# Convex Duality in Consumption-Portfolio Choice Problems with Epstein-Zin Recursive Preferences

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*"You acted unwisely," I cried, "as you see*
*By the outcome." He calmly eyed me:*
*"When choosing the course of my action," said he,*
*"I had not the outcome to guide me."*

— Ambrose Bierce, *A Lacking Factor*

# Abstract

This thesis deals with consumption-investment allocation problems with Epstein-Zin recursive utility, building upon the dualization procedure introduced by [Matoussi and Xing, 2018]. While their work exclusively focuses on truly recursive utility, we extend their procedure to include time-additive utility using results from general convex analysis. The dual problem is expressed in terms of a backward stochastic differential equation (BSDE), for which existence and uniqueness results are established. In this regard, we close a gap left open in previous works, by extending results restricted to specific subsets of parameters to cover all parameter constellations within our duality setting.

Using duality theory, we analyze the utility loss of an investor with recursive preferences, that is, her difference in utility between acting suboptimally in a given market, compared to her best possible (optimal) consumption-investment behaviour. In particular, we derive universal power utility bounds, presenting a novel and tractable approximation of the investors' optimal utility and her welfare loss associated to specific investment-consumption choices. To address quantitative shortcomings of those power utility bounds, we additionally introduce one-sided variational bounds that offer a more effective approximation for recursive utilities. The theoretical value of our power utility bounds is demonstrated through their application in a new existence and uniqueness result for the BSDE characterizing the dual problem.

Moreover, we propose two approximation approaches for consumption-investment optimization problems with Epstein-Zin recursive preferences. The first approach directly formalizes the classical concept of least favorable completion, providing an analytic approximation fully characterized by a system of ordinary differential equations. In the special case of power utility, this approach can be interpreted as a variation of the well-known Campbell-Shiller approximation, improving some of its qualitative shortcomings with respect to state dependence of the resulting approximate strategies. The second approach introduces a PDE-iteration scheme, by reinterpreting artificial completion as a dynamic game, where the investor and a dual opponent interact until reaching an equilibrium that corresponds to an approximate solution of the investors optimization problem. Despite the need for additional approximations within each iteration, this scheme is shown to be quantitatively and qualitatively accurate. Moreover, it is capable of approximating high dimensional optimization problems, essentially avoiding the curse of dimensionality and providing analytical results.

# Contents

# Chapter 1

# Introduction

Portfolio optimization plays a central role in finance. Since the pioneering works of [Markowitz, 1952], [Samuelson, 1969] and [Merton, 1971], a large volume of academic work focussed on the problem of maximizing expected time-additive utility of [Von Neumann and Morgenstern, 1944] type. Among those time-additive utilities, power utility functions with constant relative risk aversion (CRRA) are the most common, and solutions to the associated portfolio optimization problems have been found in several settings featuring complete and incomplete markets, see e.g. [Brandt, 2010] or [Wachter, 2010] for a broad review. Despite its strong presence in the literature, time-additive utility has its limitations. In particular, it imposes a strict relation between the investors attitude towards the smoothness of consumption over time and over states. This restriction in the connection between an investors risk aversion (RA) and her elasticity intertemporal substitution (EIS) becomes particularly visible in the field of equilibium asset pricing, where it produces various inconsistencies between model predictions and empirical data. Those inconsistencies lead to a rich literature on so called *asset pricing puzzles*, see e.g. [Mehra and Prescott, 1985] for the prominent *equity premium puzzle*: The excess return of stocks implied by classical asset pricing models is considerably too high for reasonable market parameters.

In order to bypass these limitations of time-additive utility, recursive utility has been developed in a discrete-time framework by [Kreps and Porteus, 1978], [Epstein and Zin, 1989], [Weil, 1990]. It dissolves the strict connection between risk aversion and elasticity of intertemporal substitution, allowing for more flexibility in the modelling of an investors preferences. [Bansal and Yaron, 2004] found that long-run risk asset pricing models featuring recursive utility fit more accurately with financial data then models based on time-additive preferences; thus recursive utility became a highly relevant tool in the asset pricing literature, see for instance [Hansen et al., 2008], [Guvenen, 2009], [Kaltenbrunner and Lochstoer, 2010], [Borovička et al., 2011], [Gabaix, 2012], [Wachter, 2013]. Almost all of those publications use the so called Epstein-Zin-Weil parametrization. Stochastic differential utility, as a continuous time analogue to recursive utility, was in a deterministic setting proposed by [Epstein, 1987] and in a stochastic setting by [Duffie and Epstein, 1992b]. The authors provide convincing arguments for the connection between recursive utility and stochastic differential utility, however their definition is axiomatic and a mathematically rigorous link between the two concepts has only much later been established by [Kraft and Seifried, 2014]. For the widely used Epstein-Zin specification of stochastic differential utility, the associated continuous-time optimal consumption and investment problems have in particular been studied in [Schroder and Skiadas, 1999], [Schroder and Skiadas, 2003], [Chacko and Viceira, 2005], [Kraft et al., 2013], [Seiferling and Seifried, 2016], [Xing, 2017], [Matoussi and Xing, 2018].

In general, there are two main approaches to obtain a solution of a consumption-investment

optimization problem. The first approach uses dynamic programming techniques, to reduce the problem to finding a solution of a certain partial differential equation (PDE), the Hamilton-Jacobi-Bellman (HJB) equation. The solution to this HJB equation characterizes not only the investors optimal utility, but also the associated investment and consumption strategy, see for example [Liu and Muhle-Karbe, 2013] for a general overview in the time-additive case or [Chacko and Viceira, 2005], [Kraft et al., 2013], [Seiferling and Seifried, 2016] in case of recursive utility. The second approach is often called the martingale approach and was first introduced by [Cox and Huang, 1989], [Karatzas et al., 1987] and [Pliska, 1986] in a time-additive framework and later also employed in problems featuring recursive preferences by [Schroder and Skiadas, 1999] and [Schroder and Skiadas, 2003]. Informally speaking, it is based on a separation of the dynamic optimization problem to a static problem and a representation problem: First, using the Lagrangian method, one determines the optimal payoff profile, then one computes the realizing strategies using martingale representation results. Thus, the optimal strategies generally have an abstract representation that can only in special cases be computed explicitly. However, the big advantage of the martingale approach over the dynamic programming approach is, that it can much easier be extended to various types of incomplete markets, such as trading constraints or undiversifiability of risk. While incompleteness of the considered market usually leads to an unsolvable HJB equation, it appears natural within the martingale method to embed the constrained optimization problem into a family of unconstrained ones, and then finding a member of this family whose optimal strategy obeys the constraints. In the case of time-additive utility, this *dual approach* was first developed in [He and Pearson, 1991], [Karatzas et al., 1991] and [Cvitanić and Karatzas, 1992].
According to [Karatzas et al., 1991], the particular member of the family of unconstrained problems, that obeys the constraints of the incomplete market, can be interpreted as the *least favorable completion* of the market. The optimization problem within this least favorably completed market is then equivalent to the initially constrained optimization problem. Finding this least favorable completion, corresponds to solving the associated *dual optimization problem.*
This idea of a dual approach had significant impact on the theory of consumption-investment optimization theory with time-additive utility. Besides far reaching theoretical implications, the concept of least favorable completion is particularly fruitful when it comes to the approximation of solutions, as the associated complete market problems are often much easier to solve. Moreover, the dual optimization problem yields an upper bound on the optimal utility. In particular, the dual approach automatically implies an upper bound on the utility loss associated to any particular consumption-investment strategy, that can be utilized to validate the accuracy of any given (numerical) approximation of the solution, without the need of a benchmark approach. This idea has prominently been employed by [Haugh et al., 2006], [Brown et al., 2010], [Brown and Smith, 2011], [Bick et al., 2013], [Kamma et al., 2020] and [Kamma and Pelsser, 2022].

Inspired by the dual approach for time-additive utility, [Matoussi and Xing, 2018], propose a dual formulation for optimization problems with Epstein-Zin preferences if the RRA ($\gamma$) and EIS ($\psi$) satisfy the restriction

$$\gamma\psi > 1,\, \psi > 1 \qquad \text{or} \qquad \gamma\psi < 1,\, \psi < 1. \tag{$\dagger$}$$

Their dual problem is characterized by the solution of a BSDE, which they call the *stochastic differential dual.* The economic interpretation of time-additive duality theory carries over to the recursive case, in particular the solution of the dual problem can be interpreted as the least favorable completion of the underlying market. [Matoussi and Xing, 2018] provide existence and uniqueness results for the stochastic differential dual, and find conditions under which the primal and dual solution actually coincide in certain Brownian models. In general, the paper of [Matoussi and Xing, 2018] provides great insights on optimization problems with Epstein-Zin utility in incomplete markets and paves the way for new methods to approach their solution. For example [Becherer et al., 2023] use the dual approach to solve a recursive optimization problem for an investor, who receives a stochastic stream of income and is faced with liquidity constraints.

Other aspects of duality theory, such as (numerical) approximations of the solution via least favorable completion, or the implied bound on the welfare loss linked to such approximations, have not been applied within the context of recursive Epstein-Zin utility thus far. Hence, the main goal of this thesis is to generalize these concepts in that regard. In pursuit of that objective, we enhance the existing theory in the multiple directions:

First, note that while time-additive power utility is a true special case of recursive Epstein-Zin utility, (†) explicitly excludes the time-additive utility case where $\gamma\psi = 1$. Hence, so far the duality methods for time-additive and recursive utility coexist. We connect both by extending the procedure proposed in [Matoussi and Xing, 2018] to include the power utility scenario as a special case. In particular, the duality theory for power utility of [He and Pearson, 1991], [Karatzas et al., 1991] and [Cvitanić and Karatzas, 1992] is a special case of our enhanced recursive dualization method, which is valid for

$$\gamma\psi \geqslant 1,\ \psi > 1 \qquad \text{or} \qquad \gamma\psi \leqslant 1,\ \psi < 1. \tag{$\star$}$$

Having established the dual optimization problem, we close a gap left open in [Becherer et al., 2023], by proving existence, uniqueness, convexity and monotonicity of the stochastic differential dual for all parameter constellations ($\star$) in a general semimartingale setting.
Next, we turn to bounding an investors welfare loss. Note that, as in the case of power utility, the stochastic differential dual yields an upper bound on an investors welfare loss associated to any admissible strategy. However, evaluating these bounds, corresponds to solving non-standard forward-backward stochastic differential equations (FBSDEs), which is in general not feasible. In order to bypass this issue, we enclose the stochastic differential dual by transformed dual power utility functions. Combining them with their primal analogoues previously derived by [Seiferling and Seifried, 2016], we introduce our universal power utility bounds on an investors utility loss. To the best of our knowledge, those provide the first tractable method in the literature, that allows to validate the accuracy of approximations to Epstein-Zin consumption-investment allocation problems, without the need of a benchmark solution. Moreover, we demonstrate the theoretical value of the power utility bounds, by utilizing them in the proof of a new existence and uniqueness result for the stochastic differential dual, that requires less restrictive integrability assumptions as our general existence result mentioned before. Thus, our power bounds are valuable both in theory and applications. However, note that their quality in measuring an investors utility loss may deteriorate when the investors preference parameters are unfavorable. We make a first step in overcoming those quantitative shortcomings by additionally introducing *variational bounds*, that are better suited to recursive utilities when RA and EIS differ significantly from the power utility case $\gamma\psi = 1$.
Finally, by combining the classical dynamic programming approach with our duality results, we develop two novel algorithms that approximate the solution to investment-consumption optimization problems with Epstein-Zin preferences in incomplete markets. Our first approach focuses on the special case of power utility, where our general algorithmic idea has already been established e.g. in [Kamma and Pelsser, 2022]. However, while previous works mainly focus on the martingale method, we utilize the analytic Campbell-Shiller (CS) approximation known from [Chacko and Viceira, 2005] to approximate the dual problem. Using this approximation to complete the market, we are able to explicitly solve the HJB equation associated to the complete

market problem. Thereby we obtain an analytic approximation of the optimal solution that is explicitly characterized by a system of ordinary differential equations. As far as we know, such an analytic approximation that takes a detour through the truly recursive utility case via the CS approximation, is not present in the literature so far.
Our second algorithm is an iterative scheme, where a primal and a dual optimizer play against each other until they find an equilibrium, that corresponds to the solution of the investors optimization problem. More precisely, the investor is allowed to trade in a complete market and therein finds her optimal strategy, possibly violating the constraints imposed by the incomplete market. On the other hand, the opposing dual optimizer will not allow for such violations and forces her strategy to follow the restrictions by changing her market conditions. This leads the investor to adjust her overall strategy and the game goes on until both are content, i.e. under the conditions set by the dual optimizer the investor maximizes her utility while respecting the constraints imposed by market incompleteness. This reinterpretation of least favorable completion and its manifestation as an algorithm to approximate optimal solutions, appears to be absent from the current literature. Note, that we solve the investors HJB equation in every iteration and in that we rely on additional approximations. Consequently the algorithm cannot converge to the *true* solution. However, both our algorithms are shown to be accurate and stable under parameter variations. They are applicable and easy to implement even in high dimensions and in particular the iterative scheme appears to essentially avoid the curse of dimensionality, despite being based on repeatedly solving high dimensional PDEs.

The remainder of this thesis is organized as follows. Within Chapter 2 we extend the duality approach by [Matoussi and Xing, 2018] to explicitly include the case of power utility. Moreover, our general existence and uniqueness result for the stochastic differential dual is established. Chapter 3 contains our main result, the derivation of the universal power utility bounds on an investor's welfare loss. We additionally establish our variational bounds and state our refined existence and uniqueness result for the stochastic differential dual. In Chapter 4 we introduce our general market model and analyze its solution using dynamic programming techniques. Chapter 5 presents our first two-step approximation approach via least favorable completion and investigates its accuracy in several numerical simulations. Our second, iterative approximation approach is introduced in Chapter 6. Multiple numerical applications in different dimensions analyze its accuracy and convergence behavior. Chapter 7 concludes this thesis.
In addition, some preliminaries on BSDEs and convex analysis are provided in Appendix A and Appendix B, respectively, and technical calculations are outsourced to Appendix C and Appendix D to improve readability of the main text.

# Chapter 2

# Duality for Recursive Systems

This first chapter is dedicated to the derivation of the dual problem associated to an Epstein-Zin investment-consumption allocation problem and its analysis. We start by introducing the general setting and the detailed optimization problem under consideration in Section 2.1. The associated dual problem is then rigorously derived in Section 2.2; it is based on the three-step approach introduced by [Matoussi and Xing, 2018]. It turns out that the dual problem (as the primal one) is characterized in terms of a nonstandard backward stochastic differential equation (BSDE), and Section 2.3 is concerned with proving existence and uniqueness of a solution to this equation, as well as several of its properties.

## 2.1 The Primal Epstein-Zin Optimization Problem

Let $\big(\Omega, \{\mathfrak{F}_t\}_{t\in[0,T]}, \mathfrak{F}, \mathbb{P}\big)$ be a filtered probability space and let the filtration $\{\mathfrak{F}\}_{t\in[0,T]}$ satisfy the usual conditions of completeness and right-continuity. Throughout this whole thesis we follow the common practice of identifying almost surely equal random variables and indistinguishable stochastic processes, respectively. Moreover, our assumptions on the underlying filtration allow us to work with suitable (in particular right-continuous) versions of stochastic processes and we do so without further mention.

We denote the class of all nonnegative and progressively measurable processes on $[0,T]\times\Omega$ by $\mathcal{C}$. A stochastic process $\{c_t\}_{t\in[0,T]}\in\mathcal{C}$ is called a *consumption stream*, where for $t<T$, $c_t$ represents the consumption stream at time $t$ and $c_T$ models the lump sum consumption at time $T$.
The subjective preferences of a representative agent can in general be described by a *utility index* functional $\nu:\mathcal{C}\to\mathbb{R}$. We say that a consumption stream $c$ is weakly preferred to $\bar{c}$, if and only if $\nu(c)\geqslant\nu(\bar{c})$. In the context of stochastic differential utility as in [Duffie and Epstein, 1992a], we define the utility index functional as

$$\nu:\mathcal{C}^a\to\mathbb{R},\ \nu(c)\triangleq V_0[c],$$

where $\mathcal{C}^a\subseteq\mathcal{C}$ is the set of *admissible* consumption streams defined below. The *utility process* $V=V[c]$ associated with a consumption stream $c\in\mathcal{C}^a$ satisfies a backward stochastic differential equation (BSDE[1]) of the form

$$V_t[c]=\mathbb{E}_t\bigg[\int_t^T f(c_s,V_s[c])\mathrm{d}s+\Phi(c_T)\bigg],\quad t\in[0,T]. \tag{2.1.1}$$

[1]A short introduction to the notion of BSDEs and a collection of results used within this thesis is provided in Appendix A.

We consider the Epstein-Zin parametrization of recursive utility as in [Epstein and Zin, 1989] and [Weil, 1990], with relative risk aversion (RRA) $0 < \gamma \neq 1$ and elasticity of intertemporal substitution (EIS) $0 < \psi \neq 1$. Define $\theta \triangleq \frac{1-\gamma}{1-\frac{1}{\psi}}$ and $\mathbb{V} \triangleq \{v \in \mathbb{R} : (1-\gamma)v > 0\}$, i.e. $\mathbb{V} = (0,\infty)$, if $\gamma < 1$ and $\mathbb{V} = (-\infty, 0)$, if $\gamma > 1$. Then the continuous-time *Epstein-Zin aggregator* $f : (0,\infty) \times \mathbb{V} \to \mathbb{R}$ reads

$$f(c,v) \triangleq \delta \frac{c^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}((1-\gamma)v)^{1-\frac{1}{\theta}} - \delta\theta v, \tag{2.1.2}$$

and the terminal utility function is given as $\Phi(c) = \varepsilon \frac{1}{1-\gamma} c^{1-\gamma}$, where the coefficients $\delta > 0$ and $\varepsilon > 0$ capture the agent's rate of time preference and weight on terminal consumption, respectively; in particular we exclude the case of zero terminal bequest $\varepsilon = 0$.
Denote by $\mathcal{S}$ the space of $(\mathfrak{F}, \mathbb{P})$-semimartingales, then the relevant class of Epstein-Zin utility processes is given by $\mathcal{V} \triangleq \{V \in \mathcal{S} : (1-\gamma)V > 0\}$. We call a consumption stream $c$ *admissible*, if $V[c]$ uniquely exists, satisfies $(1-\gamma)V[c] > 0$ and is of class (D). The class of admissible consumption streams is denoted as $\mathcal{C}^a$ and the class of corresponding recursive utilities as $\mathcal{V}^a$.

If not further specified, we assume that any $c \in \mathcal{C}^a$ at least satisfies the minimal integrability condition

$$\mathbb{E}\left[\int_0^T c_t^{1-\frac{1}{\psi}} \mathrm{d}t + c_T^{1-\gamma}\right] < \infty. \tag{2.1.3}$$

The following example shows how the classical case of power utility is incorporated by this general recursive framework.

**Example 2.1**
*When $\gamma\psi = 1$, then $f(c,v) = \delta\varphi(c) - \delta v$ and (2.1.1) reduces to the classical time-additive utility specification with constant relative risk aversion $\gamma$:*

$$\begin{aligned} V_t[c] &= \mathbb{E}_t\left[\int_t^T \delta \tfrac{1}{1-\gamma} c_s^{1-\gamma} - \delta V_s[c] \mathrm{d}s + \Phi(c_T)\right] \\ &= \mathbb{E}_t\left[\int_t^T \delta e^{-\delta(s-t)}\varphi(c_s)\mathrm{d}s + e^{-\delta(T-t)}\Phi(c_T)\right], \end{aligned} \tag{2.1.4}$$

*where $\varphi : (0,\infty) \to \mathbb{R}$, $\varphi(c) = \frac{c^{1-\gamma}}{1-\gamma}$. The second equality is due to the classical existence result for linear BSDE provided in Theorem A.6, applicable as c satisfies (2.1.3).* ○

In general the Epstein-Zin aggregator $f$ is in particular not Lipschitz continuous in the utility variable $v$, hence standard BSDE results as in Appendix A cannot be applied and existence, respectively uniqueness of a solution to (2.1.1) is a highly non-trivial question. In the special case of a Brownian framework, existence and uniqueness results have previously been established by [Schroder and Skiadas, 1999] and [Xing, 2017] as cited from [Matoussi and Xing, 2018][Proposition 2.1]:

**Proposition 2.2**
*Let $(\mathscr{F}_t^W)_{t\in[0,T]}$ be the augmented filtration generated by some Brownian motion $W$. Then the following two existence results hold:*

*(i) [Schroder and Skiadas, 1999][Theorem 1][2] When either $\gamma > 1, 0 < \psi < 1$, or $0 < \gamma < 1, \psi > 1$, then for any $c \in \mathcal{C}$ such that $\mathbb{E}\left[\int_0^T c_t^\ell \mathrm{d}t + c_T^\ell\right] < \infty$ for all $\ell \in \mathbb{R}$, there exists a*

[2] The parameter $1+\alpha$ therein is $\theta$ in our notation

*unique semimartingale* $V = V[c]$ *satisfying* (2.1.1) *such that* $\mathbb{E}\left[\operatorname{ess\,sup}_{t\in[0,T]} |V_t|^\ell\right] < \infty$ *for every* $\ell > 0$.

(ii) *[Xing, 2017][Proposition 2.2 & Proposition 2.4] When* $\gamma, \psi > 1$, *then for any* $c \in \mathcal{C}$ *such that* $\mathbb{E}\left[\int_0^T c_t^{1-\frac{1}{\psi}} \mathrm{d}t + c_T^{1-\gamma}\right] < \infty$, *there exists a unique semimartingale* $V = V[c]$ *satisfying* (2.1.1) *such that* $V$ *is of class (D).*

*In both cases* $(1-\gamma)V > 0$, *i.e.* $V \in \mathcal{V}$ *and* $V_0[c]$ *is concave in* $c$.

By Proposition 2.2 we also obtain $\mathcal{C}^a \neq \emptyset$.

[Seiferling and Seifried, 2016] were the first to prove existence and uniqueness of Epstein-Zin utility in a general semimartingale framework. Their result is further investigated in Section 2.3, see Proposition 2.17.

We construct the consumption-investment optimization problem with Epstein-Zin utility as follows. Consider a financial market $S$ where $S = (S_0, \dots, S_m)$ is a $(m+1)$-dimensional positive semimartingale. $S^0$ represents the price of the risk-less asset, whereas $S_i$, $i = 1, \dots m$ are the price processes of the $m$ risky assets. Given an initial endowment $x > 0$, a representative agent may invest in the given market by choosing a portfolio represented by a predictable $S$-integrable process $\pi = (\pi^0, \dots, \pi^m)$. Here $\pi_t^i$ is the fraction of her current wealth invested in the risky asset $S_i$ at time $t$ and $\pi_t^0 = 1 - \sum_{i=1}^m \pi_t^i$ is the fraction invested in the riskless asset. Given her initial wealth $x$, an investment strategy $\pi$ and a constumption stream $c$, the wealth process of our investor follows the dynamics

$$\mathrm{d}X_t^{(\pi,c)} = X_{t-}^{(\pi,c)} \pi_t^\top \frac{\mathrm{d}S_t}{S_{t-}} - c_t \mathrm{d}t, \quad X_0^{(\pi,c)} = x.$$

Such an investment-consumption strategy $(\pi, c)$ is called *admissible* if $c \in \mathcal{C}^a$, $X_t^{(\pi,c)} > 0$ for all $t \in [0,T]$ and $c_T = X_T^{(\pi,c)}$. We denote the class of those strategies by $\mathcal{A}$.

Then the agent's optimization problem is to

$$\text{find } (\pi^*, c^*) \in \mathcal{A} \text{ such that } \nu(c^*) = \sup_{(\pi,c)\in\mathcal{A}} \nu(c), \tag{2.1.5}$$

so she aims to maximize her utility at time $t = 0$ over all admissible investment-consumption strategies. We refer to (2.1.5) as the *primal* optimization problem.

## 2.2 The Dual Epstein-Zin Optimization Problem

Having established the primal optimization problem under consideration in (2.1.5), we now go through all necessary details to finally obtain the associated *dual* optimization problem at the end of Section 2.2.1. Plugging everything together in Section 2.2.2 we obtain the duality inequality for a consumption-investment optimization problem with recursive Epstein-Zin preferences.

Most technical computations have been outsourced to Appendix B.2 in order to make this main text as readable as possible.

### 2.2.1 Dualization - A Three-Step Procedure

Recall the BSDE characterizing the primal optimization problem

$$V_t[c] = \mathbb{E}_t\left[\int_t^T f(c_s, V_s[c])\mathrm{d}s + \Phi(c_T)\right], \quad t \in [0, T].$$

First note that, as the second argument of the Epstein-Zin aggregator $f$ from (2.1.1) depends on the whole future path of the consumption stream $c$, straight forward dualization as in [Karatzas et al., 1991] is not possible. A workaround is provided by [Matoussi and Xing, 2018] and we follow their ideas, slightly generalizing their approach in several directions. The procedure is essentially built on three separate steps as illustrated in Figure 2.1.
In a first step we reformulate the investor's recursive Epstein-Zin preferences to a variational formulation as introduced by [Geoffard, 1996]; the result is given in Lemma 2.4. Intuitively speaking, *variational utility* expresses recursive utility associated to a fixed consumption stream as a specific discounted time-additive utility of the same consumption plan, maximized over the rate of time preference. The primal problem is now formulated in a time-additive way which allows us to apply the classical approach of [Karatzas et al., 1991] using state price deflators. This is the actual dualization step and leads to a *dual variational utility*, which is again of time-additive structure. The final step, in particular Lemma 2.10, reverts the first one by reformulating the dual variational utility to what [Matoussi and Xing, 2018] call a *stochastic differential dual*, i.e. a dual utility process in recursive form.

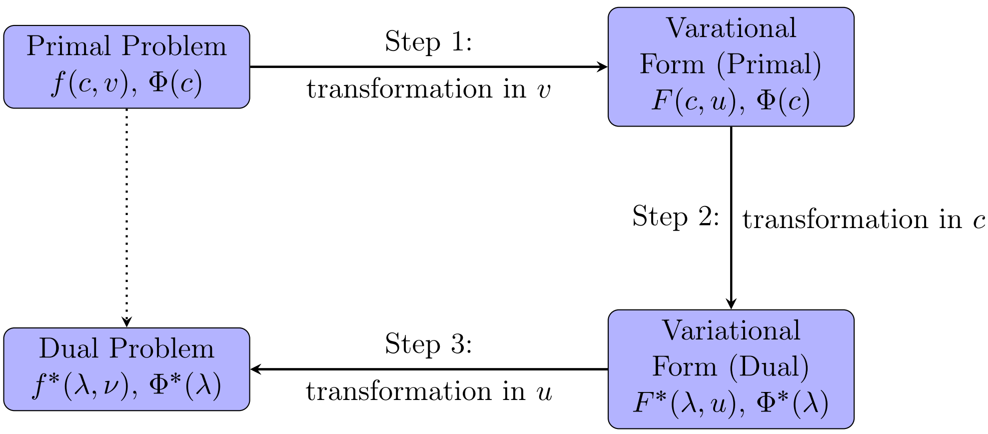



Figure 2.1: Illustration of the Dualization Procedure

We slightly generalize this three-step scheme by not only considering the aggregator $f$ as in (2.1.2), but a suitable extension in the second argument $v$ to the whole real line. The difference will show in the following way: As [Matoussi and Xing, 2018] restrict themselves to the domain $\mathbb{V}$ of $f$, which is designed specifically for recursive utility of Epstein-Zin type, they indirectly exclude the parameter constellation $\gamma\psi = 1$, i.e. power utility, from their analysis. Duality results for power utility are well known, however, as pointed out in Example 2.1, they are a true special case of recursive utility. With our extension, we are able to include the case $\gamma\psi = 1$ to our analysis and find that straight forward dualization for power utility is also just a special case of our extended dualization approach for recursive utility. Moreover, this yields a natural connection between the duality procedure for recursive utilities and the associated power utility bounds derived in Chapter 3.

The procedure heavily relies on convex (concave) conjugation of the aggregator $f$, one argument after the other. Hence, before we jump into the first step, we briefly give the definition of convex (concave) conjugates in our specific setting, just to convey the idea to the reader; for a more general treatment see Appendix B.1 based on [Rockafellar, 1997], where the whole topic is treated in close detail.

**Definition 2.3**
*Let $h : \mathbb{R} \to (-\infty, \infty]$, $x \mapsto h(x)$ be a lower-semicontinuous[3] convex function such that there exists some $x \in \mathbb{R}$ with $h(x) < \infty$. Then the function*

$$h^* : \mathbb{R} \to (-\infty, \infty],\ x^* \mapsto \sup_{x \in \mathbb{R}} \{xx^* - h(x)\} = -\inf_{x \in \mathbb{R}} \{h(x) - xx^*\}$$

*is called the* convex conjugate *of $h$ and is another lower-semicontinuous convex function. Analogously, let $g : \mathbb{R} \to [-\infty, \infty)$, $x \mapsto g(x)$ be an upper-semicontinuous concave function, such that there exists some $x \in \mathbb{R}$ with $g(x) > -\infty$. Then the function*

$$g_* : \mathbb{R} \to [-\infty, \infty),\ x_* \mapsto \inf_{x \in \mathbb{R}} \{xx_* - g(x)\}$$

*is called the* concave conjugate *of $g$ and is another upper-semicontinuous concave function.*

Figuratively speaking, the convex (concave) conjugate describes the intersection of the $y$-axis of the pointwise supremum (infimum) of affine functions majorized (minorized) by $h$. Moreover, the convex conjugate of $h^*$ equals $h$, i.e. $(h^*)^* = h$ and the analogous relation also holds for the concave conjugate. Thus the conjugacy operation implies some kind of duality relation on convex functions.[4] The convex and concave conjugate are sometimes also called *Legendre-Fenchel* transformation of $h$. We usually go by this name for simplicity and only specify the type of conjugation if necessary.

In the following we transfer the duality relation between convex (concave) functions and their conjugates just described to the notion of duality for recursive systems as in (2.1.1), thus we only allow parameter constellations where the aggregator $f$ in (2.1.2) is either convex or concave. By Lemma B.7, we know

$$f \text{ is convex in } v \quad \Leftrightarrow \quad \gamma\psi \geqslant 1, \psi > 1 \tag{1}$$
$$f \text{ is concave in } v \quad \Leftrightarrow \quad \gamma\psi \leqslant 1, \psi < 1. \tag{2}$$

Similar to [Matoussi and Xing, 2018], we now provide the detailed analysis only for the convex case $\gamma\psi \geqslant 1, \psi > 1$.[5] The case of concave aggregators as in (2) follows analogously with the appropriate adjustments, see Remark 2.13 below.

**Note:** We emphasize that during this whole thesis we assume that either

$$\gamma\psi \geqslant 1, \psi > 1 \qquad \text{or} \qquad \gamma\psi \leqslant 1, \psi < 1$$

as in (1) and (2) above holds. Naturally there might be additional restrictions, but this is the *minimum* requirement that stands behind every result, even if it might not be explicitly mentioned.

[3]Recall that a function $h : \mathbb{R} \to [-\infty, \infty]$ is said to be lower-semicontinuous at some point $x$, if $h(x) \leqslant \lim_{i\to\infty} h(x_i)$ for every sequence $(x_i)_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} x_i = x$ and $\lim_{i\to\infty} h(x_i)$ exists in $[-\infty, \infty]$. This may be expressed as $h(x) = \liminf_{y\to x} h(y)$. A function $h$ is called lower-semicontinuous, if it is lower-semicontinuous at any $x \in \mathbb{R}$. The definition of upper-semicontinuity is analogous replacing $\liminf$ by $\limsup$.

[4]This is not true in general, but only for convex functions $h$ that don't take the value $-\infty$ as we claimed in Definition 2.3. For the general case see Appendix B.1 or [Rockafellar, 1997].

[5]As mentioned earlier, [Matoussi and Xing, 2018] actually only consider the constellations $\gamma\psi > 1, \psi > 1$.

### Step 1: Transformation from Recursive to Variational Form

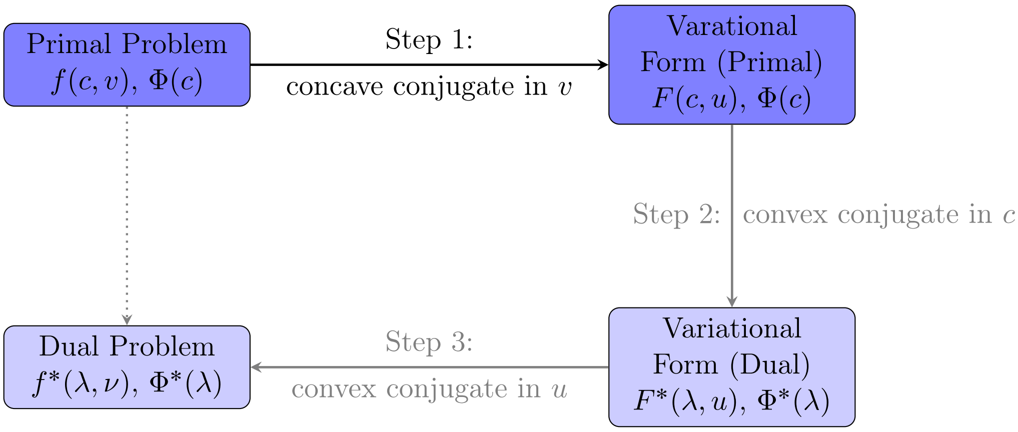



Figure 2.2: Dualization: Step 1

In a first step, we transform the utility process $V_t[c]$ to its variational representation, i.e. for a fixed consumption plan $c$ we express the associated recursive utility $V_t[c]$ as the maximum of an associated discounted time-additive utility process over future discount values. The time-additive utility within this variational representation is given exactly by the Legendre-Fenchel transformation of the recursive aggregator $f$ in the utility variable $v$. This first step is neither directly connected to any model specifications, nor to an investment optimization problem; it is merely a reformulation of the investor's utility specification based on the duality feature of convex conjugates as mentioned above. The idea of a variational representation for recursive utilities was first introduced by [Geoffard, 1996] in a deterministic setting. Later, [Karoui et al., 1997] and [Dumas et al., 1998] extended the idea to a stochastic framework assuming Lipschitz-continuity of $f$ in its second argument and certain integrability of the value function. Finally [Matoussi and Xing, 2018] generalized the approach to Epstein-Zin utility only relying on convexity (concavity) of the aggregator and the class (D) property of $V_t[c]$, but excluding the power utility case $\gamma\psi = 1$. To incorporate this special case, we define a suitable extension of the Epstein-Zin aggregator $f$ as follows.

Consider the Epstein-Zin aggregator $f$ from (2.1.2). Then the smallest lower-semicontinuous convex extension $\tilde{f}$ of $f$ is given by

$$\tilde{f}(c, v) : (0, \infty) \times \mathbb{R} \to (-\infty, \infty], (c, v) \mapsto \begin{cases} f(c, v), & (1-\gamma)v > 0 \\ f(c, 0+) + f_v(c, 0+) \cdot v, & (1-\gamma)v \leqslant 0 \end{cases}, \tag{2.2.1}$$

where $f(c, 0+) \triangleq \lim_{(1-\gamma)v \downarrow 0} f(c, v)$ and analoguously for $f_v$.

For the parameter constellations $\gamma\psi \geqslant 1, \psi > 1$, the function $\tilde{f}$ has three different forms:

$$\tilde{f}(c, v) = \begin{cases} f(c, v), & (1-\gamma)v > 0 \\ -\delta\theta v, & (1-\gamma)v \leqslant 0 \end{cases} \qquad \text{if } \gamma > 1, \psi > 1,$$

$$\tilde{f}(c, v) = \begin{cases} f(c, v), & (1-\gamma)v > 0 \\ \infty, & (1-\gamma)v \leqslant 0 \end{cases} \qquad \text{if } \gamma < 1, \gamma\psi > 1,$$

and

$$\tilde{f}(c,v) = \delta \tfrac{1}{1-\gamma} c^{1-\gamma} - \delta v \qquad \text{if } \gamma\psi = 1.$$

An illustration of the extended graphs is given in Figure 2.3. By definition $\tilde{f}(c,\cdot)$ is a lower-semicontinuous function in $v$ in all cases, convex as $f(c,\cdot)$ is convex by Lemma B.7.

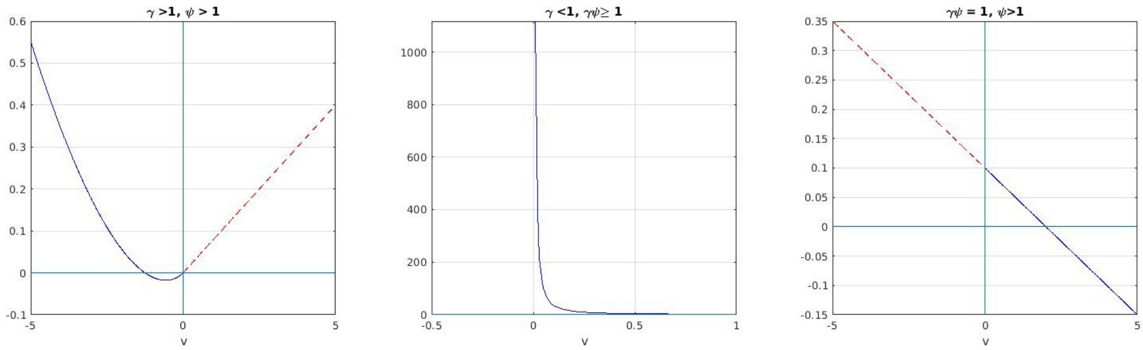



Figure 2.3: $\tilde{f}(c,v)$ for $c \equiv 1$; $f$ in blue, the extension to $(1-\gamma)v \leqslant 0$ in red dots

Now consider the concave conjugate of $-\tilde{f}$ in its second argument given by

$$F : (0,\infty) \times \mathbb{R} \to [-\infty,\infty),\ (c,u) \mapsto \inf_{v\in\mathbb{R}} \left\{ \tilde{f}(c,v) + uv \right\}. \tag{2.2.2}$$

Then $F(c,u)$ is concave in $c$ and in $u$, see Lemma B.9, and by Theorem B.3 the functions $\tilde{f}$ and $F$ are dual in the sense that $\tilde{f}$ is minus the concave conjugate of $F$ in $u$, i.e.

$$\tilde{f}(c,v) = \sup_{u\in\mathbb{R}} \{F(c,u) - uv\}. \tag{2.2.3}$$

**Note:** The concave conjugate of $-\tilde{f}$ is exactly minus the convex conjugate of $\tilde{f}$ in $-v$, which might seem like a more natural way to conjugate a convex function. However, the transformation as chosen above is notationally more convenient and in particular the dualization procedure within Step 2 below stays naturally consistent with straight forward dualization of time-additive utility as e.g. in [Karatzas et al., 1991].

We denote by $\mathcal{P}$ the class of all progressively measurable processes on $[0,T]\times\Omega$. Then for any $u \in \mathcal{P}$, $c \in \mathcal{C}^a$ and $t \in [0,T]$, define the *stochastic variational utility* by

$$U_t[c,u] \triangleq \mathbb{E}_t\left[\int_t^T \kappa^u_{t,s} F(c_s,u_s)\mathrm{d}s + \kappa^u_{t,T}\Phi_T(c_T)\right], \tag{2.2.4}$$

where $\kappa^u_{t,s} \triangleq \exp(-\int_t^s u_r dr)$, $s \geqslant t$. Note that the right hand side of (2.2.4) always exists (in $\bar{\mathbb{R}}$), see (2.2.5) and (2.2.10) below.

In Lemma 2.4 we show that the supremum of $U_t[c,u]$ over all progressively measurable processes $u$ actually equals the recursive utility process $V_t[c]$. This is how the duality between $\tilde{f}$ and $F$ from (2.2.3) transfers to the stochastic processes $U$ and $V$. The result is an extension of Lemma 2.3 in [Matoussi and Xing, 2018]. The core steps of the proof still follow their approach, although most arguments are either simplified or carried out in more detail.

**Lemma 2.4**
*For any $u \in \mathcal{P}$ and $c \in \mathcal{C}^a$, let $V[c]$ be the utility process associated with $c$ and $U[c,u]$ as given*

*in* (2.2.4). *Then for any* $t \in [0,T]$ *the recursive utility* $V_t[c]$ *can be expressed by the essential supremum*[6] *of the variational utilities* $U_t[c,u]$ *over* $u$, *i.e.*

$$V_t[c] = \operatorname*{ess\,sup}_{u \in \mathcal{P}} U_t[c,u].$$

*Moreover, the supremum is attained at*

$$u^c \triangleq -f_v(c, V[c]) = \delta c^{1-\frac{1}{\psi}}(1-\theta)((1-\gamma)V[c])^{-\frac{1}{\theta}} + \delta\theta.$$

*Proof.* First consider the case $\gamma\psi = 1, \psi > 1$. Then $\tilde{f}$ is given by $\tilde{f}(c,v) = \delta\varphi(c) - \delta v$, where $\varphi(c) = \frac{1}{1-\gamma}c^{1-\gamma}$ as in Example 2.1, thus the Legendre-Fenchel transform desintegrates to

$$F(c,u) = \begin{cases} \delta\varphi(c) & u = \delta \\ -\infty & \text{else} \end{cases},$$

see Lemma B.9. In particular

$$\begin{aligned} \sup_{u \in \mathcal{P}} U_t[c,u] &= \sup_{u \in \mathcal{P}} \mathbb{E}_t\left[\int_t^T \kappa^u_{t,s} F(c_s, u_s)\mathrm{d}s + \kappa^u_{t,T}\Phi(c_T)\right] \\ &= \mathbb{E}_t\left[\int_t^T \delta \mathrm{e}^{-\delta(s-t)}\varphi(c_s)\mathrm{d}s + \mathrm{e}^{-\delta(T-t)}\Phi(c_T)\right], \end{aligned}$$

so this case follows by (2.1.4) in Example 2.1 and Theorem A.6, respectively.
For the remaining parameter constellations $F$ is given by

$$F(c,u) = \begin{cases} \delta^\theta \frac{c^{1-\gamma}}{1-\gamma}\left(\frac{u-\delta\theta}{1-\theta}\right)^{1-\theta}, & u > \delta\theta \\ 0, & u = \delta\theta \\ -\infty, & u < \delta\theta \end{cases},$$

see again Lemma B.9. Note that it suffices to focus on $u \in \mathcal{P}$ such that $U_0[c,u] > -\infty$, so $u < \delta\theta$ is automatically excluded. Thus by introducing the space $\mathcal{U} \triangleq \{u \in \mathcal{P} : u \geqslant \delta\theta\}$, we can without loss restrict ourselves to $u \in \mathcal{U} \subseteq \mathcal{P}$. The reminder of the proof consists of three major steps:

1. *Class (D) property of* $\kappa^u_{0,\cdot}U[c,u]$: Note that

$$\{\gamma\psi > 1, \psi > 1\} = \{\gamma > 1, \psi > 1\} \cup \{0 < \gamma < 1, \gamma\psi > 1\}$$

and we split this part of the proof into those two cases.
*Case 1:* $\gamma > 1, \psi > 1$. As $\gamma > 1$ we have $\Phi < 0$ and $F \leqslant 0$, so for $u \in \mathcal{U}$ we obtain

$$\begin{aligned} \mathbb{E}\left[\left|\int_0^T \kappa^u_{0,s}F(c_s,u_s)\mathrm{d}s + \kappa^u_{0,T}\Phi(c_T)\right|\right] &\leqslant \mathbb{E}\left[\int_0^T \kappa^u_{0,s}|F(c_s,u_s)|\mathrm{d}s + \kappa^u_{0,T}|\Phi(c_T)|\right] \\ &= \mathbb{E}\left[\int_0^T \kappa^u_{0,s}(-F(c_s,u_s))\mathrm{d}s - \kappa^u_{0,T}\Phi(c_T)\right] \\ &= -U_0[c,u] < \infty. \end{aligned} \tag{2.2.5}$$

[6] A measurable random variable $Y$ is called *essential supremum* of a random family $(X^i)_{i\in I}$, if $Y \geqslant X^i$ almost surely for any $i \in I$ and $Y \leqslant Z$ almost surely for any measurable random variable $Z$ that satisfy $Z \geqslant X^i$ for any $i \in I$ almost surely. We write $Y \triangleq \operatorname{ess\,sup}_{i\in I} X^i$. Analogously, the *essential infimum* of a random family $(X^i)_{i\in I}$ is defined as $\operatorname{ess\,inf}_{i\in I} X^i \triangleq -\operatorname{ess\,sup}_{i\in I} -X^i$.

It follows that the stochastic process $\mathbb{E}_t\left[\int_0^T \kappa^u_{0,s} F(c_s,u_s)\mathrm{d}s + \kappa^u_{0,T}\Phi(c_T)\right]$ is a uniformly integrable martingale and hence of class (D). Moreover $\kappa^u$ is bounded for any $u \in \mathcal{U}$. Thus the class (D) property of $V[c]$ for $c \in \mathcal{C}^a$ implies the integrability of $\kappa^u_{0,T}\Phi(c_T)$ by choosing $\tau = T$, and hence the class (D) property of the process $\mathbb{E}_t[\kappa^u_{0,T}\Phi(c_T)]$.
Since $F \leqslant 0$ we have

$$\begin{aligned}\mathbb{E}_t\left[\int_0^T \kappa^u_{0,s} F(c_s,u_s)\mathrm{d}s + \kappa^u_{0,T}\Phi(c_T)\right] &\leqslant \mathbb{E}_t\left[\int_t^T \kappa^u_{0,s} F(c_s,u_s)\mathrm{d}s + \kappa^u_{0,T}\Phi(c_T)\right] \\ &= \kappa^u_{0,t} U_t[c,u] \\ &\leqslant \mathbb{E}_t\left[\kappa^u_{0,T}\Phi(c_T)\right]\end{aligned} \tag{2.2.6}$$

and the class (D) property of both the upper and lower bound implies the class (D) property of $\kappa^u_{0,\cdot} U[c,u]$.

*Case 2:* $0 < \gamma < 1, \gamma\psi > 1$. In this case, $F \geqslant 0$, $\Phi > 0$, so we have to show $U_0[c,u] < \infty$ first. Let $c \in \mathcal{C}^a$, $u \in \mathcal{U}$ and recall that

$$V_t[c] = \mathbb{E}_t\left[\int_t^T f(c_s, V_s[c])\mathrm{d}s + \Phi(c_T)\right], \quad t \in [0,T],$$

thus by the class (D) property of $V[c]$, the process

$$M \triangleq V[c] + \int_0^\cdot f(c_s, V_s[c])\mathrm{d}s \tag{2.2.7}$$

defines a uniformly integrable martingale. An application of Itô's formula shows

$$\begin{aligned}\mathrm{d}\left(\kappa^u_{0,t} V_t[c]\right) &= \kappa^u_{0,t}\mathrm{d}M_t - \left(\kappa^u_{0,t} f(c_t, V_t[c]) - u_t\kappa^u_{0,t}V_t[c]\right)\mathrm{d}t \\ &= \kappa^u_{0,t}\mathrm{d}M_t - \mathrm{d}A^u_t - \kappa^u_{0,t}F(c_t,u_t)\mathrm{d}t,\end{aligned}$$

where

$$\mathrm{d}A^u_t \triangleq \kappa^u_{0,t}\left(f(c_t,V_t[c]) - (F(c_t,u_t) - u_tV_t[c])\right)\mathrm{d}t. \tag{2.2.8}$$

Now by the definition of $f$ and $F$ respectively, we have $f(c,v) = \sup_{u>\delta\theta}\{F(c,u) - uv\}$ and as $\kappa^u > 0$, $A^u$ is increasing in this case. If $u = \delta\theta$, (2.2.8) simplifies to

$$\mathrm{d}A^u_t \triangleq \frac{\delta}{1-\phi}\kappa^u_{0,t}c_t^{1-\phi}\mathrm{d}t. \tag{2.2.9}$$

and as again $\kappa^u > 0$ and moreover $\phi < 1$, $A^u$ is increasing. Thus $\kappa^u_{0,\cdot}V[c] + \int_0^\cdot \kappa^u_{0,s}F(c_s,u_s)\mathrm{d}s$ is a local supermartingale. We take a localization sequence $(\tau_n)_{n\in\mathbb{N}}$, then by the supermartingale property of the stopped process we obtain

$$V_0[c] \geqslant \mathbb{E}\left[\kappa^u_{0,\tau_n\wedge T}V_{\tau_n\wedge T}[c] + \int_0^{\tau_n\wedge T}\kappa^u_{0,s}F(c_s,u_s)\mathrm{d}s\right].$$

Since $V[c]$ is of class (D) and $F \geqslant 0$, by taking the limit on the right hand side, the monotone convergence theorem implies $\mathbb{E}\left[\int_0^T \kappa^u_{0,s}F(c_s,u_s)\mathrm{d}s\right] < \infty$. As above $\mathbb{E}[\kappa^u_{0,T}\Phi(c_T)] < \infty$, so we obtain

$$U_0[c,u] < \infty. \tag{2.2.10}$$

Now similar as in (2.2.6), $F \geqslant 0$ implies

$$\begin{aligned}\mathbb{E}_t\left[\kappa^u_{0,T}\Phi(c_T)\right] &\leqslant \kappa^u_{0,t}\mathbb{E}_t\left[\kappa^u_{t,T}\Phi(c_T) + \int_t^T \kappa^u_{t,s}F(c_s, u_s)\mathrm{d}s\right] \\ &= \kappa^u_{0,t}U_t[c,u] \\ &\leqslant \mathbb{E}_t\left[\kappa^u_{0,T}\Phi(c_T) + \int_0^T \kappa^u_{0,s}F(c_s, u_s)\mathrm{d}s\right],\end{aligned}$$

therefore, $\kappa^u_{0,\cdot}U[c,u]$ is of class (D).

Concluding, if $\gamma\psi > 1, \psi > 1$, then $\kappa^u_{0,\cdot}U[c,u]$ is of class (D) and the first step of the proof is complete.

2. $V_t[c] \geqslant U_t[c,u]$ $\forall t \in [0,T]$ *a.s.*: As $U_0[c,u]$ is finite, the tower property of conditional expectation implies that the process

$$M^u \triangleq \kappa^u_{0,\cdot}U[c,u] + \int_0^{\cdot} \kappa^u_{0,s}F(c_s,u_s)\mathrm{d}s \tag{2.2.11}$$

is a martingale. Then a basic calculation using (2.2.7), (2.2.11) and Itô's formula yields

$$\mathrm{d}\left(\kappa^u_{0,t}\left(V_t[c] - U_t[c,u]\right)\right) = \mathrm{d}L^u_t - \mathrm{d}A^u_t \tag{2.2.12}$$

where $A^u_t$ as in (2.2.8) is increasing and $\mathrm{d}L^u_t \triangleq \kappa^u_{0,t}\mathrm{d}M_t - \mathrm{d}M^u_t$ is a local marginale. Hence $\kappa^u_{0,\cdot}(V[c] - U[c,u])$ is a local supermartingale. On the other hand, we have seen that $\kappa^u_{0,\cdot}U[c,u]$ is of class (D) and moreover, $\kappa^u_{0,\cdot}V[c]$ is of class (D) thanks to the boundedness of $\kappa^u$ and class (D) property of $V[c]$. Thus the local supermartingale $\kappa^u_{0,\cdot}(V[c] - U[c,u])$ is an honest supermartingale and

$$\kappa^u_{0,t}\left(V_t[c] - U_t[c,u]\right) \geqslant \mathbb{E}_t\left[\kappa^u_{0,T}(V_T[c] - U_T[c,u])\right] = \mathbb{E}_t[\kappa^u_{0,T}(\Phi(c_T) - \Phi(c_T))] = 0,$$

so as $\kappa^u_{0,t} > 0$ we have $V_t[c] \geqslant U_t[c,u]$ almost surely for all $t \in [0,T]$ and any $u \in \mathcal{U}$. By right-continuity of $V[c]$ and $U[c,u]$ it follows that $V_t[c] \geqslant U_t[c,u]$ for all $t \in [0,T]$ almost surely.

3. $V_t[c] \leqslant \operatorname{ess\,sup}_{u\in\mathcal{U}} U_t[c,u]$: To finalize the proof it suffices to find some $u^c \in \mathcal{U}$ such that $V[c] \leqslant U[c,u]$. We choose said $u^c$ by the first order condition of $f(c,v) = \sup_{u>\delta\theta}\{F(c,u) - uv\}$, more precisely

$$u^c \triangleq -f_v(c, V[c]) = \delta c^{1-\frac{1}{\psi}}(1-\theta)((1-\gamma)V[c])^{-\frac{1}{\theta}} + \delta\theta.$$

As $\theta < 1$ it follows that $u^c \in \mathcal{U}$ and we have $f(c,V[c]) = F(c,u^c) - u^c V[c]$, so clearly $A^{u^c} \equiv 0$.[7] Then by (2.2.12)

$$\mathrm{d}\left(\kappa^{u^c}_{0,t}\left(V_t[c] - U_t[c,u^c]\right)\right) = \mathrm{d}L^{u^c}_t$$

is a local martingale, bounded from above as $\kappa^{u^c} > 0$ and $V_t[c] \geqslant U_t[c,u]$ for any $u \in \mathcal{U}$. Hence the local martingale is in fact a submartingale and

$$\kappa^{u^c}_{0,t}\left(V_t[c] - U_t[c,u^c]\right) \leqslant \mathbb{E}_t\left[\kappa^{u^c}_{0,T}(V_T[c] - U_T[c,u^c])\right] = \mathbb{E}_t[\kappa^{u^c}_{0,T}(\Phi(c_T) - \Phi(c_T))] = 0$$

which implies $V_t[c] \leqslant U_t[c,u^c]$ almost surely. Again, due to right-continuity of the processes, we obtain $V_t[c] \leqslant U_t[c,u^c]$ $\forall t \in [0,T]$ almost surely.

[7] By (2.2.9) the case $u = \delta\theta$ yields $A^{u^c} > 0$, so the supremum cannot be attained there.

Combining the above we obtain $V_t[c] = \operatorname{ess\,sup}_{u \in \mathcal{U}} U_t[c,u] = \operatorname{ess\,sup}_{u \in \mathcal{P}} U_t[c,u]$.

$\square$

The crucial consequence of Lemma 2.4 is that our problem can be written as

$$V_0 = \sup_{(\pi,c)\in\mathcal{A}} V_0[c] = \sup_{(\pi,c)\in\mathcal{A}} \sup_{u\in\mathcal{P}} U_0[c,u] = \sup_{u\in\mathcal{P}} \underbrace{\sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}\left[\int_0^T \kappa_{0,s}^u F(c_s,u_s)\mathrm{d}s + \kappa_{0,T}^u \Phi(c_T)\right]}_{\triangleq (P')},$$

where the inner problem $(P')$ is a time-additive investment optimization problem. Thus, in the second step, we can dualize $(P')$ by the methods well known from the time-additive utility framework.

**Step 2: Duality for the Variational System**

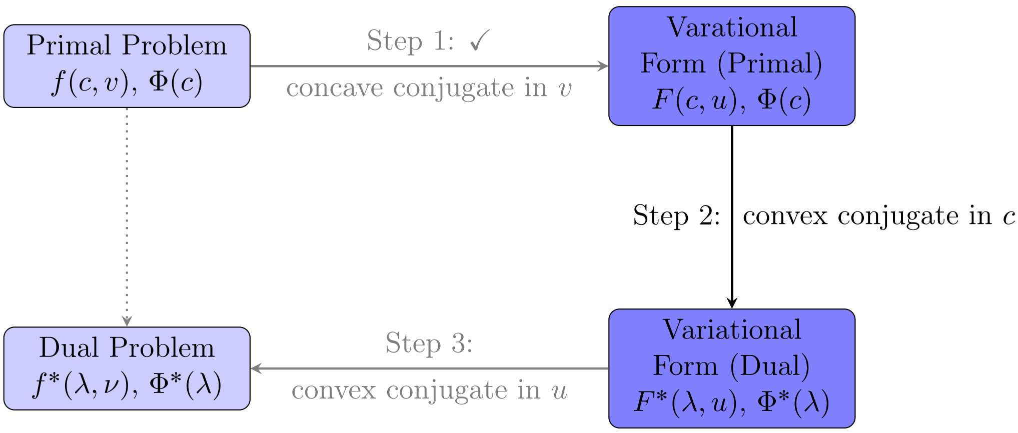



Figure 2.4: Dualization: Step 2

We now establish duality for $(P')$. While the first step was merely a reformulation of the investor's recursive utility functional, in this *true* dualization step the market model and the investor's wealth comes into play:
For a fixed $u \in \mathcal{P}$ the problem $(P')$, i.e. $\sup_{(\pi,c)\in\mathcal{A}} U_0[c,u]$, can be interpreted as an optimization problem with bequest utility $\Phi$ and a time-additive intertemporal utility function $F(c,u)$, parameterized by fictitious discount rates $u$. Thus we can dualize $(P')$ by the standard procedure for optimization problems with time-additive utility as e.g. in [Karatzas et al., 1991]. To this end, consider the Legendre-Fenchel transformations of $F$ and $\Phi$ in $c$ given by

$$F^* : (0,\infty)\times\mathbb{R} \to [-\infty,\infty), (\lambda,u) \mapsto \sup_{c>0}\{F(c,u) - \lambda c\}, \tag{2.2.13}$$
$$\Phi^* : (0,\infty) \to \mathbb{R}, \lambda \mapsto \sup_{c>0}\{\Phi(c) - \lambda c\}.$$

Then $F^*$ and $\Phi^*$ are convex in $\lambda$ and $F^*$ is concave in $u$, c.f. Lemma B.10 and Lemma B.11, respectively.

We denote the set of *state price deflators* with initial value $\lambda > 0$ as

$$\mathcal{D}_\lambda \triangleq \left\{\Lambda \in \mathcal{P} : \Lambda > 0, \Lambda_0 = \lambda, \Lambda X^{(\pi,c)} + \int_0^\cdot \Lambda_s c_s \mathrm{d}s \text{ is a supermartingale for all } (\pi,c) \in \mathcal{A}\right\} \tag{2.2.14}$$

and the set of all state price deflators

$$\mathcal{D} \triangleq \bigcup_{\lambda>0} \mathcal{D}_\lambda.$$

For notational convenience we introduce state price deflators with arbitrary positive initial values as above. Note that every $\Lambda \in \mathcal{D}$ has a trivial decomposition $\Lambda = \lambda D$ where $D \in \mathcal{D}_1$ by choosing $\lambda = \Lambda_0$. We call processes $D \in \mathcal{D}_1$ *pricing deflators* to distinguish the particularly important case where $\lambda = 1$.

From now on, we assume that at least one such state price deflator exists, i.e.

$$\mathcal{D} \neq \varnothing. \tag{2.2.15}$$

Note that (2.2.15) excludes arbitrage opportunities, c.f. [Karatzas and Kardaras, 2007].

**Remark 2.5**
*The condition in* (2.2.14) *is motivated by a particularly important special case with a nice interpretation, the so called* budget constraint*: Consider a pricing deflator $D \in \mathcal{D}_1$, then $DX^{(\pi,c)} + \int_0^\cdot D_s c_s \mathrm{d}s$ is a supermartingale and hence*

$$\mathbb{E}\left[D_T X_T^{(\pi,c)} + \int_0^T D_s c_s \mathrm{d}s\right] \leqslant x \tag{2.2.16}$$

*is satisfied for all $(\pi, c) \in \mathcal{A}$. Thus the expected discounted terminal wealth plus the expected discounted total consumption from any admissible trading strategy $(\pi, c)$ cannot exceed the initial capital $x$. The same is true for every intermediate time-step, as* (2.2.14) *implies*

$$\mathbb{E}_t\left[\int_t^T D_s c_s \mathrm{d}s + D_T X_T^{(\pi,c)}\right] = \mathbb{E}_t\left[\int_0^T D_s c_s \mathrm{d}s + D_T X_T^{(\pi,c)}\right] - \int_0^t D_s c_s \mathrm{d}s \leqslant D_t X_t^{(\pi,c)}. \tag{2.2.17}$$

△

Recalling that an admissible consumption plan $c$ satisfies $c_T = X_T^{(\pi,c)}$, the duality relation is established as follows: Consider $\Lambda \in \mathcal{D}$ and fix some $u \in \mathcal{P}$. Then[8]

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \kappa_{0,s}^u F(c_s, u_s) ds + \kappa_{0,T}^u \Phi(c_T)\right] \qquad (2.2.18)\\
&\quad = \mathbb{E}\left[\int_0^T \kappa_{0,s}^u (F(c_s, u_s) - (\kappa_{0,s}^u)^{-1}\Lambda_s c_s)\mathrm{d}s + \kappa_{0,T}^u(\Phi(c_T) - (\kappa_{0,T}^u)^{-1}\Lambda_T c_T)\right]\\
&\qquad + \mathbb{E}\left[\Lambda_T c_T + \int_0^T \Lambda_s c_s \mathrm{d}s\right]\\
&\quad \leqslant \mathbb{E}\left[\int_0^T \kappa_{0,s}^u (F(c_s, u_s) - (\kappa_{0,s}^u)^{-1}\Lambda_s c_s)\mathrm{d}s + \kappa_{0,T}^u(\Phi(c_T) - (\kappa_{0,T}^u)^{-1}\Lambda_T c_T)\right] + \lambda x\\
&\quad \leqslant \mathbb{E}\left[\int_0^T \kappa_{0,s}^u F^*((\kappa_{0,s}^u)^{-1}\Lambda_s, u_s)\mathrm{d}s + \kappa_{0,T}^u \Phi^*((\kappa_{0,T}^u)^{-1}\Lambda_T)\right] + \lambda x,
\end{aligned}$$

where the first inequality comes from the supermartingale condition in (2.2.14) and the second from the definitions of $F^*$ and $\Phi^*$ in (2.2.13). A quick calculation, see Lemma B.16, shows that $F^*$ and $\Phi^*$ satisfy the scaling property

$$\kappa_{t,s}^u F^*\left((\kappa_{t,s}^u)^{-1}\Lambda_s, u_s\right) = \kappa_{t,s}^{\frac{u}{\gamma}} F^*(\Lambda_s, u_s) \qquad \text{and} \qquad \kappa_{t,T}^u \Phi^*\left((\kappa_{t,T}^u)^{-1}\Lambda_T\right) = \kappa_{t,T}^{\frac{u}{\gamma}} \Phi^*(\Lambda_T).$$

[8]The calculation is as above if $u > \delta\theta$, becomes easier if $u = \delta\theta$ (then $F \equiv F^* \equiv 0$) and is trivial if $u < \delta\theta$ (then $F \equiv F^* \equiv -\infty$), compare Lemma B.10 and Lemma B.9, respectively.

Applying this property to the last line of (2.2.18), we introduce the *stochastic variational dual*

$$U_t^*\left[\Lambda, u\right] \triangleq \mathbb{E}_t\left[\int_t^T \kappa_{t,s}^{\frac{u}{\gamma}} F^*(\Lambda_s, u_s)\mathrm{d}s + \kappa_{t,T}^{\frac{u}{\gamma}} \Phi^*(\Lambda_T)\right]. \tag{2.2.19}$$

for all $u \in \mathcal{P}$, $\Lambda \in \mathcal{D}$ and $t \in [0,T]$. Note that analogously to the stochastic variational utility from (2.2.4), the stochastic variational dual always exists (in $\bar{\mathbb{R}}$).

By taking $\sup_{(\pi,c)\in\mathcal{A}}$ on the left-hand side and $\inf_{\Lambda\in\mathcal{D}}$ on the right-hand side in (2.2.18) and applying the scaling property from above, we obtain for any $u \in \mathcal{P}$ the duality relation

$$\sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}\left[\int_0^T \kappa_{0,s}^u F(c_s, u_s)\mathrm{d}s + \kappa_{0,T}\Phi(c_T)\right] \leqslant \underbrace{\inf_{\Lambda\in\mathcal{D}} \mathbb{E}\left[\int_0^T \kappa_{0,s}^{\frac{u}{\gamma}} F^*(\Lambda_s, u_s)\mathrm{d}s + \kappa_{0,T}^{\frac{u}{\gamma}}\Phi^*(\Lambda_T)\right] + \lambda x}_{\triangleq (D')}.$$

or in a more compact form

$$\sup_{(\pi,c)\in\mathcal{A}} U[c,u] \leqslant \inf_{\Lambda\in\mathcal{D}} \left\{U^*[\Lambda, u] + \lambda x\right\}. \tag{2.2.20}$$

Summing up our progress so far, we have transformed our initial optimization problem to a mini-max problem of the form

$$\begin{aligned}
V_0[c] &= \sup_{u\in\mathcal{P}} \sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}\left[\int_0^T \kappa_{0,s}^u F(c_s, u_s)\mathrm{d}s + \kappa_{0,T}^u\Phi(c_T)\right] \\
&\leqslant \sup_{u\in\mathcal{P}} \inf_{\Lambda\in\mathcal{D}} \mathbb{E}\left[\int_0^T \kappa_{0,s}^{\frac{u}{\gamma}} F^*(\Lambda_s, u_s)\mathrm{d}s + \kappa_{0,T}^{\frac{u}{\gamma}}\Phi^*(\Lambda_T)\right] + \lambda x \\
&\leqslant \inf_{\Lambda\in\mathcal{D}} \sup_{u\in\mathcal{P}} \mathbb{E}\left[\int_0^T \kappa_{0,s}^{\frac{u}{\gamma}} F^*(\Lambda_s, u_s)\mathrm{d}s + \kappa_{0,T}^{\frac{u}{\gamma}}\Phi^*(\Lambda_T)\right] + \lambda x,
\end{aligned}$$

or in compact form

$$V_0[c] \leqslant \inf_{\Lambda\in\mathcal{D}} \left\{\sup_{u\in\mathcal{P}} U_0^*\left[\Lambda, u\right] + \lambda x\right\}.$$

We have thus derived a dual problem, but currently in a variational formulation.

**Step 3: Retransformation from Variational to Recursive Form**

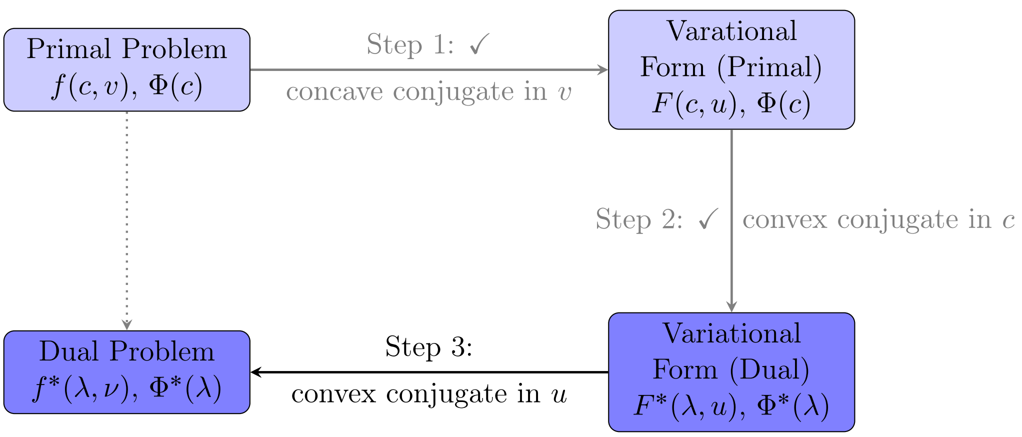



Figure 2.5: Dualization: Step 3

This last step of the dualization procedure reverses step 1, by introducing the so called *stochastic differential dual.* This process allows for a recursive formulation of the variational dual problem $\inf_{\Lambda\in\mathcal{D}}\{\sup_{u\in\mathcal{P}} U_0^*[\Lambda,u]+\lambda x\}$. Motivated by Lemma 2.4, we provide a candidate representation $V^*[\Lambda]$ for $\sup_{u\in\mathcal{U}} U_0^*[\Lambda,u]$ and prove the equality of both processes in Lemma 2.10, which is the dual analogue to Lemma 2.4.

We start reversing the Legendre-Fenchel transformation performed in step 1, by considering minus the concave conjugate of $F^*$ in $u$, i.e.

$$\tilde{f}^*:(0,\infty)\times\mathbb{R}\to(-\infty,\infty],\ (\lambda,\nu)\mapsto\sup_{u\in\mathbb{R}}\{F^*(\lambda,u)-u\nu\}. \tag{2.2.21}$$

Then $\tilde{f}^*$ reads as

$$\tilde{f}^*(\lambda,\nu)=\begin{cases}f^*(\lambda,\gamma\nu), & (1-\gamma)\nu>0\\ f^*(\lambda,0+)+f_\nu^*(\lambda,0+)\cdot\nu, & (1-\gamma)\nu\leqslant 0\end{cases}, \tag{2.2.22}$$

where $f^*$ is defined by

$$f^*:(0,\infty)\times\mathbb{V}\to\mathbb{R},\ (\lambda,\nu)\mapsto\delta^\psi\frac{\lambda^{1-\psi}}{\psi-1}\left(\frac{(1-\gamma)}{\gamma}\nu\right)^{1-\frac{\gamma\psi}{\theta}}-\frac{\delta\theta}{\gamma}\nu, \tag{2.2.23}$$

see Lemma B.14.

**Note:** The similarity of the extended primal and dual aggregators $\tilde{f}$ from (2.2.1) and $\tilde{f}^*$ is apparent, but the additional factor $\gamma$ in the second argument of $f^*$ in (2.2.22) may seem peculiar on first sight. This is mainly for notational convenience, compare Remark 2.7.

By Theorem B.3 it follows that $\tilde{f}^*$ is a lower-semicontinuous, convex function in $\nu$ and that the functions $F^*$ and $\tilde{f}^*$ are dual in the sense that $F^*$ is the concave conjugate of $-\tilde{f}^*$, i.e.

$$F^*:(0,\infty)\times\mathbb{R}\to[-\infty,\infty),\ (\lambda,u)\mapsto\inf_{\nu\in\mathbb{R}}\left\{\tilde{f}^*(\lambda,\nu)+u\nu\right\}. \tag{2.2.24}$$

We are now ready to define the stochastic differential dual as follows.

**Definition 2.6**
*An Epstein-Zin* stochastic differential dual *associated to a deflator* $\Lambda\in\mathcal{D}$ *is a semimartingale* $V^*[\Lambda]\triangleq(V_t^*[\Lambda])_{t\in[0,T]}$ *satisfying*

$$V_t^*[\Lambda]=\mathbb{E}_t\left[\int_t^T f^*\left(\Lambda_s,V_s^*[\Lambda]\right)\mathrm{d}s+\Phi^*(\Lambda_T)\right],\qquad t\in[0,T], \tag{2.2.25}$$

*where* $f^*$ *as given in* (2.2.23).

**Remark 2.7**
*Note that* $\tilde{f}^*$ *is exactly minus the concave conjugate of* $\tilde{f}$ *in* $c$, *i.e.*

$$\tilde{f}^*(\lambda,\nu)=\sup_{c>0}\left\{\tilde{f}(c,\nu)-\lambda c\right\}.$$

*This transformation has little to do with dualization though, as one doesn't address the process* $V[c]$ *and its recursive dependence on consumption, however it shows that the two variational* *transformations in step* 1 *and* 3*, respectively, cancel each other out. Hence moving to variational utility is really just a tool that allows us to apply the well known dualization in time-additive frameworks. The additional factor* $\frac{1}{\gamma}$ *in the definition of the dual aggregator* $f^*$ *is thus a byproduct of this true dualization step.* △

We denote by $\mathcal{D}^a_\lambda$ the class of state price densities $\Lambda$ with $\Lambda_0 = \lambda$, whose associated stochastic differential dual $V^*[\Lambda]$ uniquely exists, satisfies $(1-\gamma)V^*[\Lambda] > 0$ and $V^*[\Lambda]$ is of class (D). For notational convenience, we moreover set $\mathcal{D}^a = \bigcup_{\lambda>0} \mathcal{D}^a_\lambda$ as above.

Similar as in (2.1.3), we assume that $\Lambda \in \mathcal{D}^a$ satisfies the minimal integrability condition

$$\mathbb{E}\left[\int_0^T \Lambda_t^{1-\psi}\mathrm{d}t + \Lambda_T^{\frac{\gamma-1}{\gamma}}\right] < \infty.$$

We prove an existence and uniqueness result in a general semimartingale setting as in [Seiferling and Seifried, 2016] in Section 2.3. For the Brownian setting, the existence and uniqueness results from [Schroder and Skiadas, 1999] and [Xing, 2017] from Proposition 2.2 can be transferred to the stochastic differential dual as shown in [Matoussi and Xing, 2018].

**Proposition 2.8** ([Matoussi and Xing, 2018], Proposition 2.5)
*Let* $(\mathscr{F}_t^W)_{t\in[0,T]}$ *be the augmented filtration generated by some Brownian motion* $W$*. Then the following holds for the stochastic differential dual:*

*(i) When either* $\gamma > 1, 0 < \psi < 1$ *or* $0 < \gamma < 1, \psi > 1$*, then for any* $\Lambda \in \mathcal{D}$ *such that* $\mathbb{E}\left[\int_0^T \Lambda_t^\ell \mathrm{d}t + \Lambda_T^\ell\right] < \infty$ *for all* $\ell \in \mathbb{R}$*, there exists a unique semimartingale* $V^* = V^*[\Lambda]$ *satisfying* (2.2.25) *such that* $\mathbb{E}[\operatorname{ess\,sup}_{t\in[0,T]} |V_t^*|^\ell] < \infty$ *for every* $\ell > 0$*.*

*(ii) When* $\gamma, \psi > 1$*, then for any* $\Lambda \in \mathcal{D}$ *such that* $\mathbb{E}\left[\int_0^T \Lambda_t^{1-\psi}\mathrm{d}t + \Lambda_T^{\frac{\gamma-1}{\gamma}}\right] < \infty$*, there exists a unique semimartingale* $V^* = V^*[\Lambda]$ *satisfying* (2.2.25)*, such that* $V^*$ *is of class (D).*

*In both cases* $(1-\gamma)V^* > 0$*, such that* $V^* \in \mathcal{V}$*.*

In particular $\mathcal{D}^a \neq \emptyset$.

Further existence and uniqueness results for solutions of (2.2.25) in a general semimartingale setting are provided in Section 2.3 and Section 3.3.

**Example 2.9**
*As for the primal problem, if* $\gamma\psi = 1$*,* (2.2.25) *reduces to a time-additive optimization problem*

$$V_t^*[\Lambda] = \mathbb{E}_t\left[\int_t^T \delta^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}(s-t)}\varphi^*(\Lambda_s)\mathrm{d}s + e^{-\frac{\delta}{\gamma}(T-t)}\Phi^*(\Lambda_T)\right], \tag{2.2.26}$$

*where* $\varphi^* : (0,\infty) \to \mathbb{R}$*,* $\varphi^*(\lambda) = \frac{\gamma}{1-\gamma}\lambda^{\frac{\gamma-1}{\gamma}}$ *and* $\Phi^*(\lambda) = \varepsilon^{\frac{1}{\gamma}}\frac{\gamma}{1-\gamma}\lambda^{\frac{\gamma-1}{\gamma}}$ *are the Legendre-Fenchel transformations of* $\varphi$ *and* $\Phi$ *in* $c$*, respectively, see Theorem A.6. Note that this is exactly the dual utility process resulting from straightforward dualization of* $V[c]$ *in the power utility case as given in Example 2.1.* ○

The following lemma is the actual third step of the dualization procedure, i.e. it formalizes the retransformation of the dual variational utility $U^*[\Lambda, u]$ to recursive form, more precisely to the stochastic differential dual $V^*[\Lambda]$. It is a slight extension of Lemma 2.6 in [Matoussi and Xing, 2018] in the same way as Lemma 2.4 extends Lemma 2.3 therein. We only show the proof of the parameter constellation $\gamma\psi = 1$ in detail, as it emphasizes the consistency of our extended dualization procedure with straightforward dualization of the problem in the power utility case. Generally the proof is analogous to the one of Lemma 2.4.

**Lemma 2.10**
*For any $u \in \mathcal{P}$ and $\Lambda \in \mathcal{D}^a$, let $V^*[\Lambda]$ be the stochastic differential dual associated with $\Lambda$ and $U^*[\Lambda, u]$ given as in* (2.2.19)*. Then for any $t \in [0, T]$,*

$$V_t^*[\Lambda] = \operatorname*{ess\,sup}_{u \in \mathcal{P}} U_t^*[\Lambda, u].$$

*Moreover, the supremum is attained at*

$$u^\Lambda \triangleq -\tilde{f}_\nu^*(\lambda, \nu) = -\delta^\psi \tfrac{1-\gamma\psi}{\gamma(\psi-1)} \lambda^{1-\psi} \left(\tfrac{1-\gamma}{\gamma}\nu\right)^{-\frac{\gamma(\psi-1)}{1-\gamma}} + \delta\theta.$$

*Proof.* Let $\gamma\psi = 1$. Then the Legendre-Fenchel transform $F^*$ desintegrates to

$$F^*(\lambda, u) = \begin{cases} \delta^{\frac{1}{\gamma}} \frac{\gamma}{1-\gamma} \lambda^{\frac{\gamma-1}{\gamma}} & u = \delta \\ -\infty & else \end{cases},$$

see Lemma B.10. In particular

$$\begin{aligned}
\sup_{u \in \mathcal{P}} U_t^*[\Lambda, u] &= \sup_{u \in \mathcal{P}} \mathbb{E}_t \left[ \int_t^T \kappa_{t,s}^{\frac{u}{\gamma}} F^*(\Lambda_s, u_s) \mathrm{d}s + \kappa_{t,T}^{\frac{u}{\gamma}} \Phi^*(\Lambda_T) \right] \\
&= \mathbb{E}_t \left[ \int_t^T \delta^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}(s-t)} \tfrac{\gamma}{1-\gamma} \Lambda_s^{\frac{\gamma-1}{\gamma}} \mathrm{d}s + e^{-\frac{\delta}{\gamma}(T-t)} \Phi^*(\Lambda_T) \right],
\end{aligned}$$

so the proof follows by (2.2.26) in Example 2.9.
As the functions $\tilde{f}^*$ and $F^*$ have exactly the same properties as $\tilde{f}$ and $F$ in the relevant second argument in terms of sign, convexity and so on, the remaining parameter constellations are proven exactly as in Lemma 2.4. ☐

**Note:** Instead of repeating the exact same arguments from Lemma 2.4 here, we prove the analoguous statement of Lemma 2.10 for the parameter constellations $\gamma\psi \leqslant 1, \psi < 1$ in the appendix to emphasize the adjustments one needs to make in order to transfer all results from the convex to the concave case. We thereby also provide ourselves with the rigorous reference, as the concave analogue of Lemma 2.10 is used within Section 3.2.

For notational convenience we define the *dual utility index* functional as

$$\mathsf{v}^* : \mathcal{D}^a \to \mathcal{V},\ \mathsf{v}^*(\Lambda) \triangleq V_0^*[\Lambda] + \lambda x.$$

Then the agent's dual problem is to

$$\text{find } \Lambda^* \in \mathcal{D}^a \text{ such that } \mathsf{v}^*(\Lambda^*) = \inf_{\Lambda \in \mathcal{D}^a} \{V_0^*[\Lambda] + \lambda x\}. \tag{2.2.27}$$

### 2.2.2 Duality Inequality and a Simplified Version

Combining all the steps in the previous section, we obtain the final duality inequality.

**Theorem 2.11**
*Let $\gamma\psi \geqslant 1, \psi > 1$, then*[9]

$$\sup_{(\pi,c)\in\mathcal{A}} V_0[c] \leqslant \inf_{\Lambda\in\mathcal{D}^a} \{V_0^*[\Lambda] + \lambda x\}. \tag{2.2.28}$$

*Proof.*

$$\begin{aligned}
\sup_{(\pi,c)\in\mathcal{A}} V_0[c] &= \sup_{(\pi,c)\in\mathcal{A}} \sup_{u\in\mathcal{P}} U_0[c,u] && \text{(Lemma 2.4)}\\
&= \sup_{u\in\mathcal{P}} \sup_{(\pi,c)\in\mathcal{A}} U_0[c,u] \\
&\leqslant \sup_{u\in\mathcal{P}} \inf_{\Lambda\in\mathcal{D}^a} \{U_0^*[\Lambda,u] + \lambda x\} && (2.2.20)\\
&\leqslant \inf_{\Lambda\in\mathcal{D}^a} \left\{\sup_{u\in\mathcal{P}} U_0^*[\Lambda,u] + \lambda x\right\} \\
&= \inf_{\Lambda\in\mathcal{D}^a} \{V_0^*[\Lambda] + \lambda x\}. && \text{(Lemma 2.10)}
\end{aligned}$$

□

**Remark 2.12**
*With a slight abuse of notation we define*

$$\nu(c^*) \triangleq \sup_{(\pi,c)\in\mathcal{A}} V_0[c] \qquad \text{and } \nu^*(\Lambda^*) \triangleq \inf_{\Lambda\in\mathcal{D}^a} \{V_0^*[\Lambda] + \lambda x\}$$

*although the optimal strategies might not exist in general. Note that if the primal and dual optimizer exist, this definition coincides with the primal and dual utility index, respectively, as defined above.* △

**Remark 2.13**
*As already mentioned our extended procedure also carries over to the concave case with parameter constellations $\gamma\psi \leqslant 1, \psi < 1$: First, $\tilde{f}$ can be defined as above, but the image set changes from $(-\infty,\infty]$ to $[-\infty,\infty)$, however the conjugates from the convex and concave case coincide on their real domains, see Appendix B.2. As $\tilde{f}$ is now an upper-semicontinuous concave function, the suprema and infima in (2.2.2), (2.2.3), (2.2.21) and (2.2.24) become infima and suprema, respectively. The adjustment in the image set of $\tilde{f}$ naturally carries over to all conjugates. Also the essential suprema in Lemma 2.4 and Lemma 2.10 now become essential infima. In order to eludicate those adjustments formally, we have treated the concave case of Lemma 2.10 in the appendix, see Lemma B.17. The duality procedure in the case of concave aggregators can then*

[9] The result is also valid if $\gamma\psi \leqslant 1, \psi < 1$, but with a slightly different proof, see Remark 2.13

*in a compact form be written as*

$$\begin{aligned}
\sup_{(\pi,c)\in\mathcal{A}} V_0[c] &= \sup_{(\pi,c)\in\mathcal{A}} \inf_{u\in\mathcal{P}} U_0[c,u] && \text{(Remark B.18)}\\
&\leqslant \inf_{u\in\mathcal{P}} \sup_{(\pi,c)\in\mathcal{A}} U_0[c,u]\\
&\leqslant \inf_{u\in\mathcal{P}} \inf_{\Lambda\in\mathcal{D}^a} \{U_0^*[\Lambda,u] + \lambda x\} && (2.2.20)\\
&= \inf_{\Lambda\in\mathcal{D}^a} \left\{ \inf_{u\in\mathcal{P}} U_0^*[\Lambda,u] + \lambda x \right\}\\
&= \inf_{\Lambda\in\mathcal{D}^a} \{V_0^*[\Lambda] + \lambda x\}. && \text{(Lemma B.17)}
\end{aligned}$$

*In particular Theorem 2.11 stays true if* $\gamma\psi \leqslant 1, \psi < 1$. △

We now want to extend Theorem 2.11 to all $t \in [0,T]$ to obtain a dynamic duality relation of the value processes. To this end, consider for every strategy $(\pi, c) \in \mathcal{A}$ and deflator $\Lambda \in \mathcal{D}^a$, respectively, the sets

$$\begin{aligned}
\mathcal{A}(\pi,c,t) &\triangleq \{(\tilde{\pi},\tilde{c}) \in \mathcal{A} : (\tilde{\pi},\tilde{c}) = (\pi,c) \text{ on } [0,t]\},\\
\mathcal{D}^a(\Lambda,t) &\triangleq \{\tilde{\Lambda} \in \mathcal{D}^a : \tilde{\Lambda} = \Lambda \text{ on } [0,t]\},
\end{aligned}$$

and define the primal and dual value processes as

$$\mathbb{V}_t[\pi,c] \triangleq \operatorname*{ess\,sup}_{(\tilde{\pi},\tilde{c})\in\mathcal{A}(\pi,c,t)} V_t[\tilde{c}] \quad \text{and} \quad \mathbb{V}_t^*[\Lambda] \triangleq \operatorname*{ess\,inf}_{\tilde{\Lambda}\in\mathcal{D}^a(\Lambda,t)} V_t^*[\tilde{\Lambda}].$$

**Corollary 2.14**
*If* $\gamma\psi \geqslant 1, \psi > 1$ *or* $\gamma\psi \leqslant 1, \psi < 1$, *then*

$$\mathbb{V}_t[\pi,c] \leqslant \mathbb{V}_t^*[\Lambda] + \Lambda_t X_t^{(\pi,c)}, t \in [0,T]. \tag{2.2.29}$$

*Proof.* Using (2.2.17) and the scaling property of $F^*$ and $\Phi^*$ from Lemma B.16 equation (2.2.18) generalizes to

$$\mathbb{E}_t\left[\int_t^T \kappa_{t,s}^u F(\tilde{c}_s,u_s)ds + \kappa_{t,T}^u \Phi(\tilde{c}_T)\right] \leqslant \mathbb{E}_t\left[\int_t^T \kappa_{t,s}^{\frac{u}{\gamma}} F^*(\tilde{\Lambda}_s,u_s)\mathrm{d}s + \kappa_{t,T}^{\frac{u}{\gamma}} \Phi^*(\tilde{\Lambda}_T)\right] + \Lambda_t X_t^{(\pi,c)}.$$

Now, taking $\operatorname{ess\,sup}_{(\tilde{\pi},\tilde{c})\in\mathcal{A}(\pi,c,t)}$ and $\operatorname{ess\,inf}_{\tilde{\Lambda}\in\mathcal{D}^a(\Lambda,t)}$ in the above inequality we obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{(\tilde{\pi},\tilde{c})\in\mathcal{A}(\pi,c,t)} \mathbb{E}_t\left[\int_t^T \kappa_{t,s}^u F(\tilde{c}_s,u_s)\mathrm{d}s + \kappa_{t,T}\Phi(\tilde{c}_T)\right]\\
&\qquad\leqslant \operatorname*{ess\,inf}_{\tilde{\Lambda}\in\mathcal{D}^a(\Lambda,t)} \mathbb{E}_t\left[\int_t^T \kappa_{t,s}^{\frac{u}{\gamma}} F^*(\tilde{\Lambda}_s,u_s)\mathrm{d}s + \kappa_{t,T}^{\frac{u}{\gamma}} \Phi^*(\tilde{\Lambda}_T)\right] + \Lambda_t X_t^{(\pi,c)}.
\end{aligned}$$

As Lemma 2.4 and Lemma 2.10 are already stated in a dynamic manner, the result follows as in the proof of Theorem 2.11 and Remark 2.13, respectively. □

It often turns out to be convenient to reduce ourselves to pricing deflators $D \in \mathcal{D}_1^a$. The transformation of stochastic differential duals we introduce in Proposition 2.16 below, allows to transfer all results for such pricing deflators to general deflators $\Lambda = \lambda D \in \mathcal{D}^a$. The following scaling property of stochastic differential duals is the key to the proof.

**Lemma 2.15**
*The mapping* $\mathcal{D}^a \to \mathcal{V}^a$, $\Lambda \mapsto V^*[\Lambda]$ *is homothetic, i.e. for every* $k > 0$ *it holds that*

$$V^*[k\Lambda] = k^{\frac{\gamma-1}{\gamma}} V^*[\Lambda].$$

*Proof.* A small calculation reveals that $f^*\left(k\lambda, k^{\frac{\gamma-1}{\gamma}}\nu\right) = k^{\frac{\gamma-1}{\gamma}} f^*(\lambda,\nu)$, and certainly $\Phi^*(k\lambda) = k^{\frac{\gamma-1}{\gamma}}\Phi^*(\lambda)$. Then the result follows immediately by uniqueness of the BSDE solution. □

Analogously to the sets $\mathcal{A}(\pi, c, t)$ and $\mathcal{D}^a(\Lambda, t)$ above, define for any $D \in \mathcal{D}_1^a$ the set

$$\mathcal{D}_1^a(D,t) \triangleq \{\tilde{D} \in \mathcal{D}_1^a : \tilde{D} = D \text{ on } [0,t]\}.$$

Then the following duality inequality in terms of pricing deflators holds.

**Proposition 2.16**
*Let* $\Lambda = \lambda D \in \mathcal{D}^a$ *and* $\mathbb{V}^*[\Lambda]$ *be the associated dual value process. Then for any* $t \in [0,T]$ *we have*

$$\mathbb{V}_t^*[\Lambda] + \Lambda_t X_t^{(\pi,c)} = \tfrac{1}{1-\gamma}\left(D_t X_t^{(\pi,c)}\right)^{1-\gamma}\left(\tfrac{1-\gamma}{\gamma}\mathbb{V}_t^*[D]\right)^{\gamma}.$$

*In particular* (2.2.28) *can be expressed in terms of* $D^* \in \mathcal{D}_1^a$ *as*

$$\mathsf{v}(c^*) \leqslant \tfrac{1}{1-\gamma} x^{1-\gamma}\left(\tfrac{1-\gamma}{\gamma}\mathsf{v}^*(D^*)\right)^{\gamma}.$$

*Proof.* Write $\Lambda = \lambda D$ for $\Lambda \in \mathcal{D}_\lambda^a$, where $\lambda > 0$ and $D \in \mathcal{D}_1^a$ is a pricing deflator. Then the homotheticity of the stochastic differential dual Lemma 2.15 implies

$$V_t^*[\Lambda] = V_t^*[\lambda D] = \lambda^{\frac{\gamma-1}{\gamma}} V_t^*[D].$$

Inserting this equation to the duality relation (2.2.29) yields

$$\begin{aligned}
\mathbb{V}_t^*[\Lambda] + \Lambda_t X_t^{(\pi,c)} &= \operatorname*{ess\,inf}_{\tilde{\Lambda}\in\mathcal{D}^a(\Lambda,t)} \left\{V_t^*[\tilde{\Lambda}] + \Lambda_t X_t^{(\pi,c)}\right\}\\
&= \operatorname*{ess\,inf}_{\tilde{D}\in\mathcal{D}_1^a(D,t)} \inf_{\lambda>0} \left\{V_t^*[\lambda\tilde{D}] + \lambda D_t X_t^{(\pi,c)}\right\}\\
&= \operatorname*{ess\,inf}_{\tilde{D}\in\mathcal{D}_1^a(D,t)} \inf_{\lambda>0} \left\{\lambda^{\frac{\gamma-1}{\gamma}} V_t^*[\tilde{D}] + \lambda D_t X_t^{(\pi,c)}\right\}
\end{aligned}$$

The inner problem $\inf_{\lambda>0}\left\{\lambda^{\frac{\gamma-1}{\gamma}} V_t^*[\tilde{D}] + \lambda D_t X_t^{(\pi,c)}\right\}$ is convex in $\lambda$ and the first order condition yields

$$\lambda^* = \left(D_t X_t^{(\pi,c)}\right)^{-\gamma}\left(\tfrac{1-\gamma}{\gamma} V_t^*[\tilde{D}]\right)^{\gamma},$$

which is strictly positive as $(1-\gamma)V_t^*[\tilde{D}] > 0$. Inserting $\lambda^*$ to the inner problem yields the infimum

$$\inf_{\lambda>0}\left\{\lambda^{\frac{\gamma-1}{\gamma}} V_t^*[\tilde{D}] + \lambda D_t X_t^{(\pi,c)}\right\} = \tfrac{1}{1-\gamma}\left(D_t X_t^{(\pi,c)}\right)^{1-\gamma}\left(\tfrac{1-\gamma}{\gamma} V_t^*[\tilde{D}]\right)^{\gamma} \triangleq \mathfrak{d}(D_t X_t^{(\pi,c)}, V_t^*[\tilde{D}]),$$

where $\mathfrak{d}(x,\nu) = \frac{1}{1-\gamma}x^{1-\gamma}\left(\frac{1-\gamma}{\gamma}\nu\right)^{\gamma}$. Then $\mathfrak{d}_\nu(x,\nu) = x^{1-\gamma}\left(\frac{1-\gamma}{\gamma}\nu\right)^{\gamma-1} > 0$, so $\mathfrak{d}$ is strictly increasing in $\nu$ and

$$\begin{aligned}
\mathbb{V}_t^*[\Lambda] + \Lambda_t X_t^{(\pi,c)} &= \underset{\tilde{D}\in\mathcal{D}^a(D,t)}{\operatorname{ess\,inf}} \inf_{\lambda>0} \left\{\lambda^{\frac{\gamma-1}{\gamma}} V_t^*[\tilde{D}] + \lambda D_t X_t^{(\pi,c)}\right\} \\
&= \underset{\tilde{D}\in\mathcal{D}_1^a(D,t)}{\operatorname{ess\,inf}} \left\{\frac{1}{1-\gamma}\left(D_t X_t^{(\pi,c)}\right)^{1-\gamma}\left(\frac{1-\gamma}{\gamma}V_t^*[\tilde{D}]\right)^{\gamma}\right\} \\
&= \frac{1}{1-\gamma}\left(D_t X_t^{(\pi,c)}\right)^{1-\gamma}\left(\frac{1-\gamma}{\gamma}\underset{\tilde{D}\in\mathcal{D}_1^a(D,t)}{\operatorname{ess\,inf}}\left\{V_t^*[\tilde{D}]\right\}\right)^{\gamma} \\
&= \frac{1}{1-\gamma}\left(D_t X_t^{(\pi,c)}\right)^{1-\gamma}\left(\frac{1-\gamma}{\gamma}\mathbb{V}_t^*[D]\right)^{\gamma}.
\end{aligned}$$

□

Now Proposition 2.16 allows us to transfer all further investigations from pricing deflators $D$ to general deflators $\Lambda$ by applying the transformation

$$\mathfrak{d}(x,\cdot) = \frac{1}{1-\gamma}x^{1-\gamma}\left(\frac{1-\gamma}{\gamma}\cdot\right)^{\gamma}$$

to the associated stochastic differential dual $V^*[D]$. In particular Proposition 2.16 allows to determine the dual value by solely optimizing over $D \in \mathcal{D}_1^a$ as the optimization over $\lambda > 0$ is implicitly already incorporated in the above representation.

## 2.3 The Stochastic Differential Dual: Existence, Uniqueness, Convexity and Utility Gradients

The goal of this section is to answer several questions about existence, uniqueness, monotonicity and convexity of the stochastic differential dual in a general semimartingale setting, see Theorem 2.20 below. We already mentioned that with classical results the existence of a solution to the BSDE characterizing the Epstein-Zin utility process cannot be guaranteed, as the aggregator $f$ is not Lipschitz in $v$; the same is true for the BSDE characterizing the stochastic differential dual so we have to take some extra efforts.
We also treat a dual utility gradient inequality. Utility gradients and their far reaching implications are discussed in Remark 2.18.
As we have seen in Proposition 2.8, [Matoussi and Xing, 2018] already provide some general existence and uniqueness results for the associated backward stochastic differential equation in a Brownian setting. In the recent paper [Becherer et al., 2023], the authors show existence and uniqueness of stochastic differential duals in a general semimartingale setting, prove monotonicity and convexity properties and derive a dual utility gradient inequality. However, as they consider a certain modification of our classical Epstein-Zin aggregator, their result only covers the cases $\gamma < 1, \gamma\psi \geqslant 1$ and $\gamma > 1, \gamma\psi \leqslant 1$, see equation (2.3.4). Our contribution is to include the case $\gamma, \psi < 1$ and in particular the empirically relevant case $\gamma, \psi > 1$. As in [Becherer et al., 2023] we do this by transferring the respective results for stochastic differential utility to stochastic differential duals via a BSDE transformation. Hence we start briefly restating the corresponding results on the primal side provided by [Seiferling and Seifried, 2016].

Denote the set of consumption streams under consideration by

$$\mathcal{C}^\infty \triangleq \left\{c \in \mathcal{C} : \mathbb{E}\left[\int_0^T c_t^\ell \mathrm{d}t + c_T^\ell\right] < \infty \text{ for all } \ell \in \mathbb{R}\right\}$$

and the set of corresponding utility processes by

$$\mathcal{V}^{\infty} \triangleq \left\{ V \in \mathcal{V} : \mathbb{E}\left[ \sup_{t \in [0,T]} |V_t|^{\ell} \right] < \infty \text{ for all } \ell \in \mathbb{R} \right\}. \tag{2.3.1}$$

We define a partial order on $\mathcal{C}^{\infty}$ via

$$c \preceq \bar{c} \quad \text{if and only if} \quad c_t \leqslant \bar{c}_t \text{ for d}t\text{-a.e. } t \in [0,T) \text{ and } c_T \leqslant \bar{c}_T.$$

The following proposition summarizes the main results presented in [Seiferling and Seifried, 2016] which we transfer to the dual value function below.

**Proposition 2.17** ([Seiferling and Seifried, 2016], Theorem 3.1, 3.3 & 3.4)
*When $\gamma\psi \geqslant 1, \psi > 1$ or $\gamma\psi \leqslant 1, \psi < 1$, then*

*(i) for any $c \in \mathcal{C}^{\infty}$ there exists a unique semimartingale $V[c] \in \mathcal{V}^{\infty}$ that satisfies*

$$V_t[c] = \mathbb{E}_t\left[\int_t^T f(c_s, V_s[c])\mathrm{d}s + \Phi(c_T)\right] \qquad \forall t \in [0,T].$$

*In particular $\mathcal{C}^{\infty} \subseteq \mathcal{C}^a$.*

*(ii) the mapping $\mathcal{C}^{\infty} \to \mathcal{V}^{\infty}$, $c \mapsto V[c]$ is concave and increasing in the sense that if $c \preceq \bar{c}$, then $V_t[c] \leqslant V_t[\bar{c}]$ for all $t \in [0,T]$.*

*(iii) for all $c, \bar{c} \in \mathcal{C}^{\infty}$ and every $t \in [0,T]$ we have*

$$V_t[c] \leqslant V_t[\bar{c}] + \left\langle \mathfrak{m}^t(\bar{c}), c - \bar{c} \right\rangle_t \tag{2.3.2}$$

*where $\langle \mathfrak{m}, y \rangle_t \triangleq \mathbb{E}_t\left[\int_t^T \mathfrak{m}_s y_s \mathrm{d}s + \mathfrak{m}_T y_T\right]$ and the time-$t$ utility gradient $\mathfrak{m}^t(\bar{c})$ is given by*

$$\mathfrak{m}_s^t(\bar{c}) \triangleq \exp\left(\int_t^s f_v\left(\bar{c}_\tau, V_\tau[\bar{c}]\right)\mathrm{d}\tau\right)\nabla_s \quad \textit{with} \quad \nabla_s \triangleq \begin{cases} f_c\left(\bar{c}_s, V_s[\bar{c}]\right), & 0 \leqslant s < T, \\ \Phi'(\bar{c}_T), & s = T. \end{cases}$$

As we aim to prove the dual analogue to Proposition 2.17, we define

$$\mathcal{D}^{\infty} \triangleq \left\{ \Lambda \in \mathcal{D} : \mathbb{E}\left[\int_0^T \Lambda_t^{\ell}\mathrm{d}t + \Lambda_T^{\ell}\right] < \infty \text{ for all } \ell \in \mathbb{R} \right\} \tag{2.3.3}$$

and to emphasize the special importance of pricing deflators also $\mathcal{D}_1^{\infty} \triangleq \{D \in \mathcal{D}^{\infty} : D(0) = 1\}$. As above we define a partial order on $\mathcal{D}^{\infty}$ as

$$\Lambda \preceq \bar{\Lambda} \quad \text{if and only if} \quad \Lambda_t \leqslant \bar{\Lambda}_t \text{ for d}t\text{-a.e. } t \in [0,T]) \text{ and } \Lambda_T \leqslant \bar{\Lambda}_T.$$

**Remark 2.18**
*Since the pioneering work of [Duffie and Skiadas, 1994], utility gradients as in Proposition 2.17(iii) have proven to be an essential tool in optimal portfolio allocation and equilibrium asset pricing. The profound observation made by [Duffie and Skiadas, 1994] is that the first-order optimality condition can be expressed as a martingale property of prices, once normalized by the corresponding utility gradient, see also [Harrison and Kreps, 1979]. This has far reaching implications in the theory of portfolio optimization and asset pricing. On the one hand, portfolio optimization problems can now be directly addressed using the implied first-order conditions, see for example*

*[Schroder and Skiadas, 1999], [Schroder and Skiadas, 2003], [Bank and Riedel, 2001b], [Kallsen and Muhle-Karbe, 2010] and [Skiadas, 2013]. A general overview on this approach is given in [Skiadas, 2008]. On the other hand, the same line of reasoning can be applied to the representative agent's portfolio in a general equilibrium setting, which leads to the representation of the state-price deflator in the underlying economy as a utility gradient. Numerous authors have exploited this fact, see for example [Duffie and Epstein, 1992b], [Duffie et al., 1994], [Bank and Riedel, 2001a] or [Campbell, 2003] and the references therein.*
*Ultimately, utility gradients inhabit a natural connection to duality: Recall the budget constraint* (2.2.16) *from Remark 2.5*

$$\mathbb{E}\left[D_T X_T^{(\pi,c)} + \int_0^T D_s c_s \mathrm{d}s\right] \leqslant x.$$

*By the above, using the utility gradient as the deflator, the budget constraint evolves from a supermartingale property to the mentioned* martingale *property, but this means the* inequality *within the second step of our duality procedure becomes an* equality, *leaving no space for a duality gap. Thus utility gradients naturally arise as the minimizer of the dual problem, c.f. [Cox and Huang, 1989] [Karatzas et al., 1991], [Kramkov and Schachermayer, 1999], [El Karoui et al., 2001], [Matoussi and Xing, 2018].* △

In [Becherer et al., 2023][Theorem 3.8] the authors transfer all results from Proposition 2.17 to a stochastic differential dual generated by a certain modification of the dual Epstein-Zin aggregator, namely

$$(0,\infty)\times\mathbb{V}\to\mathbb{R},\ (\lambda,\nu)\mapsto \delta^{\psi}\tfrac{1}{\psi-1}\lambda^{1-\psi}\left(\tfrac{1-\gamma}{\gamma}\nu\right)^{1-\frac{\gamma\psi}{\theta}} - \frac{\zeta\delta\theta}{\gamma}\nu, \tag{2.3.4}$$

where

$$\zeta = \begin{cases} 1, & \text{if } \theta > 0 \\ -1, & \text{if } \theta < 0 \end{cases}.$$

They do so because they need always positive derivatives of the aggregator for their main results; however, this means that their existence result only includes the cases $0<\gamma<1, \gamma\psi\geqslant 1$ and $\gamma>1, \gamma\psi\leqslant 1$ of our standard Epstein-Zin aggregator, but does not take into account the cases $\gamma,\psi>1$ and $\gamma,\psi<1$. Moreover, they consider a different set of admissible dual controls than we do.
Upon close inspection one notices that their arguments, specifically a certain BSDE transformation, can be adjusted to apply to our standard stochastic differential dual, then including all those parameter constellations. We do so in Theorem 2.20, which may be seen as a ramification of [Becherer et al., 2023][Theorem 3.8].

We prepare for the proof with the following basic lemma.

**Lemma 2.19**
*Let $V^* \in \mathcal{V}^\infty$ satisfy*

$$V_t^* = \mathbb{E}_t\left[\int_t^T f^*(\Lambda_s, V_s^*)\mathrm{d}s + \Phi^*(\Lambda_T)\right], \qquad t\in[0,T]$$

*for some $\Lambda \in \mathcal{D}^\infty$. Then the process $M^*$ given by*

$$M_t^* \triangleq \mathbb{E}_t\left[\int_0^T f^*\left(\Lambda_s, V_s^*\right)\mathrm{d}s + \Phi^*(\Lambda_T)\right], \qquad t\in[0,T]$$

*is a $L^p$-martingale for all $p \geqslant 1$ and*

$$\mathrm{d}V_t^* = -f^*\left(\Lambda_t, V_t^*\right)\mathrm{d}t + \mathrm{d}M_t^*. \tag{2.3.5}$$

*Proof.* Integrability of $M_t^*$, $t \in [0,T]$, follows immediately by Hölder's inequality and the integrability assumptions in (2.3.1) and (2.3.3); the martingale property is immediate.
Moreover $V_t^* = M_t^* - \int_0^t f^*\left(\Lambda_s, V_s^*\right)\mathrm{d}s$ for any $t \in [0,T]$, which clearly implies (2.3.5). □

**Theorem 2.20**
*When $\gamma\psi \geqslant 1, \psi > 1$ or $\gamma\psi \leqslant 1, \psi < 1$, then*

*(i) for any $\Lambda \in \mathcal{D}^\infty$ there exists a unique semimartingale $V^*[\Lambda] \in \mathcal{V}^\infty$ that satisfies*

$$V_t^*[\Lambda] = \mathbb{E}_t\left[\int_t^T f^*(\Lambda_s, V_s^*[\Lambda])\mathrm{d}s + \Phi^*(\Lambda_T)\right], \qquad t \in [0,T]. \tag{2.3.6}$$

*In particular $\mathcal{D}^\infty \subseteq \mathcal{D}^a$.*

*(ii) the mapping $\mathcal{D}^\infty \to \mathcal{V}^\infty$, $\Lambda \mapsto V^*[\Lambda]$ is convex and decreasing in the sense that if $\Lambda \preceq \bar\Lambda$, then $V_t^*[\Lambda] \geqslant V_t^*[\bar\Lambda]$ for all $t \in [0,T]$.*

*(iii) for all $\Lambda, \bar\Lambda \in \mathcal{D}^\infty$ and every $t \in [0,T]$ we have*

$$V_t^*[\Lambda] \geqslant V_t^*[\bar\Lambda] - \left\langle (\mathfrak{m}^t)^*(\bar\Lambda), \Lambda - \bar\Lambda\right\rangle_t \tag{2.3.7}$$

*where $\langle \mathfrak{m}^*, y\rangle_t \triangleq \mathbb{E}_t\left[\int_t^T \mathfrak{m}_s^* y_s \mathrm{d}s + \mathfrak{m}_T^* y_T\right]$ and the time-$t$ dual utility gradient $(\mathfrak{m}^t)^*(\bar\Lambda)$ is given by*

$$(\mathfrak{m}_s^t)^*(\bar\Lambda) = \exp\left(\int_t^s f_\nu^*\left(\bar\Lambda_\tau, V_\tau^*[\bar\Lambda]\right)\mathrm{d}\tau\right)\nabla_s^*,$$

*where*

$$\nabla_s^* \triangleq \begin{cases} f_\lambda^*(\bar\Lambda_s, V_s^*[\bar\Lambda]), & 0 \leqslant s < T \\ \left(\Phi^*\right)'\left(\bar\Lambda_T\right), & s = T \end{cases}$$

*Proof.* For the first part, let $\mathcal{Y}_t \triangleq -V_t^*$. Then by an application of Itô's formula we obtain

$$\begin{aligned} \mathrm{d}\mathcal{Y}_t &= \left(\delta^\psi \tfrac{1}{\psi-1}\Lambda_t^{1-\psi}\left(\tfrac{1-\gamma}{\gamma}V_t^*\right)^{1-\frac{\gamma\psi}{\theta}} - \tfrac{\delta\theta}{\gamma}V_t^*\right)\mathrm{d}t - \mathrm{d}M_t^* \\ &= -\left(\delta^\psi \tfrac{1}{1-\psi}\Lambda_t^{1-\psi}\left(\left(1-\tfrac{1}{\gamma}\right)\mathcal{Y}_t\right)^{1-\frac{\gamma\psi}{\theta}} - \tfrac{\delta\theta}{\gamma}\mathcal{Y}_t\right)\mathrm{d}t - \mathrm{d}M_t^* \end{aligned} \tag{2.3.8}$$

and $\mathcal{Y}_T = -\Phi^*(\Lambda_T) = \varepsilon^{\frac{1}{\gamma}}\frac{\gamma}{\gamma-1}\Lambda_T^{\frac{\gamma-1}{\gamma}}$. Now setting $q^* \triangleq 1 - \frac{\gamma\psi}{\theta}$, $\hat\delta \triangleq \delta\psi$, $\hat\varepsilon \triangleq \hat\delta\psi^{\frac{\psi}{1-\psi}}\varepsilon^{\frac{1}{\gamma-1}}$, $\hat\Lambda_t \triangleq \frac{1}{\hat\delta}\psi^{\frac{\psi}{\psi-1}}\Lambda_t$, (2.3.8) transforms to

$$\begin{aligned} \mathrm{d}\mathcal{Y}_t &= -\left(\hat\delta\tfrac{1}{1-\psi}\hat\Lambda_t^{1-\psi}\left(\left(1-\tfrac{1}{\gamma}\right)\mathcal{Y}_t\right)^{q^*} - \tfrac{\hat\delta}{1-q^*}\mathcal{Y}_t\right)\mathrm{d}t + \mathrm{d}\mathcal{M}_t^*, \\ \mathcal{Y}_T &= \tfrac{1}{1-\frac{1}{\gamma}}\left(\hat\varepsilon\hat\Lambda_T\right)^{1-\frac{1}{\gamma}}, \end{aligned} \tag{2.3.9}$$

where $\mathcal{M}^* \triangleq -M^*$ is an $L^p$-martingale for all $p \geqslant 1$. Moreover, $\Lambda \in \mathcal{D}^\infty$ implies $\hat\Lambda \in \mathcal{D}^\infty \subseteq \mathcal{C}^\infty$, thus for the parameters as in Section 2.2

$$\tfrac{1}{\gamma\psi} \leqslant 1 \text{ and } \tfrac{1}{\psi} < 1 \quad \Leftrightarrow \quad \gamma\psi \geqslant 1 \text{ and } \psi > 1$$

and

$$\tfrac{1}{\gamma\psi} \geqslant 1 \text{ and } \tfrac{1}{\psi} > 1 \quad \Leftrightarrow \quad \gamma\psi \leqslant 1 \text{ and } \psi < 1.$$

the BSDE (2.3.9) is exactly the BSDE as considered in [Seiferling and Seifried, 2016] as cited in Proposition 2.17 with $\frac{1}{\psi}$, $\gamma$, $c$ and $V$ replaced by $\psi$, $\frac{1}{\gamma}$, $\hat{\Lambda}$ and $\mathcal{Y}$, respectively[10]. Thus by Proposition 2.17(i), it follows that (2.3.9) admits a unique solution

$$\mathcal{Y} \in \left\{ V \in \mathcal{S} : \left(1 - \tfrac{1}{\gamma}\right) V > 0,\, \mathbb{E}\left[ \sup_{t\in[0,T]} |V_t|^\ell \right] < \infty \text{ for all } \ell \in \mathbb{R} \right\} = -\mathcal{V}^\infty.$$

In particular, for any $\Lambda \in \mathcal{D}^\infty$ there exists a unique $V^*[\Lambda] = -\mathcal{Y} \in \mathcal{V}^\infty$.
For the second part, Proposition 2.17(ii) yields that the map $\hat{\Lambda} \mapsto \mathcal{Y}[\hat{\Lambda}]$ is concave and increasing, thus $\Lambda_t \mapsto V_t^*[\Lambda] = -\mathcal{Y}[\Lambda]$ is convex and decreasing for any $t \in [0, T]$.
Regarding (iii), we first define the aggregator and terminal utility of $\mathcal{Y}$ as

$$h(\lambda, y) \triangleq \hat{\delta}\tfrac{1}{1-\psi}\lambda^{1-\psi}\left(\left(1 - \tfrac{1}{\gamma}\right) y\right)^{q^*} - \tfrac{\hat{\delta}}{1-q^*}y \quad \text{and } H(\lambda) = \tfrac{1}{1-\frac{1}{\gamma}}\left(\hat{\varepsilon}\lambda\right)^{1-\frac{1}{\gamma}}.$$

Then by Proposition 2.17(iii), we obtain for any $\Lambda, \bar{\Lambda} \in \mathcal{D}^\infty$ and any $t \in [0, T]$

$$\mathcal{Y}_t[\Lambda] \leqslant \mathcal{Y}_t[\bar{\Lambda}] + \left\langle \mathfrak{m}^t\left(\tfrac{1}{\hat{\delta}}(\psi)^{\frac{\psi}{\psi-1}}\bar{\Lambda}\right), \tfrac{1}{\hat{\delta}}(\psi)^{\frac{\psi}{\psi-1}}\left(\Lambda - \bar{\Lambda}\right)\right\rangle_t, \tag{2.3.10}$$

where $\langle \mathfrak{m}, y\rangle_t = \mathbb{E}_t\left[\int_t^T \mathfrak{m}_s y_s \mathrm{d}s + \mathfrak{m}_T y_T\right]$ and

$$\mathfrak{m}_s^t\left(\tfrac{1}{\hat{\delta}}(\psi)^{\frac{\psi}{\psi-1}}\bar{\Lambda}\right) = \exp\left(\int_t^s h_y\left(\tfrac{1}{\hat{\delta}}(\psi)^{\frac{\psi}{\psi-1}}\bar{\Lambda}_r, \mathcal{Y}_r[\bar{\Lambda}]\right)\mathrm{d}r\right)\nabla_s$$

and

$$\nabla_s \triangleq \begin{cases} h_\lambda\left(\tfrac{1}{\hat{\delta}}(\psi)^{\frac{\psi}{\psi-1}}\bar{\Lambda}_s, \mathcal{Y}_s[\bar{\Lambda}]\right), & 0 \leqslant s < T \\ H'\left(\tfrac{1}{\hat{\delta}}(\psi)^{\frac{\psi}{\psi-1}}\bar{\Lambda}_T\right), & s = T. \end{cases}$$

We plug the parameters from above into (2.3.10) and obtain:

$$\begin{aligned} h_y\left(\tfrac{1}{\hat{\delta}}(\psi)^{\frac{\psi}{\psi-1}}\bar{\Lambda}_r, \mathcal{Y}_r[\bar{\Lambda}]\right) &= \hat{\delta}\tfrac{1-\gamma\psi}{\gamma(\psi-1)}\left(\tfrac{1}{\hat{\delta}}(\psi)^{\frac{\psi}{\psi-1}}\bar{\Lambda}_r\right)^{1-\psi}\left(\left(1 - \tfrac{1}{\gamma}\right)\mathcal{Y}_r[\bar{\Lambda}]\right)^{-\frac{\gamma\psi}{\theta}} - \hat{\delta}\tfrac{\theta}{\gamma\psi} \\ &= \delta^\psi\tfrac{1-\gamma\psi}{\gamma(\psi-1)}\left(\bar{\Lambda}_r\right)^{1-\psi}\left(\tfrac{1-\gamma}{\gamma}V_r^*[\bar{\Lambda}]\right)^{-\frac{\gamma\psi}{\theta}} - \tfrac{\delta\theta}{\gamma} \\ &= f_\nu^*\left(\bar{\Lambda}_r, V_r^*[\bar{\Lambda}]\right). \end{aligned}$$

Furthermore, for $0 \leqslant s < T$, we obtain

$$\begin{aligned} \nabla_s\tfrac{1}{\hat{\delta}}(\psi)^{\frac{\psi}{\psi-1}}(\Lambda_s - \bar{\Lambda}_s) &= h_\lambda\left(\tfrac{1}{\hat{\delta}}(\psi)^{\frac{\psi}{\psi-1}}\bar{\Lambda}_s, \mathcal{Y}_s[\bar{\Lambda}]\right)\tfrac{1}{\hat{\delta}}(\psi)^{\frac{\psi}{\psi-1}}(\Lambda_s - \bar{\Lambda}_s) \\ &= f_\lambda^*\left(\bar{\Lambda}_s, V_s^*[\bar{\Lambda}]\right)(\Lambda - \bar{\Lambda}_s) \\ &= \nabla_s^*(\Lambda_s - \bar{\Lambda}_s) \end{aligned}$$

and

$$\begin{aligned} \nabla_T\tfrac{1}{\hat{\delta}}(\psi)^{\frac{\psi}{\psi-1}}(\Lambda - \bar{\Lambda}_s) &= H'\left(\tfrac{1}{\hat{\delta}}(\psi)^{\frac{\psi}{\psi-1}}\bar{\Lambda}_T\right)\tfrac{1}{\hat{\delta}}(\psi)^{\frac{\psi}{\psi-1}}(\Lambda_T - \bar{\Lambda}_T) \\ &= (\Phi^*)'(\bar{\Lambda}_T)(\Lambda_T - \bar{\Lambda}_T) \\ &= \nabla_T^*(\Lambda_T - \bar{\Lambda}_T). \end{aligned}$$

[10]note that our $q^*$ is exactly their $q$ when one replaces $\gamma$ and $\frac{1}{\psi}$ by $\frac{1}{\gamma}$ and $\psi$, respectively.

Thus plugging in, (2.3.10) becomes

$$-V_t^*[\Lambda] \leqslant -V_t^*[\bar{\Lambda}] + \left\langle (\mathfrak{m}^t)^*(\bar{\Lambda}), \Lambda - \bar{\Lambda} \right\rangle_t$$

and the result follows. □

**Remark 2.21**
*Dual utility gradients and the associated inequality in* (2.3.7) *have not revealed as far reaching theoretical implications as the primal ones so far. Although* $\mathfrak{m}^*(\Lambda^*)$ *is intuitively connected to optimal consumption, in asset pricing consumption is a priori known and the deflator is the actual object of interest.*
*However, they could potentially be used to derive verification results for the dual value function, similar as in [Kraft et al., 2017][Theorem 5.1], where the authors use the primal gradient inequality as in* (2.3.2) *to verify their optimal strategy. Verification in turn is closely connected to duality as pointed out by [Matoussi and Xing, 2018]. This connection is shortly discussed in Section 4.4.1.* △

The assumptions on integrability imposed by Theorem 2.20 are quite strong and we make some efforts to soften them up in Section 3.3. For now we finish our analysis of the stochastic differential dual as the solution to the BSDE (2.3.6).

# Chapter 3

# Bounding the Optimality Gap

In the previous chapter we introduced the primal and dual optimization problem and established the fundamental theory of the associated value processes. However, a solution to both is in general far out of sight and has only been found under very restrictive parameter conditions, see Section 5.1 for more details.
In our later chapters we develop two numerical methods to find at least approximate solutions to the problem based on dynamic programming. However, as exact solutions are in general not available, there is no way to evaluate the accuracy of our approximations. Thus, before investigating any approximation methods, in this chapter we extend the idea of using duality theory to derive an upper bound on the welfare loss as introduced in [Haugh et al., 2006]. In other words we tackle the question

*If an investor with recursive preferences behaves suboptimally in a given market, how much worse is she off compared with optimal behavior?*

To understand our approach, denote by $c^*$ the optimal strategy associated to our consumption-investment optimization problem and by $c$ an arbitrary admissible strategy. Then the intuitive answer to the motivating question of this chapter is given by the (primal) *welfare loss* $\nu(c^*)-\nu(c)$. There are two major issues with this answer. First, the optimal strategy $c^*$ is in general not known, so there is no way to determine the associated utility $\nu(c^*)$. Second, even for simple (e.g. constant) consumption streams $c$, the associated recursive utility $\nu(c)$ is in general hard to compute, as one would have to solve the non-standard BSDE

$$V_0[c] = \mathbb{E}\left[\int_0^T f(c_s, V_s[c])\mathrm{d}s + \Phi(c_T)\right].$$

We tackle the two issues by gradually allowing for larger errors in the answer. The absence of the optimal consumption stream seems to be the biggest issue, but a solution is already at hand in terms of the duality inequality derived in Chapter 2. Denote by $\Lambda^*$ the (unknown) optimal deflator to the associated dual problem and by $\Lambda$ an arbitrary admissible deflator. Then by Theorem 2.11 (recall also Remark 2.12) we know

$$\nu(c) \leqslant \nu(c^*) \leqslant \nu^*(\Lambda^*) \leqslant \nu^*(\Lambda), \tag{3.0.1}$$

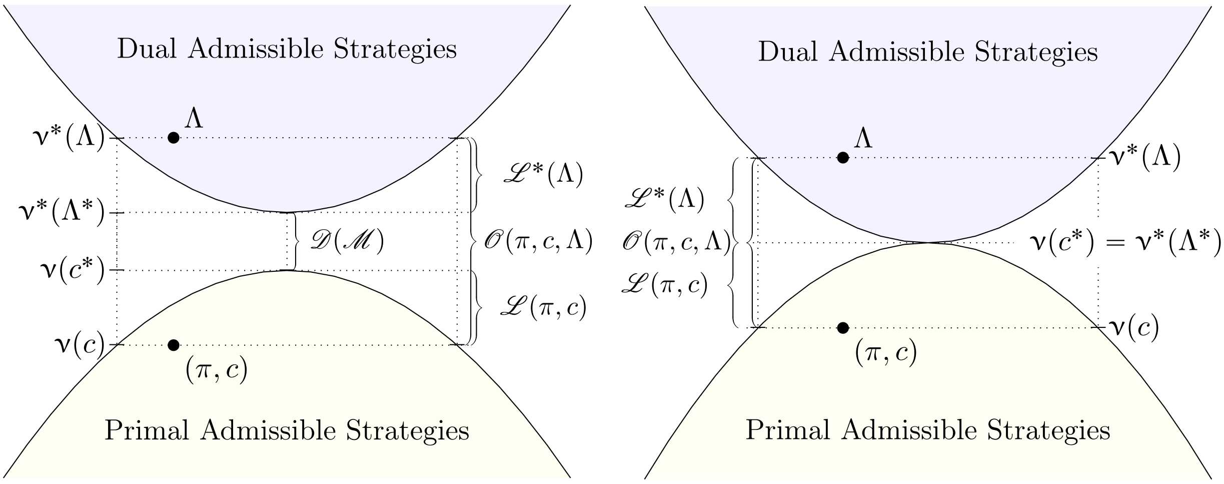
so $\nu^*(\Lambda) - \nu(c)$ is an upper bound on the welfare loss. We call this upper bounds associated to the strategies $(\pi, c)$ and $\Lambda$ the *optimality gap*. In particular one does not need the optimal primal or dual strategies to express this bound. It follows a precise definition of the concept, an illustration is given in Figure 3.1.

**Definition 3.1** (Optimality Gap)
*In a given market model* $\mathscr{M}$[1]*, let* $(\pi^*, c^*) \in \mathcal{A}$ *be an investor's optimal investment-consumption strategy and let* $\Lambda^* \in \mathcal{D}^a$ *be the optimal dual strategy. Then we define the market specific* duality gap *as* $\mathcal{D}(\mathscr{M}) \triangleq \mathsf{v}^*(\Lambda^*) - \mathsf{v}(c^*) \in [0, \infty)$.
*For arbitrary strategies* $(\pi, c) \in \mathcal{A}$ *and* $\Lambda \in \mathcal{D}^a$ *we define the investors* primal *and* dual welfare loss $\mathscr{L}(\pi, c)$ *and* $\mathscr{L}^*(\Lambda)$ *as*

$$\mathscr{L} : \mathcal{A} \to [0, \infty), (\pi, c) \mapsto \mathsf{v}(c^*) - \mathsf{v}(c) \quad \text{and} \quad \mathscr{L}^* : \mathcal{D}^a \to [0, \infty), \Lambda \mapsto \mathsf{v}^*(\Lambda) - \mathsf{v}^*(\Lambda^*).$$

*Finally, we define the* optimality gap *associated to the strategies* $(\pi, c) \in \mathcal{A}$ *and* $\Lambda \in \mathcal{D}_a$ *as*

$$\mathscr{O} : \mathcal{A} \times \mathcal{D}^a \to [0, \infty), (\pi, c, \Lambda) \mapsto \mathscr{L}(\pi, c) + \mathscr{L}^*(\Lambda) + \mathscr{D}(\mathscr{M}) = \mathsf{v}^*(\Lambda) - \mathsf{v}(c).$$




Optimality Gap including Duality Gap

Optimality Gap without Duality Gap

Figure 3.3: (Primal/Dual) Welfare Loss, Duality Gap and Optimality Gap

The optimality gap has in different appearances been used as an upper bound on the welfare loss for a while. Historically, various approximation techniques have been developed to numerically approach the solution of an intractable optimization problem in incomplete markets, e.g. the log-linear analytical approximation of [Chacko and Viceira, 2005], finite difference PDE-methods as in [Brennan and Xia, 2002] and more. However, it was usually difficult to evaluate the accuracy of the obtained approximations. [Haugh et al., 2006] were the first to come up with the idea of evaluating approximative strategies based on duality theory. Their idea was picked up and generalized to various forms of market frictions by different authors, e.g. [Brown et al., 2010], [Brown and Smith, 2011], [Bick et al., 2013], [Kamma et al., 2020] or [Kamma and Pelsser, 2022]. However, all existing results consider time-additive utility and as mentioned above, in our case of recursive utility the optimality gap itself is in general hard to evaluate, which presents us with additional issues. Our approach to bypass those, is to find upper bounds on $\mathscr{O}(\pi, c, \Lambda)$ that can easily be simulated. These upper bounds on the optimality gap are then also bounds on the duality gap and in particular on the welfare loss.

[1] We use $\mathscr{M}$ as an abstract notation for anything that might have an impact on the duality gap, in particular the specific model dynamics and the investor's risk preferences. Note that all quantities directly or indirectly depend on $\mathscr{M}$, but the dependence is omitted for notational simplicity.

## 3.1 Universal Power Utility Bounds

The first kind of bounds we establish are in terms of specifically scaled power utility functionals; hence we call them *power bounds*. It turns out that in the derivation of those bounds it is more convenient to consider the *reciprocal* of elasticity of intertemporal substitution $\phi \triangleq \frac{1}{\psi}$ instead of the EIS $\psi$ itself. In this notation the primal and dual Epstein-Zin aggregators read as

$$f(c, v) = \delta \frac{1}{1-\phi} c^{1-\phi}((1-\gamma)v)^{1-\frac{1}{\theta}} - \delta\theta v,$$
$$f^*(\lambda, \nu) = \delta^{\frac{1}{\phi}} \frac{\phi}{1-\phi} \lambda^{\frac{\phi-1}{\phi}} \left(\frac{1-\gamma}{\gamma}\nu\right)^{1-\frac{\gamma}{\theta\phi}} - \frac{\delta\theta}{\gamma}\nu,$$

where $\theta = \frac{1-\gamma}{1-\phi}$. From now on we may switch between the notation in terms of $\psi$ and $\phi$, respectively, whenever it is convenient.

### Primal Power Bounds

Upper and lower bounds for the stochastic differential utility have already been provided by [Seiferling and Seifried, 2016]: For any $c \in \mathcal{C}^\infty$ and $0 < \varrho \neq 1$ define

$$\mathrm{P}_\varrho(c) \triangleq \varphi_\gamma \circ \varphi_\varrho^{-1}(L^\varrho[c]), \tag{3.1.1}$$

where $\varphi_\varrho(c) = \frac{1}{1-\varrho}c^{1-\varrho}$, $\Phi_\varrho(c) = \varepsilon^{\frac{1-\varrho}{1-\gamma}} \frac{1}{1-\varrho} c^{1-\varrho}$, $\varepsilon > 0$ and $L^\varrho[c] = L^\varrho$ is given by[2]

$$L_t^\varrho = e^{\delta t}\mathbb{E}_t\left[\int_t^T \delta e^{-\delta s}\varphi_\varrho(c_s)\mathrm{d}s + e^{-\delta T}\Phi_\varrho(c_T)\right]. \tag{3.1.2}$$

Those transformed power utility processes are admissible in the sense of the following lemma.

**Lemma 3.2** ([Seiferling and Seifried, 2016], Lemma 4.5)
*For all $c \in \mathcal{C}^\infty$ and $1 \neq \varrho > 0$ it holds that $\mathrm{P}_\varrho(c) \in \mathcal{V}^\infty$.*

In their work, [Seiferling and Seifried, 2016] show that those power utility processes can be used as upper and lower bounds on the recursive utility associated with a fixed consumption stream $c \in \mathcal{C}^\infty$.

**Proposition 3.3** ([Seiferling and Seifried, 2016], Theorem 4.6)
*If $V[c] \in \mathcal{V}^\infty$ is a recursive utility process associated with $c \in \mathcal{C}^\infty$ we have*

$$\mathrm{P}_{\gamma\vee\phi}(c) \leqslant V[c] \leqslant \mathrm{P}_{\gamma\wedge\phi}(c).$$

Proposition 3.3 is already half the answer to our problem, as it provides a lower bound on $\mathrm{v}(c)$ that can easily be evaluated numerically; the upper bound however is not useful for our purpose. The issue is that for a non-optimal consumption stream the upper power bound provided in Proposition 3.3 might still be smaller than the primal utility associated to the optimal strategy,

$$\mathrm{P}_{\gamma\vee\phi}(c) \leqslant V[c] \leqslant \mathrm{P}_{\gamma\wedge\phi}(c) \leqslant V[c^*]. \tag{3.1.3}$$

This is certainly the case for any suboptimal strategy in the power utility case, where the first three quantities in (3.1.3) actually coincide, c.f. (3.1.1) and (2.1.4). However, it already allows us to expand our chain of inequalities from (3.0.1) on the left hand side:

$$\mathrm{P}_{\gamma\vee\phi}(c) \leqslant \mathrm{v}(c) \leqslant \mathrm{v}(c^*) \leqslant \mathrm{v}^*(\Lambda^*) \leqslant \mathrm{v}^*(\Lambda). \tag{3.1.4}$$

[2] Note that in contrast to [Seiferling and Seifried, 2016] we have to rescale the weight of terminal bequest $\varepsilon$ within $\Phi_\varrho$, as we use a different parametrisation.

Within the next section we derive similar power bounds for the stochastic differential dual associated to an arbitrary but fixed deflator process $\Lambda \in \mathcal{D}^\infty$. Then the upper bound on $\nu^*(\Lambda)$ can be used to expand (3.1.4) on the right hand side by an easily computable quantity.
In total we thereby obtain our desired bound on the optimality gap.

## Dual Power Bounds

Analogously to [Seiferling and Seifried, 2016], for any $1 \neq \varrho, \gamma > 0$, define the mappings

$$\begin{aligned} \varphi_\varrho^* &: (0,\infty) \to \mathbb{R}, \lambda \mapsto \tfrac{\varrho}{1-\varrho}\lambda^{\frac{\varrho-1}{\varrho}} \qquad \text{and} \\ \Phi_\varrho^* &: (0,\infty) \mapsto \mathbb{R}, \lambda \mapsto \varepsilon^{\frac{\varrho-1}{\varrho(\gamma-1)}}\tfrac{\varrho}{1-\varrho}\lambda^{\frac{\varrho-1}{\varrho}}, \end{aligned}$$

that is the dual time-additive utility functions with risk aversion $\varrho$. Furthermore, define for any $\Lambda \in \mathcal{D}^\infty$ the stochastic process

$$\mathrm{P}_\varrho^*(\Lambda) \triangleq \varphi_\gamma^* \circ (\varphi_\varrho^*)^{-1}(\mathcal{L}^\varrho[\Lambda]), \tag{3.1.5}$$

where $\mathcal{L}^\varrho = \mathcal{L}^\varrho[\Lambda]$ satisfies

$$\mathcal{L}_t^\varrho = \mathrm{e}^{\frac{\delta}{\varrho}t}\mathbb{E}_t\left[\int_t^T \delta^{\frac{1}{\varrho}}\mathrm{e}^{-\frac{\delta}{\varrho}s}\varphi_\varrho^*(\Lambda_s)\mathrm{d}s + \mathrm{e}^{-\frac{\delta}{\varrho}T}\Phi_\varrho^*(\Lambda_T)\right]. \tag{3.1.6}$$

Thus $\bar{\mathrm{P}}_\varrho^*(\Lambda)$ is just the dual power utility function for a given deflator $\Lambda \in \mathcal{D}^\infty$ with utility parameter $\varrho$, transformed to a $\gamma$-scale. Note that the rescaling of the weight of terminal bequest is not only a technical necessity, but also important for its interpretation as it depends on the risk preference $\gamma$, see (3.1.11) below. The rescaling could have been avoided by a different parametrization of our terminal bequest function.

Showing that those power utility functionals actually characterize an upper, respectively lower bound on the stochastic differential dual heavily relies on a so called comparison theorem. One that is general enough for our semimartingale setting is provided by [Seiferling and Seifried, 2016]. We briefly recall the prerequisites in the following definition and state the theorem thereafter without proof. For a more detailed elaboration on the topic, including a proof of the comparison result Theorem 3.5, see Appendix A.2.

**Definition 3.4** (BSDE - Sub-/Supersolutions)
*Let $g : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be $\mathcal{G} \otimes \mathbb{B}$-measurable, where $\mathcal{G}$ is the progressive $\sigma$-field and $\mathbb{B}$ is the Borel $\sigma$-field. Let $\xi \in L^1(\mathbb{P})$ and suppose $X$ is a semimartingale with $\sup_{t\in[0,T]} \mathbb{E}[|X_t|] < \infty$ and moreover $\mathbb{E}[\int_0^T |g(t,X_t)|\mathrm{d}t] < \infty$. Then we call $X$ a subsolution of the BSDE with aggregator $g$ and terminal value $\xi$, if*

$$\mathrm{d}X_t = -g(t,X_t)\mathrm{d}t + \mathrm{d}M_t - \mathrm{d}A_t, \quad X_T \leqslant \xi$$

*where $M$ is a martingale and $A$ is a decreasing and right-continuous process such that $A_0 = 0$. We say*

$$X \text{ is a subsolution of } \mathrm{BSDE}(g,\xi)$$

*for short. Analogously $X$ is supersolution of $\mathrm{BSDE}(g,\xi)$, if $X_T \geqslant \xi$ and $A$ is increasing. $X$ is a solution of $\mathrm{BSDE}(g,\xi)$ as in Definition A.1, if it is a sub- and supersolution.*
*We say that the aggregator $g$ satisfies $(M)$, if there is a constant $k > 0$ such that for $\mathrm{d}t$ a.e. $t \in [0,T]$*

$$g(\omega,t,x) - g(\omega,t,y) \leqslant k(x-y) \qquad \text{for all } x,y \in \mathbb{R} \text{ with } x \geqslant y. \tag{M}$$

The property (M) is sometimes called *monotonicity condition.*

**Note:** The primal and dual aggregators $f$ and $f^*$ both satisfy (M) as their derivatives with respect to $v$, respectively $\nu$, are bounded from above, see Lemma B.7 and Lemma B.13.

**Theorem 3.5** ([Seiferling and Seifried, 2016], Theorem 4.3)
*Suppose $X$ is a subsolution of $\mathrm{BSDE}(g,\xi)$ with $\mathbb{E}[\sup_{t\in[0,T]}|X_t|] < \infty$ and $Y$ is a supersolution of $\mathrm{BSDE}(h,\eta)$ with $\mathbb{E}[\sup_{t\in[0,T]}|Y_t|] < \infty$ where $\xi \leqslant \eta$.*

*(a) If $g(t,Y_t) \leqslant h(t,Y_t)$ for $\mathrm{d}t$ a.e. $t \in [0,T]$ and $g$ satisfies (M), then $X \leqslant Y$.*

*(b) If $g(t,X_t) \leqslant h(t,X_t)$ for $\mathrm{d}t$ a.e. $t \in [0,T]$ and $h$ satisfies (M), then $X \leqslant Y$.*

We now have everything at hand that we need to show that the rescaled power utility process as in (3.1.5) provides upper and lower bounds on the dual value process.
First observe that for $\Lambda \in \mathcal{D}^\infty$ standard representation results on linear BSDEs with BSDE$^p$-standard parameters, see e.g. Theorem A.6, yield that $\mathcal{L}^\varrho$ has a representation as

$$\mathrm{d}\mathcal{L}^\varrho_t = -\left[\delta^{\frac{1}{\varrho}}\varphi^*_\varrho(\Lambda_t) - \tfrac{\delta}{\varrho}\mathcal{L}^\varrho_t\right]\mathrm{d}t + \mathrm{d}M^\varrho_t, \qquad \mathcal{L}^\varrho_T = \Phi^*_\varrho(\Lambda_T), \tag{3.1.7}$$

where $M^\varrho$ is a $L^p$-martingale for all $p \geqslant 1$, $0 < \varrho \neq 1$. Recall also that $V^* = V^*[\Lambda]$ solves

$$\mathrm{d}V^*_t = -f^*\left(\Lambda_t, V^*_t\right)\mathrm{d}t + \mathrm{d}M^*_t, \qquad V^*_T = \Phi^*(\Lambda_T)$$

where $M^*_t = \mathbb{E}_t\left[\int_0^T f^*\left(\Lambda_s, V^*_s[\Lambda]\right)\mathrm{d}s + \Phi^*(\Lambda_T)\right]$ is a $L^p$-martingale for all $p \geqslant 1$.

Then first part of our bounds follows immediately from Theorem 3.5.

**Lemma 3.6**
*Let $V^*[\Lambda] \in \mathcal{V}^\infty$ be the stochastic differential dual associated to a pricing deflator $\Lambda \in \mathcal{D}^\infty$. If $\gamma \geqslant \phi$, then $V^*[\Lambda] \geqslant \bar{\mathrm{P}}^*_\gamma(\Lambda)$, and if $\gamma \leqslant \phi$ then $V^*[\Lambda] \leqslant \bar{\mathrm{P}}^*_\gamma(\Lambda)$.*

*Proof.* Define $\vartheta(\lambda) \triangleq \delta^{\frac{1-\gamma}{\gamma}}\frac{\gamma}{1-\gamma}\lambda^{\frac{\gamma-1}{\gamma}}$. Then a technical calculation shows

$$f^*\left(\lambda, \vartheta(\lambda)\right) = -\delta^{\frac{1}{\gamma}}\lambda^{\frac{\gamma-1}{\gamma}} \quad \text{and} \quad f^*_\nu\left(\lambda, \vartheta(\lambda)\right) = -\tfrac{\delta}{\gamma}.$$

When $\gamma \geqslant \phi$ then $f^*(\lambda,\nu)$ is convex by Lemma B.13, so we obtain the convexity inequality

$$\begin{aligned} f^*\left(\lambda,\nu\right) &\geqslant f^*\left(\lambda,\vartheta(\lambda)\right) + f^*_\nu\left(\lambda,\vartheta(\lambda)\right)\left[\nu - \vartheta(\lambda)\right] \\ &= \delta^{\frac{1}{\gamma}}\varphi^*_\gamma(\lambda) - \tfrac{\delta}{\gamma}\nu. \end{aligned}$$

The result follows from Theorem 3.5 since $V^*_T = \Phi^*(\Lambda_T) = \mathcal{L}^\gamma_T$. On the other hand, if $\gamma \leqslant \phi$ then $f^*(\lambda,\nu)$ is concave by Lemma B.13 and the result follows by the same calculations. □

The two remaining inequalities of our bounds requires more work. While the final step is once again a straightforward application of the comparison result Theorem 3.5, we need some more information on the dynamics of the involved processes first. We outsource the calculations to the following technical lemma, which is a dual analogue to Lemma 4.8 in [Seiferling and Seifried, 2016].

**Lemma 3.7**
*Let $V^* \in \mathcal{V}^\infty$ be a stochastic differential dual associated to some $\Lambda \in \mathcal{D}^\infty$ and define $\mathcal{L} = \varphi^*_\phi \circ (\varphi^*_\gamma)^{-1}(V^*)$. Then $\mathcal{L}$ satisfies*

$$\mathrm{d}\mathcal{L}_t = -\left[\delta^{\frac{1}{\phi}}\varphi^*_\phi(\Lambda_t) - \tfrac{\delta}{\phi}\mathcal{L}_t\right]\mathrm{d}t + \left(\tfrac{1-\gamma}{\gamma}V^*_{t-}\right)^{\frac{\gamma}{\theta\phi}-1}\mathrm{d}M^*_t - \mathrm{d}A_t, \tag{3.1.8}$$

*where $\left(\tfrac{1-\gamma}{\gamma}V^*_{t-}\right)^{\frac{\gamma}{\theta\phi}-1}\mathrm{d}M^*_t$ is a $L^p$-martingale for all $p \geqslant 1$ and* [3]

$$\mathrm{d}A_t = \tfrac{1}{2}\tfrac{\phi-\gamma}{\gamma}\tfrac{1}{(1-\phi)\mathcal{L}_t}\mathrm{d}[\mathcal{L}^{\mathfrak{c}}_t] + \tfrac{\gamma}{1-\gamma}\left(\tfrac{1-\phi}{\phi}\mathcal{L}_{t-}\right)^{-\frac{\theta\phi}{\gamma}}\Delta\left(\tfrac{1-\phi}{\phi}\mathcal{L}_t\right)^{\frac{\theta\phi}{\gamma}} - \Delta\mathcal{L}_t. \tag{3.1.9}$$

*is decreasing if $\gamma \geqslant \phi$ and increasing if $\gamma \leqslant \phi$.*

*Proof.* For $V^* \in \mathcal{V}^\infty$ and $\Lambda \in \mathcal{D}^\infty$, let $\mathcal{L} \triangleq \varphi^*_\phi \circ (\varphi^*_\gamma)^{-1}(V^*)$. Denote $g : \mathbb{V} \to \mathbb{R}$, $g(\nu) \triangleq \varphi^*_\phi \circ (\varphi^*_\gamma)^{-1}(\nu) = \frac{\phi}{1-\phi}\left(\frac{1-\gamma}{\gamma}\nu\right)^{\frac{\gamma}{\theta\phi}}$. Then $g'(\nu) = \left(\frac{1-\gamma}{\gamma}\nu\right)^{\frac{\gamma}{\theta\phi}-1}$ and $g''(\nu) = \frac{\gamma-\phi}{\gamma\phi}\left(\frac{1-\gamma}{\gamma}\nu\right)^{\frac{\gamma}{\theta\phi}-2}$. So $\mathcal{L} = \frac{\phi}{1-\phi}\left(\frac{1-\gamma}{\gamma}V^*\right)^{\frac{\gamma}{\theta\phi}}$ and thus we have

$$\mathcal{L} \in \left\{V \in \mathcal{S} : (1-\phi)V > 0 \text{ and } \mathbb{E}\left[\sup_{t\in[0,T]}|V_t|^\ell\right] < \infty \text{ for all } \ell \in \mathbb{R}\right\}.$$

Moreover, as $V^* \in \mathcal{V}^\infty$ and $M^* = \mathbb{E}_t\left[\int_0^T f^*(\Lambda_s, V^*_s[\Lambda])\,\mathrm{d}s + \Phi^*(\Lambda_T)\right]$ is a $L^p$-martingale for all $p \geqslant 1$, so is $\left(\frac{1-\gamma}{\gamma}V^*_{t-}\right)^{\frac{\gamma}{\theta\phi}-1}\mathrm{d}M^*_t$. An application of Ito's formula, see [Jacod and Shiryaev, 2013][Theorem 4.57], yields

$$\mathrm{d}\mathcal{L}_t = \mathrm{d}g(V^*_t) = g'(V^*_{t-})\mathrm{d}V^*_t + \tfrac{1}{2}g''(V^*_{t-})\mathrm{d}[(V^*_t)^{\mathfrak{c}}] - \mathrm{d}J_t.$$

where

$$\mathrm{d}J_t = g'(V^*_{t-})\Delta V^*_t - \Delta g(V^*_t) \tag{3.1.10}$$

A direct calculation shows that

$$g'(V^*_t)\mathrm{d}V^*_t = -\left[\delta^{\frac{1}{\phi}}\varphi^*(\Lambda_t) - \tfrac{\delta}{\phi}g(V^*_t)\right]\mathrm{d}t + \left(\tfrac{1-\gamma}{\gamma}V^*_{t-}\right)^{\frac{\gamma}{\theta\phi}-1}\mathrm{d}M^*_t,$$

so to obtain (3.1.8) it remains to show that $\frac{1}{2}g''(V^*_{t-})\mathrm{d}[(V^*_t)^{\mathfrak{c}}] - \mathrm{d}J_t = -\mathrm{d}A_t$ as in (3.1.9). Inserting $V^* = g^{-1}(\mathcal{L})$ to (3.1.10) we first compute

$$\mathrm{d}J_t = g'(V^*_{t-})\Delta V^*_t - \Delta g(V^*_t) = \tfrac{\gamma}{1-\gamma}\left(\tfrac{1-\phi}{\phi}\mathcal{L}_{t-}\right)^{-\frac{\theta\phi}{\gamma}}\Delta\left(\tfrac{1-\phi}{\phi}\mathcal{L}_t\right)^{\frac{\theta\phi}{\gamma}} - \Delta\mathcal{L}_t.$$

Moreover we have $\mathrm{d}\mathcal{L}^{\mathfrak{c}}_t = g'(V^*_{t-})\mathrm{d}(M^*_t)^{\mathfrak{c}}$, thus $\mathrm{d}[(M^*_t)^{\mathfrak{c}}] = \frac{1}{g'(V^*_{t-})^2}\mathrm{d}[\mathcal{L}^{\mathfrak{c}}_t]$ and we obtain

$$g''(\mathcal{L}_{t-})\mathrm{d}(M^*_t)^{\mathfrak{c}} = \frac{g''(V_{t-})}{g'(V^*_{t-})^2}\mathrm{d}[\mathcal{L}^{\mathfrak{c}}_t] = \tfrac{\gamma-\phi}{\gamma}\tfrac{1}{(1-\phi)\mathcal{L}_{t-}}\mathrm{d}[\mathcal{L}^{\mathfrak{c}}_t],$$

[3] For a semimartingale $X$ we denote by $X^{\mathfrak{c}}$ the unique continuous local martingale that satisfies $[X^{\mathfrak{c}}] = [X]^{\mathfrak{c}}$, [Jacod and Shiryaev, 2013][Proposition 4.27] and [Theorem 4.52].

yielding (3.1.8). We complete the proof by verifying that $A_t$ is actually increasing, if $\gamma \leqslant \phi$ and decreasing otherwise. Note that

$$g''(\nu) = \tfrac{\gamma-\phi}{\gamma\phi}\left(\tfrac{1-\gamma}{\gamma}\nu\right)^{\frac{\gamma}{\theta\phi}-2}\begin{cases}\geqslant 0, & \text{if } \gamma \geqslant \phi\\ \leqslant 0, & \text{if } \gamma \leqslant \phi\end{cases}.$$

Consider the case $\gamma \geqslant \phi$, then $g$ is convex and in particular

$$g(V^*) \geqslant g(V^*_{t-}) + g'(V^*_{t-})(V^*_t - V^*_{t-}),$$

so $J$ only has only non-positive jumps. Moreover $\gamma \geqslant \phi$ implies $\frac{1}{2}\frac{\phi-\gamma}{\gamma}\frac{1}{(1-\phi)\mathcal{L}_t} \leqslant 0$, so $A$ is decreasing. The case $\gamma \leqslant \phi$ follows analogously by the concavity of $g$ and $\frac{1}{2}\frac{\phi-\gamma}{\gamma}\frac{1}{(1-\phi)\mathcal{L}_t} \geqslant 0$. □

**Lemma 3.8**
*Let $V^*[\Lambda] \in \mathcal{V}^\infty$ be the stochastic differential dual associated to a pricing deflator $\Lambda \in \mathcal{D}^\infty$. If $\gamma \geqslant \phi$, then $V^*[\Lambda] \leqslant \bar{\mathrm{P}}^*_\phi(\Lambda)$, and if $\gamma \leqslant \phi$ then $V^*[\Lambda] \geqslant \bar{\mathrm{P}}^*_\phi(\Lambda)$.*

*Proof.* For $\Lambda \in \mathcal{D}^\infty$, let $V^* \in \mathcal{V}^\infty$ be its associated stochastic differential dual. As in Lemma 3.7, let $g : \mathbb{V} \to \mathbb{R}$, $g(\nu) \triangleq \varphi^*_\phi \circ (\varphi^*_\gamma)^{-1}(\nu)$ and $\mathcal{L} \triangleq g(V^*)$. As $g'(\nu) = \left(\frac{1-\gamma}{\gamma}\nu\right)^{\frac{\gamma}{\theta\phi}-1} > 0$, $g$ is increasing. Thus the claim is equivalent to showing $g(V^*) = \mathcal{L} \leqslant \mathcal{L}^\phi = g(\bar{\mathrm{P}}^*_\phi(\Lambda))$ if $\gamma \geqslant \phi$ and $g(\bar{\mathrm{P}}^*_\phi(\Lambda)) = \mathcal{L}^\phi \leqslant \mathcal{L} = g(V^*)$ if $\gamma \leqslant \phi$. Consider the case $\gamma \geqslant \phi$, the case $\gamma \leqslant \phi$ follows by the same arguments. First, note that

$$\mathcal{L}^\phi_T = \Phi^*_\phi(\Lambda_T) = g(\Phi^*(\Lambda_T)) = g(V^*_T) = \mathcal{L}_T, \tag{3.1.11}$$

where $\Phi^*$ is the usual dual terminal utility as in Example 2.9. Thus, if $\gamma \geqslant \phi$, $\mathcal{L}$ is a subsolution to the linear BSDE (3.1.7) by Lemma 3.7 and the result follows immediately from Theorem 3.5. □

Putting both results from above together yields the following dual power bounds.

**Proposition 3.9**
*Let $V^*[\Lambda] \in \mathcal{V}^\infty$ be the stochastic differential dual associated to a pricing deflator $\Lambda \in \mathcal{D}^\infty$. Then we have the upper and lower bounds*

$$\bar{\mathrm{P}}^*_{\gamma\vee\phi}(\Lambda) \leqslant V^*[\Lambda] \leqslant \bar{\mathrm{P}}^*_{\gamma\wedge\phi}(\Lambda).$$

Using Proposition 2.16, the reduction to pricing deflators $D \in \mathcal{D}^\infty_1$ is straight forward.

**Corollary 3.10**
*Let $V^*[\Lambda] \in \mathcal{V}^\infty$ be a utility process associated to $\Lambda \in \mathcal{D}^\infty$. Then we have*

$$\mathrm{P}^*_{\gamma\vee\phi}(D) \leqslant V^*[\Lambda] + \Lambda X^{(\pi,c)} \leqslant \mathrm{P}^*_{\gamma\wedge\phi}(D)$$

*where $\mathrm{P}^*_\varrho(D) \triangleq \mathfrak{d}\left(X^{(\pi,c)}D, \bar{\mathrm{P}}^*_\varrho(D)\right)$ and $\mathfrak{d}(x,\nu) \triangleq \frac{1}{1-\gamma}x^{1-\gamma}\left(\frac{1-\gamma}{\gamma}\nu\right)^\gamma$ as in Proposition 2.16.*

*Proof.* By Proposition 3.9 we have for $D \in \mathcal{D}^\infty_1$

$$\bar{\mathrm{P}}^*_{\gamma\vee\phi}(D) \leqslant V^*[D] \leqslant \bar{\mathrm{P}}^*_{\gamma\wedge\phi}(D). \tag{3.1.12}$$

By applying the increasing function $\mathfrak{d}(X^{(\pi,c)}D, \cdot)$ on every element of (3.1.12) the result follows as

$$V^*[\Lambda] + \Lambda X^{(\pi,c)} = \mathfrak{d}(X^{(\pi,c)}D, V^*[D]),$$

by Proposition 2.16. □

We are now ready to formulate our desired bounds on the optimality gap and more importantly, the primal welfare loss.

**Theorem 3.11**
*Let $(\pi^*, c^*) \in \mathcal{A}$ with $c^* \in \mathcal{C}^\infty$ be the optimal strategy associated to the primal problem* (2.1.5) *and $\Lambda^* = \lambda^* D^* \in \mathcal{D}^\infty$ be the optimal deflator associated to the dual problem* (2.2.27)*. Then for any strategy $(\pi, c) \in \mathcal{A}$ with $c \in \mathcal{C}^\infty$ and any pricing deflator $\Lambda = \lambda D \in \mathcal{D}^\infty$ we have*

$$\mathrm{P}_{\gamma \vee \phi}(c) \leqslant V[c^*] \leqslant \mathrm{P}^*_{\gamma \wedge \phi}(D).$$

*and in particular*

$$\mathscr{L}(\pi, c) \leqslant \mathscr{O}(\pi, c, \Lambda) \leqslant \mathrm{P}^*_{\gamma \wedge \phi}(D) - \mathrm{P}_{\gamma \vee \phi}(c)$$

*Proof.* Using Proposition 3.3 in the first, the duality relation (2.2.29) in the third and Corollary 3.10 in the last step, we obtain the first inequality in the theorem:

$$\mathrm{P}_{\gamma \vee \phi}(c) \leqslant V[c] \leqslant V[c^*] \leqslant V^*[\Lambda^*] + \Lambda^* X^{(\pi^*, c^*)} \leqslant V^*[\Lambda] + \Lambda X^{(\pi, c)} \leqslant \mathrm{P}^*_{\gamma \wedge \phi}(D).$$

The second one follows immediately as $\mathscr{O}(\pi, c, \lambda D) = \mathsf{v}^*(\Lambda) - \mathsf{v}(c) = (V_0^*[\Lambda] + \lambda x) - V_0[c]$. □

The power bounds from Theorem 3.11 are, to the best of our knowledge, the first ones in the literature that apply to recursive Epstein-Zin utility and do not rely on solving the BSDEs for $V$ and $V^*$, respectively. On the contrary they are easy to simulate and valid for every parameter constellation for which we established the duality relation (2.2.28). In particular, they are a suitable tool in the evaluation of numerical schemes that provide approximations to the optimal primal and associated dual strategies, if no other benchmark is available: The upper and lower power bound associated to the strategies provided by the algorithms' output are evaluated and yield an upper bound on the optimality gap, so if this bound is tight, the welfare loss associated to the approximate strategies must be small and the approximation must be good. However, we must mention that even for good strategies, the power utility bounds cannot be expected to provide a good bound on the optimality gap when the RRA $\gamma$ and EIS $\psi$ differ too much from the power utility case.

As a first approach to counteract those potential quantitative shortcomings, we consider a different kind of (one sided) bounds arising from our duality theory in Section 2.2.1. As part of our numerical analysis in Chapter 6 we see that they are indeed an asset when it comes to parameter constellations where the power bounds fail.

Besides providing a first general tool to measure the performance of suboptimal strategies when there is no benchmark available, dual power bounds can be a powerful tool in the theoretical treatment of stochastic differential duals. In our case we use them in Section 3.3 to soften the integrability conditions that are needed to ensure existence and uniqueness of the stochastic differential dual from Theorem 2.20.

## 3.2 Variational Utility Bounds

Our power bounds work especially well in the case of power utility; in fact, in that case they really just evaluate the considered strategy and hence are exact. However, it is intuitively clear, that their precision declines when the difference between $\gamma$ and $\phi$ becomes large. Thus this short section explains how our duality procedure from Section 2.2.1 already provided us with alternative bounds.

The following corollary captures the fact that by Lemma 2.4, Lemma 2.10 and Remark 2.13, the variational representation provides a lower, respectively upper bound on the value function, depending on whether the problem is convex or concave. Even if those bounds are one-sided, they are a first step in overcoming potential quantitative shortcomings of our power bounds.

**Corollary 3.12**
*For any investment strategy $(\pi, c) \in \mathcal{A}$ and pricing deflator $D \in \mathcal{D}^a$ such that $V[\pi, c], V^*[D] \in \mathcal{V}^a$, the following hold:*

*(L) If $\gamma, \psi \geqslant 1, \psi > 1$, then for any $u \in \mathcal{P}$*

$$U_t[c, u] \leqslant V_t[c],$$

*where* $U_t[c, u] = \mathbb{E}_t\left[\int_t^T \kappa_{t,s}^u F(c_s, u_s)\mathrm{d}s + \kappa_{t,T}^u \Phi_T(c_T)\right]$, *in particular*

$$\mathscr{L}(\pi, c) \leqslant \mathscr{O}(\pi, c, \Lambda) \leqslant \mathrm{P}^*_{\gamma\wedge\phi}(D) - U_t[c, u]$$

*(U) If $\gamma, \psi \leqslant 1, \psi < 1$, then for any $u \in \mathcal{P}$*

$$V_t^*[D] \leqslant U_t^*[D, u]$$

*where* $U_t^*[D, u] = \mathbb{E}_t\left[\int_t^T \kappa_{t,s}^{\frac{u}{\gamma}} F^*(D_s, u_s)\mathrm{d}s + \kappa_{t,T}^{\frac{u}{\gamma}} \Phi^*(D_T)\right]$, *in particular for*

$$\mathbb{U}_t^*[D, u] \triangleq \tfrac{1}{1-\gamma}\left(D_t X_t^{(\pi,c)}\right)^{1-\gamma}\left(\tfrac{1-\gamma}{\gamma} U_t^*[D, u]\right)^{\gamma},$$

*we obtain*

$$\mathscr{L}(\pi, c) \leqslant \mathscr{O}(\pi, c, \Lambda) \leqslant \mathbb{U}^*[D, u] - \mathrm{P}_{\gamma\vee\phi}(c)$$

*Proof.* The inequality in (L) follows immediately from Lemma 2.4, the inequality in (U) follows from Lemma B.17. The bounds on the optimality gap are then clear by (3.1.4) and Proposition 2.16 □

Within the proofs of Lemma 2.4 and and the concave version of Lemma 2.10 (i.e. Lemma B.17), respectively, we have seen that,

$$u^c \triangleq \operatorname*{arg\,sup}_{u\in\mathcal{U}} U_t[c, u] = -f_v(c, V[c]) \qquad \text{and} \qquad u^D \triangleq \operatorname*{arg\,sup}_{u\in\mathcal{U}} U_t^*[D, u] = -f_\nu^*(D, V^*[D])$$

Now, if we can find at least an approximate solution for the primal and dual problem and the associated strategies respectively, we also have an approximation for $u^c$ and $u^D$. As the variational representation of recursive utility is exact for exact $u^c$ and $u^D$, one would expect that good approximations of the optimal strategy, the processes $U_t[c, u]$ and $U_t^*[D, u]$ provide good lower and upper bounds, respectively. In particular those bounds are specifically designed for recursive utility and of time-additive structure, hence easily computable e.g. by Monte Carlo simulation.

Finding such approximations of the primal and dual value process and the associated primal and dual strategies is the goal of our numerical methods examined in Chapter 5 and Chapter 6, where all the above bounds on the optimality gap are put to use.

## 3.3 Application: A Refined Existence Result

The assumptions on integrability within Theorem 2.20 (i) are quite restrictive and within this section we apply our power bounds to soften them up.

Throughout this section we assume that

$$\gamma, \psi > 1 \qquad \text{such that} \qquad \theta < 0 \quad \text{and} \quad q^* \triangleq 1 - \tfrac{\gamma\psi}{\theta} > 1$$

and moreover, that for all $\gamma$, $\psi = \frac{1}{\phi} > 1$ we have

$$\mathbb{E}\left[\int_0^T \Lambda_t^{\frac{\gamma-1}{\gamma}} + \Lambda_t^{\frac{\phi-1}{\phi}} \mathrm{d}t + \Lambda_T^{\frac{\gamma-1}{\gamma}} + \Lambda_T^{\frac{\phi-1}{\phi}}\right] < \infty, \tag{3.3.1}$$

such that our dual power bounds from Proposition 3.9 exist.

We introduce the set of β-integrable semimartingales as $\mathcal{S}^{\beta} \triangleq \left\{V \in \mathcal{S} : \mathbb{E}\left[\sup_{t\in[0,T]} |V_t|^{\beta}\right] < \infty\right\}$ and the sets of relevant deflators and stochastic differential duals as

$$\begin{aligned} \mathcal{D}^{\alpha,\beta} &\triangleq \left\{\Lambda \in \mathcal{D}^a : \Lambda \text{ satisfies (3.3.1) and } \mathbb{E}\left[\int_0^T \Lambda_t^{\alpha}\mathrm{d}t + \Lambda_T^{\frac{\gamma-1}{\gamma}\beta}\right] < \infty\right\} \quad \text{and} \\ \mathcal{V}^{\beta} &\triangleq \left\{V \in \mathcal{S}^{\beta} : (1-\gamma)V > 0\right\}, \end{aligned} \tag{3.3.2}$$

where $\frac{\alpha}{1-\psi} > 1$ and $\frac{\beta}{q^*} > 1$ such that

$$\frac{1-\psi}{\alpha} + \frac{q^*}{\beta} \leqslant 1.$$

Then the existence and uniqueness result we prove within this section reads as follows.

**Theorem 3.13**
*Let $\gamma, \psi > 1$ and $\alpha, \beta$ as in* (3.3.2)*. Then for any $\Lambda \in \mathcal{D}^{\alpha,\beta}$ there exists a unique $V^* = V^*[\Lambda] \in \mathcal{V}^{\beta}$ satisfying*

$$V_t^*[\Lambda] = \mathbb{E}_t\left[\int_t^T f^*(\Lambda_s, V_s^*[\Lambda])\mathrm{d}s + \Phi^*(\Lambda_T)\right], \qquad t \in [0,T]. \tag{3.3.3}$$

The idea of the proof is similar to the approach taken by [Seiferling and Seifried, 2016][Theorem 3.1] and [Seiferling, 2016][Theorem 3.33], i.e. we transfer our results for $\Lambda \in \mathcal{D}^{\infty}$ to the $\mathcal{D}^{\alpha,\beta}$ case by using a monotone convergence result. Consequently, the steps we take in this approach are similar to the ones taken in [Seiferling and Seifried, 2016] and [Seiferling, 2016], but by making use of our dual power bounds and the duality theory from the first chapter we simplify many of the arguments therein.

The uniqueness part of Theorem 3.13 is an immediate consequence from the comparison result Theorem 3.5.

**Corollary 3.14**
*Let $V^*$, $\bar{V}^* \in \mathcal{V}^{\beta}$ be the dual value processes associated to the deflators $\Lambda \in \mathcal{D}^{\alpha,\beta}$ and $\bar{\Lambda} \in \mathcal{D}^{\alpha,\beta}$, respectively, and assume $\Lambda \preceq \bar{\Lambda}$. Then $\bar{V}^* \leqslant V^*$ and in particular for any $\Lambda \in \mathcal{D}^{\alpha,\beta}$ there exists at most one associated utility process $V \in \mathcal{D}^{\beta}$.*

*Proof.* We know from Lemma B.13 that $f^*(\lambda,\nu)$ is decreasing in $\lambda$ for all $\nu \in \mathbb{V}$, so Theorem 3.5 applied to $X = \bar{V}^*$ and $Y = V^*$ yields the result. $\square$

**Lemma 3.15**
*Let $V^* \in \mathcal{V}^\beta$ satisfy*

$$V_t^* = \mathbb{E}_t\left[\int_t^T f^*(\Lambda_s, V_s^*)\mathrm{d}s + \Phi^*(\Lambda_T)\right], \qquad t \in [0,T]$$

*for some $\Lambda \in \mathcal{D}^{\alpha,\beta}$. Then $M_t = \mathbb{E}_t\left[\int_0^T f^*(\Lambda_s, V_s^*)\,\mathrm{d}s + \Phi^*(\Lambda_T)\right]$, $t \in [0,T]$, is a uniformly integrable martingale and $V^*$ satisfies*

$$\mathrm{d}V_t^* = -f^*(\Lambda_t, V_t^*)\,\mathrm{d}t + \mathrm{d}M_t^*, \qquad V_T^* = \Phi^*(\Lambda_T). \tag{3.3.4}$$

*Proof.* Define $r^{-1} \triangleq \frac{1-\psi}{\alpha} + \frac{q^*}{\beta} \leqslant 1$. Then an application of Hölder's inequality yields

$$\left(\int_0^T |f^*(\Lambda_s, V_s^*)|^r \mathrm{d}s\right)^{\frac{1}{r}} \leqslant \tfrac{\delta\psi}{\psi-1}\left(\int_0^T \Lambda_s^\alpha \mathrm{d}s\right)^{\frac{1-\psi}{\alpha}}\left(\int_0^T \left(\tfrac{1-\gamma}{\gamma}V_s^*\right)^\beta \mathrm{d}s\right)^{\frac{q^*}{\beta}} + T^{\frac{1}{r}}\tfrac{|\delta\theta|}{\gamma}\sup_{t\in[0,T]}|V_t^*|.$$

In particular the integrability assumption (3.3.2) implies $\int_0^T |f^*(\Lambda_s, V_s^*)|\mathrm{d}s + |\Phi^*(\Lambda_T)| \in L^1(\mathbb{P})$ and $M$ is a uniformly integrable martingale; thus $V^*$ is a solution of (3.3.4). $\square$

We now want to establish a monotone convergence result on $\mathcal{V}^\beta$. We say that a sequence $(\Lambda^n)_{n\in\mathbb{N}} \subseteq \mathcal{D}^{\alpha,\beta}$ is increasing if $\Lambda^n \preceq \Lambda^{n+1}$, $n \in \mathbb{N}$ and we write $\Lambda^n \to \Lambda$ in $\mathcal{D}^{\alpha,\beta}$ if

$$\Lambda_t^n \to \Lambda_t \text{ for a.e. } t \in [0,T] \quad \text{and} \quad \Lambda_T^n \to \Lambda_T \quad \text{with} \quad \Lambda \in \mathcal{D}^{\alpha,\beta}.$$

In particular $\Lambda \in \mathcal{D}^{\alpha,\beta}$ by definition. If a sequence $(\Lambda^n)_{n\in\mathbb{N}}$ is increasing with $\Lambda^n \to \Lambda$ in $\mathcal{D}^{\alpha,\beta}$, we write $\Lambda^n \uparrow \Lambda$ in $\mathcal{D}^{\alpha,\beta}$. The decreasing case is defined analogously.

**Lemma 3.16**
*Let $(\Lambda^n)_{n\in\mathbb{N}} \subset \mathcal{D}^{\alpha,\beta}$ and $(V^{*,n})_{n\in\mathbb{N}} \subset \left\{V \in \mathcal{S}^\beta : (1-\gamma)V \geqslant 0\right\}$ such that*

$$V_t^{*,n} = \mathbb{E}_t\left[\int_t^T f^*(\Lambda_s^n, V_s^{*,n})\mathrm{d}s + \Phi(\Lambda_T^n)\right], \quad t \in [0,T], \quad n \in \mathbb{N}.$$

*If $\Lambda^n \uparrow \Lambda$ or $\Lambda^n \downarrow \Lambda$ in $\mathcal{D}^{\alpha,\beta}$, then there exists a unique $V^* \in \left\{V \in \mathcal{S}^\beta : (1-\gamma)V \geqslant 0\right\}$ with*

$$V_t^*[\Lambda] = \mathbb{E}_t\left[\int_t^T f^*(\Lambda_s, V_s^*[\Lambda])\mathrm{d}s + \Phi^*(\Lambda_T)\right] \qquad \forall t \in [0,T]$$

*and $V_t^{*,n} \to V_t^*$ for all $t \in [0,T]$.*

**Note:** Note that as $\gamma, \psi > 1$ and in particular $q^* > 1$, the dual aggregator $f^*(\lambda, \cdot)$ is well defined in zero. Including zero to the set of possible limits simplifies the proof by automatically providing an upper bound on $V^{*,n}$ for all $n \in \mathbb{N}$. We show *a posteriori* in Proposition 3.18 that $(1-\gamma)V^*[\Lambda] > 0$, so in particular $V^*[\Lambda] \in \mathcal{V}^\beta$.

*Proof.* We know from Corollary 3.14 that for every $\Lambda \in \mathcal{D}^{\alpha,\beta}$, there is at most one $V^*[\Lambda] = V^* \in \mathcal{V}^\beta$ that satisfies (3.3.3).
Also by Corollary 3.14 we have $V^{*,1} \leqslant V^{*,n}$ (resp. $V^{*,n} \leqslant V^{*,1}$) for any $n \in \mathbb{N}$, if $\Lambda^n$ is decreasing (resp. increasing). As $\gamma > 1$ we always have $V^{*,n} \leqslant 0$ for any $n \in \mathbb{N}$. It remains to find a lower

bound if $\Lambda^n$ is increasing. This bound is immediately given by the variational representation of $V^{*,n}$: By Lemma 2.10 we have

$$V_t^*[\Lambda^n] = \operatorname*{ess\,sup}_{u\in\mathcal{P}} \mathbb{E}_t\left[\int_t^T \kappa_{t,s}^{\frac{u}{\gamma}} F^*(\Lambda_s^n, u_s)\mathrm{d}s + \kappa_{t,T}^{\frac{u}{\gamma}} \Phi^*(\Lambda_T^n)\right] \geqslant \mathbb{E}_t\left[e^{-\frac{\delta\theta}{\gamma}(T-t)}\Phi^*(\Lambda_T^n)\right] \triangleq U_t^n. \tag{3.3.5}$$

The inequality in (3.3.5) is clear by choosing $u \equiv \delta\theta$, which implies $F^* \equiv 0$, see Lemma B.10. As $\Phi^*$ is decreasing by Lemma B.11, we have $U_t \triangleq \mathbb{E}_t\left[\mathrm{e}^{-\frac{\delta\theta}{\gamma}(T-t)}\Phi^*(\Lambda_T)\right] \leqslant U_t^n$. Finally $U \in \mathcal{V}^\beta$ by Doob's $L^p$-inequality and the integrability assumption (3.3.2).
Summarizing the above we always have $U \leqslant V^{*,n} \leqslant 0$ for all $n \in \mathbb{N}$. Hence we can define the stochastic process $V^*$ as the monotone pointwise limit $V_t^* \triangleq \lim_{n\to\infty} V_t^{*,n}$ for all $t \in [0,T]$. Note that for any $n \in \mathbb{N}$ and almost every $s \in [0,T]$ we have

$$|f^*(\Lambda_t^n, V_t^{*,n})| \leqslant \tfrac{\delta\psi}{\psi-1}\left(\left|\Lambda_t^1\right|^{1-\psi} + |\Lambda_t|^{1-\psi}\right)\left(\tfrac{1-\gamma}{\gamma}U_t\right)^{1-\frac{\gamma\psi}{\theta}} + \tfrac{2\delta\theta}{\gamma}U_t \triangleq B_t.$$

Then the same Hölder argument as in the proof of Lemma 3.15 shows that $B_t \in L^1(\mathbb{P}\otimes \mathrm{d}t)$. Finally we have $|\Phi^*(\Lambda^n)| \leqslant \frac{\gamma}{|1-\gamma|}\varepsilon^{\frac{1}{\gamma}}\left((\Lambda_T^1)^{\frac{\gamma-1}{\gamma}} + \Lambda_T^{\frac{\gamma-1}{\gamma}}\right) \in L^1(\mathbb{P})$ by (3.3.2), so dominated convergence yields that for all $t \in [0,T]$

$$V_t^* = \lim_{n\to\infty} \mathbb{E}_t\left[\int_t^T f^*(\Lambda_s^n, V_s^{*,n})\mathrm{d}s + \Phi^*(\Lambda_T^n)\right] = \mathbb{E}_t\left[\int_t^T f(\Lambda_s, V_s^*)\mathrm{d}s + \Phi^*(\Lambda_T)\right].$$

□

*Proof of Theorem 3.13.* Let $\Lambda \in \mathcal{D}^{\alpha,\beta}$. By monotonicity of the stochastic differential dual there is at most one process $V^* = V^*[\Lambda] \in \mathcal{V}^\beta$ that satisfies (3.3.4). To apply the monotone convergence theorem, we first consider $\bar\Lambda \in \mathcal{D}^{\alpha,\beta}$ such that $l_0 \preceq \bar\Lambda$ for some $l_0 > 0$ and define for each $n \in \mathbb{N}$ the truncated pricing deflator $\bar\Lambda_t^n \triangleq \bar\Lambda_t \wedge n$, $t \in [0,T]$. In particular $\bar\Lambda^n \in \mathcal{D}^\infty$, hence by Theorem 2.20 there exists a unique stochastic differential dual $\bar V^{*,n} = \bar V^{*,n}[\bar\Lambda^n]$. Certainly $\bar\Lambda^n \uparrow \bar\Lambda$ in $\mathcal{D}^{\alpha,\beta}$, thus Lemma 3.16 yields a unique $\bar V^* \in \left\{V \in \mathcal{S}^\beta : (1-\gamma)V \geqslant 0\right\}$ that satisfies

$$\bar V_t^* = \mathbb{E}_t\left[\int_t^T f^*(\bar\Lambda_s, \bar V_s^*)\mathrm{d}s + \Phi^*(\bar\Lambda_T)\right], \qquad t \in [0,T].$$

Now set $\Lambda^n \triangleq \Lambda + \frac{1}{n}$. Then as $0 \preceq \Lambda$, $\frac{1}{n} \preceq \Lambda^n$ and the previous argument yields an associated stochastic differential dual $V_t^{*,n} \in \left\{V \in \mathcal{S}^\beta : (1-\gamma)V \geqslant 0\right\}$ for any $n \in \mathbb{N}$. Again, since $\Lambda^n \downarrow \Lambda$, by Lemma 3.16 we know $V_t^{*,n} \to V_t^*$, $t \in [0,T]$, where $V^* \in \left\{V \in \mathcal{S}^\beta : (1-\gamma)V \geqslant 0\right\}$ satisfies

$$V_t^* = \mathbb{E}_t\left[\int_t^T f^*(\Lambda_s, V_s^*)\mathrm{d}s + \Phi^*(\Lambda_T)\right], \qquad t \in [0,T].$$

□

The following corollary states a dominated convergence result, that is used in Proposition 3.18 to show that the stochastic differential dual $V^* \in \left\{V \in \mathcal{S}^\beta : (1-\gamma)V \geqslant 0\right\}$ constructed above actually satisfies $(1-\gamma)V^* > 0$, i.e. $V^* \in \mathcal{V}^\beta$.

**Corollary 3.17**
*Suppose $(\Lambda^n)_{n\in\mathbb{N}} \subset \mathcal{D}^{\alpha,\beta}$ and there exist $\Lambda_*, \Lambda^* \in \mathcal{D}^{\alpha,\beta}$ such that $\Lambda_* \preceq \Lambda_n \preceq \Lambda^*$ for all $n \in \mathbb{N}$. If $\Lambda^n \to \Lambda$ in $\mathcal{D}^{\alpha,\beta}$, then $V_t^*[\Lambda^n] \to V_t^*[\Lambda]$ for all $t \in [0,T]$.*

*Proof.* Let $(\Lambda^n)_{n\in\mathbb{N}} \subset \mathcal{D}^{\alpha,\beta}$ and define $i_n \triangleq \inf_{k\geqslant n}\Lambda^k$ and $s_n = \sup_{k\geqslant n}\Lambda^k$. Then, as $\Lambda_* \preceq \Lambda_n \preceq \Lambda^*$ and $\Lambda_*, \Lambda^* \in \mathcal{D}^{\alpha,\beta}$ we have $(i_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}} \subset \mathcal{D}^{\alpha,\beta}$. By definition $i_n \uparrow \Lambda$ and $s_n \downarrow \Lambda$ in $\mathcal{D}^{\alpha,\beta}$, so for any $t \in [0,T]$

$$V_t^*[i_n], V_t^*[s_n] \to V_t^*[\Lambda]$$

by Lemma 3.16. On the other hand $i_n \preceq \Lambda_n \preceq s_n$ and $V^*[\Lambda]$ is decreasing in $\Lambda$, so in particular $V^*[i_n] \geqslant V^*[\Lambda] \geqslant V^*[s_n]$, $n \in \mathbb{N}$, which implies the assertion. □

**Proposition 3.18**
*Let $\Lambda \in \mathcal{D}^{\alpha,\beta}$, then the associated stochastic differential dual $V^*[\Lambda] \in \{V \in \mathcal{S}^\beta : (1-\gamma)V \geqslant 0\}$ satisfies*

$$\bar{\mathrm{P}}_\gamma^*(\Lambda) \leqslant V^*[\Lambda] \leqslant \bar{\mathrm{P}}_\phi^*(\Lambda), \tag{3.3.6}$$

*where $\bar{\mathrm{P}}_\varrho^*(\Lambda)$ is given by* (3.1.5).

*In particular we have $(1-\gamma)V^*[\Lambda] > 0$ for $\Lambda \in \mathcal{D}^{\alpha,\beta}$, i.e. $V^*[\Lambda] \in \mathcal{V}^\beta$.*

*Proof.* Consider the truncated deflator $\Lambda^n \triangleq \left(\frac{1}{n} \vee \Lambda\right) \wedge n$ for $n \in \mathbb{N}$. Note that $\Lambda^n \to \Lambda$ in $\mathcal{D}^{\alpha,\beta}$ and that $\Lambda \wedge 1 \preceq \Lambda^n \preceq \Lambda \vee 1$ for each $n \in \mathbb{N}$. Then dominated convergence (Corollary 3.17) implies

$$V_t^*[\Lambda^n] \to V_t^*[\Lambda], \qquad t \in [0,T].$$

Now consider $\mathcal{L}^\varrho[\Lambda^n]$ defined as in (3.1.6) by

$$\mathcal{L}_t^\varrho[\Lambda^n] = \mathrm{e}^{\frac{\delta}{\varrho}t}\mathbb{E}_t\left[\int_t^T \delta^{\frac{1}{\varrho}}\mathrm{e}^{-\frac{\delta}{\varrho}s}\varphi_\varrho^*(\Lambda_s^n)\mathrm{d}s + \mathrm{e}^{-\frac{\delta}{\varrho}T}\Phi_\varrho^*(\Lambda_T^n)\right].$$

Then dominated convergence yields

$$\mathcal{L}_t^\gamma[\Lambda^n] \to \mathcal{L}_t^\gamma[\Lambda] \text{ and } \mathcal{L}_t^\phi[\Lambda^n] \to \mathcal{L}_t^\phi[\Lambda] \text{ for all } t \in [0,T].$$

As $\Lambda^n \in \mathcal{D}^\infty$, Proposition 3.9 implies

$$\mathcal{L}^\gamma[\Lambda^n] = \bar{\mathrm{P}}_\gamma^*(\Lambda^n) \leqslant V^*[\Lambda^n] \leqslant \bar{\mathrm{P}}_\phi^*(\Lambda^n) = \varphi_\gamma^* \circ (\varphi_\phi^*)^{-1}(\mathcal{L}^\phi[\Lambda^n]),$$

thus sending $n \to \infty$ yields the claim.

Finally, note that by Doob's $L^p$-inequality we have

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left(\tfrac{1-\gamma}{\gamma}(\bar{\mathrm{P}}_\phi^*(\Lambda))_t\right)^{\frac{\gamma}{\theta\phi}}\right] < \infty,$$

thus (3.3.6) yields $\mathbb{E}\left[\sup_{t\in[0,T]}\left(\frac{1-\gamma}{\gamma}V_t^*[\Lambda]\right)^{\frac{\gamma}{\theta\phi}}\right] < \infty$. As $\theta < 0$ this shows $(1-\gamma)V^*[\Lambda] > 0$. □

# Chapter 4

# The Consumption-Investment Problem and its Dual Formulation

Within this chapter we describe and analyze the type of models that we consider for the rest of this thesis. To this end we first derive in detail a general multidimensional market model on a probability space with an underlying Brownian filtration. Our general market model captures several specific models that are widely used in portfolio optimization, in particular the Kim-Omberg model as in [Kim and Omberg, 1996] or [Wachter, 2002] and the Heston model as in [Heston, 1993]. As we later use those two special cases to test our algorithmic approaches to the Epstein-Zin optimization problem, they are explained in more detail.
Having introduced the general setting, we formulate and discuss an investor's primal investment-consumption optimization problem under Epstein-Zin utility and emphasize several aspects of the associated partial differential equations (PDEs). The same is done for the dual problem as derived in Section 2.2. Finally, we investigate the connection between the primal and dual problem in detail, with an emphasis on the connection of their dynamic programming equations. The insights gained from this analysis are the basis of the approximation approaches we introduce in Chapter 5 and Chapter 6.

Most technical calculations are omitted in the main text and can be found in Appendix C.

## 4.1 The General Market Model

Let $\left(\Omega, \{\mathfrak{F}_t\}_{t\in[0,T]}, \mathfrak{F}, \mathbb{P}\right)$ be a filtered probability space, where the filtration is generated by an $(m+n)$-dimensional Brownian motion $W$. We consider the consumption-optimization problem of an agent who consumes at a rate $c$. She can invest in a locally risk free money-market account $S_0$ or in $m$ risky assets $S^{\text{risky}} = (S_1, \dots, S_m)^\top$, whose price dynamics depend on an $n$-dimensional state process $Y = (Y_1, \dots, Y_n)^\top$. The first $m$ components of the Brownian motion $W$, $W^S \triangleq (W_1, \dots, W_m)^\top$ model the idiosyncratic shocks of the stocks and the correlation between the stocks and state variables, whereas the last $n$ components $W^Y \triangleq (W_{m+1}, \dots, W_{m+n})^\top$ can be thought of as the shocks driving the state variables and the correlations among the state variables. The dynamics of the risk free asset is always given as

$$\mathrm{d}S_0 = rS_0\mathrm{d}t$$

and the dynamics of the assets and state variables in terms of our $(m+n)$-dimensional Brownian motion $W$ are given as

$$\begin{aligned} \mathrm{d}S^{\text{risky}} &= \operatorname{diag}[S^{\text{risky}}]\left(\mu^S(Y)\mathrm{d}t + \Sigma^S(Y)\mathrm{d}W\right), \\ \mathrm{d}Y &= \mu^Y(Y)\mathrm{d}t + \Sigma^Y(Y)\mathrm{d}W, \end{aligned} \tag{4.1.1}$$

where $\mu^S$ and $\mu^Y$ are a $m$-dimensional, respectively $n$-dimensional vector function of $Y$ and[1]

$$\Sigma^S(Y) = \left(\hat{\Sigma}^S(Y)\,,\, \mathbf{0}_{m\times n}\right) \text{ and } \Sigma^Y(Y) = \left(\hat{\Sigma}^{SY}(Y)\,,\, \hat{\Sigma}^Y(Y)\right) \tag{4.1.2}$$

are $(m \times (n+m))$-dimensional, respectively $(n \times (n+m))$-dimensional matrix functions of $Y$, such that $\hat{\Sigma}^S(Y)$ and $\hat{\Sigma}^Y(Y)$ are invertible.
In the following we note that the above model structure is without loss of generality and on the way introduce a different notation that is more convenient in some occasions, e.g. within our numerical examples in Chapter 5 and Chapter 6. Indeed, consider a model

$$\begin{aligned} \mathrm{d}S^{\text{risky}} &= \operatorname{diag}[S^{\text{risky}}]\left(\mu^S(Y)\mathrm{d}t + \sigma^S(Y)\mathrm{d}\tilde{W}^S\right), \\ \mathrm{d}Y &= \mu^Y(Y)\mathrm{d}t + \sigma^Y(Y)\mathrm{d}\tilde{W}^Y, \end{aligned}$$

where the $(m+n)$-dimensional *correlated* Brownian motion

$$\tilde{W} = \left(\tilde{W}^S, \tilde{W}^Y\right) = \left(\tilde{W}_1^S, \dots, \tilde{W}_m^S, \tilde{W}_1^Y, \dots, \tilde{W}_n^Y\right)^\top$$

has positive definite correlation matrix $\rho$, i.e.

$$\mathrm{d}\tilde{W}\left(\mathrm{d}\tilde{W}\right)^\top = \rho \mathrm{d}t = \begin{pmatrix} \rho^S & \left(\rho^{SY}\right)^\top \\ \rho^{SY} & \rho^Y \end{pmatrix} \mathrm{d}t,$$

and $\sigma^S$, $\sigma^Y$ are invertible matrix functions of appropriate dimension.

Then $\rho$ has a unique Cholesky decomposition $\rho = LL^\top$, where $L \in \mathbb{R}^{(m+n)\times(m+n)}$ is an invertible lower triangular matrix with representation with representation

$$L = \begin{pmatrix} L^S & \mathbf{0}_{m\times n} \\ L^{SY} & L^Y \end{pmatrix}.$$

Then $W \triangleq L^{-1}\tilde{W}$ is a standard $(m+n)$-dimensional Brownian motion and defining

$$\sigma(Y) \triangleq \begin{pmatrix} \sigma^S(Y) & \mathbf{0}_{m\times n} \\ \mathbf{0}_{n\times m} & \sigma^Y(Y) \end{pmatrix}$$

and

$$\sigma(Y)L = \begin{pmatrix} \sigma^S(Y)L^S & \mathbf{0}_{m\times n} \\ \sigma^Y(Y)L^{SY} & \sigma^Y(Y)L^Y \end{pmatrix} \triangleq \begin{pmatrix} \hat{\Sigma}^S(Y) & \mathbf{0}_{m\times n} \\ \hat{\Sigma}^{SY}(Y) & \hat{\Sigma}^Y(Y) \end{pmatrix} = \begin{pmatrix} \Sigma^S(Y) \\ \Sigma^Y(Y) \end{pmatrix}, \tag{4.1.3}$$

we obtain

$$\begin{aligned} \sigma^S(Y)\mathrm{d}\tilde{W}^S &= \sigma^S(Y)\left(L\mathrm{d}W\right)^S \triangleq \Sigma^S(Y)\mathrm{d}W, \\ \sigma^Y(Y)\mathrm{d}\tilde{W}^Y &= \sigma^Y(Y)\left(L\mathrm{d}W\right)^Y \triangleq \Sigma^Y(Y)\mathrm{d}W, \end{aligned}$$

[1] We denote by $\mathbf{0}_{m\times n}$ the $(m \times n)$-dimensional matrix containing only zeros.

where $(L\mathrm{d}W)^S$ and $(L\mathrm{d}W)^Y$ denotes the first $m$ and the last $n$ coordinates of the $(m+n)$ dimensional process $L\mathrm{d}W$, respectively. In particular, basic algebra implies that $\Sigma^S(Y)$ and $\Sigma^Y(Y)$ are of the form as in (4.1.2).

Note that we avoid to write the $Y$ dependence explicitly if there is no room for confusion. Moreover, note that while our general results use the model as defined in (4.1.2), within special cases and in particular our examples we sometimes use the more specific notation as in (4.1.3) if beneficial.

With the appropriate specifications of the above parameters, our general market model captures in particular the multi-factor Ornstein-Uhlenbeck Model and the multi-factor stochastic volatility model. In the following we provide those specifications.

**Note:** We use $n$ as the dimension of a model, for example when we talk about a one-dimensional model this corresponds to $n = 1$ and *arbitrary* $m \in \mathbb{N}$, as the number of assets is of no concern when it comes to our solution approaches below.

**Example 4.1** (Multi-Factor Kim-Omberg Model)
*Define* $\mathbb{R}^p_+ \triangleq (0,\infty)^p$ *for any* $p \in \mathbb{N}$. *In the general model above, denoting by* $\mathbf{1}_m$ *the* $m$*-dimensional vector containing only ones, we choose*

$$\mu^S(y) = r\mathbf{1}_m + \bar{\lambda} + \boldsymbol{\lambda}^\top y \qquad \sigma^S(y) = \mathrm{diag}\left[\bar{\sigma}^S\right]$$
$$\mu^Y(y) = -\mathrm{diag}[\kappa]y \qquad \sigma^Y(y) = \mathrm{diag}\left[\bar{\sigma}^Y\right]$$

*where* $r \in \mathbb{R}$ *is the risk free rate,* $\bar{\lambda} \in \mathbb{R}^m$, $\boldsymbol{\lambda} \in \mathbb{R}^{n\times m}$, $\bar{\sigma}^S \in \mathbb{R}^m_+$, $\kappa \in \mathbb{R}^n_+$ *and* $\bar{\sigma}^Y \in \mathbb{R}^n_+$.
*Then the general model describes a Gaussian model with correlated Brownian motions given by the dynamics* [2]

$$\begin{aligned}\mathrm{d}S^{\text{risky}}_t &= \mathrm{diag}[S^{\text{risky}}_t]\left((r\mathbf{1}_m + \bar{\lambda} + \boldsymbol{\lambda}^\top Y_t)\mathrm{d}t + \sigma^S \mathrm{d}\tilde{W}^S_t\right),\\ \mathrm{d}Y_t &= -\mathrm{diag}[\kappa]Y_t\mathrm{d}t + \sigma^Y\mathrm{d}\tilde{W}^Y_t.\end{aligned}$$

*We rewrite the system such that we have independent Brownian motions as above. To this end let* $\Sigma^S$ *and* $\Sigma^Y$ *be given as in* (4.1.2)*, then the market dynamics are given by*

$$\begin{aligned}\mathrm{d}S^{\text{risky}}_t &= \mathrm{diag}[S^{\text{risky}}_t]\left(\left(r\mathbf{1}_m + \bar{\lambda} + \boldsymbol{\lambda}^\top Y_t\right)\mathrm{d}t + \Sigma^S\mathrm{d}W_t\right),\\ \mathrm{d}Y_t &= -\mathrm{diag}[\kappa]Y_t\mathrm{d}t + \Sigma^Y\mathrm{d}W_t,\end{aligned}$$

*where* $\Sigma^S \in \mathbb{R}^{m\times(m+n)}$ *and* $\Sigma^Y \in \mathbb{R}^{n\times(m+n)}$. *This is a straightforward extension of the classical model of mean-reverting returns as e.g. in [Wachter, 2002].* ○

**Example 4.2** (Multi-Factor Heston Model)
*In the general model above choose* $m = n$, $\rho^S = \rho^Y = \mathrm{I}_n$, *where* $\mathrm{I}_n$ *denotes the* $(n\times n)$*-dimensional identity matrix, in particular processes* $\tilde{W}^S$ *and* $\tilde{W}^Y$*, respectively, are standard Brownian motions. However, let the Brownian motions* $\tilde{W}^S$ *and* $\tilde{W}^Y$ *be mutually correlated with correlation matrix* $\rho^{SY} = \mathrm{diag}[\rho_1,\ldots,\rho_n]$. *Note that the market is incomplete if* $|\rho_i| < 1$ *for at least one* $i = 1,\ldots,n$. *Denote the set of orthogonal matrices in* $\mathbb{R}^{n\times n}$ *by* $\mathbb{R}^{n\times n}_o$ *and let*

[2] The structure of $Y$ is without loss of generality, for general mean-reversion structure consider $K = U^\top\mathrm{diag}[\kappa]U$, where $U$ is orthogonal and write $\tilde{Y} \triangleq UY$.

*$K \in \mathbb{R}_o^{n\times n}$, i.e. for $K \in \mathbb{R}_o^{n\times n}$ we have $KK^\top = \mathrm{I}_n$. Now choose*

$$\sigma^S(y) = K\mathrm{diag}[\bar\sigma^S]\, y^{\frac{1}{2}} \qquad \mu^S(y) = r\mathbf{1}_m + Ky^{\frac{1}{2}}(Ky^{\frac{1}{2}})^\top\bar\lambda$$
$$\mu^Y(y) = \bar\mu - \mathrm{diag}[\kappa]y \qquad \sigma^Y(y) = \mathrm{diag}\left[\bar\sigma^Y\right] y^{\frac{1}{2}}$$

*where $r \in \mathbb{R}$ denotes the risk free rate, $\bar\lambda \in \mathbb{R}^n$, $\bar\mu, \kappa, \bar\sigma^S, \bar\sigma^Y \in \mathbb{R}^n_+$ and for a vector $y \in \mathbb{R}^n_+$ we define $y^{\frac{1}{2}} \triangleq \mathrm{diag}[\sqrt{y_1}, \dots, \sqrt{y_n}]$.*
*Then the general model describes a multi-factor stochastic volatility model given by the dynamics*

$$\mathrm{d}S_t^{\text{risky}} = \mathrm{diag}[S_t^{\text{risky}}]\left(\left(r\mathbf{1}_m + K\mathrm{diag}[Y_t]K^\top\boldsymbol{\lambda}\right)\mathrm{d}t + K\mathrm{diag}[\bar\sigma^S]Y_t^{\frac{1}{2}}\mathrm{d}\tilde W_t^S\right),$$
$$\mathrm{d}Y_t = (\bar\mu - \mathrm{diag}[\kappa]Y_t)\,\mathrm{d}t + \mathrm{diag}[\bar\sigma^Y]Y_t^{\frac{1}{2}}\mathrm{d}\tilde W_t^Y.$$

*Note that the risky assets are uncorrelated if $K = \mathrm{I}_n$, so even though the Brownian motions $\tilde W^S$ are independent, a correlation between the assets is introduced by the matrix $K$. To ensure positivity and stationarity of the state processes $Y$, we require the Feller condition*

$$\kappa_i > 0 \quad \text{and} \quad \frac{2\bar\mu_i}{\left(\bar\sigma_i^Y\right)^2} \geqslant 1 \tag{4.1.4}$$

*to be met for all $i = 1, \dots, n$. When considering this Heston specification of our general model we always assume that the condition (4.1.4) is satisfied without further mention.*

*The model includes existing multi-factor stochastic volatility models as e.g. in [Escobar and Olivares, 2013] or [Escobar et al., 2017].*
*Again, we rewrite the system such that we have independent Brownian motions, then the dynamics read*

$$\mathrm{d}S_t^{\text{risky}} = \mathrm{diag}[S_t^{\text{risky}}]\left(\left(r\mathbf{1}_m + K\mathrm{diag}[Y_t]K^\top\bar\lambda\right)\mathrm{d}t + \Sigma^S\mathrm{d}W_t\right),$$
$$\mathrm{d}Y_t = (\bar\mu - \mathrm{diag}[\kappa]Y_t)\,\mathrm{d}t + \Sigma^Y\mathrm{d}W_t,$$

*where $\Sigma^S$ and $\Sigma^Y$ are $(n \times 2n)$-dimensional matrix functions as in (4.1.2).*

***Note:*** *The diagonal form of the covariance matrices and the fact that $KK^\top = \mathrm{I}_n$ is of particular importance here. This structure in particular implies that $L^{SY} = \rho^{SY}$ and $L^Y = \mathrm{diag}[\sqrt{1-\rho_1^2}, \dots, \sqrt{1-\rho_n^2}]$ are also diagonal matrices, which ensures that the HJB equation associated to the model separates (if modified appropriately).* ○

## 4.2 The Primal Optimization Problem

Consider a financial market $S$ where the asset pricies $S = (S_0, \dots, S_m)$ and the associated state variables $Y = (Y_1, \dots, Y_n)$ are as given in (4.1.1). Endowed with an initial capital $x > 0$, our agent may invest in the given market by choosing a portfolio represented by a predictable $S$-integrable process $\pi = (\pi^0, \dots, \pi^m)$. Here $\pi_t^i$ is the fraction of her current wealth invested in the risky asset $S_i$ at time $t$ and $\pi_t^0 = 1 - \sum_{i=1}^m \pi_t^i$ is the fraction invested in the riskless asset. Moreover, let $c \in \mathcal{C}^a$ be the investor's consumption stream. Recall that the set of admissible trading strategies was previously defined as $\mathcal{A}$. In the following we avoid to write the $Y$ dependence explicitly when there is no room for confusion.
The investor's wealth process following a certain strategy $(\pi, c) \in \mathcal{A}$ is denoted by $X^{(\pi,c)}$ and is given as

$$\mathrm{d}X_t^{(\pi,c)} = X_t^{(\pi,c)}\left(\left(r + \pi_t^\top\chi\right)\mathrm{d}t + \pi_t^\top\Sigma^S\mathrm{d}W_t\right) - c_t\mathrm{d}t, \qquad X_0^{(\pi,c)} = x,$$

where $\chi = \mu^S - r\mathbf{1}_m$ is the excess return of the risky assets. The investor now chooses her investment and consumption strategies to maximize her continuous time recursive utility, that is she aims to determine $(\pi^*, c^*) \in \mathcal{A}$ such that

$$\nu(c^*) = \sup_{(\pi,c)\in\mathcal{A}} V_0[c] = \sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}\left[\int_0^T f(c_s, V_s[c])\mathrm{d}s + \Phi(c_T)\right] \tag{4.2.1}$$

where, the Epstein-Zin aggregator $f$ is given as

$$f(c,v) = \delta\tfrac{1}{1-\phi}c^{1-\phi}((1-\gamma)v)^{1-\frac{1}{\theta}} - \delta\theta v \qquad \text{and} \qquad \Phi(c) = \varepsilon\tfrac{1}{1-\gamma}c^{1-\gamma}.$$

For more details and interpretations of the parameters, see Chapter 2.

In Appendix C.1 we show that the associated dynamic programming equation for the agent's indirect utility $V_t[c] = G(t, X_t, Y_t)$ reads

$$\begin{aligned} 0 = \sup_{(\pi,c)\in\mathcal{A}} \Big\{ & G_t + \left(x\left(r + \pi^\top\chi\right) - c\right)G_x + \left(\mu^Y\right)^\top G_y + \tfrac{1}{2}x^2\pi^\top\Sigma^S\left(\Sigma^S\right)^\top\pi G_{xx} \\ & + xG_{xy}\Sigma^Y\left(\Sigma^S\right)^\top\pi + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top G_{yy}\Sigma^Y\right] + f(c,G)\Big\} \end{aligned} \tag{4.2.2}$$

where $G(T,x,y) = \varepsilon\frac{1}{1-\gamma}x^{1-\gamma}$ with a constant weight of bequest $\varepsilon > 0$. Following [Zariphopoulou, 2001] we conjecture

$$G(t,x,y) = \tfrac{1}{1-\gamma}x^{1-\gamma}g(t,y)^k, \tag{4.2.3}$$

where $g(t,y) > 0$ for all $(t,y) \in [0,T]\times\mathbb{R}^n$ and $k \in \mathbb{R}$ is a constant yet to be determined; then the solution to (4.2.2) is summarized in the following proposition.

**Proposition 4.3**
*Assume that the agent's indirect utility has a representation as in* (4.2.3)*, then the optimal strategy reads*

$$\pi^* = \tfrac{1}{\gamma}\left(\Sigma^S\left(\Sigma^S\right)^\top\right)^{-1}\chi + \tfrac{k}{\gamma}\left(\left(\Sigma^S\right)^\top\right)^+\left(\Sigma^Y\right)^\top\frac{g_y}{g} \qquad \textit{and} \qquad \left(\frac{c}{x}\right)^* = \delta^\psi g^{-\frac{k\psi}{\theta}},$$

*where $g$ satisfies the nonlinear partial differential equation*

$$\begin{aligned} 0 = g_t &+ \tfrac{1-\gamma}{k}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\chi^\top\left(\Sigma^S\left(\Sigma^S\right)^\top\right)^{-1}\chi - \tfrac{\delta\theta}{1-\gamma}\right)g + \left(\left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\chi^\top\left(\left(\Sigma^S\right)^\top\right)^+\left(\Sigma^Y\right)^\top\right)g_y \\ &+ \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top g_{yy}\Sigma^Y\right] + \tfrac{1}{2}\frac{1}{g}(g_y)^\top\Sigma^Y\left((k-1)\mathrm{I}_{m+n} - k\tfrac{\gamma-1}{\gamma}\left(\Sigma^S\right)^\top\left(\left(\Sigma^S\right)^\top\right)^+\right)\left(\Sigma^Y\right)^\top g_y \\ &+ \tfrac{\delta^\psi\theta}{k\psi}g^{1-\frac{k\psi}{\theta}}, \end{aligned} \tag{4.2.4}$$

*with terminal condition* $g(T,y) = \varepsilon^{\frac{1}{k}}$.

*Proof.* Appendix C.1 □

**Note:** For a matrix $A^\top \in \mathbb{R}^{(m+n)\times m}$ we denote by $\left(A^\top\right)^+ \in \mathbb{R}^{m\times(m+n)}$ the Moore-Penrose inverse which is defined by $\left(A^\top\right)^+ \triangleq \left(AA^\top\right)^{-1}A$, in particular

$$\left(\left(\Sigma^S\right)^\top\right)^+ = \left(\left(\left(\sigma^S L^S\right)^\top\right)^{-1}, \mathbf{0}_{m\times n}\right). \tag{4.2.5}$$

In (4.2.4), two terms give rise to nonlinearities, but we have only one degree of freedom in terms of the constant $k$. Certainly the nonlinearity $g^{1-\frac{k\psi}{\theta}}$ could be eliminated by choosing $k$ appropriately, but in any case the nonlinearity

$$\tfrac{1}{2}\frac{1}{g}(g_y)^\top \Sigma^Y \left( (k-1)\mathrm{I}_{m+n} - k\tfrac{\gamma-1}{\gamma} \left(\Sigma^S\right)^\top \left( \left(\Sigma^S\right)^\top \right)^+ \right) \left(\Sigma^Y\right)^\top g_y \triangleq \tfrac{1}{2}\frac{1}{g}(g_y)^\top \Sigma^Y \mathcal{N} \left(\Sigma^Y\right)^\top g_y \tag{4.2.6}$$

remains: By inserting the matrices $\Sigma^Y$ and $\Sigma^S$ from (4.1.2) above, we calculate

$$\Sigma^Y \mathcal{N} \left(\Sigma^Y\right)^\top = \tfrac{k-\gamma}{\gamma} \hat{\Sigma}^{SY} \left(\hat{\Sigma}^{SY}\right)^\top + (k-1)\hat{\Sigma}^Y \left(\hat{\Sigma}^Y\right)^\top,$$

which is a system of $n$ equations, so for $n > 1$ there is in general no way to choose a scalar $k$ such that (4.2.6) disappears.

However, if the market were complete, then in particular $\left(\left(\Sigma^S\right)^\top\right)^+ = \left(\left(\Sigma^S\right)^\top\right)^{-1}$ and

$$\Sigma^Y \mathcal{N} \left(\Sigma^Y\right)^\top = \tfrac{k-\gamma}{\gamma} \Sigma^Y \left(\Sigma^Y\right)^\top,$$

and setting $k = \gamma$ eliminates the non-linearity (4.2.6). Thus, we want to complete the market using duality theory, more precisely by the notion of *least favorable completion.* The idea is classical and due to [Karatzas et al., 1991]; in the following we explain their concept.

In addition to the $m$ risky assets within our market model from Section 4.1, we introduce $n$ *artificial* assets $S^a = (S^a_1, \dots S^a_n)^\top$ with dynamics

$$\mathrm{d}S^a = \mathrm{diag}[S^a] \left(\mu^S_a(Y)\mathrm{d}t + \Sigma^S_a(Y)\mathrm{d}W\right),$$

where $\mu_a$ and $\Sigma^S_a$ are $n$-dimensional and $(n \times (m+n))$-dimensional matrix functions of the $n$-dimensional state process $Y$, respectively. Then the risky assets in the artificially completed market $S^{\mathrm{c}} = \begin{pmatrix} S^{\text{risky}} \\ S^a \end{pmatrix}$ follow the dynamics

$$\mathrm{d}S^{\mathrm{c}} = \mathrm{diag}[S^{\mathrm{c}}] \left(\mu^S_{\mathrm{c}}(Y)\mathrm{d}t + \Sigma^S_{\mathrm{c}}(Y)\mathrm{d}W\right),$$

where $\mu^S_{\mathrm{c}} = \begin{pmatrix} \mu^S \\ \mu^S_a \end{pmatrix}$ and $\Sigma^S_{\mathrm{c}} = \begin{pmatrix} \Sigma^S \\ \Sigma^S_a \end{pmatrix}$.
In order to make this artificial completion of the market meaningful, we have to choose the parameters $\mu^S_a$ and $\Sigma^S_a$ in specific ways. We certainly have to pick the matrix $\Sigma^S_a$ such that the market is actually complete, i.e. the augmented correlation matrix $\Sigma^S_{\mathrm{c}}$ is invertible. As we assumed that $\sigma^S L^S$ has full rank, this is accomplished by setting $\Sigma^S_a \triangleq \left(\mathbf{0}_{n\times m}, \rho^S_a\right)$, where $\rho^S_a$ is a $(n \times n)$-dimensional matrix with orthonormal rows. As the specific form of $\rho^S_a$ plays no role in the solution of the optimization problem, we choose $\rho^S_a = \mathrm{I}_{n\times n}$ for simplicity.
The significant choice within the completion is the drift vector $\mu^S_a$. If we would arbitrarily choose some $\mu^S_a \in \mathbb{R}^n$, the investor following her optimal strategy in the augmented market would certainly trade some of the artificial assets, so her strategy would not be admissible in our *incomplete* market setting where those assets don't actually exist. This means we have to specify $\mu^S_a$ in such a way that the investor following her optimal strategy in the completed market *chooses not to trade any artificial asset.* In that case the optimal strategy in the artificially completed market is still admissible and coincides with the optimal strategy in the original market.

This is exactly the idea of *least favorable completion* as introduced by [Karatzas et al., 1991]. As they point out, this least favorable completion is achieved by choosing $\mu_a^S$ such that the artificial assets' excess return is exactly the optimal strategy of the associated *dual* problem.

In particular, if we can solve the dual problem, we can artificially complete our model from Section 4.1 and formulate an equivalent optimization problem in a complete market. The associated complete market problem is in general more tractable, for example we can remove the nonlinearity (4.2.6). Thus our next step is to analyze the dual optimization problem associated to (4.2.1).

## 4.3 The Dual Optimization Problem

By our results from Chapter 2 the investor's dual optimization problem reads

$$\mathrm{v}^*(\Lambda^*) = \inf_{\Lambda\in\mathcal{D}^a} \{V_0^*[\Lambda] + \lambda x\} = \inf_{\Lambda\in\mathcal{D}^a} \left\{ \mathbb{E}\left[ \int_0^T f^*\left(\Lambda_s, V_s^*[\Lambda]\right) \mathrm{d}s + \Phi^*(\Lambda_T) \right] + \lambda x \right\}, \tag{4.3.1}$$

where

$$f^*(\lambda,\nu) = \delta^\psi \tfrac{1}{\psi-1} \lambda^{1-\psi} \left(\tfrac{1-\gamma}{\gamma}\nu\right)^{1-\frac{\gamma\psi}{\theta}} - \tfrac{\delta\theta}{\gamma}\nu \qquad \text{and} \qquad \Phi^*(\lambda) = \varepsilon^{\frac{1}{\gamma}} \tfrac{\gamma}{1-\gamma} \lambda^{\frac{\gamma-1}{\gamma}}.$$

As we have seen in Proposition 2.16, it suffices to solve the problem for *pricing* deflators, i.e. find $D^* \in \mathcal{D}_1^a$ such that $\mathrm{v}^*(D^*) \triangleq \inf_{D\in\mathcal{D}_1^a} V_0^*[D]$. Then by Proposition 2.16 the optimal solution to (4.3.1) is given by

$$\mathrm{v}^*\left(\Lambda^*\right) = \tfrac{1}{1-\gamma} x^{1-\gamma} \left(\tfrac{1-\gamma}{\gamma} \mathrm{v}^*(D^*)\right)^\gamma.$$

We assume that our pricing deflators have the form

$$\mathrm{d}D_t = -D_t\left(r\mathrm{d}t + \eta_t^\top \mathrm{d}W\right) \qquad \text{and} \qquad D_0 = 1, \tag{4.3.2}$$

where $\eta = \begin{pmatrix} \eta^S \\ \eta^Y \end{pmatrix}$ is a $(m+n)$-dimensional vector function of $Y$ containing the market prices of risk, hence $\eta^S = \left(\hat{\Sigma}^S\right)^{-1} \chi$ can easily be derived from the supermartingale condition in Equation (2.2.14) and is already determined by the market. Note that since a process $D$ as in (4.3.2) is a stochastic exponential, it is positive and a straightforward application of Itô's formula shows that the stochastic process

$$DX^{(\pi,c)} + \int_0^\cdot D_s c_s \mathrm{d}s$$

is a supermartingale for all $(\pi, c) \in \mathcal{A}$ and $\eta \in \mathcal{P}^{m+n}$, so $D$ is indeed a pricing deflator.

In Appendix C.2 we show that the dynamic programming equation for the indirect dual utility $V_t^*[D] = H(t, D_t, Y_t)$ reads

$$\begin{aligned} 0 = \inf_{\substack{\eta\in\mathcal{P}^{m+n} \\ \eta^S = \left(\sigma^S L^S\right)^{-1}\chi}} &\left\{ H_t - rdH_d + \left(\mu^Y\right)^\top H_y + \tfrac{1}{2} d^2 \eta^\top \eta H_{dd} \right. \\ &\left. - dH_{dy}\Sigma^Y \eta + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top H_{yy} \Sigma^Y\right] + f^*(d, H) \right\}, \end{aligned} \tag{4.3.3}$$

with terminal condition $H(T,d,y) = \varepsilon^{\frac{1}{\gamma}} \frac{\gamma}{1-\gamma} d^{\frac{\gamma-1}{\gamma}}$.

**Note:** Note that in the complete market case the dual problem is redundant as then $\eta = \eta^S$ is fully set by the market, so there is nothing to optimize.

Similar as for the primal problem, we conjecture that the agent's indirect utility has a representation

$$H(t,d,y) = \tfrac{\gamma}{1-\gamma} d^{\frac{\gamma-1}{\gamma}} h(t,y)^l, \tag{4.3.4}$$

where $h(t,y) > 0$ for all $(t,y) \in [0,T] \times \mathbb{R}^n$ and $l \in \mathbb{R}$ is a constant yet to be determined. Then the solution to (4.3.3) is summarized by the following proposition.

**Proposition 4.4**
*Assume that the agent's indirect utility has a representation as in* (4.3.4)*, then the optimal market prices of risk determining the pricing deflator read*

$$\eta^* = \begin{pmatrix} \eta^S \\ (\eta^Y)^* \end{pmatrix} = \begin{pmatrix} \left(\hat{\Sigma}^S\right)^{-1} \chi \\ -l\gamma \left(\hat{\Sigma}^Y\right)^\top \frac{h_y}{h} \end{pmatrix} \tag{4.3.5}$$

*where $h$ satisfies the nonlinear partial differential equation*

$$\begin{aligned} 0 = h_t &+ \tfrac{1-\gamma}{\gamma l}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\left(\eta^S\right)^\top \eta^S - \tfrac{\delta\theta}{1-\gamma}\right) h + \left(\left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\left(\hat{\Sigma}^{SY}\eta^S\right)^\top\right) h_y \\ &+ \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top h_{yy}\Sigma^Y\right] + \tfrac{1}{2}\frac{1}{h}(h_y)^\top\left((l-1)\Sigma^Y\left(\Sigma^Y\right)^\top - l(1-\gamma)\hat{\Sigma}^Y\left(\hat{\Sigma}^Y\right)^\top\right) h_y \\ &+ \tfrac{\delta^\psi\theta}{l\gamma\psi} h^{1-l\frac{\gamma\psi}{\theta}} \end{aligned} \tag{4.3.6}$$

*with terminal condition $h(T,y) = \varepsilon^{\frac{1}{\gamma l}}$.*

*Proof.* Appendix C.2 □

Again, two terms in (4.3.6) give rise to nonlinearities and while $h^{1-l\frac{\gamma\psi}{\theta}}$ would vanish with an appropriate choice of $l$, the nonlinearity

$$\tfrac{1}{2}\frac{1}{h}(h_y)^\top\left((l-1)\Sigma^Y\left(\Sigma^Y\right)^\top - l(1-\gamma)\hat{\Sigma}^Y\left(\hat{\Sigma}^Y\right)^\top\right) h_y,$$

which is a system of $n$ equations, cannot be eliminated by the choice of the scalar $l$.

Apparently the dual problem does not help us directly, as the dual dynamic programming equation is as hard to solve as the primal one. However, taking a closer look at the partial differential equations associated to the primal and dual problem, one notices a deeper connection also between the PDEs (4.2.4) and (4.3.6). This connection and its implications are further highlighted in the next section.

## 4.4 Primal and Dual Problem Connected

We first investigate the direct relation between the primal and dual solutions derived above. The chapter is completed with the analysis of the concept of artificial completion and in particular least favorable completion as described in Section 4.2 on PDE level. Both connections between the primal and dual problem provide valuable insights that we further exploit in our iterative solution approach.

### 4.4.1 Primal & Dual Solution and Duality

By straightforward manipulations of the partial differential equation associated to the dual problem, the direct connection between the primal and dual solutions becomes apparent: Recall the dual partial differential equation from (4.3.6) derived as

$$
\begin{aligned}
0 = h_t &+ \tfrac{1-\gamma}{\gamma l}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\left(\eta^S\right)^\top \eta^S - \tfrac{\delta\theta}{1-\gamma}\right) h + \left(\left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\left(\hat{\Sigma}^{SY}\eta^S\right)^\top\right) h_y \\
&+ \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top h_{yy}\Sigma^Y\right] + \tfrac{1}{2}\frac{1}{h}(h_y)^\top\left((l-1)\Sigma^Y\left(\Sigma^Y\right)^\top - l(1-\gamma)\hat{\Sigma}^Y\left(\hat{\Sigma}^Y\right)^\top\right) h_y \\
&+ \tfrac{\delta^\psi\theta}{l\gamma\psi} h^{1-l\frac{\gamma\psi}{\theta}}
\end{aligned}
$$

with $h(T,y) = \varepsilon^{\frac{1}{\gamma l}}$.

Plugging in the predetermined market prices of risk $\eta^S = \left(\hat{\Sigma}^S\right)^{-1}\chi$, we notice that

$$
\left(\eta^S\right)^\top \eta^S = \chi^\top\left(\Sigma^S\left(\Sigma^S\right)^\top\right)^{-1}\chi \qquad \text{and} \qquad \left(\hat{\Sigma}^{SY}\eta^S\right)^\top = \chi^\top\left(\left(\Sigma^S\right)^\top\right)^+\left(\Sigma^Y\right)^\top .
$$

Moreover a small calculation, using in particular (4.2.5), reveals

$$
(l-1)\Sigma^Y\left(\Sigma^Y\right)^\top - l(1-\gamma)\hat{\Sigma}^Y\left(\hat{\Sigma}^Y\right)^\top = \Sigma^Y\left((l\gamma-1)\mathrm{I}_{m+n} + l(1-\gamma)\left(\Sigma^S\right)^\top\left(\left(\Sigma^S\right)^\top\right)^+\right)\left(\Sigma^Y\right)^\top ,
$$

thus (4.3.6) becomes

$$
\begin{aligned}
0 = h_t &+ \tfrac{1-\gamma}{\gamma l}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\chi^\top\left(\Sigma^S\left(\Sigma^S\right)^\top\right)^{-1}\chi - \tfrac{\delta\theta}{1-\gamma}\right) h + \left(\left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\chi^\top\left(\left(\Sigma^S\right)^\top\right)^+\left(\Sigma^Y\right)^\top\right) h_y \\
&+ \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top h_{yy}\Sigma^Y\right] + \tfrac{1}{2}\frac{1}{h}(h_y)^\top\Sigma^Y\left((l\gamma-1)\mathrm{I}_{m+n} + l(1-\gamma)\left(\Sigma^S\right)^\top\left(\left(\Sigma^S\right)^\top\right)^+\right)\left(\Sigma^Y\right)^\top h_y \\
&+ \tfrac{\delta^\psi\theta}{l\gamma\psi} h^{1-l\frac{\gamma\psi}{\theta}}
\end{aligned}
$$

with $h(T,y) = \varepsilon^{\frac{1}{\gamma l}}$.
Now the primal and dual partial differential equations only differ in the parameters $k$ and $l$, respectively. Choosing $k = l\gamma$ reveals that the PDEs - and hence also their solutions $g$ and $h$ - actually coincide. This fact has interesting implications. First, assume we can verify that the strategies from Proposition 4.3 and Proposition 4.4 are indeed optimal, and let $D^*$ be the pricing deflator from (4.3.2) associated to $\left(\eta^Y\right)^*$. Then Proposition 2.16 implies

$$
\begin{aligned}
\mathbb{V}_t[\pi^*, c^*] &= \tfrac{1}{1-\gamma}\left(X_t^{(\pi^*,c^*)}\right)^{1-\gamma} g(t, Y_t)^k \\
&= \tfrac{1}{1-\gamma}\left(X_t^{(\pi^*,c^*)}\right)^{1-\gamma} h(t, Y_t)^k && (g \equiv h) \\
&= \tfrac{1}{1-\gamma}\left(D_t^* X^{(\pi^*,c^*)}\right)^{1-\gamma}\left(\left(D_t^*\right)^{\frac{\gamma-1}{\gamma}} h(t, Y_t)^l\right)^\gamma && (k = \gamma l) \\
&= \tfrac{1}{1-\gamma}\left(D_t^* X^{(\pi^*,c^*)}\right)^{1-\gamma}\left(\tfrac{1-\gamma}{\gamma}\mathbb{V}_t^*[D^*]\right)^\gamma \\
&= \mathbb{V}_t^*[\Lambda^*] + \Lambda_t^* X_t^{(\pi^*,c^*)}. && (\text{Proposition 2.16})
\end{aligned}
$$

In particular, there is no duality gap.

Moreover, since $g \equiv h$, the optimal strategies for the primal and dual problem share a simple one to one relation. By Proposition 4.3 and Proposition 4.4 we have

$$\pi^* = \tfrac{1}{\gamma}\left(\Sigma^S\left(\Sigma^S\right)^\top\right)^{-1}\chi - \tfrac{1}{\gamma}\mathcal{K}\left(\eta^Y\right)^*,$$

where

$$\mathcal{K} \triangleq \left(\left(\hat{\Sigma}^S\right)^\top\right)^{-1}\left(\hat{\Sigma}^{SY}\right)^\top\left(\left(\hat{\Sigma}^Y\right)^\top\right)^{-1}$$

is fully determined by the market and can be interpreted as a $Y$-dependent $(m \times n)$-dimensional covariation matrix. The other way around find the relation

$$\left(\eta^Y\right)^* = \mathcal{K}^+\left(\left(\Sigma^S\left(\Sigma^S\right)^T\right)^{-1}\chi - \gamma\pi^*\right).$$

In particular, every investment strategy $\pi^*$ implies certain market prices of risk $\left(\eta^Y\right)^*$, which can be used to evaluate our upper and lower bounds from the previous chapter.

From now on and for the rest of this thesis we choose $l = \frac{k}{\gamma}$ such that the primal and dual solution are characterized by the one function $g \equiv h$ and we choose $g$ to denote said solution.

### 4.4.2 Least Favorable Completion

As the reduced partial differential equations associated to the primal and dual solutions coincide, the concept of *least favorable completion* explained in the previous section can be understood directly on PDE level and without abstract arguments as provided e.g. in [Karatzas et al., 1991].

We first state the problem in an artificially completed market where the artificial assets exhibit an arbitrary market price of risk $\eta^Y$. Recall the definition of the considered market with risk free asset $S_0$, $m$ assets $S^{\text{risky}}$ and $n$ state variables $Y$ given by the dynamics

$$\begin{aligned} \mathrm{d}S^{\text{risky}} &= \operatorname{diag}[S^{\text{risky}}]\left(\mu^S(Y)\mathrm{d}t + \Sigma^S(Y)\mathrm{d}W\right), \\ \mathrm{d}Y &= \mu^Y(Y)\mathrm{d}t + \Sigma^Y(Y)\mathrm{d}W. \end{aligned}$$

Moreover, we introduce the $n$ *artificial* assets $S^a$ with dynamics

$$\mathrm{d}S^a = \operatorname{diag}[S^a]\left(\mu_a^S(Y)\mathrm{d}t + \Sigma_a^S(Y)\mathrm{d}W\right),$$

as explained in Section 4.2. Then the risky assets of the artificially completed market $S^{\mathrm{c}} = (S^{\text{risky}}, S^a)$ follows the dynamics

$$\mathrm{d}S^{\mathrm{c}} = \operatorname{diag}[S^{\mathrm{c}}]\left(\mu_{\mathrm{c}}^S(Y)\mathrm{d}t + \Sigma_{\mathrm{c}}^S(Y)\mathrm{d}W\right), \tag{4.4.1}$$

where we set $\mu_{\mathrm{c}}^S = \begin{pmatrix}\mu^S \\ r\mathbf{1}_n + \eta^Y\end{pmatrix}$ and $\Sigma_{\mathrm{c}}^S = \begin{pmatrix}\hat{\Sigma}^S & \mathbf{0}_{m\times n} \\ \mathbf{0}_{n\times m} & \mathrm{I}_{n\times n}\end{pmatrix}$.

We denote an investment strategy in this artificially completed market as $\tilde{\pi} = (\tilde{\pi}^0, \ldots, \tilde{\pi}^{m+n})$ where $\tilde{\pi}_t^i$ is the fraction of the investor's current wealth invested in the risky asset $S_i^{\mathrm{c}}$ at time $t$ and $\tilde{\pi}_t^0 = 1 - \sum_{i=1}^{m+n}\tilde{\pi}_t^i$ is the fraction invested in the riskless asset. Moreover, let $c \in \mathcal{C}^a$ be the investors consumption stream. We denote the set of admissible trading strategies in the extended market by $\tilde{\mathcal{A}}$. Then the investors wealth process following a certain strategy $(\tilde{\pi}, c) \in \mathcal{A}$ is denoted by $\tilde{X}^{(\tilde{\pi},c)}$ and is given as

$$\mathrm{d}\tilde{X}_t^{(\tilde{\pi},c)} = \tilde{X}_t^{(\tilde{\pi},c)}\left(\left(r + \tilde{\pi}_t^\top\chi_{\mathrm{c}}\right)\mathrm{d}t + \tilde{\pi}_t^\top\Sigma_{\mathrm{c}}^S\mathrm{d}W_t\right) - c_t\mathrm{d}t, \qquad \tilde{X}_0^{(\tilde{\pi},c)} = x,$$

where $\chi_{\mathrm{c}} \triangleq \begin{pmatrix}\chi \\ \chi_a\end{pmatrix} = \begin{pmatrix}\mu^S - r\mathbf{1}_m \\ \mu_a - r\mathbf{1}_n\end{pmatrix} = \begin{pmatrix}\hat{\Sigma}^S\eta^S \\ \eta^Y\end{pmatrix}$ is the excess return of the risky assets. The primal optimization problem certainly depends on the market prices of risk $\eta^Y$ used in the augmentation of the market and the optimization problem is to find $(\tilde{\pi}^*, c^*) \in \tilde{\mathcal{A}}$ such that

$$\tilde{\mathrm{v}}\left(c^*;\eta^Y\right) \triangleq \sup_{(\tilde{\pi},c)\in\tilde{\mathcal{A}}} \tilde{V}_0\left[c;\eta^Y\right] = \sup_{(\tilde{\pi},c)\in\tilde{\mathcal{A}}} \mathbb{E}\left[\int_0^T f\left(c_s, \tilde{V}_s\left[c;\eta^Y\right]\right)\mathrm{d}s + \Phi(c_T)\right].$$

Analogously to the primal problem in the incomplete market, the associated dynamic programming equation for the agent's indirect utility $\tilde{V}_t\left[c;\eta^Y\right] = \tilde{G}(t,\tilde{X}_t,Y_t)$ in the extended market reads

$$\begin{aligned} 0 = \sup_{(\tilde{\pi},c)\in\tilde{\mathcal{A}}} \Big\{ & \tilde{G}_t + \left(x\left(r + \tilde{\pi}^\top\chi_{\mathrm{c}}\right) - c\right)\tilde{G}_x + \left(\mu^Y\right)^\top \tilde{G}_y + \tfrac{1}{2}x^2\tilde{\pi}^\top\Sigma_{\mathrm{c}}^S\left(\Sigma_{\mathrm{c}}^S\right)^\top\tilde{\pi}\tilde{G}_{xx} \\ & + x\tilde{G}_{xy}\Sigma^Y\left(\Sigma_{\mathrm{c}}^S\right)^\top\tilde{\pi} + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top\tilde{G}_{yy}\Sigma^Y\right] + f(c,\tilde{G})\Big\} \end{aligned} \tag{4.4.2}$$

where $\tilde{G}(T,x,y) = \varepsilon\frac{1}{1-\gamma}x^{1-\gamma}$ with a constant weight of bequest $\varepsilon > 0$. As before we conjecture

$$\tilde{G}(t,x,y) = \tfrac{1}{1-\gamma}x^{1-\gamma}\tilde{g}(t,y)^k, \tag{4.4.3}$$

with $\tilde{g}(t,y) > 0$ for all $(t,y) \in [0,T]\times\mathbb{R}^n$ and $k \in \mathbb{R}$; then the solution to (4.4.2) is summarized in the following corollary; the proof is analogous to the one of Proposition 4.3 with the only difference being that $\Sigma_{\mathrm{c}}^S$ is now invertible.

**Corollary 4.5**
*Assume that the agent's indirect utility in the artificially completed market* (4.4.1) *has a representation as in* (4.4.3)*, then the optimal strategy reads*

$$\tilde{\pi}^* = \tfrac{1}{\gamma}\left(\Sigma_{\mathrm{c}}^S\left(\Sigma_{\mathrm{c}}^S\right)^\top\right)^{-1}\chi_{\mathrm{c}} + \tfrac{k}{\gamma}\left(\left(\Sigma_{\mathrm{c}}^S\right)^\top\right)^{-1}\left(\Sigma^Y\right)^\top\frac{\tilde{g}_y}{\tilde{g}} \qquad \text{and} \qquad \left(\frac{c}{x}\right)^* = \delta^\psi\tilde{g}^{-\frac{k\psi}{\theta}} \tag{4.4.4}$$

*where $\tilde{g}$ satisfies the nonlinear partial differential equation*

$$\begin{aligned} 0 = \tilde{g}_t &+ \tfrac{1-\gamma}{k}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\chi_{\mathrm{c}}^\top\left(\Sigma_{\mathrm{c}}^S\left(\Sigma_{\mathrm{c}}^S\right)^\top\right)^{-1}\chi_{\mathrm{c}} - \tfrac{\delta\theta}{1-\gamma}\right)\tilde{g} + \left(\left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\chi_{\mathrm{c}}^\top\left(\left(\Sigma_{\mathrm{c}}^S\right)^\top\right)^{-1}\left(\Sigma^Y\right)^\top\right)\tilde{g}_y \\ &+ \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top\tilde{g}_{yy}\Sigma^Y\right] + \tfrac{1}{2}\tfrac{k-\gamma}{\gamma}\frac{1}{\tilde{g}}(\tilde{g}_y)^\top\Sigma^Y\left(\Sigma^Y\right)^\top\tilde{g}_y + \tfrac{\delta^\psi\theta}{k\psi}\tilde{g}^{1-\frac{k\psi}{\theta}}, \end{aligned} \tag{4.4.5}$$

*with terminal condition $\tilde{g}(T,y) = \varepsilon^{\frac{1}{k}}$.*

Note that (4.4.4) in particular yields another direct relation between the fraction invested in the artificial assets and the market prices of risk $\eta^Y$. Let $\tilde{\pi}^*_{\eta^Y} \triangleq \left(\tilde{\pi}^*_{m+1},\ldots,\tilde{\pi}^*_{m+n}\right)^\top$, then

$$\tilde{\pi}^*_{\eta^Y} = \tfrac{1}{\gamma}\eta^Y + \tfrac{k}{\gamma}\left(\hat{\Sigma}^Y\right)^\top\frac{\tilde{g}_y}{\tilde{g}} \tag{4.4.6}$$

and

$$\eta^Y = \gamma\tilde{\pi}^*_{\eta^Y} - k\left(\hat{\Sigma}^Y\right)^\top\frac{\tilde{g}_y}{\tilde{g}}.$$

Moreover, when considering the completed market as above it is worth mentioning that $\mathcal{A} \subseteq \tilde{\mathcal{A}}$ implies

$$\mathrm{v}(c^*) = \sup_{(\pi,c)\in\mathcal{A}} V_0[c] \leqslant \sup_{(\tilde{\pi},c)\in\tilde{\mathcal{A}}} \tilde{V}_0\left[c;\eta^Y\right] = \tilde{\mathrm{v}}\left(c^*;\eta^Y\right).$$

Now assume that $\eta^Y$ from (4.4.1) is not arbitrary, but chosen such that the additional utility from artificial completion is minimized, i.e. we want to find

$$\tilde{\mathrm{v}}(c^*) \triangleq \inf_{\eta^Y \in \mathcal{P}^n} \tilde{\mathrm{v}}\left(c^*; \eta^Y\right).$$

We again use dynamic programming and the associated equation reads

$$0 = \inf_{\eta^Y \in \mathcal{P}^n} \Bigg\{ \tilde{G}_t + \left(x\left(r + \left(\tilde{\pi}^*\right)^\top \chi_{\mathrm{c}}\right) - c\right)\tilde{G}_x + \left(\mu^Y\right)^\top \tilde{G}_y + \tfrac{1}{2}x^2 \left(\tilde{\pi}^*\right)^\top \Sigma_{\mathrm{c}}^S \left(\Sigma_{\mathrm{c}}^S\right)^\top \tilde{\pi}^* \tilde{G}_{xx} \\ + x\tilde{G}_{xy}\Sigma^Y \left(\Sigma_{\mathrm{c}}^S\right)^\top \tilde{\pi}^* + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top \tilde{G}_{yy}\Sigma^Y\right] + f(c, \tilde{G}) \Bigg\}$$

where $\tilde{G}$ is as in (4.4.3) and $\tilde{g}$ solves (4.4.5). Then the first order condition for $\eta^Y$ yields

$$\left(\eta^Y\right)^* = -k\left(\hat{\Sigma}^Y\right)^T \frac{\tilde{g}_y}{\tilde{g}}. \tag{4.4.7}$$

Plugging $\left(\eta^Y\right)^*$ to (4.4.5) and simplifying shows that $\tilde{g}$ must actually solve (4.2.4). But this means that we have $\tilde{g} \equiv g$, hence $\tilde{G} \equiv G$ and in particular $\mathrm{v}(c^*) = \tilde{\mathrm{v}}\left(c^*; \left(\eta^Y\right)^*\right)$ where $\left(\eta^Y\right)^*$ is the solution of the dual problem as given in (4.3.5). Moreover the optimal strategy from Corollary 4.5 is admissible in our incomplete market in the sense that $\tilde{\pi}^*_{(\eta^Y)^*} = \mathbf{0}_{n\times 1}$ by (4.4.6), i.e.

$$\tilde{\pi}^* = \begin{pmatrix} \pi^* \\ \mathbf{0}_{n\times 1} \end{pmatrix}.$$

This is exactly what the abstract concept of least favorable completion is from a PDE point of view and it is the last step we need to fully understand the intuition behind our approximate solution schemes introduced in the upcoming chapters.

**Note:** As mentioned earlier, we denote the solution to the primal, respectively dual dynamic programming equation by $g$. For the complete market case we continue to use the notation $\tilde{g}$, as it is important to distinguish between those two scenarios in the following chapters.

# Chapter 5

# Approximation via Suboptimal Completion

First we considered the primal and dual problem separately. When talking about the primal problem we investigated an agent's investment-consumption decision under *given* market conditions. Similarly, when talking about the dual problem, we investigated optimal market conditions, without taking into account an investor's actions. Based on the classical idea of artificial market completion, introduced by [Karatzas et al., 1991] and [Cvitanić and Karatzas, 1992], we connected both points of view on a PDE level in Chapter 4. This connection opens the door to numerical approaches to find (approximate) solutions to the primal investment-consumption optimization problem with Epstein-Zin utility in incomplete markets, one of which we present in this chapter.

Approximations based on duality theory have often been used in the literature on time-additive utility in various incomplete market settings. One reason is that, while optimization problems in incomplete markets are usually unsolvable, their complete market analogues are generally easier to handle. Furthermore, as we have seen in Chapter 3, duality approaches directly or indirectly provide a tool to evaluate the accuracy of an approximation via the optimality gap. [Haugh et al., 2006] were the first to make use of the optimality gap, by taking existing approximations of the primal problem and evaluating the optimality gap using a dual approximation derived from the primal one. Other examples are [Brown et al., 2010] and [Brown and Smith, 2011], where the authors heuristically determine suboptimal policies whose associated utility can easily be evaluated to obtain a lower bound on the optimal utility. They then consider a frictionless variant of their model to obtain an upper bound and using certain penalties on the relaxed market frictions they gradually improve their strategies to tighten up the resulting gap until it is 'small enough'. Closest to our approximation approach explained below are[Bick, 2012], [Bick et al., 2013],[Kamma et al., 2020] and [Kamma and Pelsser, 2022]. The basic idea of the SAMS algorithm introduced in [Bick et al., 2013] is to artificially complete the underlying market and then consider only subsets of feasible strategies for which the complete market primal problem can be solved explicitly, so that the explicit solution is parameterized by the artificial completion. This parameterized optimal solution is then minimized over the subset of feasible dual strategies and projected to the set of admissible primal ones. Just recently, [Kamma and Pelsser, 2022] extended their approach. They consider the dual problem first, optimizing it over a convex subset of feasible dual strategies, using the approximate dual solution to artificially complete the market and then project the resulting primal strategies and solution to the set of admissible ones.

All those duality methods only work with time-additive utility. For the recursive case, there are some approximation approaches in the literature, e.g. the analytic approximation of [Chacko and Viceira, 2005](see Section 5.1) and a variety of algorithms to simulate the associated forward-backward SDEs, e.g. the regression methods of [Bouchard and Touzi, 2004] or [Gobet et al., 2005], the Wiener chaos expenansion method of [Briand and Labart, 2014] and the neural network approach by [Lin, 2022]. With the exception of [Lin, 2022], all the above algorithms suffer heavily from the curse of dimensionality. An iterative dynamic programming based method is introduced in [Seiferling, 2016] (see Section 5.1), however, this method is only applicable in one-dimensional models. Our goal is to develop numerical approximation algorithms based on the dynamic programming approach, that are applicable in high dimensions, do not suffer too heavily from the curse of dimensionality and allow for a direct evaluation of our power, respectively variational bounds derived in Chapter 3.

## 5.1 Existing Solutions and Approximations

The basic idea of our first approach is very similar to the one of [Kamma and Pelsser, 2022], however, instead of using the martingale method, we focus on the dynamic programming method. Thus, in order to approximate the primal optimization problem

$$\mathrm{v}(c^*) = \sup_{(\pi,c)\in\mathcal{A}} V_0[c] = \sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}\left[\int_0^T f(c_s, V_s[c])\mathrm{d}s + \Phi(c_T)\right]. \tag{5.1.1}$$

we always tackle the partial differential equation characterizing its solution, i.e.

$$\begin{aligned} 0 &= g_t + \tfrac{1-\gamma}{k}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\chi^\top\left(\Sigma^S\left(\Sigma^S\right)^\top\right)^{-1}\chi - \tfrac{\delta\theta}{1-\gamma}\right)g + \left(\left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\chi^\top\left(\left(\Sigma^S\right)^\top\right)^+\left(\Sigma^Y\right)^\top\right)g_y \\ &\quad + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top g_{yy}\Sigma^Y\right] + \tfrac{1}{2}\frac{1}{g}(g_y)^\top\Sigma^Y\left((k-1)\mathrm{I}_{m+n} - k\tfrac{\gamma-1}{\gamma}\left(\Sigma^S\right)^\top\left(\left(\Sigma^S\right)^\top\right)^+\right)\left(\Sigma^Y\right)^\top g_y \\ &\quad + \tfrac{\delta^\psi\theta}{k\psi}g^{1-\frac{k\psi}{\theta}}, \end{aligned} \tag{5.1.2}$$

subject to the terminal condition $g(T,y) = \varepsilon^{\frac{1}{k}} \triangleq \hat{\varepsilon}$.

In general the problem of finding a solution to (5.1.2) remains unsolved. However, there are scenarios where solutions can be found (numerically) and before we start our own investigation, we present some already existing results. Most cases appear in our further discussion either as benchmarks for our own solution techniques or even as part of those and it is worth to take a closer look before proceeding.

### Exact solution by [Kraft et al., 2013]

Consider our general market model in one dimension, i.e. $m = n = 1$. Choosing the parameter $k = \frac{\gamma}{\gamma+(1-\gamma)(\rho^{SY})^2}$, the nonlinearity featuring $g_y\left(\ldots\right)\frac{g_y}{g}$ vanishes. The interesting observation made in [Kraft et al., 2013] is, that by restricting to parameters $\gamma$ and $\psi$ satisfying

$$\psi = 2 - \gamma + \tfrac{(1-\gamma)^2}{\gamma}\left(\rho^{SY}\right)^2 \tag{H}$$

one can also eliminate the nonlinearity $g^{1-\frac{k\psi}{\theta}}$, which leaves the linear inhomogeneous equation

$$0 = g_t + \tfrac{1-\gamma}{k}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{\chi^2}{(\sigma^S)^2} - \tfrac{\delta\theta}{1-\gamma}\right)g + \left(\left(\mu^Y\right) + \tfrac{1-\gamma}{\gamma}\tfrac{\chi\sigma^Y\rho^{SY}}{\sigma^S}\right)g_y + \tfrac{1}{2}\left(\sigma^Y\right)^2 g_{yy} + \delta^\psi,$$

subject to the terminal condition $g(T,\cdot)=\hat{\varepsilon}$. Hence the solution is given by

$$g(t,y)=\delta^{\psi}H(t,y)+\hat{\varepsilon}h(t,y;T), \qquad \text{where} \qquad H(t,y)\triangleq\int_t^T h(t,y;s)\mathrm{d}s$$

and $h$ satisfies the associated homogeneous equation

$$0=h_t\frac{1-\gamma}{k}\left(r+\frac{1}{2}\frac{1}{\gamma}\frac{\chi^2}{(\sigma^S)^2}-\frac{\delta\theta}{1-\gamma}\right)h+\left(\left(\mu^Y\right)+\frac{1-\gamma}{\gamma}\frac{\chi\sigma^Y\rho^{SY}}{\sigma^S}\right)h_y+\frac{1}{2}\left(\sigma^Y\right)^2h_{yy},$$

on $[0,s]\times\mathbb{R}$ with terminal condition $h(s,y;s)=1$.

A remarkable feature of this method is the explicit representation of the solution and consequently the optimal investment-consumption strategy in feedback form. So far, this is the only case where an analytic solution to the PDE (5.1.2) has been found, when considering Epstein-Zin utility in incomplete markets with $\psi\neq1$. A drawback of this approach is certainly the restriction to one-dimensional affine models and in particular the parameter constellation (H).

### Fixed-Point Iteration by [Kraft et al., 2017]

Consider our general market model in one dimension, i.e. $m=n=1$. Again, by choosing the parameter $k=\frac{\gamma}{\gamma+(1-\gamma)(\rho^{SY})^2}$, the nonlinearity including $g_y(\dots)\frac{g_y}{g}$ vanishes. Setting $q\triangleq1-\frac{k\psi}{\theta}$ yields the semilinear partial differential equation

$$0=g_t+\frac{1-\gamma}{k}\left(r+\frac{1}{2}\frac{1}{\gamma}\frac{\chi^2}{(\bar{\sigma}^S)^2}-\frac{\delta\theta}{1-\gamma}\right)g+\left(\left(\mu^Y\right)+\frac{1-\gamma}{\gamma}\frac{\chi\sigma^Y\rho^{SY}}{\sigma^S}\right)g_y+\frac{1}{2}\left(\sigma^Y\right)^2g_{yy}+\frac{\delta^{\psi}}{1-q}g^q \tag{5.1.3}$$

subject to the terminal condition $g(T,\cdot)=\hat{\varepsilon}$. [Kraft et al., 2017] establish the existence and uniqueness of a solution.

**Proposition 5.1** ([Kraft et al., 2017], Theorem 4.6)
*Assume that the coefficients $r$, $\chi$, $\sigma^S$ and $\alpha$ are bounded and Lipschitz continuous and that $\sigma^Y$ is bounded and has a bounded Lipschitz continuous derivative. Moreover, assume that $\inf_{y\in\mathbb{R}}\sigma^S(y)>0$ and $\inf_{y\in\mathbb{R}}\sigma^Y(y)>0$. Then for all $\gamma,\psi,\delta>0$ with $\gamma,\psi\neq1$ there exists a unique solution $g\in C^{1,2}([0,T]\times\mathbb{R})$ to (5.1.3) and positive constants $0<\underline{g}<\bar{g}$ such that*

$$\underline{g}\leqslant g\leqslant\bar{g} \quad \text{and} \quad \|g_y\|_\infty<\infty.$$

Moreover, they show the following convergence result.

**Corollary 5.2** ([Kraft et al., 2017], Corollary 7.4)
*Let $g\in C_b^{1,2}([0,T]\times\mathbb{R})$ be the unique solution to (5.1.3). Moreover, let $g_0=\hat{\varepsilon}$ and let $g_n$ be recursively defined as the unique bounded solution to the Cauchy problem*

$$\begin{aligned}0=(g_n)_t+\frac{1-\gamma}{k}\left(r+\frac{1}{2}\frac{1}{\gamma}\frac{\chi^2}{(\sigma^S)^2}-\frac{\delta\theta}{1-\gamma}\right)g_n+\left(\left(\mu^Y\right)+\frac{1-\gamma}{\gamma}\frac{\chi\sigma^Y\rho^{SY}}{\sigma^S}\right)(g_n)_y\\+\frac{1}{2}\left(\sigma^Y\right)^2(g_n)_{yy}+\frac{\delta^{\psi}}{1-q}\left(0\vee g_{n-1}\right)^q,\end{aligned} \tag{5.1.4}$$

*subject to the terminal condition $g_n(T,\cdot)=\hat{\varepsilon}$. Then*

$$\|g_n-g\|_\infty\leqslant C\left(\frac{c}{n}\right)^n \qquad \text{for all} \qquad n>\frac{c}{\mathrm{e}}$$

*and some constants $C,c>0$.*

In particular, the following PDE-fixed point iteration converges to the true solution of (5.1.3):

**Algorithm:** (Fixed-Point Iteration)

1. *Set* $g_0 \triangleq \hat{\varepsilon}$ *and* $n \triangleq 1$
2. *Compute* $g_n$ *as the solution to the linear partial differential equation* (5.1.4)[1]
3. *If* $g_n$ *is not sufficiently close to* $g_{n-1}$*, increase* $n$ *by* 1 *and return to* 2

The big advantage of the fixed-point method by [Kraft et al., 2017], compared to the exact solution of [Kraft et al., 2013], is that it does not enforce any restrictions on the paramters $\gamma$ and $\psi$. Moreover, when solving (5.1.4), one does not rely on an affine market model, making the algorithm even more flexible. However, the method is only applicable in one-dimensional models and the accuracy of the numerical approximation is unknown.

### Campbell-Shiller Approximation à la [Chacko and Viceira, 2005]

The idea behind this specific approach is to apply a linear approximation to the optimal consumption-wealth ratio, such that the PDE (5.1.2) corresponds to a investment-consumption problem with *unit EIS*.

**Interludium**
*Investment-consumption choice problems with unit EIS are a limit case of the non-unit EIS parametrization as* $\psi \to 1$*. This is similar to logarithmic utility being the limit case of power utility when risk aversion tends to* 1*. If* $\psi = 1$*, the specification of the Epstein-Zin aggregator* $f$ *corresponds to*

$$f^1(c,v) \triangleq \delta(1-\gamma)v\left(\ln(c) - \tfrac{1}{1-\gamma}\ln((1-\gamma)v)\right).$$

*The associated dynamic programming equation for the agent's indirect utility* $V_t^1[c] = G^1(t, X_t, Y_t)$*, given as*

$$\begin{aligned}0 = \sup_{(\pi,c)\in\Gamma(x)} \Big\{ & G_t^1 + \left(x\left(r + \pi^\top \chi\right) - c\right) G_x^1 + \left(\mu^Y\right)^\top G_y^1 + \tfrac{1}{2}x^2\pi^\top \Sigma^S \left(\Sigma^S\right)^\top \pi G_{xx}^1 \\ & + xG_{xy}^1 \Sigma^Y \left(\Sigma^S\right)^\top \pi + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top G_{yy}^1 \Sigma^Y\right] + f^1(c, G^1)\Big\},\end{aligned}$$

*subject to the terminal condition* $G^1(T,x,y) = \varepsilon \frac{1}{1-\gamma} x^{1-\gamma}$*, can be solved explicitly for affine model dynamics. Using the ansatz*

$$G^1(t,x,y) = \tfrac{1}{1-\gamma}x^{1-\gamma}g^1(t,y)$$

*[Chacko and Viceira, 2005] were the first to find that the optimal strategy is of the form*

$$\pi^* = \tfrac{1}{\gamma}\left(\Sigma^S\left(\Sigma^S\right)^\top\right)^{-1}\chi + \tfrac{1}{\gamma}\left(\left(\Sigma^S\right)^\top\right)^+ \left(\Sigma^Y\right)^\top \frac{g_y^1}{g^1} \qquad \textit{and} \qquad \left(\frac{c}{x}\right)^* = \delta,$$

[1] In [Kraft et al., 2013] the authors use a semi-implicit Crank-Nicolson scheme to approximate the solution numerically.

*where* $g^1(t,y) = \varepsilon \exp\big(A(t) - y^\top B(t) - y^\top C(t)y\big)$ *and* $A$*,* $B$ *and* $C$ *are specified by a model dependent system of ODEs. Then* $g^1$ *solves*

$$\begin{aligned}0 = g^1_t &+ (1-\gamma)\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\chi^\top\left(\Sigma^S\left(\Sigma^S\right)^\top\right)^{-1}\chi + \delta\left(\ln(\delta)-1\right)\right)g^1 \\ &+ \left(\left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\chi^\top\left(\left(\Sigma^S\right)^\top\right)^+\left(\Sigma^Y\right)^\top\right)g^1_y + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top g^1_{yy}\Sigma^Y\right] \\ &+ \tfrac{1}{2}(g^1_y)^\top\Sigma^Y\left(\mathrm{I}_{m+n} - \tfrac{\gamma-1}{\gamma}\left(\Sigma^S\right)^\top\left(\left(\Sigma^S\right)^\top\right)^+\right)\left(\Sigma^Y\right)^\top\frac{g^1_y}{g^1} - \delta(1-\gamma)g^1\ln(g^1),\end{aligned} \tag{5.1.5}$$

*with terminal condition* $g^1(T,y) = \varepsilon$*.*

*Note that the PDE* (5.1.5) *is still nonlinear, but due to the exponential ansatz the equation still separates if the considered market model is affine.*

*We already excluded the case of unit EIS in our duality analysis by demanding* $\psi \neq 1$ *and it is well understood, so we will not consider it further. Besides, unit EIS is empirically less relevant and used in particular because of its tractability. We state the solution as above solely for completeness and for a better understanding of the Campbell-Shiller approximation.* ⋆

In [Chacko and Viceira, 2005], the authors use the *Campbell-Shiller approximation* to transfer their result from the case of unit EIS to general parameter constellations. The idea is to approximate the nonlinearity $g^{-\frac{k\psi}{\theta}}$ from (5.1.2) by a linear PDE corresponding to the investment-consumption choice problem under recursive utility with unit EIS. More precisely, introduce the log-linear approximation

$$\begin{aligned}\left(\frac{c}{x}\right)^*(t,y) = \exp\left(\ln\left(\left(\frac{c}{x}\right)^*(t,y)\right)\right) &\approx \mathfrak{l}(t)\left(1 - \ln(\mathfrak{l}(t)) + \ln\left(\left(\frac{c}{x}\right)^*(t,y)\right)\right) \\ &= \mathfrak{l}(t)\left(1 - \ln(\mathfrak{l}(t)) + \ln\left(\delta^\psi\right) - \tfrac{k\psi}{\theta}\ln(g(t,y))\right)\end{aligned} \tag{5.1.6}$$

of the optimal consumption rate, where $\ln(\mathfrak{l}(t)) = \mathbb{E}\left[\ln\left(\frac{c}{x}\right)^*(t,Y_\infty)\right]$ and $Y_\infty$ is a random variable that has the stationary distribution of the state process $Y$. Put differently, the Campbell-Shiller approximation uses a first-order approximation of the consumption-wealth ration around its long-term stationary value. [Chacko and Viceira, 2005] find the solution to be of a similar form as in the case with unit EIS:

When $\psi \neq 1$ there exists an approximate analytical solution to the indirect utility associated to the primal consumption-investment choice problem (5.1.1) as

$$V[c] = G(t,x,y) \approx \tfrac{1}{1-\gamma}x^{1-\gamma}g^{\mathrm{CS}}(t,y).$$

The approximate investment and consumption strategies read

$$\pi^{\mathrm{CS}} = \tfrac{1}{\gamma}\left(\Sigma^S\left(\Sigma^S\right)^\top\right)^{-1}\chi + \tfrac{k}{\gamma}\left(\left(\Sigma^S\right)^\top\right)^+\left(\Sigma^Y\right)^\top\frac{g^{\mathrm{CS}}_y}{g^{\mathrm{CS}}} \quad \text{and} \quad \left(\frac{c}{x}\right)^{\mathrm{CS}} = \delta^\psi\left(g^{\mathrm{CS}}\right)^{-\frac{k\psi}{\theta}},$$

where the function $g^{\mathrm{CS}}(t,y) = \hat{\varepsilon}\exp\big(A(t) - y^\top B(t) - y^\top C(t)y\big)$ for model specific $A$, $B$ and $C$

solves

$$
\begin{aligned}
0 = g_t^{\mathrm{CS}} &+ \tfrac{1-\gamma}{k}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\chi^\top\left(\Sigma^S\left(\Sigma^S\right)^\top\right)^{-1}\chi - \tfrac{\delta\theta}{1-\gamma}\right)g^{\mathrm{CS}} + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top g_{yy}^{\mathrm{CS}}\Sigma^Y\right] \qquad (5.1.7)\\
&+ \left(\left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\chi^\top\left(\left(\Sigma^S\right)^\top\right)^+\left(\Sigma^Y\right)^\top\right)g_y^{\mathrm{CS}} + \tfrac{\theta}{k\psi}\mathfrak{l}(t)\left(1-\ln(\mathfrak{l}(t)) + \ln\left(\delta^\psi\right) - \tfrac{k\psi}{\theta}\ln\left(g^{\mathrm{CS}}\right)\right)g^{\mathrm{CS}}\\
&+ \tfrac{1}{2}(g_y^{\mathrm{CS}})^\top\Sigma^Y\left((k-1)\mathrm{I}_{m+n} - k\tfrac{\gamma-1}{\gamma}\left(\Sigma^S\right)^\top\left(\left(\Sigma^S\right)^\top\right)^+\right)\left(\Sigma^Y\right)^\top\frac{g_y^{\mathrm{CS}}}{g^{\mathrm{CS}}},
\end{aligned}
$$

subject to the terminal condition $g^{\mathrm{CS}}(T,y) = \hat{\varepsilon}$.

**Remark 5.3**
*As pointed out by [Kraft et al., 2013], the factor $\mathfrak{l}$ should be regarded as endogenous, so in applications we determine $\mathfrak{l}(t)$ recursively: Starting from an initial function $\mathfrak{l}_0(t)$, we find the solution $g^{\mathrm{CS}}$ to* (5.1.7) *and then update the function $\mathfrak{l}_1(t)$ via*

$$\ln(\mathfrak{l}_1(t)) = \mathbb{E}\left[\ln\left(\frac{c}{x}\right)^{\mathrm{CS}}(t, Y_\infty)\right] = \ln(\delta^\psi) - \tfrac{\gamma\psi}{\theta}\mathbb{E}\left[\ln\left(g^{\mathrm{CS}}(t, Y_\infty)\right)\right]$$

*and iterate until a fixed-point is reached.* △

The advantage of the Campbell-Shiller approximation is that the PDE (5.1.7) separates if one chooses $g^{\mathrm{CS}}(t,y) = \varepsilon\exp\left(A(t) - y^\top B(t) - y^\top C(t)y\right)$ where the functions $A$, $B$ and $C$ completely characterize the solution. In particular, the approximation is applicable in any dimension $n \in \mathbb{N}$ and for any parameter $k \in \mathbb{R}\backslash\{0\}$ as long as the model under consideration is affine.

One main drawback of this method is that one cannot understand what this approximation really does to the underlying problem and how it behaves if parameters deviate from the case with unit EIS. In particular, it is not clear how the solutions relate to each other, i.e. if the approximation yields a smaller or bigger value than the true one.

Moreover, in [Kraft et al., 2013], the authors compare their exact solution to the one resulting from the approximation (5.1.6) in a Heston model similar to the one considered by [Chacko and Viceira, 2005].[2] They find the quantitative differences between the exact solution and the approximation to be small when volatility is low, but if volatility increases they become more pronounced. Quantitative measures aside, they in particular reveal several qualitative shortcomings with respect to the solution associated to the Campbell-Shiller approximation in their Heston setting. More precisely, while the consumption-wealth ration $\left(\frac{c}{x}\right)^*$ associated to the exact solution increases linearly in $y$ and the optimal investment strategy $\pi^*$ is state dependent, the consumption strategy $\left(\frac{c}{x}\right)^{\mathrm{CS}}$ associated to the Campbell-Shiller approximation increases exponentially in $y$ and the investment strategy $\pi^{\mathrm{CS}}$ is constant. While this comparison was made in a model with an infinite time horizon, the qualitative shortcomings persist in our setting with finite time. This is the second main drawback of the naive application of the Campbell-Shiller approximation.

## Special Case: Power-Utility

In the special case of power utility, i.e. $\gamma\psi = 1$ which implies $\theta = 1$, the partial differntial equation (5.1.2) simplifies accordingly. However, even in this time-additive special case of Epstein-Zin

[2] They consider the one-dimensional special case of our model from Example 4.2 in an infinite-time setting while [Chacko and Viceira, 2005] use an *inverse* Heston model.

utility the dynamic programming equation remains nonlinear and unsolvable if markets are incomplete. On the other hand, if the market is complete, i.e. $\Sigma^S$ is invertible, choosing $k = \gamma$, (5.1.2) simplifies to

$$0 = g_t + \tfrac{1-\gamma}{\gamma}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\chi^\top \left(\Sigma^S \left(\Sigma^S\right)^\top\right)^{-1}\chi - \tfrac{\delta}{1-\gamma}\right) g + \left(\left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\chi^\top \left(\left(\Sigma^S\right)^\top\right)^{-1}\left(\Sigma^Y\right)^\top\right) g_y \\ + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top g_{yy}\Sigma^Y\right] + \delta^{\frac{1}{\gamma}}, \tag{5.1.8}$$

subject to the terminal condition $g(T, y) = \varepsilon^{\frac{1}{\gamma}}$. The associated optimal strategies read

$$\pi^* = \tfrac{1}{\gamma}\left(\Sigma^S \left(\Sigma^S\right)^\top\right)^{-1}\chi + \left(\left(\Sigma^S\right)^T\right)^{-1}\left(\Sigma^Y\right)^\top \frac{g_y}{g} \qquad \text{and} \qquad \left(\frac{c}{x}\right)^* = \delta^{\frac{1}{\gamma}} g^{-1}.$$

As shown e.g. in [Liu, 2007], a solution is obtained by a separation approach similar to the one by [Kraft et al., 2013]. This is, a solution to (5.1.9) is given by

$$g(t, y) = \delta^{\frac{1}{\gamma}} \int_t^T h(t, y; s)\mathrm{d}s + \hat{\varepsilon} h(t, y; T)$$

and $h$ satisfies the associated homogeneous equation

$$0 = h_t + \tfrac{1-\gamma}{\gamma}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\chi^\top \left(\Sigma^S \left(\Sigma^S\right)^\top\right)^{-1}\chi - \tfrac{\delta}{1-\gamma}\right) h + \left(\left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\chi^\top \left(\left(\Sigma^S\right)^\top\right)^{+}\left(\Sigma^Y\right)^\top\right) h_y \\ + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top h_{yy}\Sigma^Y\right], \tag{5.1.9}$$

on $[0, s] \times \mathbb{R}$, subject to the terminal condition $h(T, y) = 1$. This is in contrast to truly recursive Epstein-Zin utility, where even in the complete market scenario the HJB equation associated to our problem in general has no known solution.

## 5.2 The ALFC-Algorithm

Consider the primal Epstein-Zin utility optimization problem

$$\nu(c^*) = \sup_{(\pi,c)\in\mathcal{A}} V_0[c].$$

In Section 4.4 we realized that the solution is characterized by three equivalent partial differential equations, which we briefly recall here for easier reference.

1. The equation associated to the primal, respectively dual, optimization problem from Proposition 4.3:

$$\begin{aligned} 0 &= g_t + \tfrac{1-\gamma}{k}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\chi^\top \left(\Sigma^S \left(\Sigma^S\right)^\top\right)^{-1}\chi - \tfrac{\delta\theta}{1-\gamma}\right) g + \left(\left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\chi^\top \left(\left(\Sigma^S\right)^\top\right)^{+}\left(\Sigma^Y\right)^\top\right) g_y \\ &\quad + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top g_{yy}\Sigma^Y\right] + \tfrac{1}{2}\frac{1}{g}(g_y)^\top\Sigma^Y\left((k-1)\mathrm{I}_{m+n} - k\tfrac{\gamma-1}{\gamma}\left(\Sigma^S\right)^\top\left(\left(\Sigma^S\right)^\top\right)^{+}\right)\left(\Sigma^Y\right)^\top g_y \\ &\quad + \tfrac{\delta^\psi\theta}{k\psi} g^{1-\frac{k\psi}{\theta}} \\ &= g_t + \tfrac{1-\gamma}{k}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\left(\eta^S\right)^\top\eta^S - \tfrac{\delta\theta}{1-\gamma}\right) g + \left(\left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\left(\hat{\Sigma}^{SY}\eta^S\right)^\top\right) g_y \\ &\quad + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top g_{yy}\Sigma^Y\right] + \tfrac{1}{2}\frac{1}{g}(g_y)^\top\left(\left(\tfrac{k}{\gamma} - 1\right)\Sigma^Y\left(\Sigma^Y\right)^\top - k\tfrac{1-\gamma}{\gamma}\hat{\Sigma}^Y\left(\hat{\Sigma}^Y\right)^\top\right) g_y \\ &\quad + \tfrac{\delta^\psi\theta}{l\gamma\psi} g^{1-k\frac{\psi}{\theta}}, \end{aligned} \tag{5.2.1}$$

subject to the terminal condition $g(T, y) = \varepsilon^{\frac{1}{k}}$.

2. The equation associated to the least favorable completion as derived in Section 4.4.2:

$$
\begin{aligned}
0 = \tilde{g}_t &+ \tfrac{1-\gamma}{k}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\chi_{\mathrm{c}}^\top \left(\Sigma_{\mathrm{c}}^S \left(\Sigma_{\mathrm{c}}^S\right)^\top\right)^{-1}\chi_{\mathrm{c}} - \tfrac{\delta\theta}{1-\gamma}\right)\tilde{g} \\
&+ \left(\left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\chi_{\mathrm{c}}^\top \left(\left(\Sigma_{\mathrm{c}}^S\right)^\top\right)^{-1}\left(\Sigma^Y\right)^\top\right)\tilde{g}_y + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top \tilde{g}_{yy}\Sigma^Y\right] \\
&+ \tfrac{1}{2}\tfrac{k-\gamma}{\gamma}\frac{1}{\tilde{g}}(\tilde{g}_y)^\top \Sigma^Y \left(\Sigma^Y\right)^\top \tilde{g}_y + \tfrac{\delta^\psi\theta}{k\psi}\tilde{g}^{1-\frac{k\psi}{\theta}},
\end{aligned}
\tag{5.2.2}
$$

subject to the terminal condition $\tilde{g}(T, y) = \varepsilon^{\frac{1}{k}}$ where the market is completed with the optimal market prices of risk

$$
\left(\eta^Y\right)^* = -k\left(\hat{\Sigma}^Y\right)^\top \frac{\tilde{g}_y}{\tilde{g}}.
$$

Note that in the above we actually have $g \equiv \tilde{g}$, but in the following it is important to notationally distinguish between the primal, respectively dual differential equation and the one associated to the artificially completed market. The idea behind the ALFC approach is to find an approximate solution $g^{\approx}$ to (5.2.1), complete the market with the associated approximation of the least favorable market prices of risk and then solve the more tractable complete market problem with $\left(\eta^Y\right)^*$ in (5.2.2) replaced by said approximation. If our approximation of the dual solution is good, then the solution $\tilde{g}$ to the equation associated with the optimization problem in a market completed with a good approximation of the least favorable market prices of risk should be a good approximation to the true solution $g$.

We put this idea to algorithmic form:

**Algorithm:** (ALFC)

1. *Find an approximation* $g^{\approx}$ *to* (5.2.1) *and set*[3]

$$
\left(\eta^Y\right)^{\approx} = -k\left(\hat{\Sigma}^Y\right)^\top \frac{g_y^{\approx}}{g^{\approx}}
$$

2. *Replace* $\left(\eta^Y\right)^*$ *in* (5.2.2) *by* $\left(\eta^Y\right)^{\approx}$ *and find a solution* $\tilde{g}$ *to the resulting partial differential equation. Then*

$$
\tilde{g} \approx g
$$

Similar to the approximations based on duality theory mentioned in the introduction, the ALFC algorithm makes use of the fact that the complete market problem is generally easier to solve than the one in incomplete markets. Note that as we complete the market only with an approximation of the dual solution, that is in particular suboptimally, the resulting strategies are in general not admissible in the actual market. The investment strategy $\tilde{\pi}$ can easily be projected on the set of feasible actions by setting $n$ entries associated to the artificial assets to zero, c.f. [Bick et al., 2013] or [Kamma and Pelsser, 2022]. If the consumption-wealth ratio is bounded, there is no issue for consumption. When considering unbounded policies, one has to verify the admissibility of the consumption stream separately.

[3] We use '$\approx$' as a placeholder for the specific approximation used by the applicant.

**Note:** During our numerical analysis we have to restrict ourselves to bounded domains by the nature of the matter. It is clear that the model parameters in our specific models are then bounded on those domains and e.g. by Proposition 5.1 we know that the solution is bounded above and below on those domains as well. Hence, we do not discuss admissiblity of consumption within our numerical analysis and generally proceed without introducing suitable truncated versions of our models as is done for example in [Kraft et al., 2017].

Suboptimal completion has nice implications as well. In particular, it implies that the result of this algorithm is automatically an upper bound on the true value, see Section 4.4.2. Moreover the algorithm is very flexible. One could theoretically use any approximation $\left(\eta^Y\right)^{\approx}$ for the market prices of risk as long as one is able to solve the resulting complete market problem.

The following exemplary approach is based on the Campbell-Shiller approximation of the dual dynamic programming equation.

### 5.2.1 Campbell-Shiller ALFC (CS-ALFC)

To obtain an approximation of the market prices of risk, we make use of the Campbell-Shiller approximation of Section 5.1:

**Algorithm:** (CS-ALFC)

1. *As in Section 5.1 an approximation* $g^{\mathrm{CS}}$ *to* (5.2.1) *and set*
$$\left(\eta^Y\right)^{\mathrm{CS}} = -k\left(\hat{\Sigma}^Y\right)^{\top}\frac{g_y^{\mathrm{CS}}}{g^{\mathrm{CS}}}$$

2. *Replace* $\left(\eta^Y\right)^*$ *in* (5.2.2) *by* $\left(\eta^Y\right)^{\mathrm{CS}}$ *and find a solution* $\tilde{g}$ *to the resulting partial differential equation. Then*
$$\tilde{g} \approx g$$

Applying the Campbell-Shiller approximation to the dual PDE instead of the primal one has several advantages. First, as the Campbell-Shiller approximation yields affine market prices of risk, the complete market PDE still has a chance to separate. Moreover, even though the approximation of MPRs is affine, the solution is of a more general structure, see (5.2.4) and (6.1.8). Finally, even if the approximation might be quantitatively good, the approximation is kind of a black box and applying it to the primal PDE completely changes the problem in an unforeseeable way. An application to the dual PDE, however, makes use of the good approximation but only for the market prices of risk used in the artificial completion. In particular, the actual investment problem on the primal side remains untouched.

For our numerical analysis of this algorithm, we restrict ourselves to the special case of power utility, i.e. $\gamma\psi = 1$. This is mainly because we focus on the dynamic programming equation, which in the power case has a closed form solution for the complete market problem as stated in Section 5.1. As there is no known solution for the PDE associated to the incomplete market problem with power utility, this case is still interesting enough and suffices our purpose to demonstrate the performance of the CS-ALFC algorithm. Also note that we have to make the detour to the truly recursive case of unit EIS to obtain our approximation of $\eta^Y$ via the Campbell-Shiller approximation. To the best of our knowledge this has not been done in the literature so far.

### 5.2.2 Numerical Results (Power Utility)

For notational convenience, we write the PDE associated to the artificially completed market in terms of arbitrary market prices of risk $\eta^Y$ and as

$$0 = \tilde{g}_t + \tilde{r}\left(\eta^Y\right)\tilde{g} + \tilde{\alpha}\left(\eta^Y\right)\tilde{g}_y + \tfrac{1}{2}\text{trace}\left[\left(\Sigma^Y\right)^\top \tilde{g}_{yy}\Sigma^Y\right] + \delta^{\frac{1}{\gamma}},$$

subject to the terminal condition $\tilde{g}(T,y) = \varepsilon^{\frac{1}{\gamma}}$, where $\tilde{r} : \mathbb{R}^n \to \mathbb{R}$ and $\tilde{\alpha} : \mathbb{R}^{n\times 1} \to \mathbb{R}^{1\times n}$ are given as

$$\tilde{r}(v) \triangleq \tfrac{1-\gamma}{\gamma}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\left(\chi^\top\left(\Sigma^S\left(\Sigma^S\right)^\top\right)^{-1}\chi + v^\top v\right) - \tfrac{\delta}{1-\gamma}\right)$$

and

$$\tilde{\alpha}(v) \triangleq \left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\left(\chi^\top\left(\left(\Sigma^S\right)^\top\right)^+\left(\Sigma^Y\right)^\top + v^\top\left(\hat{\Sigma}^Y\right)^\top\right).$$

As already mentioned, the qualitative drawbacks of the Campbell-Shiller approximation as discussed in Section 5.1 become an asset when it is used in the first step of the CS-ALFC approach. When completing the market with the resulting approximate MPRs, equation (5.2.2) separates and the solution $\tilde{g}$ is of the same structure as the one in Section 5.1 provided by [Liu, 2007].

#### Comparison with Exact Solutions

We first compare the numerical results from the CS-ALFC algorithm with the exact numerical solution provided by the fixed-point iteration algorithm of Section 5.1 in a one-dimensional Kim-Omberg model as in Example 4.1, then in a one-dimensional Heston model as in Example 4.2.

**Kim-Omberg Model:** We first state the general approximate solution and strategies provided by the algorithm. To this end, consider the model introduced in Example 4.1, i.e. for a $\mathbb{R}^{m+n}$-dimensional standard Brownian motion $W$, let the risky assets and states follow the dynamics

$$\begin{aligned} \mathrm{d}S_t^{\text{risky}} &= \text{diag}[S_t^{\text{risky}}]\left(\left(r\mathbf{1}_m + \bar{\lambda} + \boldsymbol{\lambda}^\top Y_t\right)\mathrm{d}t + \Sigma^S \mathrm{d}W_t\right), \\ \mathrm{d}Y_t &= -\text{diag}[\kappa]Y_t\mathrm{d}t + \Sigma^Y\mathrm{d}W_t, \end{aligned}$$

where $r \in \mathbb{R}$, $\bar{\lambda} \in \mathbb{R}^m$, $\boldsymbol{\lambda} \in \mathbb{R}^{m\times n}$, $\kappa \in \mathbb{R}^n_+$ and $\Sigma^S \in \mathbb{R}^{m\times(m+n)}$, $\Sigma^Y \in \mathbb{R}^{n\times(m+n)}$ are as defined in (4.1.3).
Applying the Campbell-Shiller approximation to (5.2.1) yields market prices of risk given by

$$\left(\eta^Y\right)^{\text{CS}}(t,y) \triangleq k\left(\hat{\Sigma}^Y\right)^\top\left(B(t) + 2C(t)y\right),$$

where $B$ and $C$ solve the system of ordinary differential equations (D.2.4) in Appendix D.2.1.

Introducing artificial assets with market prices of risk given by $\left(\eta^Y\right)^{\text{CS}}$, the resulting complete market problem is explicitly solvable according to [Liu, 2007] and the solution is obtained by a separation approach as

$$\tilde{g}(t,y) = \delta^\psi H(t,y) + \hat{\varepsilon}h(t,y;T), \qquad \text{where} \qquad H(t,y) \triangleq \int_t^T h(t,y;s)\mathrm{d}s, \tag{5.2.3}$$

with $h(t,y;s) = \exp\left(A(t,s) + B(t,s)y + y^\top C(t,s)y\right)$, where $\tilde{A}(t,s)$, $\tilde{B}(t,s)$ and $\tilde{C}(t,s)$ solve (D.2.6) in Appendix D.2.1. The associated approximate strategies read[4]

$$\pi(t,y) = \tfrac{1}{\gamma}\left(\hat{\Sigma}^S\left(\hat{\Sigma}^S\right)^\top\right)^{-1}(\bar{\lambda} + \boldsymbol{\lambda}^\top y) + \left(\left(\hat{\Sigma}^S\right)^\top\right)^{-1}\left(\hat{\Sigma}^{SY}\right)\frac{\tilde{g}_y(t,y)}{\tilde{g}(t,y)} \tag{5.2.4}$$
$$\left(\frac{c}{x}\right)(t,y) = \frac{\delta^{\frac{1}{\gamma}}}{\tilde{g}(t,y)} \qquad \eta^Y(t,y) = -\gamma\left(\hat{\Sigma}^Y\right)\frac{\tilde{g}_y(t,y)}{\tilde{g}(t,y)}.$$

In particular, we obtain an analytic representation of the approximating strategies and note that even if we approximated the market prices of risk linearly in the first step, all three strategies associated to the approximation are ultimately not forced to remain linear by the two step scheme and take the more general form as in [Liu, 2007]. As mentioned, this is a major distinction between the CS-ALFC approach and a naive application of the Campbell-Shiller approximation to the primal PDE.

We set $m = n = 1$ and analyze the performance of the CS-ALFC scheme by comparing its approximation with the exact result provided by the fixed-point algorithm discussed in Section 5.1.

In a meta study on the calibration of risk aversion in the literature, [Elminejad et al., 2022] find that the calibrated $\gamma$ in models that *separate* RRA and EIS is accumulated between 1 and 10. We follow [Liu and Muhle-Karbe, 2013] and choose the agent's preference parameters according to Table 5.1 and the following (monthly) model parameters (c.f. [Campbell and Viceira, 1999], [Barberis, 2000], [Wachter, 2002]).

| $\gamma$ | $\psi$ | $\delta$ | $\varepsilon$ |
|---|---|---|---|
| 5 | $\frac{1}{5}$ | 0.52% | 1 |

Table 5.1: Preference Parameters

| $T$ | $r$ | $\bar{\lambda}$ | $\bar{\sigma}^S$ | $\kappa$ | $\bar{\sigma}^Y$ | $\rho^{SY}$ |
|---|---|---|---|---|---|---|
| 20 | 0.14% | 0.34% | 4.36% | 2.26% | 0.08% | -93.5% |

Table 5.2: Model Parameters (monthly)

Moreover, we set $\boldsymbol{\lambda} = 1$. Figure 5.1 depicts the solution in a \$-scale, to maintain comparability. More precisely, in the upper left plot we see the function $\tilde{g}^\$(t,0) = \tilde{g}(t,0)^{\frac{k}{1-\gamma}}$, such that

$$\tilde{G}(t,x,0) = \frac{1}{1-\gamma}x^{1-\gamma}\tilde{g}(t,0)^k = \frac{1}{1-\gamma}\left(x\tilde{g}^\$(t,0)\right)^{1-\gamma}, \tag{5.2.5}$$

associated to the CS-ALFC scheme (red) and the exact solutions from the fixed-point algorithm of Section 5.1 that is used as benchmark (blue). The upper middle and right plot show the associated derivatives $\tilde{g}^\$_y(t,0)$ and sensitivities $\frac{\tilde{g}^\$_y(t,0)}{\tilde{g}^\$(t,0)}$ compared to the exact solution, respectively. The lower three plots show the associated strategies $\tilde{\pi}(t,0)$, $\widetilde{\left(\frac{c}{x}\right)}(t,0)$ and $\tilde{\eta}^Y(t,0)$ as given in (5.2.4) in red (dashed) and the exact solution from the fixed-point algorithm Section 5.1 in blue. Figure 5.2 shows exactly the same, only the solutions are now plotted as functions in $y$ at the fixed time $t = 0$. The dashed vertical lines indicate the 99% quantile associated to the stationary distribution of the state process $Y$.

[4] Note that setting $\pi$ immediately like this corresponds to cutting off the last $n$ coordinates of $\tilde{\pi}$ to make the investment strategy admissible in the original market.

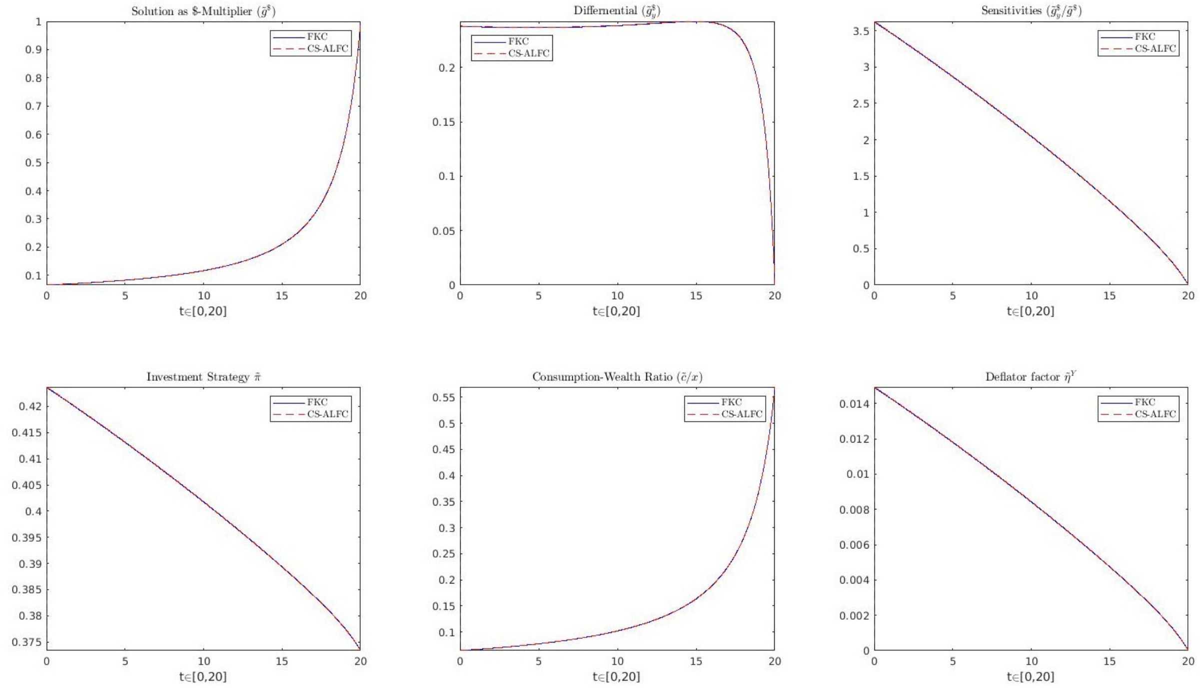



Figure 5.1: Kim-Omberg: Algorithmic solution of the FKC-algorithm from [Kraft et al., 2017] in blue and the CS-ALFC approximation in red (dashed). The preference and model parameters are given in Table 5.1 and Table 5.2. All functions are plotted as $t \mapsto \cdot(t, \bar{y})$

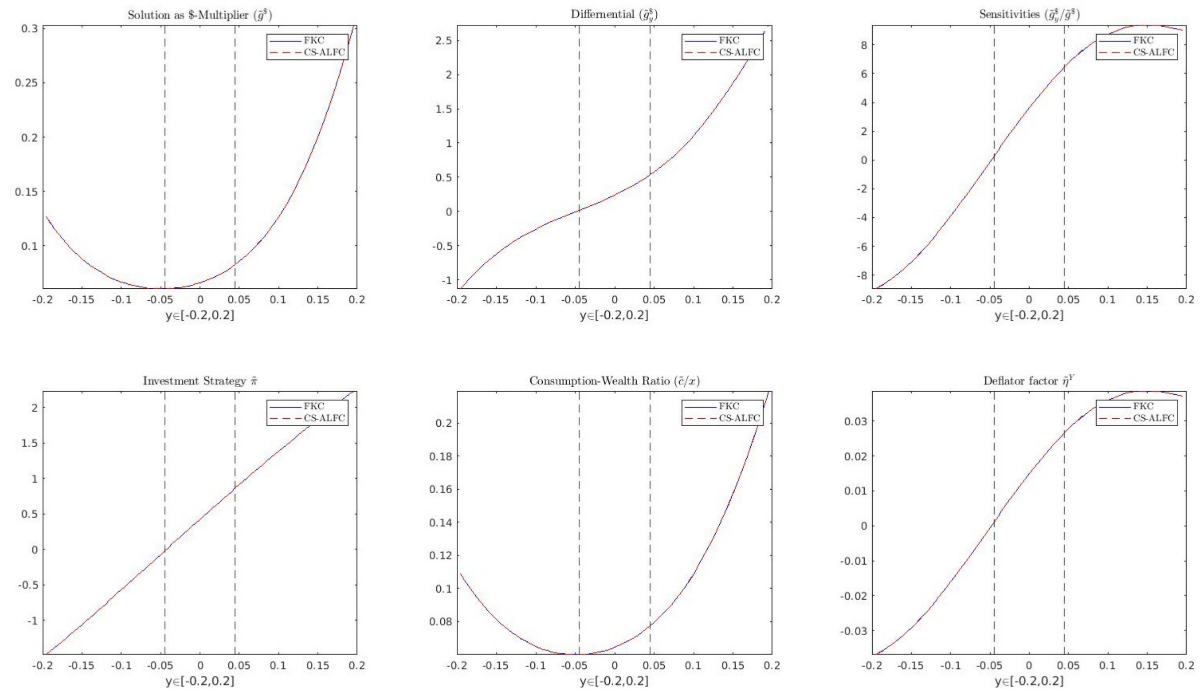



Figure 5.2: Kim-Omberg: Known algorithmic solution of the FKC-algorithm from [Kraft et al., 2017] in blue and the CS-ALFC approximation in red (dashed). All functions are plotted as $y \mapsto \cdot(0, y)$. The dashed vertical lines indicate the 99% quantile of the stationary distribution associated to the state process.

**Heston Model:** We first derive the general approximate solution and strategies. Consider the model introduced in Example 4.2, i.e. let $m = n$ and for a $\mathbb{R}^{2n}$-dimensional standard Brownian motion $W$ let the risky assets and states follow the dynamics

$$\begin{aligned} \mathrm{d}S_t^{\text{risky}} &= \operatorname{diag}[S_t^{\text{risky}}]\left(\left(r\mathbf{1}_m + K\operatorname{diag}[Y_t]K^\top\bar{\lambda}\right)\mathrm{d}t + \Sigma^S(Y_t)\mathrm{d}W_t\right), \\ \mathrm{d}Y_t &= (\bar{\mu} - \operatorname{diag}[\kappa]Y_t)\,\mathrm{d}t + \Sigma^Y(Y_t)\mathrm{d}W_t, \end{aligned}$$

where $r \in \mathbb{R}$, $\bar{\lambda} \in \mathbb{R}^n$, $\bar{\mu}, \kappa \in \mathbb{R}^n_+$, $K \in \mathbb{R}^{n\times n}_o$ and $\Sigma^S$, $\Sigma^Y$ are $(n\times 2n)$-dimensional matrix functions as in (4.1.3).
Applying the Campbell-Shiller approximation to (5.2.1) yields the market prices of risk given by

$$\left(\eta^Y\right)^{\mathrm{CS}}(t,y) \triangleq k\left(\hat{\Sigma}^Y(y)\right)^\top B(t),$$

where $B$ solves the system of ordinary differential equations (D.2.7) in Appendix D.2.2. Recall that $\hat{\Sigma}^Y(y)$ behaves as $\sqrt{y}$.
Introducing artificial assets with market prices of risk given by $\left(\eta^Y\right)^{\mathrm{CS}}$, the resulting complete market problem is explicitly solvable according to [Liu, 2007] and the solution is obtained by a separation approach as

$$\tilde{g}(t,y) = \delta^\psi \tilde{H}(t,y) + \hat{\varepsilon}\tilde{h}(t,y;T), \qquad \text{where} \qquad \tilde{H}(t,y) \triangleq \int_t^T \tilde{h}(t,y;s)\mathrm{d}s$$

with $\tilde{h}(t,y;s) = \exp\left(\tilde{A}(t,s) - y^\top\tilde{B}(t,s)\right)$, where $\tilde{A}(t,s)$ and $\tilde{B}(t,s)$ solve (D.2.9) in Appendix D.2.2. Moreover, the approximate strategies read[5]

$$\pi(t,y) = \tfrac{1}{\gamma}K\operatorname{diag}\left[\left(\bar{\sigma}^S\right)^2\right]^{-1}K^\top\bar{\lambda} + K\operatorname{diag}\left[\bar{\sigma}^Y\right]\operatorname{diag}\left[\bar{\sigma}^S\right]^{-1}\rho^{SY}\frac{\tilde{g}_y(t,y)}{\tilde{g}(t,y)}, \qquad (5.2.6)$$

$$\left(\frac{c}{x}\right)(t,y) = \frac{\delta^{\frac{1}{\gamma}}}{\tilde{g}(t,y)}, \qquad \eta^Y(t,y) = -\gamma y^{\frac{1}{2}}\operatorname{diag}[\bar{\sigma}^Y]L^Y\frac{\tilde{g}_y(t,y)}{\tilde{g}(t,y)},$$

where $\left(\bar{\sigma}^S\right)^2 \triangleq \left(\left(\bar{\sigma}^S_1\right)^2, \ldots, \left(\bar{\sigma}^S_n\right)^2\right)$. In particular, note that even if we approximate the market prices of risk by a scaled square root of $y$ in the first step, all three strategies associated to the approximation are ultimately not forced to remain of that structure by the two step scheme and are of a more general structure as in [Liu, 2007].

We set $m = n = 1$ and for our comparison with the exact solution provided by the fixed-point algorithm from Section 5.1, choose the agent's preference parameters similar to Table 5.1. Moreover, we use the following (yearly) model parameters based on [Liu and Muhle-Karbe, 2013], c.f. [Pan, 2002].

| $\gamma$ | $\psi$ | $\delta$ | $\varepsilon$ |
|---|---|---|---|
| 5 | $\frac{1}{5}$ | 0.062 | 1 |

Table 5.3: Preference Parameters

| $T$ | $r$ | $\bar{\lambda}$ | $K$ | $\bar{\sigma}^S$ | $\kappa$ | $\bar{\mu}$ | $\bar{\sigma}^Y$ | $\rho^{SY}$ |
|---|---|---|---|---|---|---|---|---|
| 10 | 0.033 | 4.4 | 1 | 1 | 5.3 | 0.13 | 0.38 | -0.57 |

Table 5.4: Model Parameters (yearly)

Figure 5.3 shows the solution $\tilde{g}$ as a \$-multiplier (i.e. $\tilde{g}^\$$, see (5.2.5)), the associated differentials, sensitivities and strategies provided by the CS-ALFC algorithm as functions in $t$ (red, dashed); the exact solution from the fixed-point algorithm is plotted in blue.

[5] Note that setting $\pi$ immediately like this corresponds to cutting off the last $n$ coordinates of $\tilde{\pi}$ to make the investment strategy admissible in the original market.

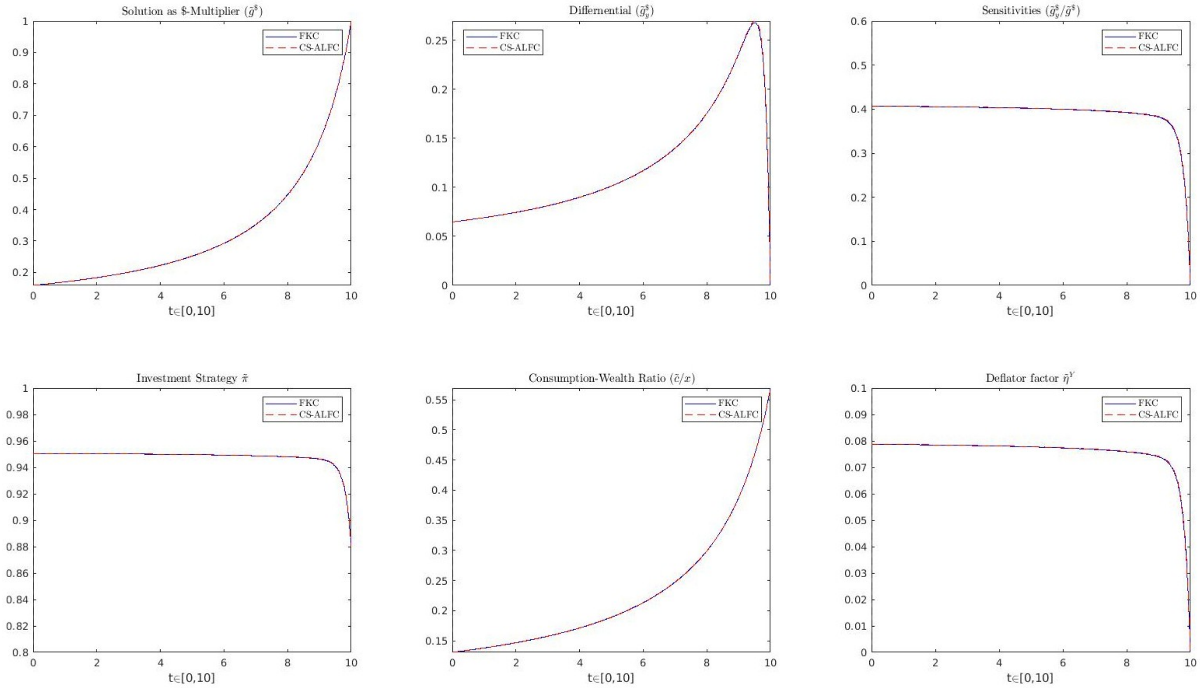



Figure 5.3: Heston: Known algorithmic solution of the FKC-algorithm from [Kraft et al., 2017] in blue and the CS-ALFC approximation in red (dashed). All functions are plotted as $t \mapsto \cdot(t, \bar{y})$

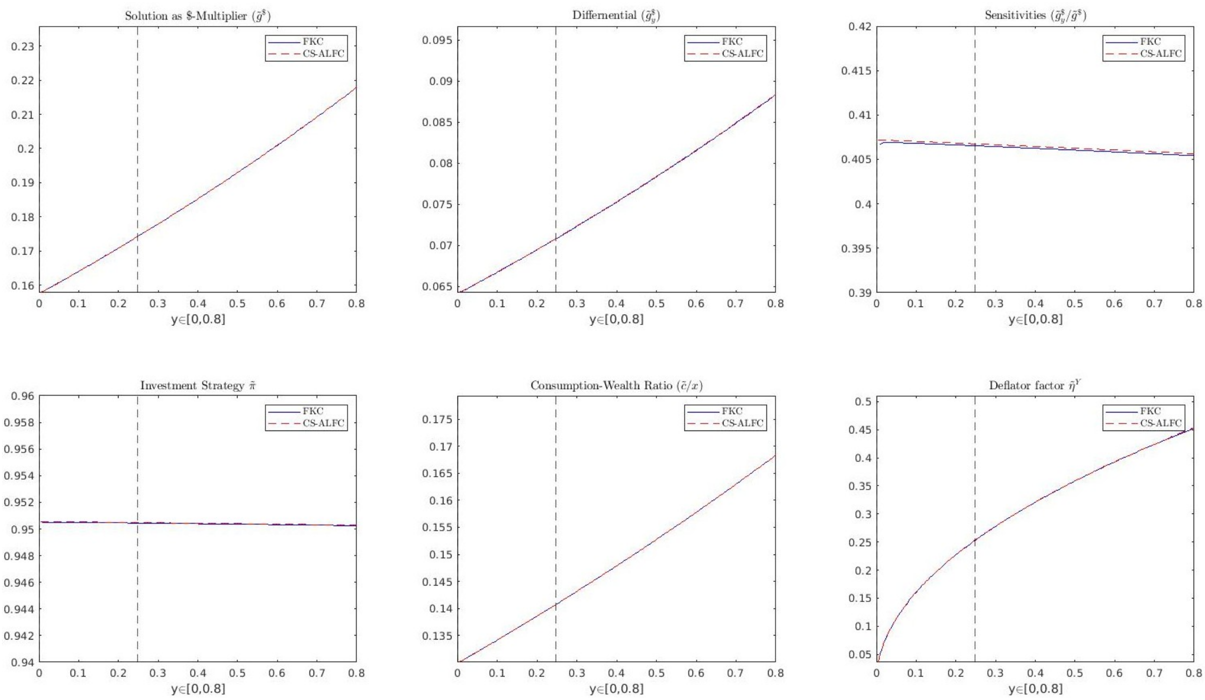



Figure 5.4: Heston: Known algorithmic solution of the FKC-algorithm from [Kraft et al., 2017] in blue and the CS-ALFC approximation in red (dashed). All functions are plotted as $y \mapsto \cdot(0, y)$. The dashed vertical lines indicate the 99% quantile of the stationary distribution associated to the state process $Y$.

Figure 5.4 shows exactly the same as Figure 5.3, only as functions in $y$ at the fixed time $t = 0$. The dotted vertical lines indicate the 99% quantile of the stationary distribution associated to the state process $Y$.

**Large Scale Model**

In order to test the CS-ALFC algorithm in higher dimensions, we construct a market that is arbitrarily scalable.

We assume that there is an underlying economy index process $S^\star$ and state of the economy process $Y^\star$, driven by one dimensional Brownian motion $W^\star$ and $\hat{W}^\star$, respectively, where $W^\star$ and $\hat{W}^\star$ are independent. Both of them are not part of the tradable market we consider, they are merely the driving forces of the overall market randomness. The tradable market is given by one risk-free asset with rate $r$ and $n$ identically distributed risky assets with dynamics

$$\mathrm{d}S_i^{\text{risky}} = S_i^{\text{risky}}\left(\left(r + \bar{\lambda} + \boldsymbol{\lambda} Y^i\right)\mathrm{d}t + \sigma^S \underbrace{\left(\sqrt{\rho^\star}\mathrm{d}W^\star + \sqrt{1-\rho^\star}\mathrm{d}\bar{W}_i\right)}_{\triangleq \mathrm{d}\tilde{W}_i^S}\right), \quad i = 1, \ldots, n,$$

where $\bar{W}_i$, $i = 1, \ldots, n$, are mutually independent Brownian Motions, also independent of $W^\star$ and $\hat{W}^\star$. In particular, every asset $S_i^{\text{risky}}$ has a correlation of $\sqrt{\rho^\star}$ with the underlying index and the return of every asset depends on a single state process $Y^i$. The processes $Y^i$ are identically distributed and each has correlation $\xi$ with the associated share $S^i$ and correlation $\sqrt{(1-\xi^2)\zeta}$ with the state of economy process, more precisely

$$\mathrm{d}Y^i = -\kappa Y^i \mathrm{d}t + \sigma^Y \underbrace{\left(\xi \mathrm{d}\tilde{W}^i + \sqrt{1-\xi^2}\left(\sqrt{\zeta}\mathrm{d}\hat{W}^\star + \sqrt{1-\zeta}\mathrm{d}\hat{W}^i\right)\right)}_{\triangleq \mathrm{d}\tilde{W}_i^Y}, \qquad i = 1, \ldots, n,$$

where $\hat{W}_i$, $i = 1, \ldots, n$, are again Brownian Motions, mutually independent and independent of $W^\star$, $\hat{W}^\star$ and all $W^i$, $i = 1, \ldots, n$. In short, we assume the overall correlation structure

$$\rho_{i,j}^S = \begin{cases} 1, & i = j \\ \rho^\star, & i \neq j \end{cases}, \qquad \rho_{i,j}^{SY} = \begin{cases} \xi, & i = j \\ \xi\rho^\star, & i \neq j \end{cases} \quad \text{and} \quad \rho_{i,j}^Y = \begin{cases} 1, & i = j \\ \xi^2\rho^\star + (1-\xi^2)\zeta, & i \neq j \end{cases}.$$

Thus, we are in a special case of the multivariate Kim-Omberg setting as in Example 4.1 and the approximated solution and strategy are given as in (5.2.3) and (5.2.4).
As we have no benchmark to compare the algorithm with in higher dimensions, now is when our bounds for the optimality gap derived in Section 3.1 come into play. In this power utility scenario we naturally stick to the power utility bounds provided in Theorem 3.11 to bound the optimality gap. Before starting the numerical analysis of the CS-ALFC algorithm with respect to its accuracy in high dimensions, we briefly discuss a better way to measure the welfare loss than using the power bounds itself.

**Remark 5.4**
*Using Theorem 3.11 one can directly bound the utility loss associated to using the suboptimal strategies c and D, respectively. However, it is also possible to transfer them to an upper bound on the* wealth equivalent loss *(WEL) as is often done in the literature, see e.g. [Bick et al., 2013]. To this end, use the classical separation Ansatz for the power utility associated to the lower bound from* (3.1.2)*, i.e.* $L_t^{\gamma\wedge\phi}[c] = \frac{1}{1-\gamma\wedge\phi}x^{1-\gamma\wedge\phi}g_L(t,y)$. *Then, for* $\bar{g}_L \triangleq g_L(0,\bar{y})$*, we have*

$$\mathrm{P}_{\gamma\vee\phi}(c) = \tfrac{1}{1-\gamma}x^{1-\gamma}\bar{g}_L^{\frac{1-\gamma}{1-\gamma\wedge\phi}} \qquad \textit{and} \qquad \mathrm{P}^*_{\gamma\wedge\phi}(D) = \tfrac{1}{1-\gamma}x^{1-\gamma}\underbrace{\left(\tfrac{1-\gamma}{\gamma}\mathrm{P}^*_{\gamma\vee\phi}(D)\right)^{\gamma}}_{\triangleq \bar{g}_U}.$$

*Then the wealth equivalent loss associated to the utility loss for the lower bound compared with the upper one corresponds to a constant $L \in [0,1]$, such that*

$$\mathrm{P}_{\gamma\vee\phi}(c) = \tfrac{1}{1-\gamma}\left(x(1-L)\right)^{1-\gamma}\bar{g}_U,$$

*which is determined by*

$$L = 1 - \bar{g}_L^{\frac{1}{1-\gamma\wedge\phi}}\bar{g}_U^{\frac{1}{\gamma-1}}.$$

*Now $L$ describes the fraction of wealth that is equivalent to the utility loss of the lower bound with respect to the upper one, and hence is an upper bound on the true wealth equivalent loss, which can be interpreted as the fraction of initial capital $x$ an investor trading with the* optimal *strategy can forego and still obtain the same utility as an investor trading suboptimally. Hence, the smaller $L$, the better the associated strategies. In order to obtain a fair idea of the stability of the approximation for different time horizons $T$, we follow [Kamma and Pelsser, 2022] by introducing an* annualized *wealth equivalent loss. We define the annualized WEL as*

$$L_T \triangleq 1 - (1-L)^{\frac{1}{T}}.$$

$\triangle$

**Remark 5.5**
*Also note that, by the representation of the value function $V_t[c] = G(t,x,y) = \frac{1}{1-\gamma}x^{1-\gamma}g(t,y)^k$ and consumption $\left(\frac{c}{x}\right)^* = \delta^{\psi}g(t,y)^{-\frac{k\psi}{\theta}}$, we can express the optimal value by the optimal consumption ratio:*

$$G(t,x,y) = \frac{1}{1-\gamma}x^{1-\gamma}\delta^{\theta}\left(\left(\frac{c}{x}\right)^*\right)^{-\frac{\theta}{\psi}}.$$

*Applying the upper and lower bounds similar as above allows us to transform the power bounds on utility to bounds on the optimal consumption, i.e.*

$$\begin{aligned}
\delta^{-\psi}\bar{g}_L^{-\frac{\psi(1-\gamma)}{\theta(1-\gamma\wedge\phi)}} \leqslant \left(\frac{c}{x}\right)^* &\leqslant \delta^{-\psi}\bar{g}_U^{-\frac{\psi}{\theta}} && \text{if } \psi < 1,\\
\delta^{-\psi}\bar{g}_U^{-\frac{\psi}{\theta}} \leqslant \left(\frac{c}{x}\right)^* &\leqslant \delta^{-\psi}\bar{g}_L^{-\frac{\psi(1-\gamma)}{\theta(1-\gamma\wedge\phi)}} && \text{if } \psi > 1.
\end{aligned}$$

$\triangle$

In our numerical simulations we use the parameters from Table 5.2 for every asset and state process, respectively. Moreover, we set $\rho^\star = 41.6\%$, i.e. we choose

| $T$ | $r$ | $\bar{\lambda}$ | $\sigma^S$ | $\kappa$ | $\sigma^Y$ | $\rho^\star$ | $\xi$ |
|---|---|---|---|---|---|---|---|
| 20 | 0.14% | 0.34% | 4.36% | 2.26% | 0.08% | 41.6% | -93.5% |

Table 5.5: Large Scale Model Parameters (monthly)

Furthermore, we set $\boldsymbol{\lambda} = 1$ and $\zeta = 0$. Table 5.6 shows the result of our bounds applied to the strategies provided by the CS-ALFC scheme in dimension $n = m = 50$ and different preference parameters $\gamma \in \{3,5,7\}$.
The first line of Table 5.6 contains the lower power bound, the second presents the upper power bound from Theorem 3.11 evaluated by Monte Carlo simulation. The resulting (annualized)

wealth equivalent loss is given in the last line. The entries presented are the average of 10 simulation with 5000 samples each. The brackets below provide the associated standard deviations.

In particular, Table 5.6 shows that even in dimension $n = 50$, the CS-ALFC algorithm provides strong upper and lower bounds on the welfare loss of less than 1% annualized WEL for each $\gamma$.

| | $n = 50$ | | |
|---|---|---|---|
| | $\gamma = 3$ | $\gamma = 5$ | $\gamma = 7$ |
| $\mathrm{P}_\gamma(c)$ | -6.47e-03 | -9.50e-05 | -2.00e-06 |
| | (1.46e-04) | (4.28e-06) | (1.99e-07) |
| $\mathrm{P}^*_\gamma(D)$ | -3.99e-03 | -3.57e-05 | -5.17e-07 |
| | (2.04e-04) | (5.34e-06) | (2.10e-07) |
| $\tilde{V}_0[c]$ | -4.84e-03 | -5.74e-05 | -1.04e-06 |
| | - | - | - |
| $L_{20}$ | **7.21e-03** | **6.25e-03** | **5.36e-03** |
| | (5.66e-04) | (5.49e-04) | (7.25e-04) |

Table 5.6: Large Scale Kim-Omberg: Accuracy of CS-ALFC algorithm in dimension $n = 50$ and investment horizon $T = 20$ years for different risk preferences $\gamma$. The bounds are the average of 10 Monte Carlo simulations of the respective expectations in Theorem 3.11 with 5.000 sample paths each. The associated standard deviations are given in brackets.

### 5.2.3 Conclusion and Extension to the Recursive Case

The comparison of the outputs provided by the CS-ALFC algorithm and the known algorithmic solution in Figure 5.1-Figure 5.4 verify that the approximation is accurate in one dimension.
In high dimensions, when there is no benchmark available, its accuracy is backed up by our bounds on the optimality gap presented in Table 5.6. Note that the CS-ALFC algorithm does suffer from the curse of dimensionality, i.e. running times increase exponentially in dimension, as the search for the fixed-point of the Campbell-Shiller approximation in Remark 5.3 becomes more and more difficult in higher dimensions. Nevertheless, as the overall approach is simple and computationally feasible, the exponential increase in running time only becomes noticeable in very high dimensions. The computation of the approximate solutions to our large scale model in dimension $n = 50$ with time horizon $T = 20$ years takes about 12 seconds.[6]

A remarkable property of the CS-ALFC algorithm is that, even if we used the affine Campbell-Shiller approximation to artificially complete the given market, the resulting strategies do in general not stay affine. This is a major qualitative distinction compared with the classical approach of applying the Campbell-Shiller approximation to the primal HJB equation. Thus, at least in our power utility setting, the analytic representation in (5.2.3) and (5.2.4) can be interpreted as a two step Campbell-Shiller approximation, improving its qualitative shortcomings identified by [Kraft et al., 2013], c.f. Section 5.1.

Conceptionally, the CS-ALFC algorithm is close to the approach of [Kamma and Pelsser, 2022], as both are based on the following general idea:

[6]Machine: Intel(R) Core$^{\mathrm{TM}}$ i7 – 8650U Processor, 1.9GHz, 16GB RAM.

**Algorithm:** (ALFC - General Idea)

1. *find an approximation to*
$$\nu^*(\Lambda^*) = \inf_{\Lambda \in \mathcal{D}^a} \{V_0^*[\Lambda] + \lambda x\} \tag{5.2.7}$$

2. *complete the market using the MPR* $\left(\eta^Y\right)^{\approx}$ *induced by the approximation of* (5.2.7) *to solve*
$$\tilde{\nu}\left(c^*; \left(\eta^Y\right)^{\approx}\right) \triangleq \sup_{(\tilde{\pi},c)\in\tilde{\mathcal{A}}} \tilde{V}_0\left[c; \left(\eta^Y\right)^{\approx}\right].$$

While we focus on dynamic programming techniques, [Kamma and Pelsser, 2022] employ the martingale method, which allows them to also treat more general (non-affine) models and more general types of utility functions. Our CS-ALFC algorithm on the other hand is conceptionally easier because it does not rely on Monte-Carlo evaluation or additional convex optimization techniques. Instead, we make use of the Campbell-Shiller approximation that is already at hand. Moreover, while [Kamma and Pelsser, 2022] are able to treat e.g. state dependent utility functions, their method heavily relies on the time-additive structure, i.e. in our setting they only treat the case $\gamma\psi = 1$. While we only tested the CS-ALFC scheme for time-additive power utility here, the algorithm can easily be extended to the recursive case. However, not without using additional approximations for the sensitivity $\frac{\tilde{g}_y}{\tilde{g}}$ in the partial differential equation (5.2.2), thereby loosing the upper bound property of the approximation. An exemplary approximation of those sensitivities is used in our second algorithm introduced in the next chapter, see Section 6.1 and in particular Remark 6.1.

# Chapter 6

# Approximation by ***Iterative*** Suboptimal Completion

After we have seen how the introduction of duality can lead to numerical approximations of optimal strategies in the last chapter, we consider the concept of least favorable completion from a slightly different angle. To begin, assume that the market is complete and that the agent trades all $m+n$ shares to maximize her utility. Denote the market prices of risk of the $n$ shares that do not actually exist in the incomplete market by $\eta^Y$, then the investor's optimal utility is given as

$$\tilde{\mathrm{v}}(c^*;\eta^Y) \triangleq \sup_{(\tilde{\pi},c)\in\tilde{\mathcal{A}}} \tilde{V}_0\left[c;\eta^Y\right].$$

Now we introduce a *price setting opponent*, who controls the market prices risk of the additional $n$ assets and will not allow the investor to trade in the artificial stocks. He determines the associated market prices of risk based on her investment choice, i.e. he sets $\eta^Y$ such that her investment in the $n$ additional assets $(\pi_{\eta^Y})$ becomes zero,

$$\pi_{\eta^Y} \stackrel{!}{=} 0.$$

This adjustment now changed the market conditions for the investor and she adapts her asset allocations appropriately, forcing the price setter to again change market conditions and so on. This is the general idea of our Primal-Dual-Iteration(PDI) algorithm:

**Algorithm:** (PDI - General Idea)

1. *initialize the market prices of risk with* $j=1$ *and* $\left(\eta^Y\right)^{(j-1)}$.
2. *solve (approximate) the* $\left(\eta^Y\right)^{(j-1)}$*-completed market problem introduced in Section 4.4.2 as*

$$\tilde{\mathrm{v}}\left(c^*;\left(\eta^Y\right)^{(j-1)}\right) \triangleq \sup_{(\tilde{\pi},c)\in\tilde{\mathcal{A}}} \tilde{V}_0\left[c;\left(\eta^Y\right)^{(j-1)}\right]$$

   *and denote the associated investment strategy by* $\tilde{\pi} = \left(\tilde{\pi}_{\eta^S}, \tilde{\pi}_{\eta^Y}\right)$.
3. *set* $\left(\eta^Y\right)^{(j)}$ *such that*

$$\tilde{\pi}_{\eta^Y} \stackrel{!}{=} 0.$$

4. • *if* $d\left(\left(\eta^Y\right)^{(j)}, \left(\eta^Y\right)^{(j-1)}\right)$ *is large: set* $j \to j+1$ *and return to 2.*

- *else:*

$$\tilde{\mathrm{v}}\left(c^{*};\left(\eta^{Y}\right)^{(j)}\right) \qquad \textit{is the approximate primal solution}$$

Speaking in terms of HJB equations, recall from Section 4.4.2 that in the $\eta^Y$-completed market we have

$$\tilde{\mathrm{v}}(c^*) = \tilde{G}(t,x,y) = \tfrac{1}{1-\gamma}x^{1-\gamma}\tilde{g}(t,y)^k,$$

where $\tilde{g}$ satisfies the nonlinear partial differential equation

$$0 = \tilde{g}_t + \tfrac{1-\gamma}{k}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\chi_{\mathrm{c}}^{\top}\left(\Sigma_{\mathrm{c}}^{S}\left(\Sigma_{\mathrm{c}}^{S}\right)^{\top}\right)^{-1}\chi_{\mathrm{c}} - \tfrac{\delta\theta}{1-\gamma}\right)\tilde{g} + \left(\left(\mu^{Y}\right)^{\top} + \tfrac{1-\gamma}{\gamma}\chi_{\mathrm{c}}^{\top}\left(\left(\Sigma_{\mathrm{c}}^{S}\right)^{\top}\right)^{-1}\left(\Sigma^{Y}\right)^{\top}\right)\tilde{g}_y$$
$$+ \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^{Y}\right)^{\top}\tilde{g}_{yy}\Sigma^{Y}\right] + \tfrac{1}{2}\tfrac{k-\gamma}{\gamma}\frac{1}{\tilde{g}}(\tilde{g}_y)^{\top}\Sigma^{Y}\left(\Sigma^{Y}\right)^{\top}\tilde{g}_y + \tfrac{\delta^{\psi}\theta}{k\psi}\tilde{g}^{1-\frac{k\psi}{\theta}}, \tag{6.0.1}$$

with terminal condition $\tilde{g}(T,y) = \varepsilon^{\frac{1}{k}}$. Moreover, the investor's optimal strategy reads

$$\tilde{\pi}^* = \tfrac{1}{\gamma}\left(\Sigma_{\mathrm{c}}^{S}\left(\Sigma_{\mathrm{c}}^{S}\right)^{\top}\right)^{-1}\chi_{\mathrm{c}} + \tfrac{k}{\gamma}\left(\left(\Sigma_{\mathrm{c}}^{S}\right)^{\top}\right)^{-1}\left(\Sigma^{Y}\right)^{\top}\frac{\tilde{g}_y}{\tilde{g}} \qquad \text{and} \qquad \left(\frac{c}{x}\right)^{*} = \delta^{\psi}\tilde{g}^{-\frac{k\psi}{\theta}}$$

as well as

$$\tilde{\pi}^{*}_{\eta^{Y}} = \tfrac{1}{\gamma}\eta^{Y} + \tfrac{k}{\gamma}\left(\hat{\Sigma}^{Y}\right)^{\top}\frac{\tilde{g}_y}{\tilde{g}}.$$

Thus, as in (4.4.7), the appropriate adjustment of the MPR is determined as

$$\left(\eta^{Y}\right)^{*} = -k\left(\hat{\Sigma}^{Y}\right)^{T}\frac{\tilde{g}_y}{\tilde{g}}.$$

Note that up to now $\tilde{\pi}^*$ is admissible after each adjustment of the MPR. Thus, if the algorithm converges, i.e. $\left(\eta^{Y}\right)^{(j)} = \left(\eta^{Y}\right)^{(j-1)}$, the output *is* admissible in the original marked as opposed to the output of the ALFC-scheme of Section 5.2.

## 6.1 Sensitivity-Approximation PDI (SA-PDI)

Consider the PDE associated to the completed market and set $k = \frac{\theta}{\psi}$, such that the nonlinearity $g^{1-\frac{k\psi}{\theta}}$ vanishes. Unfortunately, due to the nonlineariy $\frac{\tilde{g}_y}{\tilde{g}}$, we cannot solve the partial differential equation (6.0.1) and thus cannot translate the idea of the PDI algorithm to the notion of PDEs without applying additional approximations. Our approach to approximate (6.0.1) is as follows.

We represent the sensitivities $\frac{\tilde{g}_y}{\tilde{g}}$ with the optimal market prices of risk, i.e.

$$\frac{\left(\tilde{g}_y\right)^{\top}}{\tilde{g}} = -\tfrac{1}{k}\left(\eta^{Y}\right)^{T}\left(\hat{\Sigma}^{Y}\right)^{-1}, \tag{6.1.1}$$

then similar as before, we can equivalently formulate the complete market equation (5.2.2) as

$$0 = \tilde{g}_t + \tilde{r}\left(\left(\eta^{Y}\right)^{*}\right)\tilde{g} + \tilde{\alpha}\left(\left(\eta^{Y}\right)^{*}\right)\tilde{g}_y + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^{Y}\right)^{\top}\tilde{g}_{yy}\Sigma^{Y}\right] + \delta^{\psi}, \tag{6.1.2}$$

where $\tilde{r} : \mathbb{R}^n \to \mathbb{R}$ and $\tilde{\alpha} : \mathbb{R}^{n\times 1} \to \mathbb{R}^{1\times n}$ are given as

$$\tilde{r}(v) \triangleq \tfrac{1-\gamma}{k}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\left(\chi^{\top}\left(\Sigma^{S}\left(\Sigma^{S}\right)^{\top}\right)^{-1}\chi + v^{\top}v\right) - \tfrac{\delta\theta}{1-\gamma}\right)$$

and

$$\tilde{\alpha}(v) \triangleq \left(\left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\left(\chi^\top \left(\left(\Sigma^S\right)^\top\right)^+ \left(\Sigma^Y\right)^\top + v^\top \left(\hat{\Sigma}^Y\right)^\top\right) - \tfrac{1}{2}\tfrac{k-\gamma}{k\gamma} v^\top \left(\hat{\Sigma}^Y\right)^{-1} \Sigma^Y \left(\Sigma^Y\right)^\top\right).$$

We guess some initial $\eta^Y_{\approx}$, solve (6.1.2) and update $\eta^Y_{\approx}$ according to the solution by $\eta^Y_{\text{new}} = -k\left(\hat{\Sigma}^Y\right)^\top \frac{\tilde{g}_y}{\tilde{g}}$. The issue with the dynamic programming method now is that, even if the model is affine, the resulting sensitivities $\frac{\tilde{g}_y}{\tilde{g}}$ and hence market prices of risk are in general not. Thus, we *approximate* the market prices of risk $\eta^Y_{\approx} \approx \eta^Y_{\text{new}}$, such that $\eta^Y_{\approx}$ is affine and (6.1.2) separates when plugging it into the approximation, accepting the additional approximation error. Then $\eta^Y_{\approx}$ is used as initialization of the next iteration and we proceed until convergence, i.e. until the *approximations* of sensitivities coincide.

**Algorithm:** (SA-PDI)

1. *initialize the market conditions with* $j = 1$, $\left(\eta^Y_{\approx}\right)^{(j-1)} = \mathbf{0}_{n\times 1}$
2. *solve the approximated complete market problem by finding a solution* $\tilde{g}$ *to*

$$0 = \tilde{g}_t - \tilde{r}\left(\left(\eta^Y_{\approx}\right)^{(j-1)}\right)\tilde{g} + \tilde{\alpha}\left(\left(\eta^Y_{\approx}\right)^{(j-1)}\right)\tilde{g}_y + \frac{1}{2}\text{trace}\left[\left(\Sigma^Y\right)^\top \tilde{g}_{yy}\Sigma^Y\right] + \delta^\psi, \qquad (6.1.3)$$

   *subject to the terminal condition* $\tilde{g}(T,y) = \varepsilon^{\frac{1}{k}}$ *and set* $(\eta^Y)^{(j)} = -k(\hat{\Sigma}^Y)^T \frac{\tilde{g}_y}{\tilde{g}}$

3. *suitably*[1] *approximate the resulting sensitivities* $\frac{\tilde{g}_y}{\tilde{g}}$ *by some function* $\mathfrak{S} : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ *and set*

$$\left(\eta^Y_{\approx}\right)^{(j)} = -k(\hat{\Sigma}^Y)^T\mathfrak{S}.$$

4. 
   - *if* $d\left(\left(\eta^Y_{\approx}\right)^{(j-1)}, \left(\eta^Y_{\approx}\right)^{(j)}\right)$ *is large, increase* $j$ *by* 1 *and return to 2.*
   - *else: the approximated solution is characterized by* $\tilde{g}$

Note that due to the completion with the approximations of sensitivities, the resulting investment strategies are no longer admissible and have to be projected on the set of admissible ones by cutting off the last $n$ entries that are associated to the artificial assets. Moreover, besides the usual numerical inaccuracies, there are two sources of errors. The first one comes from the affine approximation of the market prices of risk, so we complete the market suboptimally. The second one is that resulting from the representation (6.1.1), we also approximate the sensitivities that do not actually correspond to the artificial completion. This certainly has an impact on the solution and, similar as the Campbell-Shiller approximation, it is not really clear what this does to the optimal value. At least, there is a probabilistic interpretation of this approximation which we briefly state in a heuristic manner:

In order to not take into account the affine approximation of $\eta^Y$, consider the partial differential equation corresponding to the exact completion (5.2.2) as above. A short manipulation shows that (5.2.2) is equivalent to

$$0 = \tilde{g}_t + \bar{r}\tilde{g} + (\tilde{g}_y)^\top\left(\mu^Y - \Sigma^Y\theta\left(\frac{\tilde{g}_y}{\tilde{g}}\right)\right) + \tfrac{1}{2}\text{trace}\left[\left(\Sigma^Y\right)^\top \tilde{g}_{yy}\Sigma^Y\right] + \delta^\psi,$$

[1] What kind of approximation is *suitable* depends on the model under consideration, see our numerical examples below.

$\tilde{g}(T, y) = \hat{\varepsilon}$, where $\bar{r} \triangleq \frac{1-\gamma}{k}\left(r + \frac{1}{2}\frac{1}{\gamma}\chi^\top\left(\Sigma^S\left(\Sigma^S\right)^\top\right)^{-1}\chi - \frac{\delta\theta}{1-\gamma}\right)$ and

$$\theta\left(\frac{\tilde{g}_y}{\tilde{g}}\right) \triangleq \frac{\gamma-1}{\gamma}\left(\Sigma^S\right)^+\chi - \frac{1}{2}\left((k-1)\mathrm{I}_{m+n} - k\frac{\gamma-1}{\gamma}\left(\Sigma^S\right)^+\Sigma^S\right)\left(\Sigma^Y\right)^\top\frac{\tilde{g}_y}{\tilde{g}}.$$

Then $\theta$ as a process of $t$ and $Y$ can be interpreted as a Girsanov kernel inducing a change of measure under which

$$\mathrm{d}W_t^\theta = \mathrm{d}W_t + \theta\left(\frac{\tilde{g}_y}{\tilde{g}}\right)\mathrm{d}t$$

is a $(m+n)$-dimensional Brownian motion, thus $\tilde{g}(t, y)$ has an implicit Feynman-Kac representation

$$\tilde{g}(t, y) = \delta^\psi\mathbb{E}_t^\theta\left[\int_t^T \mathrm{e}^{-\int_t^s \bar{r}\left(Y_u^\theta\right)\mathrm{d}u}\mathrm{d}s\right] + \hat{\varepsilon}\mathbb{E}_t^\theta\left[\mathrm{e}^{-\int_t^T \bar{r}\left(Y_u^\theta\right)\mathrm{d}u}\right],$$

where $\mathbb{E}^\theta$ is taken under the new measure of risk induced by $\theta$ and $Y^\theta$ has the dynamics

$$\mathrm{d}Y^\theta = \left(\mu^Y - \Sigma^Y\theta\left(\frac{\tilde{g}_y}{\tilde{g}}\right)\right)\mathrm{d}t + \Sigma^Y\mathrm{d}W^\theta.$$

Hence, the affine approximation of the sensitivities that are not associated to the completion of the market with $\eta_{\approx}^Y$, can be interpreted as an affine approximation of an actually nonlinear change of measure.

**Remark 6.1**
*The approximation* (6.1.1) *could also be used to extend the CS-ALFC algorithm to the recursive case:*

1. *set* $k = \frac{\theta}{\psi}$ *and determine the Campbell-Shiller approximation* $g^{\mathrm{CS}}$ *associated to the dual partial differential equation* (5.2.1)
2. *set*
$$\left(\eta^Y\right)^{\mathrm{CS}} = -k\left(\hat{\Sigma}^Y\right)\frac{g_y^{\mathrm{CS}}}{g^{\mathrm{CS}}}$$
3. *solve* (6.1.2) *with* $\left(\eta^Y\right)^{\mathrm{CS}}$ *instead of* $\left(\eta^Y\right)^*$, *then* $\tilde{g} \approx g$

*Note however, that in the truly recursive case we have no closed form solution, even for the artificially completed market problem.* △

### 6.1.1 Numerical Results

We test the SA-PDI algorithm with our multivariate Kim-Omberg and Heston model. All necessary computations can be found in Appendix D.3.

#### Comparison with Exact Solutions

First, we compare the numerical results provided by the PDI algorithm with the exact numerical solution provided by the fixed-point iteration algorithm of Section 5.1 in a one-dimensional Kim-Omberg model as in Example 4.1, then in a one-dimensional Heston model as in Example 4.2.

**Kim-Omberg Model:** Recall the model introduced in Example 4.1, where for a $\mathbb{R}^{m+n}$-dimensional standard Brownian motion $W$, the risky assets and states follow the dynamics

$$\begin{aligned} \mathrm{d}S_t^{\text{risky}} &= \mathrm{diag}[S_t^{\text{risky}}]\left(\left(r\mathbf{1}_m + \bar{\lambda} + \boldsymbol{\lambda}^\top Y_t\right)\mathrm{d}t + \Sigma^S \mathrm{d}W_t\right), \\ \mathrm{d}Y_t &= -\mathrm{diag}[\kappa]Y_t \mathrm{d}t + \Sigma^Y \mathrm{d}W_t, \end{aligned}$$

where $r \in \mathbb{R}$, $\bar{\lambda} \in \mathbb{R}^m$, $\boldsymbol{\lambda} \in \mathbb{R}^{n\times m}$, $\kappa \in \mathbb{R}^n_+$ and $\Sigma^S \in \mathbb{R}^{m\times(m+n)}$, $\Sigma^Y \in \mathbb{R}^{n\times(m+n)}$ as defined in (4.1.3). Then the algorithm in the $j$-th step behaves as follows.

Let the market prices of risk $\left(\eta^Y\right)^{(j-1)}$ from the $(j-1)$-th iteration of the form

$$\left(\eta^Y\right)^{(j-1)} \triangleq \eta_1^Y(t) + \eta_2^Y(t)y$$

be given. Plugging these into (6.1.3), a solution is given by the function

$$\tilde{g}(t,y) = \delta^\psi \int_t^T h(t,y;s)\mathrm{d}s + \hat{\varepsilon}h(t,y;T), \tag{6.1.4}$$

where

$$h(t,y;s) = \exp\left(\tilde{A}(t,s) - y^\top \tilde{B}(t,s) - y^\top \tilde{C}(t,s)y\right)$$

and $\tilde{A}(\cdot,s)$, $\tilde{B}(\cdot,s)$, $\tilde{C}(\cdot,s)$ solve (D.3.2) in Appendix D.3.1. In particular, the update for the market prices of risk are given as

$$\left(\eta^Y\right)^{(j)} = -k\left(\hat{\Sigma}^Y\right)^\top \frac{\tilde{g}_y}{\tilde{g}}.$$

We determine the approximation of sensitivities by the first-order Taylor approximation around $\bar{y} = \mathbb{E}\left[Y_\infty\right]$ as

$$\mathfrak{S}(t,y) = \frac{\tilde{g}_y(t,\bar{y})}{\tilde{g}(t,\bar{y})} + \frac{\partial}{\partial y}\left(\frac{\tilde{g}_y(t,\bar{y})}{\tilde{g}(t,\bar{y})}\right)(y-\bar{y}).$$

Then the linearized market prices of risk have a representation

$$\left(\eta^Y_{\approx}\right)^{(j-1)}(t,y) = -k\left(\hat{\Sigma}^Y\right)^\top \mathfrak{S}(t,y) \triangleq \mathfrak{S}_1(t) + \mathfrak{S}_2(t)y$$

for appropriate $\mathfrak{S}_1$ and $\mathfrak{S}_2$. Evaluate

$$d\left(\left(\eta^Y_{\approx}\right)^{(j-1)}, \left(\eta^Y_{\approx}\right)^{(j)}\right) \triangleq ||\eta_1^Y - \mathfrak{S}_1||_2 + \bar{\omega}||\eta_2^Y - \mathfrak{S}_2||_F \tag{6.1.5}$$

for some weight $\bar{\omega}$[2]. Now we are either finished if (6.1.5) is 'small', or we set $j \to j+1$ and repeat. In case we are finished, the approximate strategies read as[3]

$$\begin{aligned} \pi(t,y) &= \tfrac{1}{\gamma}\left(\hat{\Sigma}^S\left(\hat{\Sigma}^S\right)^\top\right)^{-1}\left(\bar{\lambda} + \boldsymbol{\lambda}^\top y\right) + \tfrac{k}{\gamma}\left(\left(\hat{\Sigma}^S\right)^\top\right)^{-1}\left(\hat{\Sigma}^{SY}\right)\frac{\tilde{g}_y(t,y)}{\tilde{g}(t,y)}, \\ \left(\frac{c}{x}\right)(t,y) &= \frac{\delta^\psi}{\tilde{g}(t,y)} \qquad \text{and} \qquad \eta^Y(t,y) = -k\left(\hat{\Sigma}^Y\right)\frac{\tilde{g}_y(t,y)}{\tilde{g}(t,y)}. \end{aligned} \tag{6.1.6}$$

[2]We introduce the weight $\bar{\omega}$ to compensate for potential mismatches between the two Euler and Frobenius norm. In our numerical analysis we generally do fine with $\bar{\omega} = 1$.

[3]Note that setting $\pi$ immediately like this corresponds to cutting off the last $n$ coordinates of $\tilde{\pi}$ to make the investment strategy admissible in the original market.

In particular, even though we approximated the sensitivities by the linear first order Taylor approximation, the strategies obtained from the SA-PDI algorithm are of the more general structure similar as in [Kraft et al., 2013].

As benchmark for the performance of the SA-PDI approximation, we utilize the fixed-point algorithm discussed in Section 5.1. In terms of preference parameters, we follow [Liu and Muhle-Karbe, 2013] by choosing $\gamma = 5$ and [Bansal and Yaron, 2004] by choosing $\psi = 1.5$. In particular, in our numerical analysis we focus on parameters $\gamma, \psi > 1$, which are sometimes labeled as the empirically relevant case in the literature, c.f.[Xing, 2017]. As in the CS-ALFC algorithm, we choose our (monthly) model parameters based on [Liu and Muhle-Karbe, 2013] and [Wachter, 2002], c.f. [Barberis, 2000].

| $\gamma$ | $\psi$ | $\delta$ | $\varepsilon$ |
|---|---|---|---|
| 5 | 1.5 | 0.52% | 1 |

Table 6.1: Preference Parameters

| $T$ | $r$ | $\bar{\lambda}$ | $\bar{\sigma}^S$ | $\kappa$ | $\bar{\sigma}^Y$ | $\rho^{SY}$ |
|---|---|---|---|---|---|---|
| 20 | 0.14% | 0.34% | 4.36% | 2.26% | 0.08% | -93.5% |

Table 6.2: Model Parameters (monthly)

Note that we choose our preference parameters far away from the power case, to stress the performance of our power bounds.
Finally, we set $\boldsymbol{\lambda} = 1$. Figure 6.1 show the solution $g$ as a \$-multiplier (see (5.2.5)), the associated differentials, sensitivities and approximated strategies provided by the SA-PDI algorithm in red (dashed); the exact solution from the fixed-point algorithm is plotted in blue. All graphs show the associated mappings in $t$ at the mean reversion level of the state process $\bar{y}$.

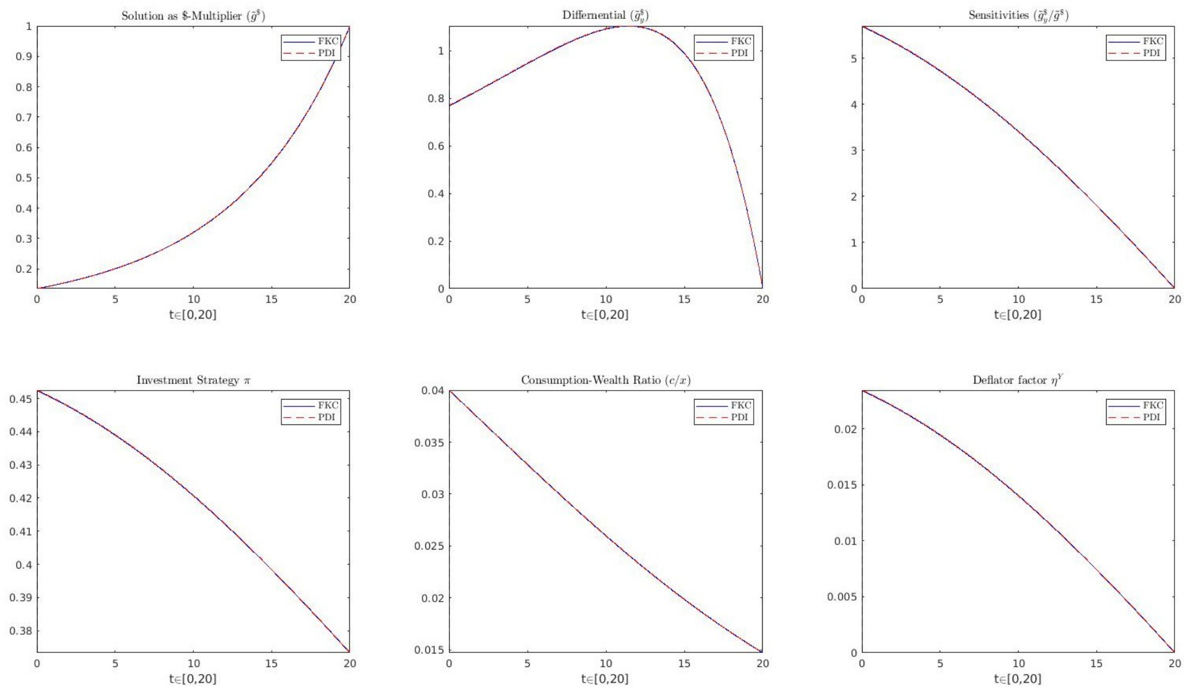



Figure 6.1: Exact solution from fixed point algorithm in blue and the SA-PDI approximation in red (dashed). Parameters are as in Table 6.1 and Table 6.2, where $T = 20$ years. All results are plotted as $t \mapsto \cdot(t, 0)$.

Figure 6.2 again shows the solution $g^{\$}$ and the associated strategies of the SA-PDI algorithm in red (dashed) and the exact solution of [Kraft et al., 2017] in blue, this time as functions in $y$ at

$t = 0$.

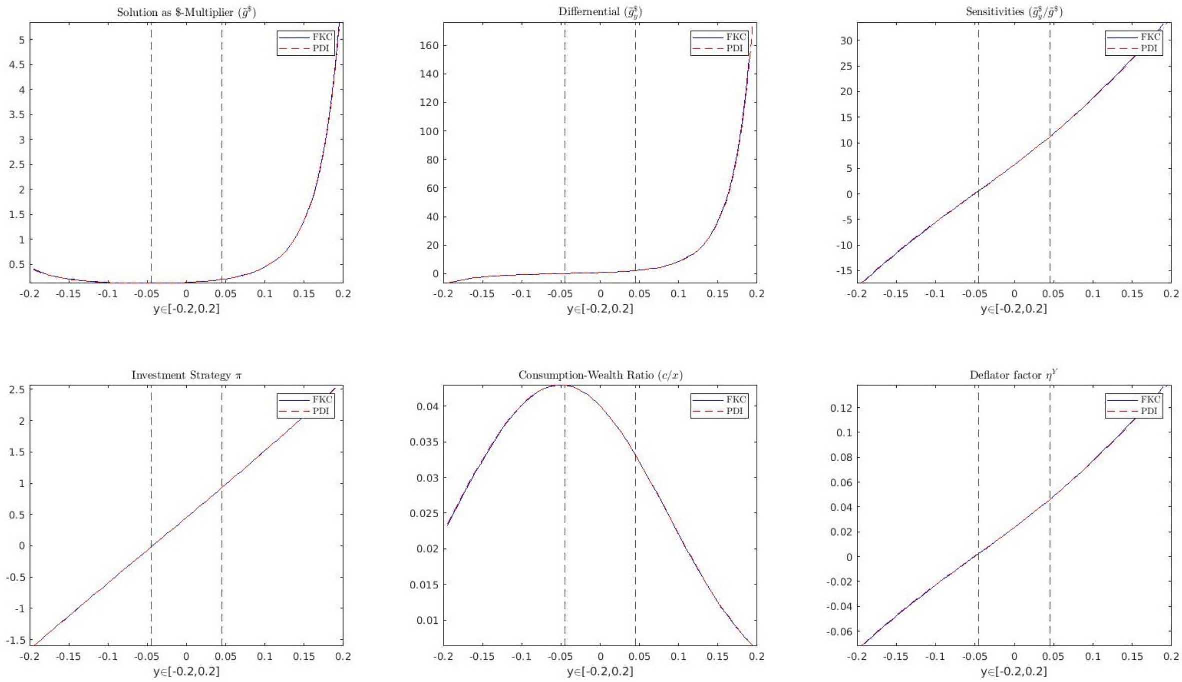



Figure 6.2: Exact solution from fixed point algorithm in blue and the SA-PDI approximation in red (dashed). Parameters are as in Table 6.1 and Table 6.2. All results are plotted as $y \mapsto \cdot(0, y)$. The dashed vertical lines indicate the 99% quantile of the state process with our given model parameters.

We compare the results in dimension $m = n = 1$ for several risk preference parameters, while the model parameters remain fixed as in Table 6.2. When iterating over $\gamma \in (0, 10]$ we follow [Bansal and Yaron, 2004] and fix $\psi = 1.5$; when iterating over $\psi \in (0, 5]$ we fix $\gamma = 5$ according to [Liu and Muhle-Karbe, 2013].[4] Note that the PDE iteration of [Kraft et al., 2017] diverges if $\psi$ gets close to 1, while the SA-PDI algorithm runs stable.

Finally, we investigate the convergence of the algorithm. Figure 6.4 shows the function $g^\$(0, y)$ of every iteration step as a function of $y$ within the 99% quantile of the given state process on the left side. The plot on the right side shows the error degression in terms of (6.1.5) on a logarithmic scale.

[4]The gaps within the graphs are the parameter constellations $\gamma, \psi$ that are out of our duality setting.

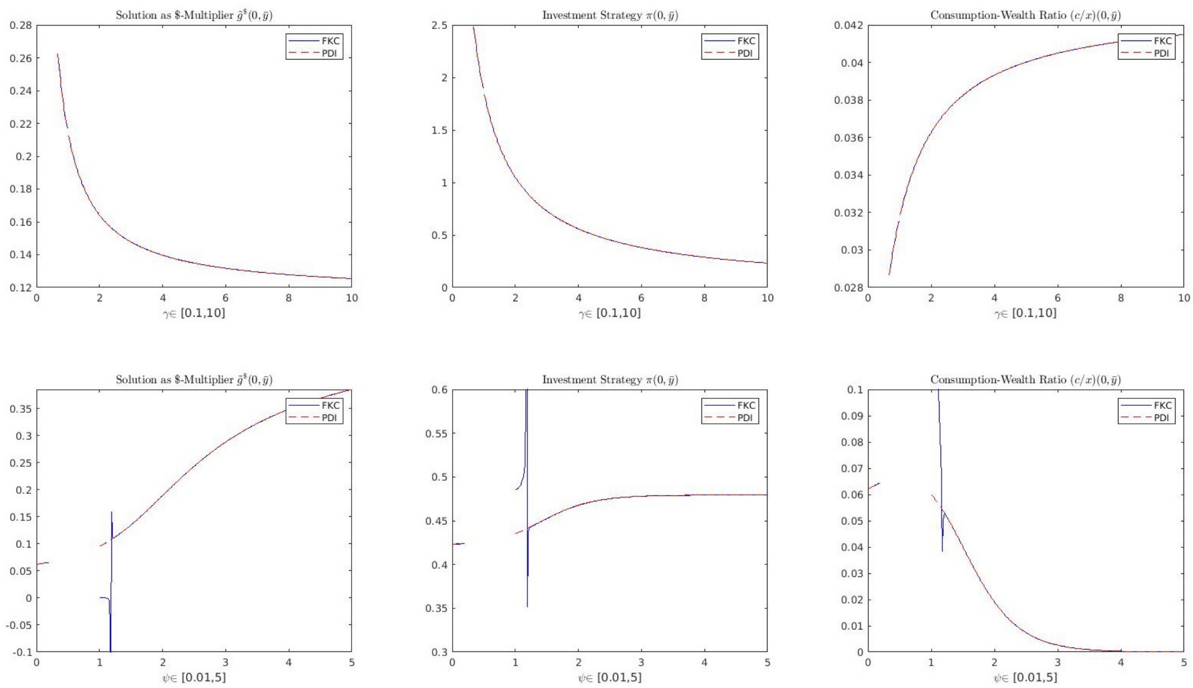



Figure 6.3: Exact solution from fixed point algorithm in blue and the SA-PDI approximation in red (dashed). Parameters are as in Table 6.1 and Table 6.2. All results are plotted at $(t, y) = (0, \bar{y})$ as functions in $\gamma$ and $\psi$ respectively.

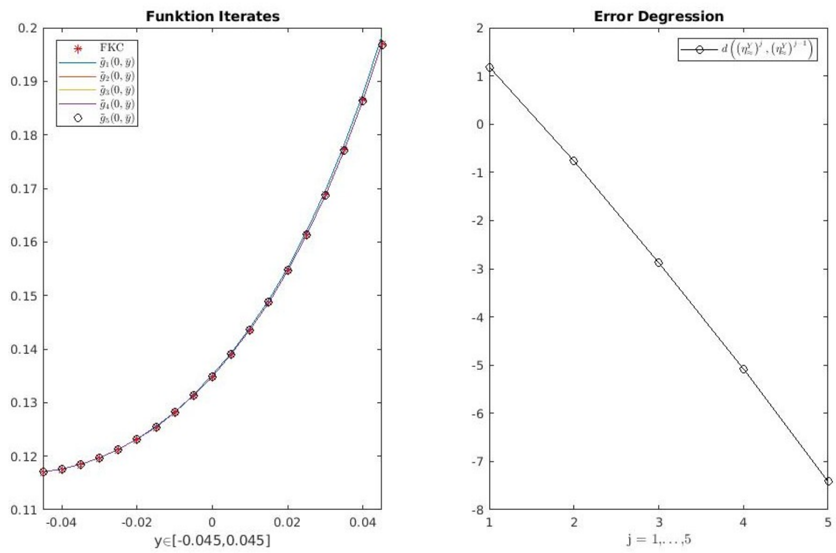



Figure 6.4: Convergence behavior of our SA-PDI scheme in the Kim-Omberg model. The function iterates on the left plot are given as $y \mapsto \cdot(0, y)$ on the 99% quantile of the state process. The error on the right hand side are on a logarithmic scale. Model parameters are as given in Table 6.2.

**Heston Model:** Consider the multivariate Heston model introduced in Example 4.2, i.e. let $m = n$ and for a $\mathbb{R}^{2n}$-dimensional standard Brownian $W$ let the assets and states follow the dynamics

$$\begin{aligned} \mathrm{d}S_t &= \operatorname{diag}[S_t]\left(\left(r + K\operatorname{diag}[Y_t]K^\top\bar{\lambda}\right)\mathrm{d}t + \Sigma^S(Y_t)\mathrm{d}W_t\right), \\ \mathrm{d}Y_t &= \left(\bar{\mu} - \operatorname{diag}[\kappa]Y_t\right)\mathrm{d}t + \Sigma^Y(Y_t)\mathrm{d}W_t, \end{aligned}$$

where $r, \bar{\lambda}, \bar{\mu}, \kappa \in \mathbb{R}^n$, $K \in \mathbb{R}_o^{n\times n}$ and $\Sigma^S$, $\Sigma^Y$ are $(n \times 2n)$-dimensional matrix functions as in (4.1.3). Then the SA-PDI algorithm in the $j$-th step behaves as follows.

Let the market prices of risk $\left(\eta^Y\right)^{(j-1)}$ of the form

$$\left(\eta^Y\right)^{(j-1)}(t,y) = -k\left(\hat{\Sigma}^Y(y)\right)\eta_1^Y(t),$$

where $\tilde{g}$ is the solution of the previous iteration. Recall that $\hat{\Sigma}^Y(y)$ behaves as $\sqrt{y}$. Plugging these into (6.1.3), a solution is given by the function

$$\tilde{g}(t,y) = \delta^\psi \int_t^T h(t,y;s)\mathrm{d}s + \hat{\varepsilon}h(t,y;T),$$

where

$$h(t,y;s) = \exp\left(\tilde{A}(t,s) - y^\top\tilde{B}(t,s)\right),$$

and $\tilde{A}(\cdot,s)$, $\tilde{B}(\cdot,s)$ solve (D.3.3) in Appendix D.3.2. In particular, the update for the market prices of risk are given as

$$\left(\eta^Y\right)^{(j)} = -k\left(\hat{\Sigma}^Y\right)^\top\frac{\tilde{g}_y}{\tilde{g}}.$$

We approximate the sensitivity $\frac{\tilde{g}_y}{\tilde{g}}$ by its the value at $\bar{y} = \mathbb{E}\left[Y_\infty\right]$ as

$$\mathfrak{S}(t) = \frac{\tilde{g}_y(t,\bar{y})}{\tilde{g}(t,\bar{y})}.$$

Then the approximated market prices of risk are of the form

$$\left(\eta_\approx^Y\right)^{(j-1)}(t,y) = -k\operatorname{diag}[\bar{\sigma}^Y]L^Y y^{\frac{1}{2}}\mathfrak{S}(t).$$

Finally, evaluate

$$d\left(\left(\eta_\approx^Y\right)^{(j-1)}, \left(\eta_\approx^Y\right)^{(j)}\right) \triangleq ||\eta^Y - \mathfrak{S}||_2 \tag{6.1.7}$$

and we are either finished if (6.1.7) is 'small' or we set $j \to j+1$ and repeat. In case we are finished, the associated approximate strategies read as[5]

$$\tilde{\pi}(t,y) = \tfrac{1}{\gamma}K\operatorname{diag}\left[\left(\bar{\sigma}^S\right)^2\right]^{-1}K^\top\bar{\lambda} + \tfrac{k}{\gamma}K\operatorname{diag}\left[\bar{\sigma}^Y\right]\operatorname{diag}\left[\bar{\sigma}^S\right]^{-1}\rho^{SY}\frac{\tilde{g}_y(t,y)}{\tilde{g}(t,y)}, \tag{6.1.8}$$

$$\left(\frac{c}{x}\right)(t,y) = \frac{\delta^\psi}{\tilde{g}(t,y)}, \qquad \eta^Y(t,y) = -ky^{\frac{1}{2}}\operatorname{diag}[\bar{\sigma}^Y]L^Y\frac{\tilde{g}_y(t,y)}{\tilde{g}(t,y)},$$

where $\left(\bar{\sigma}^S\right)^2 \triangleq \left(\left(\bar{\sigma}_1^S\right)^2, \ldots, \left(\bar{\sigma}_n^S\right)^2\right)$. In particular, even though we approximated the sensitivities with a constant function in $y$, the strategies from the SA-PDI algorithm are of the more

[5]Note that setting $\pi$ immediately like this corresponds to cutting off the last $n$ coordinates of $\tilde{\pi}$ to make the investment strategy admissible in the original market.

general form of [Kraft et al., 2013].

For our comparison, we choose $m = n = 1$ and stick to the preference parameters from Table 6.1 and for the (yearly) model parameters we follow [Liu and Muhle-Karbe, 2013], c.f. [Pan, 2002].

| $\gamma$ | $\psi$ | $\delta$ | $\varepsilon$ |
|---|---|---|---|
| 5 | 1.5 | 0.062 | 1 |

Table 6.3: Preference Parameters

| $T$ | $r$ | $\bar{\lambda}$ | $K$ | $\bar{\sigma}^S$ | $\kappa$ | $\bar{\mu}$ | $\bar{\sigma}^Y$ | $\rho^{SY}$ |
|---|---|---|---|---|---|---|---|---|
| 10 | 0.033 | 4.4 | 1 | 1 | 5.3 | 0.13 | 0.38 | -0.57 |

Table 6.4: Model Parameters (yearly)

Figure 6.5 and Figure 6.6 show the solution $g$ as a \$-multiplier (see (5.2.5)) and the associated differentials, sensitivities and approximated strategies provided by SA-PDI in red (dashed); the exact solution from the fixed-point algorithm is plotted in blue. Figure 6.5 shows all mappings as functions $t \mapsto \cdot(t, 0)$, while Figure 6.6 shows them as functions $y \mapsto \cdot(0, y)$.

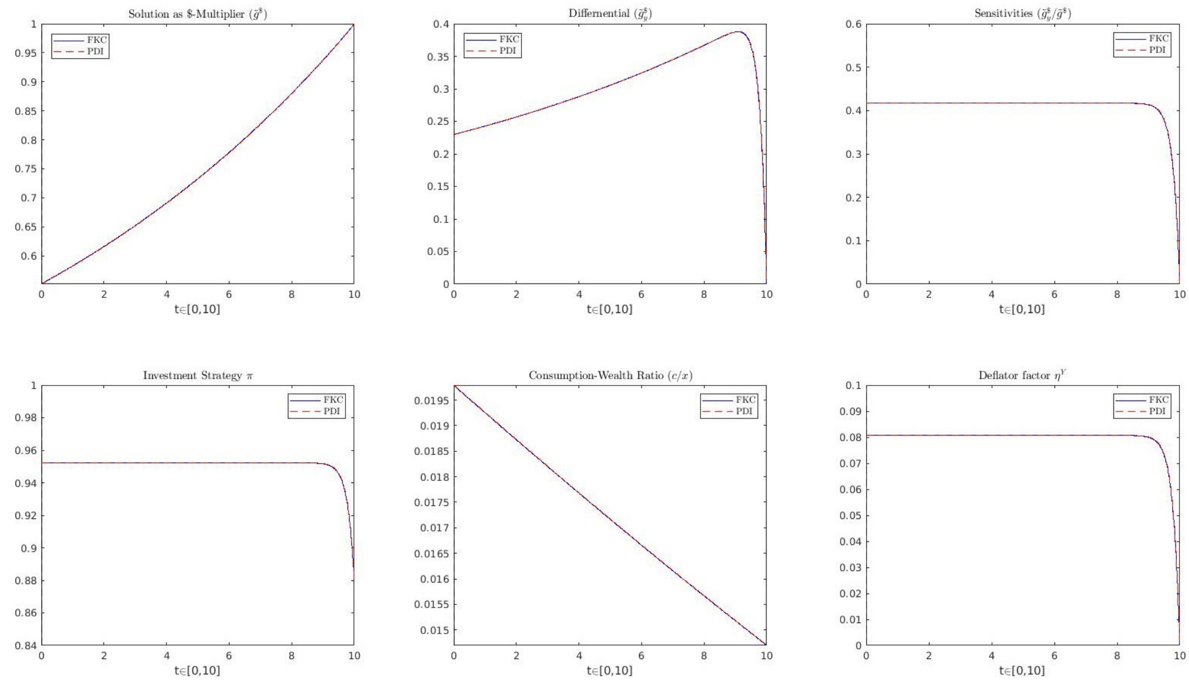



Figure 6.5: Exact solution from fixed point algorithm in blue and the SA-PDI approximation in red (dashed). Parameters are as in Table 6.3 and Table 6.4, where $T = 10$ years. All results are plotted as $t \mapsto \cdot(t, \bar{y})$.

We now compare the results for several risk preference parameters, while the model parameters remain fixed as in Table 6.4. When iterating over $\gamma \in (0, 10]$ we follow [Bansal and Yaron, 2004] and fix $\psi = 1.5$; when iterating over $\psi \in (0, 5]$ we fix $\gamma = 5$ according to [Liu and Muhle-Karbe, 2013].[6]

[6]The gaps in the graphs correspond to parameter constellations where $\gamma$, $\psi$ are out of our duality setting.

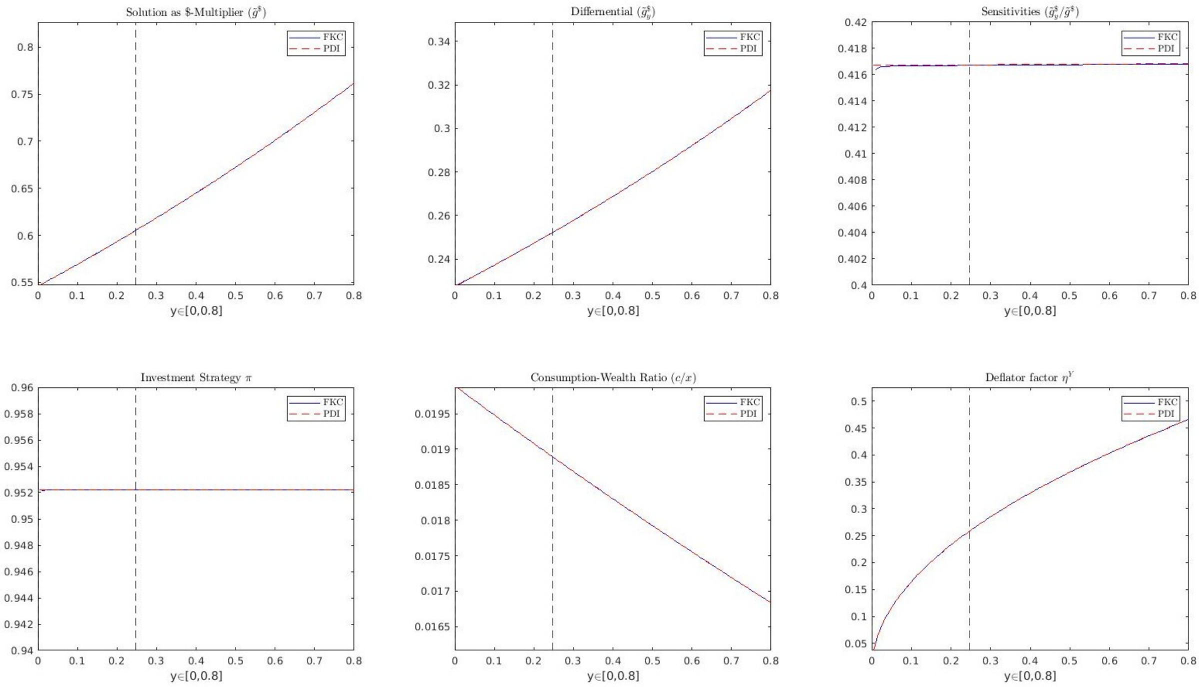



Figure 6.6: Exact solution from fixed point algorithm in blue and the SA-PDI approximation in red (dashed). Parameters are as in Table 6.3 and Table 6.4. All results are plotted as $y \mapsto \cdot(0, y)$. The dashed vertical lines indicate the 99% quantile of the state process with our given model parameters.

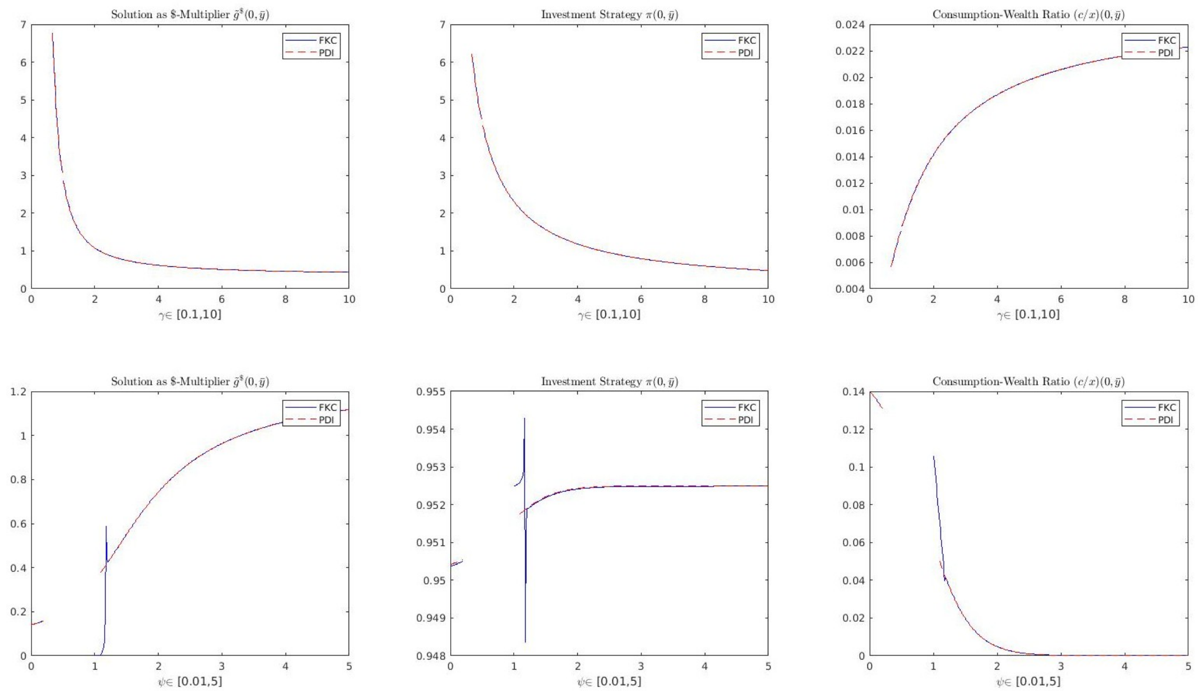



Figure 6.7: Exact solution from fixed point algorithm in blue and the SA-PDI approximation in red (dashed). Model parameters are as in Table 6.4. All results are plotted at $(t, y) = (0, \bar{y})$ as functions in $\gamma$ and $\psi$ respectively.

Concerning the convergence of the SA-PDI algorithm, Figure 6.8 shows the function $g^{\$}(0, y)$ of every iteration step as a function of $y$ within the 99% quantile of the given state process on the left side. The plot on the right side shows the error degression in terms of (6.1.7) on a logarithmic scale.

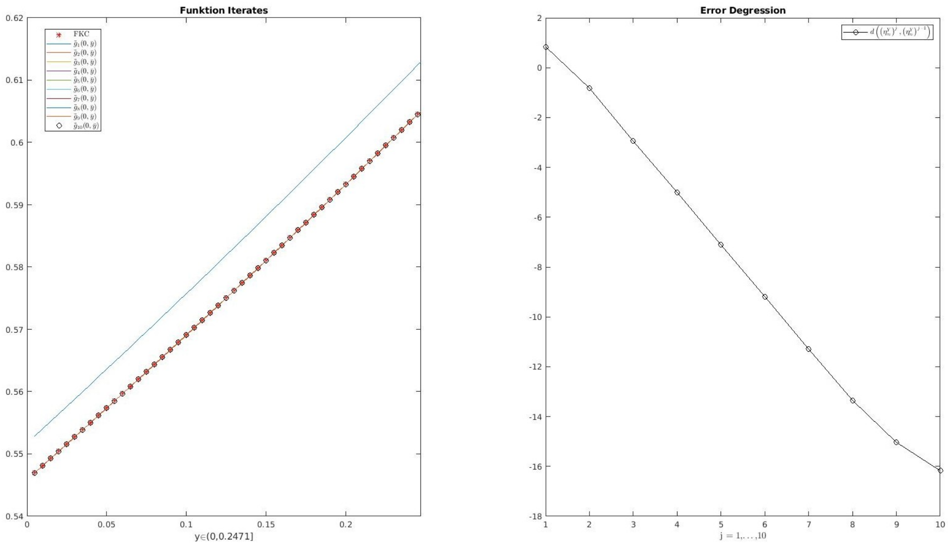



Figure 6.8: Convergence behavior of our SA-PDI scheme in the Heston model. The function iterates on the left plot are given as $y \mapsto \cdot(0, y)$ on the 99% quantile of the state process. The error on the right hand side are on a logarithmic scale. Parameters are as in Table 6.4.

**Large Scale Model**

In order to test the SA-PDI algorithm in higher dimensions, we consider our large scale market from Section 5.2.2, i.e. we assume the correlation structure

$$\rho^S_{i,j} = \begin{cases} 1, & i = j \\ \rho^\star, & i \neq j \end{cases}, \qquad \rho^{SY}_{i,j} = \begin{cases} \xi, & i = j \\ \xi\rho^\star, & i \neq j \end{cases} \quad \text{and} \quad \rho^Y_{i,j} = \begin{cases} 1, & i = j \\ \xi^2\rho^\star + (1 - \xi^2)\zeta, & i \neq j \end{cases}.$$

As mentioned above this is a special case of the multivariate Kim-Omberg setting as in Example 4.1, thus, the behavior of the algorithm in the $j$-th step is given as above and the approximated solution and strategy are given as in (6.1.4) and (6.1.6).

Again, we have no benchmark to compare the algorithm with in higher dimensions, so we have to utilize our bounds from Chapter 3. However, as we are now in a truly recursive setting, the accuracy of our power bounds from Theorem 3.11 deteriorates if $\gamma, \psi$ deviates too much from the case $\gamma\psi = 1$. While on our focus area $\gamma, \psi > 1$ we have no alternative for the upper bound, we utilize our lower variational version from Corollary 3.12 as a lower bound to evaluate the optimality gap.

Similar as in Remark 5.4, we take the annualized welfare loss as a measure of accuracy derived from the actual optimality gap. The derivation here is analogous to the derivation within Remark 5.4, with $\mathrm{P}_{\gamma\vee\phi}(c)$ from Theorem 3.11 replaced by $U(c, u)$ from Corollary 3.12.

In our numerical simulations, we set $\lambda = 1$, $\rho^\star = 41.6\%$ and $\zeta = 0$, and otherwise stick to the parameters from Table 6.2 for every asset and state process:

| $T$ | $r$ | $\bar{\lambda}$ | $\boldsymbol{\lambda}$ | $\sigma^S$ | $\kappa$ | $\sigma^Y$ | $\rho^\star$ | $\xi$ | $\zeta$ |
|---|---|---|---|---|---|---|---|---|---|
| 20 | 0.14% | 0.34% | 100% | 4.36% | 2.26% | 0.08% | 41.6% | -93.5% | 0 |

Table 6.5: Large Scale Model Parameters (monthly)

Table 6.6 shows the result of our bounds applied to the strategies provided by the SA-PDI scheme in dimension $n = m = 50$ and for different preference parameters $(\gamma, \psi) \in \{(5, 1.5), (3, 1.3), (1.1, 1.1)\}$. The first line of Table 6.6 contains the lower variational bound, the second presents the upper power bound from Theorem 3.11 evaluated by Monte Carlo simulation. The resulting (annualized) wealth equivalent loss is given in the last line. The entries presented are the average of 10 simulation with 5000 samples each. The brackets below provide the associated standard deviations.

| | $n = 50$ | | |
|---|---|---|---|
| | $(\gamma = 5,\ \psi = 1.5)$ | $(\gamma = 3,\ \psi = 1.3)$ | $(\gamma = 1.1,\ \psi = 1.1)$ |
| $U(c, \tilde{u})$ | -1.79e-06 | -9.80e-04 | -6.746 |
| | (8.95e-07) | (5.41e-05) | (0.010) |
| $\mathrm{P}^*_\gamma(D)$ | -2.83e-12 | -1.14e-05 | -6.463 |
| | (2.82e-13) | (5.69e-07) | (0.013) |
| $\tilde{V}_0[c]$ | -1.32e-06 | -8.86e-04 | -6.713 |
| | - | - | - |
| $L_{20}$ | 56.3% | 28.3% | 2.3% |
| | (9.19e-03) | (3.05e-03) | (8.01e-04) |

Table 6.6: Large Scale Kim-Omberg: Accuracy of SA-PDI algorithm in dimension $n = 50$ and investment horizon $T = 20$ years for different RRA $\gamma$ and EIS $\psi$. The bounds are the average of 10 Monte Carlo simulations of the respective expectations in Theorem 3.11 with 5000 sample paths each. The associated standard deviations are given in brackets.

It turns out that evaluating the truly recursive scenario by using our power bounds is unsatisfactory, see in particular the second row of Table 6.6 compared to the first and third. Table 6.6 verifies the intuition, that our power bounds become worse, if we deviate much from the power utility case and become better (but not great) when we are closer to $\gamma\psi = 1$.

Searching for a better way to verify the performance of the SA-PDI algorithm, its high accuracy in dimension $n = 1$, c.f. Figure 6.1-Figure 6.8, as well as the upper bound property of the value function from the suboptimal completion in every step, tempts to use $\tilde{V}_0[c]$ itself as an upper bound. However, one cannot be sure whether this upper bound property provided by the duality theory of [Cvitanić and Karatzas, 1992] was destroyed by the additional approximation of sensitivities or not. Nevertheless, as the first column of Table 6.6 provides almost no information about the performance of the algorithm, we do take $\tilde{V}_0[c]$ as an *approximate* upper bound as this is the best hint we can get, even if results might be too good (see in particular the first column of Table 6.7, where $\mathrm{v}(c^*)$ slightly exceeds $\tilde{V}_0[c]$). We denote the resulting *approximated* annualized WEL by $L_T^{\approx}$, the derivation is analogous to Remark 5.4 with $\mathbb{P}_{\gamma\wedge\phi}(D)$ replaced by $\tilde{V}_0[c]$.[7]

[7]Note that the results differ slightly from the ones in Table 6.6, as they are computed by a separate Monte Carlo simulation.

| | $(\gamma = 5,\, \psi = 1.5)$ | | $(\gamma = 3,\, \psi = 1.3)$ | | $(\gamma = 1.1,\, \psi = 1.1)$ | |
|---|---|---|---|---|---|---|
| | $n = 1$ | $n = 50$ | $n = 1$ | $n = 50$ | $n = 1$ | $n = 50$ |
| $U(c, \tilde{u})$ | -7.687e-06 | -1.602e-06 | -3.134e-03 | -1.057e-03 | -7.670 | -6.746 |
| | (1.813e-07) | (2.264e-07) | (3.102e-05) | (7.834e-05) | (9.283e-03) | (7.951e-03) |
| $\tilde{V}_0[c]$ | -7.553e-06 | -1.3251e-06 | -3.117e-03 | -8.865e-04 | -7.666 | -6.7132 |
| | - | - | - | - | - | - |
| $\nu(c^*)$ | -7.551e-06 | - | -3.117e-03 | - | -7.666 | - |
| | - | - | - | - | - | - |
| $L_{20}^{\approx}$ | 1.08e-03 | 0.0112 | 4.19e-04 | 0.0129 | 2.98e-04 | 2.74e-03 |
| | (1.04e-03) | (8.135e-03) | (4.21-04) | (5.22e-03) | (6.65e-04) | (6.44e-04) |

Table 6.7: Large Scale Kim-Omberg: Accuracy of SA-PDI algorithm in dimension $n = 50$ and investment horizon $T = 20$ years for different RRA $\gamma$ and EIS $\psi$. The lower variational bound is the average of 10 Monte Carlo simulations of the respective expectation in Corollary 3.12 with 5000 sample paths each. The associated standard deviations are given in brackets.

### 6.1.2 Conclusion and Notes on Convergence

The comparison of the outputs provided by the SA-PDI algorithm and the known algorithmic solution in Figure 6.1-Figure 6.3 and Figure 6.5-Figure 6.7, respectively, verify that the one-dimensional approximation is accurate in both, the Kim-Omberg and Heston model. Moreover, it is stable under variations of the preference parameters $\gamma$ and $\psi$, even more stable than the PDE-iteration algorithm, that diverges if $\psi$ gets close to 1. Even in higher dimensions the algorithm is fast[8] and stable under changing preference parameters. The verification of accuracy in high dimension, however, turns out to be difficult, as our power bounds become arbitrarily bad if the preference parameters deviate too much from the power utility case, recall Table 6.6. The arguably more accurate approximate bounds calculated with the direct use of the algorithms output indicates high accuracy with annualized WEL of about 1% (c.f. Table 6.7), however, those bounds have to be treated with caution, as the linear/constant approximation of sensitivities potentially destroys the upper bound property inherited by suboptimal completion. In particular, the solution provided by the SA-PDI algorithm might be smaller than the true solution and thus yield too small welfare losses, as can be seen in the first column of Table 6.7.

The SA-PDI algorithm converges fast and reliable, c.f. Figure 6.9, however it never converges to the true solution but only to the approximation of the problem with linear/constant sensitivities. It can be shown that a variation of the PDI scheme, where instead of the approximation of sensitivities, we use the Campbell-Shiller approximation in every step, converges to the direct Campbell-Shiller approximation of the primal problem.
The main drawback of the SA-PDI algorithm is its limitation to affine market models and its convergence not to the true solution but only to an approximation. The approach by [Kraft et al., 2017] is more flexible in that point of view, as by solving the associated PDE numerically with a Crank-Nicolson scheme, they don't need it to separate. However, their scheme is only applicable in one dimension, while the SA-PDI algorithm can handle high dimensions without suffering too much from the curse of dimensionality. Also the SA-PDI algorithm provides an analytical representation of the approximation, instead of a solely numerical solution. The only

[8] The running time is almost independent of the dimension. Evaluations in dimension $n = 50$ take about 12 seconds. Machine: Intel(R) Core$^{\mathrm{TM}}$ $i7 - 8650U$ Processor, 1.9GHz, 16GB RAM. Note that the computation time of the bounds on the other hand does depend on the dimension and may take several minutes.

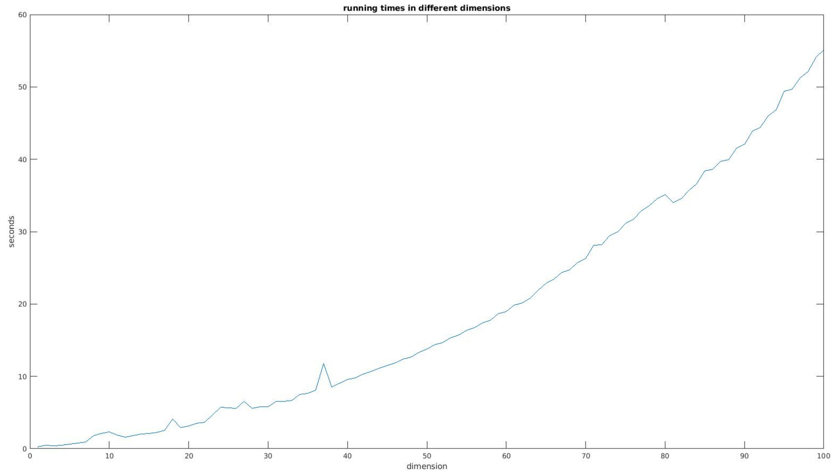



Figure 6.9: Running times of the large scale Kim-Omberg model in different dimensions in seconds.

analytical approximation to high dimensional incomplete market problems is the Campbell-Shiller approximation of [Chacko and Viceira, 2005], but as pointed out by [Kraft et al., 2013], the CS approximation implies several qualitative issues on the optimal strategy, such as exponential state dependence of consumption and no state dependence in the investment strategy. The approximate solution of our SA-PDI scheme does not inherit those shortcomings, as the solution structure coincides with the one found by [Kraft et al., 2013].

# Chapter 7

# Conclusion

Overall, this thesis extends the theory on investment-consumption optimization problems of an investor with recursive Epstein-Zin preferences by further developing the convex duality theory introduced by [Matoussi and Xing, 2018], bridging gaps in existing research, providing effective approximation methods for a agent's optimal strategies and introducing novel duality bounds on the optimality gap associated to said approximations.

More precisely, while [Matoussi and Xing, 2018] exclude power utility from their duality procedure, our extended approach derived in Section 2.2 captures power utility as a special case. As duality results for this time-additive utility specification are well known (c.f. [Pliska, 1986], [Karatzas et al., 1987], [Karatzas et al., 1991], [He and Pearson, 1991]), this extension is rather of aesthetic nature and mainly shows consistency of the two approaches within our enlarged framework. The resulting dual problem is of similar structure as the primal one, i.e. it is given as the solution to a non-standard BSDE. [Becherer et al., 2023] were the first to establish existence, uniqueness, as well as monotonicity and convexity results for solutions of this specific BSDE. However, as they consider a variation of the classical Epstein-Zin parametrization, their analysis excludes relevant parameter constellations for the RA $\gamma$ and EIS $\psi$. We close that gap in Section 2.3.

Moreover, we investigate duality bounds in terms of the optimality gap (see Definition 3.1), which have successfully been used since their introduction by [Haugh et al., 2006] in various time-additive utility settings, see e.g. [Bick et al., 2013] or [Kamma and Pelsser, 2022]. The primary benefit of those bounds is that they provide a measure for the accuracy of an approximate solution, without knowing what the exact solution is, so they can be utilized to validate numerical approximations without the need of a benchmark approach. In a time-additive setting those bounds are easy to determine by Monte Carlo simulation; in our recursive setting however, the computation of this optimality gap would correspond to solving a coupled forward-backward stochastic differential equation. Thus, in particular in high dimensions, the optimality gap itself is not a suitable tool to measure the accuracy of an approximation. We bypass this issue by introducing bounds on the optimality gap itself, in terms of transformed power utility functions in Section 3.1. To the best of our knowledge, those are the first universal bounds for the true solution of an investment-consumption optimization problem with Epstein-Zin utility, whose evaluation is feasible. However, it is later verified that those scaled power utility bounds deteriorate when the Epstein-Zin parameters vary too much from the time-additive case and are only valuable in the evaluation of numerical solutions if $\gamma\psi$ is not too far from 1. Thus, we additionally establish better suited, one-sided bounds in terms of the variational utilities, that were already introduced within the derivation of the duality inequality in Chapter 2. In particular, we obtain a well-suited lower bound in the case of a convex Epstein-Zin aggregator and a well-suited upper bound in case of a concave Epstein-Zin aggregator. In order to further

emphasize the value of our bounds, we finish this chapter with a theoretical application, i.e. we derive an existence and uniqueness result for the stochastic differential dual that allows for weaker integrability conditions than the one in Section 2.3.
Chapter 4 sets ground for our numerical approximation schemes presented in the final chapters. We introduce a general multivariate market model and embed the multivariate Kim-Omberg and Heston models later used for numerical testing. Moreover, we utilize the dynamic programming technique to characterize the optimal solutions to the primal and dual problem by their respective partial differential equations. Taking a closer look, we notice that both solutions are actually characterized by only one PDE; the same equation also characterizes the solution of the problem associated to the least favorable completion of the market.
In Chapter 5 we introduce our first approximation of the Epstein-Zin consumption investment allocation problem in terms of the CS-ALFC two-step scheme, based on said connection between the least favorable completion of the market and the primal, respectively dual problem on a PDE level. The first step employs the Campbell-Shiller transformation of the general problem to the one with unit EIS, which yields an approximation of the least favorable market prices of risk, c.f. Section 5.1. In the next step, the market is completed with said approximation and the artificially completed market problem is explicitly solved, such that the solution yields an approximation of the true solution (after projecting the resulting strategies to the admissible set). This analytic approximation is fully characterized by a system of ordinary differential equations. The accuracy of the CS-ALFC solution is validated in dimension one by direct comparison with the numerical solution of [Kraft et al., 2017] and in dimension 50 by our duality power bounds from Chapter 3, which yield an annualized wealth equivalent loss (c.f. Remark 5.4) of less than 1%. The CS-ALFC approximation can be interpreted as a variant of the classical Campbell-Shiller approximation in the incomplete market power utility scenario, improving its qualitative shortcomings identified by [Kraft et al., 2013]. To the best of our knowledge, an analytic approximation of the power utility problem, which makes use of the truly recursive CS approximation, has not been documented in the existing literature.
Chapter 6 presents our second approximation approach. The idea of the Primal-Dual-Iteration scheme is a reinterpretation of least favorable completion as a dynamic game played by the investor against an opposing price setter, that appears not to be present in the literature so far. Each iteration corresponds to the investor choosing her optimal investment strategy under given complete market conditions, while the price setter forces her strategy to remain within the constraints set by market incompleteness. This corresponds to solving the associated complete market HJB equation in every step. While we still need to approximate the sensitivities with respect to the underlying state in every iteration, the SA-PDI scheme is shown to be accurate when compared to the known algorithmic solution of [Kraft et al., 2017] in one dimensional models. Even with our one-sided variational bound, verifying the accuracy in high dimensions proves to be difficult in this truly recursive setting, as the respective other power bound may fail. Thus, lacking a suitable measure of error, we utilize the output of the SA-PDI algorithm itself as an approximate bound, even though the upper bound property inherited from suboptimal completion was generally destroyed by the approximation of sensitivities. While yielding small approximate error bounds in the area of 1% annualized WEL, those results need additional verification.

# Appendix A

# Preliminaries on Backward Stochastic Differential Equations

Within this section we give an overview on the preliminaries we need from the theory of backward stochastic differential equations (BSDEs) of the form

$$X_t = \mathbb{E}_t\left[\int_t^T g(s, X_s)\mathrm{d}s + \xi\right], \qquad t \in [0, T],$$

where $g$ and $\xi$ are specified in Definition A.2 below. The material within this section is strongly influenced and in parts borrowed from [Duffie and Epstein, 1992a], [Karoui et al., 1997],[Antonelli, 1993], [Antonelli, 1996], [Seiferling, 2016] and [Seiferling and Seifried, 2016]. In particular Appendix A.2 is due to [Seiferling and Seifried, 2016].

Let $(\Omega, \{\mathfrak{F}_t\}_{t\in[0,T]}, \mathfrak{F}, \mathbb{P})$ be a filtered probability space, where the filtration $\{\mathfrak{F}_t\}_{t\in[0,T]}$ satisfies the usual conditions of right-continuity and completeness. Denote by $\mathcal{G}$ be the $\sigma$-algebra of progressively measurable sets in $\big(\Omega, \{\mathfrak{F}_t\}_{t\in[0,T]}, \mathfrak{F}, \mathbb{P}\big)$ and by $\mathbb{B}$ the Borel-$\sigma$-field.

## A.1 Existence and Uniqueness of Solutions

We first define what we mean by a *solution* of a BSDE.

**Definition A.1** (Solution of a BSDE)
*Let $g : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be $\mathcal{G} \otimes \mathbb{B}$-measurable and $\xi \in L^1(\mathbb{P})$. Suppose $X$ is a semimartingale with $\sup_{t\in[0,T]} \mathbb{E}[|X_t|] < \infty$ and moreover $\mathbb{E}[\int_0^T |g(t, X_t)|\mathrm{d}t] < \infty$. Then we call $X$ a solution of the BSDE with aggregator $g$ and terminal value $\xi$, if $X$ satisfies*

$$X_t = \mathbb{E}_t\left[\int_t^T g(s, X_s)\mathrm{d}s + \xi\right], \qquad t \in [0, T], \tag{A.1.1}$$

*Then we say $X$ solves* $\mathrm{BSDE}(g, \xi)$ *for short.*

Of course the question whether a BSDE admits a (unique) solution strongly depends on the aggregator $g$ and the terminal value $\xi$. We now define the basic requirements for the existence and uniqueness of a solution.

**Definition A.2** (BSDE$^p$-standard parameter)
*For $p \geqslant 1$, let $\xi \in L^p(\mathbb{P})$ and $g : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be $\mathcal{G} \otimes \mathbb{B}$ measurable. If*

*(S1) $g$ is uniformly Lipschitz, i.e. there exists $L > 0$ such that*

$$|g(t,x) - g(t,y)| \leqslant L|x-y| \qquad \forall x,y \in \mathbb{R}, t \in [0,T] \qquad \textit{and}$$

*(S2) $g$ is $p$-integrable in zero, i.e. $g(t,0) \in L^p(\mathbb{P} \otimes \mathrm{d}t)$.*

*then $(g,\xi)$ is called a* BSDE$^p$-standard parameter.

**Theorem A.3**
*Let $(g,\xi)$ be a* BSDE$^p$*-standard parameter. Then there is one and only one cádlág semimartingale $X$ satisfying $\sup_{t\in[0,T]} \|X_t\|_{L^p(\mathbb{P})} < \infty$ that solves BSDE$(g,\xi)$.*
*Moreover, if $p > 1$, then $X$ satisfies $\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^p\right] < \infty$.*

*Proof.* For $p \geqslant 1$, a rather technical proof in a more general setting is provided by [Antonelli, 1993][Theorem 2.4]. An easier proof, but only for the case $p > 1$, can be found in [Duffie and Epstein, 1992a][Proposition A1]. □

**Lemma A.4**
*Let $p \geqslant 1$, $(g,\xi)$ be a* BSDE$^p$*-standard parameter and $X$ be a semimartingale with $\sup_{t\in[0,T]} \|X_t\|_{L^p(\mathbb{P})} < \infty$. Then $X$ is a solution of BSDE$(g,\xi)$, if and only if there exists a uniformly integrable martingale $M$ such that*

$$\mathrm{d}X_t = -g(t,X_t)\mathrm{d}t + \mathrm{d}M_t, \qquad X_T = \xi. \tag{A.1.2}$$

*If $p > 1$, then $M$ is a $L^p$-martingale and $\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^p\right] < \infty$.*

*Proof.* The Lipschitz condition in Definition A.2 implies that

$$\|g(t,X_t)\|_{L^p(\mathbb{P}\otimes\mathrm{d}t)} \leqslant LT^{\frac{1}{p}} \sup_{t\in[0,T]} \|X_t\|_{L^p(\mathbb{P})} + \|g(t,0)\|_{L^p(\mathbb{P}\otimes\mathrm{d}t)} < \infty.$$

As also $\xi \in L^p(\mathbb{P})$, we can define the uniformly integrable martingale $M$ via

$$M_t \triangleq \mathbb{E}_t\left[\int_0^T g(s,X_s)\mathrm{d}s + \xi\right], \qquad t \in [0,T].$$

Note that $M$ is a $L^p$-martingale if $p > 1$.
As $X$ is a solution to BSDE$(g,\xi)$, we obtain for all $t \in [0,T]$

$$X_t = \mathbb{E}_t\left[\int_t^T g(s,X_s)\mathrm{d}s + \xi\right] = -\int_0^t g(s,X_s)\mathrm{d}s + M_t$$

which certainly implies (A.1.2).
On the other hand, suppose that (A.1.2) holds and $M$ is at least uniformly integrable, so by integrating from $t$ to $T$ we obtain (A.1.1).
Finally we have

$$|X_t| \leqslant \mathbb{E}_t\left[\int_0^T |g(s,X_s)|\mathrm{d}s + |\xi|\right] \triangleq N_t,$$

where $N_t$ is a $L^p$-martingale if $p > 1$ by the same argument as above, hence Doob's $L^p$-inequality shows

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^p\right] \leqslant \mathbb{E}\left[\sup_{t\in[0,T]} |N_t|^p\right] \leqslant \left(\frac{p}{p-1}\right)^p \mathbb{E}[|\xi|^p] < \infty.$$

□

**Proposition A.5**
*Let $(g^n, \xi^n)$, $n \in \mathbb{N}$ and $(g, \xi)$ be a* BSDE$^p$*-standard parameter. Suppose that there is a constant $L > 0$ such that*

$$|g^n(t, x) - g^n(t, y)| \leqslant L|x - y| \text{ for all } x, y \in \mathbb{R},\ t \in [0, T]$$

*and all $n \in \mathbb{N}$. Finally denote by $X^n$, $n \in \mathbb{N}$ and $X$ the solutions to BSDE$(g^n, \xi^n)$, $n \in \mathbb{N}$ and BSDE$(g, \xi)$, respectively. If*

$$g^n(t, X_t) \to g(t, X_t) \text{ in } L^p(\mathbb{P} \otimes \mathrm{d}t) \text{ and } \xi^n \to \xi \text{ in } L^p(\mathbb{P}), \text{ as } n \to \infty,$$

*then $\sup_{t \in [0,T]} \|X_t - X_t^n\|_{L^p(\mathbb{P})} \to 0$ and if $p > 1$ also $\mathbb{E}\left[\sup_{t \in [0,T]} |X_t - X_t^n|^p\right] \to 0$.*

*Proof.* By applying Jensen's inequality twice, for any $0 \leqslant s \leqslant T$

$$\begin{aligned}
\|X_s - X_s^n\|_{L^p(\mathbb{P})} &\leqslant \left\| \int_s^T g(u, X_u) - g^n(u, X_u) \mathrm{d}u + (\xi - \xi^n) \right\|_{L^p(\mathbb{P})} \\
&\leqslant \mathbb{E}\left[ \int_s^T |g(u, X_u) - g^n(u, X_u)|^p \mathrm{d}u \right]^{\frac{1}{p}} + \|\xi - \xi^n\|_{L^p(\mathbb{P})} \\
&\leqslant \|g(t, X_t) - g^n(t, X_t)\|_{L^p(\mathbb{P} \otimes \mathrm{d}t)} + \|\xi - \xi^n\|_{L^p(\mathbb{P})}.
\end{aligned}$$

Thus

$$\sup_{t \in [0,T]} \|X_t - X_t^n\|_{L^p(\mathbb{P})} \leqslant \|g(t, X_t) - g^n(t, X_t)\|_{L^p(\mathbb{P} \otimes \mathrm{d}t)} + \|\xi - \xi^n\|_{L^p(\mathbb{P})}$$

and $\sup_{t \in [0,T]} \|X_s - X_s^n\|_{L^p(\mathbb{P})} \to 0$. If $p > 1$, similar as in Lemma A.4, Doob's $L^p$ inequality yields $\mathbb{E}\left[\sup_{t \in [0,T]} |X_t - X_t^n|^p\right] \to 0$. □

**Theorem A.6**
*Let $\xi \in L^p(\mathbb{P})$ and $\beta$, $\varphi$ be $\mathbb{R}$-valued and $\mathcal{G}$-measurable processes such that $\beta$ is bounded and $\|\varphi_t\|_{L^p(\mathbb{P} \otimes \mathrm{d}t)} < \infty$ for $p \geqslant 1$. Then the solution of the* linear *BSDE*

$$\mathrm{d}X_t = -(\varphi_t + \beta_t X_t)\mathrm{d}t + \mathrm{d}M_t, \qquad X_T = \xi \tag{A.1.3}$$

*is given by the closed formula*

$$X_t = \mathbb{E}\left[ \int_t^T e^{\int_t^s \beta_u \mathrm{d}u} \varphi_s \mathrm{d}s + e^{\int_t^T \beta_u \mathrm{d}u} \xi \right]. \tag{A.1.4}$$

*Proof.* As $\beta$ is bounded, the linear function $g(t, x) = \varphi_t + \beta_t x$ is uniformly Lipschitz, so $(g, \xi)$ is a BSDE$^p$-standard parameter and by Theorem A.3 the BSDE (A.1.3) has a unique solution $X$. First let $p > 1$, then Itô's formula, yields

$$\mathrm{d}e^{\int_0^t \beta_u \mathrm{d}u} X_t = -e^{\int_0^t \beta_u \mathrm{d}u} \varphi_t \mathrm{d}t + e^{\int_0^t \beta_u \mathrm{d}u} \mathrm{d}M_t.$$

As $\beta$ is bounded and $M$ is an $L^p$-martingale by Lemma A.4, so is $e^{\int_0^t \beta_u \mathrm{d}u} \mathrm{d}M_t$, so integrating by parts yields

$$e^{\int_0^t \beta_u \mathrm{d}u} X_t = \mathbb{E}_t\left[ \int_t^T e^{\int_0^s \beta_u \mathrm{d}u} \varphi_s \mathrm{d}s + e^{\int_0^T \beta_u \mathrm{d}u} \xi \right], \quad t \in [0, T]$$

which yields (A.1.4).
For the case $p = 1$, introduce the truncated parameters $\varphi^n \triangleq (-n \vee \varphi) \wedge n$ and $\xi^n \triangleq (-n \vee \xi) \wedge n$. Then for any $n \in \mathbb{N}$, the semimartingales

$$X_t^n = \mathbb{E}_t\left[ \int_t^T e^{\int_t^s \beta_u \mathrm{d}u} \varphi_s^n \mathrm{d}s + e^{\int_t^T \beta_u \mathrm{d}u} \xi^n \right]$$

are the unique solutions of $\mathrm{d}X_t^n = -(\varphi_t^n + \beta_t X_t^n)\mathrm{d}t + \mathrm{d}M_t^n$, $X_T^n = \xi^n$. Then Proposition A.5 implies that $\sup_{t\in[0,T]} \|X_t - X_t^n\|_{L^p(\mathbb{P})} \to 0$, where $X$ denotes the unique solution of eq. (A.1.3). But on the other hand, by dominated convergence we obtain

$$X_t^n = \mathbb{E}_t\left[\int_t^T e^{\int_t^s \beta_u \mathrm{d}u}\varphi_s^n \mathrm{d}s + e^{\int_t^T \beta_u \mathrm{d}u}\xi^n\right] \to \mathbb{E}_t\left[\int_t^T e^{\int_t^s \beta_u \mathrm{d}u}\varphi_s \mathrm{d}s + e^{\int_t^T \beta_u \mathrm{d}u}\xi\right] \quad \text{in } L^1,$$

so $X_t = \mathbb{E}_t\left[\int_t^T e^{\int_t^s \beta_u \mathrm{d}u}\varphi_s \mathrm{d}s + e^{\int_t^T \beta_u \mathrm{d}u}\xi\right]$. □

## A.2 Comparison Theorem

Lemma A.4 justifies to define Sub-and Supersolutions to BSDEs in the following way:

**Definition A.7** (BSDE - Sub-/Supersolutions)
*Let $g : \Omega\times[0,T]\times\mathbb{R} \to \mathbb{R}$ be $\mathcal{G}\otimes\mathbb{B}$-measurable. Let $\xi \in L^1(\mathbb{P})$ and suppose $X$ is a semimartingale with $\sup_{t\in[0,T]} \mathbb{E}[|X_t|] < \infty$ and moreover $\mathbb{E}[\int_0^T |g(t, X_t)|\mathrm{d}t] < \infty$. Then we call $X$ a subsolution of the BSDE with aggregator $g$ and terminal value $\xi$, if*

$$\mathrm{d}X_t = -g(t, X_t)\mathrm{d}t + \mathrm{d}M_t - \mathrm{d}A_t, \quad X_T \leqslant \xi \tag{A.2.1}$$

*where $M$ is a martingale and $A$ is a decreasing and right-continuous process such that $A_0 = 0$. We say*

$$X \text{ is a subsolution of } \mathrm{BSDE}(g,\xi)$$

*for short. Analogously $X$ is supersolution of $\mathrm{BSDE}(g,\xi)$, if $X_T \geqslant \xi$ and $A$ is increasing.*

Of course, $X$ is a solution of $\mathrm{BSDE}(g,\xi)$ as in Definition A.1, if it is a sub- and supersolution. We say that the aggregator $g$ satisfies $(M)$, if there is a constant $k > 0$ such that for a.e. $\omega \in \Omega$ and $\mathrm{d}t$ a.e. $t \in [0,T]$

$$g(\omega, t, x) - g(\omega, t, y) \leqslant k(x-y) \qquad \text{for all } x, y \in \mathbb{R} \text{ with } x \geqslant y. \tag{M}$$

The property (M) is sometimes called *monotonicity condition.* Note that the dual aggregator $f^*$ satisfies (M) as $f_v^*$ is bounded from above, see Lemma B.13.

Having introduced the notion of Sub-and Supersolutions of BSDEs, we are now able to state the crucial *comparison theorem.* Most comparison theorems for BSDEs require BSDE$^p$-standard parameters; however, as neither the Epstein-Zin aggregator nor the dual aggregator are uniformly Lipschitz, those wouldn't be applicable in our setting. In [Seiferling and Seifried, 2016], the authors provide a comparison theorem for BSDEs where the aggregators only have to satisfy the monotonicity contion $(M)$ above, which is the case for both, the primal and dual aggregator. For the theorem itself and the following prerequisites (in particular the stochastic Gronwall inequality Proposition A.10), we follow [Seiferling and Seifried, 2016], including the proofs.

**Lemma A.8**
*If $X$ is a subsolution of $\mathrm{BSDE}(g,\xi)$ satisfying $\mathbb{E}[\sup_{t\in[0,T]} |X_t|] < \infty$ and $\tau$ is a $[0,T]$-valued stopping time, we have*

$$\mathbb{I}_{\{\tau>t\}}X_t \leqslant \mathbb{E}_t\left[\mathbb{I}_{\{\tau>t\}}\int_t^\tau g(s, X_s)\mathrm{d}s + \mathbb{I}_{\{\tau>t\}}X_\tau\right], \qquad t \in [0,T]. \tag{A.2.2}$$

*If $X$ is a supersolution, then* (A.2.2) *holds with* $'' \geqslant''$.

*Proof.* Let $X$ be a subsolution of BSDE$(g, \xi)$ and let $\tau$ be a $[0, T]$-valued stopping time. Define $\sigma \triangleq \tau \wedge t$, integrate (A.2.1) from $\sigma$ to $\tau$ and use that fact that $A$ from (A.2.1) is decreasing to obtain

$$X_\sigma + (M_\tau - M_\sigma) \leqslant \int_\sigma^\tau g(s, X_s)\mathrm{d}s + X_\tau.$$

Taking $\mathfrak{F}_t$-conditional expectations and multplying with $\mathbb{I}_{\{\tau>t\}}$ yields (A.2.2). If $X$ is a supersolution, then $A$ is increasing and we obtain (A.2.2) with $'' \geqslant''$. ☐

**Lemma A.9**
*Let $X$ be a right-continuous adapted process, such that $\mathbb{E}[\sup_{t\in[0,T]} |X_t|] < \infty$. If there is a constant $k > 0$ such that*

$$X_t \leqslant k\mathbb{E}_t \left[ \int_t^T X_s \mathrm{d}s \right] \text{ for every } t \in [0, T]$$

*then $X_t \leqslant 0$ for all $t \in [0, T]$.*

*Proof.* See [Antonelli, 1996][Theorem 1.8]. ☐

**Proposition A.10**
*Let $\alpha$ be progressively measurable, let $X$ be right-continuous and adapted with $\mathbb{E}[\sup_{t\in[0,T]} |X_t|] < \infty$, and suppose that $\alpha$ is bounded from above on $\{X > 0\}$. If $X_T \leqslant 0$ and for every $[0, T]$-valued stopping time $\tau$ we have*

$$\mathbb{I}_{\{\tau>t\}} X_t \leqslant \mathbb{E}_t \left[ \mathbb{I}_{\{\tau>t\}} \int_t^\tau \alpha_s X_s \mathrm{d}s + \mathbb{I}_{\{\tau>t\}} X_\tau \right] \text{ for all } t \in [0, T], \tag{A.2.3}$$

*then $X_t \leqslant 0$ for all $t \in [0, T]$.*

*Proof.* Assume by contradiction that there exists some $u \in [0, T)$ such that the event $F \triangleq \{X_u > 0\}$ satisfies $\mathbb{P}(F) > 0$. We define a $[u, T]$-valued stopping time $\tau$ via

$$\tau = \mathbb{I}_F \inf \{t \geqslant u : X_t \leqslant 0\} + u\mathbb{I}_{F^c}$$

and observe that $X_s > 0$ on $\{\tau > s > u\}$, and $X_\tau \leqslant 0$ by right-continuity. By (A.2.3) we have for all $t \in [0, T]$

$$\begin{aligned} \mathbb{I}_{\{\tau>t\}} X_t &\leqslant \mathbb{E}_t \left[ \mathbb{I}_{\{\tau>t\}} \int_t^\tau \alpha_s X_s \mathrm{d}s + \mathbb{I}_{\{\tau>t\}} X_\tau \right] \\ &\leqslant \mathbb{E}_t \left[ \mathbb{I}_{\{\tau>t\}} \int_t^\tau \alpha_s^+ X_s \mathrm{d}s \right] \leqslant k\mathbb{E}_t \left[ \int_t^T \mathbb{I}_{\{\tau>s\}} X_s \mathrm{d}s \right], \end{aligned}$$

where $k$ is an upper bound for $\alpha$ on $\{X > 0\} = \{(\omega, s) \in \Omega \times T : X_s(\omega) > 0\}$. Then Lemma A.9 implies that $\mathbb{I}_{\{\tau>t\}} X_t \leqslant 0$ for all $t \in [u, T]$, and it follows that

$$0 < \mathbb{I}_F X_u = \mathbb{I}_{\{\tau>u\}} X_u \leqslant 0.$$

This is a contradiction to $\mathbb{P}(F) > 0$. ☐

**Theorem A.11** ([Seiferling and Seifried, 2016], Theorem 4.3)
*Suppose $X$ is a subsolution of* BSDE$(g, \xi)$ *with $\mathbb{E}[\sup_{t\in[0,T]} |X_t|] < \infty$ and $Y$ is a supersolution of* BSDE$(h, \eta)$ *with $\mathbb{E}[\sup_{t\in[0,T]} |Y_t|] < \infty$ where $\xi \leqslant \eta$.*

(a) *If $g(t, Y_t) \leqslant h(t, Y_t)$ for* d$t$ *a.e. $t \in [0, T]$ and $g$ satisfies $(M)$, then $X \leqslant Y$*

*(b) If* $g(t, X_t) \leqslant h(t, X_t)$ *for* $\mathrm{d}t$ *a.e.* $t \in [0, T]$ *and* $h$ *satisfies* $(M)$*, then* $X \leqslant Y$

*Proof.* Set $\Delta \triangleq X - Y$ and note that $\Delta_T = \xi - \eta$ and $\mathbb{E}[\sup_{t \in [0,T]} |\Delta_t|] < \infty$. If $\tau$ is a stopping time, Lemma A.8 implies

$$\mathbb{I}_{\{\tau > t\}} \Delta_t \leqslant \mathbb{E}_t \left[ \mathbb{I}_{\{\tau > t\}} \int_t^\tau (g(s, X_s) - h(s, Y_s))\, \mathrm{d}s + \mathbb{I}_{\{\tau > t\}} \Delta_\tau \right] \text{ for all } t \in [0, T].$$

To prove $(a)$, define a progressively measurable process $\alpha$ via

$$\alpha_s \triangleq \mathbb{I}_{\{X_s \neq Y_s\}} \frac{g(s, X_s) - h(s, Y_s)}{\Delta_s} \text{ for } s \in [0, T]$$

and note that $\alpha$ is bounded above on $\{\Delta > 0\} = \{(\omega, s) \in \Omega \times [0, T] : \Delta_s(\omega) > 0\}$ by $(M)$. Since $g(s, Y_s) \leqslant h(s, Y_s)$ for $\mathrm{d}s$-a.e. $s \in [0, T]$, it follows that

$$\mathbb{I}_{\{\tau > t\}} \Delta_t \leqslant \mathbb{E}_t \left[ \mathbb{I}_{\{\tau > t\}} \int_t^\tau \alpha_s \Delta_s \mathrm{d}s + \mathbb{I}_{\{\tau > t\}} \Delta_\tau \right] \text{ for } t \in [0, T]$$

and Proposition A.10 yields the desired conclusion. The proof of $(b)$ is analogous. $\square$

# Appendix B

# Legendre-Fenchel Dualization

## B.1 A Very Short Introduction to Conjugates of Convex Functions

In the following we provide a quick overview on the basic theory of convex functions and their conjugates based on Section 12 in [Rockafellar, 1997]. Note that after the appropriate adjustments all the results given in this section carry over to concave functions.
We follow the abstract approach of [Rockafellar, 1997] in hope to give the reader a more pictorial intuition of the concept, but restrict ourselves to the one-dimensional setting as this is all we need within this thesis. We begin with some vocabulary around the topic.
Let $h : S \to \bar{\mathbb{R}}$ be a function, where $S$ is a subset of $\mathbb{R}$ and $\bar{\mathbb{R}}$ denotes the extended real line. Then the set

$$\operatorname{epi} h = \{(x, \mu) : x \in S, \mu \in \mathbb{R}, \mu \geqslant h(x)\}$$

is called the *epigraph* of $h$. A function $h$ is said to be *convex* on $S$ if $\operatorname{epi} h$ is a convex subset of $\mathbb{R}^2$. A function $h$ is said to be *concave* on $S$ if its negative is convex. An *affine* function on $S$ is a finite, convex *and* concave function.
We define the *effective domain* of a convex function $h$ on $S$ as the projection of the epigraph of $h$ on $\mathbb{R}$, i.e.

$$\operatorname{dom} h = \{x \in S : \exists \mu, (x, \mu) \in \operatorname{epi} h\} = \{x \in S : h(x) < \infty\}.$$

A convex function $h$ is said to be *proper* if its epigraph is non-empty and contains no vertical lines, i.e. if $h(x) < \infty$ for at least one $x$ and $h(x) > -\infty$ for every $x$. The *relative interior* of a convex set $C$ in $\mathbb{R}$, which we denote by $\operatorname{ri} C$ is defined as the interior which results when $C$ is regarded as a subset of its affine hull $\operatorname{aff} C$, i.e.

$$\operatorname{ri} C = \{x \in \operatorname{aff} C : \exists \varepsilon > 0, (x + \varepsilon B) \cap (\operatorname{aff} C) \subseteq C\},$$

where $B$ is the Euclidean unit ball. Note that in this one-dimensional setting, the only convex sets are either intervals or singletons. Thus, in our simplified setting, the relative interior corresponds to either the interior of the respective interval or the singleton itself.
An extended-real-valued function $h$ is said to be lower semi-continuous at a point $x$ if

$$h(x) \leqslant \lim_{i \to \infty} h(x_i)$$

for every sequence $(x_i)_{i \in \mathbb{N}}$ such that $\lim_{i \to \infty} x_i = x$ and $\lim_{i \to \infty} h(x_i)$ exists in $\bar{\mathbb{R}}$. A function is called lower semi-continuous if the function is lower semi-continuous at every point of its domain. The importance of semi-continuity comes from its connection to closedness of the epigraph, which is pointed out by the following theorem.

**Theorem B.1** ([Rockafellar, 1997], Theorem 7.1)
*Let $h$ be an arbitrary function from $\mathbb{R}$ to $[-\infty, \infty]$. Then the following are equivalent*

*(i) $h$ is lower semi-continuous*

*(ii) The epigraph of $h$ is a closed set in $\mathbb{R}^2$*

*Proof.* Lower semi-continuity of $h$ at $x$ can be expressed as the condition that $\mu \geqslant h(x)$ whenever $\mu = \lim_{i\to\infty} \mu_i$ and $x = \lim_{i\to\infty} x_i$ for sequences $\{\mu_i\}_{i\in\mathbb{N}}$ and $\{x_i\}_{i\in\mathbb{N}}$ such that $\mu_i \geqslant h(x_i)$ for every $i$. But this is the same as closedness of the epigraph of $h$. □

Given any function $h$ on $\mathbb{R}$, there exists a greatest lower semi-continuous function majorized by $h$ namely the function whose epigraph is the closure of the epigraph of $h$ in $\mathbb{R}^2$. In general this function is called the *lower semi-continuous hull* of $h$.
The *closure* of a convex function $h$ is defined to be the lower semi-continuous hull of $h$ if $h(x) > -\infty$ for all $x \in \mathbb{R}$ and is denoted by $\operatorname{cl} h$. Then $\operatorname{cl} h$ is another convex function and a convex function is said to be *closed* if $\operatorname{cl} h = h$. For a proper convex function, closedness is thus the same as lower semi-continuity.
The following theorem provides the basis for the conjugation of convex functions.

**Theorem B.2** ([Rockafellar, 1997], Theorem 12.1)
*A closed convex function $h$ is the pointwise supremum of the collection of all affine functions $H$ such that $H \leqslant h$.*

Now one can describe the set $H^*$ consisting of all pairs $(x^*, \mu^*)$ in $\mathbb{R}^2$ such that the affine function $H(x) = xx^* - \mu^*$ is majorized by $h$. We have $H(x) \leqslant h(x)$ for every $x$ if and only if

$$\mu^* \geqslant \sup_{x\in\mathbb{R}}\{xx^* - h(x)\}.$$

Thus $H^*$ is actually the epigraph of the function $h^*$ defined by

$$h^*(x^*) = \sup_{x\in\mathbb{R}}\{xx^* - h(x)\} = -\inf_{x\in\mathbb{R}}\{h(x) - xx^*\}.$$

This $h^*$ is called the *conjugate* of $h$. It is the pointwise supremum of affine functions $g(x^*) = xx^* - \mu$ such that $(x, \mu)$ belongs to the set $\operatorname{epi} h$. Hence $h^*$ is another convex function, in fact a closed convex function. Since $h$ is the pointwise supremum of the affine functions $H(x) = xx^* - \mu^*$ such that $(x^*, \mu^*) \in \operatorname{epi} h^*$, we have

$$h(x) = \sup_{x^*\in\mathbb{R}}\{xx^* - h^*(x^*)\} = -\inf_{x^*\in\mathbb{R}}\{h^*(x^*) - xx^*\}.$$

But this means that the conjugate $h^{**}$ of $h^*$ is $h$.

We summarize those facts in the following theorem and highlight two major insights in small corollaries thereafter.

**Theorem B.3** ([Rockafellar, 1997], Theorem 12.2)
*Let $h$ be a convex function. The conjugate function $h^*$ is then a closed convex function, proper if and only if $h$ is proper. Moreover $(\operatorname{cl} h)^* = h^*$ and $h^{**} = \operatorname{cl} h$.*

In particular Theorem B.3 justifies to speak of *duality* of (proper and closed) convex functions.

**Corollary B.4** ([Rockafellar, 1997], Corollary 12.2.1)
*The conjugacy operation $h \to h^*$ induces a symmetric one-to-one correspondence in the class of all proper and closed convex functions.*

**Remark B.5**
*Note that all functions of which we consider conjugates within this thesis are closed, proper and convex. What introduces the duality for our recursive systems in Chapter 2 is thus the duality of the proper, closed and convex aggregators in the associated BSDEs.*

In fact, one can even restrict oneself to the real interior of the effective domain to conjugate a convex function $h$, which we will do in some cases within Appendix B.2 below.

**Corollary B.6** ([Rockafellar, 1997], Corollary 12.2.2)
*For any convex function $h$ on $\mathbb{R}^n$ one actually has*

$$h^*(x^*) = \sup_{x \in \operatorname{ri} \operatorname{dom} h} \{xx^* - h(x)\}.$$

## B.2 Applications during Dualization

### Convex/Concave Conjugates and Their Properties

This subsection collects all the technical computations regarding the conjugation steps in the dualitzation procedure from Chapter 2, that have been avoided in the main text for the sake of readability.

**Lemma B.7** ([Seiferling and Seifried, 2016], Lemma A.1)
*For all $c \in (0, \infty)$ and $v \in \mathbb{V} = \{v \in \mathbb{R} : (1-\gamma)v > 0\}$ the Epstein-Zin aggregator*

$$f(c, v) = \delta \frac{c^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} ((1-\gamma)v)^{1-\frac{1}{\theta}} - \delta\theta v$$

*satisfies*

$$f_c(c, v) = \delta c^{-\frac{1}{\psi}} [(1-\gamma)v]^{1-\frac{1}{\theta}} \qquad f_{cc}(c, v) = -\tfrac{1}{\psi} \delta c^{-\frac{1}{\psi}-1} [(1-\gamma)v]^{1-\frac{1}{\theta}}$$

$$f_v(c, v) = \delta \tfrac{1-\gamma\psi}{\psi-1} c^{1-\frac{1}{\psi}} [(1-\gamma)v]^{-\frac{1}{\theta}} - \delta\theta \qquad f_{vv}(c, v) = \delta \tfrac{\gamma\psi-1}{\psi} c^{1-\frac{1}{\psi}} [(1-\gamma)v]^{-\frac{1}{\theta}-1}$$

$$f_{cv}(c, v) = \delta \tfrac{1-\gamma\psi}{\psi} c^{-\phi} [(1-\gamma)v]^{-\frac{1}{\theta}}$$

*and in particular*

$$f_c > 0, \quad f_{cc} < 0, \quad \operatorname{sign}(f_{cv}) = \operatorname{sign}(1-\gamma\psi), \quad \operatorname{sign}(f_{vv}) = \operatorname{sign}(\gamma\psi - 1).$$

*Thus $f$ is always increasing and concave with respect to c; $f$ is convex with respect to $v$ if $\gamma\psi \geqslant 1$ and concave with respect to $v$ if $\gamma\psi \leqslant 1$: $f$ is (jointly) concave if and only if $\gamma\psi \leqslant 1$ and neither convex nor concave otherwise; $f_v$ is bounded above if either $\gamma\psi \geqslant 1, \psi > 1$ or $\gamma\psi \leqslant 1, \psi < 1$ and bounded below otherwise.*

**Corollary B.8**
*Let $f$ be the Epstein-Zin aggregator as in Lemma B.7. Then for all $c \in (0, \infty)$ and $v \in \mathbb{R}$ the extended Epstein-Zin aggregator*

$$\tilde{f}(c, v) = \begin{cases} f(c, v), & (1-\gamma)v > 0 \\ f(c, 0+) + f_v(c, 0+) \cdot v, & (1-\gamma)v \leqslant 0 \end{cases},$$

*where $f(c,0+) \triangleq \lim_{(1-\gamma)v\downarrow 0} f(c,v)$, is always increasing and concave with respect to c; $\tilde{f}$ is convex with respect to $v$ if $\gamma\psi \geqslant 1$ and concave with respect to $v$ if $\gamma\psi \leqslant 1$. In any case $\tilde{f}(c,\cdot)$ is proper and closed.*

*Proof.* In the easiest case $\gamma\psi = 1$ we have

$$\tilde{f}(c,v) = \delta\frac{c^{1-\gamma}}{1-\gamma} - \delta v.$$

If $\gamma > 1, \psi > 1$ or $\gamma < 1, \psi < 1$, then $\theta < 0$ and in particular $1 - \frac{1}{\theta} > 1$, which yields

$$\tilde{f}(c,v) = \begin{cases} f(c,v), & (1-\gamma)v > 0 \\ -\delta\theta v, & (1-\gamma)v \leqslant 0 \end{cases}.$$

If on the other hand $\gamma < 1, \gamma\psi > 1$ or $\gamma > 1, \gamma\psi < 1$, then $\theta \in (0,1)$ and in particular $1 - \frac{1}{\theta} < 0$, thus

$$\tilde{f}(c,v) = \begin{cases} f(c,v), & (1-\gamma)v > 0 \\ (1-\gamma)\cdot\infty, & (1-\gamma)v \leqslant 0 \end{cases}.$$

Thus all the claims follow immediately from Lemma B.7. □

As we only consider conjugates in the case $\gamma\psi \geqslant 1, \psi > 1$ in the main text, we also restrict ourselves to this case in the following lemmas. As always the case $\gamma\psi \leqslant 1, \psi < 1$ would be analogous under the necessary adjustments, see Remark B.12 below.

**Lemma B.9**
*Let $\gamma\psi \geqslant 1, \psi > 1$ and $F(c,u) = \inf_{v\in\mathbb{R}}\left\{\tilde{f}(c,v) + uv\right\}$ be the concave conjugate of $-\tilde{f}$, where $\tilde{f}$ is the extended Epstein-Zin aggregator as in Corollary B.8. If $\gamma\psi = 1$, then $F$ is given by*

$$F(c,u) = \begin{cases} \delta\frac{c^{1-\gamma}}{1-\gamma}, & u = \delta \\ -\infty, & \textit{else} \end{cases}.$$

*If $\gamma\psi > 1, \psi > 1$, then*

$$F(c,u) = \begin{cases} \delta^{\theta}\frac{c^{1-\gamma}}{1-\gamma}\left(\frac{u-\delta\theta}{1-\theta}\right)^{1-\theta}, & u > \delta\theta \\ 0, & u = \delta\theta \\ -\infty, & u < \delta\theta \end{cases}. \tag{B.2.1}$$

*In both cases $F(c,u)$ is concave in $c$ and $F(c,u)$ is concave in $u$. Moreover $F(c,\cdot)$ is proper and closed for all $c \in (0,\infty)$, in particular $\tilde{f}(c,v) = \sup_{u\in\mathbb{R}}\{F(c,u) - uv\}$.*

*Proof.* Let $\gamma\psi = 1$. Then $\tilde{f}(c,v) = \delta\frac{c^{1-\gamma}}{1-\gamma} - \delta v$ and $\inf_{v\in\mathbb{R}}\left\{\delta\frac{c^{1-\gamma}}{1-\gamma} + (u-\delta)v\right\}$ can be made arbitrarily small by taking $v$ arbitrarily small(large) if $u > \delta$ $(u < \delta)$, so the infimum there is $-\infty$. Only the case $u = \delta$ remains where the infimum is trivially given as $\delta\frac{c^{1-\gamma}}{1-\gamma}$.
Let $\gamma > 1, \psi > 1$. Then, as $\theta < 0$ and in particular $1 - \frac{1}{\theta} > 1$ the extended aggregator is given as

$$\tilde{f}(c,v) = \begin{cases} \delta\frac{c^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}((1-\gamma)v)^{1-\frac{1}{\theta}} - \delta\theta v, & v < 0 \\ -\delta\theta v, & v \geqslant 0 \end{cases},$$

Hence we investigate

$$\inf_{v\in\mathbb{R}} \begin{Bmatrix} \delta\frac{c^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}((1-\gamma)v)^{1-\frac{1}{\theta}}+(u-\delta\theta)v, & \text{if } v<0 \\ (u-\delta\theta)v, & \text{if } v\geqslant 0 \end{Bmatrix}.$$

Some elementary calculus shows that, if $u>\delta\theta$, then

$$\inf_{v>0}\left\{\delta\frac{c^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}((1-\gamma)v)^{1-\frac{1}{\theta}}+(u-\delta\theta)v\right\}=\delta^{\theta}\frac{c^{1-\gamma}}{1-\gamma}\left(\frac{u-\delta\theta}{1-\theta}\right)^{1-\theta}<0,$$

whereas $\inf_{v\geqslant 0}\{(u-\delta\theta)v\}=0$, implying the first case of (B.2.1). If $u=\delta\theta$, in the case $v\geqslant 0$ the infimum is trivially zero and the same goes for the case $v<0$ as $\frac{c^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}>0$ and $1-\frac{1}{\theta}>1$. This implies the second case from (B.2.1). If $u<\delta\theta$ then $(u-\delta\theta)v$ can be made arbitrarily small by making $v$ arbitrarily big, so the infimum is $-\infty$. Combining the above we obtain

$$F(c,u)=\begin{cases}\delta^{\theta}\frac{c^{1-\gamma}}{1-\gamma}\left(\frac{u-\delta\theta}{1-\theta}\right)^{1-\theta}, & \text{if } u>\delta\theta\\ 0, & \text{if } u=\delta\theta\\ -\infty, & \text{if } u<\delta\theta\end{cases}.$$

Finally, let $\gamma<1,\gamma\psi>1$. Then

$$\tilde{f}(c,v)=\begin{cases}\delta\frac{c^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}((1-\gamma)v)^{1-\frac{1}{\theta}}-\delta\theta v, & v>0\\ \infty, & v\leqslant 0\end{cases},$$

where now $\theta\in(0,1)$, in particular $1-\frac{1}{\theta}<0$. Thus by Corollary B.6

$$F(c,u)=\inf_{v\in\mathbb{R}}\left\{\tilde{f}(c,v)-uv\right\}=\inf_{v>0}\left\{\delta\frac{c^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}((1-\gamma)v)^{1-\frac{1}{\theta}}+(u-\delta\theta)v\right\}$$

and similar arguments as above show that

$$F(c,u)=\begin{cases}\delta^{\theta}\frac{c^{1-\gamma}}{1-\gamma}\left(\frac{u-\delta\theta}{1-\theta}\right)^{1-\theta}, & \text{if } u>\delta\theta\\ 0, & \text{if } u=\delta\theta\\ -\infty, & \text{if } u<\delta\theta\end{cases}.$$

Concavity of $F(\cdot,u)$ is immediate for $u\leqslant\delta\theta$ and follows for $u>\delta\theta$ from

$$F_{cc}(c,u)=-\gamma\delta^{\theta}c^{-\gamma-1}\left(\frac{u-\delta\theta}{1-\theta}\right)^{1-\theta}<0.$$

As $\tilde{f}(c,\cdot)$ is proper, convex and closed for all $c\in(0,\infty)$ by Corollary B.8, $F(c,\cdot)$ is proper, concave and closed by Theorem B.3. In particular Theorem B.3 implies $\tilde{f}(c,v)=\sup_{u\in\mathbb{R}}\{F(c,u)+uv\}$.

□

**Lemma B.10**
*Let $\gamma\psi\geqslant 1,\psi>1$ and $F^*(\lambda,u)=\sup_{c>0}\{F(c,u)-\lambda c\}$ for $\lambda\in(0,\infty)$ be minus the concave conjugate of $F$ in c, where $F$ is given in Lemma B.9. If $\gamma\psi=1$, then $F^*$ is given by*

$$F^*(\lambda,u)=\begin{cases}\delta^{\frac{1}{\gamma}}\frac{\gamma}{1-\gamma}\lambda^{\frac{\gamma-1}{\gamma}}, & u=\delta\\ -\infty, & \textit{else}\end{cases}.$$

*If $\gamma\psi > 1, \psi > 1$, then*

$$F^*(\lambda, u) = \begin{cases} \delta^{\frac{\theta}{\gamma}} \frac{\gamma}{1-\gamma} \lambda^{\frac{\gamma-1}{\gamma}} \left(\frac{u-\delta\theta}{1-\theta}\right)^{\frac{1-\theta}{\gamma}}, & u > \delta\theta \\ 0, & u = \delta\theta \\ -\infty, & u < \delta\theta \end{cases}.$$

*In particular $F^*(\lambda, u)$ is always convex in $\lambda$ and $F^*(\lambda, u)$ is concave in $u$.*

*Proof.* The representation of $F^*$ in the case $u > \delta\theta$ follows by basic calculus. If $u = \delta\theta$ then $\sup_{c>0}\{-\lambda c\} = 0$ is immediate as $\lambda > 0$. Finally the case $u < \delta\theta$ is trivial.
Moreover $F^*(\lambda, u)$ is convex in $\lambda$, as $-F^*$ is the concave conjugate of $F$, which is again concave by Theorem B.3. Finally $F^*(\lambda, u)$ is concave in $u > \delta\theta$ as

$$F^*_{uu}(\lambda, u) = \delta^{\frac{\theta}{\gamma}} \frac{1}{\gamma(1-\gamma\psi)} \left(\frac{u-\delta\theta}{1-\theta}\right)^{\frac{1-\theta}{\gamma}-2} < 0$$

because $\gamma\psi > 1$. □

**Lemma B.11**
*Let $\Phi^* : (0, \infty) \to \mathbb{R}, \lambda \mapsto \sup_{c>0} \{\Phi(c) - \lambda c\}$ be the Legendre-Fenchel transform of the terminal utility function $\Phi(c) = \varepsilon \frac{1}{1-\gamma} c^{1-\gamma}$. Then*

$$\Phi^*(\lambda) = \varepsilon^{\frac{1}{\gamma}} \frac{\gamma}{1-\gamma} \lambda^{\frac{\gamma-1}{\gamma}}.$$

*Moreover $\Phi^*$ is convex and decreasing in $\lambda$.*

*Proof.* The formula for $\Phi^*$ is basic calculus, convexity follows by Theorem B.3. □

**Remark B.12**
*Note that in Lemma B.9 and Lemma B.10 there doesn't change much in the concave case. If $\gamma\psi \leqslant 1, \psi < 1$, the conjugates stay essentially the same, only the $-\infty$ becomes $\infty$ and both functions are now* convex *in $u$. In particular, in the concave and convex case the conjugates coincide on their domain.*
*Finally $\Phi^*$ is the same in both the concave* and *convex case.* △

**Lemma B.13**
*For all $\lambda \in (0, \infty)$ and $\nu \in \mathbb{V}$, the dual aggregator $f^*$ given by*

$$f^*(\lambda, \nu) \triangleq \delta^{\psi} \frac{\lambda^{1-\psi}}{\psi-1} \left(\frac{(1-\gamma)}{\gamma}\nu\right)^{1-\frac{\gamma\psi}{\theta}} - \frac{\delta\theta}{\gamma}\nu.$$

*satisfies*

$$f^*_{\lambda}(\lambda, \nu) = -\delta^{\psi}\lambda^{-\psi} \left(\frac{1-\gamma}{\gamma}\nu\right)^{1-\frac{\gamma\psi}{\theta}} \qquad f^*_{\lambda\lambda}(\lambda, \nu) = \delta^{\psi}\psi\lambda^{-\psi-1} \left(\frac{1-\gamma}{\gamma}\nu\right)^{1-\frac{\gamma\psi}{\theta}}$$

$$f^*_{\nu}(\lambda, \nu) = \delta^{\psi} \frac{1-\gamma\psi}{\gamma(\psi-1)} \lambda^{1-\psi} \left(\frac{1-\gamma}{\gamma}\nu\right)^{-\frac{\gamma(\psi-1)}{1-\gamma}} - \frac{\delta\theta}{\gamma}$$

$$f^*_{\lambda\nu}(\lambda, \nu) = \delta^{\psi} \frac{\gamma\psi-1}{\gamma} \lambda^{-\psi} \left(\frac{1-\gamma}{\gamma}\nu\right)^{-\frac{\gamma(\psi-1)}{1-\gamma}} \qquad f^*_{\nu\nu}(\lambda, \nu) = \delta^{\psi} \frac{\gamma\psi-1}{\gamma} \lambda^{1-\psi} \left(\frac{1-\gamma}{\gamma}v\right)^{-\frac{\gamma(\psi-1)}{1-\gamma}-1}$$

*In particular*

$$f^*_{\lambda} < 0, \quad f^*_{\lambda\lambda} > 0, \quad \operatorname{sign}(f^*_{\lambda\nu}) = \operatorname{sign}(\gamma\psi - 1), \quad \operatorname{sign}(f^*_{\nu\nu}) = \operatorname{sign}(\gamma\psi - 1).$$

*Thus $f^*$ is always decreasing and convex with respect to $\lambda$; $f^*$ is convex with respect to $\nu$ if $\gamma\psi \geqslant 1$ and concave with respect to $\nu$ if $\gamma\psi \leqslant 1$; $f^*$ is (jointly) convex if and only if $\gamma\psi \geqslant 1$ and neither convex nor concave otherwise; $f^*_\nu$ is bounded above if either $\gamma\psi, \psi \geqslant 1$ or $\gamma\psi, \psi \leqslant 1$ and bounded below otherwise.*

*Proof.* We have

$$\left(f^*_{\lambda\lambda} \cdot f^*_{\nu\nu}\right)(\lambda,\nu) - {f^*}^2_{\lambda\nu}(\lambda,\nu) = \delta^{2\psi}\lambda^{-2\psi}\frac{\gamma\psi - 1}{\gamma^2}\left[\frac{1-\gamma}{\gamma}\nu\right]^{-2\frac{\gamma\psi}{\theta}}.$$

Since $f^*_{\lambda\lambda} > 0$, the Hessian of $f^*$ is positive semi-definite if and only if $\gamma\psi \geqslant 1$ and indefinite otherwise. Finally, $f^*_\nu \leqslant 0$ if and only if $\gamma\psi, \psi \leqslant 1$ or $\gamma\psi, \psi \geqslant 1$. □

**Lemma B.14**
*Let $\gamma\psi \geqslant 1, \psi > 1$. For all $\lambda \in (0,\infty)$ and $\nu \in \mathbb{R}$, the function $\tilde{f}^*(\lambda,\nu) = \sup_{u\in\mathbb{R}}\{F^*(\lambda,u) - u\nu\}$, i.e. minus the concave of $F^*$ as given in Lemma B.10 is given as*

$$\tilde{f}^*(\lambda,\nu) = \begin{cases} f^*(\lambda,\gamma\nu), & (1-\gamma)\nu > 0 \\ f^*(\lambda,0+) + f^*_\nu(\lambda,0+)\cdot\nu, & (1-\gamma)\nu \leqslant 0 \end{cases}, \tag{B.2.2}$$

*where*

$$f^*(\lambda,\nu) \triangleq \delta^\psi \tfrac{\lambda^{1-\psi}}{\psi-1}\left(\tfrac{(1-\gamma)}{\gamma}\nu\right)^{1-\frac{\gamma\psi}{\theta}} - \frac{\delta\theta}{\gamma}\nu.$$

*In particular, $\tilde{f}^*(\lambda,\nu)$ is convex with respect to $\lambda$, closed and convex with respect to $\nu$.*

*Proof.* Straightforward calculation similar as in Lemma B.9 shows that $\tilde{f}^*$ has three different forms:

$$\tilde{f}^*(\lambda,\nu) = \begin{cases} \delta^\psi\frac{\lambda^{1-\psi}}{\psi-1}((1-\gamma)\nu)^{1-\frac{\gamma\psi}{\theta}} - \delta\theta\nu, & (1-\gamma)\nu > 0 \\ -\delta\theta\nu, & (1-\gamma)\nu \leqslant 0 \end{cases} \qquad \text{if } \gamma > 1, \psi > 1,$$

$$\tilde{f}^*(\lambda,\nu) = \begin{cases} \delta^\psi\frac{\lambda^{1-\psi}}{\psi-1}((1-\gamma)\nu)^{1-\frac{\gamma\psi}{\theta}} - \delta\theta\nu, & (1-\gamma)\nu > 0 \\ \infty, & (1-\gamma)\nu \leqslant 0 \end{cases} \qquad \text{if } \gamma < 1, \gamma\psi > 1$$

and

$$\tilde{f}^*(\lambda,\nu) = \delta^\psi\frac{\lambda^{1-\psi}}{\psi-1} - \delta\nu \qquad \text{if } \gamma\psi = 1.$$

Then (B.2.2) follows by just comparing the limit therein with the three different cases. Now convexity in $\lambda$ is immediate by Lemma B.13; closedness and convexity in $\nu$ follows by definition of minus the concave conjugate from Theorem B.3. □

**Remark B.15**
*The representation* (B.2.2) *is the same in the case $\gamma\psi \leqslant 1, \psi < 1$.* △

## Proofs Skipped in Chapter 2

**Lemma B.16**
*For any $\Lambda \in \mathcal{D}$, $u \in \mathcal{P}$ and $s,t \in [0,T]$, $s \leqslant t$ it holds*

$$\kappa^u_{t,s}F^*((\kappa^u_{t,s})^{-1}\Lambda_s, u_s) = \kappa^{\frac{u}{\gamma}}_{t,s}F^*(\Lambda_s,u_s) \qquad \text{and} \qquad \kappa^u_{t,T}\Phi^*((\kappa^u_{t,T})^{-1}\Lambda_T) = \kappa^{\frac{u}{\gamma}}_{t,T}\Phi^*(\Lambda_T).$$

*Proof.* We only consider $\gamma\psi \geqslant 1, \psi > 1$, as by Remark B.12 the concave case $\gamma\psi \leqslant 1, \psi < 1$ is essentially the same argument.
First, let $\gamma\psi > 1$ and $\psi > 1$. Then $F^*$ is given as in Lemma B.10. If $F^* \equiv 0$ or $F^* \equiv -\infty$ the result is trivial, so it suffices to consider $u > \delta\theta$ and calculate

$$\kappa^u_{t,s} F^*((\kappa^u_{t,s})^{-1}\Lambda_s, u_s) = \kappa^u_{t,s}\delta^{\frac{\theta}{\gamma}}\frac{\gamma}{1-\gamma}\left(\frac{\Lambda_s}{\kappa^u_{t,s}}\right)^{\frac{\gamma-1}{\gamma}}\left(\frac{u_s-\delta\theta}{1-\theta}\right)^{\frac{1-\theta}{\gamma}} = \left(\kappa^u_{t,s}\right)^{\frac{1}{\gamma}} F^*(\Lambda_s, u_s) = \kappa^{\frac{u}{\gamma}}_{t,s} F^*(\Lambda_s, u_s).$$

The scaling property for $\Phi^*$ and for $F^*$ in the case where $\gamma\psi = 1$ follows by the same calculation with $\left(\frac{u-\delta\theta}{1-\theta}\right)^{\frac{1-\theta}{\gamma}}$ replaced by 1. For $\Phi^*$ consider the same calculation with $\left(\frac{u-\delta\theta}{1-\theta}\right)^{\frac{1-\theta}{\gamma}}$ replaced by 1 and $\delta^{\frac{\theta}{\gamma}}$ replaced by $\varepsilon^{\frac{1}{\gamma}}$. □

**Analogue of Dual Variational Representation in the Concave Case**

Within this paragraph we want to prove the analogoue of Lemma 2.10 in the case of concave aggregators, thus let $\gamma\psi \leqslant 1, \psi < 1$ and note that

$$\{\gamma\psi \leqslant 1, \psi < 1\} = \{\gamma < 1, \psi < 1\} \cup \{\gamma > 1, \gamma\psi \leqslant 1\}.$$

To avoid going through all computations needed for the conjugations as in the previous section, we start from the dual side, i.e. with the general form of $\tilde{f}^* : (0,\infty) \times \mathbb{R} \to [-\infty, \infty)$ given as

$$\tilde{f}^*(\lambda,\nu) = \begin{cases} f^*(\lambda, \gamma\nu), & (1-\gamma)\nu > 0 \\ f^*(\lambda, 0+) + f^*_\nu(\lambda, 0+)\cdot\nu, & (1-\gamma)\nu \leqslant 0. \end{cases}$$

Note that the only difference is that $-\infty$ is now included, whereas $\infty$ is excluded from the image set. Again, $\tilde{f}^*$ takes three different forms:

$$\tilde{f}^*(\lambda,\nu) = \begin{cases} \delta^\psi \frac{\lambda^{1-\psi}}{\psi-1}((1-\gamma)\nu)^{1-\frac{\gamma\psi}{\theta}} - \delta\theta\nu, & (1-\gamma)\nu > 0 \\ -\delta\theta\nu, & (1-\gamma)\nu \leqslant 0 \end{cases} \qquad \text{if } \gamma < 1, \psi < 1,$$

$$\tilde{f}^*(\lambda,\nu) = \begin{cases} \delta^\psi \frac{\lambda^{1-\psi}}{\psi-1}((1-\gamma)\nu)^{1-\frac{\gamma\psi}{\theta}} - \delta\theta\nu, & (1-\gamma)\nu > 0 \\ -\infty, & (1-\gamma)\nu \leqslant 0 \end{cases} \qquad \text{if } \gamma > 1, \gamma\psi < 1$$

and

$$\tilde{f}^*(\lambda,\nu) = \delta^\psi \frac{\lambda^{1-\psi}}{\psi-1} - \delta\nu \qquad \text{if } \gamma\psi = 1.$$

In particular, $\tilde{f}^*(c,\cdot)$ is now a proper, upper semi-continuous (hence closed) and concave function, hence all results from Appendix B.1 apply under the appropriate adjustments.
Then similar as in Lemma B.9 one finds that the convex conjugate of $-\tilde{f}^*$

$$F^* : (0,\infty) \times \mathbb{R} \to (-\infty, \infty],\ (\lambda, u) \mapsto \sup_{\nu\in\mathbb{R}} \left\{\tilde{f}^*(\lambda,\nu) + u\nu\right\}$$

is given by

$$F^*(\lambda, u) = \begin{cases} \delta^{\frac{1}{\gamma}} \frac{\gamma}{1-\gamma}\lambda^{\frac{\gamma-1}{\gamma}}, & u = \delta \\ \infty, & \text{else} \end{cases} \qquad \text{if } \gamma\psi = 1$$

and by

$$F^*(\lambda, u) = \begin{cases} \delta^\theta \frac{c^{1-\gamma}}{1-\gamma}\left(\frac{u-\delta\theta}{1-\theta}\right)^{1-\theta}, & u > \delta\theta \\ 0, & u = \delta\theta \\ \infty, & u < \delta\theta \end{cases} \qquad \text{if } \gamma\psi < 1, \psi < 1.$$

Again, $F^*$ and $\tilde{f}^*$ are dual in the sense that $\tilde{f}^*$ is minus the convex conjugate of $F^*$, i.e.

$$f^*(\lambda,\nu) = \inf_{u\in\mathbb{R}}\{F^*(\lambda,u) - u\nu\}. \tag{B.2.3}$$

Then the stochastic variatonal dual is defined exactly as before as

$$U_t^*\left[\Lambda,u\right] \triangleq \mathbb{E}_t\left[\int_t^T \kappa_{t,s}^{\frac{u}{\gamma}} F^*(\Lambda_s,u_s)\mathrm{d}s + \kappa_{t,T}^{\frac{u}{\gamma}}\Phi^*(\Lambda_T)\right] \tag{B.2.4}$$

for all $t \in [0,T]$, only now with $F^*$ as above and $\Phi^*(\lambda) = \varepsilon^{\frac{1}{\gamma}}\frac{\gamma}{1-\gamma}\lambda^{\frac{\gamma-1}{\gamma}}$ as before.

**Lemma B.17** (Concave analogue to Lemma 2.10)
*Let $\gamma\psi \leqslant 1, \psi < 1$ and for any $u \in \mathcal{P}$ and $\Lambda \in \mathcal{D}^a$, let $V^*[\Lambda]$ be the stochastic differential dual associated with $\Lambda$ and $U^*[\Lambda,u]$ given as in* (B.2.4)*. Then for any $t \in [0,T]$,*

$$V_t^*[\Lambda] = \operatorname*{ess\,inf}_{u\in\mathcal{P}} U_t^*\left[\Lambda,u\right].$$

*Proof.* The proof is similar to the one of lemma 2.4, but all arguments are mirrored on the concave situation. First, let $\gamma\psi = 1$. Then the Legendre-Fenchel transform $F^*$ desintegrates to

$$F^*(\lambda,u) = \begin{cases} \delta^{\frac{1}{\gamma}}\frac{\gamma}{1-\gamma}\lambda^{\frac{\gamma-1}{\gamma}} & u=\delta \\ \infty & else \end{cases},$$

see above. In particular

$$\begin{aligned}\inf_{u\in\mathcal{P}} U_t^*\left[\Lambda,u\right] &= \inf_{u\in\mathcal{P}}\mathbb{E}_t\left[\int_t^T \kappa_{t,s}^{\frac{u}{\gamma}} F^*(\Lambda_s,u_s)\mathrm{d}s + \kappa_{t,T}^{\frac{u}{\gamma}}\Phi^*(\Lambda_T)\right] \\ &= \mathbb{E}_t\left[\int_t^T \delta^{\frac{1}{\gamma}} e^{-\frac{\delta}{\gamma}(s-t)}\frac{\gamma}{1-\gamma}\Lambda_s^{\frac{\gamma-1}{\gamma}}\mathrm{d}s + e^{-\frac{\delta}{\gamma}(T-t)}\Phi^*(\Lambda_T)\right],\end{aligned}$$

so this case follows by (2.2.26), respectively standard results on linear BSDE as Example 2.9. For the remaining parameter constellations $F^*$ is given by

$$F^*(\lambda,u) = \begin{cases} \delta^{\theta}\frac{c^{1-\gamma}}{1-\gamma}\left(\frac{u-\delta\theta}{1-\theta}\right)^{1-\theta}, & u > \delta\theta \\ 0, & u = \delta\theta \\ \infty, & u < \delta\theta \end{cases}.$$

Note that it suffices to focus on $u \in \mathcal{P}$ such that $U_0^*[\Lambda,u] < \infty$, so $u < \delta\theta$ is automatically excluded and we can without loss restrict ourselves to the space $\mathcal{U} = \{u \in \mathcal{P} : u \geqslant \delta\theta\}$. We divide the proof into three major steps:

1. *Class (D) property of $\kappa_{0,\cdot}^{\frac{u}{\gamma}}U^*[\Lambda,u]$:* We have

$$\{\gamma\psi < 1, \psi < 1\} = \{\gamma < 1, \psi < 1\} \cup \{\gamma > 1, \gamma\psi < 1\}.$$

and we split this part into two cases.
*Case 1: $\gamma < 1, \psi < 1$.* As $\gamma < 1$ we have $\Phi^* > 0$ and $F^* \geqslant 0$, so for $u \in \mathcal{U}$ we obtain

$$\mathbb{E}\left[\left|\int_0^T \kappa_{0,s}^{\frac{u}{\gamma}} F^*(\Lambda_s,u_s)\mathrm{d}s + \kappa_{0,T}^{\frac{u}{\gamma}}\Phi^*(\Lambda_T)\right|\right] = U_0^*[\Lambda,u] < \infty,$$

so the process $\mathbb{E}_t\left[\int_0^T \kappa_{0,s}^{\frac{u}{\gamma}} F^*(\Lambda_s, u_s)\mathrm{d}s + \kappa_{0,T}^{\frac{u}{\gamma}}\Phi^*(\Lambda_T)\right]$ is a uniformly integrable martingale, hence of class (D). As $\kappa^{\frac{u}{\gamma}}$ is bounded for any $u \in \mathcal{U}$, the class (D) property of $V^*[\Lambda]$ for $\Lambda \in \mathcal{D}^a$ implies the integrability of $\kappa_{0,T}^{\frac{u}{\gamma}}\Phi^*(\Lambda)$ and thus the class (D) property of $\mathbb{E}_t\left[\kappa_{0,T}^{\frac{u}{\gamma}}\Phi^*(\Lambda_T)\right]$. Then the inequality

$$\mathbb{E}\left[\kappa_{0,T}^{\frac{u}{\gamma}}\Phi^*(\Lambda_T)\right] \leqslant \kappa_{0,t}^{\frac{u}{\gamma}} U_t^*[\Lambda, u] \leqslant \mathbb{E}_t\left[\int_0^T \kappa_{0,s}^{\frac{u}{\gamma}} F^*(\Lambda_s, u_s)\mathrm{d}s + \kappa_{0,T}^{\frac{u}{\gamma}}\Phi^*(\Lambda_T)\right]$$

implies the class (D) property of $\kappa_{0,\cdot}^{\frac{u}{\gamma}} U^*[\Lambda, u]$.

*Case 2:* $\gamma > 1, \gamma\psi < 1$. Now $\Phi^* < 0$ and $F^* \leqslant 0$, and we have to show $U^*[\Lambda, u] > -\infty$. To this end let $\Lambda \in \mathcal{D}^a$ and $u \in \mathcal{U}$ and recall that

$$V^*[\Lambda] = \mathbb{E}_t\left[\int_t^T f^*(\Lambda_s, V_s[c])\mathrm{d}s + \Phi^*(\Lambda_T)\right], \qquad t \in [0, T],$$

where $f^*(\lambda, \nu) = \tilde{f}^*\big|_{(1-\gamma)v>0}\left(\lambda, \frac{1}{\gamma}\nu\right)$. Thus by the class (D) property of $V^*[\Lambda]$, the process

$$M^* \triangleq V^*[\Lambda] + \int_0^\cdot f^*(\Lambda_s, V_s^*[\Lambda])\mathrm{d}s \tag{B.2.5}$$

defines a uniformly integrable martingale. An application of Itô's formula yields

$$\begin{aligned}
\mathrm{d}\left(\kappa_{0,t}^{\frac{u}{\gamma}} V_t^*[\Lambda]\right) &= \kappa_{0,t}^{\frac{u}{\gamma}}\mathrm{d}M_t^* - \left(\kappa_{0,t}^{\frac{u}{\gamma}} f^*(\Lambda_t, V_t^*[\Lambda]) - \tfrac{u_t}{\gamma}\kappa_{0,t}^{\frac{u}{\gamma}} V_t^*[\Lambda]\right)\mathrm{d}t \\
&= \kappa_{0,t}^{\frac{u}{\gamma}}\mathrm{d}M_t^* - \mathrm{d}A_t^{\frac{u}{\gamma}} - \kappa_{0,t}^{\frac{u}{\gamma}} F^*(\Lambda_t, u_t)\mathrm{d}t,
\end{aligned}$$

where

$$\mathrm{d}A_t^{\frac{u}{\gamma}} \triangleq \kappa_{0,t}^{\frac{u}{\gamma}}\left(f^*(\Lambda_t, V_t^*[\Lambda]) - (F^*(\Lambda_t, u_t) - \tfrac{u_t}{\gamma}V_t^*[\Lambda]\right)\mathrm{d}t. \tag{B.2.6}$$

By the definition of $f^*$ and $F^*$ respectively, we have $f^*(\lambda, \nu) = \inf_{u > \delta\theta}\left\{F^*(\lambda, u) - u\frac{\nu}{\gamma}\right\}$. As $\kappa^{\frac{u}{\gamma}} > 0$ for all $u \in \mathcal{U}$, this implies that $A^{\frac{u}{\gamma}}$ is decreasing, so $\kappa_{0,\cdot}^{\frac{u}{\gamma}} V^*[\Lambda] + \int_0^\cdot \kappa_{0,s}^{\frac{u}{\gamma}} F^*(\Lambda_s, u_s)\mathrm{d}s$ is a local submartingale. Taking a localizing sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times, the submartingale property of the stopped process implies

$$V_0^*[\Lambda] \leqslant \mathbb{E}\left[\kappa_{0,\tau_n\wedge T}^{\frac{u}{\gamma}} V_{\tau_n\wedge T}^*[\Lambda] + \int_0^{\tau_n \wedge T} \kappa_{0,s}^{\frac{u}{\gamma}} F^*(\Lambda_s, u_s)\mathrm{d}s\right].$$

Since $V^*[\Lambda]$ is of class (D) and $F^* < 0$, taking the limit on the right hand side yields

$$\mathbb{E}\left[\int_0^T \kappa_{0,s}^{\frac{u}{\gamma}} F^*(\Lambda_s, u_s)\mathrm{d}s\right] > -\infty$$

by the monotone convergence theorem. As in the first case, the class (D) property of $\kappa_{0,t}^{\frac{u}{\gamma}} V^*[\Lambda]$ implies $\mathbb{E}\left[\kappa_{0,T}^{\frac{u}{\gamma}}\Phi^*(\Lambda_T)\right] > -\infty$, so in total we obtain $U_0^*[\Lambda] > -\infty$.
Now similar as in the first case we obtain

$$\mathbb{E}_t\left[\int_0^T \kappa_{0,s}^{\frac{u}{\gamma}} F^*(\Lambda_s, u_s)\mathrm{d}s + \kappa_{0,T}^{\frac{u}{\gamma}}\Phi^*(\Lambda_T)\right] \leqslant \kappa_{0,t}^{\frac{u}{\gamma}} U_t^*[\Lambda, u] \leqslant \mathbb{E}_t\left[\kappa_{0,T}^{\frac{u}{\gamma}}\Phi^*(\Lambda_T)\right]$$

and therefore the class (D) property of $\kappa_{0,\cdot}^{\frac{u}{\gamma}} U^*[\Lambda, u]$.

2. $V_t^*[\Lambda] \leqslant U_t^*[\Lambda, u] \, \forall t \in [0,T]$ *a.s.:* The tower property of conditional expectation implies that the process

$$M^{\frac{u}{\gamma}} \triangleq \kappa_{0,\cdot}^{\frac{u}{\gamma}} U^*[\Lambda, u] + \int_0^\cdot \kappa_{0,s}^{\frac{u}{\gamma}} F^*(\Lambda_s, u_s) \mathrm{d}s \tag{B.2.7}$$

is a martingale. Then a basic calculations using (B.2.5), (B.2.7) and Itô's formula yields

$$\mathrm{d}\left(\kappa_{0,t}^{\frac{u}{\gamma}} \left(V_t^*[\Lambda] - U_t^*[\Lambda, u]\right)\right) = \mathrm{d}L_t^{\frac{u}{\gamma}} - \mathrm{d}A_t^{\frac{u}{\gamma}},$$

where $\mathrm{d}L_t^{\frac{u}{\gamma}} \triangleq \kappa_{0,t}^{\frac{u}{\gamma}} \mathrm{d}M_t^* - \mathrm{d}M_t^{\frac{u}{\gamma}}$ is a local martingale and $A^{\frac{u}{\gamma}}$ as in (B.2.6) is decreasing. It follows that $\kappa_{0,\cdot}^{\frac{u}{\gamma}}(V^*[\Lambda] - U^*[\Lambda, u])$ is a local submartingale, hence a true submartingale by the class (D) property of $\kappa_{0,\cdot}^{\frac{u}{\gamma}} V^*[\Lambda]$ and $\kappa_{0,\cdot}^{\frac{u}{\gamma}} U^*[\Lambda, u]$ and it follows that

$$\kappa_{0,t}^{\frac{u}{\gamma}}(V_t^*[\Lambda] - U_t^*[\Lambda, u]) \leqslant \mathbb{E}_t\left[\kappa_{0,T}^{\frac{u}{\gamma}}(V_T^*[\Lambda] - U_T^*[\Lambda, u])\right] = \mathbb{E}_t\left[\kappa_{0,T}^{\frac{u}{\gamma}}(\Phi^*(\Lambda_T) - \Phi^*(\Lambda_T))\right] = 0.$$

As $\kappa_{0,t}^{\frac{u}{\gamma}} > 0$ for all $t \in [0,T]$, this implies $V_t^*[\Lambda] \leqslant U_t^*[\Lambda, u]$ almost surely for all $t \in [0,T]$ and any $u \in \mathcal{U}$. By right-continuity of the processes it further follows that $V_t^*[\Lambda] \leqslant U_t^*[\Lambda, u]$ for all $t \in [0,T]$ almost surely.

3. $V_t^*[\Lambda] \geqslant \operatorname{ess\,inf}_{u \in \mathcal{U}} U_t^*[\Lambda, u]$*:* To finalize the proof it suffices to identify $u \in \mathcal{U}$ such that $V^*[\Lambda] \geqslant U^*[\Lambda, u]$. Motivated by (B.2.3) we choose

$$u^\Lambda \triangleq -\frac{\partial}{\partial \nu} \tilde{f}^*\Big|_{(1-\gamma)\nu > 0}\left(\lambda, \tfrac{1}{\gamma}\nu\right) = -\delta^\psi \frac{1-\gamma\psi}{\gamma(\psi-1)} \lambda^{1-\psi} \left(\tfrac{1-\gamma}{\gamma}\nu\right)^{-\frac{\gamma(\psi-1)}{1-\gamma}} + \delta\theta$$

and as $\gamma\psi < 1$ and $\psi > 1$ we find $u^\Lambda \in \mathcal{U}$. Moreover $f^*(\Lambda, V^*[\Lambda]) = F^*(\Lambda, u^\Lambda) - \frac{u^\Lambda}{\gamma} V^*[\Lambda]$ by the first order conditions in (B.2.3), so clearly $A^{\frac{u^\Lambda}{\gamma}} \equiv 0$. Thus similar as above

$$\mathrm{d}\left(\kappa_{0,t}^{\frac{u^\Lambda}{\gamma}} \left(U_t^*[\Lambda, u^\Lambda] - V_t^*[\Lambda]\right)\right) = \mathrm{d}L_t^{\frac{u^\Lambda}{\gamma}}$$

is a local martingale, bounded from above since $U_t^*[\Lambda, u] \geqslant V_t^*[\Lambda]$. Hence it is in fact a submartingale and it follows that

$$\kappa_{0,t}^{\frac{u}{\gamma}}(U_t^*[\Lambda, u^\Lambda] - V_t^*[\Lambda]) \leqslant \mathbb{E}_t\left[\kappa_{0,T}^{\frac{u}{\gamma}}(\Phi^*(\Lambda_T) - \Phi^*(\Lambda_T))\right] = 0$$

and it follows that $V_t^*[\Lambda] \geqslant U_t^*[\Lambda, u^\Lambda]$ almost surely, $t \in [0,T]$. Again, due to right-continuity of the processes we obtain $V_t^*[\Lambda] \geqslant U_t^*[\Lambda, u^\Lambda]$ for all $t \in [0,T]$ almost surely.

□

**Remark B.18**
*Certainly, as the proof of Lemma 2.4 carries over to Lemma 2.10, the proof of Lemma B.17 carries over to the concave analogue of Lemma 2.4 under reversed adjustments. As the second step in the duality procedure remains the same in the concave aggregator setting, we have all together shown the validity of the whole duality scheme in both cases,* $\gamma\psi \geqslant 1, \psi > 1$ *and* $\gamma\psi \leqslant 1, \psi < 1$. △

# Appendix C

# Hamilton-Jacobi-Bellman Equations

Within this part of the appendix we perform all calculations necessary to find and solve the Hamiton-Jacobi-Bellman equations for the general primal and dual optimization problem from Section 4.2 and Section 4.3, respectively.

## C.1 Primal Hamilton-Jacobi-Bellman Equation

Recall that the dynamics of the assets and underlying state processes are in general form given as

$$\mathrm{d}S^{\text{risky}} = \operatorname{diag}[S^{\text{risky}}] \left(\mu^S \mathrm{d}t + \Sigma^S \mathrm{d}W\right),$$

$$\mathrm{d}Y = \mu^Y \mathrm{d}t + \Sigma^Y \mathrm{d}W,$$

where $W$ is an $(m+n)$-dimensional Brownian motion, $\mu^S$, $\mu^Y$, $\Sigma^S$ and $\Sigma^Y$ are matrix functions of $Y$ in the appropriate dimension (c.f. (4.1.1)). We avoid to write the $Y$ dependence explicitly to keep notation simple.
Moreover the investors wealth process $X^{(\pi,c)}$ for any strategy $(\pi, c) \in \mathcal{A}$ is given by

$$\mathrm{d}X_t^{(\pi,c)} = X_t^{(\pi,c)} \left(\left(r + \pi_t^\top \chi\right) \mathrm{d}t + \pi_t^\top \Sigma^S \mathrm{d}W_t\right) - c_t \mathrm{d}t, \qquad X_0^{(\pi,c)} = x,$$

where $\pi$ denotes the proportions of the investors total wealth invested in the risky securities, $c$ is her consumption rate and $\chi = \mu^S - r\mathbf{1}_m$ is the excess return of the risky assets. To simplify notation during our calculations below, we will just write $X$ for the investors wealth, suppressing the dependence on a particular strategy $(\pi, c)$.

The investor chooses between investment and consumption to maximize her continuous time recursive utility

$$V_0 = \sup_{(\pi,c)\in\mathcal{A}} V_0[c] = \sup_{(\pi,c)\in\mathcal{A}} \mathbb{E}\left[\int_0^T f(c_s, V_s[c])\mathrm{d}s + \Phi(c_T)\right]$$

where, the Epstein-Zin aggregator $f$ is given as

$$f(c,v) = \delta \tfrac{1}{1-\phi} c^{1-\phi} ((1-\gamma)v)^{1-\frac{1}{\theta}} - \delta\theta v \qquad \text{and} \qquad \Phi(c) = \varepsilon \tfrac{1}{1-\gamma} c^{1-\gamma}.$$

We define the $(1+n)$-dimensional process $Z = (X, Y)^\top$ with dynamics given by

$$\mathrm{d}Z = \operatorname{diag}[Z] \left(\mu^Z \mathrm{d}t + \Sigma^Z \mathrm{d}W\right),$$

where
$$\mu^Z = \begin{pmatrix} X(r+\pi^\top\chi)-c \\ \mu^Y \end{pmatrix} \qquad \text{and} \qquad \Sigma^Z = \begin{pmatrix} X\pi^\top\Sigma^S \\ \Sigma^Y \end{pmatrix}.$$
Considering the indirect utility $V_t = G(t, X_t, Y_t)$, the dynamic programming equation for the agents optimization problem reads
$$0 = \sup_{(\pi,c)\in\Gamma(x)} \left\{ G_t + \left(\mu^Z\right)^\top G_z + \tfrac{1}{2}\text{trace}\left[\left(\Sigma^Z\right)^\top G_{zz}\Sigma^Z\right] + f(c,G) \right\} \tag{C.1.1}$$
with $G_z = (G_x, G_{y_1}, \ldots, G_{y_n})^\top$ and $G_{zz} = \begin{pmatrix} G_{xx} & G_{xy} \\ G_{yx} & G_{yy}\end{pmatrix}$ where
$$G_{yx} = \begin{pmatrix} G_{y_1x} \\ G_{y_2x} \\ \vdots \\ G_{y_nx} \end{pmatrix} = (G_{xy})^\top \qquad \text{and} \qquad G_{yy} = \begin{pmatrix} G_{y_1y_1} & \cdots & G_{y_1y_n} \\ \vdots & \ddots & \vdots \\ G_{y_ny_1} & \cdots & G_{y_ny_n} \end{pmatrix}.$$
By inserting the matrices of differentials and the definition of $\Sigma^Z$ we calculate
$$\text{trace}\left[(\Sigma^Z)^\top G_{zz}\Sigma^Z\right] = x^2 G_{xx}\pi^\top\Sigma^S\left(\Sigma^S\right)^\top\pi + 2xG_{xy}\Sigma^Y\left(\Sigma^S\right)^\top\pi + \text{trace}\left[\left(\Sigma^Y\right)^\top G_{yy}\Sigma^Y\right],$$
where we in particular used the invariance of the trace operator under circular shifts. Thus (C.1.1) unfolds to
$$\begin{aligned} 0 = \sup_{(\pi,c)\in\Gamma(x)} \Big\{ & G_t + \left(x(r+\pi^\top\chi)-c\right)G_x + (\mu^Y)^\top G_y + \tfrac{1}{2}x^2\pi^\top\Sigma^S\left(\Sigma^S\right)^\top\pi G_{xx} \\ & + xG_{xy}\Sigma^Y\left(\Sigma^S\right)^\top\pi + \tfrac{1}{2}\text{trace}\left[\left(\Sigma^Y\right)^\top G_{yy}\Sigma^Y\right] + f(c,G) \Big\} \end{aligned} \tag{C.1.2}$$
with terminal condition $G(T,x,y) = \varepsilon\frac{1}{1-\gamma}x^{1-\gamma}$.

*Proof of Proposition 4.3:* Using the Ansatz $G(t,x,y) = \frac{1}{1-\gamma}x^{1-\gamma}g(t,y)^k$ for some $k\in\mathbb{R}$ we obtain the differentials
$$\begin{aligned} G_t &= k\tfrac{1}{1-\gamma}x^{1-\gamma}g(t,y)^{k-1}g_t & G_x &= x^{-\gamma}g(t,y)^k \\ G_{xx} &= -\gamma x^{-\gamma-1}g(t,y)^k & G_y &= k\tfrac{1}{1-\gamma}x^{1-\gamma}g(t,y)^{k-1}g_y \\ G_{xy} &= kx^{-\gamma}g(t,y)^{k-1}(g_y)^\top = (G_{yx})^\top & G_{yy} &= k\tfrac{1}{1-\gamma}x^{1-\gamma}g(t,y)^{k-1}\left((k-1)\frac{g_y(g_y)^T}{g} + g_{yy}\right). \end{aligned}$$
Moreover $g(T,y) = \varepsilon^{\frac{1}{k}}$. Inserting the differentials to (C.1.2), dividing by $k\frac{1}{1-\gamma}x^{1-\gamma}g(t,y)^{k-1}$ and simplifying yields
$$\begin{aligned} 0 = \sup_{(\pi,c)\in\Gamma(x)} \Big\{ & g_t + \tfrac{1-\gamma}{k}\left(r+\pi^\top\chi - \frac{c}{x} - \tfrac{\delta\theta}{1-\gamma}\right)g + \left(\mu^Y\right)^\top g_y - \tfrac{1}{2k}(1-\gamma)\gamma\pi^T\Sigma^S\left(\Sigma^S\right)^\top\pi g \\ & + (1-\gamma)(g_y)^\top\Sigma^Y\left(\Sigma^S\right)^\top\pi + \tfrac{1}{2}\text{trace}\left[\left(\Sigma^Y\right)^\top g_{yy}\Sigma^Y\right] + \tfrac{k-1}{2}\frac{1}{g}(g_y)^\top\Sigma^Y\left(\Sigma^Y\right)^\top g_y \\ & + \tfrac{\delta\theta}{k}\left(\frac{c}{x}\right)^{1-\phi}g^{1-\frac{k}{\theta}} \Big\}. \end{aligned}$$

The first order conditions for $c$ and $\pi$ imply[1]

$$\pi^* = \tfrac{1}{\gamma}\left(\Sigma^S\left(\Sigma^S\right)^\top\right)^{-1}\left(\chi + k\Sigma^S\left(\Sigma^Y\right)^\top \frac{g_y}{g}\right) \qquad \text{and} \qquad \left(\frac{c}{x}\right)^* = \delta^\psi g^{-\frac{k\psi}{\theta}}.$$

Inserting and simplifying again we obtain the partial differential equation for $g$:

$$\begin{aligned}
0 = g_t &+ \tfrac{1-\gamma}{k}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\chi^\top\left(\Sigma^S\left(\Sigma^S\right)^\top\right)^{-1}\chi - \tfrac{\delta\theta}{1-\gamma}\right)g + \left(\left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\chi^\top\left(\left(\Sigma^S\right)^\top\right)^+\left(\Sigma^Y\right)^\top\right)g_y \\
&+ \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top g_{yy}\Sigma^Y\right] + \tfrac{1}{2}\frac{1}{g}(g_y)^\top\Sigma^Y\left((k-1)\mathrm{I}_{m+n} + k\tfrac{1-\gamma}{\gamma}\left(\Sigma^S\right)^\top\left(\left(\Sigma^S\right)^\top\right)^+\right)\left(\Sigma^Y\right)^\top g_y \\
&+ \tfrac{\delta^\psi\phi\theta}{k}g^{1-\frac{k\psi}{\theta}}
\end{aligned}$$

where $\left(\left(\Sigma^S\right)^\top\right)^+$ is the Moore-Penrose inverse of $\left(\Sigma^S\right)^\top$ given by

$$\left(\left(\Sigma^S\right)^\top\right)^+ = \left(\Sigma^S\left(\Sigma^S\right)^\top\right)^{-1}\Sigma^S.$$

□

## C.2 Dual Hamilton-Jacobi-Bellman Equation

Recall that the dynamics of the pricing deflators and state processes of our underlying market are in general form given as

$$\begin{aligned}
\mathrm{d}D_t &= -D_t\left(r\mathrm{d}t + \eta^\top\mathrm{d}W\right) \\
\mathrm{d}Y &= \mu^Y\mathrm{d}t + \Sigma^Y\mathrm{d}W,
\end{aligned}$$

where $W$ is an $(m+n)$-dimensional Brownian motion, $r$, $\eta$, $\mu^Y$ and $\Sigma^Y$ are matrix (functions) of $Y$ in the appropriate dimension (compare (4.1.1) and (4.3.2), respectively). We avoid to write the $Y$ dependence explicitly to keep notation simple. Moreover, recall that

$$\eta = \begin{pmatrix}\eta^S \\ \eta^Y\end{pmatrix}$$

denotes the market prices of risk, so we must have $\chi = \sigma^S L^S \eta^S$ or equivalently

$$\eta^S = \left(\hat{\Sigma}^S\right)^{-1}\chi. \tag{C.2.1}$$

We define the $(1+n)$-dimensional process $Z = (D, Y)^\top$ with dynamics given by

$$\mathrm{d}Z = \mathrm{diag}[Z]\left(\mu^Z\mathrm{d}t + \Sigma^Z\mathrm{d}W\right)$$

where

$$\mu^Z = \begin{pmatrix}-Dr \\ \mu^Y\end{pmatrix} \qquad \text{and} \qquad \Sigma^Z = \begin{pmatrix}-D\eta^\top \\ \Sigma^Y\end{pmatrix}.$$

Considering the indirect dual utility $V_t^*[D] = H(t, D, Y)$, the dynamic programming equation for the dual optimization problem reads

$$0 = \inf_{\substack{\eta\in\mathcal{P}^{m+n} \\ \eta^S = \left(\hat{\Sigma}^S\right)^{-1}\chi}}\left\{H_t + \left(\mu^Z\right)^\top H_z + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Z\right)^\top H_{zz}\Sigma^Z\right] + f^*(d, H)\right\}, \tag{C.2.2}$$

[1]The matrix $\Sigma^S\left(\Sigma^S\right)^\top$ is invertible by the assumption that $\sigma^S L^S$ is of full rank.

with $H_z = (H_d, H_{y_1}, \dots, H_{y_n})^\top$ and $H_{zz} = \begin{pmatrix} H_{dd} & H_{dy} \\ H_{yd} & H_{yy} \end{pmatrix}$ where

$$H_{yd} = \begin{pmatrix} H_{y_1 d} \\ \vdots \\ H_{y_n d} \end{pmatrix} = (H_{dy})^\top \qquad \text{and} \qquad H_{yy} = \begin{pmatrix} H_{y_1 y_1} & \cdots & H_{y_1 y_n} \\ \vdots & \ddots & \vdots \\ H_{y_n y_1} & \cdots & H_{y_n y_n} \end{pmatrix}.$$

Inserting the matrices of differentials and the definition of $\Sigma^Z$ we calculate

$$\mathrm{trace}\left[\left(\Sigma^Z\right)^\top H_{zz}\Sigma^Z\right] = d^2 H_{dd}\eta^\top\eta - 2dH_{dy}\Sigma^Y\eta + \mathrm{trace}\left[\left(\Sigma^Y\right)^\top H_{yy}\Sigma^Y\right],$$

where we used the invariance of the trace operator under circular shifts. Thus (C.2.2) unfolds to

$$0 = \inf_{\substack{\eta\in\mathcal{P}^{m+n} \\ \eta^S = \left(\sigma^S L^S\right)^{-1}\chi}} \left\{ H_t - rdH_d + \left(\mu^Y\right)^\top H_y + \tfrac{1}{2}d^2 H_{dd}\eta^\top\eta - dH_{dy}\Sigma^Y\eta + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top H_{yy}\Sigma^Y\right] + f^*(d,H)\right\},$$

with terminal condition $H(T,d,y) = \varepsilon^{\frac{1}{\gamma}}\frac{\gamma}{1-\gamma}d^{\frac{\gamma-1}{\gamma}}$.

*Proof of Proposition 4.4:* First notice that $\eta^S$ is already determined by (C.2.1), so it suffices optimize over all $\eta^Y \in \mathcal{P}^n$ and we obtain

$$\begin{aligned} 0 = \inf_{\eta^Y\in\mathcal{P}^n} \Big\{ & H_t - rdH_d + \left(\mu^Y\right)^\top H_y + \tfrac{1}{2}d^2 H_{dd}\left(\left(\eta^S\right)^\top\eta^S + \left(\eta^Y\right)^\top\eta^Y\right) \\ & - dH_{dy}\left(\hat{\Sigma}^{SY}\eta^S + \hat{\Sigma}^Y\eta^Y\right) + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top H_{yy}\Sigma^Y\right] + f^*(d,H)\Big\}. \end{aligned}$$

Now using the Ansatz $H(t,d,y) = \frac{\gamma}{1-\gamma}d^{\frac{\gamma-1}{\gamma}}h(t,y)^l$ we calculate the differentials

$$\begin{aligned} H_t &= l\tfrac{\gamma}{1-\gamma}d^{\frac{\gamma-1}{\gamma}}h(t,y)^{l-1}h_t & H_y &= l\tfrac{\gamma}{1-\gamma}d^{\frac{\gamma-1}{\gamma}}h(t,y)^{l-1}h_y \\ H_d &= -d^{-\frac{1}{\gamma}}h(t,y)^l & H_{dd} &= \tfrac{1}{\gamma}d^{-\frac{1}{\gamma}-1}h(t,y)^l \\ H_{dy} &= -ld^{-\frac{1}{\gamma}}h(t,y)^{l-1}(h_y)^\top = (H_{yd})^\top & H_{yy} &= l\tfrac{\gamma}{1-\gamma}d^{\frac{\gamma-1}{\gamma}}h(t,y)^{l-1}\left[\tfrac{l-1}{h}h_y(h_y)^\top + h_{yy}\right]. \end{aligned}$$

Moreover, $h(T,y) = \varepsilon^{\frac{1}{l}}$. Inserting the differentials, dividing by $l\frac{\gamma}{1-\gamma}d^{\frac{\gamma-1}{\gamma}}h(t,y)^{l-1}$ and simplifying yields

$$\begin{aligned} 0 = \inf_{\eta^Y\in\mathcal{P}^n} \Big\{ & h_t + \tfrac{1-\gamma}{\gamma l}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\left(\left(\eta^S\right)^\top\eta^S + \left(\eta^Y\right)^\top\eta^Y\right) - \tfrac{\delta\theta}{1-\gamma}\right)h \\ & + \left(\left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\left(\left(\hat{\Sigma}^Y\eta^Y\right)^\top + \left(\hat{\Sigma}^{SY}\eta^S\right)^\top\right)\right)h_y + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top h_{yy}\Sigma^Y\right] \\ & + \tfrac{1}{2}\frac{l-1}{h}(h_y)^\top\Sigma^Y\left(\Sigma^Y\right)^\top h_y + \tfrac{\delta^\psi\theta}{l\gamma\psi}h^{1-l\frac{\gamma\psi}{\theta}}\Big\} \end{aligned} \tag{C.2.3}$$

Now the first order condition for $\eta^Y$ implies

$$\left(\eta^Y\right)^* = -\gamma l\left(\hat{\Sigma}^Y\right)^\top\frac{h_y}{h}.$$

Inserting $\left(\eta^Y\right)^*$ to (C.2.3) yields the partial differential equation for $h$:

$$
\begin{aligned}
0 = h_t &+ \tfrac{1-\gamma}{\gamma l}\left(r + \tfrac{1}{2\gamma}\left(\eta^S\right)^\top \eta^S - \tfrac{\delta\theta}{1-\gamma}\right) h + \left(\left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\left(\hat{\Sigma}^{SY}\eta^S\right)^\top\right) h_y \\
&+ \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top h_{yy}\Sigma^Y\right] + \tfrac{1}{2}\frac{1}{h}(h_y)^\top\left((l-1)\Sigma^Y\left(\Sigma^Y\right)^\top - l(1-\gamma)\hat{\Sigma}^Y\left(\hat{\Sigma}^Y\right)^\top\right) h_y \\
&\quad + \tfrac{\delta\psi\theta}{l\gamma\psi} h^{1-l\frac{\gamma\psi}{\theta}}
\end{aligned}
$$

$\square$

# Appendix D

# Explicit and Approximate Solutions

## D.1 Exact Solutions in One Dimension

Let $n = 1$ and $k = \frac{\gamma}{\gamma+(1-\gamma)(\rho^{SY})^2}$. Moreover let

$$\psi = 2 - \gamma + \tfrac{(1-\gamma)^2}{\gamma}\left(\rho^{SY}\right)^2.$$

Then as already mentioned in Section 5.1, the primal HJB equation (4.2.4) simplifies to

$$0 = g_t + \tfrac{1-\gamma}{k}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{\chi^2}{(\sigma^S)^2} - \tfrac{\delta\theta}{1-\gamma}\right)g + \left(\mu^Y + \tfrac{1-\gamma}{\gamma}\tfrac{\chi\sigma^Y\rho^{SY}}{\sigma^S}\right)g_y + \tfrac{1}{2}\left(\sigma^Y\right)^2 g_{yy} + \delta^\psi, \qquad \text{(D.1.1)}$$

see [Kraft et al., 2013].

### Kim-Omberg Model

The 1-dimensional special case ($m = n = 1$) of our multivariate Kim-Omberg model introduced in Example 4.1 reads

$$\begin{aligned}
\mathrm{d}S_t &= S_t\left((r + \bar{\lambda} + \boldsymbol{\lambda}Y_t)\mathrm{d}t + \sigma^S\mathrm{d}W_t^S\right)\\
\mathrm{d}Y_t &= -\kappa Y_t\mathrm{d}t + \sigma^Y\left(\rho^{SY}\mathrm{d}W_t^S + \sqrt{1-(\rho^{SY})^2}\mathrm{d}W_t^Y\right)
\end{aligned}$$

Let $A(\cdot, s)$, $B(\cdot, s)$ and $C(\cdot, s)$ be given by

$$\begin{aligned}
C_t &= 2\left(\kappa - \tfrac{1-\gamma}{\gamma}\boldsymbol{\lambda}\tfrac{\sigma^Y\rho^{SY}}{\sigma^S}\right)C + \tfrac{1-\gamma}{k}\tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{\boldsymbol{\lambda}^2}{(\sigma^S)^2} + 2\left(\sigma^Y\right)^2 C^2,\\
B_t &= \left(\kappa - \tfrac{1-\gamma}{\gamma}\boldsymbol{\lambda}\tfrac{\sigma^Y\rho^{SY}}{\sigma^S} + 2\left(\sigma^Y\right)^2 C\right)B - 2\tfrac{1-\gamma}{\gamma}\bar{\lambda}\tfrac{\sigma^Y\rho^{SY}}{\sigma^S}C + \tfrac{1-\gamma}{k}\tfrac{1}{\gamma}\tfrac{\bar{\lambda}\boldsymbol{\lambda}}{(\sigma^S)^2}\\
A_t &= \tfrac{1-\gamma}{\gamma}\bar{\lambda}\frac{\sigma^Y\rho^{SY}}{\sigma^S}B - \tfrac{1-\gamma}{k}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{(\bar{\lambda})^2}{(\sigma^S)^2} - \tfrac{\delta\theta}{1-\gamma}\right) + \tfrac{1}{2}\left(\sigma^Y\right)^2(2C - B^2),
\end{aligned}$$

and $A(s, s) = B(s, s) = C(s, s) = 0$. Then

$$h(t, y; s) \triangleq \exp\left(A(t, s) + B(t, s)y + C(t, s)y^2\right)$$

satisfies the linear homogeneous partial differential equation

$$0 = h_t + \tfrac{1-\gamma}{k}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\tfrac{1}{(\sigma^S)^2}\left(\bar{\lambda} + \boldsymbol{\lambda}y\right)^2 - \tfrac{\delta\theta}{1-\gamma}\right)h + \left(-\kappa y + \tfrac{1-\gamma}{\gamma}\left(\bar{\lambda} + \boldsymbol{\lambda}y\right)\tfrac{\sigma^Y\rho^{SY}}{\sigma^S}\right)h_y + \tfrac{1}{2}\left(\sigma^Y\right)^2 h_{yy}$$

on $[0, s] \times \mathbb{R}$ and a solution to (D.1.1) is given by

$$g(t, y) = \delta^{\frac{1}{\psi}}\int_t^T h(t, y; s)\mathrm{d}s + \hat{\varepsilon}h(T, y; T).$$

### Heston-Model

The 1-dimensional special case ($m = n = 1$) of our multivariate Heston model introduced in Example 4.2 reads

$$\begin{aligned} \mathrm{d}S_t &= S_t\left((r+\bar{\lambda}Y_t)\mathrm{d}t + \bar{\sigma}^S\sqrt{Y_t}\mathrm{d}W_t^S\right)\\ \mathrm{d}Y_t &= (\bar{\mu}-\kappa Y_t)\mathrm{d}t + \bar{\sigma}^Y\sqrt{Y_t}\left(\rho^{SY}\mathrm{d}W_t^S + \sqrt{1-(\rho^{SY})^2}\mathrm{d}W_t^Y\right). \end{aligned}$$

Let $A(\cdot,s)$ and $B(\cdot,s)$ be given by

$$\begin{aligned} B_t &= \left(\kappa - \tfrac{1-\gamma}{\gamma}\bar{\lambda}\tfrac{\bar{\sigma}^Y\rho^{SY}}{\bar{\sigma}^S}\right)B + \tfrac{1-\gamma}{k}\tfrac{1}{2}\tfrac{1}{\gamma}\bar{\lambda}^2 + \tfrac{1}{2}\left(\bar{\sigma}^Y\right)^2 B^2\\ A_t &= \bar{\mu}B - \tfrac{1-\gamma}{k}\left(r - \tfrac{\delta\theta}{1-\gamma}\right), \end{aligned}$$

where $A(s,s) = B(s,s) = 0$. Then

$$h(t,y;s) \triangleq \exp(A(t,s) - B(t,s)y)$$

satisfies the linear homogeneous partial differential equation

$$0 = h_t + \tfrac{1-\gamma}{k}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\bar{\lambda}^2 y - \tfrac{\delta\theta}{1-\gamma}\right)h + \left(\bar{\mu} - \kappa y + \tfrac{1-\gamma}{\gamma}\bar{\lambda}\tfrac{\sigma^Y\rho^{SY}}{\sigma^S}\right)h_y + \tfrac{1}{2}\left(\sigma^Y\right)^2 y h_{yy}$$

on $[0,s]\times\mathbb{R}$ and a solution to (D.1.1) is given by

$$g(t,y) = \delta^{\frac{1}{\psi}}\int_t^T h(t,y;s)\mathrm{d}s + \hat{\varepsilon}h(T,y;T).$$

## D.2 CS-ALFC Algorithm

The Campbell-Shiller approximation associated to the dual HJB equation (5.2.1) in a model with power utility is in general form given as

$$\begin{aligned} 0 &= g_t^{\mathrm{CS}} + \tfrac{1-\gamma}{\gamma}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\left(\eta^S\right)^\top\eta^S - \tfrac{\delta}{1-\gamma}\right)g^{\mathrm{CS}} + \left(\left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\left(\eta^S\right)^\top\left(\hat{\Sigma}^{SY}\right)^\top\right)g_y^{\mathrm{CS}}\\ &\quad - \tfrac{1}{2}(1-\gamma)\left(g_y^{\mathrm{CS}}\right)^\top\hat{\Sigma}^Y\left(\hat{\Sigma}^Y\right)^\top\frac{g_y^{\mathrm{CS}}}{g^{\mathrm{CS}}} + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top g_{yy}^{\mathrm{CS}}\Sigma^Y\right]\\ &\quad + \mathfrak{l}(t)\left(1 - \ln(\mathfrak{l}(t)) + \ln(\delta^{\frac{1}{\gamma}}) - \ln\left(g^{\mathrm{CS}}\right)\right)g^{\mathrm{CS}} \end{aligned} \tag{D.2.1}$$

subject to the terminal condition $g^{\mathrm{CS}}(T,y) = \hat{\varepsilon}$, where $\ln(\mathfrak{l}(t)) = \mathbb{E}\left[\ln\left(\frac{c}{x}\right)^*(t,Y_\infty)\right]$ and $Y_\infty$ is a random variable that has the stationary distribution of the process $Y$.

We already mentioned that the factor $\mathfrak{l}$ should be regarded as endogenous, so we determine $\mathfrak{l}(t)$ recursively: Starting with an initial function $\mathfrak{l}_0(t)$, find the solution $h^{\mathrm{CS}}$ to (5.1.7) and then update the function $\mathfrak{l}_1(t)$ via

$$\ln(\mathfrak{l}_1(t)) = \mathbb{E}\left[\ln\left(\frac{c}{x}\right)^{\mathrm{CS}}(t,Y_\infty)\right] = \ln(\delta^{\frac{1}{\gamma}}) - \mathbb{E}\left[\ln\left(g^{\mathrm{CS}}(t,Y_\infty)\right)\right] \tag{D.2.2}$$

and iterate until a fixed-point is reached.

Moreover, recall the partial differential equation associated to the artificially completed market with arbitrary $\eta^Y$ and power utility by

$$0 = \tilde{g}_t + \tilde{r}\left(\eta^Y\right)\tilde{g} + \tilde{\alpha}\left(\eta^Y\right)\tilde{g}_y + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top \tilde{g}_{yy}\Sigma^Y\right] + \delta^{\frac{1}{\gamma}}, \tag{D.2.3}$$

subject to the terminal condition $\tilde{g}(T,y) = \varepsilon^{\frac{1}{\gamma}}$, where $\tilde{r} : \mathbb{R}^n$ and $\tilde{\alpha} : \mathbb{R}^{n\times 1} \to \mathbb{R}^{1\times n}$ are given as

$$\tilde{r}(v) \triangleq \tfrac{1-\gamma}{\gamma}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\left(\chi^\top\left(\Sigma^S\left(\Sigma^S\right)^\top\right)^{-1}\chi + v^\top v\right) - \tfrac{\delta}{1-\gamma}\right)$$

and

$$\tilde{\alpha}(v) \triangleq \left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\left(\chi^\top\left(\left(\Sigma^S\right)^\top\right)^+\left(\Sigma^Y\right)^\top + v^\top\left(\hat{\Sigma}^Y\right)^\top\right).$$

### D.2.1 Multivariate Kim-Omberg Model

Recall the model introduced in Example 4.1, where for a $\mathbb{R}^{m+n}$-dimensional standard Brownian motion $W$, let the assets and states follow the dynamics

$$\begin{aligned}\mathrm{d}S_t &= \mathrm{diag}[S_t]\left(\left(r + \bar{\lambda} + \boldsymbol{\lambda}^\top Y_t\right)\mathrm{d}t + \Sigma^S \mathrm{d}W_t\right),\\ \mathrm{d}Y_t &= -\mathrm{diag}[\kappa]Y_t\mathrm{d}t + \Sigma^Y\mathrm{d}W_t,\end{aligned}$$

where $r, \bar{\lambda} \in \mathbb{R}^m$, $\boldsymbol{\lambda} \in \mathbb{R}^{n\times m}$, $\kappa \in \mathbb{R}^n$ and $\Sigma^S \in \mathbb{R}^{m\times(m+n)}$, $\Sigma^Y \in \mathbb{R}^{n\times(m+n)}$ as defined in (4.1.3).

#### Campbell-Shiller Approximation of the Dual HJB Equation

Let $A$, $B$ and $C$ be given by

$$\begin{aligned}C'(t) &= \left(\mathfrak{l}(t) + 2\mathrm{diag}[\kappa] - 2\tfrac{1-\gamma}{\gamma}\boldsymbol{\lambda}\Gamma\right)C(t) + \tfrac{1-\gamma}{\gamma}\tfrac{1}{2}\tfrac{1}{\gamma}\boldsymbol{\lambda}\Sigma\boldsymbol{\lambda}^\top + 2C(t)^\top\Xi C(t)\\ B'(t) &= \left(l(t) + \mathrm{diag}[\kappa] - \tfrac{1-\gamma}{\gamma}\boldsymbol{\lambda}\Gamma\right)B(t) + \tfrac{1-\gamma}{\gamma}\tfrac{1}{\gamma}\boldsymbol{\lambda}\Sigma\bar{\lambda} - 2\tfrac{1-\gamma}{\gamma}C(t)^\top\Gamma^\top\bar{\lambda} + 2C(t)^\top\Xi B(t)\\ A'(t) &= l(t)A(t) - \mathfrak{l}(t)\left(1 - \ln(\mathfrak{l}(t)) - \ln(\hat{\varepsilon}) + \ln\left(\delta^{\frac{1}{\gamma}}\right)\right) - \tfrac{1-\gamma}{\gamma}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\bar{\lambda}^\top\Sigma\bar{\lambda} - \tfrac{\delta}{1-\gamma}\right)\\ &\quad + \tfrac{1-\gamma}{\gamma}\bar{\lambda}^\top\Gamma B(t) - \tfrac{1}{2}B(t)^\top\Xi B(t) + \mathrm{trace}\left[\left(\Sigma^Y\right)^\top C(t)\Sigma^Y\right],\end{aligned} \tag{D.2.4}$$

where $A(T) = 0$, $B(T) = \mathbf{0}_{n\times 1}$, $C(T) = \mathbf{0}_{n\times n}$ and $\Xi$, $\Gamma$, $\Sigma$ are given by

$$\begin{aligned}\Xi &\triangleq \hat{\Sigma}^{SY}\left(\hat{\Sigma}^{SY}\right)^\top + \gamma\hat{\Sigma}^Y\left(\hat{\Sigma}^Y\right)^\top\\ \Gamma &\triangleq \left(\hat{\Sigma}^S\right)^{-\top}\left(\hat{\Sigma}^{SY}\right)^\top\\ \Sigma &\triangleq \left(\hat{\Sigma}^S\left(\hat{\Sigma}^S\right)^\top\right)^{-1}\end{aligned}$$

Then the function

$$g^{\mathrm{CS}}(t,y) = \hat{\varepsilon}\exp\left(A(t) - y^\top B(t) - y^\top C(t)y\right),$$

solves (D.2.1) and

$$\left(\eta^Y\right)^{\mathrm{CS}} \triangleq \eta_1^Y(t) + \eta_2^Y(t)y \tag{D.2.5}$$

with $\eta_1^Y(t) = \gamma\left(\hat{\Sigma}^Y\right)^\top B(t)$ and $\eta_2^Y(t) = 2\gamma\left(\hat{\Sigma}^Y\right)^\top C(t)$ are the associated (linear) market prices of risk.

**Remark D.1**
*For the recursive definition of* $\mathfrak{l}$ *from* (D.2.2) *we need to compute*

$$\mathbb{E}[Y_\infty^\top B(t)] \quad \textit{and} \quad \mathbb{E}[Y_\infty^\top C(t) Y_\infty]$$

*in every step. Following [Meucci, 2009], the multivariate Ornstein-Uhlenbeck process has a stationary distribution which is multivariate normal with parameters*

$$\mu_\infty = 0 \quad \textit{and} \quad \mathrm{vec}(\Sigma_\infty) = (\mathrm{diag}[\kappa] \oplus \mathrm{diag}[\kappa])^{-1} \mathrm{vec}\left(\Sigma^Y \left(\Sigma^Y\right)^\top\right),$$

*where* $\oplus$ *is the Kronecker sum which for* $M, N \in \mathbb{R}^{n\times n}$ *defined via the Kronecker product as*

$$M \oplus N = M \otimes \mathrm{I}_{n\times n} + \mathrm{I}_{n\times n} \otimes N$$

*and* vec *is the stack operator that transforms a* $m \times n$ *matrix to a* $mn \times 1$ *vector. Thus, for every* $t \in [0, T]$ *we have* $\mathbb{E}[Y_\infty^\top B(t)] = 0$ *and*

$$\begin{aligned}
\mathbb{E}[Y_\infty^\top C(t) Y_\infty] &= \sum_{i=1}^n \mathbb{E}[(Y_\infty)_i (C(t)Y_\infty)_i] = \sum_{i=1}^n \mathbb{E}\left[(Y_\infty)_i \sum_{j=1}^n C(t)_{ij} (Y_\infty)_j\right] \\
&= \sum_{i=1}^n \sum_{j=1}^n C(t)_{ij} \mathbb{E}[(Y_\infty)_i (Y_\infty)_j] = \sum_{i=1}^n \sum_{j=1}^n C(t)_{ij} \mathrm{Cov}[(Y_\infty)_i, (Y_\infty)_j] \\
&= \sum_{i=1}^n \sum_{j=1}^n C(t)_{ij} (\Sigma_\infty)_{ij} = \mathrm{trace}[C(t)\Sigma_\infty],
\end{aligned}$$

*as* $\Sigma_\infty$ *is symmetric.* $\triangle$

**Exact Solution for Power Utility in Completed Markets**

We choose $\eta^Y = \left(\eta^Y\right)^{\mathrm{CS}}$ as given in (D.2.5) and write $\left(\eta^Y\right)^{\mathrm{CS}}(t, y) \triangleq \eta_1^Y(t) + \eta_2^Y(t) y$ for short, omitting the $t$ and $y$ dependence for brevity of notation below. Then the function

$$h(t, y; s) = \exp\left(\tilde{A}(t,s) - y^\top \tilde{B}(t,s) - y^\top \tilde{C}(t,s) y\right)$$

with $\tilde{A}(\cdot, s)$, $\tilde{B}(\cdot, s)$, $\tilde{C}(\cdot, s)$ being the solutions of the ODE system

$$\begin{aligned}
\tilde{C}_t &= 2\left(\mathrm{diag}[\kappa] - \tfrac{1-\gamma}{\gamma}(\boldsymbol{\lambda}\Gamma + \Xi_2)\right)\tilde{C} + \tfrac{1-\gamma}{\gamma}\tfrac{1}{2}\tfrac{1}{\gamma}\left(\boldsymbol{\lambda}\Sigma\lambda^\top + \left(\eta_2^Y\right)^\top \eta_2^Y\right) + 2\tilde{C}^\top \Sigma^Y \left(\Sigma^Y\right)^\top \tilde{C} \\
\tilde{B}_t &= \left(\mathrm{diag}[\kappa] - \tfrac{1-\gamma}{\gamma}(\boldsymbol{\lambda}\Gamma + \Xi_2)\right)\tilde{B} + \tfrac{1-\gamma}{\gamma}\tfrac{1}{\gamma}\left(\boldsymbol{\lambda}\Sigma\bar{\lambda} + \left(\eta_2^Y\right)^\top \eta_1^Y\right) \\
&\quad - 2\tfrac{1-\gamma}{\gamma}\tilde{C}^\top\left(\Gamma^\top\bar{\lambda} + \Xi_1^\top\right) + 2\tilde{C}^\top\Sigma^Y\left(\Sigma^Y\right)^\top\tilde{B} \\
\tilde{A}_t &= -\tfrac{1-\gamma}{\gamma}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\left(\bar{\lambda}^\top\Sigma\bar{\lambda} + \left(\eta_1^Y\right)^\top\eta_1^Y\right) - \tfrac{\delta}{1-\gamma}\right) + \tfrac{1-\gamma}{\gamma}\left(\bar{\lambda}^\top\Gamma + \Xi_1\right)\tilde{B} \\
&\quad + \mathrm{trace}\left[\left(\Sigma^Y\right)^\top\tilde{C}\Sigma^Y\right] - \frac{1}{2}\tilde{B}^\top\Sigma^Y\left(\Sigma^Y\right)^\top\tilde{B},
\end{aligned} \tag{D.2.6}$$

where

$$\begin{aligned}
\Xi_i &\triangleq \left(\eta_i^Y\right)^\top\left(\hat{\Sigma}^Y\right)^\top, \qquad i = 1, 2 \\
\Gamma &\triangleq \left(\hat{\Sigma}^S\right)^{-\top}\left(\hat{\Sigma}^{SY}\right)^\top \\
\Sigma &\triangleq \left(\hat{\Sigma}^S\left(\hat{\Sigma}^S\right)^\top\right)^{-1}
\end{aligned}$$

$\tilde{A}(s,s) = 0$, $\tilde{B}(s,s) = \mathbf{0}_{n\times 1}$, $\tilde{C}(s,s) = \mathbf{0}_{n\times n}$ solves the linear homogeneous partial differential equation

$$0 = h_t + \tilde{r}\left(\left(\eta^Y\right)^{\mathrm{CS}}\right) h + \tilde{\alpha}\left(\left(\eta^Y\right)^{\mathrm{CS}}\right) h_y + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top h_{yy}\Sigma^Y\right],$$

on $[0,s]\times\mathbb{R}$ subject to $h(s,y;s) = 1$, and

$$\tilde{g}(t,y) = \delta^{\frac{1}{\gamma}}\int_t^T h(t,y;s)\mathrm{d}s + \hat{\varepsilon}h(t,y;T),$$

solves (D.2.3), see Section 5.1.

### D.2.2 Multivariate Heston-Model

Recall the model from Example 4.2, where $m = n$ and for a $\mathbb{R}^{2n}$-dimensional standard Brownian $W$ let the assets and states follow the dynamics

$$\begin{aligned}
\mathrm{d}S_t &= \mathrm{diag}[S_t]\left(\left(r + K\mathrm{diag}[Y_t]K^\top\bar{\lambda}\right)\mathrm{d}t + \Sigma^S(Y_t)\mathrm{d}W_t\right),\\
\mathrm{d}Y_t &= \left(\bar{\mu} - \mathrm{diag}[\kappa]Y_t\right)\mathrm{d}t + \Sigma^Y(Y_t)\mathrm{d}W_t,
\end{aligned}$$

where $r,\bar{\lambda},\bar{\mu},\kappa\in\mathbb{R}^n$, $K\in\mathbb{R}_o^{n\times n}$ and $\Sigma^S$, $\Sigma^Y$ are $(n\times 2n)$-dimensional matrix functions as in (4.1.3).

#### Campbell-Shiller Approximation of the Dual HJB Equation

Let $A$ and $B = (B_1,\ldots,B_n)^\top$ be given by

$$\begin{aligned}
B_i'(t) &= \left(\mathfrak{l}(t) + \kappa_i - \tfrac{1-\gamma}{\gamma}\frac{\bar{\sigma}_i^Y\rho_i}{\bar{\sigma}_i^S}\sum_{j=1}^n K_{ji}\bar{\lambda}_j\right)B_i(t) + \tfrac{1-\gamma}{\gamma}\tfrac{1}{2}\tfrac{1}{\gamma}\left(\sum_{j=1}^n\frac{K_{ji}\bar{\lambda}_j}{\bar{\sigma}_i^S}\right)^2 \qquad \text{(D.2.7)}\\
&\quad - \tfrac{1}{2}\left(\bar{\sigma}_i^Y\right)^2\left((1-\gamma)(1-\rho_i^2) - 1\right)B_i(t)^2\\
A'(t) &= \mathfrak{l}(t)A(t) + \bar{\mu}^\top B(t) - \tfrac{1-\gamma}{\gamma}\left(r - \tfrac{\delta}{1-\gamma}\right) - \mathfrak{l}(t)\left(1 - \ln(\mathfrak{l}(t)) + \ln\left(\delta^{\frac{1}{\gamma}}\right) - \ln(\hat{\varepsilon})\right)
\end{aligned}$$

where $A(T) = 0$ and $B(T) = \mathbf{0}_{n\times 1}$. Then the function

$$g^{\mathrm{CS}}(t,y) = \varepsilon\exp\left(A(t) - y^\top B(t)\right),$$

solves (D.2.1) and

$$\left(\eta^Y\right)^{\mathrm{CS}}(t,y) = \gamma\left(\hat{\Sigma}^Y(y)\right)^\top B(t) \qquad \text{(D.2.8)}$$

are the associated market prices of risk. Recall that $\hat{\Sigma}^Y(y)$ behaves as $\sqrt{y}$.

**Remark D.2**
*For the recursive definition of $\mathfrak{l}$ from* (D.2.2) *we need to compute $\mathbb{E}[Y_\infty^\top B(t)]$. The single coordinate processes $Y^i$, $i = 1,\ldots,n$ have the dynamics*

$$\mathrm{d}Y_t^i = (\bar{\mu}_i - \kappa_i Y_t^i)\mathrm{d}t + \bar{\sigma}_i^Y\sqrt{Y_t^i}\left(\rho_i\mathrm{d}W_t^i + \sqrt{1-\rho_i^2}\mathrm{d}W_t^{2i}\right).$$

*As the Brownian motions $W^i$ and $W^{2i}$ are independent, this is a classical CIR process and it is well known that the asymptotic distribution of such is a gamma distribution with expectation $\frac{\bar{\mu}_i}{\kappa_i}$, see e.g.[Cox et al., 2005]. Thus*

$$\mathbb{E}\left[Y_\infty^\top B(t)\right] = \mathbb{E}\left[Y_\infty^\top\right]B(t) = \bar{\mu}^\top\left(\mathrm{diag}[\kappa]\right)^{-1}B(t).$$

**Exact Solution for Power Utility in Completed Markets**

We choose $\eta^Y = \left(\eta^Y\right)^{\mathrm{CS}} = \gamma\left(\hat{\Sigma}^Y\right)^\top B$ as given in (D.2.8) and omit the $t$ and $y$ dependence for brevity of notation below. Then the function

$$h(t,y;s) = \exp\left(\tilde{A}(t,s) - y^\top \tilde{B}(t,s)\right)$$

with $\tilde{A}(\cdot,s)$ and $\tilde{B}(\cdot,s) = (\tilde{B}^1(\cdot,s),\ldots,\tilde{B}^n(\cdot,s))^\top$ being the solutions of the ODE system

$$\begin{aligned}\tilde{B}^i_t &= \left(\kappa_i - \tfrac{1-\gamma}{\gamma}\left(\frac{\bar{\sigma}^Y_i\rho_i}{\bar{\sigma}^S_i}\sum_{j=1}^n K_{ji}\bar{\lambda}_j + \gamma B_i\left(\bar{\sigma}^Y_i\right)^2(1-\rho_i^2)\right)\right)\tilde{B}^i \\ &\quad + \tfrac{1-\gamma}{\gamma}\tfrac{1}{2}\tfrac{1}{\gamma}\left(\left(\sum_{j=1}^n \frac{K_{ji}\bar{\lambda}_j}{\bar{\sigma}^S_i}\right)^2 + \gamma^2\left(\bar{\sigma}^Y_i\right)^2(1-\rho_i^2)B_i^2\right) + \tfrac{1}{2}\left(\bar{\sigma}^Y_i\right)^2\left(\tilde{B}^i\right)^2 \\ \tilde{A}_t &= \bar{\mu}^\top\tilde{B} - \tfrac{1-\gamma}{\gamma}\left(r - \tfrac{\delta}{1-\gamma}\right)\end{aligned} \tag{D.2.9}$$

$\tilde{A}(s,s) = 0$, $\tilde{B}(s,s) = \mathbf{0}_{n\times 1}$ solves the linear homogeneous partial differential equation

$$0 = h_t + \tilde{r}\left(\left(\eta^Y\right)^{\mathrm{CS}}\right)h + \tilde{\alpha}\left(\left(\eta^Y\right)^{\mathrm{CS}}\right)h_y + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top h_{yy}\Sigma^Y\right],$$

on $[0,s]\times\mathbb{R}$ subject to $h(s,y;s) = 1$, and

$$\tilde{g}(t,y) = \delta^{\frac{1}{\gamma}}\int_t^T h(t,y;s)\mathrm{d}s + \hat{\varepsilon}h(t,y;T),$$

solves (D.2.3), see Section 5.1.

## D.3 SA-PDI Algorithm

Assume that from the $(j-1)$-th iteration we obtained some market prices of risk $\left(\eta^Y\right)^{(j-1)}$. We choose $k = \frac{\theta}{\psi}$, then the partial differential equation associated to the $\left(\eta^Y\right)^{(j-1)}$-completed market reads

$$0 = \tilde{g}_t + \tilde{r}\left(\left(\eta^Y\right)^{(j-1)}\right)\tilde{g} + \tilde{\alpha}\left(\left(\eta^Y\right)^{(j-1)}\right)\tilde{g}_y + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top\tilde{g}_{yy}\Sigma^Y\right] + \delta^\psi, \tag{D.3.1}$$

where $\tilde{r}:\mathbb{R}^n\to\mathbb{R}$ and $\tilde{\alpha}:\mathbb{R}^{n\times 1}\to\mathbb{R}^{1\times n}$ are given as

$$\tilde{r}(v) \triangleq \tfrac{1-\gamma}{k}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\left(\chi^\top\left(\Sigma^S\left(\Sigma^S\right)^\top\right)^{-1}\chi + v^\top v\right) - \tfrac{\delta\theta}{1-\gamma}\right)$$

and

$$\tilde{\alpha}(v) \triangleq \left(\left(\mu^Y\right)^\top + \tfrac{1-\gamma}{\gamma}\left(\chi^\top\left(\left(\Sigma^S\right)^\top\right)^+\left(\Sigma^Y\right)^\top + v^\top\left(\sigma^Y L^Y\right)^\top\right) - \tfrac{1}{2}\tfrac{k-\gamma}{k\gamma}v^\top\left(\hat{\Sigma}^Y\right)^{-1}\Sigma^Y\left(\Sigma^Y\right)^\top\right).$$

### D.3.1 Multivariate Kim-Omberg Model

Recall the model introduced in Example 4.1, where for a $\mathbb{R}^{m+n}$-dimensional standard Brownian motion $W$, let the assets and states follow the dynamics

$$\begin{aligned} \mathrm{d}S_t &= \mathrm{diag}[S_t]\left(\left(r+\bar{\lambda}+\boldsymbol{\lambda}^\top Y_t\right)\mathrm{d}t + \Sigma^S \mathrm{d}W_t\right), \\ \mathrm{d}Y_t &= -\mathrm{diag}[\kappa]Y_t\mathrm{d}t + \Sigma^Y \mathrm{d}W_t, \end{aligned}$$

where where $r, \bar{\lambda} \in \mathbb{R}^m$, $\boldsymbol{\lambda} \in \mathbb{R}^{n\times m}$, $\kappa \in \mathbb{R}^n$ and $\Sigma^S \in \mathbb{R}^{m\times(m+n)}$, $\Sigma^Y \in \mathbb{R}^{n\times(m+n)}$ as defined in (4.1.3).
Let the market prices of risk $\left(\eta^Y\right)^{(j-1)}$ from the $(j-1)$-th iteration be given and determine the approximation of the associated sensitivities by their first-order Taylor approximation around $\bar{y} = \mathbb{E}\left[Y_\infty\right]$ as

$$\mathfrak{S}(t,y) = \frac{g_y(t,\bar{y})}{g(t,\bar{y})} + \frac{\partial}{\partial y}\left(\frac{g_y(t,\bar{y})}{g(t,\bar{y})}\right)(y-\bar{y}).$$

Then the linearly approximated market prices of risk have a representation

$$\left(\eta^Y_{\approx}\right)^{(j-1)}(t,y) = -k\left(\hat{\Sigma}^Y\right)^\top \mathfrak{S}(t,y) \triangleq \eta^Y_1(t) + \eta^Y_2(t)y$$

for some $\eta^Y_1$ and $\eta^Y_2$.
Let $\tilde{A}(\cdot,s)$, $\tilde{B}(\cdot,s)$, $\tilde{C}(\cdot,s)$ be given by the solutions of the ODE system

$$\begin{aligned} \tilde{C}_t &= 2\left(\mathrm{diag}[\kappa] - \tfrac{1-\gamma}{\gamma}\left(\boldsymbol{\lambda}\Gamma + \Xi_2\right) + \tfrac{1}{2}\tfrac{k-\gamma}{k\gamma}\left(\eta^Y_2\right)^\top \Psi\right)\tilde{C} \\ &\quad + \tfrac{1-\gamma}{k}\tfrac{1}{2}\tfrac{1}{\gamma}\left(\boldsymbol{\lambda}\Sigma\boldsymbol{\lambda}^\top + \left(\eta^Y_2\right)^\top \eta^Y_2\right) + 2\tilde{C}^\top\Sigma^Y\left(\Sigma^Y\right)^\top\tilde{C} \\ \tilde{B}_t &= \left(\mathrm{diag}[\kappa] - \tfrac{1-\gamma}{\gamma}\left(\boldsymbol{\lambda}\Gamma + \Xi_2\right) + \tfrac{1}{2}\tfrac{k-\gamma}{k\gamma}\left(\eta^Y_2\right)^\top \Psi\right)\tilde{B} + \tfrac{1-\gamma}{k}\tfrac{1}{\gamma}\left(\boldsymbol{\lambda}\Sigma\bar{\lambda} + \left(\eta^Y_2\right)^\top\eta^Y_1\right) \\ &\quad + 2\tilde{C}^\top\Sigma^Y\left(\Sigma^Y\right)^\top\tilde{B} - 2\tilde{C}^\top\left(\tfrac{1-\gamma}{\gamma}\left(\Gamma^\top\bar{\lambda} + \Xi_1^\top\right) - \tfrac{1}{2}\tfrac{k-\gamma}{k\gamma}\Psi^\top\eta^Y_1\right) \\ \tilde{A}_t &= -\tfrac{1-\gamma}{k}\left(r + \tfrac{1}{2}\tfrac{1}{\gamma}\left(\bar{\lambda}^\top\Sigma\bar{\lambda} + \left(\eta^Y_1\right)^\top\eta^Y_1\right) - \tfrac{\delta\theta}{1-\gamma}\right) + \mathrm{trace}\left[\left(\Sigma^Y\right)^\top\tilde{C}\Sigma^Y\right] \\ &\quad - \left(\tfrac{1}{2}\tfrac{k-\gamma}{k\gamma}\left(\eta^Y_1\right)^\top\Psi - \tfrac{1-\gamma}{\gamma}\left(\bar{\lambda}^\top\Gamma + \Xi_1\right)\right)\tilde{B} - \frac{1}{2}\tilde{B}^\top\Sigma^Y\left(\Sigma^Y\right)^\top\tilde{B}, \end{aligned} \tag{D.3.2}$$

$\tilde{A}(s,s) = 0$, $\tilde{B}(s,s) = \mathbf{0}_{n\times 1}$, $\tilde{C}(s,s) = \mathbf{0}_{n\times n}$, where

$$\begin{aligned} \Xi_i &\triangleq \left(\eta^Y_i\right)^\top\left(\hat{\Sigma}^Y\right)^\top, \qquad i = 1,2 \\ \Gamma &\triangleq \left(\hat{\Sigma}^S\right)^{-\top}\left(\hat{\Sigma}^{SY}\right)^\top \\ \Sigma &\triangleq \left(\hat{\Sigma}^S\left(\hat{\Sigma}^S\right)^\top\right)^{-1} \\ \Psi &\triangleq \left(\hat{\Sigma}^Y\right)^{-1}\Sigma^Y\left(\Sigma^Y\right)^\top. \end{aligned}$$

Then the function

$$h(t,y;s) = \exp\left(\tilde{A}(t,s) - y^\top\tilde{B}(t,s) - y^\top\tilde{C}(t,s)y\right)$$

solves the linear homogeneous partial differential equation

$$0 = h_t + \tilde{r}\left(\left(\eta^Y_{\approx}\right)^{(j-1)}\right)h + \tilde{\alpha}\left(\left(\eta^Y_{\approx}\right)^{(j-1)}\right)h_y + \tfrac{1}{2}\mathrm{trace}\left[\left(\Sigma^Y\right)^\top h_{yy}\Sigma^Y\right],$$

on $[0, s] \times \mathbb{R}$ subject to $h(s, y; s) = 1$, and

$$\tilde{g}(t, y) = \delta^{\frac{1}{\gamma}} \int_t^T h(t, y; s) \mathrm{d}s + \hat{\varepsilon} h(t, y; T),$$

solves (D.3.1).

### D.3.2 Multivariate Heston Model

Consider the model introduced in Example 4.2, i.e. let $m = n$ and for a $\mathbb{R}^{2n}$-dimensional standard Brownian $W$ let the assets and states follow the dynamics

$$\begin{aligned} \mathrm{d}S_t &= \mathrm{diag}[S_t] \left( \left( r + K \mathrm{diag}[Y_t] K^\top \bar{\lambda} \right) \mathrm{d}t + \Sigma^S(Y_t) \mathrm{d}W_t \right), \\ \mathrm{d}Y_t &= \left( \bar{\mu} - \mathrm{diag}[\kappa] Y_t \right) \mathrm{d}t + \Sigma^Y(Y_t) \mathrm{d}W_t, \end{aligned}$$

where $r, \bar{\lambda}, \bar{\mu}, \kappa \in \mathbb{R}^n$, $K \in \mathbb{R}_o^{n \times n}$ and $\Sigma^S$, $\Sigma^Y$ are $(n \times 2n)$-dimensional matrix functions as in (4.1.3).

Let the market prices of risk $\left(\eta^Y\right)^{(j-1)}$ from the $(j-1)$-th iteration be given and determine the approximation of the associated sensitivities by their value at $\bar{y} = \mathbb{E}\left[Y_\infty\right]$, i.e.

$$\mathfrak{S}(t) = \frac{g_y(t, \bar{y})}{g(t, \bar{y})}.$$

Then the approximated market prices of risk are of the form

$$\left(\eta^Y_{\approx}\right)^{(j-1)}(t, y) = -k \hat{\Sigma}^Y(y) \mathfrak{S}(t).$$

Recall that $\hat{\Sigma}^Y(y)$ behaves as $\sqrt{y}$. Then the function

$$h(t, y; s) = \exp\left( \tilde{A}(t, s) - y^\top \tilde{B}(t, s) \right)$$

with $\tilde{A}(\cdot, s)$ and $\tilde{B}(\cdot, s) = (\tilde{B}^1(\cdot, s), \ldots, \tilde{B}^n(\cdot, s))^\top$ being the solutions of the ODE system

$$\begin{aligned} \tilde{B}^i_t &= \left( \kappa_i - \tfrac{1-\gamma}{\gamma} \left( \frac{\bar{\sigma}^Y_i \rho_i}{\bar{\sigma}^S_i} \sum_{j=1}^n K_{ji} \bar{\lambda}_j - k \mathfrak{S}_i \left(\bar{\sigma}^Y_i\right)^2 (1 - \rho_i^2) \right) - \tfrac{1}{2} \tfrac{k-\gamma}{\gamma} \left(\bar{\sigma}^Y_i\right)^2 \mathfrak{S}_i \right) \tilde{B}^i \qquad \text{(D.3.3)} \\ &\quad + \tfrac{1-\gamma}{k} \tfrac{1}{2} \tfrac{1}{\gamma} \left( \left( \sum_{j=1}^n \frac{K_{ji} \bar{\lambda}_j}{\bar{\sigma}^S_i} \right)^2 + k^2 \left(\bar{\sigma}^Y_i\right)^2 (1 - \rho_i^2) \mathfrak{S}_i^2 \right) + \tfrac{1}{2} \left(\bar{\sigma}^Y_i\right)^2 \left(\tilde{B}^i\right)^2 \\ \tilde{A}_t &= \bar{\mu}^\top \tilde{B} - \tfrac{1-\gamma}{k} \left( r - \tfrac{\delta \theta}{1-\gamma} \right) \end{aligned}$$

$\tilde{A}(s, s) = 0$, $\tilde{B}(s, s) = \mathbf{0}_{n \times 1}$ solves the linear homogeneous partial differential equation

$$0 = h_t + \tilde{r}\left( \left(\eta^Y_{\approx}\right)^{(j-1)} \right) h + \tilde{\alpha}\left( \left(\eta^Y_{\approx}\right)^{(j-1)} \right) h_y + \frac{1}{2} \mathrm{trace}\left[ \left(\Sigma^Y\right)^\top h_{yy} \Sigma^Y \right],$$

on $[0, s] \times \mathbb{R}$ subject to $h(s, y; s) = 1$, and

$$\tilde{g}(t, y) = \delta^{\frac{1}{\gamma}} \int_t^T h(t, y; s) \mathrm{d}s + \hat{\varepsilon} h(t, y; T),$$

solves (D.3.1).

# Bibliography


[Antonelli, 1993] Antonelli, F. (1993). Backward-forward stochastic differential equations. *The Annals of Applied Probability*, pages 777–793.

[Antonelli, 1996] Antonelli, F. (1996). Stability of backward stochastic differential equations. *Stochastic processes and their applications*, 62(1):103–114.

[Bank and Riedel, 2001a] Bank, P. and Riedel, F. (2001a). Existence and structure of stochastic equilibria with intertemporal substitution. *Finance and Stochastics*, 5:487–509.

[Bank and Riedel, 2001b] Bank, P. and Riedel, F. (2001b). Optimal consumption choice with intertemporal substitution. *The Annals of Applied Probability*, 11(3):750–788.

[Bansal and Yaron, 2004] Bansal, R. and Yaron, A. (2004). Risks for the long run: A potential resolution of asset pricing puzzles. *The journal of Finance*, 59(4):1481–1509.

[Barberis, 2000] Barberis, N. (2000). Investing for the long run when returns are predictable. *The Journal of Finance*, 55(1):225–264.

[Becherer et al., 2023] Becherer, D., Kuissi-Kamdem, W., and Menoukeu-Pamen, O. (2023). Optimal consumption with labor income and borrowing constraints for recursive preferences.

[Bick, 2012] Bick, B. (2012). *Essays on Continuous-Time Portfolio Optimization and Credit Risk*. doctoralthesis, Goethe Universität Frankfurt am Main.

[Bick et al., 2013] Bick, B., Kraft, H., and Munk, C. (2013). Solving constrained consumption–investment problems by simulation of artificial market strategies. *Management Science*, 59(2):485–503.

[Borovička et al., 2011] Borovička, J., Hansen, L. P., Hendricks, M., and Scheinkman, J. A. (2011). Risk-price dynamics. *Journal of Financial Econometrics*, 9(1):3–65.

[Bouchard and Touzi, 2004] Bouchard, B. and Touzi, N. (2004). Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations. *Stochastic Processes and their applications*, 111(2):175–206.

[Brandt, 2010] Brandt, M. W. (2010). Portfolio choice problems. In *Handbook of financial econometrics: Tools and techniques*, pages 269–336. Elsevier.

[Brennan and Xia, 2002] Brennan, M. J. and Xia, Y. (2002). Dynamic asset allocation under inflation. *The journal of finance*, 57(3):1201–1238.

[Briand and Labart, 2014] Briand, P. and Labart, C. (2014). Simulation of BSDEs by Wiener chaos expansion. *The Annals of Applied Probability*, pages 1129–1171.

[Brown and Smith, 2011] Brown, D. B. and Smith, J. E. (2011). Dynamic portfolio optimization with transaction costs: Heuristics and dual bounds. *Management Science*, 57(10):1752–1770.

[Brown et al., 2010] Brown, D. B., Smith, J. E., and Sun, P. (2010). Information relaxations and duality in stochastic dynamic programs. *Operations research*, 58(4-part-1):785–801.

[Campbell, 2003] Campbell, J. Y. (2003). Consumption-based asset pricing. *Handbook of the Economics of Finance*, 1:803–887.

[Campbell and Viceira, 1999] Campbell, J. Y. and Viceira, L. M. (1999). Consumption and portfolio decisions when expected returns are time varying. *The Quarterly Journal of Economics*, 114(2):433–495.

[Chacko and Viceira, 2005] Chacko, G. and Viceira, L. M. (2005). Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets. *The Review of Financial Studies*, 18(4):1369–1402.

[Cox and Huang, 1989] Cox, J. C. and Huang, C.-f. (1989). Optimal consumption and portfolio policies when asset prices follow a diffusion process. *Journal of economic theory*, 49(1):33–83.

[Cox et al., 2005] Cox, J. C., Ingersoll Jr, J. E., and Ross, S. A. (2005). A theory of the term structure of interest rates. In *Theory of valuation*, pages 129–164. World Scientific.

[Cvitanić and Karatzas, 1992] Cvitanić, J. and Karatzas, I. (1992). Convex duality in constrained portfolio optimization. *The Annals of Applied Probability*, pages 767–818.

[Duffie and Epstein, 1992a] Duffie, D. and Epstein, L. G. (1992a). Appendix C with Costis Skiadas, Stochastic differential utility. *Econometrica*, 60(2):353–394.

[Duffie and Epstein, 1992b] Duffie, D. and Epstein, L. G. (1992b). Asset pricing with stochastic differential utility. *The Review of Financial Studies*, 5(3):411–436.

[Duffie et al., 1994] Duffie, D., Geoffard, P.-Y., and Skiadas, C. (1994). Efficient and equilibrium allocations with stochastic differential utility. *Journal of Mathematical Economics*, 23(2):133–146.

[Duffie and Skiadas, 1994] Duffie, D. and Skiadas, C. (1994). Continuous-time security pricing: A utility gradient approach. *Journal of Mathematical Economics*, 23(2):107–131.

[Dumas et al., 1998] Dumas, B., Uppal, R., and Wang, T. (1998). Efficient intertemporal allocations with recursive utility.

[El Karoui et al., 2001] El Karoui, N., Peng, S., and Quenez, M. C. (2001). A dynamic maximum principle for the optimization of recursive utilities under constraints. *Annals of Applied Probability*, pages 664–693.

[Elminejad et al., 2022] Elminejad, A., Havranek, T., and Irsova, Z. (2022). Relative risk aversion: a meta-analysis.

[Epstein, 1987] Epstein, L. G. (1987). The global stability of efficient intertemporal allocations. *Econometrica: Journal of the Econometric Society*, pages 329–355.

[Epstein and Zin, 1989] Epstein, L. G. and Zin, S. E. (1989). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica*, 57(4):937–969.

[Escobar et al., 2017] Escobar, M., Ferrando, S., and Rubtsov, A. (2017). Optimal investment under multi-factor stochastic volatility. *Quantitative Finance*, 17(2):241–260.

[Escobar and Olivares, 2013] Escobar, M. and Olivares, P. (2013). Pricing of mountain range derivatives under a principal component stochastic volatility model. *Applied Stochastic Models in Business and Industry*, 29(1):31–44.

[Gabaix, 2012] Gabaix, X. (2012). Variable rare disasters: An exactly solved framework for ten puzzles in macro-finance. *The Quarterly Journal of Economics*, 127(2):645–700.

[Geoffard, 1996] Geoffard, P.-Y. (1996). Discounting and optimizing: Capital accumulation problems as variational minmax problems. *Journal of Economic Theory*, 69(1):53–70.

[Gobet et al., 2005] Gobet, E., Lemor, J.-P., and Warin, X. (2005). A regression-based monte carlo method to solve backward stochastic differential equations. *Annals of Applied Probability*, pages 2172–2202.

[Guvenen, 2009] Guvenen, F. (2009). A parsimonious macroeconomic model for asset pricing. *Econometrica*, 77(6):1711–1750.

[Hansen et al., 2008] Hansen, L. P., Heaton, J. C., and Li, N. (2008). Consumption strikes back? measuring long-run risk. *Journal of Political Economy*, 116(2):260–302.

[Harrison and Kreps, 1979] Harrison, J. M. and Kreps, D. M. (1979). Martingales and arbitrage in multiperiod securities markets. *Journal of Economic theory*, 20(3):381–408.

[Haugh et al., 2006] Haugh, M. B., Kogan, L., and Wang, J. (2006). Evaluating portfolio policies: A duality approach. *Operations Research*, 54(3):405–418.

[He and Pearson, 1991] He, H. and Pearson, N. D. (1991). Consumption and portfolio policies with incomplete markets and short-sale constraints: the finite-dimensional case 1. *Mathematical Finance*, 1(3):1–10.

[Heston, 1993] Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The review of financial studies*, 6(2):327–343.

[Jacod and Shiryaev, 2013] Jacod, J. and Shiryaev, A. (2013). *Limit theorems for stochastic processes*, volume 288. Springer Science & Business Media.

[Kallsen and Muhle-Karbe, 2010] Kallsen, J. and Muhle-Karbe, J. (2010). Utility maximization in models with conditionally independent increments.

[Kaltenbrunner and Lochstoer, 2010] Kaltenbrunner, G. and Lochstoer, L. A. (2010). Long-run risk through consumption smoothing. *The Review of Financial Studies*, 23(8):3190–3224.

[Kamma and Pelsser, 2022] Kamma, T. and Pelsser, A. (2022). Near-optimal asset allocation in financial markets with trading constraints. *European Journal of Operational Research*, 297(2):766–781.

[Kamma et al., 2020] Kamma, T., Pelsser, A., and Day, N. P. (2020). Closed-form approximations to optimal investment policies in markets with frictions.

[Karatzas and Kardaras, 2007] Karatzas, I. and Kardaras, C. (2007). The numéraire portfolio in semimartingale financial models. *Finance and Stochastics*, 11:447–493.

[Karatzas et al., 1987] Karatzas, I., Lehoczky, J. P., and Shreve, S. E. (1987). Optimal portfolio and consumption decisions for a “small investor” on a finite horizon. *SIAM journal on control and optimization*, 25(6):1557–1586.

[Karatzas et al., 1991] Karatzas, I., Lehoczky, J. P., Shreve, S. E., and Xu, G.-L. (1991). Martingale and duality methods for utility maximization in an incomplete market. *SIAM Journal on Control and Optimization*, 29(3):702–730.

[Karoui et al., 1997] Karoui, N. E., Peng, S., and Quenez, M.-C. (1997). Backward stochastic differential equations in finance. *Mathematical Finance*, 7(1):1–71.

[Kim and Omberg, 1996] Kim, T. S. and Omberg, E. (1996). Dynamic nonmyopic portfolio behavior. *The Review of Financial Studies*, 9(1):141–161.

[Korn and Korn, 2001] Korn, R. and Korn, E. (2001). *Option pricing and portfolio optimization: modern methods of financial mathematics*, volume 31. American Mathematical Soc.

[Kraft et al., 2017] Kraft, H., Seiferling, T., and Seifried, F. T. (2017). Optimal consumption and investment with Epstein–Zin recursive utility. *Finance and Stochastics*, 21(1):187–226.

[Kraft and Seifried, 2014] Kraft, H. and Seifried, F. T. (2014). Stochastic differential utility as the continuous-time limit of recursive utility. *Journal of Economic Theory*, 151:528–550.

[Kraft et al., 2013] Kraft, H., Seifried, F. T., and Steffensen, M. (2013). Consumption-portfolio optimization with recursive utility in incomplete markets. *Finance and Stochastics*, 17(1):161–196.

[Kramkov and Schachermayer, 1999] Kramkov, D. and Schachermayer, W. (1999). The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, pages 904–950.

[Kreps and Porteus, 1978] Kreps, D. M. and Porteus, E. L. (1978). Temporal resolution of uncertainty and dynamic choice theory. *Econometrica: journal of the Econometric Society*, pages 185–200.

[Lin, 2022] Lin, K. (2022). *Essays on consumption portfolio optimization with recursive utility in multi-asset markets*. PhD thesis, Boston University.

[Liu, 2007] Liu, J. (2007). Portfolio selection in stochastic environments. *The Review of Financial Studies*, 20(1):1–39.

[Liu and Muhle-Karbe, 2013] Liu, R. and Muhle-Karbe, J. (2013). Portfolio choice with stochastic investment opportunities: A user’s guide. *arXiv preprint arXiv:1311.1715*.

[Markowitz, 1952] Markowitz, H. (1952). Portfolio selection, The Journal of Finance. 7 (1). *N*, 1:71–91.

[Matoussi and Xing, 2018] Matoussi, A. and Xing, H. (2018). Convex duality for Epstein–Zin stochastic differential utility. *Mathematical Finance*, 28(4):991–1019.

[Mehra and Prescott, 1985] Mehra, R. and Prescott, E. C. (1985). The equity premium: A puzzle. *Journal of Monetary Economics*, 15(2):145–161.

[Merton, 1971] Merton, R. C. (1971). Optimum consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory*, 3(4):373–413.

[Meucci, 2009] Meucci, A. (2009). Review of statistical arbitrage, cointegration, and multivariate ornstein-uhlenbeck. *Available at SSRN: https://ssrn.com/abstract=1404905 or http://dx.doi.org/10.2139/ssrn.1404905.*

[Pan, 2002] Pan, J. (2002). The jump-risk premia implicit in options: Evidence from an integrated time-series study. *Journal of Financial Economics*, 63(1):3–50.

[Pliska, 1986] Pliska, S. R. (1986). A stochastic calculus model of continuous trading: optimal portfolios. *Mathematics of Operations Research*, 11(2):371–382.

[Rockafellar, 1997] Rockafellar, R. T. (1997). *Convex analysis*, volume 11. *Princeton University Press.*

[Samuelson, 1969] Samuelson, P. A. (1969). Lifetime portfolio selection by dynamic stochastic programming. *The Review of Economics and Statistics*, 51(3):239–246.

[Schroder and Skiadas, 1999] Schroder, M. and Skiadas, C. (1999). Optimal consumption and portfolio selection with stochastic differential utility. *Journal of Economic Theory*, 89(1):68–126.

[Schroder and Skiadas, 2003] Schroder, M. and Skiadas, C. (2003). Optimal lifetime consumption-portfolio strategies under trading constraints and generalized recursive preferences. *Stochastic Processes and Their Applications*, 108(2):155–202.

[Seiferling, 2016] Seiferling, T. (2016). Recursive Utility and Stochastic Differential Utility: From Discrete to Continuous Time. doctoralthesis, Technische Universität Kaiserslautern.

[Seiferling and Seifried, 2016] Seiferling, T. and Seifried, F. T. (2016). Epstein-Zin Stochastic Differential Utility: Existence, uniqueness, concavity, and utility gradients. *Uniqueness, Concavity, and Utility Gradients (May 23, 2016).*

[Skiadas, 2008] Skiadas, C. (2008). Dynamic portfolio choice and risk aversion. *Handbooks in Operations Research and Management Science*, 15:789–843.

[Skiadas, 2013] Skiadas, C. (2013). Scale-invariant asset pricing and consumption/portfolio choice with general attitudes toward risk and uncertainty. *Mathematics and Financial Economics*, 7:431–456.

[Von Neumann and Morgenstern, 1944] Von Neumann, J. and Morgenstern, O. (1944). Theory of games and economic behavior.

[Wachter, 2002] Wachter, J. A. (2002). Portfolio and consumption decisions under mean-reverting returns: An exact solution for complete markets. *Journal of Financial and Quantitative Analysis*, 37(1):63–91.

[Wachter, 2010] Wachter, J. A. (2010). Asset allocation. *Annual Review of Financial Economics*, 2(1):175–206.

[Wachter, 2013] Wachter, J. A. (2013). Can time-varying risk of rare disasters explain aggregate stock market volatility? *The Journal of Finance*, 68(3):987–1035.

[Weil, 1990] Weil, P. (1990). Nonexpected utility in macroeconomics. *The Quarterly Journal of Economics*, 105(1):29–42.

[Xing, 2017] Xing, H. (2017). Consumption–investment optimization with Epstein–Zin utility in incomplete markets. *Finance and Stochastics*, 21:227–262.

[Zariphopoulou, 2001] Zariphopoulou, T. (2001). A solution approach to valuation with unhedgeable risks. *Finance and Stochastics*, 5:61–82.

# List of Figures

# List of Tables

# Overview of Scientific Career

| | |
|---|---|
| | **Research Associate** |
| 04/2019 – 03/2024 | Trier University |
| | **Master of Science, Applied Mathematics** |
| 04/2017 – 02/2019 | Trier University |
| 08/2018 – 12/2018 | University of Oslo |
| | **Bachelor of Science, Business Mathematics** |
| 10/2013 – 09/2016 | Trier University |
| | **Abitur** |
| 09/2004 – 03/2013 | Gymnasium Konz |