixed point theorem applied to the collection of non-empty compact subsets of X endowed with the Hausdorff metric (cf.* [**Kig01**, Theorem 1.1.7]*). Moreover, the IFS describes a cell structure via the map*

$$w = w_1 \dots w_m \longmapsto F_w(K) := F_{w_1} \circ F_{w_2} \circ \dots \circ F_{w_m}(K),$$

*where $w \in W_m := \{1, \dots, N\}^m$ is a word of length m over the alphabet $\{1, \dots, N\}$. We call $F_w(K)$ an m-*cell*. Then a (canonical) filtration is given by $\mathcal{A}_m = W_m$.*

We call K post-critically finite self-similar fractal (pcf fractal) *if K is connected, self-similar and if there exists a finite set $V_0 \subset K$ such that*

$$F_w(K) \cap F_{w'}(K) = F_w(V_0) \cap F_{w'}(V_0)$$

for any pair of distinct words w and w' of the same length m. The set V_0 is called boundary of K and V_0 is usually a subset of the fixed points of the IFS. Note that V_0 is not necessarily the topological boundary of K.

We easily verify that a post-critically finite self-similar fractal is a finitely ramified fractal if $|V_0| > 1$ (cf. [**Tep08**, Prop. 7.2]*). By the definition of pcf fractals it is clear that in any point $p \in V_\infty$ at most finitely many distinct m-cells can intersect. However, this might not be true anymore for an arbitrary finitely ramified fractal (e.g. the Diamond lattice fractal introduced in Subsection 3.4.3).*

As mentioned in the example above, the vertex boundary V_α of the cell K_α does not necessarily agree with the topological boundary of the cell K_α in (K, d) but one can show that the boundary of K_α is contained in V_α (cf. [**Tep08**, Prop. 2.7]). The vertex boundary V_0 of a general finitely ramified fractal K itself is arbitrary (cf. [**Tep08**, Rem. 2.4] for a more detailed discussion). However, if K is a post-critically finite self-similar fractal then V_0 is determined as the post-critical set (cf. [**Kig01**, Section 1.3]). Moreover, it is understood that objects like a Laplacian do actually depend on the choice of the boundary we make here.

The filtration $\mathcal{A}$ is also not unique in general. The objects we are interested in like resistance forms and their associated Laplacians do not depend on the particular choice of a filtration (cf. [**Tep08**, Rem. 2.5]). We tacitly assume that a filtration is fixed.

Let us briefly mention some typical examples satisfying the definition of finitely ramified fractals. Later, we discuss some of them in greater detail in Section 3.4. As mentioned before, the collection of *post-critically finite self-similar fractals* supporting a resistance form in the sense of [**Kig01**] is one of the major classes for examples. The most prominent representatives are the unit interval and the Sierpiński gasket. The *diamond lattice fractal*, described in Subsection 3.4.3, is an example for a self-similar finitely ramified fractal which is self-similar (i.e., described by an IFS) but is not post-critically finite. Another interesting example is a *metric graph*. We will remind the definition and give a detailed explanation later in Section 3.4.1.

Example 3.2.4 (Unit interval). *The unit interval* $I = [0,1]$ *is the self-similar fractal associated with the two contractions* $F_1, F_2\colon \mathbb{R} \longrightarrow \mathbb{R}$, *defined by*

$$F_1(x) = \frac{1}{2}\cdot x \qquad \text{and} \qquad F_2(x) = \frac{1}{2}\cdot(x+1),$$

i.e., I *is the unique non-empty compact subset of* $\mathbb{R}$ *such that* $I = F(I) \coloneqq F_1(I) \cup F_2(I)$ *(cf.* [**Kig01**, Thm. 1.2.4 and Ex. 1.2.7]*). A natural filtration* $\mathcal{A} = \{\mathcal{A}_m\}_{m\in\mathbb{N}_0}$ *is given by the words* W_m *of length* m *over the alphabet* $\{1,2\}$, *i.e., we define* $\mathcal{A}_m \coloneqq W_m = \{1,2\}^m$. *Moreover, the contractions describe a cell structure* $\{F_w(I)\}_{w\in\mathcal{A}}$ *by the map*

$$\mathcal{A}_m \ni w = w_1 \dots w_m \longmapsto F_w(I) \coloneqq F_{w_1} \circ \dots \circ F_{w_m}(I)$$

and the set of all m*-cell boundary points* V_m *is given by*

$$V_m = \bigcup_{w\in\mathcal{A}_m} F_w(V_0) \qquad \text{and} \qquad V_0 \coloneqq \{0,1\}.$$

Then $V_m = \{\, k2^{-m} \mid k = 0,1,\dots,2^m \,\}$ *is the set of* m*-dyadic numbers in* I *and two distinct* m*-cells can only intersect in these points. That is, for two distinct words* w *and* w' *of the same length* m, *we have*

$$F_w(I) \cap F_{w'}(I) = F_w(V_0) \cap F_{w'}(V_0).$$

The cell boundary of an m*-cell* $F_w(I)$ *is given by* $V_w = F_w(I) \cap V_m$. *Note that the vertex-boundary* V_0 *from the interval coincides with the topological boundary but this is not necessarily the case as the next example shows.*

Example 3.2.5 (Sierpiński gasket). *The Sierpiński gasket is a more convenient example of a post-critically finite self-similar fractal. Let* p_1, p_2 *and* p_3 *be the vertices of an equilateral triangle in the plain. Then the Sierpiński gasket* K *is described by the three contractions*

$$F_j\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad F_j(x) = \frac{1}{2}(x - p_j) + p_j, \qquad j = 1,2,3.$$

A canonical choice for a filtration $\mathcal{A} = \{\mathcal{A}_m\}_{m\in\mathbb{N}_0}$ *is again* $\mathcal{A}_m \coloneqq W_m = \{1,2,3\}^m$. *As in the case of the interval, the map* $w \longmapsto F_w(K)$ *also define a cell structure and a boundary structure, where* $V_0 \coloneqq \{p_1,p_2,p_3\}$ *is the vertex-boundary of the Sierpiński gasket. Note that* V_0 *is* not *the topological boundary of* K *as subset of* $\mathbb{R}^2$ *because the Sierpiński gasket has empty interior.*

3.2.2. Resistance forms on finitely ramified fractals. Let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be a resistance form on V_∞ (defined in Definition 3.1.1). The trace of $\mathcal{E}$ on V_m is given by

$$\mathcal{E}_{V_m}(f) = \frac{1}{2}\sum_{x,y\in V_m} \gamma_{\{x,y\},m}\bigl|f(x) - f(y)\bigr|^2$$

for any function $f \in \ell(V_m)$, where

$$\gamma_{\{x,y\},m} = \mathcal{E}(\psi_x^{V_m}, \psi_y^{V_m}) \geq 0 \tag{3.2.2}$$

denotes the *conductance* between two distinct vertices x and y in V_m. In the above equation, $\psi_x^{V_m}$ denotes the m-harmonic function with boundary values $\mathbb{1}_{\{x\}}$ on V_m, defined in Definition 3.1.4.

REMARK 3.2.6 (Resistance forms as discrete graph energies). *The existence of such a resistance form is not guaranteed in general. But if it exists, it also allows us to define an increasing sequence $\{G_m\}_{m\in\mathbb{N}}$ of (finite) discrete graphs $G_m = (V_m, E_m)$, where the vertex set V_m is as defined above and with edge set, given by*

$$E_m := \{\, e \,|\, e = \{x, y\} \subset V_m \text{ such that } \mathcal{E}(\psi_x^{V_m}, \psi_y^{V_m}) > 0 \,\}.$$

Moreover, we will introduce a canonical vertex measure $\mu_m = \{\mu_m(v)\}_{v\in V_m}$ later and an edge weight is given by the conductance $\gamma_m = \{\gamma_{e,m}\}_{e\in E_m}$. Hence, we can interpret $\mathcal{E}_{V_m}$ also as the canonical energy form on the graph G_m.

For $\alpha \in \mathcal{A}_m$, we can restrict $\mathcal{E}_{V_m}$ to the boundary vertices V_α of the cell K_α by

$$\mathcal{E}_{V_\alpha}(f) = \frac{1}{2} \sum_{x,y\in V_\alpha} \gamma_{\{x,y\},m} \big| f(x) - f(y) \big|^2.$$

The form $\mathcal{E}_{V_\alpha}$ is only affected by the edges inside K_α. Hence, $\mathcal{E}_{V_m}$ decomposes with respect to the cell structure, i.e.,

$$\mathcal{E}_{V_m}(f) = \sum_{\alpha\in\mathcal{A}_m} \mathcal{E}_{V_\alpha}(f),$$

for any $f \in \ell(V_m)$. The same holds true for the energy on V_∞.

LEMMA 3.2.7. *There is a decomposition of the quadratic form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ on V_∞ according to the cell structure, i.e.*

$$\mathcal{E}(u) = \sum_{\alpha\in\mathcal{A}_m} \mathcal{E}_{V_{\infty,\alpha}}(u\restriction_{V_{\infty,\alpha}})$$

for any $u \in \operatorname{dom}\mathcal{E}$ and where $V_{\infty,\alpha} := V_\infty \cap K_\alpha$.

PROOF. Let $\alpha \in \mathcal{A}_m$. Since $\mathcal{E}$ is a resistance form on V_∞ and $V_m \cap K_\alpha \subset V_{m+1} \cap K_\alpha$, the sequence $\{(V_m \cap K_\alpha, \mathcal{E}_{V_m\cap K_\alpha})\}_{m\in\mathbb{N}_0}$ is compatible. Hence, by Proposition 3.1.6, $\mathcal{E}_{V_{\infty,\alpha}}$ is a resistance form on $V_{\infty,\alpha}$. The claim follows by summation over all m-cells and the fact that there are no edges between $V_m \cap K_\alpha$ and $V_m \cap K_\beta$ for $\alpha, \beta \in \mathcal{A}_m$ with $\alpha \neq \beta$. □

Moreover, the existence of a cell structure allows us to concretise the Hölder estimate in (3.1.1).

LEMMA 3.2.8. *Let $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ be a resistance form on V_∞ and $\alpha \in \mathcal{A}_m$. Then*

$$|u(x) - u(y)|^2 \leq R(x, y)\mathcal{E}_{V_{\infty,\alpha}}(u\restriction_{V_{\infty,\alpha}}) \leq \frac{1}{\gamma_{-,m}} \mathcal{E}_{V_{\infty,\alpha}}(u\restriction_{V_{\infty,\alpha}})$$

for $u \in \operatorname{dom}\mathcal{E}$ and $x, y \in V_{\infty,\alpha}$ and where $\gamma_{-,m}$ is defined in (3.2.3).

Proof. First, we estimate the R-diameter of an m-cell. Let $\alpha \in \mathcal{A}_m$ and $x, y \in V_\alpha$. Then

$$R(x,y) \coloneqq \bigl(\min\{\, \mathcal{E}(u) \,|\, u \in \operatorname{dom}\mathcal{E}, u(x)=1, u(y)=0 \,\}\bigr)^{-1} \le \frac{1}{\gamma_{-,m}},$$

where we used (3.1.3), i.e., $\mathcal{E}(u) \ge \mathcal{E}_{V_m}(u\restriction_{V_m}) \ge \gamma_{-,m}$ for each $u \in \operatorname{dom}\mathcal{E}$ and where

$$\gamma_{-,m} \coloneqq \min\{\, \gamma_{\{x,y\},m} \,|\, x,y \in V_m, x \ne y, \gamma_{\{x,y\}} > 0 \,\}. \tag{3.2.3}$$

Since the cells are nested by Definition 3.2.1 (v), we have $\operatorname{diam}_R(V_{\infty,\alpha}) \le 1/\gamma_{-,m}$ and by Lemma 3.2.7 we conclude the claimed estimate. □

Example 3.2.9 (Standard energy form on the unit interval, [**Str06**]). *Let us continue the example of the unit interval $I = [0,1]$ introduced in Example 3.2.4. In order to define a resistance form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ on I it is sufficient to start with a compatible sequence $\{(V_m, \mathcal{E}_m)\}_{m\in\mathbb{N}_0}$ by Proposition 3.1.6.*

Let $G_0 = (V_0, E_0)$ be the complete graph with vertex set $V_0 = \{0,1\}$. We define a sequence of discrete graphs $G_m = (V_m, E_m)$ inductively as the line graphs with vertices V_m, i.e.,

$$V_m \coloneqq \{\, k2^{-m} \,|\, k = 0,1,\dots,2^m \,\} \quad \textit{and} \quad E_m \coloneqq \{\, \{x,y\} \subset V_m \,|\, |x-y| = 2^{-m} \,\}.$$

Then, the renormalised graph energy form $\mathcal{E}_m$ on G_m is given by

$$\mathcal{E}_m(f) \coloneqq \sum_{e=\{x,y\}\in E_m} 2^m \bigl|f(x) - f(y)\bigr|^2 = \sum_{k=1}^{2^m} 2^m \bigl|f\bigl(k2^{-m}\bigr) - f\bigl((k-1)2^{-m}\bigr)\bigr|^2, \tag{3.2.4}$$

which is defined for any $f\colon V_m \longrightarrow \mathbb{C}$. The renormalisation factor $\gamma_{\{x,y\},m} = 2^{-m}$ *is chosen such that* (3.1.5) *holds, i.e., it guarantees that the sequence $\{(V_m, \mathcal{E}_m)\}_{m\in\mathbb{N}_0}$ is compatible. Hence, the limit exists and we can define a resistance form $(\mathcal{E}_\infty, \operatorname{dom}\mathcal{E}_\infty)$ on V_∞.*

The space of harmonic functions is a 2-dimensional subspace of $\operatorname{dom}\mathcal{E}_\infty$*, spanned by $\psi_1(x) = 1 - x$ and $\psi_2(x) = x$, i.e., harmonic functions are just affine linear functions. A function $u \in \operatorname{dom}\mathcal{E}_\infty$ is m-harmonic, if and only if $u \circ F_w$ is harmonic for each word $w \in \mathcal{A}_m$. That means that an m-harmonic function is defined by fixing boundary values on V_m and extending affine linear to the m-cells.*

The resistance metric R coincides with the Euclidean metric. Let x and y be two points in V_∞ and such that $y < x$. Then there exists $m \in \mathbb{N}_0$ such that $x, y \in V_m$. Define a function $f = f_{x,y}$ by $f = 0$ on $[0, y)$, $f = 1$ on $(x, 1]$ and affine linear on $[y, x]$. Then f realises the maximum in Definition 3.1.2 and we conclude

$$R(x,y) = |x - y|$$

for all $x, y \in V_\infty$. Hence, the completion of V_∞ with respect to the resistance metric coincides with I and by Proposition 3.1.6, we can extend $(\mathcal{E}_\infty, \operatorname{dom}\mathcal{E}_\infty)$ to a resistance form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ on I.

Since we need a suitable Hilbert space structure to define the associated energy or Dirichlet form on I, we first need to introduce a suitable measure. Let $\mathcal{B}(I)$ denote *the Borel σ-field (with respect to the Euclidian metric). The* (standard) self-similar probability measure μ with weights $(1/2, 1/2)$ *on* I *is defined by the equation*

$$\mu(A) = \frac{1}{2}\mu(F_1^{-1}(A)) + \frac{1}{2}\mu(F_2^{-1}(A))$$

for any Borel set $A \in \mathcal{B}(I)$ *(see e.g.* [**Str06**, Sec. 1.2] *for more details). Note that* μ *coincides with the restriction of the Lebesgue measure to* I*. As a Hilbert space structure on* I *we choose* $\mathsf{L}_2(I,\mu)$*, i.e., the space of square integrable functions with respect to the measure* μ*. Then a natural energy form* $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ *in* $\mathsf{L}_2(I,\mu)$ *is given by*

$$\mathcal{E}(u) = \|u'\|^2_{\mathsf{L}_2(I,\mu)} = \int_0^1 \big|u'(x)\big|^2 \,\mathrm{d}\mu(x), \tag{3.2.5}$$

which is defined for each weakly differentiable $u \in \mathsf{L}_2(I,\mu)$ *with* $u' \in \mathsf{L}_2(I,\mu)$*, i.e.,* $\operatorname{dom}\mathcal{E} = \mathsf{H}^1(I,\mu)$*. It is a classical result in Sobolev space theory that* $\mathsf{H}^1(I,\mu)$ *is the space of all absolutely continuous and square integrable functions on* I*.*

The energy form $\mathcal{E}$ *is indeed the limit of the sequence* $\{\mathcal{E}_m\}_{m\in\mathbb{N}_0}$*. Let* u *be a continuously differentiable function on* I*. Then by the mean value theorem, there exists an* $x_k \in (k-1/2^m, k/2^m)$ *for each* $k = 1, 2, \dots, 2^m$*, such that*

$$\mathcal{E}_m(u\restriction_{V_m}) = \sum_{k=1}^{2^m} \Big|\frac{u\big(k2^{-m}\big) - u\big((k-1)2^{-m}\big)}{2^{-m}}\Big|^2 \cdot \frac{1}{2^m} = \sum_{k=1}^{2^m} \big|u'(x_k)\big|^2 \cdot \frac{1}{2^m}$$

and hence, in this case, the claim is standard (cf. [**Str06**, Sec. 1.3] *for more details).*

Example 3.2.10 (Standard Dirichlet form on the Sierpiński gasket, [**Str06**]). *In the case of the Sierpiński gasket* K*, introduced in Example 3.2.5, the strategy to define a resistance form is the same. Let* $G_0 = (V_0, E_0)$ *be the complete graph and again, we define* $G_m = (V_m, E_m)$*, where*

$$E_m := \big\{ \{x,y\} \,\big|\, x,y \in V_m, x \neq y \text{ and there exists } w \in \mathcal{A}_m \text{ such that } x,y \in F_w(K) \big\}.$$

Then we define a compatible sequence $\{(V_m, \mathcal{E}_m)\}_{m\in\mathbb{N}_0}$ *as follows. Let* $f\colon V_m \longrightarrow \mathbb{C}$*. Then*

$$\mathcal{E}_m(f) = \Big(\frac{5}{3}\Big)^m \sum_{\{x,y\}\in E_m} \big|f(x) - f(y)\big|^2.$$

This is indeed a compatible sequence which one easily checks by verifying (3.1.5) *(see e.g.* [**Str06**, Sec. 1.3] *for a direct proof). Hence, we define the limit form* $(\mathcal{E}_\infty, \operatorname{dom}\mathcal{E}_\infty)$*. One can show that the topology induced by the resistance metric and the relative topology (that* K *inherits as subset of* $\mathbb{R}^2$*) are (topologically) equivalent — but not metrically — (cf.* [**Str06**, Sec. 1.6] *or* [**Kig01**, Thm.3.3.4] *for a general statement), i.e., we can identify* Ω *with* K*. Thus, by* (3.1.1)*, we can identify any function in the domain of* $\mathcal{E}_\infty$ *uniquely with a continuous function on* K *(a direct proof is also given in* [**Str06**, Sec. 1.4]*).*

Moreover, from the aforementioned proof we can extract the so called $1/5 - 2/5$ *rule to compute the values of harmonic functions recursively. The values of an harmonic function* h *on* $V_\star$ *can be computed explicitly by iteration: If the values in the vertices of* V_m *are known, then, for each vertex* $y \in V_{m+1} \setminus V_m$*, there exists a unique* m*-cell* *that contains y; the value* $h(y)$ *is given by* $1/5$ *times the value at the vertex in* V_m *opposite to y in the* m*-cell plus* $2/5$ *times the values of h at the vertices (of* V_m*) adjacent to y in the same* m*-cell (cf.* [**Str06**, Sec. 1.3]*).*

A convenient choice for a measure on the Sierpiński gasket K is the (standard) self-similar measure μ with weights $(1/3, 1/3, 1/3)$, *defined by*

$$\mu(A) = \frac{1}{3}\mu(F_1^{-1}(A)) + \frac{1}{3}\mu(F_2^{-1}(A)) + \frac{1}{3}\mu(F_3^{-1}(A))$$

for any Borel set $A \subset K$*. Then the resistance form* $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ *induces a regular Dirichlet or energy form in* $\mathsf{L}_2(K, \mu)$*.*

In order to construct magnetic potentials as in [**IRT12**] and in order to apply the concept of quasi-unitary equivalence of [**Pos12**] we make the following assumption. If no magnetic potential is involved only the closeness of $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ in $\mathsf{L}_2(K, \mu)$ is required.

Assumption 3.2.11. *We assume the following:*

- *Let* μ *be a finite Borel regular measure on K such that any non-empty open set has strictly positive measure.*
- *The resistance form* $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ *induces a local regular Dirichlet form in* $\mathsf{L}_2(K, \mu)$*, in particular,* $\mathcal{E}$ *is a closed quadratic form.*
- *For* $m \in \mathbb{N}_0$ *and distinct* $x, y \in V_m$, $\langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathsf{L}_2(K_\alpha, \mu)} \neq 0$ *implies* $c_{\{x,y\},m} > 0$.

Remark 3.2.12. *In general, a resistance form induces a closed metric space* (Ω, R) *as in Proposition 3.1.6, but* (Ω, R) *may be a proper subset of* (K, d)*. As described in Subsection 3.1.2 we can always construct a local regular Dirichlet form* $\mathcal{E}$ *in* $\mathsf{L}_2(\Omega, \mu)$*, where* μ *is a Borel regular measure on* (Ω, R) *which is strictly positive on any non-empty open subset. Moreover, if all m-harmonic functions are continuous in the topology of K, then there is a continuous injective map from* (Ω, R) *into* (K, d) *which is the identity on* V_∞ *(see* [**Tep08**, Thm. 3.9]*).*

For our approach, it does not matter whether the Dirichlet form lives on (K, d) *or on* (Ω, R)*; the proofs are essentially the same. In what follows we hence write K instead of distinguishing between* Ω *and K.*

If K is a self-similar finitely ramified fractal and $\mathcal{E}$ *is a self-similar resistance form which is regular (i.e., the energy renormalisation factors are greater than* 1*) then* $\Omega = K$*, cf.* [**Tep08**, Thm. 7.9] *and* [**Kig01**]*. Even if* $\mathcal{E}$ *is not regular,* $\mathcal{E}$ *still might induce a local Dirichlet form in* $\mathsf{L}_2(K, \mu)$ *as e.g. in the case of the diamond lattice fractal discussed in Subsection 3.4.3. For more details see also* [**HK10**, Ch. 4] *where the authors extended results from* [**Kig01**, Ch. 3.4].

The last point in Assumption 3.2.11 enters in the proof of Theorem 3.3.4.

3.2.3. Resistance forms and magnetic potentials. Let K be a finitely ramified fractal with regular resistance form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ as described in Section 3.2 of this chapter and let μ be a Borel measure on K. Moreover, we assume that the first and the second point of Assumption 3.2.11 hold.

In (3.2.5) we introduced the classical example of a Drichlet form on the unit interval. In this section, we give an answer to the question whether we get a similar representation for a more general Dirichlet form. More precisely, we ask whether there exist a suitable space $\mathcal{H}$ and a map $\partial\colon \operatorname{dom}\mathcal{E} \longrightarrow \mathcal{H}$ (satisfying the Leibniz rule) such that $\mathcal{E}(u) = \|\partial u\|^2_{\mathcal{H}}$ for all $u \in \operatorname{dom}\mathcal{E}$. This turns out to be the key step to define magnetic potentials and perturbed energy forms.

In this subsection, we follow the construction from [**IRT12**]. The general theory was developed in [**CS03**]. For more information about the energy measure associated with a Dirichlet form, we refer to [**FOT94**, Sec. 3] and references therin.

For any two functions a and b in $\operatorname{dom}\mathcal{E}$, there exists a finite (signed) Borel measure $\nu_{a,b}$ such that

$$\frac{1}{2}\big(\mathcal{E}(au, b) + \mathcal{E}(bu, a) - \mathcal{E}(ab, u)\big) = \int_K u(x)\,\mathrm{d}\nu_{a,b}(x)$$

for all $u \in \operatorname{dom}\mathcal{E}$. Moreover, if $a, b, c \in \operatorname{dom}\mathcal{E}$, then we have a Leibniz rule via

$$\mathrm{d}\nu_{ab,c} = a\,\mathrm{d}\nu_{b,c} + b\,\mathrm{d}\nu_{a,c}.$$

If $a = b$ then $\nu_a := \nu_{a,a}$ is non-negative, non-atomic (cf. [**IRT12**, Thm. 2.6]) and $\mathcal{E}(a) = \nu_a(F)$. We call ν_a the *energy measure of* a. We define a non-negative bilinear form by

$$\langle a \otimes b, c \otimes d\rangle_{\mathcal{H}} := \int_K b(x)d(x)\,\mathrm{d}\nu_{a,c}(x) \tag{3.2.6}$$

for all $a, b, c, d \in \operatorname{dom}\mathcal{E}$.

Definition 3.2.13 (Hilbert module of 1-forms). *The* Hilbert module of 1-forms $\mathcal{H}$ *associated with the Dirichlet form* $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ *is obtained by factoring* $\operatorname{dom}\mathcal{E} \otimes \operatorname{dom}\mathcal{E}$ *with the norm zero subspace and taking the completion of this quotient with respect to the semi norm* $\|.\|_{\mathcal{H}} := \sqrt{\langle ., .\rangle_{\mathcal{H}}}$.

We define left and right action of $\operatorname{dom}\mathcal{E}$ on the product $\operatorname{dom}\mathcal{E} \otimes \operatorname{dom}\mathcal{E}$ by

$$a(b \otimes c) = (ab) \otimes c - a \otimes (bc) \qquad \text{and} \qquad (b \otimes c)d = b \otimes (cd)$$

and extended this action onto $\mathcal{H}$ by continuity and density. In our setting, left and right action commute by [**IRT12**, Thm. 2.7] and by the completeness of $\mathcal{H}$ we can extend the definition of $a \otimes b$ to functions $a \in \operatorname{dom}\mathcal{E}$ and $b \in \mathsf{L}_\infty(K, \mu)$. More precisely, that means that left action coincides with the restriction of right action to functions in $\operatorname{dom}\mathcal{E}$. We define a derivation

$$\partial\colon \operatorname{dom}\mathcal{E} \longrightarrow \mathcal{H}, \qquad a \longmapsto a \otimes \mathbb{1}.$$

which satisfies $(\partial a)b = (a \otimes \mathbb{1})b = a \otimes b$ for all $a, b \in \operatorname{dom}\mathcal{E}$ and

$$\mathcal{E}(u) = \|\partial u\|^2_{\mathcal{H}} = \|u \otimes \mathbb{1}\|^2_{\mathcal{H}} \tag{3.2.7}$$

for any $u \in \operatorname{dom}\mathcal{E}$. Moreover, for $a \in \mathcal{H}$ and $u \in \mathsf{L}_\infty(K, \mu)$, we have

$$\|au\|_{\mathcal{H}} \le \|a\|_{\mathcal{H}}\|u\|_\infty. \tag{3.2.8}$$

For $m \in \mathbb{N}$ and $\alpha \in \mathcal{A}_m$ denote by $H_m(K_\alpha)$ the space of m-harmonic functions on K_α. We then define

$$\mathcal{H}_m := \Big\{ \sum_{\alpha \in \mathcal{A}_m} \partial h_\alpha \mathbb{1}_{K_\alpha} \,\Big|\, h_\alpha \in H_m(K) \Big\} \subset \mathcal{H}, \tag{3.2.9}$$

the space of piecewise harmonic (and not necessarily continuous) functions. The elements of $\mathcal{H}_m \setminus \mathcal{H}_{m-1}$ are called *locally exact* 1*-forms* or 1*-forms exact at level* m. These spaces are closed (because they are finite dimensional) and we have $\mathcal{H}_m \subset \mathcal{H}_{m+1}$. Moreover, the union $\bigcup_{m \in \mathbb{N}} \mathcal{H}_m$ is dense in $\mathcal{H}$. Hence, by the spectral theorem, we conclude the following: Let $a \in \mathcal{H}$ and let a_m be the orthogonal projection of a onto $\mathcal{H}_m$. Then

$$\lim_{m \to \infty} \|a - a_m\|_{\mathcal{H}} \to 0. \tag{3.2.10}$$

We are now prepared to define *magnetic* resistance forms in the Hilbert spaces $\mathsf{L}_2(K, \mu)$ and $\ell_2(V_m, \mu_m)$, respectively. Magnetic perturbations of Dirichlet forms have already been treated in [**HT13, HR16, HKM**$^{+}$**17**] (see also references therein).

Definition 3.2.14 (Magnetic resistance form). *Let $a \in \mathcal{H}$ be a real valued. We call the non-negative quadratic form $\mathcal{E}^a$ defined by*

$$\mathcal{E}^a(u) := \big\|(\partial + \mathrm{i}a)u\big\|_{\mathcal{H}}^2 \tag{3.2.11}$$

with $\operatorname{dom}\mathcal{E}^a = \operatorname{dom}\mathcal{E}$ *the* magnetic resistance form on K *(associated with $\mathcal{E}$) with* magnetic potential a. *In particular, if $a \in \mathcal{H}_m$ for some $m \geq 0$ then we call the magnetic potential* locally exact.

If $a_m = \sum_{\alpha \in \mathcal{A}_m} A_\alpha \otimes \mathbb{1}_{K_\alpha} \in \mathcal{H}_m$ is real valued then $\mathcal{E}^{a_m}$ decomposes with respect to the cell structure, i.e.,

$$\begin{aligned} \mathcal{E}^{a_m}(u) &= \sum_{\alpha \in \mathcal{A}_m} \big\|(\partial u)\mathbb{1}_{K_\alpha} + \mathrm{i}(\partial A_\alpha)u\mathbb{1}_{K_\alpha}\big\|_{\mathcal{H}}^2 \\ &= \sum_{\alpha \in \mathcal{A}_m} \big\|\mathrm{e}^{-\mathrm{i}A_\alpha}\partial(u\mathrm{e}^{\mathrm{i}A_\alpha})\mathbb{1}_{K_\alpha}\big\|_{\mathcal{H}}^2 = \sum_{\alpha \in \mathcal{A}_m} \mathcal{E}_{K_\alpha}(u\mathrm{e}^{\mathrm{i}A_\alpha}), \end{aligned}$$

where we used the Leibniz rule and the decomposition of the norm $\|.\|_{\mathcal{H}}$ as sum of energies on the cells.

Definition 3.2.15 (Discrete magnetic form). *For real valued $a_m = \sum_{\alpha \in \mathcal{A}_m} A_\alpha \otimes \mathbb{1}_{K_\alpha} \in \mathcal{H}_m$ we define the* magnetic resistance form *(at scale m) associated with $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ by*

$$\mathcal{E}_{V_m}^{a_m}(f) = \frac{1}{2} \sum_{\alpha \in \mathcal{A}_m} \sum_{x,y \in V_\alpha} \gamma_{\{x,y\},m} \big|f_\alpha^{a_m}(x) - f_\alpha^{a_m}(y)\big|^2 \tag{3.2.12}$$

for $f \colon V_m \longrightarrow \mathbb{C}$, where $f_\alpha^{a_m}(z) := f(z)\mathrm{e}^{\mathrm{i}A_\alpha(z)}$.

From (3.2.12) we conclude that $\mathcal{E}_{V_m}^{a_m}$ also decomposes with respect to the cell structure:

$$\mathcal{E}_{V_m}^{a_m}(f) = \sum_{\alpha \in \mathcal{A}_m} \mathcal{E}_{V_\alpha}(f\mathrm{e}^{\mathrm{i}A_\alpha}).$$

Let us end this section by studying the structure of $\mathcal{H}$ in the case of the interval.

EXAMPLE 3.2.16 (Unit interval, [**IRT12**, Sec. 5.1]). *Let us continue Example 3.2.9: $I = [0,1]$ with Borel σ-field $\mathcal{B}(I)$ and μ is the Lebesgue measure on I. We define a map between $\mathcal{H}$ and $\mathsf{L}_2(I,\mu)$ via*

$$\mathcal{H} \ni a \otimes b \longmapsto (x \mapsto a'(x)b(x)) \in \mathsf{L}_2(I,\mu). \tag{3.2.13}$$

This is well-defined since $a \in \operatorname{dom}\mathcal{E} = \mathsf{H}^1(I,\mu)$ and $b \in \mathsf{L}_\infty(I,\mu)$. Moreover, this map defines an isometry because

$$\|a' \cdot b\|^2_{\mathsf{L}_2(I,\mu)} = \int_0^1 |a'(x)b(x)|^2 \,\mathrm{d}\mu(x) = \int_0^1 |b(x)|^2 \cdot \underbrace{|a'(x)|^2 \,\mathrm{d}\mu(x)}_{= \mathrm{d}\nu_a(x)} = \|a \otimes b\|^2_{\mathcal{H}}.$$

Studying the inverse of the map defined in (3.2.13) will help us to understand the action of the derivation associated with the Dirichlet form. Let $u \in \mathsf{L}_2(I,\mu)$ be arbitrary. Then u is μ-integrable and the function

$$x \longmapsto u(0) + \int_0^x u(y)\,\mathrm{d}\mu(y)$$

is absolutely continuous. Hence, there exist $A \in \operatorname{dom}\mathcal{E}$ such that $A' = u$ a.e. and the inverse is given by $u \longmapsto A \otimes \mathbb{1}$. Thus we can identify the space of 1-forms $\mathcal{H}$ with $\mathsf{L}_2(I,\mu)$. Moreover, $\partial\colon \operatorname{dom}\mathcal{E} \longrightarrow \mathsf{L}_2(I,\mu)$ acts as the usual derivative.

Let a_m be an arbitrary element of $\mathcal{H}_m$. Then, on each subinterval $[k/2^m, k+1/2^m)$, there exist an affine linear function A_k, where $k = 0,1\ldots,2^m-1$, such that

$$a_m = \sum_{k=0}^{2^m-1} A_k \otimes \mathbb{1}_{[\frac{k}{2^m},\frac{k+1}{2^m})}.$$

In particular, the isometry between $\mathcal{H}$ and $\mathsf{L}_2(I,\mu)$, defined in (3.2.13), maps $\mathcal{H}_m$ onto the space of simple functions being constant on each subinterval $[k/2^m, k+1/2^m)$. Applying the Leibniz rule, we conclude for $a_m \in \mathcal{H}_m$ as above, that

$$\begin{aligned}\mathcal{E}^a(u) = \|(\partial + \mathrm{i}a_m)u\|^2_{\mathcal{H}} &= \sum_{k=0}^{2^m-1} \left\|(\mathrm{e}^{\mathrm{i}A_k}\partial u + \mathrm{i}\partial A_k \mathrm{e}^{\mathrm{i}A_k} u)\mathbb{1}_{[\frac{k}{2^m},\frac{k+1}{2^m})}\right\|^2_{\mathcal{H}} \\ &= \sum_{k=0}^{2^m-1} \left\|\partial(\mathrm{e}^{\mathrm{i}A_k}u)\mathbb{1}_{[\frac{k}{2^m},\frac{k+1}{2^m})}\right\|^2_{\mathcal{H}} = \sum_{k=0}^{2^m-1} \int_{k2^{-m}}^{(k+1)2^{-m}} \left|u'(x) + \mathrm{i}A_k'(x)u(x)\right|^2 \mathrm{d}\mu(x),\end{aligned}$$

for each $u \in \operatorname{dom}\mathcal{E}$.

3.3. Quasi-unitary equivalence of resistance forms with magnetic potentials and their associated discrete graph energies

We assume the situation from Section 3.1, i.e., K is a finitely ramified fractal with cell structure $\{K_\alpha\}_{\alpha\in\mathcal{A}}$ and boundary vertex structure $\{V_\alpha\}_{\alpha\in\mathcal{A}}$ and compatible resistance form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$. Let μ be a Borel finite measure as in Assumption 3.2.11. We set $\mathscr{H}_\infty = \mathsf{L}_2(K,\mu)$ (i.e., the space of square integrable functions $u\colon K \longrightarrow \mathbb{C}$ w.r.t. the measure μ with the usual L_2-norm).

On each V_m, we define the (discrete) measure $\mu_m = \{\mu_m(x)\}_{x\in V_m}$ associated with μ by

$$\mu_m(x) := \int_K \psi_{x,m}\,\mathrm{d}\mu,$$

where $\psi_{x,m} = \psi_x^{V_m}$ denotes the m-harmonic function with boundary values $\mathbb{1}_{\{x\}}$ on V_m. The sequence of Hilbert spaces $\{\mathscr{H}_m\}_{m\in\mathbb{N}_0}$ is defined by $\mathscr{H}_m := \ell_2(V_m, \mu_m)$, i.e., the space of square integrable functions $f\colon V_m \longrightarrow \mathbb{C}$ w.r.t. the measure μ_m and with norm

$$\|f\|^2_{\ell_2(V_m,\mu_m)} = \sum_{x\in V_m} |f(x)|^2 \mu_m(x).$$

3.3.1. Locally exact magnetic potentials. Before we state our main theorem, we need to fix some notation in order to define our identification operators. Let $a = \sum_{\alpha\in\mathcal{A}_m} A_\alpha \otimes \mathbb{1}_{K_\alpha} \in \mathcal{H}_m$ be real valued. For $x \in V_m$, we define

$$\widetilde{\psi}^a_{x,m} := \sum_{\alpha\in\mathcal{A}_{x,m}} \mathrm{e}^{\mathrm{i}A_\alpha(x)-\mathrm{i}A_\alpha} \psi_{x,m}\restriction_{K_\alpha\setminus V_\alpha},$$

where $\mathcal{A}_{x,m} := \{\, \alpha \mid \alpha \in \mathcal{A}_m \text{ and } x \in K_\alpha \,\}$.

Lemma 3.3.1. *The function $\widetilde{\psi}^a_{x,m}$ extends uniquely to an R-continuous function $\psi^a_{x,m}$ into $\cup_{\alpha\in\mathcal{A}_{x,m}} V_\alpha$ such that $\psi^a_{x,m}(x) = 1$ and $\psi^a_{x,m}(y) = 0$ if $y \in V_m \setminus \{x\}$ and $\psi^a_{x,m} = 0$ on each m-cell that does not contain x. Moreover, $\psi^a_{x,m} \in \operatorname{dom}\mathcal{E}$ and $\psi^a_{x,m}$ is m-harmonic with respect to $\mathcal{E}^a$, i.e., $\mathcal{E}^a(\psi^a_{x,m}) = \mathcal{E}^a_{V_m}(\psi^a_{x,m}\restriction_{V_m})$.*

Proof. Since the magnetic resistance form $\mathcal{E}^a$ and the discrete magnetic form $\mathcal{E}^a_{V_m}$ decompose with respect to the cell structure and because $\psi_{x,m}$ is m-harmonic with respect to $\mathcal{E}$,

$$\mathcal{E}^a(\psi^a_{x,m}) = \sum_{\alpha\in\mathcal{A}_{x,m}} \mathcal{E}_{K_\alpha}(\psi_{x,m}\mathrm{e}^{\mathrm{i}A_\alpha(x)}) = \sum_{\alpha\in\mathcal{A}_{x,m}} \mathcal{E}_{V_\alpha}(\psi_{x,m}\mathrm{e}^{\mathrm{i}A_\alpha(x)}) = \mathcal{E}^a_{V_m}(\psi^a_{x,m}\restriction_{V_m}).$$

The rest of the claim is obviously satisfied. □

By Assumption 3.2.11, the quadratic form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ defines a closed form in $\mathsf{L}_2(K,\mu)$. If we assume that the magnetic potential a is in $\mathcal{H}_m$, this is also true for $\mathcal{E}^a$ by the following two lemmata.

Lemma 3.3.2. *Let $\alpha \in \mathcal{A}_m$ and $x \in K_\alpha$ be arbitrary. Then*

$$|u(x)|^2 \le \frac{2}{\gamma_{-,m}} \mathcal{E}_{K_\alpha}(u) + \frac{2}{\mu_{-,m}} \|u\|^2_{\mathsf{L}_2(K_\alpha,\mu)} \le C_m \|u\|^2_{\mathcal{E}_{K_\alpha}}, \quad C_m := \max\Big\{\frac{2}{\gamma_{-,m}}, \frac{2}{\mu_{-,m}}\Big\} \tag{3.3.1}$$

for $u \in \operatorname{dom}\mathcal{E}$, where $\mu_{-,m} := \min\{\, \mu(K_\alpha) \mid \alpha \in \mathcal{A}_m \,\}$ and where $\gamma_{-,m}$ is defined in (3.2.3).

Proof. Let $\alpha \in \mathcal{A}_m$. Then, for any $x \in K_\alpha$, we have

$$|u(x)|^2 \le 2\Big(\frac{1}{\mu(K_\alpha)} \int_{K_\alpha} |u(x) - u(y)| \,\mathrm{d}\mu(y)\Big)^2 + 2\Big(\frac{1}{\mu(K_\alpha)} \int_{K_\alpha} u(y) \,\mathrm{d}\mu(y)\Big)^2$$
$$\le \frac{2}{\gamma_{-,m}} \mathcal{E}_{K_\alpha}(u) + \frac{2}{\mu(K_\alpha)} \|u\|^2_{\mathsf{L}_2(K_\alpha,\mu)},$$

where we applied Lemma 3.2.8 to the first summand and the Cauchy-Schwarz inequality to the second one. □

Lemma 3.3.3. *Let $a \in \mathcal{H}_m$ be real valued. Then there exists a constant $\hat{C}_m = \hat{C}_m(a,\mu) > 0$ such that*

$$\frac{1}{\hat{C}_m} \|u\|^2_{\mathcal{E}} \le \|u\|^2_{\mathcal{E}^a} \le \hat{C}_m \|u\|^2_{\mathcal{E}}$$

for all $u \in \operatorname{dom} \mathcal{E}$, i.e., $(\mathcal{E}^a, \operatorname{dom} \mathcal{E})$ is a closed *quadratic form in $\mathsf{L}_2(K,\mu)$.*

Proof. Let $u, v \in \operatorname{dom} \mathcal{E}$. Then we have $\mathcal{E}(uv)^{1/2} \le \|u\|_\infty \mathcal{E}(v)^{1/2} + \|v\|_\infty \mathcal{E}(u)^{1/2}$ by [**IRT12**, Lem. 2.1]. We apply this estimate actually for the energy form on K_α instead of K. Moreover, by Lemma 3.3.2 we have, for any $u \in \operatorname{dom} \mathcal{E}$ and $\alpha \in \mathcal{A}_m$,

$$\|u\|^2_{\infty,K_\alpha} := \sup\big\{\, |u(x)|^2 \,\big|\, x \in K_\alpha \,\big\} \le C_m \|u\|^2_{\mathcal{E}_{K_\alpha}}.$$

Hence, for each $a = \sum_{\alpha \in \mathcal{A}_m} A_\alpha \otimes \mathbb{1}_{K_\alpha} \in \mathcal{H}_m$, by the above we have

$$\|u\|^2_{\mathcal{E}} = \mathcal{E}(u) + \|u\|^2_{\mathsf{L}_2(K,\mu)} = \sum_{\alpha \in \mathcal{A}_m} \Big(\mathcal{E}_{K_\alpha}(\mathrm{e}^{-\mathrm{i}A_\alpha} \mathrm{e}^{\mathrm{i}A_\alpha} u) + \|u\|^2_{\mathsf{L}_2(F_\alpha,\mu)} \Big)$$
$$\le 2 \sum_{\alpha \in \mathcal{A}_m} \Big(\mathcal{E}_{K_\alpha}(u\mathrm{e}^{\mathrm{i}A_\alpha}) + \|u\|^2_{\mathsf{L}_2(K_\alpha,\mu)} + C_m \mathcal{E}_{K_\alpha}(\mathrm{e}^{-\mathrm{i}A_\alpha}) \|u\mathrm{e}^{\mathrm{i}A_\alpha}\|^2_{\mathcal{E}_{K_\alpha}} \Big)$$
$$\le 2 \max_{\alpha \in \mathcal{A}_m} \big(1 + C_m \mathcal{E}_{V_\alpha}(A_\alpha)\big) \|u\|^2_{\mathcal{E}^a} =: \hat{C}_m \|u\|^2_{\mathcal{E}^a},$$

because $\mathcal{E}_{K_\alpha}(\mathrm{e}^{-\mathrm{i}A_\alpha}) = \|A_\alpha \otimes \mathbb{1}_{K_\alpha}\|^2_{\mathcal{H}} = \mathcal{E}_{V_\alpha}(A_\alpha)$ by (3.2.6). The missing inequality holds with the same constant and follows in the same manner. □

We now state our first main result in this chapter:

Theorem 3.3.4. *For each real valued $a \in \mathcal{H}_m$, the corresponding magnetic energy form $(\mathcal{E}^a, \operatorname{dom} \mathcal{E})$ in $\mathsf{L}_2(K,\mu)$ and the discrete magnetic form $(\mathcal{E}^a_{V_m}, \ell(V_m))$ on $\ell_2(V_m,\mu_m)$ are δ_m-quasi-unitarily equivalent, where*

$$\delta_m = \big(1 + \sqrt{N_m}\big) \sqrt{\frac{\mu_{+,m}}{\gamma_{-,m}}} \qquad \text{and} \qquad N_m := \sup\big\{\, |V_\alpha| \,\big|\, \alpha \in \mathcal{A}_m \,\big\}. \tag{3.3.2}$$

We begin by choosing the identification operators on the level of Hilbert spaces. Let

$$J_m \colon \ell_2(V_m,\mu_m) \longrightarrow \mathsf{L}_2(K,\mu), \qquad J_m f := \sum_{x \in V_m} f(x) \psi^a_{x,m}.$$

Note that, using the cell structure, we can rewrite the sum in the definition of J_m as

$$\sum_{x \in V_m} f(x) \psi^a_{x,m} = \sum_{\alpha \in \mathcal{A}_m} \sum_{x \in V_\alpha} f(x) \mathrm{e}^{\mathrm{i}A_\alpha(x) - \mathrm{i}A_\alpha} \psi_{x,m}\restriction_{K_\alpha \setminus V_\alpha}$$

for each $f \in \ell_2(V_m, \mu_m)$. The adjoint $J_m^*\colon \mathsf{L}_2(K,\mu) \longrightarrow \ell_2(V_m,\mu_m)$ of J_m is given by

$$J_m^* u(y) = \frac{1}{\mu_m(y)} \langle u, \psi_{y,m}^a \rangle_{\mathsf{L}_2(K,\mu)} = \frac{1}{\mu_m(y)} \sum_{\alpha \in \mathcal{A}_{y,m}} \mathrm{e}^{-\mathrm{i}A_\alpha(y)} \langle u, \mathrm{e}^{-\mathrm{i}A_\alpha} \psi_{y,m} \rangle_{\mathsf{L}_2(K_\alpha,\mu)},$$

for any $u \in \mathsf{L}_2(K,\mu)$ and $y \in V_m$. Moreover, we define the identification operators on the form domains as follows: Let $J_m^1\colon \ell_2(V_m,\mu_m) \longrightarrow \operatorname{dom}\mathcal{E}$ be given by $J_m^1 := J_m\restriction_{\operatorname{dom}\mathcal{E}}$ and $J_m'^1\colon \operatorname{dom}\mathcal{E} \longrightarrow \ell_2(V_m,\mu_m)$ be the evaluation in points of V_m, i.e., $J_m'^1 u(y) = u(y)$ for any $y \in V_m$. This choice actually makes sense, since $\psi_{x,m}^a \in \operatorname{dom}\mathcal{E}$ and hence $J_m f \in \operatorname{dom}\mathcal{E}$. Moreover, functions in $\operatorname{dom}\mathcal{E}$ can be evaluated in points, see Lemma 3.3.2.

Let $x, y \in V_m$ be two distinct points. If there is no m-cell that contains both points x and y, then

$$\langle \psi_{x,m}^a, \psi_{y,m}^a \rangle_{\mathsf{L}_2(K,\mu)} = 0.$$

Otherwise, there exists a unique $\alpha \in \mathcal{A}_m$ such that $x, y \in V_\alpha$ and

$$\langle \psi_{x,m}^a, \psi_{y,m}^a \rangle_{\mathsf{L}_2(K,\mu)} = \mathrm{e}^{\mathrm{i}A_\alpha(x) - \mathrm{i}A_\alpha(y)} \langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathsf{L}_2(K_\alpha,\mu)}, \tag{3.3.3}$$

because $\operatorname{supp}\psi_{x,m} \cap \operatorname{supp}\psi_{y,m} \subseteq K_\alpha$ and A_α is uniquely determined on this m-cell. Moreover, if $x = y$, then we have

$$\langle \psi_{x,m}^a, \psi_{x,m}^a \rangle_{\mathsf{L}_2(K,\mu)} = \|\psi_{x,m}\|^2_{\mathsf{L}_2(K,\mu)}.$$

Note that, $\langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathsf{L}_2(K_\alpha,\mu)} = 0$ implies that the associated conductance also vanishes, i.e., $\gamma_{\{x,y\},m} = 0$.

Proof of Theorem 3.3.4. We verify the conditions from Definition 2.2.1 with the identification operators defined above.

Applying the Cauchy-Young inequality, we verify that

$$\begin{aligned}
\|J_m f\|^2_{\mathsf{L}_2(K,\mu)} &= \sum_{x \in V_m} \sum_{y \in V_m} f(x)\overline{f(y)} \underbrace{\langle \psi_{x,m}^a, \psi_{y,m}^a \rangle_{\mathsf{L}_2(K,\mu)}}_{\geq 0} \\
&\leq \frac{1}{2} \sum_{x \in V_m} |f(x)|^2 \sum_{y \in V_m} \langle \psi_{x,m}^a, \psi_{y,m}^a \rangle_{\mathsf{L}_2(K,\mu)} \\
&\qquad + \frac{1}{2} \sum_{y \in V_m} |f(y)|^2 \sum_{x \in V_m} \langle \psi_{x,m}^a, \psi_{y,m}^a \rangle_{\mathsf{L}_2(K,\mu)} \\
&= \sum_{x \in V_m} |f(x)|^2 \underbrace{\langle \psi_{x,m}^a, \mathbb{1}_K \rangle_{\mathsf{L}_2(K,\mu)}}_{=\mu_m(x)} = \|f\|^2_{\ell_2(V_m,\mu_m)}
\end{aligned}$$

for any $f \in \ell_2(V_m,\mu_m)$. Hence, J_m is bounded by 1.

Next, we examine the first inequality from (2.2.2b). By (3.3.3), we have

$$J_m^* J_m f(y) = \frac{1}{\mu_m(y)} \sum_{\alpha \in \mathcal{A}_{y,m}} \sum_{x \in V_\alpha} f(x) \mathrm{e}^{\mathrm{i}A_\alpha(x) - \mathrm{i}A_\alpha(y)} \langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathsf{L}_2(K_\alpha,\mu)}$$

for any $f \in \ell_2(V_m, \mu_m)$ and $y \in V_m$. Moreover, we can rewrite $f \in \ell_2(V_m, \mu_m)$ as

$$f(y) = \frac{1}{\mu_m(y)} \sum_{\alpha \in \mathcal{A}_{y,m}} \sum_{x \in V_\alpha} f(y) \langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathsf{L}_2(K_\alpha,\mu)},$$

because $\{\psi_{x,m}\}_{x \in V_m}$ is a partition of unity. Combining the above equations, we conclude

$$f(y) - J_m^* J_m f(y) = \frac{1}{\mu_m(y)} \sum_{\alpha \in \mathcal{A}_{y,m}} \sum_{x \in V_\alpha} \bigl(f(y) - f(x) \mathrm{e}^{\mathrm{i}A_\alpha(x) - \mathrm{i}A_\alpha(y)}\bigr) \langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathsf{L}_2(K_\alpha,\mu)}$$

and applying the Cauchy-Schwarz inequality, we estimate in the norm of $\ell_2(V_m, \mu_m)$,

$$\begin{aligned}
&\bigl\| f - J_m^* J_m f \bigr\|^2_{\ell_2(V_m,\mu_m)} \\
&= \sum_{y \in V_m} \frac{1}{\mu_m(y)} \Bigl| \sum_{\alpha \in \mathcal{A}_{y,m}} \sum_{x \in V_\alpha} \bigl(f(y) - f(x) \mathrm{e}^{\mathrm{i}A_\alpha(x) - \mathrm{i}A_\alpha(y)}\bigr) \langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathsf{L}_2(K_\alpha,\mu)} \Bigr|^2 \\
&\le \sum_{y \in V_m} \frac{1}{\mu_m(y)} \Bigl(\sum_{\alpha \in \mathcal{A}_{y,m}} \sum_{x \in V_\alpha} \frac{\bigl| \langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathsf{L}_2(K_\alpha,\mu)} \bigr|^2}{\gamma_{\{x,y\},m}} \Bigr) \sum_{\alpha \in \mathcal{A}_{y,m}} \sum_{x \in V_\alpha} \gamma_{\{x,y\},m} \bigl| f^a_\alpha(y) - f^a_\alpha(x) \bigr|^2 \\
&\le \frac{1}{\gamma_{-,m}} \max_{y \in V_m} \frac{1}{\mu_m(y)} \Bigl(\sum_{\alpha \in \mathcal{A}_{y,m}} \sum_{x \in V_\alpha} \bigl| \langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathsf{L}_2(K_\alpha,\mu)} \bigr|^2 \Bigr) \cdot \mathcal{E}^a_{V_m}(f),
\end{aligned}$$

where $\gamma_{-,m}$ is defined in (3.2.3) and where $f^a_\alpha(z) := f(z) \mathrm{e}^{\mathrm{i}A_\alpha(z)}$. Note that this is well defined because $\langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathsf{L}_2(K_\alpha,\mu)} \neq 0$ implies $\gamma_{\{x,y\},m} > 0$ for any distinct $x, y \in V_m$ (by the last part of Assumption 3.2.11). We estimate the maximum in the above equation by

$$\begin{aligned}
\max_{y \in V_m} \frac{1}{\mu_m(y)} \sum_{\alpha \in \mathcal{A}_{y,m}} \sum_{x \in V_\alpha} \bigl| \langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathsf{L}_2(K_\alpha,\mu)} \bigr|^2 &\le \max_{y \in V_m} \sum_{\alpha \in \mathcal{A}_{y,m}} \sum_{x \in V_\alpha} \langle \psi_{x,m}, \psi_{y,m} \rangle_{\mathsf{L}_2(K_\alpha,\mu)} \\
&= \max_{y \in V_m} \mu_m(y) \le \mu_{+,m},
\end{aligned}$$

where we define $\mu_{+,m}$ as follows (this choice will become clear after the next step)

$$\mu_{+,m} := \max \Bigl\{ \max_{x \in V_m} \mu_m(x), \max_{\alpha \in \mathcal{A}_m} \mu(K_\alpha) \Bigr\}.$$

In particular, the first inequality of (2.2.2b) is fulfilled with $\delta_m = \sqrt{\mu_{+,m}/\gamma_{-,m}}$.

Instead of proving (2.2.2b) we will verify the assumption from Lemma 2.2.5. Let $u \in \operatorname{dom} \mathcal{E}$. Again by the partition of unity property of the family $\{\psi_{x,m}\}_{x \in V_m}$, we get

$$u - J_m J'^1_m u = \sum_{\alpha \in \mathcal{A}_m} \sum_{x \in V_\alpha} \bigl(u - u(x) \mathrm{e}^{\mathrm{i}A_\alpha(x) - \mathrm{i}A_\alpha}\bigr) \psi_{x,m} \restriction_{K_\alpha \setminus V_\alpha}$$

and hence, by Lemma 3.2.8, we estimate in the corresponding L_2-norm,

$$\begin{aligned}\|u - J_m J'^1_m u\|^2_{\mathsf{L}_2(K,\mu)} &= \sum_{\alpha\in\mathcal{A}_m} \|u - J_m J'^1_m u\|^2_{\mathsf{L}_2(K_\alpha,\mu)}\\ &\le \sum_{\alpha\in\mathcal{A}_m} \max_{z_1,z_2\in K_\alpha} \big|u^a_\alpha(z_1) - u^a_\alpha(z_2)\big|^2 \underbrace{\sum_{x,y\in V_\alpha} \langle\psi_{x,m},\psi_{y,m}\rangle_{\mathsf{L}_2(K_\alpha,\mu)}}_{=\mu(K_\alpha)\le\mu_{+,m}}\\ &\le \frac{\mu_{+,m}}{\gamma_{-,m}} \sum_{\alpha\in\mathcal{A}_m} \mathcal{E}_{K_\alpha}(u\mathrm{e}^{\mathrm{i}A_\alpha}\restriction_{F_\alpha}) = \frac{\mu_{+,m}}{\gamma_{-,m}}\mathcal{E}^a(u),\end{aligned}$$

where again $u^a_\alpha(z) = u(z)\mathrm{e}^{\mathrm{i}A_\alpha(z)}$. Thus, (2.2.2b) is also fulfilled with $\delta_m = \sqrt{\mu_{+,m}/\gamma_{-,m}}$.

The first inequality from (2.2.2c) holds true for $\delta_m = 0$ by the choice of the identification operators. Let $y \in V_m$ and $u \in \operatorname{dom}\mathcal{E}$ be arbitrary. By Lemma 3.2.8 and the Cauchy-Schwarz inequality,

$$\begin{aligned}\big|J^*_m u(y) - J'^1_m u(y)\big|^2 &\le \Big(\frac{1}{\mu_m(y)} \sum_{\alpha\in\mathcal{A}_{y,m}} \int_{K_\alpha} \big|u^a_\alpha - u^a_\alpha(y)\big|\psi_{y,m}\,\mathrm{d}\mu\Big)^2\\ &\le \frac{1}{\gamma_{-,m}} \sum_{\alpha\in\mathcal{A}_{y,m}} \mathcal{E}_{K_\alpha}(u\mathrm{e}^{\mathrm{i}A_\alpha}\restriction_{K_\alpha})\end{aligned}$$

and hence, applying the Cauchy-Schwarz inequality again, we estimate

$$\begin{aligned}\big\|J^*_m u - J'^1_m u\big\|^2_{\ell_2(V_m,\mu_m)} &\le \frac{1}{\gamma_{-,m}} \sum_{y\in V_m} \mu_m(y) \sum_{\alpha\in\mathcal{A}_{y,m}} \mathcal{E}_{K_\alpha}(u\,\mathrm{e}^{\mathrm{i}A_\alpha}\restriction_{K_\alpha})\\ &\le \frac{1}{\gamma_{-,m}} \sum_{\alpha\in\mathcal{A}_m} \mathcal{E}_{K_\alpha}(u\,\mathrm{e}^{\mathrm{i}A_\alpha}\restriction_{K_\alpha}) \sum_{y\in V_\alpha} \mu_m(y) \le \frac{N_m\mu_{+,m}}{\gamma_{-,m}}\mathcal{E}^a(u)\end{aligned} \tag{3.3.4}$$

where $N_m \coloneqq \sup\{\,|V_\alpha| \mid \alpha\in\mathcal{A}_m\,\}$. Thus (2.2.2c) holds with $\delta_m = \sqrt{N_m\mu_{+,m}/\gamma_{-,m}}$.

The last inequality (2.2.2d) is actually an equality, i.e., it holds with $\delta_m = 0$. This follows from (3.1.4) because the functions $\psi^a_{x,m}$ are m-harmonic with respect to $\mathcal{E}^a$ for each $x \in V_m$ by Lemma 3.3.1. Hence,

$$\mathcal{E}^a_{V_m}(f, J'^1_m u) = \sum_{x\in V_m} f(x)\mathcal{E}^a_{V_m}(\psi^a_{x,m}\restriction_{V_m}, u\restriction_{V_m}) = \sum_{x\in V_m} f(x)\mathcal{E}^a(\psi^a_{x,m}, u) = \mathcal{E}^a(J^1_m f, u)$$

for any $f\colon V_m \longrightarrow \mathbb{C}$ and $u \in \operatorname{dom}\mathcal{E}$.

We apply Lemma 2.2.5 with $\delta_a = 0$, $\delta' = \sqrt{\mu_{+,m}/\gamma_{-,m}}$ and $\delta_c = \sqrt{N_m\mu_{+,m}/\gamma_{-,m}}$. This implies that (2.2.2b) holds and in particular, $\mathcal{E}^a_m$ and $\mathcal{E}^a$ are δ_m-quasi-unitary equivalent with $\delta_m = (1+\sqrt{N_m})\sqrt{\mu_{+,m}/\gamma_{-,m}}$. ☐

From Theorem 3.3.4, we also derive the convergence of the free Dirichlet form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ (without a magnetic potential) and its traces $(\mathcal{E}_{V_m}, \ell(V_m))$ by choosing $a = 0$. Note that the error δ_m in Theorem 3.3.4 does not refer to the magnetic potential $a \in \mathcal{H}_m$.

COROLLARY 3.3.5. *The Dirichlet form* $(\mathcal{E}, \operatorname{dom} \mathcal{E})$ *in* $\mathsf{L}_2(F, \mu)$ *and its trace* $(\mathcal{E}_{V_m}, \mu_m)$ *on* $\ell_2(V_m, \mu_m)$ *are* δ_m*-quasi-unitarily equivalent where* $\delta_m = (1 + \sqrt{N_m})\sqrt{\mu_{+,m}/c_{-,m}}$.

The above corollary is of course also valid for a post-critically finite self-similar fractal K. In particular, the above coincides with the main result from [**PS18a**], where the authors proved the result in the special case of a post-critically finite self-similar fractal.

3.3.2. General magnetic potentials. In Theorem 3.3.4 we assumed an quite simple magnetic potential, i.e. we assumed that the potential is in $\mathcal{H}_m$. We will generalise this to arbitrary potentials by applying the transitivity of quasi-unitary equivalence from Proposition 2.2.4. Let $a \in \mathcal{H}$ be real valued. The following assumption will be a key step in this section (see e.g. [**HR16**] for a discussion of sufficient conditions on the measure μ and Lemma 3.3.10):

ASSUMPTION 3.3.6. *We assume that* $(\mathcal{E}^a, \operatorname{dom} \mathcal{E})$ *is a* closed *quadratic form in* $\mathsf{L}_2(K, \mu)$.

Let $u \in \operatorname{dom} \mathcal{E}$. Applying Lemma 3.3.2 to the supremum norm we estimate

$$\mathcal{E}^a(u) \le 2\mathcal{E}(u) + 2\|a\|_{\mathcal{H}}^2 \|u\|_\infty^2 \le 2\Big(1 + \frac{2\|a\|_{\mathcal{H}}^2}{\gamma_{-,0}}\Big)\mathcal{E}(u) + \frac{4\|a\|_{\mathcal{H}}^2}{\mu(K)}\|u\|_{\mathsf{L}_2(K,\mu)}^2,$$

i.e., there exists a constant $\widetilde{C} := \widetilde{C}(a, \mu) > 0$ such that $\|u\|_{\mathcal{E}^a}^2 \le \widetilde{C}\|u\|_{\mathcal{E}}^2$. Hence, both norms are equivalent by the inverse mapping theorem, i.e., there are constants $\widetilde{c} > 0$ and $\widetilde{C} > 0$ such that

$$\frac{1}{\widetilde{c}}\|u\|_{\mathcal{E}}^2 \le \|u\|_{\mathcal{E}^a}^2 \le \widetilde{C}\|u\|_{\mathcal{E}}^2, \qquad \widetilde{C} := \max\Big\{2\Big(1 + \frac{2\|a\|_{\mathcal{H}}^2}{\gamma_{-,0}}\Big), 1 + \frac{4\|a\|_{\mathcal{H}}^2}{\mu(K)}\Big\}. \tag{3.3.5}$$

The constant $\widetilde{c}$ is not explicitly given, but in some examples, we can specify it (see Lemma 3.3.10).

Before showing our second main result, we need the following estimate:

LEMMA 3.3.7. *Let* $a \in \mathcal{H}$ *be real valued and* a_m *be its projection onto* $\mathcal{H}_m$. *Then we have*

$$\big|\mathcal{E}^{a_m}(u) - \mathcal{E}^a(u)\big| \le \widetilde{\delta}_m \|u\|_{\mathcal{E}^a}^2 \tag{3.3.6}$$

for all $u \in \operatorname{dom} \mathcal{E}$, *where* $\widetilde{\delta}_m$ *is defined in* (3.3.7).

PROOF. Let $a_m = \sum_{\alpha \in \mathcal{A}_m} A_\alpha \otimes \mathbb{1}_{F_\alpha}$ and $u \in \operatorname{dom} \mathcal{E}$. Then we have

$$\begin{aligned}
&\big|\mathcal{E}^a(u) - \mathcal{E}^{a_m}(u)\big|^2 \\
&\qquad = \big(\|(\partial + \mathrm{i}a)u\|_{\mathcal{H}} + \|(\partial + \mathrm{i}a_m)u\|_{\mathcal{H}}\big)\big|\|(\partial + \mathrm{i}a)u\|_{\mathcal{H}} - \|(\partial + \mathrm{i}a_m)u\|_{\mathcal{H}}\big| \\
&\qquad \le \big(2\|\partial u\|_{\mathcal{H}} + \|au\|_{\mathcal{H}} + \|a_m u\|_{\mathcal{H}}\big)\|(a - a_m)u\|_{\mathcal{H}} \\
&\qquad \le 2\big(\sqrt{\mathcal{E}(u)} + \|a\|_{\mathcal{H}}\|u\|_\infty\big)\|a - a_m\|_{\mathcal{H}}\|u\|_\infty,
\end{aligned}$$

where we used that $\|a_m\|_{\mathcal{H}} \le \|a\|_{\mathcal{H}}$ (a_m is the orthogonal projection of a onto $\mathcal{H}_m$), (3.2.7) and several times the inequality (3.2.8). By Lemma 3.3.2 (for $m = 0$) and by (3.3.5), we conclude

$$\|u\|_\infty^2 \le C_0 \|u\|_{\mathcal{E}}^2 \le C_0 \widetilde{c} \|u\|_{\mathcal{E}^a}^2$$

and hence

$$\left|\mathcal{E}^a(u) - \mathcal{E}^{a_m}(u)\right|^2 \le 2\widetilde{c}\sqrt{C_0}\big(1 + \|a\|_{\mathcal{H}}\sqrt{C_0}\big)\|a - a_m\|_{\mathcal{H}}\|u\|_{\mathcal{E}^a}^2,$$

i.e., inequality (3.3.6) holds with

$$\widetilde{\delta}_m = 2\widetilde{c}\sqrt{C_0}\big(1 + \|a\|_{\mathcal{H}}\sqrt{C_0}\big)\|a - a_m\|_{\mathcal{H}} \tag{3.3.7}$$

□

Lemma 3.3.8. *We have* $\|J_m^1 f\|_{\mathcal{E}^{a_m}} \le \|f\|_{\mathcal{E}_{V_m}^{a_m}}$ *for* $f \in \ell(V_m)$.

Proof. Let $f \in \ell(V_m)$. Since $\psi_{x,m}$ is m-harmonic with respect to $\mathcal{E}$, we easily verify that $\mathcal{E}^{a_m}(J_m^1 f) = \mathcal{E}_{V_m}^{a_m}(f)$ and the estimate $\|J_m^1 f\|_{\mathsf{L}_2(K,\mu)} \le \|f\|_{\ell_2(V_m,\mu_m)}$ was already shown in the proof of Theorem 3.3.4. □

Using the transitivity Proposition 2.2.4 we conclude from Theorem 3.3.4 and the above lemma our second main result:

Theorem 3.3.9. *Let* $a \in \mathcal{H}$ *be real valued and* a_m *be its projection onto* $\mathcal{H}_m$. *Moreover, assume that* $m_0 \in \mathbb{N}$ *is large enough such that* $\widetilde{\delta}_m < 1$ *for* $m \ge m_0$ *(see (3.3.7)). Then the magnetic resistance forms* $\mathcal{E}^a$ *in* $\mathsf{L}_2(K,\mu)$ *and* $\mathcal{E}_{V_m}^{a_m}$ *in* $\ell_2(V_m,\mu)$ *are* $\hat{\delta}_m$*-quasi-unitarily equivalent, where* $\hat{\delta}_m = \mathrm{O}(\delta_m) + \mathrm{O}(\|a - a_m\|_{\mathcal{H}})$ *with* δ_m *from Theorem 3.3.4.*

Proof. By Theorem 3.3.4, $\mathcal{E}^{a_m}$ in $\mathsf{L}_2(K,\mu)$ and $\mathcal{E}_{V_m}^{a_m}$ in $\ell_2(V_m,\mu_m)$ are δ_m-quasi-unitarily equivalent with δ_m given in (3.3.2). We then apply Proposition 2.2.4 with $\widetilde{\mathcal{E}} = \mathcal{E}^{a_m}$, $\hat{\mathcal{E}} = \mathcal{E}^a$ and $\widetilde{\delta} = \widetilde{\delta}_m$ as in Lemma 3.3.7. The claim then follows with

$$\hat{\delta}_m = \widetilde{\delta}_m \cdot \frac{1 + \delta_m}{1 - \widetilde{\delta}_m} + \delta_m(1 + \widetilde{\delta}_m). \tag{3.3.8}$$

where δ_m is defined in (3.3.2) and $\widetilde{\delta}_m$ is defined in (3.3.7). (The fact that J_m^1 is energy norm bounded by 1 and not by $1 + \delta_m$ as required in (2.2.8b) does not significantly simplify the error $\hat{\delta}_m$.) □

In the special situation of the following lemma we can specify the constant $\widetilde{c}$ in (3.3.5): We already know how to estimate the supremum norm (cf. Lemma 3.3.2) so we only need to find an upper bound for the energy norm. The assumption made will be satisfied in most of our examples. In particular, we do not need to assume *a priori* that $\mathcal{E}^a$ is a *closed* form.

Lemma 3.3.10. *If there exists* $m \ge 0$ *such that* $4\|a\|_{\mathcal{H}}^2 < \gamma_{-,m}$ *then*

$$\|u\|_{\mathcal{E}}^2 \le \Big(1 - \frac{4\|a\|_{\mathcal{H}}^2}{\gamma_{-,m}}\Big)^{-1}\Big(2\mathcal{E}^a(u) + \Big(1 + \frac{4\|a\|_{\mathcal{H}}^2}{\mu_{-,m}}\Big)\|u\|_{\mathsf{L}_2(K,\mu)}^2\Big),$$

i.e., we can choose

$$\widetilde{c} = \Big(1 - \frac{4\|a\|_{\mathcal{H}}^2}{\gamma_{-,m}}\Big)^{-1} \max\Big\{2, 1 + \frac{4\|a\|_{\mathcal{H}}^2}{\mu_{-,m}}\Big\}$$

in (3.3.5). *In particular,* $(\mathcal{E}^a, \operatorname{dom}\mathcal{E})$ *is a* closed *quadratic form in* $\mathsf{L}_2(K,\mu)$.

Proof. We have

$$\mathcal{E}(u) = \|\partial u\|_{\mathcal{H}}^2 \le 2\big(\|(\partial + \mathrm{i}a)u\|_{\mathcal{H}}^2 + \|\mathrm{i}au\|_{\mathcal{H}}^2\big) \le 2\mathcal{E}^a(u) + 2\|a\|_{\mathcal{H}}^2\|u\|_\infty^2.$$

using (3.2.6), (3.2.8) and (3.2.11). Since all cells K_α are compact, there is $x \in K_\alpha$ with $|u(x)| = \|u\|_\infty$ for some $\alpha \in \mathcal{A}_m$. Hence, by Lemma 3.3.2, we have

$$\|u\|_\infty^2 = |u(x)|^2 \le \frac{2}{\gamma_{-,m}}\mathcal{E}(u) + \frac{2}{\mu_{-,m}}\|u\|_{\mathsf{L}_2(K,\mu)}^2$$

for all $u \in \operatorname{dom}\mathcal{E}$ using also $\mathcal{E}_{K_\alpha}(u) \le \mathcal{E}(u)$ and $\|u\|_{\mathsf{L}_2(K_\alpha,\mu)} \le \|u\|_{\mathsf{L}_2(K,\mu)}$. Combining the last two estimates, we conclude

$$\Big(1 - \frac{4\|a\|_{\mathcal{H}}^2}{\gamma_{-,m}}\Big)\mathcal{E}(u) \le 2\mathcal{E}^a(u) + \frac{4\|a\|_{\mathcal{H}}^2}{\mu_{-,m}}\|u\|_{\mathsf{L}_2(K,\mu)}^2.$$

The result then follows. □

3.4. Examples

3.4.1. Approximating metric graphs by subdivision graphs. As an example, we consider metric graphs here and briefly introduce the necessary notation; for more details, we refer to [**Pos12, BK13**].

Let $G = (V, E, \partial)$ be a discrete graph where $\partial\colon E \longrightarrow V \times V$ defines an orientation. Moreover, let $\ell\colon E \longrightarrow (0,\infty)$ be a function, where $\ell_e := \ell(e)$ is interpreted as *length* of the edge e. Then the pair (G, ℓ) describes a *metric graph* as a topological space

$$M := \biguplus_{e\in E} M_e/\omega,$$

where $\omega\colon \biguplus_{e\in E} \partial M_e \longrightarrow V$ identifies the endpoints of the intervals $M_e := [0, \ell_e]$ according to the graph's structure, namely $\omega(0) = \partial_- e$ and $\omega(\ell_e) = \partial_+ e$ if $0, \ell_e \in \partial M_e$. The metric on M is described by the length of the shortest path $\gamma_{x,y}$ between two points x and y. A canonical measure μ is given by the sum of the Lebesgue measures μ_e on M_e. Note that the Lebesgue measure on an interval is a self-similar Hausdorff measure.

Moreover, a Dirichlet form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ in $\mathsf{L}_2(M,\mu)$ is given by

$$\mathcal{E}(u) = \int_M |u'(t)|^2 \,\mathrm{d}\mu(t) \tag{3.4.1}$$

with domain $\operatorname{dom}\mathcal{E} = \bigoplus_{e\in E} \mathsf{H}^1(M_e) \cap \mathsf{C}(M)$.

For $m \in \mathbb{N}$, we cut each interval M_e into 2^{m-1} pieces of equal length $\ell_{m,e} := 2^{-(m-1)}\ell_e$. We start at $v_0 = \partial_- e$ and add new vertices $v_1, \dots, v_{2^{m-1}-1}$ with edges e_i from $v_i = \partial_{m,-} e_i$ to $v_{i+1} = \partial_{m,+} e_i$ and end with $v_{2^{m-1}} = \partial_+ e$. Of course, other subdivisions of the edges are also possible, as far as the maximal and minimal subdivision lengths tend to 0 uniformly as $m \to \infty$. Let V_m and E_m be the collection of all these vertices and edges, respectively. In particular, $(V_{m+1}, E_{m+1}, \partial_{m+1})$ is the subdivision graph of (V_m, E_m, ∂_m) and we have $V_1 = V$ and $E_1 = E$. Using the notation of Subsection 3.2.1, we have $\mathcal{A}_0 = \{0\}$ and $\mathcal{A}_m = E_m$ ($m \in \mathbb{N}$); moreover, V_m, as defined in Definition 3.2.1, consists of the vertices of the m-th subdivision graph of G, and cells $F_\alpha = M_\alpha$ are just intervals corresponding to an edge $\alpha \in \mathcal{A}$.

The finite dimensional Dirichlet form on V_m is given by

$$\mathcal{E}_{V_m}(f) = \sum_{e \in E_m} \frac{1}{\ell_{m,e}} \bigl|f(\partial_{m,+}e) - f(\partial_{m,-}e)\bigr|^2 = \sum_{e \in E} \frac{2^{m-1}}{\ell_e} \sum_{i=0}^{2^{m-1}-1} \bigl|f_e(v_{i+1}) - f_e(v_i)\bigr|^2$$

for $f \in \operatorname{dom} \mathcal{E}_{V_m} = \ell(V_m)$.

In this example, we have $\gamma_{-,m} = 2^{m-1}/\ell_+$ and $\mu_{+,m} = \deg_+ \ell_+ / 2^{m-1}$, where $\deg_+ := \max\{ \deg v \mid v \in V \}$ and $\ell_+ = \max_{e \in E} \ell_e$ denote the maximal degree and length, respectively. By Corollary 3.3.5 the Dirichlet form $(\mathcal{E}, \mathsf{H}^1(M))$ in $\mathsf{L}_2(M, \mu)$ and the finite dimensional Dirichlet form $(\mathcal{E}_{V_m}, \ell(V_m))$ in $\ell_2(V_m, \mu_m)$ are δ_m-quasi unitarily equivalent with

$$\delta_m = \bigl(1 + \sqrt{2}\bigr) \sqrt{\deg_+ \frac{\ell_+}{2^{m-1}}}. \tag{3.4.2}$$

We can also understand this result as a Riemann sum approximation of the energy integral (3.4.1).

Before dealing with magnetic potentials, we first want to understand how the space $\mathcal{H}$ looks like in the case of metric graphs which was discussed in [**IRT12**, Sec. 5.1]. Note that the situation here is basically the same as in Example 3.2.16 since edges are just intervals. The map $a \otimes b \longmapsto (x \mapsto a'(x)b(x))$ defines an isometry between $\mathcal{H}$ and $\mathsf{L}_2(M, \mu)$. Thus

$$\int_M |a'(x)b(x)|^2 \, \mathrm{d}\mu(x) = \|a \otimes b\|^2_{\mathcal{H}}.$$

The inverse is given by $u \longmapsto \sum_{e \in E} A_e \otimes \mathbb{1}_{M_e}$, where A_e is absolutely continuous and $A'_e = u_e$ on each edge $e \in E$. Hence, $\partial \colon \operatorname{dom} \mathcal{E} \longrightarrow \mathsf{L}_2(M, \mu)$ acts as the usual derivative. Moreover, the space $\mathcal{H}_m$ is given by functions of the form $\sum_{e \in E_m} A_e \otimes \mathbb{1}_{M_e}$ where A_e is an affine linear function on M_e. In particular, the above-mentioned isometry between $\mathcal{H}$ and $\mathsf{L}_2(M, \mu)$ maps $\mathcal{H}_m$ onto the step functions being constant on each edge. For $a_m = \sum_{e \in E_m} A_e \otimes \mathbb{1}_{M_e} \in \mathcal{H}_m$ we then have

$$\mathcal{E}^{a_m}(u) = \sum_{e \in E_m} \mathcal{E}_{M_e}(u \mathrm{e}^{\mathrm{i}A_e}) = \sum_{e \in E_m} \bigl\| u \mathrm{e}^{\mathrm{i}A_e} \otimes \mathbb{1}_{M_e} \bigr\|^2_{\mathcal{H}} = \sum_{e \in E_m} \int_{M_e} |u'_e + \mathrm{i}A'_e u_e|^2 \, \mathrm{d}\mu_e$$

by the Leibniz rule; note that here A'_e is just a constant. In particular, for any such a_m the resistance form $(\mathcal{E}^{a_m}, \operatorname{dom} \mathcal{E})$ in $\mathsf{L}_2(M, \mu)$ and the discrete resistance form $(\mathcal{E}^{a_m}_{V_m}, \ell(V_m))$ in $\ell_2(V_m, \mu_m)$ are δ_m-quasi-unitarily equivalent with the same error term as for the non-magnetic form defined in (3.4.2), in particular, $\delta_m = \mathrm{O}(2^{-m})$.

Let $a \in \mathcal{H}$ be real valued and a_m its projection onto $\mathcal{H}_m$. Actually, writing $a_m = \sum_{e \in E_m} A_{m,e} \otimes \mathbb{1}_{M_e}$, the function $A_{m,e}$ is just the affine linear function on the interval M_e with boundary values given by a_e. By Theorem 3.3.9, the magnetic resistance form $\mathcal{E}^a$ and the discrete magnetic resistance form $\mathcal{E}^{a_m}_{V_m}$ are $\hat{\delta}_m$-quasi-unitarily equivalent with $\hat{\delta}_m = \mathrm{O}(2^{-m}) + \widetilde{\delta}_m$, where $\widetilde{\delta}^2_m$ is (roughly) the error of the Riemann sum approximation of the integral $\int_M |a'(x)|^2 \,\mathrm{d}\mu(x)$.

If we fix $m_0 \geq 0$ such that $\ell_+ \|a\|^2_{\mathcal{H}} < 2^{m_0 - 3}$, then we conclude from Lemma 3.3.10 that $(\mathcal{E}^a, \operatorname{dom} \mathcal{E})$ is closed if $a \in \mathcal{H}_m$ if $m \geq m_0$. In the metric graph case, we can even say more: as we can concentrate the magnetic potential on one of the subdivision graph edges only (say, one adjacent to an original vertex $v \in V$), preserving the total flux along the original edge.

One can even put the magnetic potential in the vertex condition and one obtains a unitarily equivalent *closed* quadratic form (but then the functions in the quadratic form domain are not necessarily continuous any more). This argument shows that the form $(\mathcal{E}^a, \operatorname{dom} \mathcal{E})$ is closed for *any* magnetic potential $a \in \mathcal{H}$.

3.4.2. Self-similar finitely ramified fractals. Let K be a post-critically finite self-similar fractal described by the IFS $\{F_j\}_{j=1,\dots,N}$ as defined in Example 3.2.3.

Let $\mathcal{E}$ be a resistance form on V_∞ as in Subsection 3.2.1. We call $\mathcal{E}$ *self-similar* with *energy renormalisation factors* $\gamma = (\gamma_1, \dots, \gamma_m)$ if for any $u \in \operatorname{dom} \mathcal{E}$,

$$\mathcal{E}(u) = \sum_{j=1}^{N} \gamma_j \mathcal{E}(u \circ F_j). \tag{3.4.3}$$

Moreover, we call a self-similar resistance form $\mathcal{E}$ *regular* if all the energy renormalisation factors are greater than 1, i.e., if $\gamma_j > 1$ for all $j \in \{1, \dots, N\}$. In this case the R-completion Ω of V_∞ and K coincide by [**Tep08**, Thm. 7.9].

Let μ be a finite Borel regular measure on K such that any open subset of K has strictly positive measure. Then the (self-similar) resistance form $(\mathcal{E}, \operatorname{dom} \mathcal{E})$ induces a Dirichlet form in $\mathsf{L}_2(K, \mu)$ (denoted by the same symbol) and $\mathcal{E}$ and its trace $(\mathcal{E}_{V_m}, \ell(V_m))$ in $\ell_2(V_m, \mu_m)$ are δ_m-quasi-unitarily equivalent, with error

$$\delta_m = \bigl(1 + \sqrt{V_0}\bigr) \sqrt{\mu_{+,m} / (\gamma_0 \gamma_-^m)},$$

where $\gamma_- := \min\{\, \gamma_j \,|\, j = 1, \dots, N \,\} > 1$ and where $\gamma_0 > 0$ is the minimal conductance at generation 0. The same holds true for the magnetic forms $\mathcal{E}^{a_m}$ and $\mathcal{E}^{a_m}_{V_m}$ whenever $a_m \in \mathcal{H}_m$ is real valued.

Let $a \in \mathcal{H}$ be real valued. If $\mathcal{E}$ is regular, then $\gamma_j > 1$ for each j and there exists $m_0 \geq 0$ such that $4\|a\|^2_{\mathcal{H}} < \gamma_0 \gamma_-^{m_0}$. By Lemma 3.3.10 the magnetic energy form $\mathcal{E}^a$ is closed in $\mathsf{L}_2(K, \mu)$. Applying Theorem 3.3.9, we conclude that $\mathcal{E}^a$ and $\mathcal{E}^{a_m}_{V_m}$ are $\hat{\delta}_m$-quasi-unitarily equivalent where $\hat{\delta}_m$ is specified in (3.3.7) and (3.3.8) and the constant $\widetilde{c}$ of Lemma 3.3.2 is given by

$$\widetilde{c} = \Bigl(1 - \frac{4\|a\|^2_{\mathcal{H}}}{\gamma_0 \gamma_-^{m_0}}\Bigr)^{-1} \max\Bigl\{2, 1 + \frac{4\|a\|^2_{\mathcal{H}}}{\mu_{-,m}}\Bigr\}.$$

Note that in the case of a non-regular resistance form, we need to make sure that $\mathcal{E}^a$ is closed, e.g., a situation as in [**HR16**] or that the magnetic potential a is small enough.

The definition of self-similar finitely ramified fractals also includes the class of self-similar post-critically finite fractals, introduced in Example 3.2.3. For a more detailed discussion of concrete examples like e. g. the Sierpiński gasket and the hexagasket, we refer to [**PS18a**].

3.4.3. The diamond lattice fractal. The example of the diamond lattice fractal is particularly interesting because it is a self-similar finitely ramified fractal where all the renormalisation factors equal 1. It may happen as in this example that Ω is a proper subset of K. Nevertheless, we can verify Assumption 3.2.11 by introducing a local regular Dirichlet form as introduced in [**HK10**].

Let (X, d) be a compact metric space containing two points labelled by 0 and 1. Let $F = \{F_j\colon X \longrightarrow X\}_{j=1,\dots,4}$ be a family of contractions with ratio 1/2 and such that

$$F_1(0) = F_2(0) = 0 \qquad F_1(1) = F_3(0)$$
$$F_2(1) = F_4(0) \qquad F_3(1) = F_4(1) = 1$$

and

$$F_j(X \setminus \{0,1\}) \cap F_k(X \setminus \{0,1\}) = \emptyset$$

whenever $j \neq k$. The *diamond lattice fractal* is defined as the unique non-empty compact subset D of X that satisfies

$$D = F(D) \coloneqq F_1(D) \cap F_2(D) \cap F_3(D) \cap F_4(D).$$

Since D is a self-similar finitely ramified fractal, it satisfies Definition 3.2.1 (see Subsection 3.4.2). We set $V_0 = \{0,1\}$ and define a finite-dimensional Dirichlet form on V_0 by

$$\mathcal{E}_{V_0}(f,g) = \frac{1}{2}\big(f(0) - f(1)\big)\overline{\big(g(0) - g(1)\big)}$$

for any $f, g\colon V_0 \longrightarrow \mathbb{C}$ and on $V_m = \bigcup_{j=1}^4 F_j(V_{m-1})$, we define

$$\mathcal{E}_{V_m}(f,g) = \sum_{j=1}^{4} \mathcal{E}_{V_{m-1}}(f \circ F_j, g \circ F_j),$$

for $f\colon V_m \longrightarrow \mathbb{C}$. One easily verifies that

$$\inf\big\{\, \mathcal{E}_{V_1}(f) \,\big|\, f\colon V_1 \longrightarrow \mathbb{C}, f\restriction_{V_0} = \varrho \,\big\} = \mathcal{E}_{V_0}(\varrho)$$

for any $\varrho\colon V_0 \longrightarrow \mathbb{C}$ and hence $\{(V_m, \mathcal{E}_{V_m})\}_{m\in\mathbb{N}_0}$ defines a compatible sequence. By Proposition 3.1.6 we can define the limit form

$$\mathcal{E}_{V_\infty}(f,g) = \lim_{m\to\infty} \mathcal{E}_{V_m}(f\restriction_{V_m}, g\restriction_{V_m})$$

with domain

$$\operatorname{dom} \mathcal{E}_{V_\infty} = \Big\{\, f \,\Big|\, f\colon V_\infty \longrightarrow \mathbb{C}, \sup_{m\in\mathbb{N}_0} \mathcal{E}_{V_m}(f\restriction_{V_m}) < \infty \,\Big\}.$$

Note that the diamond lattice fractal D is *not* post-critically finite and the harmonic structure is not regular because $\gamma_j = 1$ for each $j = 1, \ldots, 4$. However, if μ is the homogeneous self-similar Hausdorff measure on D then $(\mathcal{E}_\infty, \operatorname{dom} \mathcal{E}_\infty)$ induces a local regular Dirichlet form $(\mathcal{E}, \operatorname{dom} \mathcal{E})$ in $\mathsf{L}_2(D, \mu)$ (see [**HK10**, Thm. 4.3] where the authors extend results from [**Kig01**, Ch. 3.4] to the special needs of this example). Hence, $(\mathcal{E}_{V_m}, \ell(V_m))$ and $(\mathcal{E}, \operatorname{dom} \mathcal{E})$ are δ_m-quasi-unitarily equivalent with

$$\delta_m = \frac{1 + \sqrt{2}}{2^m},$$

because $\gamma_-^m = 1$, $|V_\alpha| = 2$ and $\mu(D_\alpha) = 4^{-m}$ for any $\alpha \in \mathcal{A}_m$. Moreover, if $a_m \in \mathcal{H}_m$, the magnetic forms $\mathcal{E}^{a_m}$ and $\mathcal{E}_{V_m}^{a_m}$ are δ_m-quasi-unitarily equivalent with the same error.

For a general potential $a \in \mathcal{H}$, we need to assume that $\mathcal{E}^a$ is closed in $\mathsf{L}_2(D, \mu)$. This is for example the case, if the potential is small enough by Lemma 3.3.10 and hence, $(\mathcal{E}^a, \operatorname{dom} \mathcal{E})$ and $(\mathcal{E}_{V_m}^{a_m}, \ell(V_m))$, where a_m denotes the projection of a in $\mathcal{H}$ onto $\mathcal{H}_m$, are $\hat{\delta}_m$-quasi-unitarily equivalent, where the error is specified in Subsection 3.3.2.

CHAPTER 4

Approximation of fractals by graph-like manifolds

The goal of this section is to approximate suitable fractals by graph-like manifolds (defined in Definition 4.1.1). We begin with a brief introduction of the notion of graph-like manifolds in Section 4.1. In Section 4.2 we first prove that the corresponding energy forms on a discrete weighted graph and an associated graph-like manifold are δ-quasi-unitarily equivalent provided the associated weight functions are compatible (in the sense of Definition 4.2.2). Then, we again use the transitivity of the notion of quasi-unitary equivalence Proposition 2.2.3 and our previous results to conclude the closeness of the (standard) energy form on the fractal and its associated graph-like manifold.

This chapter is mainly based on results from [**PS18b**]. The results from the aforementioned article were reformulated in the preprint [**PS19c**], where the authors presented the topic in a more structured way which – to their believes – also improves the applicability. Moreover, the concrete example of the Sierpiński gasket is discussed therein.

4.1. Graph-like manifolds and their energy forms

In this section, we briefly introduce the notion of *graph-like manifolds* which is a special cases of a metric measure space. For more details we refer to [**Pos12**] and references therein.

Let $G = (V, E, \partial)$ be a simple discrete graph with (finite or countable) vertex set V and edge set E and where $\partial \colon E \longrightarrow V \times V$, $e \longmapsto (\partial_- e, \partial_+ e)$ assignes to each edge e an orientation, i.e., an initial vertex $\partial_- e$ and a terminal vertex $\partial_+ e$. We define

$$E_v \coloneqq E_v^+ \cup E_v^- \quad \text{where} \quad E_v^\pm \coloneqq \{\, e \in E \mid \partial_\pm e = v \,\}.$$

Moreover, let $\mu \colon V \longrightarrow (0, \infty)$ be a *vertex weight function* and $\gamma \colon E \longrightarrow (0, \infty)$ an *edge weight function*. Then there is a natural Hilbert space structure on G given by

$$\ell_2(V, \mu) \coloneqq \Big\{\, f \,\Big|\, f \colon V \longrightarrow \mathbb{C} \text{ such that } \sum_{v \in V} \mu(v) |f(v)|^2 < \infty \,\Big\}$$

with norm $\|f\|^2_{\ell_2(V,\mu)} \coloneqq \sum_{v \in V} \mu(v) |f(v)|^2$ and a non-negative quadratic form

$$\mathcal{E}_G(f) \coloneqq \sum_{e \in E} \gamma_e |(df)_e|^2 \quad \text{where} \quad (df)_e = f(\partial_+ e) - f(\partial_- e),$$

with associated non-negative self-adjoint operator

$$\Delta_G f(v) = \frac{1}{\mu(v)} \sum_{e \in E_v} \gamma_e (f(v) - f(v_e)),$$

where v_e denotes the vertex on e opposite to v. If G is an infinite graph, we assume that the graph energy $\mathcal{E}_G$ is bounded by $2 \sup_{v \in V} \sum_{e \in E_v} \gamma_e / \mu(v)$. This is equivalent to the boundedness of the relative weights

$$\varrho(v) \coloneqq \frac{\sum_{e \in E_v} \gamma_e}{\mu(v)}.$$

A *graph-like manifold* associated with the discrete graph $G = (V, E, \partial)$ is a Riemannian manifold of dimension $d \geq 2$ glued together from *vertex neighbourhoods* and *edge neighbourhoods* respecting the structure of the underlying graph. The following definition makes this precise.

DEFINITION 4.1.1 (Graph-like manifold). *We say that X is a* graph-like manifold *with associated discrete graph $G = (V, E)$ if there are compact and connected subsets $\{\check{X}_v\}_{v \in V}$ and $\{X_e\}_{e \in E}$ of X with the following properties:*

(i) $X = \bigcup_{v \in V} \check{X}_v \cup \bigcup_{e \in E} X_e$ is connected and $\check{X}_v \cap X_e \neq \emptyset$ if and only if $e \in E_v$; all other sets $\check{X}_v$ and X_e are pairwise disjoint;

(ii) there exists a function $\ell\colon E \longrightarrow (0, \infty)$, $e \longmapsto \ell_e$ (assigning a length ℓ_e to each edge $e \in E$) and each X_e is isometric with $M_e \times Y_e$, where $M_e = [0, \ell_e]$ and where Y_e is a $(d-1)$-dimensional Riemannian manifold;

(iii) there exists a function $\kappa\colon E \longrightarrow (0, \infty)$, $e \longmapsto \kappa_e$ such that $\partial_e \check{X}_v := \check{X}_v \cap X_e$ (being isometric with Y_e by (ii)*) has a κ_e-collar neighbourhood $X_{v,e}$ inside $\check{X}_v$, i.e., $X_{v,e}$ is isometric with $[0, \kappa_e] \times Y_e$; we assume that the edge neighbourhoods $\{X_{v,e}\}_{e \in E_v}$ are pairwise disjoint for each vertex $v \in V$.*

We call $\check{X}_v$ the core vertex neighbourhood *of $v \in V$ and X_e the* edge neighbourhood *of $e \in E$. We call Y_e the* transversal manifold *of e. Moreover, we call $X_v := \check{X}_v \cup \bigcup_{e \in E_v} X_e$ the* (enlarged) vertex neighbourhood *of v. We sometimes refer to the data $(X, \ell, \{\check{X}_v\}_{v \in V}, \kappa, \{Y_e\}_{e \in E})$ as a graph-like manifold.*

The decomposition into vertex neighbourhoods $\check{X}_v$ and edge neighbourhoods X_e is not unique. Moreover, in the special case of a compact graph-like manifold, the third condition in the above definition follows from the first and the second one: For each edge $e \in E_v$, we take away a little piece from X_e and add it to $\check{X}_v$. This, of cause, results in a shorter edge length ℓ_e.

If we already have a metric graph (see Subsection 3.4.1), we can construct a graph-like manifold by using the same length function ℓ. Then the associated graph-like manifold X is defined as an abstract space. However, if the metric graph is embedded into $\mathbb{R}^d$, we get the following convenient example.

EXAMPLE 4.1.2 (Thickened metric graph, [**PS18b**, Example 4.2]). *Let M be a compact metric graph embedded into $\mathbb{R}^d$ in such a way that the edges M_e are line segments. Then the vertex set V of M is a subset of $\mathbb{R}^d$ and the lengths ℓ_e of an edge e is given by $|\partial_+ e - \partial_- e|$. For an arbitrary $\varepsilon > 0$, we denote by X the closed ε-neighbourhood of M in $\mathbb{R}^d$. If we choose ε small enough, it can be seen that X is a graph-like manifold. In particular, one can choose $\ell_e = (1 - 2\varepsilon)|\partial_+ e - \partial_- e|$ as the length of the edge e. The factor $(1 - 2\varepsilon)$ is necessary, since we need to dedicate some space of order $\varepsilon \ell_e$ for the collar neighbourhoods $X_{v,e}$.*

Metric graphs with edges embedded as curved segments can also be treated as a perturbation of abstract metric graphs (not necessarily embedded) with straight edges, see [**Pos12**, Sec. 5.4 and 6.7]).

In our application we are interested in graph-like manifolds beging *scaled* (in transversal direction) by a scaling parameter $\varepsilon > 0$. The notation we use here is as follows: Let X be a graph-like manifold with Riemannian metric g and $\varepsilon > 0$. Then X_ε denotes the Riemannian manifold X with metric $\varepsilon^2 g$. Note that the ε-dependence only enters in the metric and hence εX and X have the same underlying manifold.

DEFINITION 4.1.3 (Scaled graph-like manifolds). *Let $\varepsilon > 0$. We call X_ε a* (transversally) ε-scaled graph-like manifold *if X_ε is a graph-like manifold described by the data $(X_\varepsilon, \ell, \{\varepsilon \check{X}_v\}_{v\in V}, \varepsilon\kappa, \{\varepsilon Y_e\}_{e\in E})$.*

The Hilbert space on a graph-like manifold X, we consider here is the usual space of square integrable functions $\mathsf{L}_2(X, \nu)$ with respect to the Riemannian measure ν and with the usual norm denoted by $\|.\|^2_{\mathsf{L}_2(X,\nu)}$. The energy form on X is given by

$$\mathcal{E}_X(u) = \int_X |\nabla u(x)|^2_x \, \mathrm{d}\nu(x) \tag{4.1.1}$$

for each $u \in \mathsf{H}^1(X)$, i.e., the domain of $\mathcal{E}_X$ is the closure of Lipschitz continuous functions with compact support in X with respect to the energy norm defined by

$$\|u\|^2_{\mathsf{H}^1(X)} = \|u\|^2_{\mathsf{L}_2(X)} + \mathcal{E}_X(u). \tag{4.1.2}$$

Here, ∇ denotes the gradient and $|.|_x$ is the norm induced by the Riemannian metric tensor at a point $x \in X$.

4.2. Compatibility of graph-like manifolds and weighted discrete graphs

Before we formulate our first main result about the comparison with graph-like manifolds in the next section, stating the quasi-unitary equivalence of an energy form on a graph-like manifold and the energy on the underlying discrete weighted graph, we need to fix some more notation first.

Let $G = (V, E)$ be a discrete graph. Let X be the associated graph-like manifold with Riemannian measure ν. We first define a partition of unity $\{\varphi_v\}_{v\in V}$ on X that respects the structure of the associated discrete graph. The idea in this section is quite similar to the approach we chose when we compared a finitely ramified fractal with its associated family of approximating discrete graphs in Subsection 3.3.1.

We fix an arbitrary vertex $v \in V$ of the graph G. Let $\varphi_v \colon X \longrightarrow [0, 1]$ be given by

$$\varphi_v(x) = \begin{cases} 1 & \text{if } x \in \check{X}_v, \\ \dfrac{t}{\ell_e} & \text{if } x = (t, y) \in M_e \times Y_e, \\ 0 & \text{if } x \in X \setminus X_v. \end{cases} \tag{4.2.1}$$

Note that $\varphi_v \in \mathsf{H}^1(X)$ for all $v \in V$, because these functions are Lipschitz continuous on X and (piecewise) harmonic since φ_v is constant on the vertex cores $\check{X}_{v'}$ ($v' \in V$), and affine linear in longitudinal and constant in transversal direction on the edge neighbourhoods X_e ($e \in E$). The above definition is also viable for an ε-scaled graph-like manifold X_ε, since the scaling parameter only enters in the metric, as discussed before.

Moreover, the partition of unity allows us to define a vertex measure $\{\nu(v)\}_{v\in V}$ on G inherited by the Riemannian measure on X. We define $\nu\colon V \longrightarrow (0,\infty)$ by

$$\nu(v) := \int_X \varphi_v(x)\,\mathrm{d}\nu(x) = \operatorname{vol}\check{X}_v + \sum_{e\in E_v} \operatorname{vol} X_e = \operatorname{vol}\check{X}_v + \frac{1}{2}\sum_{e\in E_v} \ell_e \operatorname{vol} Y_e. \quad (4.2.2)$$

Remark 4.2.1. *Of course, $\{\nu(v)\}_{v\in V}$ would be the natural choice for a vertex measure on the discrete graph G, which we want to compare with the graph-like manifold X. But keep in mind that in our application we need to compare a suitable fractal with its associated graph-like manifold. Thus, we already have a given vertex weight $\mu = \{\mu(v)\}_{v\in V}$ and edge weight $\gamma = \{\gamma_e\}_{e\in E}$ on G. Hence, we need to make sure that the weights μ and γ induced by the fractal and the once induced by the graph-like manifold are compatible.*

Definition 4.2.2 (Compatibility of the weights). *Let X be a graph-like manifold as in Definition 4.1.1 and let $G = (V, E)$ be its underlying discrete graph. Let $\mu\colon V \longrightarrow [0,\infty)$ and $\gamma\colon E \longrightarrow [0,\infty)$ be a vertex weight and edge weight respectively.*

(i) We say that X and (G,μ,γ) are compatible *if there exist two constants $c > 0$ and $\tau > 0$ such that the edge length function ℓ, the weights $\mu = \{\mu(v)\}_{v\in V}$ and $\gamma = \{\gamma_e\}_{e\in E}$ and the transversal volumes $\{\operatorname{vol} Y_e\}_{e\in E}$ fulfil*

$$\frac{1}{2\mu(v)}\sum_{e\in E_v} \ell_e \operatorname{vol} Y_e = \frac{1}{c^2} \qquad \text{and} \qquad \frac{\gamma_e \ell_e}{\operatorname{vol} Y_e} = c^2\tau, \quad (4.2.3a)$$

for all vertices $v \in V$ and edges $e \in E$.

(ii) We say that X has uniformly small (core) vertex neighbourhoods *if there are constants $\alpha_0 \in (0,1]$ and $\alpha_\infty > 0$ such that*

$$\alpha_0 \le \alpha(v) := \frac{2\operatorname{vol}\check{X}_v}{\sum_{e\in E_v} \operatorname{vol} X_e} \le \alpha_\infty, \quad (4.2.3b)$$

for all $v \in V$, and in addition, if

$$K_\infty := \sup_{v\in V} \max_{e\in E_v} \Big(\kappa_e + \frac{2}{\kappa_e \lambda_2(\check{X}_v)}\Big) < \infty, \quad (4.2.3c)$$

where $\lambda_2(\check{X}_v) > 0$ denotes the second (first non-zero) Neumann eigenvalue on $\check{X}_v$.

(iii) We say that X has uniform transversal volume, *if*

$$0 < \operatorname{vol}_0 := \inf_{e\in E} \operatorname{vol} Y_e \le \operatorname{vol}_\infty := \sup_{e\in E} \operatorname{vol} Y_e < \infty, \quad (4.2.3d)$$

where $\operatorname{vol} Y_e$ is the volume of the $(d-1)$-dimensional manifold Y_e (and d is the dimension of X).

Remark 4.2.3. *(i) Note that in the situation of Example 4.1.2, i.e., if we consider an embedded metric graph M with edge length function ℓ together with a small ε-neighbourhood as graph-like manifold X, then the edge length* *of X is $(1-2\varepsilon)\ell$; the common factor $(1-2\varepsilon)$ does not destroy the compatibility in the sense of the above Definition 4.2.2 — it just changes the constants c and τ slightly.*

(ii) There is still some freedom in the choice of the parameters ℓ_e, $\operatorname{vol} Y_e$, τ and c, as they only need to satisfy the two equations (4.2.3a). *We easily derive*

$$c = \Big(\frac{1}{2\mu(v)} \sum_{e \in E_v} \ell_e \operatorname{vol} Y_e\Big)^{-1/2} \qquad \textit{and} \qquad \tau = \frac{1}{2\mu(v)} \sum_{e \in E_v} \ell_e^2 \gamma_e$$

from the aforementioned equations. We stress that these constants c and τ are independent of the vertex v and the edge e.

(iii) Moreover, the compatibility conditions in (4.2.3a) *give a restriction on the vertex weights μ and edge weights γ of the weighted graph G and the transversal volume* $\operatorname{vol} Y_e$*, namely that*

$$\frac{1}{2\mu(v)} \sum_{e \in E_v} \frac{(\operatorname{vol} Y_e)^2}{\gamma_e} = \frac{1}{c^4 \tau}$$

is independent of the vertex $v \in V$.

Definition 4.2.4 (Uniform compatibility of a graph-like manifold and a discrete graph). *We say that a graph-like manifold X and a discrete weighted graph (G, μ, γ) are* uniformly compatible*, if they are compatible and if X has uniformly small vertex neighbourhoods and uniform transversal volume.*

Remark 4.2.5. *The various constants introduced in Definition 4.2.2 jar well with the scaling parameter ε. For a d-dimensional compact Riemannian manifold X, we have the scaling behavior*

$$\operatorname{vol}(X_\varepsilon) = \varepsilon^d X \qquad \textit{and} \qquad \lambda_k(X_\varepsilon) = \frac{1}{\varepsilon^2} \lambda_k(X),$$

where $\lambda_k(X)$ denotes the k-th eigenvalue of its Laplacian (without boundary, or Neumann boundary conditions). From the definition it also follows directly that an ε-scaled graph-like manifold X_ε is uniformly compatible with (G, μ, γ) if the following holds: τ is independent of ε and

$$\begin{aligned} c_\varepsilon &= \varepsilon^{-(d-1)/2} c, & K_{\varepsilon,\infty} &= \varepsilon K_\infty, \\ \alpha_{\varepsilon,0} &= \varepsilon \alpha_0, & \alpha_{\varepsilon,\infty} &= \varepsilon \alpha_\infty, \\ \operatorname{vol}_{\varepsilon,0} &= \varepsilon^{d-1} \operatorname{vol}_0, & \operatorname{vol}_{\varepsilon,\infty} &= \varepsilon^{d-1} \operatorname{vol}_\infty, \end{aligned}$$

where d is the dimension of X. The quantities with subscript ε refer to X_ε and the ones without to the unscaled manifold $X = X_1$.

The energy $\mathcal{E}_X$ (and also the rescaled energy $\widetilde{\mathcal{E}}_X := \tau \mathcal{E}_X$) on a graph-like manifold also respects the decomposition into building blocks in the following sense:

Definition 4.2.6 (Decomposition of the energy). *Let $v \in V$ be an arbitrary vertex. We define a quadratic form in $\mathsf{L}_2(X_v, \nu)$, where X_v is the corresponding* *vertex neighbourhood of* v *by*

$$\mathcal{E}_{X_v}(u) = \int_{X_v} |\nabla u(x)|_x^2 \,\mathrm{d}\nu(x), \qquad u \in \operatorname{dom}\mathcal{E}_{X_v} := \{\, u{\restriction}_{X_v} \mid u \in \mathsf{H}^1(X) \,\}.$$

This allows us to decompose the energy with respect to the building blocks. We have

$$\mathcal{E}_X(u) \le \sum_{v\in V} \mathcal{E}_{X_v}(u{\restriction}_{X_v}) \le 2\mathcal{E}_X(u) \tag{4.2.4}$$

for each $u \in \mathsf{H}^1(X)$. This is true because we assumed that the graph is simple, thus there is at most one edge between any two distinct vertices. Moreover, the quadratic form $(\mathcal{E}_{X_v}, \operatorname{dom}\mathcal{E}_{X_v})$ is closable in the weighted Hilbert space

$$\mathsf{L}_2(X_v, \varphi_v) := \Big\{\, u \,\Big|\, \|u\|^2_{\mathsf{L}_2(X_v,\varphi_v)} := \int_{X_v} |u(x)|^2 \varphi_v(x)\,\mathrm{d}\nu(x) < \infty \Big\}. \tag{4.2.5}$$

The first eigenvalue $\lambda_1(X_v, \varphi_v) = 0$ is simple with constant eigenfunction $\mathbb{1}_{X_v}$ and the second eigenvalue fulfills $0 < \lambda_2(X_v, \varphi_v)$. This is true because X_v is connected.

The following lemma is a handy result that we use to simplify the error δ in our main result where some of the estimates rely on a Poincaré type estimate, i.e., they depend on a corresponding second eigenvalue. By the following lemma, we do not have to specifically deal with bounds for the weighted eigenvalues in the weighted Hilbert spaces because we can estimate them against the unweighted eigenvalue.

Lemma 4.2.7 ([**PS18b**, Lemma 2.6]). *Let* $\Phi_{2,v}$ *be a normalised eigenfunction associated with* $\lambda_2(X_v, \varphi_v)$*, the first non-zero eigenvalue of* $\mathcal{E}_{X_v}$ *in the weighted Hilbert space* $\mathsf{L}_2(X_v, \varphi_v)$*. Moreover, denote by* $\lambda_2(X_v) := \lambda_2(X_v, \nu)$ *the second (first non-zero) eigenvalue of* $\mathcal{E}_{X_v}$ *in the unweighted Hilbert space* $\mathsf{L}_2(X_v, \nu)$*. Then* $\Phi_{2,v} \in \mathsf{L}_2(X_v, \nu)$ *and*

$$\lambda_2(X_v, \varphi_v) \ge \lambda_2(X_v).$$

Proof. Let $v \in V$ be arbitrary. Since $0 \le \varphi_v(x) \le 1$ for all $x \in X$, we have

$$\|u\|^2_{\mathsf{L}_2(X_v,\varphi_v)} = \int_{X_v} |u(x)|^2 \varphi_v(x)\,\mathrm{d}\nu(x) \le \int_{X_v} |u(x)|^2 \,\mathrm{d}\nu(x) = \|u\|_{\mathsf{L}_2(X_v,\nu)}.$$

Hence, $\mathsf{L}_2(X_v, \nu) \subset \mathsf{L}_2(X_v, \varphi_v)$. Using the min-max characterisation of the second eigenvalue, we can estimate the unweighted eigenvalue by

$$\lambda_2(X_v) = \inf_{D_2} \sup_{u\in D_2\setminus\{0\}} \frac{\mathcal{E}_{X_v}(u)}{\|u\|^2_{\mathsf{L}_2(X_v,\nu)}} \le \sup_{u\in D_2\setminus\{0\}} \frac{\mathcal{E}_{X_v}(u)}{\|u\|^2_{\mathsf{L}_2(X_v,\nu)}} \le \sup_{u\in D_2\setminus\{0\}} \frac{\mathcal{E}_{X_v}(u)}{\|u\|^2_{\mathsf{L}_2(X_v,\varphi_v)}},$$

where D_2 runs through all two-dimensional subspaces of $\mathsf{L}_2(X_v, \nu) \cap \operatorname{dom}\mathcal{E}_{X_v}$. Note that the first eigenfunction for both eigenvalue problems is the constant $\mathbb{1}_{X_v}$ on the vertex neighbourhood X_v.

Since φ_v is harmonic on X (affine linear in longitudinal direction M_e and constant in transversal direction Y_e), the second eigenfunction is an Airy function on X_e (in longitudinal direction), i.e., it is continuous and thus in the unweighted Hilbert space $\mathsf{L}_2(X_v, \nu)$. Hence, we can choose $D_2 = \mathbb{C}\mathbb{1}_{X_v} + \mathbb{C}\Phi_{2,v}$ and the latter Rayleigh quotient in the equation above becomes $\lambda_2(X_v, \varphi_v)$. ☐

In the case of an ε-scaled graph-like manifold, we can estimate the weighted eigenvalue on $X_{v,\varepsilon}$ by applying a simple scaling argument:

LEMMA 4.2.8 ([**PS18b**, Proposition B.3]). *Let X_ε be an ε-scaled graph-like manifold which is uniformly compatible with a discrete weighted graph (G, μ, γ) such that there are constants $\ell_0, \ell_\infty \in (0, \infty)$, $\check{\lambda}_2 > 0$ and $\lambda_2^{\pitchfork} > 0$ with*

$$\ell_0 \leq \ell_e \leq \ell_\infty, \qquad \lambda_2(Y_e) \geq \lambda_2^{\pitchfork} \qquad \text{and} \qquad \lambda_2(\check{X}_v) \leq \check{\lambda}_2,$$

for all $e \in E$ and $v \in V$. Then there exists a constant $C > 0$ depending only on ℓ_∞/ℓ_0, $\lambda_2^{\pitchfork}$ and $\check{\lambda}_2$ (from the unscaled graph-like manifold $X = X_1$) such that for all $v \in V$,

$$\lambda_2(X_{v,\varepsilon}) \geq \frac{1}{\ell_\infty^2} \qquad \text{for all} \qquad 0 < \varepsilon \leq \varepsilon_0 := \frac{\ell_0}{C^2}. \tag{4.2.6}$$

PROOF. Note that we can replace the parameters ℓ_e and ε by $\ell_e/\ell_0 \in [1, \ell_\infty/\ell_0]$ and ε/ℓ_0 using a simple scaling argument. Let $\ell_0^{-1} X_{v,\varepsilon}$ denote the scaled graph-like manifold with metric $\ell_0^{-2} g_{v,\varepsilon}$ and similarly denote by $\ell_0^{-1} M_v$ the associated metric graph with edge length ℓ_e/ℓ_0. Then we can apply the convergence result for graph-like manifolds (see e.g. [**EP05**]) and we get

$$\left|\lambda_k(\ell_0^{-1} X_{v,\varepsilon}) - \lambda_k(\ell_0^{-1} M_v)\right| \leq C \sqrt{\frac{\varepsilon}{\ell_0}}. \tag{4.2.7}$$

The error estimate $(\varepsilon/\ell_0)^{1/2}$ is proven in [**Pos12**, Theorem 6.4.1 and Theorem 4.6.4]. Since

$$\lambda_k(\ell_0^{-1} X_{v,\varepsilon}) = \ell_0^2 \lambda_k(X_{v,\varepsilon}) \qquad \text{and} \qquad \lambda_k(\ell_0^{-1} M_v) = \ell_0^2 \lambda_k(M_v)$$

we obtain from (4.2.7) by the triangle inequality,

$$\lambda_2(X_{v,\varepsilon}) \geq \lambda_2(M_v) - \frac{C}{\ell_0^2} \sqrt{\frac{\varepsilon}{\ell_0}} \geq \frac{2}{\ell_\infty^2} - \frac{C}{\ell_0^2} \sqrt{\frac{\varepsilon}{\ell_0}}$$

using [**PS18b**, Lemma 3.4] for the last estimate. We choose $0 < \varepsilon \leq \varepsilon_0 := \ell_0/C^2$ and because $\ell_0 \leq \ell_\infty$ the claim follows. ☐

Combining the two results above, Lemma 4.2.7 and Lemma 4.2.8, we conclude that the weighted eigenvalue problem on the scaled vertex neighbourhood $X_{v,\varepsilon}$ satisfies

$$\frac{1}{\ell_\infty^2} \leq \lambda_2(X_{v,\varepsilon}, \varphi_v), \tag{4.2.8}$$

for all vertices $v \in V$ and scaling parameter $\varepsilon \in (0, \varepsilon_0]$ as in the latter lemma.

4.3. Quasi-unitary equivalence of the energy forms on a graph-like manifold and its related discrete graph

The goal of this section is to prove that the canonical energy form on a weighted discrete graph and its associated graph-like manifold are unitarily equivalent. These results have already been stated in [**PS18b**] (see in particular Sections 2 and 4) — except the simplified proof of Proposition 4.3.5 and Theorem 4.3.1 which lead to a slightly better and more convenient constant in the error δ. These improvements have been reformulated in the recent preprint by the same authors [**PS19c**].

Let X be a graph-like manifold with Riemannian measure ν and uniformly spectrally small core vertex neighbourhoods, i.e.,the equations (4.2.3a)–(4.2.3c) are fulfilled. Let (G, μ, γ) be the underlying discrete weighted graph with compatible weights (in the sense of (4.2.3a)).

THEOREM 4.3.1 ([**PS18b**, Theorem 4.9]). *Assume that (G, μ, γ) is a weighted discrete graph with weights fulfilling*

$$\mu_\infty := \sup_{v \in V} \mu(v) < \infty \qquad \textit{and} \qquad 0 < \gamma_0 := \inf_{e \in E} \gamma_e \leq \gamma_\infty := \sup_{e \in E} \gamma_e < \infty. \tag{4.3.1}$$

Assume in addition that X is a graph-like manifold with underlying graph G and that X and (G, μ, γ) are uniformly compatible. Then $\ell_0 := \inf_{e \in E} \ell_e > 0$ and $\ell_\infty := \sup_{e \in E} \ell_e < \infty$. If finally,

$$0 < \frac{1}{\ell_\infty^2} \leq \lambda_2(X_v, \varphi_v), \tag{4.3.2}$$

then the discrete energy form $\mathcal{E}_G$ on the weighted discrete graph and the rescaled energy form $\widetilde{\mathcal{E}}_X := \tau \mathcal{E}_X$ with domain $\operatorname{dom} \widetilde{\mathcal{E}}_X = \mathsf{H}^1(X)$ in $\mathsf{L}_2(X, \nu)$ are δ-quasi-unitarily equivalent with

$$\delta^2 = \max\Big\{\alpha_\infty^2, \frac{4}{\alpha_0 \cdot d_0}\Big(\frac{\mathrm{vol}_\infty}{\mathrm{vol}_0}\Big)^2 \frac{\gamma_\infty}{\gamma_0} \cdot \frac{\mu_\infty}{\gamma_0}, \frac{4K_\infty}{\ell_0}\Big\}, \tag{4.3.3}$$

where $d_0 := \min_{v \in V} \deg v$ is the minimal degree and where α_0, α_∞, K_∞, vol_0 and vol_∞ are defined in Definition 4.2.2 (Compatibility of the weights).

The idea behind the proof to the above theorem is similar to the one that stated the quasi-unitarily equivalence of a finitely ramified fractal with its approximating family of discrete graphs. For a more clear arrangement of the arguments, we split the proof into several propositions following the structure of Definition 2.2.1. However, let us first make some comments on the error above.

REMARK 4.3.2 ([**PS18b**]). *The error terms have the following meaning:*

(i) The first term containing α_∞ ensures that the volumes of the core vertex neighbourhoods of the graph-like manifold are small compared to the volumes of the corresponding edge neighbourhoods.

(ii) The second term is quite similar to the one in the result for fractals Theorem 3.3.4 and in particular Corollary 3.3.5 since both contain the fraction μ_∞/γ_0. This is not surprising since a metric graph is one example for a finitely ramified fractal and metric graphs and graph-like manifolds are similar somehow. However, the factor $1/\alpha_0$ is expected to be big and hence, the term will make the error worse.

(iii) The last term solely comes from the comparison of the energy forms (see Proposition 4.3.6). It is best to be understood in the setting of an ε-scaled graph-like manifolds defined in Definition 4.1.3: As discussed in Remark 4.2.5, the vertex neighbourhoods scale as $\varepsilon \check{X}_v$, thus $\lambda_2(\varepsilon \check{X}_v) = \varepsilon^{-2} \lambda_2(\check{X}_v)$. Hence, the entire expression is of order ε/ℓ_0.

From the above theorem, we can immediately conclude the result, we are actually interested in, namely, the convergence in the case of a shrinking family of graph-like manifolds. The following corollary is an easy consequence from the above and the concrete error can be verified by the scaling behavior of the relevant parameters, which we specified in Remark 4.2.5.

COROLLARY 4.3.3 ([**PS18b**, Corollary 4.10]). *Let us assume the situation from the above Theorem 4.3.1, i.e.,* (G,μ,γ) *is a weighted discrete graph fulfilling* (4.3.1), *and* X_ε *is an* ε*-scaled graph-like manifold being uniformly compatible with* (G,μ,γ). *Then there is* $\varepsilon_0 := \ell_0/C > 0$ *with* C *depending only on* ℓ_∞/ℓ_0, $\lambda_2^{\pitchfork} := \inf_{e\in E}\lambda_2(Y_e)$ *and* $\check\lambda_2 := \inf_{e\in E}\lambda_2(\check X_v)$ *such the the discrete graph energy form* $\mathcal{E}_G$ *on the weighted graph and the rescaled energy form* $\widetilde{\mathcal{E}}_{X_\varepsilon} := \tau\mathcal{E}_{X_\varepsilon}$ *on* X_ε *are* δ_ε*-quasi-unitarily equivalent with*

$$\delta_\varepsilon^2 = \max\Big\{\varepsilon^2\alpha_\infty^2, \frac{4}{\varepsilon\alpha_0\cdot d_0}\Big(\frac{\mathrm{vol}_\infty}{\mathrm{vol}_0}\Big)^2\frac{\gamma_\infty}{\gamma_0}\cdot\frac{\mu_\infty}{\gamma_0}, \frac{4\varepsilon K_\infty}{\ell_0}\Big\} \tag{4.3.4}$$

for all $\varepsilon\in(0,\varepsilon_0]$.

Note that the error δ_ε in (4.3.4) only vanishes (as $\varepsilon\to 0$) if and only if $\mu_\infty/\varepsilon\gamma_0 \to 0$ (as $\varepsilon\to 0$). We will discuss this late in our application in Section 4.4, where we consider a family of discrete weighted graphs $\{(G_m,\mu_m,\gamma_m)\}_{m\in\mathbb{N}}$ and where not just the graph itself depends on m but also the weights μ_m and γ_m which we indicate by the subscript m.

We begin by defining the identification operators needed in Definition 2.2.1: We want to emphasise that the idea behind this approach is not new: In the case of a graph-like manifold and its associated discrete graph, the operators J and J' are also called *smoothening* and *discretisation* operator in the literature (see e.g. [**Cha01**, Sec. VI.5]). In order to simplify the notation, we will omit the generation of the approximating graph (i.e. the subscript m). As smoothening operator $J := J_m$, we choose

$$J\colon \ell_2(V,\mu) \longrightarrow \mathsf{L}_2(X,\nu), \quad Jf = c\cdot\sum_{v\in V} f(v)\varphi_v$$

where $c>0$ is a constant, specified by the compatibility of the weighted discrete graph and the graph-like manifold in (4.2.3a). Let $J' := J'_m$ be the adjoint of J, that is,

$$J'\colon \mathsf{L}_2(X,\nu) \longrightarrow \ell_2(V,\mu), \quad J'u(v) = \frac{1}{c}\cdot\frac{1}{\nu(v)}\langle u,\varphi_v\rangle_{\mathsf{L}_2(X,\nu)},$$

for each $v\in V$. Moreover, let $J^1 := J^1_m\colon \operatorname{dom}\mathcal{E}_G \longrightarrow \operatorname{dom}\mathsf{H}^1(X)$, $J^1 := J\restriction_{\mathsf{H}^1(X)}$ and

$$J'^1 := J'^1_m\colon \mathsf{H}^1(X) \longrightarrow \operatorname{dom}\mathcal{E}_G = \ell_2(V,\mu), \qquad J'^1u(v) = \frac{1}{c}\cdot\frac{1}{\operatorname{vol}\check X_v}\int_{\check X_v} u\,\mathrm{d}\nu$$

for each $v\in V$.

PROPOSITION 4.3.4 ([**PS18b**, Proposition 2.8]). *Let* X *be a graph-like manifold with uniformly spectrally small vertex neighbourhoods and compatible discrete graph* (G,μ,γ). *Then the operator* J *is* δ*-quasi-unitary with* δ*-quasi-adjoint* J' *(for the* *energy forms* $\mathcal{E}_G$ *and* $\widetilde{\mathcal{E}}_X := \tau\mathcal{E}_X$*), where* $\delta := \max\{\delta_a, \delta_b\}$*. In particular,* (2.2.2a) *holds with error* δ_a *and* (2.2.2b) *with error* δ_b*, given by*

$$\delta_a = \alpha_\infty \qquad \text{and} \qquad \delta_b^2 = \max\Big\{\frac{2\mu_\infty}{\gamma_0}, \frac{2}{\tau\lambda_2}\Big\},$$

where α_∞ *is defined in* (4.2.3b)*,* μ_∞ *and* γ_0 *are defined in* (4.3.1) *and where* $\lambda_2 := \inf_{v\in V}\lambda_2(X_v, \varphi_v)$*.*

Note that by Lemma 4.2.7, we can estimate the weighted eigenvalue against the unweighted one.

Proof. First, we verify (2.2.2a) and we begin with the boundedness of J and J'. For all $f \in \ell_2(V, \mu)$ we estimate, applying the Cauchy-Young inequality,

$$\begin{aligned}\|Jf\|^2_{\mathsf{L}_2(X,\nu)} &= \sum_{v\in V}\sum_{v'\in V} c^2 f(v)\overline{f(v')}\langle\varphi_v, \varphi_{v'}\rangle_{\mathsf{L}_2(X,\nu)} \\ &\le c^2\sum_{v\in V}|f(v)|^2\sum_{v'\in V}\langle\varphi_v, \varphi_{v'}\rangle_{\mathsf{L}_2(X,\nu)} \le \sup_{v\in V}\frac{c^2\nu(v)}{\mu(v)}\|f\|^2_{\ell_2(V,\mu)},\end{aligned}$$

where we used that $\varphi_v \ge 0$ for $v \in V$. Thus J is bounded by $1+\alpha_\infty$. Let $u \in \mathsf{L}_2(X, \nu)$. Then, by the Cauchy-Schwarz inequality and because $\varphi_v \le 1$, for all $v \in V$, we have

$$\begin{aligned}\|J'u\|^2_{\ell_2(V,\mu)} &= \sum_{v\in V}\frac{\mu(v)}{c^2\nu(v)^2}\big|\langle u, \varphi_v\rangle_{\mathsf{L}_2(X,\nu)}\big|^2 \le \sum_{v\in V}\frac{\mu(v)}{c^2\nu(v)}\|u\|^2_{\mathsf{L}_2(X,\varphi_v)} \\ &\le \sup_{v\in V}\frac{1}{1+\alpha(v)}\|u\|^2_{\mathsf{L}_2(X,\nu)} \le \|u\|^2_{\mathsf{L}_2(X,\nu)}\end{aligned}$$

where we used that $\alpha(v) \ge 0$ in the last estimate. Hence, J' is also bounded by 1. We still need to verify the second condition in (2.2.2a). Since the weights are just compatible and not the same this requires some more technique here: We define a function $\Xi\colon (0,\infty) \longrightarrow \mathbb{R}$ by

$$\Xi(\xi) := \Big|\sqrt{\xi} - \frac{1}{\sqrt{\xi}}\Big| = 2\sinh\Big|\frac{1}{2}\log\xi\Big|, \tag{4.3.5}$$

where the latter equality follows directly from the definition of the function sinh. Then $\Xi(1) = 0$, $\Xi(1/\xi) = \Xi(\xi)$ and $0 \le \Xi(\xi) \le \xi - 1$ for $\xi \ge 1$. Hence,

$$\begin{aligned}&\big|\langle Jf, u\rangle_{\mathsf{L}_2(X,\nu)} - \langle f, J'u\rangle_{\ell_2(V,\mu)}\big| \\ &\quad= \Big|\sum_{v\in V}\Big(c - \frac{\mu(v)}{c\nu(v)}\Big)f(v)\langle\varphi_v, u\rangle_{\mathsf{L}_2(X,\nu)}\Big| \\ &\quad\le \sup_{v\in V}\Xi\Big(\frac{c^2\nu(v)}{\mu(v)}\Big)\sum_{v\in V}\Big|\sqrt{\mu(v)}f(v)\frac{1}{\sqrt{\nu(v)}}\langle\varphi_v, u\rangle_{\mathsf{L}_2(X,\nu)}\Big| \\ &\quad\le \sup_{v\in V}\Xi\Big(\frac{c^2\nu(v)}{\mu(v)}\Big)\|f\|_{\ell_2(V,\mu)}\|u\|_{\mathsf{L}_2(X,\nu)},\end{aligned}$$

where we used the Cauchy-Schwarz inequality in the last estimate. We can further estimate the supremum by

$$\sup_{v\in V} \Xi\Big(\frac{c^2\nu(v)}{\mu(v)}\Big) = \sup_{v\in V} \Xi\big(1+\alpha(v)\big) \le \Xi\big(1+\alpha_\infty\big).$$

Since $\sqrt{1+\alpha_\infty}-1 \le \alpha_\infty$ and $\Xi(1+\alpha_\infty) \le \alpha_\infty$, we conclude that (2.2.2a) holds with $\delta_a = 2\alpha_\infty$.

Next, we check the estimates in (2.2.2b). For the first one, let $f \in \operatorname{dom}\mathcal{E}_G$. Then,

$$f(v) - J'Jf(v) = \frac{1}{\nu(v)} \sum_{v'\in V} \big(f(v)-f(v')\big) \langle \varphi_{v'}, \varphi_v\rangle_{\mathsf{L}_2(X,\nu)},$$

for each $v \in V$. Thus, estimating in norm, we get

$$\begin{aligned} \|f - J'Jf\|^2_{\ell_2(V,\mu)} &= \sum_{v\in V} \frac{\mu(v)}{\nu(v)^2} \Big| \sum_{v'\in V} \big(f(v)-f(v')\big)\langle \varphi_{v'}, \varphi_v\rangle_{\mathsf{L}_2(X,\nu)}\Big|^2 \\ &\le \sum_{v\in V} \frac{\mu(v)}{\nu(v)^2} \sum_{e\in E_v} \gamma_e^{-1} \big|\langle \varphi_{v_e}, \varphi_v\rangle_{\mathsf{L}_2(X,\nu)}\big|^2 \sum_{e\in E_v} \gamma_e \big|f(v)-f(v_e)\big|^2, \end{aligned}$$

where we used the Cauchy-Schwartz inequality. Note that φ_v is non-negative, and by the definition of these functions, we have that $\gamma_e > 0$ if and only if $\langle \varphi_{v_e}, \varphi_v\rangle_{\mathsf{L}_2(X,\nu)} > 0$. Let us further estimate the sum in the middle as follows

$$\sum_{e\in E_v} \gamma_e^{-1} \big|\langle \varphi_{v_e}, \varphi_v\rangle_{\mathsf{L}_2(X,\nu)}\big|^2 \le \sum_{e\in E_v} \gamma_e^{-1} \langle \varphi_{v_e}, \varphi_v\rangle_{\mathsf{L}_2(X,\nu)} \cdot \nu(v) \le \frac{\nu(v)^2}{\gamma_0},$$

and we conclude

$$\|f - J'Jf\|^2_{\ell_2(V,\mu)} \le \sum_{v\in V} \frac{\mu(v)}{\gamma_0} \sum_{e\in E_v} \gamma_e \big|f(v)-f(v_e)\big|^2 \le \frac{2\mu_\infty}{\gamma_0} \cdot \mathcal{E}_G(f).$$

Hence, the first inequality in (2.2.2b) holds with $\delta_b = {}^{2\mu_\infty}/_{\gamma_0}$.

For the second inequality in (2.2.2b), we first compute

$$u - JJ'u = \sum_{v\in V} u\varphi_v - \sum_{v\in V} \frac{1}{\nu(v)} \langle u, \varphi_v\rangle_{\mathsf{L}_2(X,\nu)} \cdot \varphi_v,$$

where we used that the family $\{\varphi_v\}_{v\in V}$ is a partition of unity. Hence,

$$\begin{aligned} \|u - JJ'u\|^2_{\mathsf{L}_2(X,\nu)} &= \int_X \Big| \sum_{v\in V} \Big(u(x) - \frac{1}{\nu(v)}\langle u, \varphi_v\rangle_{\mathsf{L}_2(X,\nu)}\Big) \cdot \varphi_v(x)\Big|^2 \,\mathrm{d}\nu(x) \\ &\le \int_X \sum_{v\in V} \Big| u(x) - \frac{1}{\nu(v)}\langle u, \varphi_v\rangle_{\mathsf{L}_2(X,\nu)}\Big|^2 \cdot \varphi_v(x) \sum_{v\in V} \varphi_v(x)\,\mathrm{d}\nu(x) \\ &= \sum_{v\in V} \int_X \Big| u(x) - \frac{1}{\nu(v)}\langle u, \varphi_v\rangle_{\mathsf{L}_2(X,\nu)}\Big|^2 \cdot \varphi_v(x)\,\mathrm{d}\nu(x), \end{aligned}$$

where we again used the Cauchy-Schwarz inequality. Since $\|\mathbb{1}_{X_v}\|^2_{\mathsf{L}_2(X_v,\varphi_v)} = \nu(v)$,

$$u - \frac{1}{\nu(v)}\langle u, \mathbb{1}_{X_v}\rangle_{\mathsf{L}_2(X_v,\varphi_v)}\mathbb{1}_{X_v}$$

is the projection onto the orthogonal complement of the first eigenspace $\mathbb{C}\mathbb{1}_{X_v}$ and because $\lambda_2(X_v, \varphi_v) > 0$ and the energy form $\mathcal{E}_{X_v}$ is closable in $\mathsf{L}_2(X_v, \varphi_v)$, the min-max characterisation of eigenvalues implies that

$$\int_X \Big| u(x) - \frac{1}{\nu(v)}\langle u, \varphi_v\rangle_{\mathsf{L}_2(X,\nu)}\Big|^2 \cdot \varphi_v(x)\,\mathrm{d}\nu(x) \leq \frac{1}{\lambda_2(X_v, \varphi_v)}\mathcal{E}_{X_v}(u). \tag{4.3.6}$$

Thus, we can further estimate

$$\|u - JJ'u\|^2_{\mathsf{L}_2(X,\nu)} \leq \sum_{v\in V}\frac{1}{\lambda_2(X_v,\varphi_v)}\mathcal{E}_{X_v}(u{\restriction}_{X_v}) \leq \frac{2}{\lambda_2}\mathcal{E}_X(u) \leq \frac{2}{\tau\lambda_2}\widetilde{\mathcal{E}}_X(u),$$

where $\lambda_2 := \inf_{v\in V}\lambda_2(X_v, \varphi_v) > 0$ and where we applied (4.2.4) for the last inequality. Hence, the second inequality in 2.2.2b is fulfilled with $\delta_b = {}^2/_{\tau\lambda_2}$. □

The following proposition has already been stated in [**PS18b**] — see in particular Theorem 2.10 and Lemma 4.5 therein. The proof of the Lemma given there, relied on standard estimates for graph-like manifolds (see [**PS18b**, Lemma B.1]). Here we give an easier proof by using the properties of the partition of unity $\{\varphi_v\}_{v\in V}$. This argument leads to a simpler, more convenient error estimate.

PROPOSITION 4.3.5. *Let X be a graph-like manifold with uniformly spectrally small vertex neighbourhoods and compatible discrete graph (G, μ, γ). Then the operators J^1 and J'^1 are δ_c-compatible with the identification operators J and J', i.e., the inequalities in* (2.2.2c) *hold with*

$$\delta_c^2 = \frac{2}{\alpha_0\tau\lambda_2},$$

where α_0 is defined in (4.2.3b) *and* $\lambda_2 := \inf_{v\in V}\lambda_2(X_v, \varphi_v)$.

PROOF. Since $J^1 = J$, the first estimate in (2.2.2c) is trivially fulfilled with $\delta_c = 0$. For the second inequality, we fix $u \in \mathsf{H}^1(X)$. Then we have

$$J'u(v) - J'^1u(v) = \frac{1}{c}\Big(\frac{1}{\nu(v)}\langle u, \varphi_v\rangle_{\mathsf{L}_2(X,\nu)} - \frac{1}{\operatorname{vol}\check{X}_v}\langle u, \mathbb{1}_{\check{X}_v}\rangle_{\mathsf{L}_2(X,\nu)}\Big)$$

for all $v \in V$ and thus, we can estimate in norm

$$\begin{aligned}
&\|J'u - J'^1 u\|^2_{\ell_2(V,\mu)} \\
&= \sum_{v\in V} \frac{\mu(v)}{c^2} \Big| \frac{1}{\nu(v)} \langle u, \varphi_v \rangle_{\mathsf{L}_2(X,\nu)} - \frac{1}{\operatorname{vol} \check{X}_v} \langle u, \mathbb{1}_{\check{X}_v} \rangle_{\mathsf{L}_2(X,\nu)} \Big|^2 \\
&= \sum_{v\in V} \frac{\mu(v)}{c^2} \Big| \frac{1}{\operatorname{vol} \check{X}_v} \int_{\check{X}_v} \Big(u(x) - \frac{1}{\nu(v)} \langle u, \varphi_v \rangle_{\mathsf{L}_2(X,\nu)} \Big) \,\mathrm{d}\nu(x) \Big|^2 \\
&\le \sum_{v\in V} \frac{\mu(v)}{(c \cdot \operatorname{vol} \check{X}_v)^2} \int_{\check{X}_v} \Big| u(x) - \frac{1}{\nu(v)} \langle u, \varphi_v \rangle_{\mathsf{L}_2(X,\nu)} \Big|^2 \,\mathrm{d}\nu(x) \cdot \operatorname{vol} \check{X}_v \\
&\le \sum_{v\in V} \frac{\mu(v)}{c^2 \operatorname{vol} \check{X}_v} \int_X \Big| u(x) - \frac{1}{\nu(v)} \langle u, \varphi_v \rangle_{\mathsf{L}_2(X,\nu)} \Big|^2 \cdot \varphi_v(x) \,\mathrm{d}\nu(x) \\
&\le \sum_{v\in V} \frac{1}{\alpha(v)} \cdot \frac{1}{\lambda_2(X_v, \varphi_v)} \mathcal{E}_{X_v}(u) \le \frac{2}{\alpha_0 \tau \lambda_2} \widetilde{\mathcal{E}}_X(u),
\end{aligned}$$

where we used the Cauchy-Schwarz inequality in the first estimate, the definition of the functins φ_v in the second one (i.e., that φ_v is equal to $\mathbb{1}_{\check{X}_v}$ on the vertex core $\check{X}_v$ and non-negative on X) and (4.3.6) and (4.2.3a) in the third estimate. □

Proposition 4.3.6 ([**PS18b**, Proposition 2.10]). *Let X be a graph-like manifold with compatible discrete graph (G, μ, γ). The discrete energy form $\mathcal{E}_G$ and the rescaled energy form on the graph-like manifold $\widetilde{\mathcal{E}}_X := \tau \mathcal{E}_X$ are δ_d-close (with respect to J^1 and J'^1), i.e.,* (2.2.2d) *holds with error term*

$$\delta_d^2 = \frac{4K_\infty}{\ell_0},$$

where K_∞ is defined in (4.2.3c) *and where $\ell_0 := \inf_{e\in E} \ell_e$.*

Proof. Let $f \in \operatorname{dom} \mathcal{E}_G$ and $u \in \mathsf{H}^1(X)$. Then, on the discrete weighted graph, we have

$$\mathcal{E}_G(f, J'^1 u) = \frac{1}{c} \sum_{e\in E} \gamma_e \big(f(\partial_+ e) - f(\partial_- e) \big) \Big(\frac{1}{\operatorname{vol} \check{X}_{\partial_+ e}} \int_{\check{X}_{\partial_+ e}} \overline{u} \,\mathrm{d}\nu - \frac{1}{\operatorname{vol} \check{X}_{\partial_- e}} \int_{\check{X}_{\partial_- e}} \overline{u} \,\mathrm{d}\nu \Big)$$

and for the rescaled energy form $\widetilde{\mathcal{E}}_X := \tau \mathcal{E}_X$ on the graph-like manifold,

$$\begin{aligned}
\widetilde{\mathcal{E}}_X(Jf, u) &= c\tau \sum_{v\in V} f(v) \int_X \langle \nabla \varphi_v(x), \nabla u(x) \rangle_x \,\mathrm{d}\nu(x) \qquad (4.3.7) \\
&= c\tau \sum_{v\in V} f(v) \sum_{e\in E_v} \frac{1}{\ell_e} \int_0^{\ell_e} \int_{Y_e} \overline{u}'_e(t, y) \,\mathrm{d}t \,\mathrm{d}y \\
&= c\tau \sum_{v\in V} f(v) \sum_{e\in E_v} \frac{1}{\ell_e} \int_{Y_e} \big(\overline{u}_e(\ell_e, y) - \overline{u}_e(0_e, y) \big) \,\mathrm{d}y \\
&= \frac{1}{c} \sum_{e\in E} \gamma_e \big(f(\partial_+ e) - f(\partial_- e) \big) \Big(\frac{1}{\operatorname{vol} Y_e} \int_{\partial_e \check{X}_{\partial_+ e}} \overline{u} \,\mathrm{d}y - \frac{1}{\operatorname{vol} Y_e} \int_{\partial_e \check{X}_{\partial_- e}} \overline{u} \,\mathrm{d}y \Big)
\end{aligned}$$

where we used that φ_v is supported on X_v, constant on $\check{X}_v$ and $\varphi_v(x) = {}^t\!/\!_{\ell_e}$ on each edge $e \in E_v$, for $x = (t, y) \in [0, \ell_e] \times Y_e$. In the last equation, we rearranged the sum via $\sum_{v \in V} \sum_{e \in E_v} = \sum_{e \in E} \sum_{v = \partial_\pm e}$ and used the choice of the edge weight in (4.2.3a) to replace ${}^{c\tau}\!/\!_{\ell_e}$ by ${}^{\gamma_e}\!/\!_{c \operatorname{vol} Y_e}$. Combining the above equations and rearranging the integral terms we get — applying the Cauchy-Schwarz inequality again — that

$$\begin{aligned}&\big|\mathcal{E}_G(f, J'^1 u) - \widetilde{\mathcal{E}}_X(Jf, u)\big|^2 \\ &\qquad \leq \frac{2}{c^2} \cdot \mathcal{E}_G(f) \sum_{e \in E} \sum_{v = \partial_\pm e} \gamma_e \Big| \frac{1}{\operatorname{vol} \check{X}_v} \int_{\check{X}_v} u \,\mathrm{d}\nu - \frac{1}{\operatorname{vol} Y_e} \int_{\partial_e \check{X}_v} u \,\mathrm{d}y \Big|^2.\end{aligned}$$

In order to estimate the sum in the above inequality, we need the following standard estimates; a min-max and a Sobolev trace estimate:

$$\Big\| u - \frac{1}{\operatorname{vol} \check{X}_v} \int_{\check{X}_v} u \,\mathrm{d}\nu \Big\|^2_{\mathsf{L}_2(X_{v,e})} \leq \frac{1}{\lambda_2(\check{X}_v)} \|du\|^2_{\mathsf{L}_2(\check{X}_v)} \tag{4.3.8}$$

$$\int_{\partial_e \check{X}_v} |u(y)|^2 \,\mathrm{d}y \leq \kappa_e \|du\|^2_{\mathsf{L}_2(X_{v,e})} + \frac{2}{\kappa_e} \|u\|^2_{\mathsf{L}_2(X_{v,e})}, \tag{4.3.9}$$

see e.g. [**Pos12**, Proposition 5.1.1 and Corollary A.2.12]. Then

$$\begin{aligned}&\sum_{e \in E_v} \gamma_e \Big| \frac{1}{\operatorname{vol} \check{X}_v} \int_{\check{X}_v} u \,\mathrm{d}\nu - \frac{1}{\operatorname{vol} Y_e} \int_{\partial_e \check{X}_v} u \,\mathrm{d}y \Big|^2 \qquad (4.3.10) \\ &\quad = \sum_{e \in E_v} \gamma_e \Big| \frac{1}{\operatorname{vol} Y_e} \int_{\partial_e \check{X}_v} \Big(u(y) - \frac{1}{\operatorname{vol} \check{X}_v} \int_{\check{X}_v} u \,\mathrm{d}\nu \Big) \mathrm{d}y \Big|^2 \\ &\quad \leq \sum_{e \in E_v} \frac{\gamma_e}{\operatorname{vol} Y_e} \int_{\partial_e \check{X}_v} \Big| u(y) - \frac{1}{\operatorname{vol} \check{X}_v} \int_{\check{X}_v} u \,\mathrm{d}\nu \Big|^2 \mathrm{d}y \\ &\quad \leq \sum_{e \in E_v} \frac{\gamma_e}{\operatorname{vol} Y_e} \Big(\kappa_e \|du\|^2_{\mathsf{L}_2(X_{v,e})} + \frac{2}{\kappa_e} \Big\| u - \frac{1}{\operatorname{vol} \check{X}_v} \int_{\check{X}_v} u \,\mathrm{d}\nu \Big\|^2_{\mathsf{L}_2(X_{v,e})} \Big) \\ &\quad \leq c^2 \tau \max_{e \in E_v} \frac{1}{\ell_e} \Big(\kappa_e + \frac{2}{\kappa_e^2 \lambda_2(\check{X}_v)} \Big) \cdot \|du\|^2_{\mathsf{L}_2(\check{X}_v)},\end{aligned}$$

where we used the Cauchy-Schwarz inequality in the first estimate, (4.3.9) in the second, and (4.3.8) and (4.2.3a) for the last one. Hence, by the above and (4.2.4), we conclude

$$\big|\mathcal{E}_G(f, J'^1 u) - \widetilde{\mathcal{E}}_X(Jf, u)\big|^2 \leq \tau \cdot \frac{4K_\infty}{\ell_0} \cdot \mathcal{E}_G(f) \cdot \mathcal{E}_X(u)$$

and thus the claim follows. □

We are now prepared to prove the quasi-unitary equivalence of the canonical energy form on the weighted discrete graph (G, μ, γ) and the (rescaled) standard energy form on the associated graph-like manifold, stated in Theorem 4.3.1.

PROOF OF THEOREM 4.3.1. By the second equation of (4.2.3a), we have

$$\ell_e = \frac{c^2 \tau \operatorname{vol} Y_e}{\gamma_e} \begin{cases} \le \dfrac{c^2 \tau \operatorname{vol}_\infty}{\gamma_0} =: \ell_\infty < \infty \\ \ge \dfrac{c^2 \tau \operatorname{vol}_0}{\gamma_\infty} =: \ell_0 > 0. \end{cases}$$

Now, collecting all the individual error terms for the identification operators from Proposition 4.3.4, Proposition 4.3.5 and Proposition 4.3.6, we see that

$$\delta^2 \ge \max\Big\{\alpha_\infty^2, \frac{2\mu_\infty}{\gamma_0}, \frac{2}{\tau\lambda_2}, \frac{2}{\alpha_0 \tau \lambda_2}, \frac{4K_\infty}{\ell_0}\Big\}.$$

We can further estimate the three terms in the middle as follows: The term ${}^2/_{\tau\lambda_2}$ appears in δ_b and in δ_c and it can be estimated by

$$\frac{2}{\tau\lambda_2} \le \frac{2\ell_\infty^2}{\tau} = \frac{2c^2 \operatorname{vol}_\infty}{\gamma_0} \cdot \ell_\infty = \frac{2 \operatorname{vol}_\infty}{\operatorname{vol}_0} \cdot \frac{\ell_\infty}{\ell_0} \cdot \frac{d_0 c^2 \ell_0 \operatorname{vol}_0}{d_0 \gamma_0} \le \frac{4 \operatorname{vol}_\infty}{\operatorname{vol}_0} \cdot \frac{\ell_\infty}{\ell_0} \cdot \frac{\mu_\infty}{d_0 \gamma_0}. \quad (4.3.11)$$

In the first step above, we used our assumption (4.3.2) and Lemma 4.2.7. Next, we use the definition of ℓ_∞. The last estimate follows from the first equation in (4.2.3a) because

$$2\mu_\infty \ge 2\mu(v) = c^2 \sum_{e \in E_v} \ell_e \operatorname{vol} Y_e \ge d_0 c^2 \ell_0 \operatorname{vol}_0, \quad (4.3.12)$$

for each $v \in V$. Finally, using the definition of ℓ_0 and ℓ_∞ and because $\alpha_0 \le 1$, the claim follows. □

From the above discussion, in particular from Proposition 4.3.4, we can extract the following lemma, which allows us to compare two different weights on the same discrete graph.

LEMMA 4.3.7. *Let $G = (V, E, \partial)$ be a simple discrete graph. Let $\nu, \widetilde{\nu} \colon V \longrightarrow (0,\infty)$ be two vertex weight functions and moreover, let $\gamma, \widetilde{\gamma} \colon E \longrightarrow (0,\infty)$ be two edge weight functions. Then the energy forms $\mathcal{E}_G$ associated with the edge weight γ in $\ell_2(V,\nu)$ and $\widetilde{\mathcal{E}}_G$ associated with $\widetilde{\gamma}$ in $\ell_2(V,\widetilde{\nu})$ are δ-quasi-unitarily equivalent with*

$$\delta = \max\Big\{\sup_{v \in V} \Xi\Big(\frac{\widetilde{\nu}(v)}{\nu(v)}\Big), \sup_{e \in E} \Xi\Big(\frac{\widetilde{\gamma}_e}{\gamma_e}\Big)\Big\}. \quad (4.3.13)$$

where Ξ is defined in (4.3.5)*.*

PROOF. We choose the identification operators J, J', J^1 and J'^1 to be the corresponding identity operators. Then (2.2.2b) and (2.2.2c) are trivially fulfilled with $\delta_{b/c} = 0$. The two inequalities in (2.2.2a) are satisfied with $\delta_a = \sup_{v \in V} \Xi({}^{\widetilde{\nu}(v)}/_{\nu(v)})$ which follows as in the first part of the proof from Proposition 4.3.4. Let us now check (2.2.2d). Using the same arguments as before, we get

$$\big|\widetilde{\mathcal{E}}_G(Jf, g) - \mathcal{E}_G(f, J'g)\big| = \Big|\sum_{e \in E} (\widetilde{\gamma}_e - \gamma_e)(df)_e(\overline{dg})_e\Big| \le \sup_{e \in E} \Xi\Big(\frac{\widetilde{\gamma}_e}{\gamma_e}\Big) \mathcal{E}_G(f) \widetilde{\mathcal{E}}_G(g),$$

for all $f, g \colon V \longrightarrow \mathbb{C}$. □

We end this section with the following lemma stating that the identification operators J^1 and J'^1, acting on the domains of the energy forms, are also bounded with respect to the corresponding energy norms and the error is also given by δ. This is later needed if we want to use the transitivity Proposition 2.2.3 of the notion of quasi-unitary equivalence.

LEMMA 4.3.8 ([**PS18b**, Proposition 2.12]). *Assume the situation from Theorem 4.3.1. The operators J^1 and J'^1 are bounded with respect to the corresponding energy norms. More precisely, we have*

$$\|J^1 f\|_{\widetilde{\mathcal{E}}_X} \le (1+\delta)\|f\|_{\mathcal{E}_G} \qquad \text{and} \qquad \|J'^1 u\|_{\mathcal{E}_G} \le (1+\delta)\|u\|_{\widetilde{\mathcal{E}}_X},$$

for $f \in \ell_2(V,\mu)$ and $u \in \mathsf{H}^1(X)$, where δ is specified in (4.3.3) and where $\widetilde{\mathcal{E}}_X := \tau\mathcal{E}_X$.

PROOF. Let $f \in \ell_2(V,\mu)$. Since $J^1 f = Jf$, we already proved in Proposition 4.3.4, that

$$\|J^1 f\|^2_{\mathsf{L}_2(X,\nu)} \le (1+\alpha_\infty)\|f\|^2_{\ell_2(V,\mu)},$$

and in energy norm, we use the calculation made earlier in the proof of Proposition 4.3.6; more precisely, we use (4.3.7). In particular, we have

$$\widetilde{\mathcal{E}}_X(J^1 f) = \tau c \sum_{v\in V}\sum_{v'\in V} f(v)\int_X \langle \nabla\varphi_v(x), \nabla J^1 f(x)\rangle_x \,\mathrm{d}\nu(x) = \sum_{v\in V}\gamma_e\big|f(v)-f(v_e)\big|^2,$$

Combining both estimates, the first inequality follows. In particular, the first inequality holds with $\delta = \alpha_\infty$.

Next, let us check the second inequality. Let $u \in \operatorname{dom}\mathcal{E}_X = \mathsf{H}^1(X)$. Then, we can estimate J'^1 in the Hilbert space norm by

$$\begin{aligned}
\|J'^1 u\|^2_{\ell_2(V,\mu)} &\le \Big(1+\frac{1}{\delta_c}\Big)\|J'^1 u - J'u\|^2_{\ell_2(V,\mu)} + (1+\delta_c)\|J'u\|^2_{\ell_2(V,\mu)}\\
&\le \delta_c^2\Big(1+\frac{1}{\delta_c}\Big)\widetilde{\mathcal{E}}_X(u) + (1+\delta_c)\sum_{v\in V}\mu(v)\Big|\frac{1}{c}\cdot\frac{1}{\nu(v)}\langle u,\varphi_v\rangle_{\mathsf{L}_2(X,\nu)}\Big|^2\\
&\le \delta_c^2\Big(1+\frac{1}{\delta_c}\Big)\widetilde{\mathcal{E}}_X(u) + (1+\delta_c)\sum_{v\in V}\frac{\mu(v)}{c^2\nu(v)}\int_X |u|^2\varphi_v\,\mathrm{d}\nu\\
&\le \delta_c(1+\delta_c)\widetilde{\mathcal{E}}_X(u) + (1+\delta_c)\|u\|^2_{\mathsf{L}_2(X,\nu)},
\end{aligned}$$

where we used the Cauchy-Young inequality in the first estimate. Then, we apply Proposition 4.3.5 to estimate the first summand (in the second line). In the third step, we again use the Cauchy-Schwarz inequality — note that this also cancels one factor $\nu(v)$. For the final line we use the partition of unity property of the family $\{\varphi_v\}_{v\in V}$ and $\mu(v)/c^2\nu(v) \le 1$ for $v \in V$. The latter follows from (4.2.3a) and the definition of $\nu(v)$ in (4.2.2), because

$$\frac{1}{c^2} = \frac{1}{2\mu(v)}\sum_{e\in E_v}\ell_e \operatorname{vol} Y_e \le \frac{1}{\mu(v)}\Big(\operatorname{vol}\check{X}_v + \frac{1}{2}\sum_{e\in E_v}\ell_e\operatorname{vol} Y_e\Big) = \frac{\nu(v)}{\mu(v)}$$

which is independent of the vertex $v \in V$. Moreover, we have

$$\begin{aligned}\mathcal{E}_G(J'^1 u) &= \frac{1}{c^2} \cdot \frac{1}{2} \sum_{v \in V} \sum_{e \in E_v} \gamma_e \Big| \frac{1}{\operatorname{vol} \check{X}_v} \int_{\check{X}_v} u \, \mathrm{d}\nu - \frac{1}{\operatorname{vol} \check{X}_{v_e}} \int_{\check{X}_{v_e}} u \, \mathrm{d}\nu \Big|^2 \\ &\le \frac{1}{c^2} \sum_{v \in V} \sum_{e \in E_v} \gamma_e \Big(2 \Big(1 + \frac{1}{\delta_d} \Big) \Big| \frac{1}{\operatorname{vol} \check{X}_v} \int_{\check{X}_v} u \, \mathrm{d}\nu - \frac{1}{\operatorname{vol} Y_e} \int_{\partial_e \check{X}_{v_e}} u \, \mathrm{d}y \Big|^2 \\ &\qquad + \frac{1 + \delta_d}{2} \Big| \frac{1}{\operatorname{vol} Y_e} \int_{\partial_e \check{X}_v} u \, \mathrm{d}y - \frac{1}{\operatorname{vol} Y_e} \int_{\partial_e \check{X}_{v_e}} u \, \mathrm{d}y \Big|^2 \Big) \\ &\le \Big(\delta_d^2 \Big(1 + \frac{1}{\delta_d} \Big) + 1 + \delta_d \Big) \widetilde{\mathcal{E}}_X(u) = (1 + \delta_d)^2 \widetilde{\mathcal{E}}_X(u),\end{aligned}$$

using the Cauchy-Young inequality in the first estimate. In the second line, we use the calculation (4.3.10) from the proof of Proposition 4.3.6 to estimate the first summand. Finally, the estimate on the latter summand follows from

$$\begin{aligned}\frac{1}{c^2} \sum_{e \in E_v} \gamma_e \Big| \frac{1}{\operatorname{vol} Y_e} \int_{\partial_e \check{X}_v} u \, \mathrm{d}y - \frac{1}{\operatorname{vol} Y_e} \int_{\partial_e \check{X}_{v_e}} u \, \mathrm{d}y \Big|^2 &= \sum_{e \in E_v} \frac{\tau}{\ell_e \operatorname{vol} Y_e} \Big| \int_0^{\ell_e} \int_{Y_e} u'(t, y) \, \mathrm{d}t \, \mathrm{d}y \Big|^2 \\ &\le \tau \sum_{e \in E_v} \| u'_e \|^2_{\mathsf{L}_2(X_e)} = \tau \mathcal{E}_{X_v}(u),\end{aligned}$$

where the first equality follows from the fundamental theorem of calculus and (4.2.3a) and the final line by the Cauchy-Schwarz inequality (note that we also use the decomposition property of the energy form (4.2.4)). Combining both estimates, we conclude

$$\| J'^1 u \|^2_{\mathcal{E}_G} \le (1 + \delta_c) \| u \|^2_{\mathsf{L}_2(X, \nu)} + \delta_c (1 + \delta_c) \widetilde{\mathcal{E}}_X(u) + (1 + \delta_d)^2 \widetilde{\mathcal{E}}_X(u) \le (1 + \delta)^2 \| u \|^2_{\widetilde{\mathcal{E}}_X},$$

where $\delta = \max\{\delta_c, \delta_d\}$ and the claim follows. □

4.4. Approximation of energy forms on fractals by rescaled energy forms on graph-like manifolds

In this section we apply Corollary 4.3.3 to the case of the family of discrete weighted graphs $\{(G_m, \mu_m, \gamma_m)\}_{m \in \mathbb{N}_0}$ associated with a subclass of the class of finitely ramified fractal, namely the so-called *post-critically finite self-similar fractals* in the sense of [**Kig01**]. We already discussed two examples — the unit interval and the Sierpiński gasket. The goal of this section is to approximate a symmetric post-critically finite self-similar fractal (defined below in Definition 4.4.2) by a proper family of graph-like manifolds $\{X_m\}_{m \in \mathbb{N}_0}$. Although we already introduced post-critically finite self-similar fractals in Example 3.2.3, we start here by giving a precise definition. For simplicity, we also assume that the fractal is a subset of $\mathbb{R}^d$.

DEFINITION 4.4.1 (Post-critically finite self-similar fractals, [**Kig01**]). *Let K be the self-similar set associated with an iterated function system $\{F_j \colon \mathbb{R}^d \longrightarrow \mathbb{R}^d\}_{j=1,\dots,N}$*

(as defined in Subsection 3.4.2). We call K a post-critically finite self-similar fractal *if there exist a finite set $V_0 \subseteq K$ such that*

$$F_w(K) \cap F_{w'}(K) = F_w(V_0) \cap F_{w'}(V_0)$$

for any two distinct words $w \neq w'$, $w, w' \in W_m := \{1, \dots, N\}^m$ of the same length $m \in \mathbb{N}$. The set V_0 is called the boundary of K.

By the above definition at most finitely many cells can intersect in an arbitrary point of $V_\star \setminus V_0$ — these points where cells intersect are called *junction points*. Hence, the degree $\deg v$ of each vertex v is bounded (independent of the generation m). This is not the case for an arbitrary finitely ramified fractal, e.g. in the case of the Diamond lattice, the degree of a vertex depends on the generation of the graph and is of order $\mathrm{O}(2^m)$. We will not treat the most general situation here (as described in Subsection 3.2.1) – but we comment on this in the next section. This will be a topic for a forthcoming publication.

In order to get a concrete error estimate and formulas for the associated constants, we will assume a certain symmetry. More precisely, we make the following definition.

Definition 4.4.2 (Symmetric fractals). *Let $K \subset \mathbb{R}^d$ be a post-critically finite self-similar fractal with iterated function system $\{F_j\}_{j=1,\dots,N}$, self-similar Hausdorff measure μ and regular self-similar energy form $(\mathcal{E}_K, \operatorname{dom} \mathcal{E}_K)$ (defined in (3.4.3)).*

(i) We say that K is symmetric *if the contraction ratio ϑ is the same for all functions in the iterated function system, i.e. if*

$$\bigl|F_j(x) - F_j(y)\bigr| = \vartheta \bigl|x - y\bigr|, \text{ for all } x, y \in \mathbb{R}^d, j = 1, \dots, N. \tag{4.4.1a}$$

(ii) We say that a regular self-similar energy form $\mathcal{E}_K$ is symmetric *if the (energy) renormalisation parameters $\varrho_1, \dots, \varrho_N$ fulfil $\varrho_j > 1$ for all j and if $\varrho_j = \varrho_i$ is the same for all $i, j \in \{1, \dots, N\}$. Then, we set $\varrho := \varrho_j$ and*

$$\mathcal{E}_K(u, v) = \sum_{j=1}^N \varrho\, \mathcal{E}_K(u \circ F_j, v \circ F_j), \quad \text{for all } u, v \in \operatorname{dom} \mathcal{E}_K. \tag{4.4.1b}$$

(iii) We say that a self-similar measure is symmetric *(or* homogeneous*) if each m-cell has the same measure, i.e., if*

$$\mu(F_w(K)) = \frac{1}{N^m} \cdot \mu(K), \quad \text{for all } w \in W_m \text{ and } m \in \mathbb{N}_0. \tag{4.4.1c}$$

(iv) We say that the boundary V_0 of K is symmetric *if $\mu_0 = \{\mu_0(v)\}_{v \in V_0}$ gives the same mass to all boundary points, i.e., if there is a natural number N_0 such that $\mu_0(v) = 1/N_0$ for all $v \in V_0$ and if there is a constant $C_0 > 0$ such that*

$$\sum_{e_0 \in E_v(G_0)} \frac{1}{\gamma_{0,e_0}} = C_0, \qquad \text{for all } v \in V_0. \tag{4.4.1d}$$

We call a post-critically finite self-similar fractal K symmetric *if all of the four above conditions* (4.4.1a)–(4.4.1d) *hold. Moreover, we call (G_m, μ_m, γ_m) where $G_m := (V_m, E_m)$ the m-*th approximation *of K.*

Let us recall that there is an edge between two distinct vertices v_1 and v_2 in V_m if and only if there is a word w of length m such that v_1 and v_2 are contained in the same cell $F_w(V_0)$, i.e. the edge set of the graph G_m is given by

$$E_m := \left\{ e = (v_1, v_2) \,\middle|\, v_1 \neq v_2 \in V_m \text{ and there exists } w \in W_m \text{ s.th. } v_1, v_2 \in F_w(V_0) \right\}.$$

Let $X_{\varepsilon,m}$ be an ε-scaled graph-like manifold with underlying discrete graph G_m. Recall that the scaling parameter ε_m scales the transversal manifold $Y_{m,e}$ and the vertex manifold $\check{X}_{m,v}$ as

$$Y_{m,e} = \varepsilon_m Y_e \qquad \text{and} \qquad \check{X}_{m,v} = \varepsilon_m \check{X}_v, \tag{4.4.2}$$

for all $v \in V_m$ and $e \in E_m$. Let us determine the parameters of the graph-like manifold exponentially depending on the generation m, namely

$$\ell_{m,e} = \frac{\ell_{0,0}}{\gamma_{e_0,0}} \Lambda^m \qquad \text{and} \qquad \varepsilon_m = \varepsilon_0 \mathrm{E}^m, \qquad \text{where} \qquad 0 < \mathrm{E} < \Lambda < 1, \tag{4.4.3}$$

where $e = F_w(e_0)$ for some word $w \in W_m$ and where $\ell_{0,0} > 0$ and $\varepsilon_0 > 0$ are some constants. We will specify the parameters Λ and E later.

Before we state our next result about the quasi-unitary equivalence of the energy forms on an approximating discrete graph of a symmetric post-critically finite self-similar fractal and the associated graph-like manifold, we first prove the following lemma.

Lemma 4.4.3 ([**PS18b**, Lemma 5.5]). *Assume that K is a symmetric post-critically finite self-similar fractal, that (G_m, μ_m, γ_m) is a corresponding member of the approximating sequence of weighted graphs and that X_m is a graph-like manifold according to G_m with edge length $\{\ell_{m,e}\}_{e \in E_m}$ and transversal manifolds $\{Y_{m,e}\}_{e \in E_m}$ fulfilling* (4.4.2) *and* (4.4.3). *Then the graph-like manifold and the weighted discrete graph (G_m, μ_m, γ_m) are compatible, i.e., there are constants $c_m > 0$ and $\tau_m > 0$ such that both equations in* (4.2.3a) *hold. In particular, the constants are given by*

$$c_m^2 = \frac{2}{\ell_{0,0}\varepsilon_0 C_0 N_0} \cdot \Big(\frac{1}{\mathrm{E}^{d-1} N \Lambda}\Big)^m \qquad \textit{and} \qquad \tau_m = \frac{\ell_{0,0}^2 C_0 N_0}{2} \cdot \big(\varrho N \Lambda^2\big)^m. \tag{4.4.4}$$

Proof. By the definition of the vertex measures $\{\mu_m(v)\}_{v \in V_m}$ and by the symmetry of the post-critically finite self-similar fractal K, we have

$$\begin{aligned} \mu_m(v) &= \int_K \psi_{m,v}(x)\,\mathrm{d}\mu(x) = \sum_{w \in W_{m,v}} \int_{F_w(K)} \psi_{m,v}(x)\,\mathrm{d}\mu(x) \\ &= \sum_{w \in W_{m,v}} \frac{1}{N^m} \int_K \psi_{m,F_w^{-1}(v)}(x)\,\mathrm{d}\mu(x) = \sum_{w \in W_{v,m}} \frac{1}{N_0 N^m} = \frac{|W_{v,m}|}{N_0 N^m}, \end{aligned}$$

where we used $\operatorname{supp}\psi_{m,v} = \cup_{w \in W_{m,v}} F_w(K)$ in the second equality, the symmetry of the measure μ (see (4.4.1c)) in the third equality and the symmetry of the boundary in the fourth one. By our assumptions (4.4.2) and (4.4.3), we have

$$\begin{aligned}\frac{1}{2}\sum_{e\in E_v(G_m)} \ell_{m,e} \operatorname{vol} Y_{m,e} &= \frac{\ell_{0,0}\varepsilon_0}{2}\cdot (\mathrm{E}^{d-1}\Lambda)^m \sum_{w\in W_{v,m}} \sum_{e_0\in E_{F_w^{-1}(v)}(G_0)} \frac{1}{\gamma_{0,e_o}}\\ &= \frac{\ell_{0,0}\varepsilon_0 C_0}{2}\cdot |W_{v,m}|\cdot (\mathrm{E}^{d-1}\Lambda)^m,\end{aligned}$$

using (4.4.1d). Thus, (4.2.3a) is fulfilled with

$$\frac{1}{c_m^2} = \frac{1}{2\mu_m(v)}\sum_{e\in E_v(G_m)} \ell_{m,e}\operatorname{vol} Y_{m,e} = \frac{\ell_{0,0}\varepsilon_0 C_0 N_0}{2}\cdot (\mathrm{E}^{d-1}\Lambda N)^m.$$

Hence, the vertex weights are compatible and we can compute the constant τ_m using (4.2.3a) again and $\gamma_{m,e} = \gamma_{0,F_w^{-1}(e)}\cdot \varrho^m$ (for each $e\in E_m$), i.e.,

$$c_m^2\tau_m = \frac{\gamma_{m,e}\ell_{m,e}}{\operatorname{vol} Y_{m,e}} = \frac{\gamma_{0,e_0}\cdot\varrho^m\cdot \frac{\ell_{0,0}}{\gamma_{0,e_0}}\cdot\Lambda^m}{\varepsilon_0\cdot \mathrm{E}^{(d-1)m}} = \frac{\ell_{0,0}}{\varepsilon_0}\cdot\left(\frac{\varrho\Lambda}{\mathrm{E}^{d-1}}\right)^m,$$

which is independent of the edge $e\in E_m$. Thus, the claim follows. □

Next, we assume that Y_e is isometric with a fixed manifold Y_0 with $\operatorname{vol} Y_0 = 1$, e.g., $Y_0 := [-1/2, 1/2]$ if the dimension of X_m is $d=2$, or Y_0 is a circle of circumference 1 if $d=2$.

Moreover, in our situation of a symmetric post-critically finite self-similar fractal, we only need to work with a finite number of (appropriately scaled) different building blocks $\check{X}_1,\dots,\check{X}_N$ for the core vertex neighbourhoods $\check{X}_v$ — independent of the generation m. In particular, the (ε-scaled) graph-like manifold X_m has uniformly small vertex neighbourhoods: The inequality (4.2.3b) holds because

$$\frac{\operatorname{vol}\check{X}_v}{\ell_\infty\cdot\deg v}\cdot\Lambda^{-m}\le \alpha_m(v)\le \frac{\operatorname{vol}\check{X}_v}{\ell_0\cdot\deg v}\cdot\Lambda^{-m},$$

and the degree $\deg v$ is uniformly bounded for each m. Since the eigenvalues $\lambda_2(Y_e) = \lambda_2(Y_0)$ and $\lambda_2(\check{X}_v)$ only achieve a finite set of numbers for $e\in E_m$ and $v\in V_m$ *independently* of the generation $m\in\mathbb{N}_0$. Thus (4.2.3c) is also fulfilled. In particular, the constant C in Lemma 4.2.8 is independent of m, too.

THEOREM 4.4.4. *Let (G_m,μ_m,γ_m) be the m-th approximation of a symmetric post-critically finite self-similar fractal K. Moreover, let X_m be a (transversally ε_m-scaled) graph-like manifold with underlying discrete graph G_m and with edge length $\{\ell_{m,e}\}_{e\in E_m}$, transversal manifolds $\{Y_{m,e}\}_{e\in E_m}$ and core vertex neighbourhoods $\{\check{X}_{m,v}\}_{v\in V_m}$, fulfilling* (4.4.2) *and* (4.4.3). *Finally, we assume*

$$\frac{1}{\varrho N}\cdot\Lambda < \mathrm{E} < \Lambda.$$

Then the energy form $\mathcal{E}_{G_m}$ in $\ell_2(V_m, \mu_m)$ and the rescaled energy form $\widetilde{\mathcal{E}}_{X_m} := \tau_m \mathcal{E}_{X_m}$ in $\mathsf{L}_2(X_m, \nu_m)$ are δ_m-quasi-unitarily equivalent with

$$\delta_m = \max\Big\{\mathrm{O}\Big(\Big(\frac{\mathrm{E}}{\Lambda}\Big)^{m/2}\Big), \mathrm{O}\Big(\Big(\frac{\Lambda}{\mathrm{E}} \cdot \frac{1}{\varrho N}\Big)^{m/2}\Big)\Big\} \tag{4.4.5}$$

and with isometric and energy rescaling factors given in (4.4.4)*. In particular, if we choose* $\mathrm{E} = (1/\varrho N)^{1/2}\Lambda$ *the error is of order* $\delta_m = \mathrm{O}((1/\varrho N)^{m/4})$ *which is the best possible choice.*

Remark 4.4.5. *Note that* $\mathrm{E} = (1/\varrho N)^{1/2}\Lambda$ *is the number where both terms agree: If we choose* $\mathrm{E} \le (1/\varrho N)^{1/2}\Lambda$*, then*

$$\Big(\frac{E}{\Lambda}\Big)^{m/2} \le \Big(\frac{\Lambda}{(\varrho N)^{1/2}} \cdot \frac{1}{\Lambda}\Big)^{m/2} = \frac{1}{(\varrho N)^{m/4}}$$

and

$$\Big(\frac{\Lambda}{E} \cdot \frac{1}{\varrho N}\Big)^{m/2} \ge \Big(\Lambda \cdot \frac{(\varrho N)^{1/2}}{\Lambda} \cdot \frac{1}{\varrho N}\Big)^{m/2} = \frac{1}{(\varrho N)^{m/4}}.$$

Hence, in this situation the second term in (4.4.5) *wins. If, on the other hand, we choose* $\mathrm{E} \ge (1/\varrho N)^{1/2}\Lambda$*, then the inequalities just switch and the first term determines the error* δ_m*. In particular, we come to the following conclusion: If* $\Lambda/\varrho N < \mathrm{E} \le \Lambda/(\varrho N)^{1/2}$ *then* $\delta_m = \mathrm{O}((\Lambda/\mathrm{E} \cdot 1/\varrho N)^{m/2})$ *and else, if* $\Lambda/(\varrho N)^{1/2} \le \mathrm{E} < \Lambda$*, then* $\delta_m = \mathrm{O}((\mathrm{E}/\Lambda)^{m/2})$*.*

Proof of Theorem 4.4.4. The compatibility of the weighted discrete graph (G_m, μ_m, γ_m) and the graph-like manifold follows from Lemma 4.4.3. Thus, the assumptions in Corollary 4.3.3 are fulfilled and

$$\delta_m^2 = \max\Big\{\varepsilon_m^2\alpha_{m,\infty}^2, \frac{4}{\varepsilon_m\alpha_{m,0} \cdot d_{m,0}} \cdot \frac{\gamma_{m,\infty}}{\gamma_{m,0}} \cdot \frac{\mu_{m,\infty}}{\gamma_{m,0}}, \frac{4\varepsilon_m K_{m,\infty}}{\ell_{m,0}}\Big\}.$$

Note first that the factors $\mathrm{vol}_\infty/\mathrm{vol}_0 = 1$ and $\gamma_\infty/\gamma_0 = 1$ are independent of the generation m. Moreover, note that the minimal degree $d_{m,0}$ and $K_{m,\infty}$ are also independent of m in our situation here, i.e., they are constant once $m \ge 1$. Hence, we can precisely give the error by

$$\delta_m^2 = \max\Big\{\Big(\frac{\varepsilon_0 \operatorname{vol}_\infty \check{X}_v}{\ell_0 d_0}\Big)^2\Big(\frac{\mathrm{E}}{\Lambda}\Big)^{2m}, \frac{4\ell_\infty d_\infty}{\varepsilon_0\gamma_0 d_0 \operatorname{vol}_0 \check{X}_v}\Big(\frac{\Lambda}{\varrho N \mathrm{E}}\Big)^m, \frac{4\varepsilon_0 K_\infty}{\ell_0}\Big(\frac{\mathrm{E}}{\Lambda}\Big)^m\Big\}, \tag{4.4.6}$$

where $\operatorname{vol}_0 \check{X}_v := \min_{v \in V_m} \operatorname{vol} \check{X}_v$, $\operatorname{vol}_\infty \check{X}_v := \max_{v \in V_m} \operatorname{vol} \check{X}_v$, $\ell_0 := \min_{e \in E_0} \ell_{0,0}/\gamma_{e_0,0}$ and $\ell_\infty := \max_{e \in E_0} \ell_{0,0}/\gamma_{e_0,0}$. Thus, the error is of the claimed order. □

Remark 4.4.6 ([**PS18b**, Remark 5.12]). *We would like to stress here, that* $\Lambda = \mathrm{E}$ *is not admissible in the situation above. In our situation, where we assume a self-similar fractal, i.e., the existence of an iterated function system, this would allow us to apply the IFS directly to a suitable (compact) graph-like manifold* X_0 *associated with the discrete graph* $G_0 = (V_0, E_0)$ *with shrinking parameters* $\Lambda = \mathrm{E} = \vartheta$ *(where the latter is the contraction ratio of the IFS). In* [**PS18b**, Remark 5.12]*, the authors conjectured that the above mentioned choice should be viable. However, the problem* *is not only the compatibility of the weights μ_m and ν_m but also the δ-closeness of the associated energy forms (see Proposition 4.3.6).*

COROLLARY 4.4.7 ([**PS18b**, Corollary 5.13]). *Let K be a symmetric post-critically finite self-similar fractal. Moreover, let X_m be a graph-like manifold associated with the m-th approximation (G_m, μ_m, γ_m) and rescaled energy form $\widetilde{\mathcal{E}}_{X_m} = \tau_m \mathcal{E}_{X_m}$ as in Theorem 4.4.4. Then $\widetilde{\mathcal{E}}_{X_m}$ and $\mathcal{E}_K$ are δ_m-quasi-unitarily equivalent where the error is of the same order as in* (4.4.5)*. In particular, $\mathcal{E}_{X_m}$ converges to $\mathcal{E}_K$ in the sense of Definition 2.2.1.*

The proof is again a direct consequence of the transitivity of the notion of quasi-unitary equivalence Proposition 2.2.3 and our previous results. Also, note that by the symmetry of the finitely ramified fractal, we have $\mu_{+,m} = \mu(K)/N^m$ and $\gamma_{-,m} = \varrho^m$. Since $\mathrm{E} < \Lambda$ by our assumption made in (4.4.3) and N_m is a constant, the error in (3.3.2) is already covered by the the latter term in (4.4.5).

EXAMPLE 4.4.8 (Unit interval). *Let us first discuss our easiest example $K = [0, 1]$ with self-similar measure μ (which is the Lebesgue measure μ restricted onto K). Obviously, K is symmetric in the sense of Definition 4.4.2: Both functions of the iterated function system have the same contraction ratio $\vartheta = 1/2$. The canonical energy form (defined in* (3.2.5)*) is self-similar with energy renormalisation factor $\varrho = 2$ and discrete approximation* (3.2.4)*. Moreover, we easily check that the measure is symmetric and $N = 2$. Finally, $\mu_0(0) = \mu_0(1) = 1/2$, i.e., $N_0 = 2$ and since $|E_0| = 1$, we have $C_0 = 1$.*

Let us now define a suitable graph-like manifold X_m of dimension $d = 2$. We need to choose a length scaling Λ and an edge scaling E. It is convenient to set $\Lambda = 1/2$ since this is the right *choice from a geometrical point of view. In order to get the best error possible, we fix $\mathrm{E} = 1/2\sqrt{4}$ and hence, the energy form $\widetilde{\mathcal{E}}_{X_m} := \tau_m \mathcal{E}_{X_m}$ on the associated (ε_m-scaled) graph-like manifold X_m and the discrete energy form $\mathcal{E}_{G_m}$ on the weighted discrete graph (G_m, μ_m, γ_m) are δ_m-quasi-unitarily equivalent with*

$$\delta_m = \mathrm{O}\Big(\Big(\frac{1}{4}\Big)^{m/4}\Big), \qquad c_m = \big(2\sqrt[3]{4}\big)^{m/2} = 32^{m/6} \qquad \text{and} \qquad \tau_m = 1$$

using (4.4.5) *and* (4.4.4) *with dimension $d = 2$, $\ell_{0,0} = 1$, $\varepsilon_0 = 1$, $C_0 = 1$ and $N_0 = 2$. Moreover, the energy form $\widetilde{\mathcal{E}}_{X_m} = \mathcal{E}_{X_m}$ on the family of graph-like manifolds X_m converges to the energy form $\mathcal{E}_K$ on the interval K in the sense of Definition 2.2.1 with convergence speed $\delta_m = \mathrm{O}(4^{-m/4})$.*

EXAMPLE 4.4.9 (Sierpiński gasket, [**PS18b**, Example 5.11]). *We easily verify, that the Sierpiński gasket is also a symmetric fractal. The relevant parameters are $N = 3$ and $\varrho = 5/3 > 1$. Moreover, $\mu_0(v) = 1/3$ for all $v \in V_0$, hence, $N_0 = 3$ and since $|E_0| = 3$, we have $C_0 = 3$.*

A natural choice for the length scaling factor is again the contraction ratio of the IFS, i.e., $\Lambda = 1/2$. Since the transversal manifold has to shrink faster, we set $\mathrm{E} = 1/2\sqrt{5}$. Again, this is the choice that leads to the best error δ_m — as stated in Theorem 4.4.4, other choices are also possible.

By Theorem 4.4.4, the energy form $\mathcal{E}_{G_m}$ on the weighted discrete graph (G_m, μ_m, γ_m) associated with the Sierpiński gasket K and the rescaled energy form $\widetilde{\mathcal{E}}_{X_m} := \tau_m \mathcal{E}_{X_m}$ on the (ε_m-scaled) graph-like manifold X_m are δ_m-quasi-unitarily equivalent with

$$\delta_m = \mathrm{O}\Big(\Big(\frac{1}{5}\Big)^{m/4}\Big), \qquad c_m^2 = \frac{2}{9} \cdot \Big(\frac{4\sqrt[3]{5}}{3}\Big)^m, \qquad \textit{and} \qquad \tau_m = \frac{9}{2} \cdot \Big(\frac{5}{4}\Big)^m,$$

where we used (4.4.4) *again to calculate the latter expressions and where $d = 2$, $\ell_{0,0} = 1$, $\varepsilon_0 = 1$, $C_0 = 3$ and $N_0 = 3$. Moreover, the energy form $\widetilde{\mathcal{E}}_{X_m} = \tau_m \mathcal{E}_{X_m}$ converges to $\mathcal{E}_K$ in the sense of Definition 2.2.1 with convergence speed of order $\delta_m = \mathrm{O}(5^{-m/4})$.*

4.5. Outlook: Where do we go from here?

In the previous Section 4.4, we applied our main results from this chapter, Theorem 4.3.1 and Corollary 4.3.3 to a very specific subclass of finitely ramified fractals, namely, symmetric post-critically finite self-similar fractals (see Definition 4.4.1 and Definition 4.4.2). The symmetry of the fractal seems not to be required and is mainly assumed for the purpose of getting a concrete error term and simplifying our proofs. But does our approach really rely on the self-similarity or the pcf property? Also, in the case of an approximation by graph-like manifolds, we do not allow a perturbation by a magnetic potential as we do in Chapter 3. Is there a solution, too? Let us end this chapter by commenting briefly on these questions.

4.5.1. Approximating magnetic energy forms on graph-like manifolds. Our guess, how to treat this case is to employ or modify the ideas, introduced in [**EP13**]. In particular, we are interested in a version of Proposition 3.1 and Proposition 3.2, from the aforementioned reference. Roughly speaking, in the first proposition, the authors showed that a magnetic energy form $\mathcal{E}^a$ is relatively form bounded w.r.t. the free form $\mathcal{E}$ with relative bound zero. More precisely, this means that for an arbitrary number $\eta > 0$, there is a constant C_η, depending only on η, the dimension d of the manifold, and the magnetic potential a, such that

$$|\mathcal{E}^a(u) - \mathcal{E}(u)| \le \eta \mathcal{E}(u) + C_\eta \|u\|^2,$$

for all $u \in \operatorname{dom} \mathcal{E} = \operatorname{dom} \mathcal{E}^a$. In particular, we can conclude from the above inequality, that $\mathcal{E}^a$ is also closed and, moreover, we have

$$\mathcal{E}(u) \le 2\big(\mathcal{E}^a(u) + C_{1/2}\|u\|^2\big),$$

for all $u \in \operatorname{dom} \mathcal{E}$. The authors then allowed a shrinking parameter $\varepsilon > 0$ (similar as we did in the previous sections of this chapter) in Proposition 3.2 and prove a similar result but w.r.t. a shrinking family of manifolds X_ε. We conjecture that we can conclude our desired result, applying similar techniques as in the cited article, together with our results and the transitivity of the notion of quasi-unitary equivalence. More precisely, we conjecture the following:

Conjecture 4.5.1. *Let $a \in \mathcal{H}$ be a real-valued magnetic potential associated with the energy form $(\mathcal{E}, \operatorname{dom} \mathcal{E})$ on a symmetric post-critically finite self-similar fractal K. Then the energy forms $\mathcal{E}^a$ in $\mathsf{L}_2(K, \mu)$ and $\widetilde{\mathcal{E}}^a$ in $\mathsf{L}_2(X_m, \nu_m)$ are δ'_m-quasi-unitarily* *equivalent. The order of the error δ'_m depends on $\hat{\delta}_m$ (see Theorem 3.3.9) and δ_m (see Theorem 4.4.4), respectively.*

Moreover, we are optimistic, that the above result also holds true in the case of certain finitely ramified fractals such as the Diamond lattice fractal (see also the next section below). However, since general finitely ramified fractals can have a very complicated structure (which will result in many variables one has to control in order to get a shrinking error term – see also the next section) we suggest to work with concrete examples first. Here, in our opinion the Diamond lattice fractal is a natural choice, since it is symmetric and behaves very nice, but with the only real constraint, that in each junction point infinitely many cells intersect. This is a topic for a future publication.

4.5.2. Approximating finitely ramified fractals by graph-like manifolds. One also might wonder, why we did not apply Corollary 4.3.3 to compare a finitely ramified fractal with its associated sequence of approximating manifolds. The answer lies in the nature of the spaces: In the case of a post-critically finite self-similar fractal, the degree of the vertices is uniformly bounded, more precisely, it is independent of the generation m. In contrast, the degrees of the vertices in the case of an arbitrary finitely ramified fractal are not necessarily bounded. We saw in the example of the Diamond lattice fractal (introduced in Subsection 3.4.3), that the maximal degree in generation m is $\deg_{m,\infty} = 2^m$ (see Figure 1: In each step, any edge is replaced by four edges forming a parallelogram and two new vertices are added). The degree of the existing vertices is doubled and the new vertices have degree two, which is also going to be doubled in the next iteration step).

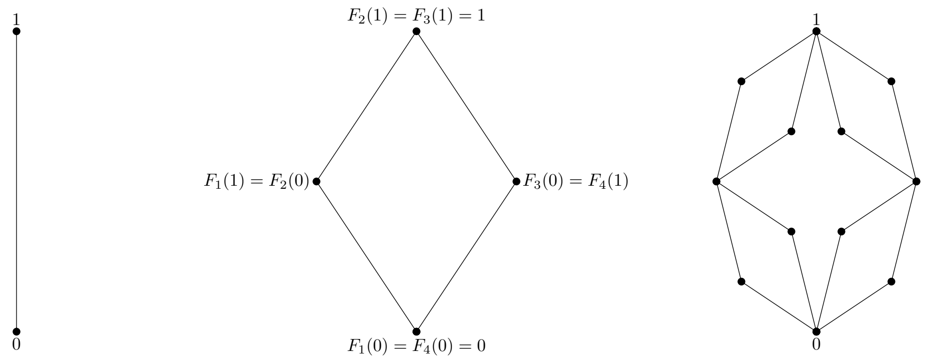

FIGURE 1. The first three discrete graphs G_m ($m = 0, 1, 2$) of the approximating sequence for the diamond lattice fractal. The figure in the middle illustrates the cell structure: At any iteration step, any edge is split up into four new edges of the same length, two new vertices are added and the pieces are arranged in such a way that they form a parallelogramm.

This becomes a problem, because we require *uniformly small (core) vertex neighbourhoods* in Theorem 4.3.1 and in Corollary 4.3.3. Thus, we need to discuss the two estimates (4.2.3b) and (4.2.3c) again.

Assumption 4.5.2. *We assume that* $d \coloneqq \dim X = 2$.

In the first inequality, we defined $\alpha_{m,0}$ and $\alpha_{m,\infty}$ as the minimum and maximum of $\alpha_m(v)$, respectively, where

$$\alpha_m(v) \coloneqq \frac{2 \operatorname{vol} \check{X}_{m,v}}{\sum_{e \in E_{m,v}} \operatorname{vol} X_{m,e}}, \tag{4.5.1}$$

for all $v \in V_m$. The core vertex neighbourhood $\check{X}_{m,v}$ of an arbitrary vertex $v \in V_m$ behaves roughly like a regular polygon with $n = \deg v$ many sides of the same length $a = 1$. Thus, the volume $\operatorname{vol} \check{X}_{m,v}$ should behave like the area of a regular polygon,

$$\operatorname{Area}(n) = \frac{na^2}{4} \cdot \cot \frac{\pi}{n}.$$

Since we are many interested in the corner cases $\deg v \in \{2, 2^m\}$, the unscaled vertex core neighbourhood $\check{X}_{v,m}$ for such a vertex in the expression $\alpha_m(v)$ roughly scales as

$$\operatorname{vol} \check{X}_{v,m} = \mathrm{O}\big(1\big) \tag{4.5.2}$$

if $v \in V_m$ with $\deg v = 2$ and else

$$\operatorname{vol} \check{X}_{v,m} = \mathrm{O}\big((\deg v)^2\big), \tag{4.5.3}$$

at least for large degrees, because $\cot z = {}^{\cos z}/_{\sin z}$ and $\cos z \approx 1$ and $\sin z \approx z$ for $0 < z \ll 1$.

Let us use the above to estimate $\alpha_{m,0}$ and $\alpha_{m,\infty}$. In order to estimate the denominator in (4.5.1), we use that all the edges $e \in E_m$ have the same length $\ell_{m,0} = \ell_0 \Lambda^m$ and that $\operatorname{vol} Y_{m,0} = 1$ (for all $e \in E_m$). Thus

$$\alpha_m(v) \coloneqq \frac{2 \operatorname{vol} \check{X}_{m,v}}{\sum_{e \in E_{m,v}} \operatorname{vol} X_{m,e}} = \frac{2 \operatorname{vol} \check{X}_{m,v}}{\deg v \cdot \ell_{m,0} \cdot \operatorname{vol} Y_{m,0}} = \frac{2}{\ell_0} \cdot \frac{1}{\deg v} \cdot \frac{\operatorname{vol} \check{X}_{m,v}}{\Lambda^m} \approx \frac{\deg v}{\Lambda^m}.$$

Since $\deg v \in \{\, 2^k \mid k = 1, \dots, m \,\}$ (whenever $v \in V_m$) we find an upper bound $\alpha_{m,\infty}$ by choosing a vertex $v \in V_m$ with maximal degree $\deg v = 2^m$. In this case, the core vertex neighbourhood scales as in (4.5.3) and we get

$$\alpha_{m,\infty} = \max_{v \in V_m} \alpha_m(v) = \frac{2}{\ell_0} \cdot \Big(\max_{v \in V_m} \frac{1}{\deg v} \cdot \frac{\operatorname{vol} \check{X}_{m,v}}{\Lambda^m} \Big) = \mathrm{O}\Big(\Big(\frac{2}{\Lambda} \Big)^m \Big). \tag{4.5.4}$$

On the other hand, the minimum $\alpha_{m,0}$ is achieved, when we choose a vertex $v \in V_m$ with minimal degree $\deg v = 2$. The core vertex neighbourhood then scales as a polygon with $n = 5$ sides of length $a = 1$ and hence,

$$\alpha_{m,0} = \min_{v \in V_m} \alpha_m(v) = \frac{2}{\ell_0} \cdot \Big(\min_{v \in V_m} \frac{1}{\deg v} \cdot \frac{\operatorname{vol} \check{X}_{m,v}}{\Lambda^m} \Big) = \mathrm{O}\Big(\Big(\frac{1}{\Lambda} \Big)^m \Big). \tag{4.5.5}$$

What does this mean for the error in (4.3.4)? The first term satisfies

$$\varepsilon_m^2 \alpha_{m,\infty}^2 = \mathrm{O}\Big(\Big(\frac{2\mathrm{E}}{\Lambda} \Big)^{2m} \Big)$$

and does not cause any problems if we assume

$$0 < \mathrm{E} < \frac{1}{2} \cdot \Lambda.$$

For the second term, we first note that in our particular example, where all the edges have the same lengths $\ell_{m,0}$, we can do better by replacing the minimal degree $d_{m,0} := \min_{v \in V_m} \deg v = 2$ in the estimate (4.3.4) by the maximal degree

$$d_{m,\infty} := \max_{v \in V_m} \deg v = 2^m.$$

This step is admissible, because we actually have the following equality (instead of the estimate used in (4.3.12)):

$$2\mu_{m,\infty} = 2 \max_{v \in V_m} \mu_m(v) = c_m^2 \cdot \max_{v \in V_m} \sum_{e \in E_{m,v}} \ell_{m,e} \operatorname{vol} Y_{m,e} = c_m^2 d_{m,\infty} \varepsilon_0 \ell_0 (\Lambda \mathrm{E})^m.$$

By (4.5.5), we get

$$\frac{4}{\varepsilon_m \alpha_{m,0} \cdot d_{m,\infty}} \Big(\frac{\mathrm{vol}_{m,\infty}}{\mathrm{vol}_{m,0}} \Big)^2 \frac{\gamma_{m,\infty}}{\gamma_{m,0}} \cdot \frac{\mu_{m,\infty}}{\gamma_{m,0}} = \frac{1}{\varepsilon_m \alpha_{m,0} \cdot d_{m,\infty}} \cdot \mu_{m,\infty} = \mathrm{O}\Big(\Big(\frac{\Lambda}{2\mathrm{E}} \Big)^m \mu_{m,\infty} \Big),$$

where we used that $\gamma_m = 1$ (for all $m \geq 0$) and where we assumed that $\mathrm{vol}_0 = \mathrm{vol}_\infty$. Moreover, we can estimate $\mu_{m,\infty}$ easily by applying the same arguments as in the proof of Lemma 4.4.3. We get

$$\mu_m(v) = \int_K \psi_{v,m}(x) \, \mathrm{d}\mu(x) = \frac{1}{2} \cdot \frac{1}{4^m} \cdot \deg v \leq \frac{1}{2^{m+1}}, \tag{4.5.6}$$

because $\deg v \leq 2^m$ for all $v \in V_m$. Now we are prepared to estimate the second term in (4.3.4). Combining the above estimates and equations, we get

$$\frac{4}{\varepsilon_m \alpha_{m,0} \cdot d_{m,\infty}} \Big(\frac{\mathrm{vol}_{m,\infty}}{\mathrm{vol}_{m,0}} \Big)^2 \frac{\gamma_{m,\infty}}{\gamma_{m,0}} \cdot \frac{\mu_{m,\infty}}{\gamma_{m,0}} = \mathrm{O}\Big(\Big(\frac{\Lambda}{2\mathrm{E}} \Big)^m \cdot \frac{1}{2^m} \Big) = \mathrm{O}\Big(\Big(\frac{\Lambda}{4\mathrm{E}} \Big)^m \Big).$$

Hence, we get a lower bound for E (depending on our choice for Λ):

$$\frac{1}{4} \cdot \Lambda < \mathrm{E} \Big(< \frac{1}{2} \cdot \Lambda \Big).$$

In the second equation (4.2.3c), we defined

$$K_{m,\infty} := \sup_{v \in V_m} \max_{e \in E_{m,v}} \Big(\kappa_{m,e} + \frac{2}{\kappa_{m,e} \lambda_2(\check{X}_{m,v})} \Big)$$

and required $K_{m,\infty} < \infty$. Thus, the second (first non-zero) Neumann eigenvalue $\lambda_2(\check{X}_{m,v})$ on $\check{X}_{m,v}$ enters into the error δ_m in Corollary 4.3.3. In our example of the Diamond lattice fractal (which we consider a simple example for a non-pcf self-similar fractal), this also becomes a problem: As discussed above, the core vertex neighbourhood $\check{X}_{m,v}$ of a vertex $v \in V_m$ behaves as a polygon with $\deg v \in \{ 2^k \mid k = 1, \dots, m \}$ many sides. This means, that we cannot treat the eigenvalue $\lambda_2(\check{X}_{m,v})$ as a constant number anymore. In fact the number of different building blocks and corresponding eigenvalues depends on the generation $m \geq 0$. In order to ensure, that the error δ_m (defined in (4.3.4)) decreases as $m \to \infty$, we need to specify suitable scaling parameters and ensure, that the other calculations still hold true for that scaling.

We conjecture the following:

CONJECTURE 4.5.3. *For an arbitrary vertex $v \in V_m$, the second (first non-zero) Neumann eigenvalue $\lambda_2(\check{X}_{m,v}) > 0$ on the core vertex neighbourhood $\check{X}_{m,v}$ scales as the eigenvalue on a regular polygon with* $\deg v \in \{\, 2^k \mid k = 1, \dots, m \,\}$ *many sides and for large generation $m \gg 0$ the eigenvalues $\lambda_2(\check{X}_{m,v}) > 0$ are of order* $\mathrm{O}(1/4^m)$*, i.e. as the ones of a circle with radius $r_m = 2^m$.*

By the above conjecture, we conclude that $K_{m,\infty} = \mathrm{O}(4^m)$ and hence, the missing term is of order

$$\frac{4\varepsilon_m K_{m,\infty}}{\ell_{m,e}} = \frac{4\varepsilon_0}{\ell_0} \cdot \Big(\frac{\mathrm{E}}{\Lambda}\Big)^m \cdot K_{m,\infty} = \mathrm{O}\Big(\Big(\frac{4\mathrm{E}}{\Lambda}\Big)^m\Big).$$

In order to get a decreasing sequence $\{\delta_m\}_{m\in\mathbb{N}_0}$, we would need to require

$$\frac{1}{4} \cdot \Lambda < \mathrm{E} < \frac{1}{4} \cdot \Lambda$$

and the above does not have a solution $\mathrm{E}, \Lambda > 0$.

However, we hope that it is still possible to find a converging sequence of approximating graph-like manifolds by re-thinking the estimates and trying to improve some of them with this concrete example in mind. Also we would like to emphasis that in this section, our approximations are very rough and a more careful approach might already result in a solvable condition. If that is the case, we should treat the above topic in a forthcoming publication.

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