The Hadamard product and universal power series

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Preface

Universality has been fascinating since the first third of the twentieth century. Many types of this phenomenon have been elaborated since. The universal objects that will be studied in this thesis are universal power series whose universal property is defined by means of overconvergence. Their definition will be given in the fifth chapter.

But studying universal power series and their properties for their own sake is neither the major nor the minor concern—much work has been devoted to this task in the past. A central part of this thesis is the following question: What happens if a given universal power series is modified? Is the result of this modification still universal? If this is the case, one speaks about derived universality. This topic will be treated in detail in the sixth chapter.

Before exploring derived universality, it has to be described how to modify a given power series. This modification is realized by the Hadamard product, which is the other central part of this thesis. First introduced for power series, it was later generalized to functions holomorphic at the origin. But it turns out that considerations concerning universality make it necessary to define a Hadamard product for functions holomorphic in open sets that do not contain the origin. In the third chapter, a definition of a Hadamard product for this situation will be given. Since this new version of the Hadamard product is defined by means of a parameter integral, its definition requires appropriate integration curves. These objects and their properties (especially their existence) will be studied in the second chapter.

The Hadamard product can be regarded as a bilinear and continuous operator between Fréchet spaces. These properties can be exploited to prove the Hadamard multiplication theorem and the Borel-Okada theorem. Furthermore, the new version of the Hadamard product makes it possible to state generalizations of these two theorems that will be proved in the fourth chapter.

There is an intimate link between the Hadamard product and Euler differential operators. This connection will be revealed in the seventh chapter.

In the first chapter, notations and conventions will be introduced. Furthermore, the star product, the set on which the Hadamard product is defined, will be defined.

In the last chapter, open problems concerning universal power series will be posed.

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Contents

Chapter 1

Notations and preliminaries

In the first section, we will set the topological stage upon which the holomorphic functions act.

In the second section, we will introduce the notation concerning the analytic surroundings. The main purpose is to have a mutual language—this is necessary because the concepts are not consistently used throughout the literature.

In the third section, we will define the star product, and we will prove properties of the star product. The star product plays an important role in the definition of the Hadamard product (see section 3.4).

1.1 Topological preliminaries

In this section, we want to summarize essential topological concepts. The notions of topological vector spaces refer to [Rud1].

The set of complex numbers will be endowed with the topology $\mathscr{T}_{\mathbb{C}}$ consisting of all subsets of the complex plane that are open with respect to the norm induced by the absolute value. This is a locally compact topological Hausdorff space, that is not compact. By means of the Alexandrov compactification we obtain the extended complex plane $\mathbb{C}_\infty$ and its topology $\mathscr{T}_\infty$. Equipped with the new topology, we get a compact topological Hausdorff space, that is metrizable—for instance by the chordal metric.

Let $M \subset \mathbb{C}$ and $S \subset \mathbb{C}_\infty$ be sets. By $\overline{M}$ we denote the closure with respect to $\mathscr{T}_{\mathbb{C}}$, and by $\overline{S}^{\infty}$ we denote the closure with respect to $\mathscr{T}_\infty$. By ∂M we denote the boundary with respect $\mathscr{T}_{\mathbb{C}}$, and by $\partial_\infty S$ we denote the boundary with respect to $\mathscr{T}_\infty$. The symbol S^C refers to the complement with respect to the extended complex plane. The complement with respect to the complex plane will be denoted by the symbol $\mathbb{C} \setminus S$. Continuity of functions has to be understood as continuity with respect to the these topologies. By $C(S)$ we denote the linear space of all complex-valued continuous functions on S.

A connected and open subset of the (extended) complex plane is called a domain.

For $\zeta \in S$, we denote by S_ζ the component of S that contains the element ζ.

A subset of the (extended) complex plane is called simply connected if its complement with respect to the extended plane is connected in the extended plane. It can be shown that an open set is simply connected if and only if all of its components are simply connected (domains).

For $\zeta \in \mathbb{C}$, $r > 0$, and $\varrho \geq 0$ we define

$$\begin{aligned}
U_r(\zeta) &:= \{z \in \mathbb{C} : |z - \zeta| < r\}, \\
U_r[\zeta] &:= \{z \in \mathbb{C} : |z - \zeta| \leq r\}, \\
\mathbb{T}_r(\zeta) &:= \{z \in \mathbb{C} : |z - \zeta| = r\}, \\
U_\varrho(\infty) &:= \{z \in \mathbb{C} : |z| > \varrho\} \cup \{\infty\}, \\
U_\varrho[\infty] &:= \{z \in \mathbb{C} : |z| \geq \varrho\} \cup \{\infty\}.
\end{aligned}$$

Since disks and circles around the origin appear frequently, we additionally define $\mathbb{D}_r := U_r(0)$ and $\mathbb{T}_r := \mathbb{T}_r(0)$.

Let Ω be a non-empty open subset of the complex plane. For each $n \in \mathbb{N}$ we define the set

$$K_n(\Omega) := \left\{ z \in \Omega : |z| \leq n, \ \operatorname{dist}(z, \partial\Omega) \geq \frac{1}{n} \right\}. \tag{1.1}$$

If it is clear which open set is meant, we simply write K_n instead of $K_n(\Omega)$. The family $(K_n(\Omega))_{n\in\mathbb{N}}$ forms a compact exhaustion of Ω. Each component of the set $\mathbb{C}_\infty \setminus K_n(\Omega)$ contains a component of $\mathbb{C}_\infty \setminus \Omega$. In particular, if Ω is simply connected, then all the sets $\mathbb{C}_\infty \setminus K_n(\Omega)$ are connected. In this case we call $(K_n(\Omega))_{n\in\mathbb{N}}$ a compact exhaustion with connected complements. Here, we would like to mention one more thing: For a plane compact set K the set $\mathbb{C} \setminus K$ is connected if and only if the set $\mathbb{C}_\infty \setminus K$ is connected.

It will also be important to study subsets of the extended complex plane. Let Ω be an open subset of the extended complex plane containing the point at infinity. For $n \in \mathbb{N}$ the (countable) family consisting of the sets

$$K_n(\Omega) \ := \ \left\{ z \in \Omega \cap \mathbb{C} : \operatorname{dist}\big(z, \partial(\Omega \cap \mathbb{C})\big) \geq \frac{1}{n} \right\} \cup \{\infty\} \tag{1.2}$$

forms a compact exhaustion of Ω.

For a non-empty open subset Ω of the (extended) plane, a non-empty compact set $K \subset \Omega$, and $f : K \to \mathbb{C}$ continuous we define

$$\|f\|_K \ := \ \max\{|f(z)| : z \in K\}. \tag{1.3}$$

With the exhaustion (1.1) or respectively (1.2), the vector space topology induced by the family $\big(\| \cdot \|_{K_n(\Omega)}\big)_{n \in \mathbb{N}}$ is called the topology of compact convergence or the compact-open topology. The linear space $C(\Omega)$ will always be endowed with this topology, that makes it a Fréchet space.

1.2 Analytical preliminaries

Let Ω be a non-empty open subset of the extended complex plane. A function $f : \Omega \to \mathbb{C}$ is called holomorphic in Ω if f is continuous on Ω and $f\big|_{\Omega \cap \mathbb{C}}$ is holomorphic in $\Omega \cap \mathbb{C}$. If Ω does not contain the point at infinity, this definition coincides with the usual definition of holomorphy. If Ω contains the point at infinity, we set $\tilde{\Omega} := \{1/\omega : \omega \in \Omega, \, \omega \neq 0\}^\dagger$ and associate with f a new function $\tilde{f} : \tilde{\Omega} \to \mathbb{C}$ defined by $\tilde{f}(z) := f(1/z)$. It can be shown that f is holomorphic in Ω if and only if $f\big|_{\Omega \cap \mathbb{C}}$ is holomorphic in $\Omega \cap \mathbb{C}$ and $\tilde{f}$ is holomorphic at 0. The derivatives of f at the point at infinity are defined by

$$f^{(k)}(\infty) \ := \ \tilde{f}^{(k)}(0) \qquad (k \in \mathbb{N}_0).$$

For $k \in \mathbb{N}$ we have

$$\begin{aligned} f^{(k)}(\infty) \ &= \ \lim_{z \to 0} \frac{f^{(k-1)}(1/z) - f^{(k-1)}(\infty)}{z} \\ &= \ \lim_{w \to \infty} w \cdot \Big(f^{(k-1)}(w) - f^{(k-1)}(\infty) \Big). \end{aligned}$$

† We set $1/\infty := 0$.

Moreover, f has a power series expansion around the point at infinity of the form

$$f(z) \;=\; \sum_{\nu=0}^{\infty} \frac{f^{(\nu)}(\infty)}{\nu!} \cdot \frac{1}{z^\nu} \qquad \big(z \in U_d(\infty)\big)$$

with $d := \big(\operatorname{dist}(0, \partial\tilde{\Omega})\big)^{-1}$. If $R > 0$ satisfies $U_R[\infty] \subset \Omega$, the coefficients in this expansion can be represented in the form

$$\frac{f^{(\nu)}(\infty)}{\nu!} \;=\; \frac{1}{2\pi i} \int\limits_{|\zeta|=R} f(\zeta)\, \zeta^{\nu-1} d\zeta \qquad \big(\nu \in \mathbb{N}_0\big).$$

We remark that in general the derivatives are not continuous at the point at infinity, as the next example shows.

1.2.1 Example:
For the function $f : \mathbb{C}_\infty \setminus \{1\} \to \mathbb{C}$ defined by $f(z) := 1/(1-z)$ we have

$$f(z) \;=\; \sum_{\nu=1}^{\infty} \frac{-1}{z^\nu} \qquad \big(z \in U_1(\infty)\big).$$

Thus, we get $f^{(k)}(\infty) = -(k!)$, but $\lim\limits_{z\to\infty} f^{(k)}(z) = 0$ for all $k \in \mathbb{N}$. ◇

A function f referred to as holomorphic in Ω has to be understood as a function $f : \Omega \to \mathbb{C}$ that is holomorphic in Ω. If f is holomorphic in an open set containing the origin, then we denote by $\sum\limits_{\nu=0}^{\infty} f_\nu z^\nu$ its local power series expansion around the origin. Similarly, if f is holomorphic in an open set containing the point at infinity, we denote its power series expansion around the point at infinity by $\sum\limits_{\nu=0}^{\infty} f_\nu / z^\nu$.

Next, we define several sets of holomorphic functions that will frequently appear.

1.2.2 Definition:
Let $\Omega \subset \mathbb{C}_\infty$ be a non-empty open set and $k \in \mathbb{N}_0$. If Ω contains the point at infinity, we define

$$H^{(k)}(\Omega) \;:=\; \Big\{ f \in \mathbb{C}^\Omega : f \text{ holomorphic and } f^{(p)}(\infty) = 0 \ (0 \le p \le k) \Big\}.$$

Moreover, by $H(\Omega)$ we denote the set $H^{(0)}(\Omega)$. If Ω does not contain the point at infinity, we denote by $H(\Omega)$ or by $H^{(k)}(\Omega)$ the set of all functions holomorphic in Ω. ◇

The linear space $H^{(k)}(\Omega)$ will always be endowed with the compact-open topology. Since it is a closed linear subspace of $C(\Omega)$, it is a Fréchet space.

If K is a non-empty compact subset of the complex plane, we denote by $A(K)$ the subset of $C(K)$ that consists of all functions whose restrictions to the interior of K are holomorphic. It is admissible that the interior of K is the empty set; in this case the set $A(K)$ coincides with $C(K)$. The linear space $A(K)$ will always be endowed with the uniform norm, that makes it a Banach space.

Next, we will introduce an important class of operators. To this end, let $\xi \in \{0, \infty\}$. We consider the germs of holomorphic functions at ξ: On the set

$$\Big\{\big(f, U\big) : U \text{ neighborhood of } \xi,\ f \in H(U)\Big\}$$

the relation

$$\big(f, U\big) \sim_{\{\xi\}} \big(g, V\big) \quad \text{if } f^{(k)}(\xi) = g^{(k)}(\xi) \text{ for all } k \in \mathbb{N}_0$$

defines an equivalence relation. The equivalence class $[(f, U)]_{\sim_{\{\xi\}}}$ is called the germ of f (at ξ). By $H\big(\{\xi\}\big)$ we denote the corresponding quotient space. Moreover, we identify a germ with each of its representatives.

1.2.3 Remark:
Let $\Omega \subset \mathbb{C}_\infty$ be a non-empty open set, $\xi \in \{0, \infty\}$, and $T : H(\{\xi\}) \to H(\Omega)$ linear. The continuity of T can be characterized in the following way (cf. [Köthe, pp. 375]):

(a) Let $\xi = 0$. Then T is continuous if and only if for every $K \subset \Omega$ compact and every $m \in \mathbb{N}$ there exists a $C > 0$ so that $\|T(f)\|_K \leq C \cdot \|f\|_{U_{1/m}[0]}$ for every $f \in A\big(U_{1/m}[0]\big)$.

(b) Let $\xi = \infty$. Then T is continuous if and only if for every $K \subset \Omega$ compact and every $m \in \mathbb{N}$ there exists a $C > 0$ so that $\|T(f)\|_K \leq C \cdot \|f\|_{U_m[\infty]}$ for every $f \in A\left(U_m[\infty]\right)$. $\diamond$

An infinite matrix A is a mapping $A : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{C}$. We usually write $A = (a_{n\nu})_{(n,\nu)\in\mathbb{N}_0\times\mathbb{N}_0} = (a_{n\nu})$ with $a_{n\nu} := A(n, \nu)$ $(n, \nu \in \mathbb{N}_0)$. Furthermore, in the whole thesis, we assume that each infinite Matrix $A = (a_{n\nu})$ satisfies the following condition:

$$\lim_{\nu\to\infty} \sqrt[\nu]{|a_{n\nu}|} = 0 \quad (n \in \mathbb{N}_0), \tag{1.4}$$

i.e. each row of the matrix consists of the Taylor coefficients of an entire function.

Using infinite matrices, we can now define the operators mentioned above.

1.2.4 Definition:
Let A be an infinite matrix. For $n \in \mathbb{N}_0$ we define $\sigma_n^A : H(\{0\}) \to H(\mathbb{C})$ by

$$\sigma_n^A(f)(z) := \sum_{\nu=0}^{\infty} a_{n\nu}\, f_\nu\, z^\nu \qquad \bigl(z \in \mathbb{C}\bigr).$$

The sequence of operators $(\sigma_n^A)_{n\in\mathbb{N}_0}$ is called *A-transformation*, and the sequence $(\sigma_n^A(f))_{n\in\mathbb{N}_0}$ is called the *A-transform of f*. Moreover, we define $s_n : H\bigl(\{0\}\bigr) \to H(\mathbb{C})$ by

$$s_n(f)(z) := \sum_{\nu=0}^{n} f_\nu\, z^\nu \qquad \bigl(z \in \mathbb{C}\bigr).$$

The operator s_n is called the *n-th partial sum operator.* ◇

1.2.5 Remarks:

1. If there is no confusion to be expected, we simply write σ_n instead of σ_n^A.
2. If we consider the infinite matrix A defined by

   $$a_{n\nu} := \begin{cases} 1, & \nu \le n \\ 0, & \nu > n \end{cases} \qquad (n, \nu \in \mathbb{N}_0),$$

   then we have $\sigma_n^A = s_n$ for all $n \in \mathbb{N}_0$. ◇

The partial sum operators s_n are applied to germs of holomorphic functions at the origin. We will also need operators that are applied to germs holomorphic at the point at infinity (see the Hadamard multiplication theorem at infinity).

1.2.6 Definition:
For $n \in \mathbb{N}$ we define $s_n^\infty : H(\{\infty\}) \to H(\mathbb{C}_\infty \setminus \{0\})$ by

$$s_n^\infty(f)(z) := \sum_{\nu=1}^{n} f_\nu\, z^{-\nu} \qquad \bigl(z \in \mathbb{C}_\infty \setminus \{0\}\bigr).$$

The operator s_n^∞ is called the *n-th partial sum operator at infinity.* ◇

The operators $\sigma_n^A : H(\{0\}) \to H(\mathbb{C})$ have the following two important properties.

1.2.7 Proposition:
Let A be an infinite matrix and $n \in \mathbb{N}_0$. Then $\sigma_n^A : H\big(\{0\}\big) \to H(\mathbb{C})$ is linear and continuous.

Proof: The linearity follows from the associativity and commutativity property of series.

Let $m \in \mathbb{N}$, $K \subset \mathbb{C}$ compact, $r \in (0, 1/m)$, and $M := \max\{|z| : z \in K\} + 1$. Furthermore, choose $\varepsilon := r/(2M)$. By (1.4) there exists $\nu_0 \in \mathbb{N}_0$ so that $|a_{n\nu}| \le \varepsilon^\nu$ for all $\nu > \nu_0$. If we set $T := \sigma_n^A\big|_{A(\overline{\mathbb{D}}_{1/m})}$, we get for every $f \in A\big(\overline{\mathbb{D}}_{1/m}\big)$ and every $z \in K$:

$$\begin{aligned} |T(f)(z)| \;&=\; \left| \sum_{\nu=0}^{\infty} a_{n\nu} \cdot \left(\frac{1}{2\pi i} \int\limits_{|\zeta|=r} \frac{f(\zeta)}{\zeta^{\nu+1}}\, d\zeta \right) \cdot z^\nu \right| \\ &\le\; \left\{ \sum_{\nu=0}^{\nu_0} |a_{n\nu}| \cdot \left(\frac{M}{r}\right)^{\nu} + \sum_{\nu=\nu_0+1}^{\infty} \left(\frac{\varepsilon M}{r}\right)^{\nu} \right\} \cdot \|f\|_{\overline{\mathbb{D}}_{1/m}} \;=:\; C \cdot \|f\|_{\overline{\mathbb{D}}_{1/m}}. \end{aligned}$$

(Notice that C is independent of z and of f.) Since $z \in K$ was arbitrary, this yields

$$\|T(f)\|_K \;\le\; C \cdot \|f\|_{\overline{\mathbb{D}}_{1/m}} \qquad \Big(f \in A\big(\overline{\mathbb{D}}_{1/m}\big)\Big).$$

According to Remark 1.2.3, T is continuous. □

From this proposition, we immediately get the following consequence.

1.2.8 Corollary:
For each $n \in \mathbb{N}_0$ the operator $s_n : H\big(\{0\}\big) \to H(\mathbb{C})$ is linear and continuous.

Proof: Apply Proposition 1.2.7 to the matrix of Remark 1.2.5.2. □

1.2.9 Remark:
It can also be shown that each operator $s_n^\infty : H(\{\infty\}) \to H(\mathbb{C}_\infty \setminus \{0\})$ is linear and continuous. ◇

1.3 The star product

The star product naturally emerges in the context of the Borel-Okada theorem and the Hadamard multiplication theorem. For the original statements we refer to [Bo99], [Ok25], or [Ha99].

Working in the extended complex plane, makes it necessary to agree upon some arithmetical rules concerning the point at infinity. For this purpose, we set

$$\frac{1}{0} := \infty, \quad \frac{1}{\infty} := 0, \quad \frac{\infty}{0} := \infty, \quad \frac{0}{\infty} := 0$$

as well as

$$\zeta \cdot \infty \ := \ \infty \cdot \zeta \ := \ \infty \qquad \big(\zeta \in \mathbb{C}_\infty \setminus \{0\}\big). \tag{1.5}$$

Expressions of the form "$0 \cdot \infty$" and "$\infty \cdot 0$" will be excluded from our considerations.

Let A and B be non-empty subsets of the extended plane in such a way that $0 \notin A$ if $\infty \in B$ and $0 \notin B$ if $\infty \in A$. In this case we say that the algebraic product

$$A \cdot B \ := \ \big\{ab : a \in A, b \in B\big\}$$

of A and B is well defined. For the empty set we define

$$A \cdot \emptyset \ := \ \emptyset \cdot A \ := \ \emptyset \cdot \emptyset \ := \ \emptyset.$$

Furthermore, we set

$$\frac{1}{A} \ := \ 1/A \ := \ A^{-1} \ := \ \left\{\frac{1}{a} : a \in A\right\}.$$

1.3.1 Example:
Let $\zeta \in \mathbb{C}$ and $r > 0$ so that $r \neq |\zeta|$. Then the relation

$$\frac{1}{\mathbb{T}_r(\zeta)} \ = \ \left\{ z \in \mathbb{C} : \left| z - \frac{\overline{\zeta}}{|\zeta|^2 - r^2} \right| = \frac{r}{\big||\zeta|^2 - r^2\big|} \right\}$$

holds, i.e. $1/\mathbb{T}_r(\zeta)$ is also a circle.‡ ◇

‡ Which is not really surprising since inversion is a Möbius transformation.

For a non-empty set $S \subsetneq \mathbb{C}_\infty$ the following important property holds:

$$\left(\frac{\zeta}{S}\right)^C = \frac{\zeta}{S^C} \qquad \big(\zeta \in \mathbb{C} \setminus \{0\}\big). \tag{1.6}$$

We proceed with a property of the algebraic product.

1.3.2 Lemma:
Let $A, B \subset \mathbb{C}_\infty$ compact sets in such a way that $A \cdot B$ is well defined. Then $A \cdot B$ is compact, too.

Proof: 1. Assume, without loss of generality, that $A \neq \emptyset$ and $B \neq \emptyset$. The multiplication mapping

$$\big(A \times B,\, (\mathscr{T}_\infty \cap A) \times (\mathscr{T}_\infty \cap B)\big) \to \big(\mathbb{C}_\infty, \mathscr{T}_\infty\big), \quad (z, w) \mapsto z \cdot w$$

is continuous. (The multiplication involving the point at infinity has to be understood in the sense of (1.5).) To see this, let $(z, w) \in A \times B$ and $((z_n, w_n))_{n \in \mathbb{N}}$ a sequence in $A \times B$ with $(z_n, w_n) \to (z, w)$ in the product topology as $n \to \infty$. Denote by χ the chordal metric on $\mathbb{C}_\infty$. If $z, w \in \mathbb{C}$, the assertion follows from the continuity of the multiplication in $\mathbb{C}$. Now let $z = \infty$. (Hence, we get $|z_n| \to \infty$ as $n \to \infty$.) Since $A \cdot B$ is well defined, we have $w \neq 0$. Without loss of generality, we can assume that $z_n \neq \infty$ and $w_n \neq \infty$ for all $n \in \mathbb{N}$. We obtain

$$\chi(z_n \cdot w_n, \infty) \;=\; \frac{1}{\sqrt{1 + |z_n \cdot w_n|^2}} \;\leq\; \frac{1}{\sqrt{1 + C^2 \cdot |z_n|^2}} \;\xrightarrow{n \to \infty}\; 0.$$

From this we get the continuity. (The case $w = \infty$ is handled analogously.)

2. Since the multiplication mapping is continuous, and since $A \times B$ is compact, the set $A \cdot B$, i.e. the image of $A \times B$ under the multiplication mapping, is also compact. □

Now, we define the star product.

1.3.3 Definition:
Let $A, B \subset \mathbb{C}_\infty$ be non-empty sets. If the two conditions

- $0 \in A$ if $\infty \in \mathbb{C}_\infty \setminus B$,
- $0 \in B$ if $\infty \in \mathbb{C}_\infty \setminus A$

are satisfied, we say that A and B satisfy the *star condition*. If A and B satisfy the star condition, then the set

$$A * B \;:=\; \mathbb{C}_\infty \setminus \big((\mathbb{C}_\infty \setminus A) \cdot (\mathbb{C}_\infty \setminus B)\big)$$

is called the *star product of A and B*. If in addition $A \neq \mathbb{C}_\infty$, $B \neq \mathbb{C}_\infty$, and $A * B \neq \emptyset$, then A and B are called *star-eligible*. Furthermore, we define

$$S^* \;:=\; \frac{1}{\mathbb{C}_\infty \setminus S}$$

for each set $S \subsetneq \mathbb{C}_\infty$. ◇

1.3.4 Remark:
Notice that the undefined expressions "$0 \cdot \infty$" and "$\infty \cdot 0$" do not appear in $(\mathbb{C}_\infty \setminus A) \cdot (\mathbb{C}_\infty \setminus B)$ if A and B satisfy the star condition. ◇

In some papers and monographs only plane sets are considered, and the complement is taken with respect to the plane instead of the extended plane. For star-eligible sets the result is the same (see Proposition 1.3.10.4). In this setting the set $(\mathbb{C} \setminus M)^{-1} \cup \{0\}$ appears for a set $M \subsetneq \mathbb{C}$ with $0 \in M$ (see [Mü92]). But $(\mathbb{C} \setminus M)^{-1} \cup \{0\} = M^*$ holds for every such M.

We give some first examples.

1.3.5 Examples:

1. If $A \subset \mathbb{C}_\infty$ is not empty, then A and $\mathbb{C}_\infty$ satisfy the star condition, and the relation $\mathbb{C}_\infty * A = A * \mathbb{C}_\infty = \mathbb{C}_\infty$ holds.
2. For $r, s > 0$ we have $U_r(0) * U_s(0) = U_{rs}(0)$.
3. For $R, S \geq 0$ we have $U_R(\infty) * U_S(\infty) = U_{RS}(\infty)$.
4. For $A := \mathbb{C}_\infty \setminus \{e^{it} : 0 \leq t \leq \pi\}$ and $B := \mathbb{C}_\infty \setminus [-\infty, 0]$ we have $A * B = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. ◇

1.3.6 Example:
Let $A \subset \mathbb{C}_\infty$ and $B \subset \mathbb{C} \setminus \{0\}$ be non-empty and star-eligible sets. Then the relation

$$A * (\mathbb{C}_\infty \setminus B) \;=\; \bigcap_{b \in B} b \cdot A$$

holds. ◇

1.3.7 Example:
Let $A \subset \mathbb{C}_\infty$ be a non-empty set and $\eta \in \mathbb{C}_\infty$.

(a) If $\eta \notin \{0, \infty\}$, then A and $\mathbb{C}_\infty \setminus \{\eta\}$ are star-eligible, and the relation

$$A * \big(\mathbb{C}_\infty \setminus \{\eta\}\big) \;=\; \eta \cdot A$$

holds. In particular, $\mathbb{C}_\infty \setminus \{1\}$ is a neutral element for the star product.

(b) If $\eta = 0$ and $\infty \in A$, then A and $\mathbb{C}_\infty \setminus \{0\}$ are star-eligible, and the relation

$$A * \big(\mathbb{C}_\infty \setminus \{0\}\big) \;=\; \mathbb{C}_\infty \setminus \{0\}$$

holds.

(c) If $\eta = \infty$ and $0 \in A$, then A and $\mathbb{C}_\infty \setminus \{\infty\} = \mathbb{C}$ are star-eligible, and the relation

$$A * \mathbb{C} \;=\; \mathbb{C}$$

holds. ◇

1.3.8 Example:
Let $\theta \in \mathbb{R}$, $p \in \mathbb{N}$, and

$$\xi_{\theta,j}^{(p)} \;:=\; \exp\!\left(\frac{2\pi(j-1)+\theta}{p}\, i\right) \qquad (1 \le j \le p).$$

If $\Omega_1 \subset \mathbb{C}_\infty$ is a non-empty open set, and if $\Omega_2 := \mathbb{C}_\infty \setminus \{\xi_{\theta,j}^{(p)} : 1 \le j \le p\}$, then Ω_1 and Ω_2 are star-eligible, and we have

$$\Omega_1 * \Omega_2 \;=\; \bigcap_{j=1}^{p} \xi_{\theta,j}^{(p)} \cdot \Omega_1$$

according to Example 1.3.6. ◇

Definition 1.3.3 gives rise to the question whether two sets that satisfy the star condition are necessarily star-eligible. The next example shows that this is not the case.

1.3.9 Examples:

1. Let $\Omega_1 := \mathbb{C}_\infty \setminus [0,\infty]$ and $\Omega_2 := \mathbb{C}_\infty \setminus \mathbb{T}_1$. The sets Ω_1 and Ω_2 satisfy the star condition. But we have $\Omega_1 * \Omega_2 = ([0,\infty]\cdot\mathbb{T}_1)^C = \emptyset$.
2. Let $\theta \in \mathbb{R}$ and $p \in \mathbb{N}\setminus\{1\}$. For $\xi_{\theta,j}^{(p)}$, Ω_1, and Ω_2 as in Example 1.3.8, we denote by

$$W_{\theta,j}^{(p)} \;:=\; \left\{ te^{i\beta} : t>0,\; \frac{2\pi(j-1)+\theta}{p} \;<\; \beta \;<\; \frac{2\pi j+\theta}{p} \right\}$$

be the angle induced by the two rays starting in the origin and emanating through the points $\xi_{\theta,j}^{(p)}$ and $\xi_{\theta,j+1}^{(p)}$ $(1\le j\le p)$. If there exist $j\in\mathbb{N}$, $j\le p$, and $\theta\in\mathbb{R}$ so that $\Omega_1^C \supset W_{\theta,j}^{(p)}$, then we have $\Omega_1 * \Omega_2 = \emptyset$. ◇

Next, we summarize some properties of the star product.

1.3.10 Proposition:
Let $A, B \subset \mathbb{C}_\infty$ be star-eligible sets, $\zeta \in \mathbb{C}\setminus\{0\}$, and $M \subset \mathbb{C}_\infty$ in such a way that $M\cdot B^*$ is well defined. Then the following assertions hold:

1. $A*B = B*A$.
2. $0 \in A*B$ *if and only if* $0 \in A\cap B$.
3. $\infty \in A*B$ *if and only if* $\infty \in A\cap B$.
4. *If* $A,B\subset\mathbb{C}$, *then* $A*B = \mathbb{C}\setminus\big((\mathbb{C}\setminus A)\cdot(\mathbb{C}\setminus B)\big)$.
5. $M\subset A*B$ *if and only if* $M\cdot B^*\subset A$.
6. $\zeta\in A*B$ *if and only if* $A^C\subset \zeta/B$.
7. *If in addition B is an open set and if $K\subset A*B$ is a compact set, then $K\cdot B^*$ is a compact subset of A.*

Proof: *ad 1.:* For complex numbers this follows from the commutativity of the multiplication; for the point at infinity this follows from (1.5).

ad 2.: We have $0\in A*B$ if and only if $0\notin A^C\cdot B^C$, which is the case if and only if $0\notin A^C$ and $0\notin B^C$.

ad 3.: We have $\infty\in A*B$ if and only if $\infty\notin A^C\cdot B^C$, which is the case if and only if $\infty\notin A^C$ and $\infty\notin B^C$.

ad 4.: Since A and B are plane sets, we get

$$\big((\mathbb{C}\setminus A)\cdot(\mathbb{C}\setminus B)\big)\cup\{\infty\}=(\mathbb{C}_\infty\setminus A)\cdot(\mathbb{C}_\infty\setminus B).$$

This implies

$$\mathbb{C}\setminus\big((\mathbb{C}\setminus A)\cdot(\mathbb{C}\setminus B)\big)=\mathbb{C}_\infty\setminus\big((\mathbb{C}_\infty\setminus A)\cdot(\mathbb{C}_\infty\setminus B)\big)=A*B.$$

ad 5.: (i) Let $M\subset A*B$ and $z\in M\cdot B^*$. Then there are elements $a\in M$ and $b\in B^*$ so that $z=ab$.

Case 1: $a=0$. By part 2 we get $0\in A\cap B$, and hence $z\in A$.

Case 2: $b=0$. Thus $\infty\notin B$. According to the star condition, we get $z\in A$.

Case 3: $a=\infty$. By part 3 we get $\infty\in A\cap B$, and hence $z\in A$.

Case 4: $b=\infty$. Thus $0\notin B$. According to the star condition, we get $z\in A$.

Case 5: $a\notin\{0,\infty\}$ and $b\notin\{0,\infty\}$. Assume that $z\notin A$. Then we get $a=zb^{-1}\in A^CB^C=(A*B)^C$, which is a contradiction.

(ii) Let $M\cdot B^*\subset A$ and $z\in M$.

Case 1: $z=0$. Thus $0\in A$. Since the algebraic product is well defined, we have $\infty\notin B^*$, and hence $0\in B$. Therefore, we have $z\in A*B$.

Case 2: $z=\infty$. Thus $\infty\in A$. Since the algebraic product is well defined, we have $0\notin B^*$, and hence $\infty\in B$. Therefore, we have $z\in A*B$.

Case 3: $z\neq 0$ and $z\neq\infty$. Assume that $z\notin A*B$. Then there exist $a\in A^C$ and $b\in B^C$ so that $z=ab$. Thus, we get $a=zb^{-1}\in M\cdot B^*\subset A$, which is a contradiction.

ad 6.: Part 5 yields $\zeta\in A*B$ if and only if $\zeta\cdot B^*\subset A$. By taking complements, we get $\zeta\cdot B^*\subset A$ if and only if $A^C\subset(\zeta\cdot B^*)^C=(\zeta/B^C)^C=\zeta/B$, where the last equality is furnished by (1.6).

ad 7.: Because of part 5 the set $K\cdot B^*$ is contained in A. The sets K and B^* are compact subsets of the extended plane. According to Lemma 1.3.2, $K\cdot B^*$ is compact, too. □

1.3.11 Remark:

If $A,B\subset\mathbb{C}_\infty$ both contain the origin or both contain the point at infinity, then A and B are star-eligible. (The assumption implies that A and B satisfy the star condition; the non-emptiness is guaranteed by Proposition 1.3.10 part 2 or 3, respectively.) We will use this property of the star product tacitly throughout the rest of the thesis. $\diamond$

Next, we will prove some topological properties of the star product.

1.3.12 Proposition:
If $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ are open sets that satisfy the star condition, then $\Omega_1 * \Omega_2$ is an open set.

Proof: Since Ω_1 and Ω_1 satisfy the star condition, $\Omega_1^C \cdot \Omega_2^C$ is well defined. According to the assumption, Ω_1^C and Ω_2^C are compact, and hence $\Omega_1^C \cdot \Omega_2^C$ is also compact by Lemma 1.3.2. This implies that $\Omega_1 * \Omega_2$ is open. $\square$

We remark that the star product can be the empty set (see Example 1.3.9). As the next example shows, Proposition 1.3.12 is not true in general for the plane version of the star product.

1.3.13 Example:
Let $\Omega_1 := \mathbb{C} \setminus [0, \infty)$ and $\Omega_2 := \mathbb{C} \setminus \{i + e^{it} : -\pi/2 \leq t \leq \pi/2\}$. Here we have

$$(\mathbb{C} \setminus \Omega_1) \cdot (\mathbb{C} \setminus \Omega_2) \;=\; \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0, \mathrm{Im}(z) > 0\} \cup \{0\}$$

which is neither closed in the plane nor in the extended plane. $\diamond$

According to Proposition 1.3.12, the star product of open sets that satisfy the star condition is open (maybe empty). What can we say about other topological properties? Is the star product of connected star-eligible sets connected again? The next example shows that this is not the case.

1.3.14 Example:
The sets

$$\Omega_1 \;:=\; \Omega_2 \;:=\; \mathbb{C}_\infty \setminus \left(\{e^{it} : -\pi/2 \leq t \leq \pi/2\} \cup [1, \infty]\right)$$

are star-eligible, simply connected plane domains. The star product is the set

$$\Omega_1 * \Omega_2 \;=\; \mathbb{D} \cup \{z \in \mathbb{C} : \mathrm{Re}(z) < 0,\; |z| > 1\}$$

which is not connected. $\diamond$

Notice that the star product in Example 1.3.14 is simply connected. This is true for a more general situation.

1.3.15 Proposition:
Let $A \subset \mathbb{C}_\infty$ and $B \subset \mathbb{C}$ satisfy the star condition. If B is simply connected, then $A * B$ is also simply connected.

Proof: Without loss of generality, we may assume that the star product is not the empty set. Moreover, we have

$$(A * B)^C \ = \bigcup_{w \in A^C} w \cdot B^C.$$

Since $w \cdot B^C$ is connected and $\infty \in w \cdot B^C$ for all $w \in A^C$, the set $(A * B)^C$ is connected. Hence, $A * B$ is simply connected. □

Let $0 \leq s_1 < s_2 \leq \infty$ and $0 < \alpha \leq \pi$. The set

$$G\,(s_1, s_2; \alpha) \ := \ \left\{ se^{i\theta} : s_1 < s < s_2,\ |\theta| < \alpha \right\}$$

is called an annular sector (with respect to s_1, s_2, and α). Such an annular sector is a simply connected domain in the plane.

Let $0 < r_1 \leq r_2 < \infty$ and $0 \leq \beta < \pi$. Then we define the sets

$$B\,(r_1, r_2; \beta) \ := \ \left\{ re^{i\theta} : r_1 \leq r \leq r_2,\ |\theta| \leq \beta \right\}$$

and

$$\Omega\,(r_1, r_2; \beta) \ := \mathbb{C}_\infty \setminus B(r_1, r_2; \beta).$$

The set $\Omega(r_1, r_2; \beta)$ is called a complemented annular sector (with respect to r_1, r_2, and β). A complemented annular sector is a domain in the plane.

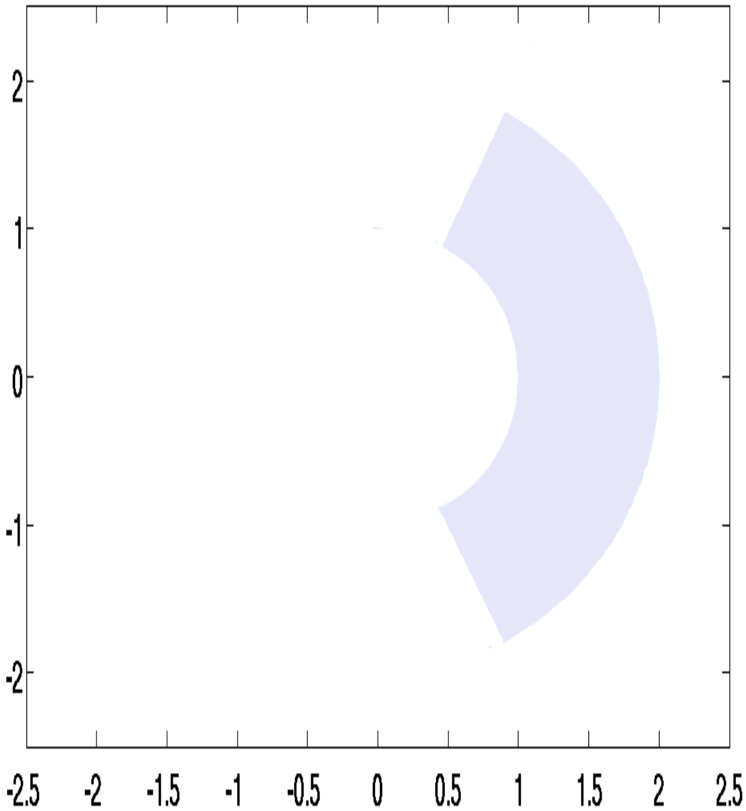

Figure 1.1: The domain $G(1, 2; \pi/4)$

1.3.16 Proposition:
For annular sectors and complemented annular sectors the relation

$$\Omega\,(r_1, r_2; \beta) * G\,(s_1, s_2; \alpha) \;=$$

$$= \begin{cases} G\,(s_1 r_2,\, r_1 s_2;\, \alpha - \beta) & , \text{ if } \alpha > \beta \text{ and } s_1 r_2 < r_1 s_2 \\ \emptyset & , \text{ else} \end{cases}$$

holds.

Proof: According to Example 1.3.6, we have

$$\Omega(r_1, r_2; \beta) * G(s_1, s_2; \alpha) \;=\; \bigcap_{b \in B(r_1, r_2; \beta)} b \cdot G(s_1, s_2; \alpha)$$

$$= \bigcap_{\substack{r_1 \le r \le r_2 \\ |\delta| \le \beta}} re^{i\delta} \cdot G(s_1, s_2; \alpha) \qquad (\star)$$

$$= \bigcap_{\substack{r_1 \le r \le r_2 \\ |\delta| \le \beta}} \left\{ rse^{i(\delta+\theta)} : s_1 < s < s_2,\ |\theta| < \alpha \right\}.$$

(i) Let $\alpha > \beta$ and $s_1 r_2 < r_1 s_2$. Then the star product equals

$$\left\{ te^{i\theta} : s_1 r_2 < t < r_1 s_2,\ |\theta| < \alpha - \beta \right\} = G(s_1 r_2,\ r_1 s_2;\ \alpha - \beta).$$

(ii) Let $s_1 r_2 \ge r_1 s_2$. Then we get

$$(r_1 \cdot G(s_1, s_2; \alpha)) \cap (r_2 \cdot G(s_1, s_2; \alpha)) \;=\; \emptyset,$$

which implies the emptiness of the star product.

(iii) Now let $s_1 r_2 < r_1 s_2$ and $\alpha \le \beta$. According to $(\star)$, the star product must be confined to the annulus $A := \{z \in \mathbb{C} : s_1 r_2 < |z| < r_1 s_2\}$. Let $\zeta = |\zeta| e^{i\varphi} \in A$. For reasons of symmetry, we can assume that $\varphi \in [0, \pi]$. Moreover, since only rotation arguments are used, we can assume $|\zeta| = 1$ without loss of generality. Denote by $b(\alpha)$ the arc $\{e^{i\theta} : |\theta| < \alpha\}$. If $e^{i\varphi} \notin b(\alpha)$, then we get $e^{i\varphi} \notin \Omega(r_1, r_2; \beta) * G(s_1, s_2; \alpha)$. If $e^{i\varphi} \in b(\alpha)$, then we have $\varphi - \alpha \in [-\beta, 0]$. Thus, we obtain $e^{i\varphi} \notin e^{i(\varphi-\alpha)} \cdot b(\alpha)$, i.e. $e^{i\varphi} \notin \Omega(r_1, r_2; \beta) * G(s_1, s_2; \alpha)$. These considerations show that for each $\zeta \in A$ there exists a $\delta \in [-\beta, \beta]$ so that $\zeta \notin \{te^{i(\delta+\theta)} : s_1 r_2 < t < r_1 s_2,\ |\theta| < \alpha\}$. Hence, the star product is the empty set. □

1.3.17 Example:
Consider the domains $\Omega\left(\frac{1}{2}, 6; \frac{\pi}{2}\right)$ and $G\left(\frac{1}{4}, 4; \frac{3}{4}\pi\right)$. According to Proposition 1.3.6, they are star-eligible, and we have

$$\Omega\left(\frac{1}{2}, 6; \frac{\pi}{2}\right) * G\left(\frac{1}{4}, 4; \frac{3}{4}\pi\right) \;=\; G\left(\frac{3}{2}, 2; \frac{\pi}{4}\right)$$

for their star product. ◇

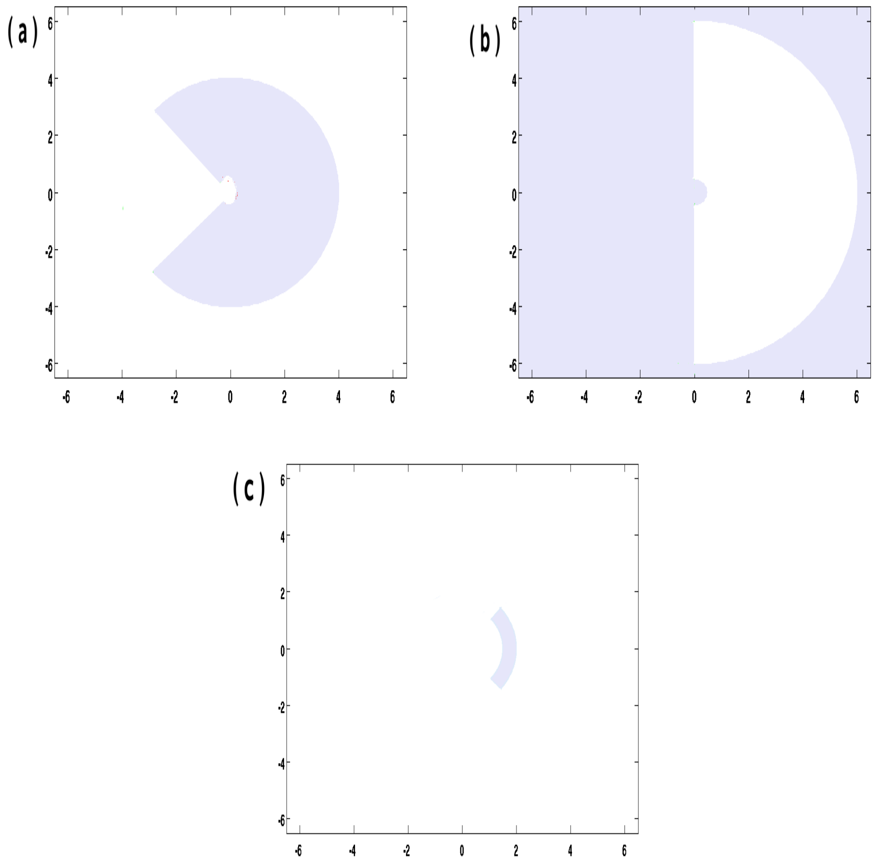

Figure 1.2: The sets $G\left(\frac{1}{4},4;\frac{3}{4}\pi\right)$ (a), $\Omega\left(\frac{1}{2},6;\frac{\pi}{2}\right)$ (b), and $G\left(\frac{1}{4},4;\frac{3}{4}\pi\right) * \Omega\left(\frac{1}{2},6;\frac{\pi}{2}\right) = G\left(\frac{3}{2},2;\frac{\pi}{4}\right)$ (c)

At the end of this section, we want to mention another topic that will emerge later on (see sections 4.2 and 6.5): solving equations for the star product.

1.3.18 Problem:
Let $A, B \subset \mathbb{C}_\infty$ be given sets. Under what conditions does there exist a set $X \subset \mathbb{C}_\infty$ in such a way that A and X satisfy the star condition and that $A * X = B$? $\diamond$

There are some cases that can be treated at once: a) If $A = \mathbb{C}_\infty \setminus \{1\}$ and $B \neq \emptyset$, then $A * B = B$. b) If $A = \mathbb{C} \setminus [1, \infty)$ and B star-like with respect to the origin, then $A * B = B$. c) If $A = \Omega\left(\frac{1}{2},6;\frac{\pi}{2}\right)$ and $B = G\left(\frac{3}{2},2;\frac{\pi}{4}\right)$, the set $X = G\left(\frac{1}{4},4;\frac{3}{4}\pi\right)$ satisfies our equation (cf. Example 1.3.16).

Chapter 2

Cycles

In this chapter, we will introduce the notion of Cauchy cycles, anti-Cauchy cycles, and Hadamard cycles. The latter type is essential for the definition of the Hadamard product (see chapter 3), that is defined by a parameter integral. The crucial point is to find appropriate integration curves for this parameter integral. In a special case, viz. plane open sets both containing the origin, these curves are Cauchy cycles (cf. [GE93]). Since we are interested in a version of the Hadamard product that is defined for a more general setting, we have to enlarge this class of integration curves. The Hadamard cycles will serve this need.

In the first section, we will define the concepts of Cauchy cycles, anti-Cauchy cycles, and Hadamard cycles. As already mentioned above, the last type will be the one that is used to define the Hadamard product (see section 3.4).

In the second section, we will prove properties of Cauchy cycles, anti-Cauchy cycles, and Hadamard cycles. Furthermore, we will prove an auxiliary result that enables us to evaluate a line integral by means of another line integral (see Lemma 2.2.6 and Remark 2.1.8)—a result that will be very helpful in subsequent chapters

One of the most important things is the existence of Hadamard cycles: If we could not guarantee it, we would not be able to define a Hadamard product. In the third section, we will prove existence results for Cauchy cycles and anti-Cauchy cycles. These results will allow us to prove an existence theorem for Hadamard cycles.

2.1 The definition of cycles

The concepts of line integrals will be adopted from [Rud2].

2.1.1 Definition:
Let $[a,b] \subset \mathbb{R}$ be an interval and $\gamma : [a,b] \to \mathbb{C}$ a mapping. The set $|\gamma| := \gamma([a,b])$ is called the *trace of* γ. The mapping $\gamma^- : [a,b] \to \mathbb{C}$ that is defined by $\gamma^-(t) := \gamma(a+b-t)$ is called the *reverse of* γ. $\diamond$

2.1.2 Remark:
For a mapping $\gamma : [a,b] \to \mathbb{C}$ we have $(\gamma^-)^- = \gamma$ and $|\gamma| = |\gamma^-|$. If in addition $0 \notin |\gamma|$, we have $(1/\gamma)^- = 1/(\gamma^-)$. $\diamond$

Using this notation, we can prove the following transformation rule that will be applied frequently.

2.1.3 Proposition:
Let $\gamma : [a,b] \to \mathbb{C} \setminus \{0\}$ be a continuous and piecewise continuously differentiable mapping, and let $f : |\gamma| \to \mathbb{C}$ be continuous. Then

$$\alpha \int\limits_{\gamma} \frac{f(w)}{w^2}\, dw \;=\; \int\limits_{\alpha/\gamma^-} f\left(\frac{\alpha}{\zeta}\right) d\zeta \tag{2.1}$$

for every complex number α. $\diamond$

With these concepts, we now define an important class of mappings.

2.1.4 Definition:
Let $N \in \mathbb{N}$, $[a_j, b_j] \subset \mathbb{R}$ intervals for $1 \le j \le N$, $S \subset \mathbb{C}$ a non-empty set, and $\gamma_j : [a_j, b_j] \to S$ continuous, piecewise continuously differentiable mappings with $\gamma_j(a_j) = \gamma_j(b_j)$ $(1 \le j \le N)$. Furthermore, let $\varphi_j : \left[\frac{j-1}{N}, \frac{j}{N}\right) \to [a_j, b_j)$ be the uniquely determined bijective and affine-linear mapping with the property $\varphi_j\left(\frac{j-1}{N}\right) = a_j$ $(1 \le j \le N)$. The mapping $\gamma : [0,1] \to S$ defined by

$$\gamma(t) \;:=\; \begin{cases} \gamma_j(\varphi_j(t)) & , \text{ if } \frac{j-1}{N} \le t < \frac{j}{N} \\ \gamma_N(b_N) & , \text{ if } t = 1 \end{cases}$$

is called a *cycle in S* and will be denoted by

$$\gamma \;=:\; \bigoplus_{j=1}^{N} \gamma_j .$$

For every $\kappa \in \mathbb{C} \setminus |\gamma|$ the number

$$\operatorname{ind}(\gamma, \kappa) \;:=\; \sum_{j=1}^{N} \frac{1}{2\pi i} \int_{\gamma_j} \frac{1}{\zeta - \kappa}\, d\zeta$$

is called the *index of γ with respect to κ*. In addition, we define

$$\operatorname{ind}(\gamma, \infty) \;:=\; 0$$

for each cycle γ. The number

$$L(\gamma) \;:=\; \sum_{j=1}^{N} \int_{a_j}^{b_j} |\gamma_j'(t)|\, dt$$

is called the *length of γ*. ◇

Let γ be a cycle as in Definition 2.1.4. Its trace is given by

$$|\gamma| \;=\; \bigcup_{j=1}^{N} |\gamma_j|.$$

For $\kappa \in \mathbb{C}_\infty \setminus |\gamma|$, we have

$$\operatorname{ind}\big(\gamma^-, \kappa\big) \;=\; -\operatorname{ind}(\gamma, \kappa) .$$

2.1.5 Remark:
The parametrization interval of a cycle need not be the unit interval; every compact interval serves the same purpose.† Therefore, we will usually omit the explicit specification of the interval in the notation of cycles and simply speak of a cycle γ in S. ◇

Some cycles that frequently appear are the standard parametrizations of circles. For this reason, we introduce a special notation: For $\zeta \in \mathbb{C}$ and $r > 0$ we define the mappings

$$\tau_r(\zeta) : [0, 2\pi] \to \mathbb{C}, \quad t \mapsto \zeta + re^{it}$$

† This can be achieved by a reparametrization.

and

$$\tau_r^-(\zeta) : [0, 2\pi] \longrightarrow \mathbb{C}, \quad t \mapsto \zeta + re^{-it}.$$

If $\zeta = 0$, we simply write $\tau_r := \tau_r(0)$ and $\tau_r^- := \tau_r^-(0)$.

2.1.6 Definition:
Let $\Omega \subset \mathbb{C}_\infty$ be a non-empty open set, $K \subset \Omega$ a non-empty compact set, and γ a cycle in $\Omega \setminus \big(K \cup \{0, \infty\}\big)$. If $\infty \notin K$ and

$$\operatorname{ind}(\gamma, \kappa) \quad = \quad \begin{cases} 1 & , \kappa \in K \\ 0 & , \kappa \in \mathbb{C}_\infty \setminus \Omega \end{cases},$$

then γ is called a *Cauchy cycle for K in Ω*. If $\infty \in \Omega$ and

$$\operatorname{ind}(\gamma, \kappa) \quad = \quad \begin{cases} 0 & , \kappa \in K \\ -1 & , \kappa \in \mathbb{C}_\infty \setminus \Omega \end{cases},$$

then γ is called an *anti-Cauchy cycle for K in Ω*. ◇

Cauchy cycles and anti-Cauchy cycles neither contain the origin nor the point at infinity in their traces.

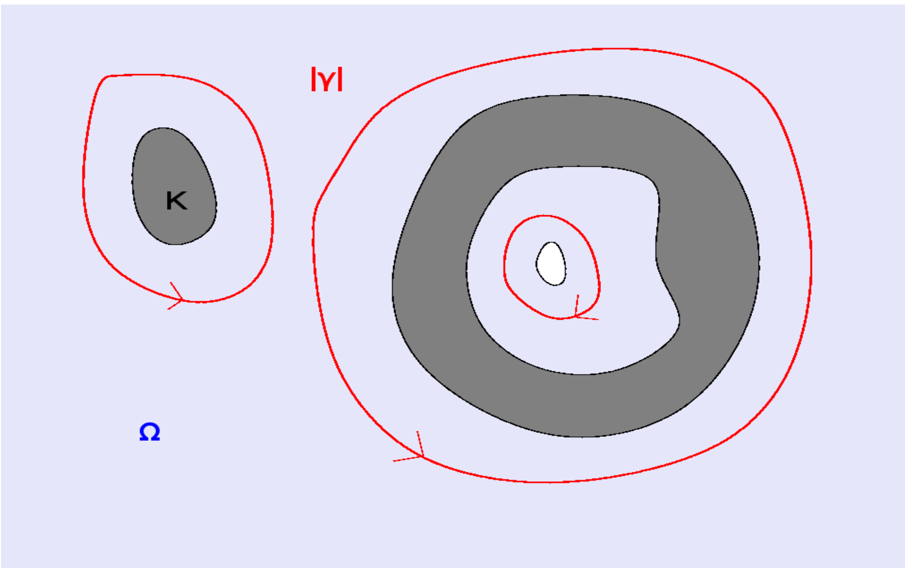

Figure 2.1: A Cauchy cycle γ for K in Ω

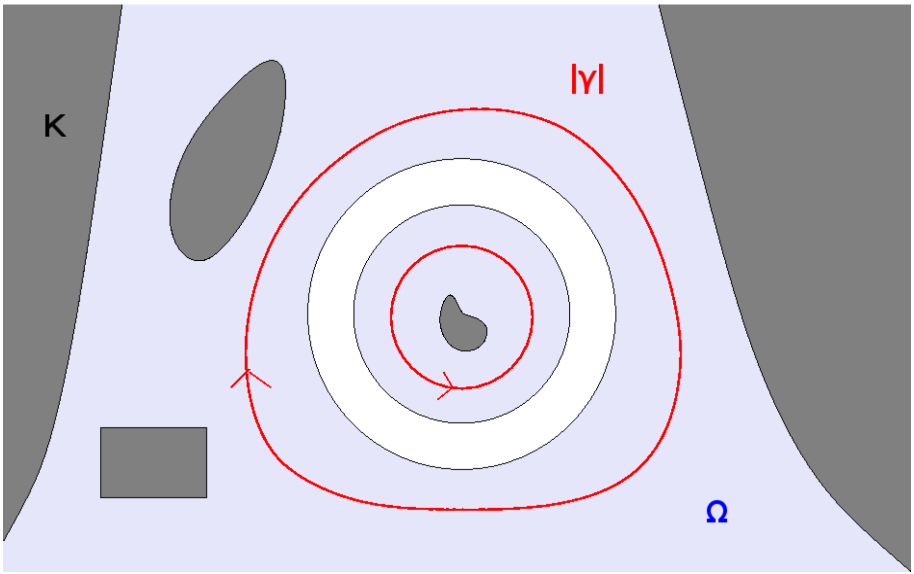

Figure 2.2: An anti-Cauchy cycle γ for K in Ω

2.1.7 Definition:
Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open and star-eligible sets, and let $z \in \Omega_1 * \Omega_2$. Furthermore, let γ be a cycle in $\Omega_1 \setminus (z \cdot \Omega_2^*)$ with $0 \notin |\gamma|$, $\infty \notin |\gamma|$, and the following property:

1. If $0 \in \Omega_1 \cap \Omega_2$ and $z = 0$, let γ be a Cauchy cycle for $\{0\}$ in Ω_1.
2. If $\infty \in \Omega_1 \cap \Omega_2$ and $z = \infty$, let γ be an anti-Cauchy cycle for $\{\infty\}$ in Ω_1.
3. If $z \neq 0$ and $z \neq \infty$, let γ be
   - a Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 with $\operatorname{ind}(\gamma, 0) = 1$ if $0 \in \Omega_1 \cap \Omega_2$ and $\infty \notin \Omega_1 \cap \Omega_2$,
   - an anti-Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 with $\operatorname{ind}(\gamma, 0) = -1$ if $0 \notin \Omega_1 \cap \Omega_2$ and $\infty \in \Omega_1 \cap \Omega_2$,
   - a Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 with $\operatorname{ind}(\gamma, 0) = 1$ or an anti-Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 with $\operatorname{ind}(\gamma, 0) = -1$ if $0 \in \Omega_1 \cap \Omega_2$ and $\infty \in \Omega_1 \cap \Omega_2$,
   - a Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 if $0 \in \Omega_2 \setminus \Omega_1$ and $\infty \in \Omega_2 \setminus \Omega_1$,
   - an anti-Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 if $0 \in \Omega_1 \setminus \Omega_2$ and $\infty \in \Omega_1 \setminus \Omega_2$.

Then γ is called a *Hadamard cycle for $z \cdot \Omega_2^*$ in Ω_1*. $\diamond$

2.1.8 Remarks:

1. In the last two cases of part three the index condition $\operatorname{ind}(\gamma, 0) = 0$ must necessarily be satisfied.
2. The conditions $\operatorname{ind}(\gamma, 0) = 1$ in the first case and $\operatorname{ind}(\gamma, 0) = -1$ in the second case of the definition seem to be quite artificial. To see why they are maintained, compare Example 2.2.4 and Example 2.2.5. $\diamond$

The following table gives a survey of the Hadamard cycles. "cc" stands for Cauchy cycle; "acc" for anti-Cauchy cycle; "cc$^+$" for Cauchy cycle with $\operatorname{ind}(\gamma, 0) = 1$; and "acc$^-$" for anti-Cauchy cycle with $\operatorname{ind}(\gamma, 0) = -1$. A "/" means that this case cannot occur. The elements in the first row and the first column tell us which of these elements are in Ω_1 and Ω_2, respectively.

Ω_2 \ Ω_1	$0, \infty$	∞	0	
$0, \infty$	cc$^+$ *or* acc$^-$	acc$^-$	cc$^+$	cc
∞	acc$^-$	acc$^-$	/	/
0	cc$^+$	/	cc$^+$	/
	acc	/	/	/

Table 2.1: Hadamard cycles for $z \cdot \Omega_2^*$ in Ω_1 $(z \notin \{0, \infty\})$

2.1.9 Examples:
Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open and star-eligible sets, and let $z \in (\Omega_1 * \Omega_2) \setminus \{0, \infty\}$.

1. If $0 \in \Omega_1 \cap \Omega_2$ and $r > 0$ so that $U_r[0] \subset \Omega_1$, then τ_r is a Hadamard cycle for $\{0\}$ in Ω_1.
2. If $\infty \in \Omega_1 \cap \Omega_2$ and $R > 0$ so that $U_R[\infty] \subset \Omega_1$, then τ_R^- is a Hadamard cycle for $\{\infty\}$ in Ω_1.
3. Let $\Omega_2 = \mathbb{C}_\infty \setminus \{1\}$, $0 \in \Omega_1$, and $\infty \notin \Omega_1$. Here we have $\Omega_1 * \Omega_2 = \Omega_1$. For $r, s \in (0, |z|/2)$ with $U_r[z] \subset \Omega_1$ and $U_s[0] \subset \Omega_1$ the cycle $\tau_r(z) \oplus \tau_s$ is a Hadamard cycle for $z \cdot \Omega_2^* = \{z\}$ in Ω_1.
4. Let $\Omega_2 = \mathbb{C}_\infty \setminus \{1\}$, $0 \notin \Omega_1$, and $\infty \in \Omega_1$. Here we have $\Omega_1 * \Omega_2 = \Omega_1$. For $r > 0$ with $U_r[z] \subset \Omega_1$ and $S > |z| + r$ with $U_S[\infty] \subset \Omega_1$ the cycle $\tau_r(z) \oplus \tau_S^-$ is a Hadamard cycle for $z \cdot \Omega_2^* = \{z\}$ in Ω_1.

5. Let Let $\Omega_2 = \mathbb{C}_\infty \setminus \{1\}$, $0 \in \Omega_1$, and $\infty \in \Omega_1$. Here we have $\Omega_1 * \Omega_2 = \Omega_1$. In this case each of the cycles of the last two examples is a Hadamard cycle for $z \cdot \Omega_2^* = \{z\}$ in Ω_1. ◇

2.2 Properties of cycles

In this section, we are concerned with properties that will be used in the work with cycles. The first proposition accentuates the interplay between Cauchy cycles and anti-Cauchy cycles.

2.2.1 Proposition:
Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open and star-eligible with $\infty \in \Omega_1$. Furthermore, let $z \in (\Omega_1 * \Omega_2) \setminus \{0, \infty\}$ and γ a cycle. Then the following assertions are equivalent:

(a) *γ is a Cauchy cycle for Ω_1^C in z/Ω_2.*

(b) *γ^- is an anti-Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1.*

Proof: Using (1.6), we get

$$\Omega_1 \setminus (z \cdot \Omega_2^*) \;=\; \Omega_1 \cap (z \cdot \Omega_2^*)^C \;=\; \Omega_1 \cap \frac{z}{\Omega_2} \;=\; \frac{z}{\Omega_2} \setminus \big(\Omega_1^C\big). \qquad (\star)$$

1. Let γ be a Cauchy cycle for Ω_1^C in z/Ω_2. According to relation $(\star)$, we obtain $|\gamma^-| = |\gamma| \subset (z/\Omega_2) \setminus \big(\Omega_1^C\big) = \Omega_1 \setminus (z \cdot \Omega_2^*)$. Moreover, we have

$$\operatorname{ind}\big(\gamma^-, \kappa\big) \;=\; -\operatorname{ind}(\gamma, \kappa) \;=\; \begin{cases} 0 & , \ \kappa \in (z/\Omega_2)^C \;=\; z \cdot \Omega_2^* \\ -1 & , \ \kappa \in \Omega_1^C \end{cases}.$$

Hence, γ^- is an anti-Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1.

2. Now let γ^- be an anti-Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1. Again by relation $(\star)$ we obtain $|\gamma| = |\gamma^-| \subset \Omega_1 \setminus (z \cdot \Omega_2^*) = (z/\Omega_2) \setminus \big(\Omega_1^C\big)$. Moreover, we have

$$\operatorname{ind}(\gamma, \kappa) \;=\; -\operatorname{ind}\big(\gamma^-, \kappa\big) \;=\; \begin{cases} 0 & , \ \kappa \in z \cdot \Omega_2^* = (z/\Omega_2)^C \\ 1 & , \ \kappa \in \Omega_1^C \end{cases}.$$

But this just means that γ is a Cauchy cycle for Ω_1^C in z/Ω_2. □

By applying the transformation rule of Proposition 2.1.3, the integration cycle γ is substituted by α/γ^-. Let us study the following situation: Given a Hadamard cycle γ for $z \cdot \Omega_2^*$ in Ω_1; what can we say about z/γ^- with respect to $z \cdot \Omega_1^*$ and Ω_2? The next theorem is concerned with the answer of this question. Before stating it, we prove a lemma that will be used throughout the proof of the theorem.

2.2.2 Lemma:
Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open and star-eligible sets, $z \in (\Omega_1 * \Omega_2) \setminus \{0, \infty\}$, and γ a cycle in $\Omega_1 \setminus (z \cdot \Omega_2^*)$ so that $0 \notin |\gamma|$ and $\infty \notin |\gamma|$. Then z/γ^- is a cycle in $\Omega_2 \setminus (z \cdot \Omega_1^*)$ with

$$\operatorname{ind}\left(\frac{z}{\gamma^-}, \kappa\right) \;=\; \operatorname{ind}(\gamma, 0) \;-\; \operatorname{ind}\left(\gamma, \frac{z}{\kappa}\right)$$

for all $\kappa \in \Omega_2^C \cup (z \cdot \Omega_1^*)$.

Proof: From $|\gamma| \subset \Omega_1 \cap (z \cdot \Omega_2^*)^C = \Omega_1 \cap (z/\Omega_2)$ we get $z/|\gamma^-| \subset \Omega_2 \cap (z/\Omega_1) = \Omega_2 \cap (z \cdot \Omega_1^*)^C$, i.e. z/γ^- is a cycle in $\Omega_2 \setminus (z \cdot \Omega_1^*)$.
For $\kappa \in \Omega_2^C \cup (z \cdot \Omega_1^*)$, $\kappa \neq 0$, $\kappa \neq \infty$ we get

$$\begin{aligned}
\operatorname{ind}\left(\frac{z}{\gamma^-}, \kappa\right) &= \frac{1}{2\pi i} \int_{z/\gamma^-} \frac{1}{\zeta - \kappa}\, d\zeta \;=\; \frac{1}{2\pi i} \int_{z/\gamma^-} \frac{1}{z} \frac{1}{\frac{1}{z/\zeta} - \frac{\kappa}{z}}\, d\zeta \\
&\overset{(2.1)}{=} \frac{1}{2\pi i} \int_\gamma \frac{z}{w(z - \kappa w)}\, dw \;=\; \frac{1}{2\pi i} \int_\gamma \left(\frac{1}{w} - \frac{1}{w - \frac{z}{\kappa}}\right) dw \\
&= \operatorname{ind}(\gamma, 0) - \operatorname{ind}\left(\gamma, \frac{z}{\kappa}\right).
\end{aligned}$$

For $\kappa = 0$—if this case occurs at all—, we get

$$\operatorname{ind}\left(\frac{z}{\gamma^-}, 0\right) \overset{(2.1)}{=} \frac{1}{2\pi i} \int_\gamma \frac{1}{w}\, dw \;=\; \operatorname{ind}(\gamma, 0) - \operatorname{ind}\left(\gamma, \frac{z}{0}\right).$$

For $\kappa = \infty$—if this case occurs at all—, we get

$$\operatorname{ind}\left(\frac{z}{\gamma^-}, \infty\right) \;=\; 0 \;=\; \operatorname{ind}(\gamma, 0) - \operatorname{ind}\left(\gamma, \frac{z}{\infty}\right).$$

These prove the lemma. $\square$

Next, we present the theorem mentioned above.

2.2.3 Theorem:
Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ *be open and star-eligible sets,* $z \in (\Omega_1 * \Omega_2) \setminus \{0, \infty\}$*, and* γ *a cycle. Then* γ *is a Hadamard cycle for* $z \cdot \Omega_2^*$ *in* Ω_1 *if and only if* z/γ^- *is a Hadamard cycle for* $z \cdot \Omega_1^*$ *in* Ω_2*.*

Proof: The following equivalences hold:

$$\begin{aligned} \kappa \in \Omega_2^C \quad &\text{if and only if} \quad z/\kappa \in z \cdot \Omega_2^*. \\ \kappa \in z \cdot \Omega_1^* \quad &\text{if and only if} \quad z/\kappa \in \Omega_1^C. \end{aligned} \tag{2.2}$$

We abbreviate $\Gamma := z/\gamma^-$ for the rest of the proof.

1. Let γ be a Hadamard cycle for $z \cdot \Omega_2^*$ in Ω_1.

Case 1: $0 \in \Omega_1 \cap \Omega_2$ and $\infty \notin \Omega_1 \cap \Omega_2$. By the definition of Hadamard cycles we have $\operatorname{ind}(\gamma, 0) = 1$. Lemma 2.2.2 and (2.2) yield

$$\operatorname{ind}(\Gamma, \kappa) \quad = \quad \begin{cases} 0 & , \ \kappa \notin \Omega_2 \\ 1 & , \ \kappa \in z \cdot \Omega_1^* \end{cases}.$$

Hence, Γ is a Cauchy cycle for $z \cdot \Omega_1^*$ in Ω_2 with $\operatorname{ind}(\Gamma, 0) = 1$.

Case 2: $0 \notin \Omega_1 \cap \Omega_2$ and $\infty \in \Omega_1 \cap \Omega_2$. By the definition of Hadamard cycles we have $\operatorname{ind}(\gamma, 0) = -1$. Lemma 2.2.2 and (2.2) now show

$$\operatorname{ind}(\Gamma, \kappa) \quad = \quad \begin{cases} -1 & , \ \kappa \notin \Omega_2 \\ 0 & , \ \kappa \in z \cdot \Omega_1^* \end{cases}.$$

Thus, Γ is an anti-Cauchy cycle for $z \cdot \Omega_1^*$ in Ω_2 with $\operatorname{ind}(\Gamma, 0) = -1$.

Case 3: $0, \infty \in \Omega_1 \cap \Omega_2$. If γ is a Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 satisfying $\operatorname{ind}(\gamma, 0) = 1$, we get

$$\operatorname{ind}(\Gamma, \kappa) \quad = \quad \begin{cases} 0 & , \ \kappa \notin \Omega_2 \\ 1 & , \ \kappa \in z \cdot \Omega_1^* \end{cases}.$$

If γ is an anti-Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 with $\operatorname{ind}(\gamma, 0) = -1$, we get

$$\operatorname{ind}(\Gamma, \kappa) \quad = \quad \begin{cases} -1 & , \ \kappa \notin \Omega_2 \\ 0 & , \ \kappa \in z \cdot \Omega_1^* \end{cases}.$$

Consequently, Γ is a Cauchy cycle for $z \cdot \Omega_1^*$ in Ω_2 with $\operatorname{ind}(\Gamma, 0) = 1$ or an anti-Cauchy cycle for $z \cdot \Omega_1^*$ in Ω_2 with $\operatorname{ind}(\Gamma, 0) = -1$, respectively.

Case 4: $0, \infty \in \Omega_2 \setminus \Omega_1$. Here we get $\operatorname{ind}(\gamma, 0) = 0$, and hence by Lemma 2.2.2 and (2.2)

$$\operatorname{ind}(\Gamma, \kappa) \;=\; \begin{cases} -1 & , \; \kappa \notin \Omega_2 \\ 0 & , \; \kappa \in z \cdot \Omega_1^* \end{cases}.$$

Thus, Γ is an anti-Cauchy cycle for $z \cdot \Omega_1^*$ in Ω_2.

Case 5: $0, \infty \in \Omega_1 \setminus \Omega_2$. In this case we have $\operatorname{ind}(\gamma, 0) = 0$. By Lemma 2.2.2 and (2.2) we obtain

$$\operatorname{ind}(\Gamma, \kappa) \;=\; \begin{cases} 0 & , \; \kappa \notin \Omega_2 \\ 1 & , \; \kappa \in z \cdot \Omega_1^* \end{cases}.$$

This shows that Γ is a Cauchy cycle for $z \cdot \Omega_1^*$ in Ω_2.

According to Definition 2.1.7, in each of the cases listed above Γ is a Hadamard cycle for $z \cdot \Omega_1^*$ in Ω_2.

2. Now let Γ be a Hadamard cycle for $z \cdot \Omega_1^*$ in Ω_2. Since $z/\Gamma^- = \gamma$, the reverse implication follows from the already proved part. □

With regard to Theorem 2.2.3, the index condition $\operatorname{ind}(\gamma, 0) = 1$ in the case "$0 \in \Omega_1 \cap \Omega_2$ and $\infty \notin \Omega_1 \cap \Omega_2$" makes sense. To see why, we present the following example.

2.2.4 Example:
Let us consider $\Omega_1 := \mathbb{D}_4$, $\Omega_2 := \mathbb{C}_\infty \setminus \{1\}$, and $z := 2$. Here we get $z \cdot \Omega_1^* = \overline{\mathbb{D}}_{1/2}$ and $z \cdot \Omega_2^* = \{2\}$. Furthermore, $\tau_1(2)$ is a Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 with $\operatorname{ind}(\tau_1(2), 0) = 0$. By Example 1.3.1 we get $z/\tau_1^-(2) = \tau_{2/3}^-(4/3)$. But this is not a Cauchy cycle for $z \cdot \Omega_1^*$ in Ω_2. ◇

The next example shows that the index condition $\operatorname{ind}(\gamma, 0) = -1$ in the case "$\infty \in \Omega_1 \cap \Omega_2$ and $0 \notin \Omega_1 \cap \Omega_2$" should not be dropped either.

2.2.5 Example:
Let $\Omega_1 := \mathbb{C}_\infty \setminus \{2\}$, $\Omega_2 := \mathbb{C}_\infty \setminus \overline{\mathbb{D}}_{1/2}$, and $z := 2$. Here we have $z \cdot \Omega_1^* = \{1\}$ and $z \cdot \Omega_2^* = \mathbb{C}_\infty \setminus \mathbb{D}_4$. Moreover, $\tau_1^-(2)$ is an anti-Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 with $\operatorname{ind}\big(\tau_1^-(2), 0\big) = 0$. But $z/\tau_1(2) = \tau_{2/3}(4/3)$ is not an anti-Cauchy cycle for $z \cdot \Omega_1^*$ in Ω_2. ◇

The next lemma provides a key tool for the work with line integrals along Cauchy cycles. If we are given an open set, a function which is holomorphic in this set, and two homotopic cycles, then the corresponding line integrals have the same values. For Cauchy cycles we can prove a similar result that enables us to evaluate the function at the point at infinity by the sum of two line integrals.

2.2.6 Lemma:
Let $K, L \subset \mathbb{C}$ be disjoint compact sets, $f : \mathbb{C}_\infty \setminus (K \cup L) \to \mathbb{C}$ holomorphic, γ a Cauchy cycle for K in $\mathbb{C}_\infty \setminus L$, and Γ a Cauchy cycle for L in $\mathbb{C}_\infty \setminus K$. Then we have

$$f(\infty) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta}\,d\zeta \;+\; \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta}\,d\zeta$$

if one of the following conditions holds:

(i) $0 \in K \cup L$;

(ii) $0 \notin K \cup L$ *and* $\operatorname{ind}(\gamma \oplus \Gamma, 0) = 1$;

(iii) $0 \notin K \cup L$ *and* $f(0) = 0$.

Proof: Let $R > \max\{|z| : z \in K \cup L\}$. In the first two cases the mapping $\zeta \mapsto f(\zeta)/\zeta$ defines a function that is holomorphic in $\mathbb{C}_\infty \setminus (K \cup L \cup \{0\})$, and we have $\operatorname{ind}(\tau_R, w) = \operatorname{ind}(\gamma \oplus \Gamma, w)$ for all $w \in K \cup L \cup \{0\}$. In the third case this function is holomorphic in $\mathbb{C}_\infty \setminus (K \cup L)$, and we have $\operatorname{ind}(\tau_R, w) = \operatorname{ind}(\gamma \oplus \Gamma, w)$ for all $w \in K \cup L$. Thus, we get

$$\int_{|\zeta|=R} \frac{f(\zeta)}{\zeta}\,d\zeta \;=\; \int_{\gamma\oplus\Gamma} \frac{f(\zeta)}{\zeta}\,d\zeta \;=\; \int_\gamma \frac{f(\zeta)}{\zeta}\,d\zeta \;+\; \int_\Gamma \frac{f(\zeta)}{\zeta}\,d\zeta.$$

For each such R there exists a $\zeta_R \in \mathbb{T}_R$ so that

$$\max_{\zeta\in\mathbb{T}_R} |f(\zeta) - f(\infty)| \;=\; |f(\zeta_R) - f(\infty)|.$$

Therefore, we get

$$\left|\frac{1}{2\pi i}\int_{|\zeta|=R} \frac{f(\zeta)}{\zeta}\,d\zeta \;-\; f(\infty)\right| \;=\; \left|\frac{1}{2\pi i}\int_{|\zeta|=R} \frac{f(\zeta)-f(\infty)}{\zeta}d\zeta\right|$$

$$\leq\; \max_{\zeta\in\mathbb{T}_R} |f(\zeta) - f(\infty)\,| \;=\; |f(\zeta_R) - f(\infty)\,| \;\xrightarrow{R\to\infty}\; 0$$

since $\zeta_R \to \infty$ as $R \to \infty$. □

2.2.7 Remarks:

1. In the first case of Lemma 2.2.6 the index condition $\operatorname{ind}(\gamma \oplus \mathit{\Gamma}, 0) = 1$ is automatically satisfied.
2. We would like to stress an important special case: Let, in addition, f vanish at infinity. Moreover, let $\mathit{\Gamma}^-$ (instead of $\mathit{\Gamma}$) be a Cauchy cycle. Then we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta}\,d\zeta \;=\; \frac{1}{2\pi i}\int_{\mathit{\Gamma}} \frac{f(\zeta)}{\zeta}\,d\zeta.$$

We will use this relation tacitly. $\diamond$

If the origin is contained in one (and then in only one) of the compact sets, we need no additional constraints on the cycles or the function. Otherwise, the condition in part two or three of the above lemma is necessary for the conclusion to be true; the next example shows why.

2.2.8 Example:
Let $K := \{1\}$, $L := \{-1\}$, and $f : \mathbb{C}_\infty \setminus \{-1, 1\} \to \mathbb{C}$ defined by $f(z) := 1/(z^2 - 1)$. Furthermore, we regard the Cauchy cycles $\gamma, \mathit{\Gamma} : [0, 2\pi] \to \mathbb{C}$ given by $\gamma(t) := 1 + \frac{1}{2}e^{it}$ and $\mathit{\Gamma}(t) := -1 + \frac{1}{2}e^{it}$. Here we have $\operatorname{ind}(\gamma \oplus \mathit{\Gamma}, 0) \neq 1$ and $f(0) \neq 0$. Moreover, we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta}\,d\zeta + \frac{1}{2\pi i}\int_{\mathit{\Gamma}} \frac{f(\zeta)}{\zeta}\,d\zeta \;=\; 1 \;\neq\; 0 \;=\; f(\infty)$$

by Cauchy's integral formula. $\diamond$

2.3 On the existence of Hadamard cycles

The existence of a Cauchy cycle under the preconditions in Definition 2.1.4 is guaranteed by the following result.

2.3.1 Lemma:
Let $\Omega \subset \mathbb{C}_\infty$ be a non-empty open set and $K \subset \Omega$ a non-empty compact set with $\infty \notin K$. Then there exists a Cauchy cycle for K in Ω.

Proof: (1) If $\infty \notin \Omega$, see [Rud2, p. 269].

(2) If $\infty \in \Omega$, then, according to (1), there exists a Cauchy cycle for K in $\Omega \setminus \{\infty\}$. This one is also a Cauchy cycle for K in Ω because its index with respect to the point at infinity is 0 by definition. □

From this lemma, we immediately get an existence result for Cauchy cycles.

2.3.2 Proposition (Existence of Cauchy cycles):
Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open and star-eligible sets with $0 \in \Omega_2$ and $K \subset \Omega_1 * \Omega_2$ a non-empty compact set with $\infty \notin K$. Then there exists a cycle that is a Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 for every $z \in K$.

Proof: According to Proposition 1.3.10.7, $K \cdot \Omega_2^*$ is a compact subset of Ω_1. Since $0 \in \Omega_2$, we have $\infty \notin \Omega_2^*$, and hence $\infty \notin K \cdot \Omega_2^*$. According to Lemma 2.3.1, there exists a Cauchy cycle for $K \cdot \Omega_2^*$ in Ω_1. This one is also a Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 for every $z \in K$. □

Moreover, we need the following existence result for anti-Cauchy cycles.

2.3.3 Proposition (Existence of anti-Cauchy cycles):
Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open and star-eligible sets with $\infty \in \Omega_1$ and $K \subset \Omega_1 * \Omega_2$ a non-empty compact set. Then there exists a cycle that is an anti-Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 for every $z \in K$.

Proof: According to Proposition 1.3.10.7, $K \cdot \Omega_2^*$ is a compact subset of Ω_1. By taking complements, we obtain $\Omega_1^C \subset (K \cdot \Omega_2^*)^C$. Since Ω_1^C is a plane compact subset of (the open set) $(K \cdot \Omega_2^*)^C$, there exists a Cauchy cycle γ for Ω_1^C in $(K \cdot \Omega_2^*)^C$ according to Lemma 2.3.1. Furthermore, we have

$$|\gamma^-| \;=\; |\gamma| \;\subset\; (K \cdot \Omega_2^*)^C \setminus (\Omega_1^C) \;=\; \Omega_1 \cap (K \cdot \Omega_2^*)^C \;=\; \Omega_1 \setminus (K \cdot \Omega_2^*).$$

Therefore, γ^- is a cycle in $\Omega_1 \setminus (K \cdot \Omega_2^*)$. The index property of γ yields

$$\operatorname{ind}(\gamma, \kappa) \;=\; \begin{cases} 1 & , \kappa \in \Omega_1^C \\ 0 & , \kappa \in K \cdot \Omega_2^* \end{cases},$$

whence we get

$$\operatorname{ind}\big(\gamma^-, \kappa\big) \;=\; -\operatorname{ind}(\gamma, \kappa) \;=\; \begin{cases} -1 & , \kappa \in \Omega_1^C \\ 0 & , \kappa \in K \cdot \Omega_2^* \end{cases}.$$

Hence, γ^- is an anti-Cauchy cycle for $K\cdot\Omega_2^*$ in Ω_1, and thus also an anti-Cauchy cycle for $z\cdot\Omega_2^*$ in Ω_1 for every $z\in K$. □

This section's results culminate in the following existence theorem.

2.3.4 Existence theorem for Hadamard cycles:
Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open and star-eligible sets, and let $K \subset \Omega_1 * \Omega_2$ be a non-empty compact set. Then there exists a cycle that is a Hadamard cycle for $z\cdot\Omega_2^*$ in Ω_1 for every $z\in K$.

Proof: *Case 1:* $0\in\Omega_1\cap\Omega_2$ and $\infty\notin\Omega_1\cap\Omega_2$. According to Proposition 2.3.2, there exists a cycle that is a Cauchy cycle for $z\cdot\Omega_2^*$ in Ω_1 for every $z\in K$. Since $0\in\Omega_1$, we can even find such a cycle whose index with respect to 0 equals 1. (Notice that this cycle is also suitable if $z=0$).

Case 2: $0\notin\Omega_1\cap\Omega_2$ and $\infty\in\Omega_1\cap\Omega_2$. According to Proposition 2.3.3, there exists a cycle that is an anti-Cauchy cycle for $z\cdot\Omega_2^*$ in Ω_1 for every $z\in K$. Since $\infty\in\Omega_2$, we can even find such a cycle whose index with respect to 0 equals -1. (Notice that this cycle is also suitable if $z=\infty$).

Case 3: $0,\infty\in\Omega_1\cap\Omega_2$. If $\infty\notin K$, we can argue in the same way as in the first case. If $\infty\in K$, we can argue in the same way as in the second case.

Case 4: $0,\infty\in\Omega_2\setminus\Omega_1$. According to Proposition 2.3.2, there exists a cycle that is a Cauchy cycle for $z\cdot\Omega_2^*$ in Ω_1 for every $z\in K$.

Case 5: $0,\infty\in\Omega_1\setminus\Omega_2$. According to Proposition 2.3.3, there exists a cycle that is an anti-Cauchy cycle for $z\cdot\Omega_2^*$ in Ω_1 for all $z\in K$.

Thus, in all possible cases there exists a cycle that is a Hadamard cycle for $z\cdot\Omega_2^*$ in Ω_1 for every $z\in K$. □

Chapter 3

The Hadamard product

The notion of the Hadamard product is fairly old. It appeared in J. S. Hadamard's 1899 paper *Théorème sur les séries entières* ([Ha99]). G. Pólya investigated it thoroughly in his well-known paper *Untersuchungen über Lücken und Singularitäten von Potenzreihen* ([Pó33]) from 1933. Two other sources are S. Schottlaender's *Der Hadamardsche Multiplikationssatz und weitere Kompositionssätze der Funktionentheorie* ([Sch54]) from 1954 and the book *Analytische Fortsetzung* ([Bieb]) by L. Bieberbach, that appeared one year later. E. Hille's textbook *Analytic Function Theory* ([Hille]; first edition issued in 1959) devotes one section to so-called composition theorems under which Hadamard's multiplication theorem can be subsumed. The objects studied in these works are power series with center zero or their analytic continuation into the corresponding Mittag-Leffler stars.

Approximately three decades later, J. Müller's *The Hadamard Multiplication Theorem and Applications in Summability Theory* ([Mü92]) and K.-G. Große-Erdmann's *On the Borel-Okada Theorem and the Hadamard Multiplication Theorem* ([GE93]) were published in the consecutive years 1992 and 1993. In contrast to the works mentioned in the first paragraph, Müller and Große-Erdmann no longer restricted their attention to power series, they rather studied functions holomorphic in open sets containing the origin.

All of these studies on the Hadamard product have one thing in common: the origin is involved. No matter if the factors of the Hadamard product are power series or holomorphic functions, the open sets on which they are examined contain the origin. In this chapter, we shall generalize this condition in such a way that the factors need not be holomorphic at zero.

In the first three sections, we will give a brief outline of the Hadamard product hitherto existing. We will call this product the *plane version* of the Hadamard product. We will commence with power series, and then consider holomorphic functions. In the third section we will state the Hadamard multiplication theorem.

In the fourth section, we will define a Hadamard product for functions holomorphic in open and star-eligible sets (see Definition 3.4.4). We will call this product the *extended version* of the Hadamard product. To this end, the Hadamard cycles introduced in chapter two are of great importance. The Hadamard product will be defined by a parameter integral along Hadamard cycles. We will show that the values of these integrals do not depend on the Hadamard cycle (see Lemma 3.4.2). Furthermore, we will calculate the Hadamard product of important functions (see examples 3.4.6, 3.4.7, 3.4.8, and 3.4.9).

In the fifth section, we will show that the Hadamard product defined in Definition 3.4.4 coincides with the old Hadamard product in the case of plane open sets containing the origin (see Proposition 3.5.1). Closely connected with this property is the question of how the Hadamard product behaves when the factors are restricted to subsets.

In the sixth section, we will prove algebraic and analytic properties of the Hadamard product. These properties are already known for the plane version of the Hadamard product, and is is expected that they hold for the extended version as well.

In the seventh section, we consider the Hadamard product from a functional analytic point of view. We will prove an essential continuity result (see 3.7.4).

All power series emerging in this chapter are supposed to have positive radii of convergence (infinity is not excluded).

3.1 The Hadamard product of power series

We are concerned with two power series

$$\sum_{\nu=0}^{\infty} a_\nu z^\nu \quad \text{and} \quad \sum_{\nu=0}^{\infty} b_\nu z^\nu \tag{3.1}$$

whose radii of convergence are denoted by r_a and r_b, respectively.

The usual fashion to multiply them is by means of the Cauchy product. Another possibility is to multiply them coefficient by coefficient (which resembles the addition of power series). In this case, the power series

$$\sum_{\nu=0}^{\infty} a_\nu \, b_\nu \, z^\nu$$

is called the Hadamard product series of the power series in (3.1). Using the submultiplicativity of the upper limit, it turns out that the radius of convergence r of the Hadamard product series satisfies

$$r \;\geq\; r_a \cdot r_b.$$

If, in particular, one of the power series in (3.1) defines an entire function, then the Hadamard product series defines an entire function, too.

Let $f : \mathbb{D}_{r_a} \to \mathbb{C}$ and $g : \mathbb{D}_{r_b} \to \mathbb{C}$ be the functions defined by the power series in (3.1). For $\rho \in (0, r_a)$, we obtain the following integral representation—known as the Parseval integral representation—of the Hadamard product series:

$$\sum_{\nu=0}^{\infty} a_\nu \, b_\nu \, z^\nu \;=\; \frac{1}{2\pi i} \int_{|\zeta|=\rho} \frac{f(\zeta)}{\zeta} \, g\!\left(\frac{z}{\zeta}\right) d\zeta \tag{3.2}$$

which holds for all $z \in \mathbb{C}$ with $|z| < \rho \cdot r_b$. The integral on the right-hand side of (3.2) is often called the Parseval integral (of f and g). To prove this representation, write the coefficients a_ν by means of Cauchy's integral formula and notice that the second series in (3.1) converges uniformly on $\{z/\zeta : \zeta \in \mathbb{T}_\rho\}$ for each $z \in \mathbb{C}$ satisfying $|z| < \rho \cdot r_b$. This yields

$$\begin{aligned}\sum_{\nu=0}^{\infty} a_\nu \, b_\nu \, z^\nu \;&=\; \sum_{\nu=0}^{\infty} \left\{ \frac{1}{2\pi i} \int\limits_{|\zeta|=\rho} \frac{f(\zeta)}{\zeta^{\nu+1}} \, d\zeta \right\} b_\nu z^\nu \\ &=\; \frac{1}{2\pi i} \int\limits_{|\zeta|=\rho} \left\{ \frac{f(\zeta)}{\zeta} \underbrace{\sum_{\nu=0}^{\infty} b_\nu \cdot \left(\frac{z}{\zeta}\right)^{\nu}}_{=\, g(z/\zeta)} \right\} d\zeta \;=\; \frac{1}{2\pi i} \int\limits_{|\zeta|=\rho} \frac{f(\zeta)}{\zeta} \, g\!\left(\frac{z}{\zeta}\right) d\zeta\end{aligned}$$

for all $z \in \mathbb{C}$ with $|z| < \rho \cdot r_b$.

The geometric series has a specific role. Since all its coefficients equal 1, it is a neutral element with respect to Hadamard multiplication of power series in the following sense: If $f(z) := \sum_{\nu=0}^{\infty} a_\nu z^\nu$ has positive radius of convergence, then the Hadamard product series of $f(z)$ and the geometric series is $f(z)$.

3.2 The Hadamard product of holomorphic functions

The right-hand side of (3.2) need not be restricted to functions defined by power series; it works in a much more general context. Let Ω_1 and Ω_2 be plane open sets both containing the origin, f in $H(\Omega_1)$, and g in $H(\Omega_2)$. Then the set $\Omega_1 * \Omega_2$ is open and contains the origin (cf. Proposition 1.3.12 and Proposition 1.3.10.2), i.e. the sets Ω_1 and Ω_2 are star-eligible. Recall that Proposition 1.3.10.4 guarantees $\mathbb{C} \setminus \big((\mathbb{C} \setminus \Omega_1) \cdot (\mathbb{C} \setminus \Omega_2)\big) = \Omega_1 * \Omega_2$. For every $z \in \Omega_1 * \Omega_2$ the set $z \cdot \Omega_2^*$ is a compact subset of Ω_1 (cf. Proposition 1.3.10.7). According to Proposition 2.3.2 there exists a Cauchy cycle γ_z for $z \cdot \Omega_2^*$ in Ω_1. Since

$$\operatorname{ind}(\gamma_z, \kappa) \;=\; \operatorname{ind}(\tilde{\gamma}_z, \kappa) \qquad \big(\kappa \in (\mathbb{C} \setminus \Omega_1) \cup (z \cdot \Omega_2^*)\big)$$

for every other Cauchy cycle $\tilde{\gamma}_z$ for $z \cdot \Omega_2^*$ in Ω_1, we get

$$\int_{\gamma_z} \frac{f(\zeta)}{\zeta}\, g\!\left(\frac{z}{\zeta}\right) d\zeta \;=\; \int_{\tilde{\gamma}_z} \frac{f(\zeta)}{\zeta}\, g\!\left(\frac{z}{\zeta}\right) d\zeta$$

by [Rud2, 10.35]. This implies that the value of the Parseval integral is independent of the Cauchy cycle. (Notice that γ_z and $\tilde{\gamma}_z$ are also Hadamard cycles for $z \cdot \Omega_2^*$ in Ω_1.) For a Cauchy cycle γ for $z \cdot \Omega_2^*$ in Ω_1, we define a new function $f *_{\Omega_1,\Omega_2} g : \Omega_1 * \Omega_2 \to \mathbb{C}$ by

$$(f *_{\Omega_1,\Omega_2} g)(z) \;:=\; \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta}\, g\!\left(\frac{z}{\zeta}\right) d\zeta. \tag{3.3}$$

This function is called the Hadamard product of f and g.

It can be shown that $f *_{\Omega_1,\Omega_2} g$ is holomorphic in $\Omega_1 * \Omega_2$ (see for instance Proposition 3.6.4).

In the previous section, we mentioned that the geometric series plays the role of a neutral element with respect to Hadamard multiplication of power series. Is there an analogue in this setting? The answer is yes. In the unit disk, the geometric series represents the function $\Theta\big|_{\mathbb{D}}$. By means of Cauchy's integral formula it can be verified that $\Theta\big|_{\mathbb{C}\setminus\{1\}}$ is a neutral element in the following sense: If $\Omega \subset \mathbb{C}$ is an open set containing the origin and if $f \in H(\Omega)$, then we have $f *_{\Omega,\,\mathbb{C}\setminus\{1\}} (\Theta\big|_{\mathbb{C}\setminus\{1\}})) = f$.

3.3 The Hadamard multiplication theorem

The question of the connection between the Hadamard product of power series and of holomorphic functions immediately arises. An answer to this question is given by the Hadamard multiplication theorem which reveals that these two concepts are intimately linked.

3.3.1 Hadamard multiplication theorem:
Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be open sets both containing the origin, $f \in H(\Omega_1)$, and $g \in H(\Omega_2)$. Then

$$(f *_{\Omega_1,\Omega_2} g)\,(z) \;=\; \sum_{\nu=0}^{\infty} f_\nu\, g_\nu\, z^\nu$$

for all $z \in \mathbb{C}$ with $|z| < \operatorname{dist}\big(0, \partial(\Omega_1 * \Omega_2)\big)$. ◇

This theorem shows that the Hadamard product series of the functions' power series expansions is exactly the local power series expansion of $f *_{\Omega_1,\Omega_2} g$ around zero. Proofs can be found in [Mü92, Theorem H] or [GE93, Theorem 2.3].

If f and g are defined by power series, and if we denote by $S[f]$ the Mittag-Leffler star of the function f, then Hille (cf. [Hille, Theorem 11.6.1]) states the theorem in the form $S[f] * S[g] \subset S[h]$, where h is the function defined by the Hadamard product series.

At the end of this section, we provide an application of the Hadamard multiplication theorem. For $k \in \mathbb{N}_0$ we denote by $P_k : \mathbb{C} \to \mathbb{C}$ the monomial defined by

$$P_k(z) \;:=\; z^k.$$

For $\eta \in \mathbb{C}$ we define the function $\Theta_\eta : \mathbb{C}_\infty \setminus \{\eta\} \to \mathbb{C}$ by

$$\Theta_\eta\,(z) := \frac{1}{\eta - z}. \tag{3.4}$$

We emphasize an important special case by setting

$$\Theta \;:=\; \Theta_1. \tag{3.5}$$

Let $\Omega \subset \mathbb{C}_\infty$ be a non-empty open set, $k \in \mathbb{N}_0$, $f \in H^{(k)}(\Omega)$, and $\eta \in \mathbb{C} \setminus \{0\}$. If $\infty \notin \Omega$, the function $P_k \cdot f$ is holomorphic in Ω. Now let $\infty \in \Omega$. Since

$f \in H^{(k)}(\Omega)$, there exists an $R > 1$ in such a way that we have

$$f(z) = \sum_{\nu=k+1}^{\infty} \frac{f_\nu}{z^\nu} \qquad \big(z \in U_R[\infty]\big),$$

which implies $\lim\limits_{z\to\infty} z^k \cdot f(z) = 0$. Therefore, $P_k \cdot f : \Omega \to \mathbb{C}$ defined by

$$(P_k \cdot f)(z) := \begin{cases} z^k \cdot f(z) & , \quad z \in \Omega \setminus \{\infty\} \\ 0 & , \quad z = \infty \end{cases}$$

is continuous. Moreover, $(P_k \cdot f)\big|_{\Omega\cap\mathbb{C}}$ is holomorphic. Thus, $P_k \cdot f$ belongs to $H(\Omega)$, and we have

$$(P_k \cdot f)(z) = \sum_{\nu=1}^{\infty} \frac{f_{\nu+k}}{z^\nu} \qquad \big(z \in U_R[\infty]\big).$$

This yields

$$(P_k \cdot f)^{(p)}(\infty) = \frac{p!}{(k+p)!} \cdot f^{(k+p)}(\infty) \qquad \big(p \in \mathbb{N}_0\big). \tag{3.6}$$

Hence, if we even have $f \in H^{(k+p)}(\Omega)$ for a $p \in \mathbb{N}$, we get $P_k \cdot f \in H^{(p)}(\Omega)$.

Based on these considerations, we have the following situation: Let $\Omega \subset \mathbb{C}_\infty$ be a non-empty open set, $k \in \mathbb{N}_0$, $f \in H^{(k)}(\Omega)$, and $\eta \in \mathbb{C} \setminus \{0\}$. If we set $\eta \cdot \Omega := \{\eta\omega : \omega \in \Omega\}^\dagger$, we associate with f a new function $f_{\eta,k} : \eta \cdot \Omega \to \mathbb{C}$ defined by

$$f_{\eta,k}(z) := \frac{1}{k!} \cdot \big(P_k \cdot f\big)^{(k)}\left(\frac{z}{\eta}\right). \tag{3.7}$$

In particular, we have

$$f_{\eta,k}(0) = f(0) \tag{3.8}$$

and

$$f_{\eta,0}(z) = f\left(\frac{z}{\eta}\right) \qquad \big(z \in \eta \cdot \Omega\big). \tag{3.9}$$

The function $f_{\eta,k}$ is holomorphic in $\eta \cdot \Omega$. But in general it does not vanish at infinity. From (3.6) we obtain

$$f_{\eta,k}(z) = \sum_{\nu=0}^{\infty} f_{2k+\nu} \cdot \binom{k+\nu}{k} \cdot \eta^\nu \cdot z^{-\nu} \quad \big(z \in U_{|\eta|R}[\infty]\big). \tag{3.10}$$

† We define $\eta \cdot \infty := \infty$.

This shows that $f_{\eta,k} \in H(\eta \cdot \Omega)$ if in addition $f \in H^{(2k)}(\Omega)$.

Example 1.2.1 shows that $\Theta \in H(\mathbb{C}_\infty \setminus \{1\})$, but $\Theta \notin H^{(1)}(\mathbb{C}_\infty \setminus \{1\})$. Thus, we have no definition for $\Theta_{\eta,k}$ if $k \geq 1$. On the other hand, the Leibniz rule for differentiating a product yields

$$\begin{aligned} \frac{1}{k!} \cdot (P_k \cdot \Theta)^{(k)}\left(\frac{z}{\eta}\right) &= \frac{1}{k!} \cdot \sum_{\nu=0}^{k} \binom{k}{\nu} \cdot P_k^{(k-\nu)}\left(\frac{z}{\eta}\right) \cdot \Theta^{(\nu)}\left(\frac{z}{\eta}\right) \\ &= \frac{1}{1 - z/\eta} \cdot \sum_{\nu=0}^{k} \binom{k}{\nu} \cdot \left(\frac{z/\eta}{1 - z/\eta}\right)^{\nu} = \frac{1}{(1 - z/\eta)^{k+1}} = \Theta_{\eta,0}^{k+1}(z) \end{aligned}$$

for all $z \in \mathbb{C} \setminus \{\eta\}$. Moreover, we have

$$\Theta_{\eta,0}^{k+1}(z) = (\eta \cdot \Theta_\eta)^{k+1}(z) \qquad \big(z \in \mathbb{C}_\infty \setminus \{\eta\}\big).$$

Since $\Theta_{\eta,0}^{k+1}(\infty) = 0$, we define the function $\Theta_{\eta,k} : \mathbb{C}_\infty \setminus \{\eta\} \to \mathbb{C}$ by

$$\Theta_{\eta,k}(z) := (\eta \cdot \Theta_\eta)^{k+1}(z). \tag{3.11}$$

The function $\Theta_{\eta,k}$ belongs to $H^{(k)}(\mathbb{C}_\infty \setminus \{\eta\})$ and has the power series expansion

$$\Theta_{\eta,k} = (-1)^{k+1} \cdot \sum_{\nu=k+1}^{\infty} \binom{\nu-1}{k} \cdot \eta^{\nu} \cdot z^{-\nu} \quad \big(z \in U_{|\eta|}(\infty)\big)$$

around the point at infinity.

3.3.2 Example:

Let $D \subset \mathbb{C}$ be a domain with $0 \in D$, $f \in H(D)$, $\eta \in \mathbb{C} \setminus \{0\}$, and $k \in \mathbb{N}_0$. (In this example we denote the restriction of Θ onto the set $\mathbb{C} \setminus \{\eta\}$ by Θ for the sake of brevity.) We have

$$\Theta^{k+1}(z) = \sum_{n=0}^{\infty} \binom{k+n}{k} \cdot z^n \qquad (z \in \mathbb{D}). \tag{3.12}$$

From (3.11) and (3.12) we get

$$\Theta_{\eta,k}(z) = \sum_{n=0}^{\infty} \binom{k+n}{k} \cdot \left(\frac{z}{\eta}\right)^n \qquad (z \in \eta \cdot \mathbb{D}).$$

According to the Hadamard multiplication theorem, we obtain

$$\big(f *_{D,\,\mathbb{C}\setminus\{\eta\}} \Theta_{\eta,\,k}\big)(z) \;=\; \sum_{n=0}^{\infty} \frac{f_n}{\eta^n} \cdot \binom{k+n}{k} \cdot z^n$$

for all z with small modulus. A direct calculation shows

$$\begin{aligned}
f_{\eta,\,k}\,(z) \;&=\; \frac{1}{k!} \cdot (P_k \cdot f)^{(k)}\left(\frac{z}{\eta}\right) \;=\; \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{1}{\nu!} \left(\frac{z}{\eta}\right)^{\nu} f^{(\nu)}\left(\frac{z}{\eta}\right) \\
&=\; \sum_{\nu=0}^{k} \sum_{n=\nu}^{\infty} \frac{k!\cdot n!}{\nu!\cdot(k-\nu)!\cdot\nu!\cdot(n-\nu)!} \frac{f_n}{\eta^n}\, z^n \;=\; \sum_{\nu=0}^{k} \sum_{n=0}^{\infty} \binom{n}{\nu} \binom{k}{\nu} \frac{f_n}{\eta^n}\, z^n \\
&=\; \sum_{n=0}^{\infty} \left\{ \sum_{\nu=0}^{k} \binom{n}{\nu} \binom{k}{k-\nu} \right\} \frac{f_n}{\eta^n}\, z^n \;=\; \sum_{n=0}^{\infty} \frac{f_n}{\eta^n} \cdot \binom{k+n}{k} \cdot z^n
\end{aligned}$$

for all z with small modulus. By comparing coefficients, we get the relation $(f *_{D,\,\mathbb{C}\setminus\{\eta\}} \Theta_{\eta,\,k})(z) \;=\; f_{\eta,\,k}\,(z)$ in a disk around the origin. Since $\eta \cdot D$ is a domain, we get

$$f *_{D,\,\mathbb{C}\setminus\{\eta\}} \Theta_{\eta,\,k} \;=\; f_{\eta,\,k} \tag{3.13}$$

on $D * (\mathbb{C} \setminus \{\eta\}) = \eta \cdot D$ by the identity theorem. $\diamond$

We shall show that (3.13) is also true if we replace the domain by an open set that does not necessarily contain the origin (see Example 3.4.8).

3.4 The extended Hadamard product

In the first two sections of this chapter, we recalled the notions of the Hadamard product for power series and for holomorphic functions. In both cases, it was required that the involved functions are holomorphic at the origin.

3.4.1 Example:
Consider the polynomial $P := \sum_{\nu=0}^{N} a_\nu P_\nu$, two non-empty open sets $\Omega' \subset \mathbb{C}\setminus\mathbb{D}$ and $\Omega := \mathbb{D} \cup \Omega'$, as well as two functions $\Phi \in H(\Omega)$ and $\Psi \in H(\mathbb{C}_\infty \setminus \{1\})$ so that $\Phi(z) = P(z)$ on Ω'. If $\psi := \Psi\big|_{\mathbb{C}\setminus\{1\}}$, the Hadamard multiplication

theorem yields

$$(\Phi *_{\Omega,\, \mathbb{C}\setminus\{1\}} \psi)(z) \;=\; \frac{1}{2\pi i} \int\limits_{\tau_{(|z|+1)/2}} \frac{\Phi(\zeta)}{\zeta}\, \psi\!\left(\frac{z}{\zeta}\right) d\zeta \;=\; \sum_{\nu=0}^{N} \Phi_\nu\, \psi_\nu\, z^\nu \quad (z \in \mathbb{D}).$$

(Notice that $(\Omega * (\mathbb{C} \setminus \{1\}))_0 = \Omega_0 = \mathbb{D}$.) But what can we say about the Hadamard product on Ω', the component of Ω that does not contain the origin? Since $\Psi \in H(\mathbb{C}_\infty \setminus \{1\})$, there exists an entire function g with $g(0) = 0$ and

$$\Psi(z) \;=\; g\!\left(\frac{1}{1-z}\right) \qquad \big(z \in \mathbb{C}_\infty \setminus \{1\}\big).$$

Let $z \in \Omega'$ and $r > 0$ so that $U_r[z] \subset \Omega'$. Then $\gamma_z := \tau_r(z)$ is no Cauchy cycle for $z\cdot(\mathbb{C}\setminus\{1\})^* = \{z, 0\}$ in Ω; but it is a Cauchy cycle for $z\cdot(\mathbb{C}_\infty\setminus\{1\})^* = \{z\}$ in Ω. Furthermore, we have

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\gamma_z} \frac{\Phi(\zeta)}{\zeta}\, \Psi\!\left(\frac{z}{\zeta}\right) d\zeta \;&=\; \frac{1}{2\pi i}\int_{\gamma_z} \frac{\Phi(\zeta)}{\zeta}\, g\!\left(\frac{1}{1-z/\zeta}\right) d\zeta \\
&=\; \frac{1}{2\pi i}\int_{\gamma_z} \left(\sum_{\nu=0}^{N} a_\nu\, \zeta^{\nu-1}\right) \cdot g\!\left(\frac{1}{1-z/\zeta}\right) d\zeta \\
&=\; \sum_{\nu=0}^{N} \Bigg(a_\nu \cdot \underbrace{\frac{1}{2\pi i}\int_{\gamma_z} \zeta^{\nu-1}\, g\!\left(\frac{1}{1-z/\zeta}\right) d\zeta}_{=:\, b_{\nu,z}}\Bigg).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
b_{\nu,z} \;&=\; \frac{1}{2\pi i}\int_{\gamma_z} \left(\zeta^{\nu-1} \sum_{k=1}^{\infty} g_k \cdot \left(\frac{1}{1-z/\zeta}\right)^{k}\right) d\zeta \\
&=\; \sum_{k=1}^{\infty} \left(g_k \cdot \frac{1}{2\pi i}\int_{\gamma_z} \frac{P_{k+\nu-1}(\zeta)}{(\zeta-z)^k}\, d\zeta\right) \;=\; \sum_{k=1}^{\infty} g_k \cdot \frac{P_{k+\nu-1}^{(k-1)}(z)}{(k-1)!} \\
&=\; \sum_{k=1}^{\infty} g_k \cdot \frac{(k+\nu-1)!}{\nu!(k-1)!} \cdot z^\nu \;=\; z^\nu \cdot \sum_{k=1}^{\infty} g_k \cdot \binom{k+\nu-1}{\nu}.
\end{aligned}$$

The Laurent series expansion of Ψ around 1 is given by

$$\Psi(z) \;=\; \sum_{k=1}^{\infty} \frac{b_k}{(z-1)^k} \qquad (z \in \mathbb{C}_\infty \setminus \{1\}) \tag{3.14}$$

for certain complex numbers b_k $(k \in \mathbb{N})$. It follows

$$\Psi^{(\nu)}(z) \;=\; \sum_{k=1}^{\infty} \frac{(-1)^\nu \cdot b_k \cdot (k+\nu-1)!}{(k-1)!} \cdot \frac{1}{(z-1)^{k+\nu}}$$

for all $z \in \mathbb{C} \setminus \{1\}$ and for all $\nu \in \mathbb{N}_0$. This implies

$$\begin{aligned} \Psi_\nu \;&=\; \frac{\Psi^{(\nu)}(0)}{\nu!} \;=\; \sum_{k=1}^{\infty} (-1)^k \cdot b_k \cdot \frac{(k+\nu-1)!}{\nu!(k-1)!} \\ &=\; \sum_{k=1}^{\infty} (-1)^k \cdot b_k \cdot \binom{k+\nu-1}{\nu} \end{aligned} \tag{3.15}$$

for all $\nu \in \mathbb{N}_0$. For $z \in \mathbb{C}_\infty \setminus \{1\}$, we get

$$\Psi(z) \;=\; g\left(\frac{1}{1-z}\right) \;=\; \sum_{k=1}^{\infty} \frac{(-1)^k \cdot g_k}{(z-1)^k}. \tag{3.16}$$

The uniqueness of the Laurent expansion, together with (3.14) and (3.16), now implies

$$b_k \;=\; (-1)^k \cdot g_k \qquad (k \in \mathbb{N}).$$

Inserting this into (3.15) yields

$$\Psi_\nu \;=\; \sum_{k=1}^{\infty} g_k \cdot \binom{k+\nu-1}{\nu} \qquad (\nu \in \mathbb{N}_0).$$

These considerations show that we have

$$\frac{1}{2\pi i} \int_{\gamma_z} \frac{\Phi(\zeta)}{\zeta}\, \Psi\left(\frac{z}{\zeta}\right) d\zeta \;=\; \sum_{\nu=0}^{N} a_\nu\, \Psi_\nu\, z^\nu$$

for all $z \in \Omega'$. ◇

The previous example shows that, according to the Hadamard multiplication theorem, the local expansion of the Hadamard product is the Hadamard product series of the local expansions of the factors. But let us examine this example a little more conscientiously. It also shows that the expansion holds in the components that do not contain the origin. In this situation, it does not matter how the function Φ is defined on the unit disk. We merely need a definition of Φ in the unit disk to have the origin involved. But this artificial definition of Φ around the origin is not very satisfactory. Therefore, it would be desirable to have a Hadamard product for functions that are not necessarily holomorphic at the origin. Such a definition is the aim of this section. As Example 3.4.1 shows, we have to study the extended complex plane instead of the complex plane (otherwise, γ_z would not be a Cauchy cycle).

Now, we are concerned with open subsets Ω_1 and Ω_2 of the extended plane that satisfy the star condition. Furthermore, we have functions f in $H(\Omega_1)$ and g in $H(\Omega_2)$. In this case, the star eligibility is not guaranteed at all! The star product of Ω_1 and Ω_2 can be the empty set (see Example 1.3.9). (However, the set $\Omega_1 * \Omega_2$ is open according to Proposition 1.3.12.) In the plane case, this cannot happen because the star product of two plane open sets that both contain the origin also contains the origin (see Proposition 1.3.10.2).

The idea how to define a Hadamard product in the new situation is the same as in the well-known case: via a Parseval integral as in (3.3). The bottleneck is to find adequate integration cycles. If $0 \notin \Omega_2$, then $\infty \in \Omega_2^*$. Therefore, we are not able to find an appropriate Cauchy cycle. At this point, anti-Cauchy cycles come into play: Since $0 \notin \Omega_2$ implies $\infty \in \Omega_1$, the set Ω_1^C is a plane compact set. According to Proposition 2.2.1, for each Cauchy cycle for Ω_1^C in z/Ω_2 its reverse cycle is an anti-Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1, and vice versa. This relation provides the key tool for the definition of the Hadamard product. Equipped with the notion of Hadamard cycles, we would like to define the Hadamard product by means of a Parseval integral such as in the familiar situation.

At this juncture, several questions arise:

1. Is the value of the Parseval integral independent of the Hadamard cycle?

2. In what way is the new definition consistent with the former one?

We want to tackle the first question and will give a response to it. The other question will be answered after the definition of the Hadamard product (see section 3.5).

Let us now address to the first question. We shall show that the value of Parseval integral does not depend on the Hadamard cycle.

3.4.2 Lemma:
Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open and star-eligible sets, $z \in \Omega_1 * \Omega_2$, $f \in H(\Omega_1)$, and $g \in H(\Omega_2)$. If γ and Γ are Hadamard cycles for $z \cdot \Omega_2^*$ in Ω_1, then

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta}\, g\!\left(\frac{z}{\zeta}\right) d\zeta \;=\; \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta}\, g\!\left(\frac{z}{\zeta}\right) d\zeta.$$

Proof: 1. Let us first assume that $0, \infty \in \Omega_1 \cap \Omega_2$, $z \notin \{0, \infty\}$, γ is a Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 with $\operatorname{ind}(\gamma, 0) = 1$, and Γ is an anti-Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 with $\operatorname{ind}(\Gamma, 0) = -1$. Then the mapping $\zeta \mapsto f(\zeta)g(z/\zeta)$ defines a function that belongs to $H\big(\mathbb{C}_\infty \setminus (\Omega_1^C \cup (z \cdot \Omega_2^*))\big)$ and vanishes at zero. Moreover, γ is a Cauchy cycle for $z \cdot \Omega_2^*$ in $\mathbb{C}_\infty \setminus \big(\Omega_1^C\big) = \Omega_1$, and, according to Proposition 2.2.1, Γ^- is a Cauchy cycle for Ω_1^C in $\mathbb{C}_\infty \setminus (z \cdot \Omega_2^*) = z/\Omega_2$. Thus, by Lemma 2.2.6 (iii) the assertion follows.

2. In all the other cases, the index relation

$$\operatorname{ind}(\gamma, \kappa) \;=\; \operatorname{ind}(\Gamma, \kappa) \qquad \big(\kappa \in \Omega_1^C \cup (z \cdot \Omega_2^*)\big) \tag{3.17}$$

holds. According to Theorem 10.35 of [Rud2], the assertion follows. $\square$

Part two of the proof shows that the independence of the Pareseval's integral value can still be guaranteed if $0 \notin \Omega_1 \cap \Omega_2$ or $\infty \notin \Omega_1 \cap \Omega_2$. On the other hand, if $0 \in \Omega_1 \cap \Omega_2$ and $\infty \in \Omega_1 \cap \Omega_2$, it is necessary that f and g vanish at infinity. The next example shows why.

3.4.3 Example:
Let $\Omega_1 := \Omega_2 := \mathbb{C}_\infty \setminus \{1\}$, $z \in (\Omega_1 * \Omega_2) \setminus \{0, \infty\} = \mathbb{C} \setminus \{0, 1\}$, γ a Cauchy cycle for $z \cdot \Omega_2^* = \{z\}$ in Ω_1 with $\operatorname{ind}(\gamma, 0) = 1$, and Γ an anti-Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 with $\operatorname{ind}(\gamma, 0) = -1$. Furthermore, let $f : \Omega_1 \to \mathbb{C}$ and $g : \Omega_2 \to \mathbb{C}$.

1. Define $f(z) := g(z) := 1$. Then we have

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta}\, g\!\left(\frac{z}{\zeta}\right) d\zeta \;=\; \operatorname{ind}(\gamma, 0) \;=\; 1,$$

but

$$\frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta}\, g\!\left(\frac{z}{\zeta}\right) d\zeta \;=\; \operatorname{ind}(\Gamma, 0) \;=\; -1.$$

2. Define $f(z) := 1$ and $g(z) := 1/(1-z)$. Then we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta}\, g\left(\frac{z}{\zeta}\right) d\zeta \;=\; \operatorname{ind}(\gamma, z) \;=\; 1,$$

but

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta}\, g\left(\frac{z}{\zeta}\right) d\zeta \;=\; \operatorname{ind}(\Gamma, z) \;=\; 0.$$

3. Define $f(z) := 1/(z-1)$ and $g(z) := 1$. Then we have

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta}\, g\left(\frac{z}{\zeta}\right) d\zeta \;=\; \operatorname{ind}(\gamma, 1) - \operatorname{ind}(\gamma, 0) \;=\; -1,$$

but

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta}\, g\left(\frac{z}{\zeta}\right) d\zeta \;=\; \operatorname{ind}(\Gamma, 1) - \operatorname{ind}(\Gamma, 0) \;=\; 0.$$

The first example shows that the Parseval's integral value depends on the Hadamard if neither f nor g vanishes at infinity. But even if only one of them does not vanish at infinity, the independence cannot be guaranteed any longer, as the last two examples show. ◇

We therefore answered the first question on page 43 in the affirmative. Moreover, this lemma justifies the following definition.

3.4.4 Definition:
Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open and star-eligible sets, $f \in H(\Omega_1)$, $g \in H(\Omega_2)$, $z \in \Omega_1 * \Omega_2$, and γ a Hadamard cycle for $z \cdot \Omega_2^*$ in Ω_1. Then the function $f *_{\Omega_1,\Omega_2} g : \Omega_1 * \Omega_2 \to \mathbb{C}$ defined by

$$(f *_{\Omega_1,\Omega_2} g)(z) := \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta}\, g\left(\frac{z}{\zeta}\right) d\zeta \tag{3.18}$$

is called the *Hadamard product of f and g.* ◇

As a first property, we present how the value of the Hadamard product at the origin and at the point at infinity can be evaluated.

3.4.5 Lemma:
Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open and star-eligible sets, $f \in H(\Omega_1)$, and $g \in H(\Omega_2)$.

1. *If* $0 \in \Omega_1 * \Omega_2$, *then* $(f *_{\Omega_1,\Omega_2} g)(0) = f(0) \cdot g(0)$.

2. *If* $\infty \in \Omega_1 * \Omega_2$, *then* $(f *_{\Omega_1,\Omega_2} g)(\infty) = 0$.

Proof: *ad 1.:* If $r > 0$ satisfies $U_r[0] \subset \Omega_1$, then τ_r is a Hadamard cycle for $\{0\}$ in Ω_1. Hence, we get

$$(f *_{\Omega_1,\Omega_2} g)(0) \;=\; \frac{1}{2\pi i} \int_{\tau_r} \frac{f(\zeta)}{\zeta}\, d\zeta \cdot g(0) \;=\; f(0) \cdot g(0).$$

ad 2.: If γ is a Hadamard cycle for $\{\infty\}$ in Ω_1, then we have

$$(f *_{\Omega_1,\Omega_2} g)(\infty) \;=\; \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)}{\zeta}\, d\zeta \cdot g(\infty) \;=\; 0$$

since $g(\infty) = 0$. □

At the end of this section, we would like to present some examples. We recall that $P_k(z) = z^k$.

3.4.6 Example:
Let $\Omega \subsetneq \mathbb{C}_\infty$ be an open set with $0 \in \Omega$, and let $f \in H(\Omega)$. Let $r > 0$ so that $U_r[0] \subset \Omega$. If $z \in \Omega * \mathbb{C} = \mathbb{C}$, then τ_r is a Hadamard cycle for $z \cdot \mathbb{C}^* = \{0\}$ in Ω. We get

$$(f *_{\Omega,\mathbb{C}} P_k)(z) \;=\; \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta^{k+1}}\, d\zeta \cdot z^k \;=\; \frac{f^{(k)}(0)}{k!} \cdot z^k.$$

Thus, we have

$$f *_{\Omega,\mathbb{C}} P_k \;=\; \frac{f^{(k)}(0)}{k!} \cdot P_k,$$

i.e. $f *_{\Omega,\mathbb{C}} P_k$ is a multiple of P_k. ◇

For $k \in \mathbb{N}$ we denote by $R_k : \mathbb{C}_\infty \setminus \{0\} \to \mathbb{C}$ the rational function defined by

$$R_k(z) \;:=\; z^{-k}.$$

3.4.7 Example:
Let $\Omega \subsetneq \mathbb{C}_\infty$ be an open set with $\infty \in \Omega$, and let $f \in H(\Omega)$. Let $R > 0$ so

that $U_R[\infty] \subset \Omega$. If $z \in \Omega * (\mathbb{C}_\infty \setminus \{0\}) = \mathbb{C}_\infty \setminus \{0\}$, then τ_R^- is a Hadamard cycle for $z \cdot (\mathbb{C}_\infty \setminus \{0\})^* = \{\infty\}$ in Ω. We get

$$\big(f *_{\Omega,\, \mathbb{C}_\infty \setminus \{0\}} R_k\big)(z) \;=\; \frac{1}{2\pi i} \int_{\tau_R^-} f(\zeta)\, \zeta^{k-1}\, d\zeta \cdot z^{-k} \;=\; -\,\frac{f^{(k)}(\infty)}{k!} \cdot z^{-k}.$$

Thus, we have

$$f *_{\Omega,\, \mathbb{C}_\infty \setminus \{0\}} R_k \;=\; -\,\frac{f^{(k)}(\infty)}{k!} \cdot R_k,$$

i.e. $f *_{\Omega,\, \mathbb{C}_\infty \setminus \{0\}} R_k$ is a multiple of R_k. ◇

3.4.8 Example:
Let $\Omega \subsetneq \mathbb{C}_\infty$ be a non-empty open set, $\eta \in \mathbb{C} \setminus \{0\}$, $k \in \mathbb{N}_0$, and $f \in H^{(2k)}(\Omega)$. The sets Ω and $\mathbb{C}_\infty \setminus \{\eta\}$ are star-eligible with $\Omega * (\mathbb{C}_\infty \setminus \{\eta\}) = \eta \cdot \Omega$. Now let $z \in \eta \cdot \Omega$ and γ a Hadamard cycle for $z \cdot (\mathbb{C}_\infty \setminus \{\eta\})^* = \{z/\eta\}$ in Ω.

(i) If $z \notin \{0, \infty\}$, then we have

$$\big(f *_{\Omega,\, \mathbb{C}_\infty \setminus \{\eta\}} \Theta_{\eta,\, k}\big)(z) \;=\; \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta) \cdot \zeta^k}{(\zeta - z/\eta)^{k+1}}\, d\zeta.$$

Case 1: $0 \in \Omega$ and $\infty \notin \Omega$. Here, γ must be a Cauchy cycle for $\{z/\eta\}$ in Ω with $\operatorname{ind}(\gamma, 0) = 1$. Since $\operatorname{ind}(\gamma, z/\eta) = 1$, the value of the integral on the right-hand side equals $f_{\eta,\, k}(z)$.

Case 2: $0 \notin \Omega$ and $\infty \in \Omega$. Here, γ must be an anti-Cauchy cycle for $\{z/\eta\}$ in Ω with $\operatorname{ind}(\gamma, 0) = -1$. Since the function

$$\Omega \setminus \{z/\eta\} = \mathbb{C}_\infty \setminus \big(\Omega^C \cup \{z/\eta\}\big) \to \mathbb{C}, \quad \zeta \mapsto \frac{f(\zeta) \cdot \zeta^{k+1}}{(\zeta - z/\eta)^{\,k+1}}$$

is holomorphic and vanishes at infinity, we get, according to Lemma 2.2.6,

$$\frac{1}{2\pi i} \int\limits_\gamma \frac{f(\zeta) \cdot \zeta^k}{(\zeta - z/\eta)^{\,k+1}}\, d\zeta \;=\; \frac{1}{2\pi i} \int\limits_{\tau_r(z/\eta)} \frac{f(\zeta) \cdot \zeta^k}{(\zeta - z/\eta)^{\,k+1}}\, d\zeta$$

for an $r > 0$ small enough. The value of the second integral in the last equation equals $f_{\eta,\, k}(z)$.

Case 3: $0, \infty \in \Omega$. Here, γ must be a Cauchy cycle for $\{z/\eta\}$ in Ω with $\operatorname{ind}(\gamma, 0) = 1$ or an anti-Cauchy cycle for $\{z/\eta\}$ in Ω with $\operatorname{ind}(\gamma, 0) = -1$. In either situation, we get the same value as in the first two cases.

Case 4: $0, \infty \notin \Omega$. Here, γ must be a Cauchy cycle for $\{z/\eta\}$ in Ω. Since $\operatorname{ind}(\gamma, z/\eta) = 1$, Cauchy's formula once more establishes the same result.

(ii) If $z = 0$, we get, according to Lemma 3.4.5.1 and (3.8),

$$(f *_{\Omega,\, \mathbb{C}_\infty \setminus \{\eta\}} \Theta_{\eta,\, k})(0) \;=\; f(0) \cdot \Theta_{\eta,\, k}(0) \;=\; f(0) \;=\; f_{\eta,\, k}(0).$$

(iii) If $z = \infty$, we get

$$(f *_{\Omega,\, \mathbb{C}_\infty \setminus \{\eta\}} \Theta_{\eta,\, k})(\infty) \;=\; 0 \;=\; f_{\eta,\, k}(\infty)$$

since $f \in H^{(2k)}(\Omega)$.

Thus, in all possible cases we have

$$f *_{\Omega,\, \mathbb{C}_\infty \setminus \{\eta\}} \Theta_{\eta,\, k} \;=\; f_{\eta,\, k}. \tag{3.19}$$

This is a generalization of the result in Example 3.3.2. Moreover, we get

$$f *_{\Omega,\, \mathbb{C}_\infty \setminus \{\eta\}} \Theta_\eta^{k+1} \;=\; \eta^{-(k+1)} \cdot f_{\eta,\, k} \tag{3.20}$$

by means of (3.11). ◇

3.4.9 Example:
Let $p \in \mathbb{N}$. Denote by $\xi_j := \exp(2\pi(j-1)i/p)$ $(1 \le j \le p)$ the pth roots of unity. For $U := \mathbb{C}_\infty \setminus \{\xi_j : 1 \le j \le p\}$, we define $g : U \to \mathbb{C}$ by $g(z) := 1/(1 - z^p)$. Moreover, let $\Omega \subset \mathbb{C}_\infty$ be a non-empty open set so that Ω and U are star-eligible, and let $f \in H(\Omega)$. According to Example 1.3.8, we get

$$\Omega * U \;=\; \bigcap_{j=1}^{p} \xi_j \cdot \Omega.$$

Now let $z \in \Omega * U$ and $A_j := (p \cdot \xi_j^{p-1})^{-1}$ $(1 \le j \le p)$. The partial fractions decomposition of g is given by

$$\frac{1}{1 - z^p} \;=\; \sum_{j=1}^{p} \frac{A_j}{\xi_j - z} \;=\; \sum_{j=1}^{p} A_j \cdot \Theta_{\xi_j}(z) \qquad (z \in U).$$

According to (3.11), (3.20), and (3.9), the Hadamard product of f and g is

$$\begin{aligned}(f *_{\Omega,U} g)(z) &= \sum_{j=1}^{p} A_j \cdot \Big(f *_{\Omega,U} \big(\Theta_{\xi_j}\big|_U\big)\Big)(z) \\ &= \sum_{j=1}^{p} \frac{A_j}{\xi_j} \cdot \Big(f *_{\Omega,U} \big(\Theta_{\xi_j,0}\big|_U\big)\Big)(z) \;=\; \frac{1}{p} \cdot \sum_{j=1}^{p} f_{\xi_j,0}(z) \\ &= \frac{1}{p} \cdot \sum_{j=1}^{p} f\left(\frac{z}{\xi_j}\right).\end{aligned}$$

In particular, we have

$$(f *_{\Omega,U} g)(z) \;=\; \frac{f(z)+f(-z)}{2} \qquad \big(z \in \Omega \cap (-\Omega)\big)$$

for the special case $p = 2$. ◇

3.5 The Hadamard product on subsets

The second question on page 43 is concerned with the compatibility of the plane and the extended version of the Hadamard product. To be a reasonable definition, the extended version of the Hadamard product should coincide with the plane one in the case of plane open sets both containing the origin. This is indeed true, as the next proposition shows.

3.5.1 Proposition:
Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be open sets both containing the origin, $f \in H(\Omega_1)$, and $g \in H(\Omega_2)$. Then the parameter integrals in (3.3) and (3.18) define the same function.

Proof: Since Ω_1 and Ω_2 contain the origin, they are star-eligible. Moreover, we have $\mathbb{C} \setminus \big[(\mathbb{C} \setminus \Omega_1) \cdot (\mathbb{C} \setminus \Omega_2)\big] = \Omega_1 * \Omega_2$ according to Proposition 1.3.10.4. For $z \in \Omega_1 * \Omega_2$ we have $0 \in z \cdot \Omega_2^*$, and every Hadamard cycle for $z \cdot \Omega_2^*$ in Ω_1 is a Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 with $\operatorname{ind}(\gamma, 0) = 1$. This shows that the index relation

$$\operatorname{ind}(\gamma, \kappa) \;=\; \operatorname{ind}(\tilde{\gamma}, \kappa) \qquad \big(\kappa \in \Omega_1^C \cup (z \cdot \Omega_2^*)\big)$$

holds for all Cauchy cycles γ and $\tilde{\gamma}$ for $z \cdot \Omega_2^*$ in Ω_1. Thus, the integrals in (3.3) and (3.18) define the same function. □

Another question is how the Hadamard product acts on subsets. Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open and star-eligible sets; $D_1 \subset \Omega_1$ and $D_2 \subset \Omega_2$ open and star-eligible sets; $f \in H(\Omega_1)$; and $g \in H(\Omega_2)$. Is then

$$\left(f *_{\Omega_1,\Omega_2} g\right)\big|_{D_1 * D_2} \;=\; \left(f\big|_{D_1}\right) *_{D_1,D_2} \left(g\big|_{D_2}\right)?$$

To answer this question, we consider a Hadamard cycle γ for $z \cdot D_2^*$ in D_1. What we have to check is whether γ is also a Hadamard cycle for $z \cdot \Omega_2^*$ in Ω_1. (This need not always be true; in such a case we will replace γ by a suitable Hadamard cycle for $z \cdot \Omega_2^*$ in Ω_1 in such a way that the corresponding Parseval integrals have the same value.) Several cases can occur, depending on which of the sets contain the origin and the point at infinity (see Appendix B). We can prove the following result.

3.5.2 Compatibility theorem:
Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open and star-eligible sets, let $D_1 \subset \Omega_1$ and $D_2 \subset \Omega_2$ be open and star-eligible sets. Furthermore, let $f \in H(\Omega_1)$ and $g \in H(\Omega_2)$. Then we have

$$\left(f *_{\Omega_1,\Omega_2} g\right)\big|_{D_1 * D_2} \;=\; \left(f\big|_{D_1}\right) *_{D_1,D_2} \left(g\big|_{D_2}\right).$$

Proof: 1. Let $z \in (D_1 * D_2) \setminus \{0, \infty\}$ and γ a Hadamard cycle for $z \cdot D_2^*$ in D_1. Since $D_j \subset \Omega_j$, we obtain the inclusions

$$\Omega_j^C \subset D_j^C \quad \text{and} \quad z \cdot \Omega_j^* \subset z \cdot D_j^* \qquad (j = 1; 2). \tag{3.21}$$

The function

$$\varphi : \Omega_1 \cap \frac{z}{\Omega_2} \to \mathbb{C}, \quad \varphi(\zeta) \;:=\; f(\zeta) \cdot g\left(\frac{z}{\zeta}\right)$$

is holomorphic. If we abbreviate $\tilde{f} := f\big|_{D_1}$ and $\tilde{g} := g\big|_{D_2}$, we get

$$\left(\tilde{f} *_{D_1,D_2} \tilde{g}\right)(z) \;=\; \frac{1}{2\pi i} \int_\gamma \frac{\varphi(\zeta)}{\zeta}\, d\zeta.$$

Case 1: $0 \in D_1 \cap D_2$ and $\infty \notin D_1 \cap D_2$. Here, γ is a Cauchy cycle for $z \cdot D_2^*$ in D_1 with $\operatorname{ind}(\gamma, 0) = 1$. Since $0 \in D_1 \cap D_2$, we immediately get $0 \in \Omega_1 \cap \Omega_2$. We have

$$\operatorname{ind}(\gamma, \kappa) \;=\; \begin{cases} 1, & \kappa \in z \cdot \Omega_2^* \\ 0, & \kappa \in \Omega_1^C \end{cases}$$

according to (3.21), i.e. γ is a Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 with $\operatorname{ind}(\gamma, 0) = 1$. Thus, we get $(\tilde{f} *_{D_1,D_2} \tilde{g})(z) = (f *_{\Omega_1,\Omega_2} g)(z)$.

Case 2: $0 \notin D_1 \cap D_2$ and $\infty \in D_1 \cap D_2$. Here, γ is an anti-Cauchy cycle for $z \cdot D_2^*$ in D_1 with $\operatorname{ind}(\gamma, 0) = -1$. Since $\infty \in D_1 \cap D_2$, we immediately get $\infty \in \Omega_1 \cap \Omega_2$. We have

$$\operatorname{ind}(\gamma, \kappa) \;=\; \begin{cases} 0, & \kappa \in z \cdot \Omega_2^* \\ -1, & \kappa \in \Omega_1^C \end{cases}$$

according to (3.21), i.e. γ is an anti-Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 with $\operatorname{ind}(\gamma, 0) = -1$. Thus, we get $(\tilde{f} *_{D_1,D_2} \tilde{g})(z) = (f *_{\Omega_1,\Omega_2} g)(z)$.

Case 3: $0, \infty \in D_1 \cap D_2$. We get $0, \infty \in \Omega_1 \cap \Omega_2$, so that γ is also a Hadamard cycle for $z \cdot \Omega_2^*$ in Ω_1. Thus, we get $(\tilde{f} *_{D_1,D_2} \tilde{g})(z) = (f *_{\Omega_1,\Omega_2} g)(z)$.

Case 4: $0, \infty \in D_2 \setminus D_1$. We get $0, \infty \in \Omega_2$. Here, γ is a Cauchy cycle for $z \cdot D_2^*$ in D_1 with $\operatorname{ind}(\gamma, 0) = 0$ (cf. Remark 2.1.8). According to (3.21), γ is a Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 with $\operatorname{ind}(\gamma, 0) = 0$. Now, we have to distinguish four subcases:

4.1: $0, \infty \notin \Omega_1$. Here, we need a Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1. The cycle γ is suitable.

4.2: $0, \infty \in \Omega_1$. Here, we need a Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 whose index with respect to 0 equals 1, or we need an anti-Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 whose index with respect to 0 equals -1. Let $r > 0$ sufficiently small and so that $U_r[0] \subset \Omega_1 \cap (z/\Omega_2)$. Then $\Gamma := \gamma \oplus \tau_r$ is a Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 with $\operatorname{ind}(\gamma, 0) = 1$. Since, in this case, φ is holomorphic at the origin and vanishes there, we get $\int_{\tau_r} \varphi(\zeta)/\zeta \, d\zeta = 0$, and hence

$$\int_\Gamma \frac{\varphi(\zeta)}{\zeta} \, d\zeta \;=\; \int_\gamma \frac{\varphi(\zeta)}{\zeta} \, d\zeta.$$

Thus, the cycle Γ is suitable.

4.3: $0 \in \Omega_1$ and $\infty \notin \Omega_1$. Here, we need a Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 whose index with respect to 0 equals 1. Let r be as in subcase 4.2. Then the same strategy works here, too.

4.4: $0 \notin \Omega_1$ and $\infty \in \Omega_1$. Here, we need an anti-Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 whose index with respect to 0 equals -1. Since γ is a Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 with $\operatorname{ind}(\gamma, 0) = 0$, there is a Cauchy cycle Γ for Ω_1^C in $\mathbb{C}_\infty \setminus (z \cdot \Omega_2^*) = z/\Omega_2$ in such a way that Lemma 2.2.6 (condition (i)) can be applied to φ, Ω_1^C, and

$z \cdot \Omega_2^*$. According to Proposition 2.2.1, Γ^- is an anti-Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1. Moreover, it satisfies $\operatorname{ind}(\Gamma^-, 0) = -1$. Thus, the cycle Γ^- is suitable.

In all four subcases, we get $(\tilde{f} *_{D_1,D_2} \tilde{g})(z) = (f *_{\Omega_1,\Omega_2} g)(z)$.

Case 5: $0, \infty \in D_1 \setminus D_2$. This case can be treated like case 4.

2. If $z = 0$, we necessarily have $0 \in D_1 \cap D_2$, and hence $0 \in \Omega_1 \cap \Omega_2$. Thus, we get $(\tilde{f} *_{D_1,D_2} \tilde{g})(0) = \tilde{f}(0) \cdot \tilde{g}(0) = f(0) \cdot g(0) = (f *_{\Omega_1,\Omega_2} g)(0)$ according to Lemma 3.4.5.1.

3. If $z = \infty$, we necessarily have $\infty \in D_1 \cap D_2$, and hence $\infty \in \Omega_1 \cap \Omega_2$. Thus, we get $(\tilde{f} *_{D_1,D_2} \tilde{g})(\infty) = 0 = (f *_{\Omega_1,\Omega_2} g)(\infty)$ according to Lemma 3.4.5.2. □

3.5.3 Remark:
Now that we know that the Hadamard product does not depend on the underlying open sets, we could omit them in the notation and simply write $f * g$ instead of $f *_{\Omega_1,\Omega_2} g$. However, we will not do this for the sake of clarity. ◇

3.6 Algebraic and analytic properties of the Hadamard product

Whenever we encounter a product, we ask if it commutes. It is already known that the plane version of the Hadamard product has this property (cf. [GE93]). The next theorem shows that this is also true for the extended version.

3.6.1 Proposition:
Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open and star-eligible sets, $f \in H(\Omega_1)$, and $g \in H(\Omega_2)$. Then we have $f *_{\Omega_1,\Omega_2} g = g *_{\Omega_2,\Omega_1} f$.

Proof: Let $z \in \Omega_1 * \Omega_2$ and γ a Hadamard cycle for $z \cdot \Omega_2^*$ in Ω_1.

For $z = 0$—if at all—the commutativity follows from Lemma 3.4.5.1.

For $z = \infty$—if at all—the commutativity follows from Lemma 3.4.5.2.

Now let $z \neq 0$ and $z \neq \infty$. Proposition 2.1.3 yields

$$(f *_{\Omega_1,\Omega_2} g)(z) \;=\; \frac{1}{2\pi i} \int_{z/\gamma^-} \frac{g(\zeta)}{\zeta} \, f\!\left(\frac{z}{\zeta}\right) d\zeta.$$

According to Theorem 2.2.3, the cycle z/γ^- is a Hadamard cycle for $z\cdot\Omega_1^*$ in Ω_2. Therefore, the value of the integral on the right-hand side equals $(g *_{\Omega_2,\Omega_1} f)(z)$. This proves the commutativity. □

Next, we devote ourselves to neutral elements. We already know that the geometric series serves as a type of neutral element with respect to Hadamard multiplication of power series (see section 3.1). For the plane version of the Hadamard product, this role is played by the function $\Theta\big|_{\mathbb{C}\setminus\{1\}}$ (see section 3.2). What can be said about the extended version of the Hadamard product? Indeed, Θ is a neutral element in the following sense.

3.6.2 Proposition:
If $\Omega \subset \mathbb{C}_\infty$ is a non-empty open set and $f \in H(\Omega)$, then $f *_{\Omega,\,\mathbb{C}_\infty\setminus\{1\}} \Theta = f$.

Proof: Example 3.4.8 can be applied with $\eta = 1$ and $k = 0$. According to (3.20) and (3.9), we obtain the assertion. □

Combining this proposition with the results of section 3.5, we can deduce the following corollary.

3.6.3 Corollary:
Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open an star-eligible sets with $1 \notin \Omega_2$, and let $f \in H(\Omega_1)$. Then we have

$$f\,\big|_{\Omega_1 * \Omega_2} \;=\; f *_{\Omega_1,\Omega_2} \Big(\Theta\big|_{\Omega_2}\Big).$$

Proof: By the compatibility theorem we get

$$f *_{\Omega_1,\Omega_2} \big(\Theta\big|_{\Omega_2}\big) = \big(f *_{\Omega_1,\mathbb{C}_\infty\setminus\{1\}} \Theta\big)\big|_{\Omega_1 * \Omega_2}.$$

The assertion then follows by means of Proposition 3.6.2. □

At last, we shall show that the Hadamard product is a holomorphic function.

3.6.4 Proposition:
Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open and star-eligible sets, $f \in H(\Omega_1)$, and $g \in H(\Omega_2)$. Then we have $f *_{\Omega_1,\,\Omega_2} g \in H(\Omega_1 * \Omega_2)$. Furthermore, its derivatives are given by

$$(f *_{\Omega_1,\Omega_2} g)^{(k)}(z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta^{k+1}}\, g^{(k)}\!\left(\frac{z}{\zeta}\right) d\zeta$$

for all $z \in (\Omega_1 * \Omega_2) \setminus \{\infty\}$ *and all* $k \in \mathbb{N}_0$.‡

Proof: Let $z_0 \in (\Omega_1 * \Omega_2) \setminus \{\infty\}$ and $r > 0$ in such a way that $U_{2r}[z_0] \subset \Omega_1 * \Omega_2$. According to the existence theorem for Hadamard cycles, there exists a cycle γ that is a Hadamard cycle for $z \cdot \Omega_2^*$ in Ω_1 for every $z \in U_{2r}[z_0]$. The holomorphy of $(f *_{\Omega_1,\Omega_2} g)\big|_{U_r(z_0)}$ and the formula for the derivatives in z_0 follow from well-known results on parameter integrals. Therefore, the function $(f *_{\Omega_1,\Omega_2} g)\big|_{(\Omega_1 * \Omega_2)\,\cap\,\mathbb{C}}$ is holomorphic.

Now assume that we have $\infty \in \Omega_1 * \Omega_2$ in addition. It remains to show that f is continuous at the point at infinity. According to Lemma 3.4.5.2, we have $(f *_{\Omega_1,\Omega_2} g)(\infty) = 0$. Since $\infty \in \Omega_1 \cap \Omega_2$, there exist $R, S > 0$ so that $U_R[\infty] \subset \Omega_1$ and so that τ_R^- is a Hadamard cycle for $z \cdot \Omega_2^*$ in Ω_1 for every $z \in U_S(\infty)$. For each such z we get

$$\begin{aligned}
\big|(f *_{\Omega_1,\Omega_2} g)(z)\big| &= \left| \frac{1}{2\pi i} \int_{\tau_R^-} \frac{f(\zeta)}{\zeta}\, g\!\left(\frac{z}{\zeta}\right) d\zeta \right| \leq \|f\|_{\mathbb{T}_R} \cdot \|g\|_{z/\mathbb{T}_R} \\
&= \|f\|_{\mathbb{T}_R} \cdot \max_{w \in z/\mathbb{T}_R} |g(w)| \xrightarrow{z\to\infty} 0.
\end{aligned}$$

This proves the proposition. □

The evaluation of the derivatives of the Hadamard product at the point at infinity will be postponed (see Example 4.1.4).

3.7 The Hadamard product as an operator between Fréchet spaces

As usual, Ω_1 and Ω_2 are open and star-eligible subsets of the extended complex plane, f is in $H(\Omega_1)$, and g is in $H(\Omega_2)$.

In contrast to the last sections where we considered the Hadamard product as a function of a complex variable, we now consider the Hadamard product as an operator that maps an element of the set $H(\Omega_1) \times H(\Omega_2)$ to an element of the set $H(\Omega_1 * \Omega_2)$. To this end, we define the operator

$$T : H(\Omega_1) \times H(\Omega_2) \to H(\Omega_1 * \Omega_2), \quad (f, g) \mapsto f *_{\Omega_1,\Omega_2} g.$$

‡ Here, γ is a Hadamard cycle for $z \cdot \Omega_2^*$ in Ω_1.

The vector space $H(\Omega_1) \times H(\Omega_2)$ is supposed to carry the product topology. For $\varphi \in H(\Omega_1)$ and $\psi \in H(\Omega_2)$, we define

$$T_\psi : H(\Omega_1) \to H(\Omega_1 * \Omega_2), \quad f \mapsto f *_{\Omega_1,\Omega_2} \psi,$$

$$T^\varphi : H(\Omega_2) \to H(\Omega_1 * \Omega_2), \quad g \mapsto \varphi *_{\Omega_1,\Omega_2} g.$$

We want to prove two essential properties of the these operators. The idea of the proofs follows the idea of the proofs in [GE93].

3.7.1 Lemma:
The operators T_ψ and T^φ are linear and continuous.

Proof: The linearity follows immediately from the linearity of the integral.

We only show that T_ψ is continuous. To this end, let $K \subset \Omega_1 * \Omega_2$ be a non-empty compact set. According to the existence theorem for Hadamard cycles, there exists a cycle γ that is a Hadamard cycle for $z \cdot \Omega_2^*$ in Ω_1 for every $z \in K$. For $z \in K$ and $f \in H(\Omega_1)$ we get

$$|T_\psi(f)(z)| \;\leq\; \frac{1}{2\pi} \int_\gamma \frac{|f(\zeta)|}{|\zeta|} \left|\psi\left(\frac{z}{\zeta}\right)\right| |d\zeta| \;\leq\; C \cdot \|f\|_{|\gamma|}$$

with $C := \frac{1}{2\pi} \cdot L(\gamma) \cdot \max\limits_{\zeta \in |\gamma|} \frac{1}{|\zeta|} \cdot \max\limits_{\zeta \in K/|\gamma|} |\psi(\zeta)|$. (Notice that C is independent of z and of f.) Hence, we get

$$\|T_\psi(f)\|_K \;\leq\; C \cdot \|f\|_{|\gamma|} \qquad \big(f \in H(\Omega_1)\big).$$

But this proves the continuity of T_ψ (see e.g. [Yos, Theorem 1]). $\square$

3.7.2 Example:
Let $\Omega \subsetneq \mathbb{C}_\infty$ be a non-empty open set and $f \in H(\Omega)$.

(a) If $0 \in \Omega$ and $k \in \mathbb{N}_0$, then, according to Example 3.4.6, P_k is an eigenfunction for $T^f : H(\mathbb{C}) \to H(\mathbb{C})$ with corresponding eigenvalue $f^{(k)}(0)/k!$.

(b) If $\infty \in \Omega$ and $k \in \mathbb{N}$, then, according to Example 3.4.7, R_k is an eigenfunction for $T^f : H(\mathbb{C}_\infty \backslash \{0\}) \to H(\mathbb{C}_\infty \backslash \{0\})$ with corresponding eigenvalue $-f^{(k)}(\infty)/k!$. $\diamond$

Next, we prove the corresponding properties of the operator T.

3.7.3 Proposition:
The operator T is bilinear and continuous.

Proof: The bilinearity is a consequence of Lemma 3.7.1.

Again by Lemma 3.7.1 the operators T_ψ and T^φ are continuous. Since $H(\Omega_1)$, $H(\Omega_2)$, and $H(\Omega_1 * \Omega_2)$ are Fréchet spaces, the continuity of T follows by means of [Rud1, Theorem 2.17]. □

A universal power series is defined by a certain property of its partial sums (see section 5.2). If we are interested in the question whether the Hadamard product inherits this universal property, we have to know how the partial sums behave under Hadamard multiplication. The next result answers this question for a more general setting.

3.7.4 Continuity theorem:
Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open sets with $0 \in \Omega_1 \cap \Omega_2$, and let $G \subset \mathbb{C}$ be an open set that is star-eligible to Ω_1. Furthermore, let $f \in H(\Omega_1)$, $g \in H(\Omega_2)$, $h \in H(G)$ as well as $L_n : H(\{0\}) \to H(\mathbb{C})$ for $n \in \mathbb{N}$ so that $L_n(g) \to h$ compactly on G as $n \to \infty$.

1. *Then $f *_{\Omega_1, \mathbb{C}} \big(L_n(g)\big) \to f *_{\Omega_1, G} h$ compactly on $\Omega_1 * G$ as $n \to \infty$.*
2. *If in addition $L_n(f *_{\Omega_1, \Omega_2} g) = f *_{\Omega_1, \mathbb{C}} \big(L_n(g)\big)$ for all $n \in \mathbb{N}$, then we have $L_n(f *_{\Omega_1,\Omega_2} g) \to f *_{\Omega_1,G} h$ compactly on $\Omega_1 * G$ as $n \to \infty$.*

Proof: Let $K \subset \Omega_1 * G$ be a compact set. Proposition 3.7.3 guarantees the existence of two compact sets $K' \subset \Omega_1$ and $K'' \subset G$ as well as a constant $C > 0$ in such a way that we have

$$\begin{aligned}\left\| f *_{\Omega_1,G} \left(L_n(g)\big|_G\right) - f *_{\Omega_1,G} h \right\|_K &= \left\| f *_{\Omega_1,G} \left(\left(L_n(g)\big|_G\right) - h \right) \right\|_K \\ &\le C \cdot \|f\|_{K'} \cdot \|L_n(g) - h\|_{K''} .\end{aligned}$$

Since $\Omega_1 * G \subset \Omega_1 * \mathbb{C}$, we get

$$\big(f *_{\Omega_1, \mathbb{C}} (L_n(g))\big)\big|_{\Omega_1 * G} = f *_{\Omega_1, G} \left(L_n(g)\big|_G\right) \qquad (\star)$$

according to the compatibility theorem.

ad 1.: Since $\|L_n(g) - h\|_{K''} \to 0$ as $n \to \infty$, the first assertion follows.

ad 2.: From $(\star)$ and the assumption on L_n, we get

$$\big(L_n(f *_{\Omega_1,\Omega_2} g)\big)\big|_{\Omega_1 * G} \;=\; f *_{\Omega_1,G} \big(L_n(g)\big|_G\big).$$

By means of part one the second assertion follows. $\square$

3.7.5 Remark:
The continuity theorem remains true if we replace 0 by ∞ and if we replace $\mathbb{C}$ by $\mathbb{C}_\infty \setminus \{0\}$. $\diamond$

Notice that $L_n(f *_{\Omega_1,\Omega_2} g) \in H(\mathbb{C})$ and, since $\Omega_1 * \mathbb{C} = \mathbb{C}$, we also have $f *_{\Omega_1,\mathbb{C}} L_n(g) \in H(\mathbb{C})$. Thus, the additional condition in part two of the continuity theorem makes sense.

Chapter 4

Applications

In this chapter, we will use the Hadamard product and its properties to deduce two classical results: the Hadamard multiplication theorem and the Borel-Okada theorem. In both cases, the continuity of the Hadamard operator is the decisive property.

In the first section, we will prove two versions of the Hadamard multiplication theorem. The first one is a generalization of the Hadamard multiplication theorem (see section 3.3). In this version the functions under consideration are allowed to be holomorphic at the point at infinity. The second version is an analogous result for the local expansion around the point at infinity. At the end of the section, we will use the generalization of the Hadamard multiplication theorem to show that the partial sum operators satisfy the condition of the operators in part two of the continuity theorem (see Proposition 4.1.5).

In the second section, we will devote ourselves to the Borel-Okada theorem. The classical version of this theorem considers domains containing the origin (see [Bo96], [Ok25], [GT76]). Later, it was shown that it also works for open sets both containing the origin (see [Mü92], [GE93]). It will turn out that this finding can be generalized (see 4.2.3).

4.1 Extended versions of the Hadamard multiplication theorem

The first result shows that the Hadamard multiplication theorem is true for open subsets of the extended plane that both contain the origin.

4.1.1 Hadamard multiplication theorem:
Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open sets both containing the origin, $f \in H(\Omega_1)$, and $g \in H(\Omega_2)$. Then we have

$$(f *_{\Omega_1,\Omega_2} g)(z) \;=\; \sum_{\nu=0}^{\infty} f_\nu \, g_\nu \, z^\nu$$

for all $z \in \mathbb{C}$ with $|z| < \operatorname{dist}\big(0, \partial((\Omega_1 * \Omega_2) \cap \mathbb{C})\big)$.

Proof: Let $r, s > 0$ so that $\mathbb{D}_r \subset \Omega_1$ and $\mathbb{D}_s \subset \Omega_2$. Furthermore, let $f_r := f\big|_{\mathbb{D}_r}$ and $g_s := g\big|_{\mathbb{D}_s}$. According to the compatibility theorem, we have

$$(f *_{\Omega_1,\Omega_2} g)\big|_{\mathbb{D}_r * \mathbb{D}_s} \;=\; f_r *_{\mathbb{D}_r,\mathbb{D}_s} g_s.$$

By (3.2) we get

$$(f_r *_{\mathbb{D}_r,\mathbb{D}_s} g_s)(z) \;=\; \sum_{\nu=0}^{\infty} f_\nu \, g_\nu \, z^\nu$$

for all $z \in \mathbb{D}_r * \mathbb{D}_s = \mathbb{D}_{rs}$. The assertion follows by means of the identity theorem. $\square$

We proceed with an example.

4.1.2 Example:
Using the Hadamard multiplication theorem, we are able to express the derivatives $(f *_{\Omega_1,\Omega_2} g)^{(k)}(0)$ by $f^{(k)}(0)$ and $g^{(k)}(0)$ (cf. Proposition 3.6.4):

$$(f *_{\Omega_1,\Omega_2} g)^{(k)}(0) \;=\; \frac{f^{(k)}(0) \cdot g^{(k)}(0)}{k!}$$

for all $k \in \mathbb{N}_0$. $\diamond$

It is also possible to state a Hadamard multiplication theorem for the point at infinity.

4.1.3 Hadamard multiplication theorem at infinity:
Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open sets both containing the point at infinity, $f \in H(\Omega_1)$, and $g \in H(\Omega_2)$. Then we have

$$(f *_{\Omega_1,\Omega_2} g)(z) \;=\; -\sum_{\nu=1}^{\infty} \frac{f_\nu \cdot g_\nu}{z^\nu}$$

for all $z \in U_d(\infty)$ with $d := \inf\big\{t > 0 : U_t(\infty) \subset \Omega_1 * \Omega_2\big\}$.

Proof: Let $R, S > 0$ according to $U_R := U_R(\infty) \subset \Omega_1$ and $U_S := U_S(\infty) \subset \Omega_2$, let $f_R := f\big|_{U_R}$ and $g_S := g\big|_{U_S}$. Then we have $s_n^\infty(g_S) \to g_S$ compactly on U_S as $n \to \infty$. According to the compatibility theorem, we have the relation $(f *_{\Omega_1,\Omega_2} g)\big|_{U_R * U_S} = f_R *_{U_R,U_S} g_S$. By Remark 3.7.5, Proposition 3.7.3, and Example 3.4.7 we obtain

$$\begin{aligned}
(f *_{\Omega_1,\Omega_2} g)(z) \;&=\; (f_R *_{U_R,U_S} g_S)(z) \;=\; \lim_{n\to\infty} \big(f_R *_{U_R,\mathbb{C}_\infty\setminus\{0\}} (s_n^\infty(g_S))\big)(z) \\
&= \lim_{n\to\infty} \left(f_R *_{U_R,\mathbb{C}_\infty\setminus\{0\}} \left(\sum_{\nu=1}^{n} g_\nu\, R_\nu\right)\right)(z) \\
&= \lim_{n\to\infty} \sum_{\nu=1}^{n} g_\nu \cdot (f_R *_{U_R,\mathbb{C}_\infty\setminus\{0\}} R_\nu)(z) \;=\; \lim_{n\to\infty} \sum_{\nu=1}^{n} -\frac{f_\nu \cdot g_\nu}{z^\nu} \\
&= -\sum_{\nu=1}^{\infty} \frac{f_\nu \cdot g_\nu}{z^\nu}
\end{aligned}$$

for all $z \in U_R * U_S = U_{RS}(\infty)$. The assertion follows by means of the identity theorem. □

4.1.4 Example:
Using the Hadamard multiplication theorem at infinity, we are able to express the derivatives $(f *_{\Omega_1,\Omega_2})^{(k)}(\infty)$ by $f^{(k)}(\infty)$ and $g^{(k)}(\infty)$:

$$(f *_{\Omega_1,\Omega_2} g)^{(k)}(\infty) \;=\; -\frac{f^{(k)}(\infty) \cdot g^{(k)}(\infty)}{k!}$$

for all $k \in \mathbb{N}_0$. ◇

The negative sign in the Hadamard multiplication theorem at infinity seems a little odd if we compare this result with the Hadamard multiplication theorem. Recall that Θ is a neutral element for the Hadamard product (in the sense of Proposition 3.6.2); its power series expansion around the point at infinity is given by

$$\Theta(z) \;=\; \sum_{\nu=1}^{\infty} \frac{-1}{z^{\nu}} \qquad \big(z \in U_1(\infty)\big).$$

If $\Omega \subset \mathbb{C}_\infty$ is an open set containing the point at infinity, and if $f \in H(\Omega)$, then, according to the Hadamard multiplication theorem at infinity, we get

$$\left(f *_{\Omega,\mathbb{C}_\infty\setminus\{1\}} \Theta\right)(z) \;=\; -\sum_{\nu=1}^{\infty} \frac{f_\nu \cdot (-1)}{z^{\nu}} \;=\; f(z) \qquad \big(z \in U_R(\infty)\big)$$

for a suitable $R > 0$. Hence, the negative sign in the Hadamard multiplication theorem at infinity is quite natural.

As an application of the Hadamard multiplication theorem we can prove the following result that shows how the operators σ_n^A act upon the Hadamard product.

4.1.5 Proposition:
Let A be an infinite matrix, $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ open sets both containing the origin, $f \in H(\Omega_1)$, and $g \in H(\Omega_2)$. Then we have

$$\sigma_n^A\left(f *_{\Omega_1,\Omega_2} g\right) \;=\; \big(\sigma_n^A(f)\big) *_{\mathbb{C},\Omega_2} g \;=\; f *_{\Omega_1,\mathbb{C}} \big(\sigma_n^A(g)\big)$$

for all $n \in \mathbb{N}_0$.

Proof: By Proposition 3.6.4 and the definition of the A-transformation all the functions appearing in the conclusion are entire. According to the Hadamard multiplication theorem, there exists an $r > 0$ in such a way that we have

$$(f *_{\Omega_1,\Omega_2} g)(z) = \sum_{\nu=0}^{\infty} f_\nu\, g_\nu\, z^{\nu} \qquad \big(z \in U_r(0)\big),$$

whence follows

$$\sigma_n^A\left(f *_{\Omega_1,\Omega_2} g\right)(z) = \sum_{\nu=0}^{\infty} a_{n\nu} f_\nu\, g_\nu\, z^{\nu} \qquad (z \in \mathbb{C}).$$

Applying the Hadamard multiplication theorem again, we obtain

$$\big(\big(\sigma_n^A(f)\big) *_{\mathbb{C},\Omega_2} g\big)(z) \;=\; \sum_{\nu=0}^{\infty} a_{n\nu}\, f_\nu\, g_\nu\, z^\nu \qquad \big(z \in \mathbb{C}\big)$$

and

$$\big(f *_{\Omega_1,\mathbb{C}} \big(\sigma_n^A(g)\big)\,\big)(z) \;=\; \sum_{\nu=0}^{\infty} f_\nu\, a_{n\nu}\, g_\nu\, z^\nu \qquad \big(z \in \mathbb{C}\big).$$

This proves the proposition. □

4.2 The Borel-Okada theorem

The Borel-Okada theorem allows the analytic continuation of a given function by means of summability methods. It is possible to derive the Borel-Okada theorem from the Hadamard multiplication theorem; the decisive point is the continuity theorem. For the plane case this was already done in [Mü92] and [GE93]. As in the case of the Hadamard multiplication theorem, all involved sets had to contain the origin. The question now is whether a corresponding result also holds in a more general context. The answer is yes.

Before stating the result, we will recall the notion of a summability method. Let A be an infinite matrix, $\Omega \subset \mathbb{C}_\infty$ an open set containing the origin, $f \in H(\Omega)$, and $\Omega' \subset \Omega \cap \mathbb{C}$ a non-empty and open set. We say that f is compactly A-summable on Ω' if $\sigma_n^A(f) \to f$ compactly on Ω' as $n \to \infty$.

4.2.1 Borel-Okada theorem:
Let $\Omega, G \subset \mathbb{C}$ be open sets with $0 \in \Omega \cap G$ and $1 \notin G$. Furthermore, let $f \in H(\Omega)$ and A an infinite matrix. If Θ is compactly A-summable on G, then f is compactly A-summable on $\Omega * G$. ◇

Proofs can be found in [Mü92] or [GE93]. Moreover, the Borel-Okada theorem can be deduced from from the extended Borel-Okada theorem (see 4.2.3).

4.2.2 Remark:
The Borel-Okada theorem states that if the "test function" Θ is compactly A-summable on G, then f is compactly A-summable on $\Omega * G$. (Notice that $\Omega * G \subset \Omega$ since $1 \notin G$.) This can be exploited to continue f analytically across Ω: If f is defined by a power series with radius of convergence $r \in (0, \infty)$, and

if A can be chosen in such a way that there is a subsequence $(\sigma_{n_k}^A(f))_{k\in\mathbb{N}_0}$ that converges compactly on a domain $\tilde{\Omega} \supset U_r(0)$, then the limit is the analytic continuation of f into $\tilde{\Omega}$. $\diamond$

In the above version of the Borel-Okada theorem, the set G has to contain the origin. The next theorem shows that this condition can be dropped.

4.2.3 Extended Borel-Okada theorem:
Let $\Omega \subset \mathbb{C}_\infty$ and $G \subset \mathbb{C}$ be open and star-eligible sets with $1 \notin G$. Furthermore, let $f \in H(\Omega)$ and A an infinite matrix. If Θ is compactly A-summable on G, then f is compactly A-summable on $\Omega * G$.

Proof: According to Proposition 4.1.5, the operators σ_n satisfy the condition of the second part of the continuity theorem. Thus, by the continuity theorem we get $\sigma_n(f *_{\Omega,\mathbb{C}_\infty\setminus\{1\}} \Theta) \to f *_{\Omega,G} \left(\Theta\big|_G\right)$ compactly on $\Omega * G$. Moreover, $f *_{\Omega,\mathbb{C}_\infty\setminus\{1\}} \Theta = f$ and $f *_{\Omega,G} \left(\Theta\big|_G\right) = f\big|_{\Omega * G}$ by Proposition 3.6.2 and Corollary 3.6.3. This proves the theorem. $\square$

The (extended) Borel-Okada theorem states that—under the conditions given there—compact A-summability of Θ on G implies compact A-summability of f on $\Omega * G$. Thus, the set on which f is compactly A-summable depends not only on G but also on Ω. Sometimes, we want to sum a function f compactly (with respect to an infinite matrix A) on a given set G. If we want to apply the (extended) Borel-Okada theorem, we have to find a set $\tilde{G}$ with $\Omega * \tilde{G} = G$. Then we obtain that f is compactly A-summable on $\Omega * \tilde{G} = G$. This takes us back to Problem 1.3.18. Our question is the following.

4.2.4 Problem:
Let $\Omega \subset \mathbb{C}_\infty$ and $G \subset \Omega \cap \mathbb{C}$ be non-empty open sets, and let $f \in H(\Omega)$. Can we find an infinite matrix A in such a way that f is compactly A-summable on the set G? $\diamond$

Chapter 5

Universal power series

In the first section, we will recapitulate the notion of universality. Furthermore, we will state the universality criterion, that gives a sufficient condition for the existence of universal elements.

In the second section, we will introduce universal power series as a specification of universal elements introduced in the first section.

In the third section, we will recapitulate Runge's and Mergelyan's approximation theorems in the language introduced in section 1.1.

In the fourth section, we will prove properties of the set of universal power series. The main result is that they form a dense G_δ-set (see Theorem 5.4.7; see also [MNP97, Theorem 3.2] and [CV06, Theorem 1.2]).

5.1 The universality criterion

Let X be a non-empty set, $(Y, \mathscr{T})$ a topological space, I a non-empty index set, and $(T_\iota)_{\iota \in I}$ a family of operators from X into Y. We shall now introduce the notion of universal elements and universal families (see [GE99]).

5.1.1 Definition:
An element $x \in X$ is called *universal for* $(T_\iota)_{\iota \in I}$ if the set $\{T_\iota(x) : \iota \in I\}$ is dense in Y. The family $(T_\iota)_{\iota \in I}$ of operators is called *universal* if there exists a universal element for this family. $\diamond$

Instantaneously, the question arises if there exist universal elements for a given family of operators. The universality criterion (see also [GE99, Theorem 1]) not only guarantees the existence but also a plethora[†] of universal elements—provided that the topological spaces and the operators satisfy certain conditions.

5.1.2 Universality criterion:
Let $(X, \mathscr{S})$ be a Baire space, $(Y, \mathscr{T})$ second-countable, and T_ι continuous for each $\iota \in I$. Denote by $\mathscr{U}$ the set of universal elements for $(T_\iota)_{\iota \in I}$. Then the following assertions are equivalent:

(i) *$\mathscr{U}$ is residual in $(X, \mathscr{S})$.*

(ii) *$\mathscr{U}$ is dense in $(X, \mathscr{S})$.*

(iii) *If $U \subset X$ and $V \subset Y$ are non-empty open sets, then there exists an index $\kappa \in I$ so that $T_\kappa(U) \cap V \neq \emptyset$.*

If one of these conditions holds, then $\mathscr{U}$ is a dense G_δ-subset of X. $\diamond$

A proof of a slightly more general version of this theorem can be found in [GE87, Satz 1.1.7].

5.1.3 Remark:
In a topological space a residual set is a set whose complement is of the first category. Sometimes residuality is referred to sets that contain a dense G_δ-set. However, in Baire spaces these two definitions are equivalent. $\diamond$

In order to apply the universality criterion, we have to specify the spaces $(X, \mathscr{S})$ and $(Y, \mathscr{T})$ as well as the family of the operators T_ι (see section 6.3). Useful will be the following result: If $\Omega \subset \mathbb{C}$ is a non-empty open set, then $H(\Omega)$ is a second-countable Baire space.

5.2 O-universality

The concept of universal power series uses the partial sum operators (see section 1.2). A power series converges compactly on its disk of convergence; it is divergent in each point in the complement of this disk's closure. But it might

[†] In the sense of Baire's categories.

happen that a subsequence converges compactly on a domain that contains the disk of convergence. This phenomenon is called overconvergence.

The type of universality we study is called O-universality or universality with respect to overconvergence.

In order to have a handy language, we will introduce two systems of sets in the complex plane whose elements frequently occur in combination with the approximation theorem of Mergelyan and universal functions. Let $A \subset \mathbb{C}$ be a set. By $\mathcal{M}(A)$ we denote the set consisting of all non-empty compact sets $K \subset \mathbb{C} \setminus A$ so that $\mathbb{C} \setminus K$ is connected. Furthermore, $\mathcal{M}(\emptyset)$ is denoted by $\mathcal{M}$. By $\mathcal{G}(A)$ we denote the set of all non-empty, open simply connected subsets of $\mathbb{C} \setminus A$. Furthermore, $\mathcal{G}(\emptyset)$ is denoted by $\mathcal{G}$.

5.2.1 Definition:
For an open simply connected set $D \subsetneq \mathbb{C}$ containing the origin, we define the following sets of universal functions:

- For $G \in \mathcal{G}(D)$ we denote by $\mathscr{U}(D,G)$ the set of all $\varphi \in H(D)$ so that for each $f \in H(G)$ there exists a subsequence of $(s_n(\varphi))_{n\in\mathbb{N}_0}$ that converges to f compactly on G.

- By $\mathscr{U}(D)$ we denote the set of all $\varphi \in H(D)$ so that for each $G \in \mathcal{G}(D)$ and each $f \in H(G)$ there exists a subsequence of $(s_n(\varphi))_{n\in\mathbb{N}_0}$ that converges to f compactly on G.

- For $K \in \mathcal{M}(\overline{D})$ we denote by $\tilde{\mathscr{U}}(D,K)$ the set of all $\varphi \in H(D)$ so that for each $f \in A(K)$ there exists a subsequence of $(s_n(\varphi))_{n\in\mathbb{N}_0}$ that converges to f uniformly on K.

- By $\tilde{\mathscr{U}}(D)$ we denote the set of all $\varphi \in H(D)$ so that for each $K \in \mathcal{M}(\overline{D})$ and each $f \in A(K)$ there exists a subsequence of $(s_n(\varphi))_{n\in\mathbb{N}_0}$ that converges to f uniformly on K.

Furthermore, we define
$$\mathscr{U}(D,\emptyset) := H(D)$$
and
$$\tilde{\mathscr{U}}(D,\emptyset) := H(D).$$

An element of the set $\mathscr{U}(D,G)$ is called an *O-universal function with respect to G* or just a universal function with respect to G. An element of the set $\mathscr{U}(D)$

is called an *O-universal function* or just a universal function. The power series expansion around the origin of an element of $\mathscr{U}(D)$ is called a *universal power series*. $\diamond$

5.2.2 Remark:
The above sets of universal functions can be written as

$$\mathscr{U}(D,G) = \left\{ \varphi \in H(D) : \overline{\{ s_n(\varphi)\big|_G : n \in \mathbb{N}_0 \}}^{H(G)} = H(G) \right\},$$

$$\tilde{\mathscr{U}}(D,K) = \left\{ \varphi \in H(D) : \overline{\{ s_n(\varphi)\big|_K : n \in \mathbb{N}_0 \}}^{A(K)} = A(K) \right\},$$

$$\mathscr{U}(D) = \left\{ \varphi \in H(D) : \overline{\{ s_n(\varphi)\big|_G : n \in \mathbb{N}_0 \}}^{H(G)} = H(D) \text{ for all } G \in \mathcal{G}(D) \right\},$$

$$\tilde{\mathscr{U}}(D) = \left\{ \varphi \in H(D) : \overline{\{ s_n(\varphi)\big|_K : n \in \mathbb{N}_0 \}}^{A(K)} = A(K) \text{ for all } K \in \mathcal{M}(\overline{D}) \right\}.$$

Moreover, we get

$$\mathscr{U}(D) = \bigcap_{G \in \mathcal{G}(D)} \mathscr{U}(D,G) \tag{5.1}$$

and

$$\tilde{\mathscr{U}}(D) = \bigcap_{K \in \mathcal{M}(\overline{D})} \tilde{\mathscr{U}}(D,K) \tag{5.2}$$

which will be used later on. $\diamond$

For the existence of universal functions and universal power series see for instance [Luh70], [CP71], [Luh86], or [GLM00].

5.2.3 Remark:
Let $D \subsetneq \mathbb{C}$ be an open simply connected set containing the origin, and let $G \in \mathcal{G}(D)$. The condition that G is simply connected is necessary for the existence of universal functions with respect to G: If G was not simply connected, then $H(\mathbb{C})$ would not be dense in $H(G)$. Thus, the set $\mathscr{U}(D,G)$ is empty. $\diamond$

5.3 Approximation by polynomials

Two of the most important results concerning the approximation of holomorphic functions by polynomials are Runge's and Mergelyan's approximation theorems (see for instance [Rud2]). In the terminology used so far, Runge's theorem reads as follows:

$$\textit{If } \Omega \in \mathcal{G}, \textit{ then } \overline{\mathscr{P}_{|\Omega}}^{H(\Omega)} = H(\Omega).$$

Mergelyan's theorem can be stated as follows:

$$\textit{If } K \in \mathcal{M}, \textit{ then } \overline{\mathscr{P}_{|K}}^{A(K)} = A(K).$$

We remark that the connectivity assumption is necessary in both theorems.

5.4 Properties of universal functions

Sometimes it is useful to know in which way universality is "inherited" to the results of certain algebraic operations. If we are given two universal functions, we can ask whether the sum, the product, or other algebraic operations performed on these functions will produce a new universal function or not. For the sum of two universal functions the answer is negative: If φ is in $\mathscr{U}(\mathbb{D})$, then $-\varphi$ also belongs to $\mathscr{U}(\mathbb{D})$ (see Lemma 5.4.1). But their sum is not universal at all. As a consequence, the set of universal functions is not a linear space.

The situation is totally different when one (and only one) of the two functions is entire. As the next lemma shows, universality is invariant under addition of entire functions and under scalar multiplication with scalars different from zero.

5.4.1 Lemma:
Let $D \subsetneq \mathbb{C}$ be an open simply connected set containing the origin, $\varphi \in \mathscr{U}(D)$, $g \in H(\mathbb{C})$, and $\lambda \in \mathbb{C} \setminus \{0\}$. Then $g\big|_D + \varphi$ and $\lambda \cdot \varphi$ also belong to $\mathscr{U}(D)$.

Proof: Let $G \in \mathcal{G}(D)$ and $f \in H(G)$. The functions $g\big|_D + \varphi$ and $\lambda \cdot \varphi$ are holomorphic in D.

1. There exists a strictly monotonic increasing sequence $(n_k)_{k \in \mathbb{N}_0}$ in $\mathbb{N}_0$ so that $s_{n_k}(\varphi) \to f - g$ compactly on G as $k \to \infty$. Since the partial sum

operators are linear and since $s_{n_k}(g) \to g$ compactly on $\mathbb{C}$ as $k \to \infty$, we obtain $s_{n_k}\left(g\big|_D + \varphi\right) \to g + (f - g) = f$ compactly on G as $k \to \infty$

2. Furthermore, there exists a strictly monotonic increasing sequence $(n_\ell)_{\ell \in \mathbb{N}_0}$ in $\mathbb{N}_0$ so that $s_{n_\ell}(\varphi) \to \lambda^{-1} \cdot f$ compactly on G as $\ell \to \infty$. Once more, the linearity yields the desired result. $\square$

The rest of this section is devoted to the relations between the different kinds of universal sets. We shall show that the sets $\tilde{\mathscr{U}}(D)$ and $\mathscr{U}(D)$ are equal for open simply connected sets D with the property that $\mathbb{C}_\infty \setminus \overline{D}^\infty$ is a simply connected domain (see Proposition 5.4.5). Moreover, we shall show that the universal functions form a dense G_δ-subset of the set of holomorphic functions (see Theorem 5.4.7). We commence with a result on the exhaustion by simply connected domains.

5.4.2 Lemma:
Let $D \subsetneq \mathbb{C}$ be a non-empty open set so that $\mathbb{C}_\infty \setminus \overline{D}^\infty$ is a simply connected domain. Then there exists a family $(G_n)_{n \in \mathbb{N}}$ of simply connected domains in $\mathbb{C} \setminus D$ with the following property: For each $K \in \mathcal{M}(\overline{D})$ there exists an $N \in \mathbb{N}$ so that $K \subset G_N$.

Proof: 1. The set $\mathcal{A}$ of all polygonal Jordan arcs in $\{1\} \cup \overline{\mathbb{D}}^C$ that connect 1 and the point at infinity and whose vertices have rational real and imaginary parts is countable. For every $\mathit{\Gamma} \in \mathcal{A}$, the set $G'_{\mathit{\Gamma}} := \mathbb{C} \setminus \left(\overline{\mathbb{D}} \cup \mathit{\Gamma}\right)$ is a simply connected domain in $\mathbb{C} \setminus \mathbb{D}$. Let $(G'_n)_{n \in \mathbb{N}}$ be an enumeration of the countable set $\{G'_{\mathit{\Gamma}} : \mathit{\Gamma} \in \mathcal{A}\}$. If $K \in \mathcal{M}(\overline{\mathbb{D}})$, then there exists an arc $\mathit{\Gamma} \in \mathcal{A}$ so that $\mathit{\Gamma} \cap K = \emptyset$. Hence, we have $K \subset G'_{\mathit{\Gamma}}$.

2. Assume that D is bounded. Since $\mathbb{C}_\infty \setminus \overline{D}$ is a simply connected domain, there exists, according to the Riemann mapping theorem, a conformal map $F : \mathbb{C}_\infty \setminus \overline{\mathbb{D}} \to \mathbb{C}_\infty \setminus \overline{D}$ with $F(\infty) = \infty$. For $n \in \mathbb{N}$, we set $G_n := F(G'_n)$ with the sets G'_n from part one of the proof. Then G_n is a simply connected domain in $\mathbb{C} \setminus D$ for all $n \in \mathbb{N}$. Now let K be in $\mathcal{M}(\overline{D})$. Then the set $F^{-1}(K)$ belongs to $\mathcal{M}(\overline{\mathbb{D}})$. By part one of the proof there exists an $N \in \mathbb{N}$ so that $F^{-1}(K) \subset G'_N$, whence we get $K \subset G_N$.

3. Assume now that D is unbounded. According to the Riemann mapping theorem, there exists a conformal map $F : \mathbb{D} \to \mathbb{C}_\infty \setminus \overline{D}^\infty$. For $n \in \mathbb{N}$ we set $G''_n := U_{1-\frac{1}{n+1}}(0)$ and $G_n := F(G''_n)$. Then G_n is a simply connected domain

in $\mathbb{C} \setminus D$ for all $n \in \mathbb{N}$. If $K \in \mathcal{M}(\overline{D})$, then $F^{-1}(K)$ is a compact subset of $\mathbb{D}$. Hence, there exists an $N \in \mathbb{N}$ so that $F^{-1}(K) \subset G''_N$. From this, we obtain $K \subset G_N$. This completes the proof of the lemma. □

5.4.3 Remark:
If D is unbounded, the proof of Lemma 5.4.2 shows that the domains can be chosen in such a way that $(G_n)_{n\in\mathbb{N}}$ is isotonic. This is in general not possible if D is bounded. ◇

The next lemma provides an auxiliary result to prove the above mentioned equality of the universal sets.

5.4.4 Lemma:
Let $D \subsetneq \mathbb{C}$ be an open simply connected set containing the origin and so that $\mathbb{C}_\infty \setminus \overline{D}^\infty$ is a simply connected domain. Furthermore, let $G \in \mathcal{G}(D)$, $K \in \mathcal{M}(\overline{D})$ with $K \subset G$, and $(G_n)_{n\in\mathbb{N}}$ as in Lemma 5.4.2. Then the following assertions hold:

1. $\mathscr{U}(D, G) \subset \tilde{\mathscr{U}}(D, K)$.

2. $\tilde{\mathscr{U}}(D) \subset \mathscr{U}(D, G)$.

3. $\tilde{\mathscr{U}}(D) = \bigcap_{n\in\mathbb{N}} \mathscr{U}(D, G_n)$.

Proof: *ad 1.:* Let $\varphi \in \mathscr{U}(D, G)$, $f \in A(K)$, and $\varepsilon > 0$ be given. By Mergelyan's theorem there exists a polynomial P so that

$$\max_{z\in K} |P(z) - f(z)| < \frac{\varepsilon}{2}.$$

Since $P|_G$ is holomorphic and $\varphi \in \mathscr{U}(D, G)$, there is an $N \in \mathbb{N}$ with

$$\max_{z\in K} |s_N(\varphi)(z) - P(z)| < \frac{\varepsilon}{2}.$$

The triangle inequality now yields

$$\max_{z\in K} |s_N(\varphi)(z) - f(z)| < \varepsilon.$$

Hence, φ belongs to $\tilde{\mathscr{U}}(D, K)$.

ad 2.: Let $\varphi \in \tilde{\mathscr{U}}(D)$, $(K_n(G))_{n\in\mathbb{N}}$ the exhaustion of G according to (1.1), and $f \in H(G)$. Then there exists an $n_1 \in \mathbb{N}_0$ so that

$$\max_{z\in K_1(G)} |s_{n_1}(\varphi)(z) - f(z)| < 1 .$$

Now let $n_j \in \mathbb{N}_0$ $(1 \leq j \leq k-1)$ be already defined for a $k \in \mathbb{N}_0$, $k \geq 2$. We can find an $n_k \in \mathbb{N}_0$ so that $n_k > n_{k-1}$ and

$$\max_{z\in K_k(G)} |s_{n_k}(\varphi)(z) - f(z)| < \frac{1}{k} . \tag{5.3}$$

Inductively, we get a strictly monotonic increasing sequence of non-negative integers. For every compact set $K \subset G$, there exists an $N \in \mathbb{N}$ so that

$$K \subset K_k(G) \quad (k \geq N). \tag{5.4}$$

The relations (5.3) and (5.4) yield

$$\max_{z\in K} |s_{n_k}(\varphi)(z) - f(z)| \leq \max_{z\in K_k(G)} |s_{n_k}(\varphi)(z) - f(z)| < \frac{1}{k}$$

for all $k \geq N$. This implies the assertion.

ad 3.: For every $K \in \mathcal{M}(\overline{D})$ there exists a positive integer $N = N(K)$ so that $\mathscr{U}(D, G_N) \subset \tilde{\mathscr{U}}(D, K)$ (part 1). Since $\bigcap\limits_{n\in\mathbb{N}} \mathscr{U}(D, G_n) \subset \mathscr{U}(D, G_N)$ for all $N \in \mathbb{N}$, we get

$$\bigcap_{n\in\mathbb{N}} \mathscr{U}(D, G_n) \subset \bigcap_{K\in\mathcal{M}(\overline{D})} \tilde{\mathscr{U}}(D, K) \overset{(5.2)}{=} \tilde{\mathscr{U}}(D) .$$

The reverse inclusion follows from (5.1) and the second part. □

Altogether, we get the following result.

5.4.5 Proposition:
Let $D \subsetneq \mathbb{C}$ be an open simply connected set containing the origin and so that $\mathbb{C}_\infty \setminus \overline{D}^\infty$ is a simply connected domain. Then we have

$$\tilde{\mathscr{U}}(D) = \mathscr{U}(D) .$$

Proof: Let $(G_n)_{n\in\mathbb{N}}$ according to Lemma 5.4.2. Applying Lemma 5.4.4.3, we get

$$\tilde{\mathscr{U}}(D) \;=\; \bigcap_{n\in\mathbb{N}} \mathscr{U}(D,G_n) \;\supset\; \bigcap_{G\in\mathcal{G}(D)} \mathscr{U}(D,G) \;\overset{(5.1)}{=}\; \mathscr{U}(D)\,.$$

According to Lemma 5.4.4.2, we have $\tilde{\mathscr{U}}(D) \subset \mathscr{U}(D,G)$ for every $G\in\mathcal{G}(D)$. We get

$$\tilde{\mathscr{U}}(D) \;\subset\; \bigcap_{G\in\mathcal{G}(D)} \mathscr{U}(D,G) \;\overset{(5.1)}{=}\; \mathscr{U}(D)\,.$$

This completes the proof. □

As a last step before showing that the universal functions are dense, we prove another lemma.

5.4.6 Lemma:
Let $D\subset\mathbb{C}$ be an open simply connected set containing the origin, and let $G\subset\mathbb{C}\setminus D$ be an open simply connected set. Then $\mathscr{U}(D,G)$ is a dense G_δ-set in $H(D)$.

Proof: If $G=\emptyset$, then we have $\mathscr{U}(D,G)=H(D)$. Now, let $G\neq\emptyset$. Since $H(D)$ is a Baire space and $H(G)$ is second-countable, it suffices, according to the universality criterion, to show that for given non-empty open sets $U\subset H(D)$ and $V\subset H(G)$ there exists an $N\in\mathbb{N}_0$ so that $s_N(U)\cap V\neq\emptyset$. Without loss of generality, we can assume that U and V are of the form $U=g+V_j$, $V=h+W_j$ with $g\in H(D), h\in H(G)$, and

$$V_j \;=\; \left\{f\in H(D) : \|f\|_{K_j(D)} < \frac{1}{j}\right\},\;\; W_j \;=\; \left\{f\in H(G) : \|f\|_{K_j(G)} < \frac{1}{j}\right\}.$$

Since $K_j(D)\cup K_j(G)$ is a compactum with connected complement, Runge's approximation theorem guarantees the existence of a polynomial P so that

$$\max_{z\in K_j(D)} |P(z)-g(z)| \;<\; \frac{1}{j} \quad\text{and}\quad \max_{z\in K_j(G)} |P(z)-h(z)| \;<\; \frac{1}{j}\,.$$

For $N>\deg(P)$ we obtain $s_N(P)=P$. Therefore, we get $P\big|_D\in g+V_j$ and $P\big|_G\in h+W_j$. This shows that $P\big|_G = s_N(P\big|_D)\big|_G\in s_N(U)\cap V$. □

Combining Lemma 5.4.4, Proposition 5.4.5, and Lemma 5.4.6, we gain the following result.

5.4.7 Theorem:
Let $D \subset \mathbb{C}$ be an open simply connected set containing the origin and so that $\mathbb{C}_\infty \setminus \overline{D}^\infty$ is a simply connected domain. Then $\tilde{\mathscr{U}}(D)$ and $\mathscr{U}(D)$ are dense G_δ-sets in $H(D)$.

Proof: Let $(G_n)_{n\in\mathbb{N}}$ be as in Lemma 5.4.2. By Lemma 5.4.6 $\mathscr{U}(D, G_n)$ is a dense G_δ-set in $H(D)$ for all $n \in \mathbb{N}$. Since $H(D)$ is a Baire space, the intersection of all these universal sets must be a dense G_δ-set in $H(D)$, too. According to Proposition 5.4.4, $\tilde{\mathscr{U}}(D)$ equals this intersection. Consequently, $\tilde{\mathscr{U}}(D)$ is a dense G_δ-set in $H(D)$. The other claim follows with the help of Proposition 5.4.5. $\square$

Chapter 6

Derived universality

What is derived universality? Let be given a universal element and a mapping. If the image of the universal element under this mapping is also universal, we speak about derived universality. Our main interest focuses on Hadamard multiplication. Is the Hadamard product of a universal function with a given function universal again? To answer this question, it must be specified which functions are apt to be multiplied with the universal function, on which set the product is holomorphic, and with respect to which set it is universal (if universal at all).

In the first section, we will specify the notion of derived universality. Furthermore, we will give a sufficient condition for derived universality (see Proposition 6.1.2) and a partial converse (see Proposition 6.1.4).

In the second section, we will link universality with lacunary polynomials.

In the third section, we will study universal Hadamard products. We will present a theorem that links derived universality for the Hadamard product with the density of the range of the corresponding Hadamard operator (see 6.3.5), and we will present a theorem that gives a necessary and sufficient condition for the Hadamard operator having dense range (see Theorem 6.3.7). Moreover, we will prove a theorem that provides a necessary and sufficient condition for the universality of the Hadamard product (see Theorem 6.3.10).

In the fourth section, we will use the condition of Theorem 6.3.7 to present functions that serve as universality preserving factors (see Theorem 6.4.3).

In the fifth section, we will give some examples.

In the whole chapter, let $D \subsetneq \mathbb{C}$ be an open simply connected set containing the origin, $\Omega \subsetneq \mathbb{C}_\infty$ a non-empty open set, and $\psi \in H(\Omega)$.

6.1 The general setting

In this section, we adopt the following notation: Let X_1 and X_2 be non-empty sets; $(Y_1, \mathscr{T}_1)$ and $(Y_2, \mathscr{T}_2)$ topological spaces; $\Phi : X_1 \to X_2$ and $\Psi : Y_1 \to Y_2$ mappings; I a non-empty index set; $T_\iota^{(1)} : X_1 \to Y_1$ and $T_\iota^{(2)} : X_2 \to Y_2$ mappings for each $\iota \in I$. For each $\iota \in I$, these objects are supposed to be linked by the following diagram.

$$\begin{tikzcd} X_1 \arrow[r, "T_\iota^{(1)}"] \arrow[d, "\Phi"'] & Y_1 \arrow[d, "\Psi"] \\ X_2 \arrow[r, "T_\iota^{(2)}"] & Y_2 \end{tikzcd}$$

Diagram 6.1: The general setting

We are interested in the following question.

6.1.1 Problem:
Let $x \in X_1$ be a universal element for $(T_\iota^{(1)})_{\iota \in I}$. Under what conditions is $\Phi(x)$ a universal element for $(T_\iota^{(2)})_{\iota \in I}$? $\diamond$

The next proposition gives a sufficient condition.

6.1.2 Proposition:
Assume that the following conditions hold:

(i) $T_\iota^{(2)} \circ \Phi = \Psi \circ T_\iota^{(1)}$ *for every* $\iota \in I$*;*

(ii) Ψ *is continuous;*

(iii) Ψ *has dense range.*

Then the following assertion holds: If $x \in X_1$ *is universal for* $(T_\iota^{(1)})_{\iota \in I}$*, then* $\Phi(x)$ *is universal for* $(T_\iota^{(2)})_{\iota \in I}$*.*

Proof: Let $\Omega \subset Y_2$ be a non-empty open set. Since Ψ has dense range and is continuous, the set $U := \Psi^{-1}(\Omega)$ is a non-empty open subset of Y_1. By the universality of x the set $\{T_\iota^{(1)}(x) : \iota \in I\} \cap U$ is not the empty set. Let y be an element of this intersection. Then there exists an index $\kappa \in I$ so that $T_\kappa^{(1)}(x) = y$. Using condition (i), we get

$$\begin{aligned} \Psi(y) &\in \Psi\left(\{T_\iota^{(1)}(x) : \iota \in I\} \cap U\right) \subset \Psi(U) \cap \Psi\left(\{T_\iota^{(1)}(x) : \iota \in I\}\right) \\ &\subset \Omega \cap \left\{T_\iota^{(2)}(\Phi(x)) : \iota \in I\right\}. \end{aligned}$$

But this means that $\left\{T_\iota^{(2)}(\Phi(x)) : \iota \in I\right\}$ is dense in Y_2, and hence $\Phi(x)$ is universal for $(T_\iota^{(2)})_{\iota \in I}$. □

6.1.3 Remark:
The first condition in Proposition 6.1.2 states that for every $\iota \in I$ the corresponding diagram commutes. ◇

Proposition 6.1.2 admits a partial converse concerning the density of the range of Ψ.

6.1.4 Proposition:
Assume that the first condition of Proposition 6.1.2 is satisfied. If there exists an $x \in X_1$ so that $\Phi(x)$ is universal for $(T_\iota^{(2)})_{\iota \in I}$, then Ψ has dense range.

Proof: Let $\Omega \subset Y_2$ be a non-empty open set. By the universality of $\Phi(x)$ there exists an index $\kappa \in I$ with $T_\kappa^{(2)}(\Phi(x)) \in \Omega$. Since all diagrams commute, we have $T_\kappa^{(2)}(\Phi(x)) = \Psi(T_\kappa^{(1)}(x))$. This implies $T_\kappa^{(2)}(\Phi(x)) \in \Omega \cap \Psi(Y_1)$. Thus, the mapping Ψ has dense range. □

6.2 Universality and lacunary polynomials

In this section, we will provide an auxiliary result concerning the connection between lacunary polynomials and universal functions.

Let M be a non-empty subset of the complex plane. By $\mathscr{P}$ we denote the set of all complex polynomials; and by $\mathscr{P}_{|M}$ the set consisting of the restrictions of the functions in $\mathscr{P}$ onto the set M.

For a non-empty set $\Lambda \subset \mathbb{N}_0$ we denote by $\mathscr{P}^\Lambda$ the set of all complex polynomials that are linear combinations of monomials whose exponents belong to Λ. Moreover, we denote by $\mathscr{P}^\Lambda_{|M}$ the set consisting of the restrictions of the functions in $\mathscr{P}^\Lambda$ onto the set M.

Let f be holomorphic in a non-empty open set, and let a be an element of this set. We define

$$\Lambda_a(f) \;:=\; \Big\{\nu \in \mathbb{N}_0 : f^{(\nu)}(a) \neq 0\Big\}.$$

If $a = 0$, we simply write $\Lambda(f)$. The elements of $\mathbb{N}_0 \setminus \Lambda_a(f)$ are called the gaps of f (with respect to a).

6.2.1 Remark:
Under the assumptions of the Hadamard multiplication theorem, we get the relation $\Lambda(f *_{\Omega_1,\Omega_2} g) = \Lambda(f) \cap \Lambda(g)$. $\diamond$

6.2.2 Proposition:
Let $G \in \mathcal{G}(D)$. If $\varphi \in \mathscr{U}(D, G)$, then $\mathscr{P}^{\Lambda(\varphi)}_{|G}$ is dense in $H(G)$.

Proof: Since φ is universal with respect to G, the set $\{s_n(\varphi)\big|_G : n \in \mathbb{N}_0\}$ is dense in $H(G)$. Since

$$\Big\{s_n(\varphi)\big|_G : n \in \mathbb{N}_0\Big\} \;\subset\; \mathscr{P}^{\Lambda(\varphi)}_{|G},$$

the proposition is proved. $\square$

6.2.3 Remark:
The converse of Proposition 6.2.2 is false in general: Let $\varphi := \Theta\big|_{\mathbb{D}}$ and $G \in \mathscr{U}(\mathbb{D}, G)$. Then $\mathscr{P}^{\Lambda(\varphi)}_{|G}$ is dense in $H(G)$ according to Runge's theorem. But φ is not universal with respect to G. $\diamond$

6.3 Derived universality with respect to simply connected sets

In this section, we study the question whether universality is preserved under Hadamard multiplication. This cannot be true for all functions ψ (the zero function, for instance, serves as a counterexample). Therefore, derived universality must be linked to a certain condition on ψ.

At first, we will specify the spaces and mappings of the diagram in Diagram 6.1. For a non-empty, plane open set M we define the restriction map $r_M : H(\mathbb{C}) \to H(M)$ by $r_M(f) := f\big|_M$.

6.3.1 Remark:
If $G \in \mathcal{G}(D)$ in such a way that Ω and G are star-eligible, then Ω and D are star-eligible, too. $\diamond$

The specification of the diagram in Diagram 6.1 is the following one.

$$\begin{tikzcd} H(D) \arrow[r, "r_G \circ s_n"] \arrow[d, "\psi *_{\Omega,D} \bullet"'] & H(G) \arrow[d, "\psi *_{\Omega,G} \bullet"] \\ H(\Omega * D) \arrow[r, "r_{\Omega * G} \circ s_n"] & H(\Omega * G) \end{tikzcd}$$

Diagram 6.2: The specified setting

The Hadamard product $\psi *_{\Omega,D} \varphi$ is defined on $\Omega * D$. If it is universal, we expect the universality with respect to $\Omega * G$. Therefore, $\Omega * G$ must be a subset of $\mathbb{C} \setminus (\Omega * D)$; this is indeed the case. Moreover, we would like to notice that, according to Lemma 1.3.15, the sets $\Omega * D$ and $\Omega * G$ are simply connected.

The specification of Problem 6.1.1 is the following question.

6.3.2 Problem:
Let $\varphi \in \mathscr{U}(D, G)$. Under what conditions is $\psi *_{\Omega,D} \varphi \in \mathscr{U}(\Omega * D, \Omega * G)$? $\diamond$

What can we say about the conditions in Proposition 6.1.2 so far?

6.3.3 Lemma:
Let $G \in \mathcal{G}(D)$ so that Ω and G are star-eligible. Then the diagram in Diagram 6.2 commutes for every $n \in \mathbb{N}_0$.

Proof: Let $n \in \mathbb{N}_0$ and $f \in H(D)$. By Proposition 4.1.5 we get

$$\big(\psi *_{\Omega,\,\mathbb{C}} (s_n(f))\big)\big|_{\Omega * G} = \big(s_n(\psi *_{\Omega,D} f)\big)\big|_{\Omega * G} = \big(r_{\Omega * G} \circ s_n \circ (\psi *_{\Omega,D} \cdot)\big)(f).$$

By the compatibility theorem we get

$$\big(\psi *_{\Omega,\,\mathbb{C}} (s_n(f))\big)\big|_{\Omega * G} = \psi *_{\Omega,G} \big(s_n(f)\big|_G\big) = \big((\psi *_{\Omega,G} \cdot) \circ r_G \circ s_n\big)(f).$$

From these two identities, the commutativity follows. □

6.3.4 Remarks:

1. The above lemma shows that the first condition in Proposition 6.1.2 is satisfied.
2. The continuity of $\psi *_{\Omega,G} \cdot$ (i.e. the second condition in Proposition 6.1.2) is guaranteed by Lemma 3.7.1. ◇

According to Remark 6.3.4, the first two conditions of Proposition 6.1.2 are satisfied. If we can verify that the operator $\psi *_{\Omega,G} \cdot : H(G) \to H(\Omega * G)$ has dense range, then universality is inherited under Hadamard multiplication, i.e. $\psi *_{\Omega,G} \varphi$ is universal with respect to $\Omega * G$ for every φ that is universal with respect to G.

The next theorem is a central result concerning derived universality.

6.3.5 Universality preservation theorem:
Let $G \in \mathcal{G}(D)$ so that Ω and G are star-eligible. Then the following assertions hold:

1. *If the operator $\psi *_{\Omega,\,G} \cdot : H(G) \to H(\Omega * G)$ has dense range, then $\psi *_{\Omega,D} \varphi \in \mathscr{U}(\Omega * D, \Omega * G)$ for every $\varphi \in \mathscr{U}(D, G)$.*
2. *If $\psi *_{\Omega,\,D} f \in \mathscr{U}(\Omega * D, \Omega * G)$ for an $f \in H(D)$, then the operator $\psi *_{\Omega,\,G} \cdot : H(G) \to H(\Omega * G)$ has dense range.*

Proof: *ad 1.:* Let $\psi *_{\Omega,\,G} \cdot : H(G) \to H(\Omega * G)$ have dense range. Since the conditions of Proposition 6.1.2 are satisfied, we get $\psi *_{\Omega,D} \varphi \in \mathscr{U}(\Omega * D, \Omega * G)$ for every $\varphi \in \mathscr{U}(D, G)$.

ad 2.: Let $\psi *_{\Omega,\,D} f \in \mathscr{U}(\Omega * D, \Omega * G)$ for an $f \in H(D)$. Since the conditions of Proposition 6.1.4 are satisfied, the operator $\psi *_{\Omega,\,G} \cdot : H(G) \to H(\Omega * G)$ has dense range. □

6.3.6 Remark:
Let $G \in \mathcal{G}(D)$ so that Ω and G are star-eligible, and let $\varphi \in \mathscr{U}(D, G)$. By the universality preservation theorem we get the following characterization: The operator $\psi *_{\Omega, G} \cdot : H(G) \to H(\Omega * G)$ has dense range if and only if $\psi *_{\Omega, D} \varphi \in \mathscr{U}(\Omega * D, \Omega * G)$. $\diamond$

The next theorem links the lacunary structure of ψ with the range of the Hadamard operator.

6.3.7 Theorem:
Let $G \in \mathcal{G}(D)$ so that Ω and G are star-eligible. Then the set $\mathscr{P}^{\Lambda(\psi)}_{|\Omega * G}$ is dense in $H(\Omega * G)$ if and only if $\psi *_{\Omega, G} \cdot : H(G) \to H(\Omega * G)$ has dense range.

Proof: 1. Let $\mathscr{P}^{\Lambda(\psi)}_{|\Omega * G}$ be dense in $H(\Omega * G)$. Since $\psi *_{\Omega,G} \cdot$ is linear, it suffices to show that the image of $\psi *_{\Omega,G} \cdot$ contains the monomials in $\mathscr{P}^{\Lambda(\psi)}_{|\Omega * G}$. To this end, let $k \in \Lambda(\psi)$. We define a polynomial P by

$$P(z) \ := \ \psi_k^{-1} \cdot z^k \qquad (z \in \mathbb{C}).$$

By the compatibility theorem we have $(\psi *_{\Omega,\mathbb{C}} P)\big|_{\Omega * G} = \psi *_{\Omega,G} \big(P\big|_G\big)$. By the Hadamard multiplication theorem we get

$$\psi *_{\Omega, \mathbb{C}} P \ = \ \psi_k \cdot \psi_k^{-1} \cdot P_k \ = \ P_k\,,$$

and thus

$$P_k\big|_{\Omega * G} \ = \ \psi *_{\Omega,G} \big(P\big|_G\big).$$

2. Let $\psi *_{\Omega, G} \cdot : H(G) \to H(\Omega * G)$ have dense range. By the universality preservation theorem the set $\big\{s_n(\psi *_{\Omega, D} \varphi)\big|_{\Omega * G} : n \in \mathbb{N}_0\big\}$ is dense in $H(\Omega * G)$ for every $\varphi \in \mathscr{U}(D, G)$. Since $s_n(\psi *_{\Omega,D} \varphi) \in \mathscr{P}^{\Lambda(\psi)}$ for every $n \in \mathbb{N}_0$ (and every $\varphi \in \mathscr{U}(D, G)$), the assertion follows. $\square$

We get the following corollary.

6.3.8 Corollary:
Let $G \in \mathcal{G}(D)$ so that Ω and G are star-eligible. Then the following assertions hold:

1. *If $\mathscr{P}^{\Lambda(\psi)}_{|\Omega * G}$ is dense in $H(\Omega * G)$, then $\psi *_{\Omega,D} \varphi \in \mathscr{U}(\Omega * D, \Omega * G)$ for every $\varphi \in \mathscr{U}(D, G)$.*

2. *If* $\psi *_{\Omega,D} f \in \mathscr{U}(\Omega * D, \Omega * G)$ *for an* $f \in H(D)$*, then* $\mathscr{P}^{\Lambda(\psi)}_{|\,\Omega * G}$ *is dense in* $H(\Omega * G)$*.*

Proof: The assertions follow by Theorem 6.3.7 and the universality preservation theorem. □

6.3.9 Remark:
Corollary 6.3.8 states that an alternative approach to derived universality is to study under what conditions the set $\mathscr{P}^{\Lambda(\psi)}_{|\,\Omega * G}$ of lacunary polynomials is dense in $H(\Omega * G)$. For the latter property, there is an extensive amount of sufficient conditions in the literature (see e.g. [DK77], [Mar84], [LMM98a], [LMM98b], [LMM02]). ◇

It is possible to state a modification of Theorem 6.3.7 if we replace the set $\Lambda(\psi)$ by $\Lambda(\psi *_{\Omega,\,D} \varphi)$.

6.3.10 Theorem:
Let $G \in \mathcal{G}(D)$ *so that* Ω *and* G *are star-eligible, and let* $\varphi \in \mathscr{U}(D, G)$*. Then* $\mathscr{P}^{\Lambda(\psi *_{\Omega,\,D} \varphi)}_{|\,\Omega * G}$ *is dense in* $H(\Omega * G)$ *if and only if* $\psi *_{\Omega,D} \varphi \in \mathscr{U}(\Omega * D, \Omega * G)$*.*

Proof: (i) Assume that $\mathscr{P}^{\Lambda(\psi *_{\Omega,\,D} \varphi)}_{|\,\Omega * G}$ is dense in $H(\Omega * G)$. The Hadamard multiplication theorem yields $\Lambda(\psi *_{\Omega,D} \varphi) = \Lambda(\psi) \cap \Lambda(\varphi)$. Thus, we get

$$\mathscr{P}^{\Lambda(\psi *_{\Omega,\,D} \varphi)}_{|\,\Omega * G} \subset \mathscr{P}^{\Lambda(\psi)}_{|\,\Omega * G}.$$

By Corollary 6.3.8 we get $\psi *_{\Omega,D} \varphi \in \mathscr{U}(\Omega * D, \Omega * G)$.

(ii) Now assume that $\psi *_{\Omega,D} \varphi \in \mathscr{U}(\Omega * D, \Omega * G)$. By Proposition 6.2.2 the set $\mathscr{P}^{\Lambda(\psi *_{\Omega,\,D} \varphi)}_{|\,\Omega * G}$ is dense in $H(\Omega * G)$. □

At the end of this section, we would like to recapitulate the universality preservation theorem. Let us recall the setting. We are given an element G of $\mathcal{G}(D)$, an open set Ω star-eligible to G—thus, Ω must contain the origin and the point at infinity—, a function φ in $\mathscr{U}(D, G)$, and a function ψ in $H(\Omega)$. There are two objects of interest:

1. The gaps of the power series expansion of ψ around the origin.

2. The geometry of Ω.

The first item deals with the set $\mathbb{N}_0 \setminus \Lambda(\psi)$; it has an intimate connection with the set $\mathscr{P}^{\Lambda(\psi)}_{|\,\Omega * G}$ whose density in $H(\Omega * G)$ is necessary and sufficient for derived universality. For further studies we refer to section 6.4.

The second item is concerned with the star eligibility of the sets Ω and G. In section 6.6, we will discuss this item for special sets.

6.4 Lacuna conditions

At the beginning of this section, we briefly recall the notion of Pólya density for subsets of the set of non-negative integers. For detailed information and proofs, we refer to [Pó29] and [Pó33]. Let $\Lambda \subset \mathbb{N}_0$.[†] Then the numbers

$$\underline{\Delta}(\Lambda) := \varliminf_{n\to\infty} \frac{\big|\Lambda \cap [0,n]\big|}{n} \quad \text{and} \quad \overline{\Delta}(\Lambda) := \varlimsup_{n\to\infty} \frac{\big|\Lambda \cap [0,n]\big|}{n}$$

are called the lower and upper density of Λ, respectively. If both numbers are the same, we say that Λ is measurable and call the limit the density of Λ; in this case we write $\Delta(\Lambda)$.

Let us summarize some properties.

$$0 \le \underline{\Delta}(\Lambda) \le \overline{\Delta}(\Lambda) \le 1;$$

$$\underline{\Delta}(\Lambda) + \overline{\Delta}(\mathbb{N}_0 \setminus \Lambda) = 1;$$

$$\Delta(\Lambda) + \Delta(\mathbb{N}_0 \setminus \Lambda) = 1, \text{ if } \Lambda \text{ is measurable.}$$

The first item on the list on page 81 deals with the set $\mathbb{N}_0 \setminus \Lambda(\psi)$, that is to say with the lacunary structure of the local Taylor expansion of ψ. According to Corollary 6.3.8, a sufficient condition for derived universality is that $\mathscr{P}^{\Lambda(\psi)}_{|\,\Omega * G}$ is dense in $H(\Omega * G)$. This requires the application of lacunary versions of Runge-type approximation theorems.

The next challenge is to find necessary and sufficient conditions under which the set $\mathscr{P}^{\Lambda(\psi)}_{|\,\Omega * G}$ is dense in $H(\Omega * G)$. The following lemma gives a sufficient one.

[†] This time, the empty set is not excluded from considerations.

6.4.1 Lemma:
Let $\Lambda \subset \mathbb{N}_0$ *and* $G \in \mathcal{G}(\{0\})$*. If* $\overline{\Delta}(\Lambda) = 1$*, then* $\mathscr{P}^{\Lambda}_{|G}$ *is dense in* $H(G)$*.*

Proof: Let $K \subset G$ be a compactum with connected complement, $f \in H(G)$, and $\varepsilon > 0$. For a given real number δ satisfying $0 < \delta < \mathrm{dist}(0, K)$, the set $L := U_\delta[0] \cup K$ is a compactum with connected complement and $L_0 = U_\delta[0]$. The function $F : L \to \mathbb{C}$ with

$$F(z) := \begin{cases} f(z) & , \ z \in K \\ 0 & , \ z \in U_\delta[0] \end{cases}$$

is holomorphic on L. According to the Lemma in [LMM02, p. 203], there exists a lacunary polynomial $P \in \mathscr{P}^{\Lambda}_{|G}$ so that $\|F - P\|_L < \varepsilon$, whence we get $\|f - P\|_K < \varepsilon$. Since G can be exhausted by compact sets having connected complements, the assertion follows. □

Next, we present a class of holomorphic functions whose local expansions around the origin satisfy the density condition of Lemma 6.4.1.

6.4.2 Lemma:
If $f \in H(\mathbb{C}_\infty \setminus \{1\})$ *and* $f \neq 0$*, then* $\Delta(\Lambda(f)) = 1$*.*

Proof: By Wigert's theorem (see e.g. [Hille, Theorems 11.2.1 and 11.2.2]) there exists an entire function g of exponential type zero so that

$$f(z) = \sum_{\nu=0}^{\infty} g(\nu)\, z^\nu$$

holds for all $z \in \mathbb{D}$. According to Theorems 2.5.12 and 2.5.13 of [Boas], the set $\Lambda(f)$ has unit density. □

If $\Omega = \mathbb{C}_\infty \setminus \{1\}$, we are able to prove the following important result.

6.4.3 Theorem:
Let $G \in \mathcal{G}(D)$*. If* $\psi \in H(\mathbb{C}_\infty \setminus \{1\})$ *and* $\psi \neq 0$*, then* $\psi *_{\mathbb{C}_\infty \setminus \{1\}, D} \varphi \in \mathscr{U}(D, G)$ *for all* $\varphi \in \mathscr{U}(D, G)$*.*

Proof: Notice that $\mathbb{C}_\infty \setminus \{1\}$ and G are star-eligible and $(\mathbb{C}_\infty \setminus \{1\}) * G = G$. According to Lemma 6.4.2, we have $\Delta(\Lambda(\psi)) = 1$. By Lemma 6.4.1 the set $\mathscr{P}^{\Lambda(\psi)}_{|G}$ is dense in $H(G)$. The conclusion follows from Corollary 6.3.8. □

6.5 Derived universality without further restrictions

In section 6.3, we considered functions universal with respect to some open simply connected set G and asked if their Hadamard products were universal with respect to $\Omega * G$. In this section, we study functions in $\mathscr{U}(D)$. We are interested in the following question.

6.5.1 Problem:
Let $\varphi \in \mathscr{U}(D)$. Under what conditions is $\psi *_{\Omega,D} \varphi \in \mathscr{U}(\Omega * D)$? $\diamond$

In special situations, we have the following result.

6.5.2 Proposition:
Let for every $\alpha \in (0,\pi)$ there exist a $\xi \in \mathbb{C} \setminus \{0\}$ with

$$\left(\xi \cdot G\left(e^{-\pi}, e^{\pi}; \alpha\right)\right) \cap D \;=\; \emptyset. \tag{6.1}$$

If $\varphi \in \mathscr{U}(D)$, then $\overline{\Delta}(\Lambda(\varphi)) = 1$.

Proof: The proof is a slight modification of the proof of Theorem 2.4 in [MM06]. Assume that $d := \overline{\Delta}(\Lambda(\varphi)) < 1$. Choose $\alpha \in \big(\pi d, \pi\big)$. Then we have

$$G \;:=\; G\big(e^{-\pi}, e^{\pi}; \alpha\big) \;\supset\; B\big(e^{-\pi d}, e^{\pi d}; \pi d\big).$$

According to our precondition, there exists a $z_0 \in \mathbb{C} \setminus \{0\}$ in such a way that $(z_0 \cdot G) \cap D = \emptyset$, and hence $z_0 \cdot G \in \mathcal{G}(D)$. Since $\varphi \in \mathscr{U}(D)$, there exists a strictly monotonic increasing sequence $(n_k)_{k\in\mathbb{N}_0}$ in $\mathbb{N}_0$ so that $s_{n_k}(\varphi) \to 0$ compactly on $z_0 \cdot G$ as $k \to \infty$. By Theorem 1.1 and Remark 1.2 of [MM06] we obtain $\varphi_\nu = 0$ for all $\nu \in \mathbb{N}_0$. But this is a contraction. $\square$

6.5.3 Remarks:

1. If D is bounded, then condition (6.1) is satisfied.
2. If $\overline{\Delta}(\Lambda(\varphi)) = 1$, then φ need not belong to $\mathscr{U}(D)$: Consider for instance $\varphi := \Theta\big|_{\mathbb{D}}$. Then (6.1) is satisfied, and we have $\Lambda(\varphi) = \mathbb{N}_0$. But φ is not universal.

3. If we want to solve Problem 6.5.1 for $\Omega * D$ satisfying (6.1), we necessarily have $\overline{\Delta}(\Lambda(\psi *_{\Omega,D} \varphi)) = 1$, and hence $\overline{\Delta}(\Lambda(\psi)) = 1$. $\diamond$

Under a certain condition on the sets in $\mathcal{G}(\Omega * D)$, we can prove the following result.

6.5.4 Strong universality preservation theorem:
Let Ω contain the origin and the point at infinity. Furthermore, assume that for every $G \in \mathcal{G}(\Omega * D)$ there exists a $\tilde{G} \in \mathcal{G}(D)$ so that $\Omega * \tilde{G} = G$. Then the following assertions hold:

1. *If for every $G' \in \mathcal{G}(D)$ the operator $\psi *_{\Omega,G'} \cdot : H(G') \to H(\Omega * G')$ has dense range, then $\psi *_{\Omega,D} \varphi \in \mathscr{U}(\Omega * D)$ for every $\varphi \in \mathscr{U}(D)$.*
2. *If $\psi *_{\Omega,D} f \in \mathscr{U}(\Omega * D)$ for an $f \in H(D)$, then for every $G' \in \mathcal{G}(D)$ star-eligible to Ω the operator $\psi *_{\Omega,G'} \cdot : H(G') \to H(\Omega * G')$ has dense range.*

Proof: *ad 1.:* Let $\varphi \in \mathscr{U}(D)$ and $G \in \mathcal{G}(\Omega * D)$. Then there is a $\tilde{G} \in \mathcal{G}(D)$ with $\Omega * \tilde{G} = G$. Since $\varphi \in \mathscr{U}(D, \tilde{G})$ and $\psi *_{\Omega,\tilde{G}} \cdot : H(\tilde{G}) \to H(\Omega * \tilde{G})$ has dense range, we get $\psi *_{\Omega,D} \varphi \in \mathscr{U}(\Omega * D, \Omega * \tilde{G}) = \mathscr{U}(\Omega * D, G)$ by the universality preservation theorem. Thus, $\psi *_{\Omega,D} \varphi \in \mathscr{U}(\Omega * D)$.

ad 2.: Let $\psi *_{\Omega,D} f \in \mathscr{U}(\Omega * D)$ for an $f \in H(D)$, and let $G' \in \mathcal{G}(D)$ be star-eligible to Ω. Since $\Omega * G \in \mathcal{G}(\Omega * D)$, we get $\psi *_{\Omega,D} f \in \mathscr{U}(\Omega * D, \Omega * G)$. Thus, the operator $\psi *_{\Omega,G'} \cdot : H(G') \to H(\Omega * G')$ has dense range by the universality preservation theorem. $\square$

6.5.5 Remark:
If $\Omega = \mathbb{C}_\infty \setminus \{1\}$, then we have $\Omega * G = G$, and we can choose $\tilde{G} = G$. Thus, the condition on the sets G in the strong universality preservation theorem is satisfied. $\diamond$

What can we say if $\Omega \neq \mathbb{C}_\infty \setminus \{1\}$? In this case, we are faced with Problem 1.3.18. It is not clear at all if for every $G \in \mathcal{G}(\Omega * D)$ there exists a $\tilde{G} \in \mathcal{G}(D)$ in such a way that $\Omega * \tilde{G} = G$. But even if this was not true, the two parts of the strong universality preservation theorem could be valid anyway. Our proof uses the universality preservation theorem. There could be an alternative proof that does not need the universality preservation theorem.

6.6 Examples

At first, we are concerned with annular sectors (see section 1.3). To this end, let $0 < r_1 \leq r_2 < \infty$, $0 \leq \beta < \pi$, $0 \leq s_1 < s_2 \leq \infty$, and $0 < \alpha \leq \pi$ so that $G(s_1, s_2; \alpha) \in \mathcal{G}(D)$. According to Proposition 1.3.16, the sets $G(s_1, s_2; \alpha)$ and $\Omega(r_1, r_2; \beta)$ are star-eligible if and only if $\alpha > \beta$ and $s_1 r_2 < s_2 r_1$. Assume that these inequalities hold.

6.6.1 Example:
If $\mathscr{P}^{\Lambda(\psi)}_{|G(s_1r_2,\, s_2r_1;\, \alpha-\beta)}$ is dense in $H(G(s_1r_2,\, s_2r_1;\, \alpha-\beta))$, then

$$\psi *_{\Omega(r_1,\, r_2;\, \beta),D} \varphi \in \mathscr{U}(\Omega(r_1,\, r_2;\, \beta) * D, G(s_1r_2,\, s_2r_1;\, \alpha-\beta))$$

for each $\varphi \in \mathscr{U}(D, G(s_1,\, s_2;\, \alpha))$ according to Corollary 6.3.8. $\diamond$

We remark the following: If in addition $r_1 \leq 1 \leq r_2$, i.e. $1 \notin \Omega(r_1, r_2; \beta)$, then $G(s_1r_2,\, s_2r_1;\, \alpha-\beta) \subset G(s_1, s_2; \alpha)$. This means that the set with respect to which the Hadamard product $\psi *_{\Omega,D} \varphi$ is universal is “smaller” than the set with respect to which φ is universal.

Let now $\Omega := \mathbb{C}_\infty \setminus \{1, -1\}$. Next, we study the rational function

$$\psi : \Omega \to \mathbb{C}, \quad \psi(z) := \frac{1}{1-z^2}.$$

If there is a $\theta \in [0, \pi)$ in such a way that G^C contains a half plane induced by the line $\{te^{i\varphi} : t \geq 0,\, \varphi - \theta \in \{0, \pi\}\}$, then $\Omega * G = \emptyset$, i.e. Ω and G are not star-eligible (cf. Example 1.3.9.2).

6.6.2 Example:
If $G \in \mathcal{G}(D)$ is star-eligible to Ω we have (see Example 1.3.8)

$$\Omega * G \;=\; G \cap (-G)$$

and (see Example 3.4.9)

$$(\psi *_{\Omega,D} \varphi)(z) \;=\; \frac{\varphi(z) + \varphi(-z)}{2} \qquad \big(z \in D \cap (-D)\big),$$

i.e. $\Omega * G$ is radial symmetric to the origin, and $\psi *_{\Omega,D} \varphi$ is an even function. If G lies in a half plane, then $G \cap (-G) = \emptyset$, i.e. Ω and G are not star-eligible. If $G \cap (-G) \neq \emptyset$, we can only approximate even functions by the partial sums $s_n\, (\psi *_{\Omega,D} \varphi)$. Thus, the Hadamard product cannot be universal with respect to $G \cap (-G)$. $\diamond$

Chapter 7

The Hadamard product and Euler differential operators

In this chapter, we will expose the connection between the Hadamard product and Euler differential operators. We will use our customary notation. The results already appeared in chapter eleven of [Hille]. The purpose of this chapter is to link these two topics.

In the first section, we will introduce a differential operator whose iterates are used to define the Euler differential operators.

In the second section, we will introduce the Euler differential operators (see Definition 7.2.1).

In the third section, we will prove the already mentioned connection between the Hadamard product and the Euler differential operators (see Proposition 7.3.2).

7.1 A lacuna preserving derivative operator

The derivative operator on the set of functions holomorphic in a given open set is a linear self map. Thus, the operator we shall now define is well defined, too.

7.1.1 Definition:
Let $\Omega \subset \mathbb{C}$ be a non-empty open set. Then we define $\vartheta_\Omega : H(\Omega) \to H(\Omega)$ by

$$\vartheta_\Omega(f)(z) \;:=\; z \cdot f'(z) \qquad (z \in \Omega).$$

By

$$\vartheta_\Omega^0 \;:=\; \mathrm{id}\big|_{H(\Omega)} \quad \text{and} \quad \vartheta_\Omega^{k+1} \;:=\; \vartheta_\Omega \circ \vartheta_\Omega^k \;\; (k \in \mathbb{N}_0).$$

we define the iterates of this operator. ◇

7.1.2 Remarks:

1. The operator ϑ_Ω—and hence each ϑ_Ω^k—is linear and continuous.
2. How does ϑ_Ω act upon subsets? Let $D \subset \Omega$ be a non-empty open set and $r : H(\Omega) \to H(D)$ the restriction map defined by $r(f) := f\big|_D$. Then we have $\vartheta_D \circ r = r \circ \vartheta_\Omega$. ◇

The next example shows the connection between the power series expansion of a holomorphic function and the image under the differential operator.

7.1.3 Example:
Let $\Omega \subset \mathbb{C}$ be a non-empty open set with $0 \in \Omega$, and let $f \in H(\Omega)$. For each $k \in \mathbb{N}_0$ we have

$$\vartheta_\Omega^k(f)(z) \;=\; \sum_{\nu=0}^{\infty} \nu^k \cdot \frac{f^{(\nu)}(0)}{\nu!} \cdot z^\nu$$

compactly on $\{z \in \mathbb{C} : |z| < \mathrm{dist}(0, \partial\Omega)\}$. ◇

7.1.4 Remark:
Example 7.1.3 shows that $\Lambda(\vartheta_\Omega^k(f)) = \Lambda(f)\backslash\{0\}$ for all $k \in \mathbb{N}$, i.e. ϑ_Ω^k preserves gaps (possibly 0 is added to the gaps). ◇

For the iterates of ϑ_Ω we have the following representation.

7.1.5 Lemma:
Let $k \in \mathbb{N}$. Then there exist $\alpha_{\mu,k} \in \mathbb{N}$ $(1 \le \mu \le k)$ with $\alpha_{1,k} = \alpha_{k,k} = 1$ and so that for each non-empty open set $\Omega \subset \mathbb{C}$ and for each $f \in H(\Omega)$ we have

$$\vartheta_\Omega^k(f)(z) \;=\; \sum_{\mu=1}^{k} \alpha_{\mu,k} \cdot f^{(\mu)}(z) \cdot z^\mu$$

for all $z \in \Omega$.

Proof: (i) Let $k = 1$. If we define $\alpha_{1,1} := 1$, we have $\vartheta_\Omega(f)(z) = zf'(z)$ for each non-empty open set $\Omega \subset \mathbb{C}$, each $f \in H(\Omega)$, and each $z \in \Omega$.

(ii) Now let the identity hold for a $k \in \mathbb{N}$. Define $\alpha_{1,k+1} := \alpha_{k+1,k+1} := 1$ and $\alpha_{\mu,k+1} := \mu\,\alpha_{\mu,k} + \alpha_{\mu-1,k}$ $(2 \le \mu \le k)$. Then for each non-empty open set $\Omega \subset \mathbb{C}$, each $f \in H(\Omega)$, and each $z \in \Omega$ we have

$$\begin{aligned}\vartheta_\Omega^{k+1}(f)(z) \;&=\; z \cdot \big(\vartheta_\Omega^k(f)\big)'(z)\\ &=\; \sum_{\mu=1}^{k} \Big(\mu\,\alpha_{\mu,k}\,z^\mu\,f^{(\mu)}(z) \;+\; \alpha_{\mu,k}\,z^{\mu+1}\,f^{(\mu+1)}(z)\Big)\\ &=\; \alpha_{1,k}\,z\,f'(z) \;+\; \alpha_{k,k}\,z^{k+1}\,f^{(k+1)}(z) \;+\; \sum_{\mu=2}^{k} (\mu\,\alpha_{\mu,k} + \alpha_{\mu-1,k})\,z^\mu\,f^{(\mu)}(z)\\ &=\; \sum_{\mu=1}^{k+1} \alpha_{\mu,k+1}\,f^{(\mu)}(z)\,z^\mu.\end{aligned}$$

Thus, the identity also holds for $k + 1$. □

7.1.6 Remark:
We would like to stress that the numbers $\alpha_{\mu,k}$ in Lemma 7.1.5 neither depend on Ω nor on f. ◇

7.2 Euler differential operators

In order to introduce Euler differential operators, we need entire functions of exponential type zero. We denote by EXP(0) the vector space of all entire functions of exponential type zero. For $f \in \mathrm{EXP}(0)$ and $n \in \mathbb{N}$ we define

$$p_n(f) \;:=\; \sup_{z \in \mathbb{C}} \;|f(z)| \cdot e^{-\frac{1}{n}\cdot|z|}.$$

Each p_n is a norm on EXP(0). Thus, the family $(p_n)_{n\in\mathbb{N}}$ induces a locally convex vector space topology on EXP(0).

Let g be an entire function of exponential type zero. According to Wigert's theorem, the power series $\sum_{\nu=0}^{\infty} g(\nu)\,z^\nu$ can be analytically continued to a function ψ in $H(\mathbb{C}_\infty \setminus \{1\})$ so that

$$\psi(z) \;=\; \begin{cases} \sum\limits_{\nu=0}^{\infty} g(\nu) z^{\nu} & , \quad z \in \mathbb{D} \\ -\sum\limits_{\nu=1}^{\infty} g(-\nu) z^{-\nu} & , \quad z \in \mathbb{C}_\infty \setminus \overline{\mathbb{D}} \end{cases} \tag{7.1}$$

holds. Thus, $\mathfrak{M}^{-1} : \mathrm{EXP}(0) \to H(\mathbb{C}_\infty \setminus \{1\})$ defined by $\mathfrak{M}^{-1}(g) := \psi$ is well defined. It can be shown that $\mathfrak{M}^{-1}$ is a linear, bijective continuous operator whose inverse is also continuous. (The inverse $\mathfrak{M}$ is called Mellin transformation.)

Let $\Omega \subset \mathbb{C}$ be a non-empty open set, ψ in $H(\mathbb{C}_\infty \setminus \{1\})$, and g an entire function of exponential type zero. If f is in $H(\Omega)$, the series $\sum_{k=0}^{\infty} g_k \vartheta_\Omega^k(f)$ converges compactly on Ω (cf. [Hille, Theorem 11.2.3]). (Actually, the result is shown for domains instead of open sets; but since only local arguments are used, it also works for open sets.) Hence, it defines a function holomorphic in Ω. Moreover, $\psi *_{\mathbb{C}_\infty \setminus \{1\},\Omega} f$ is in $H(\Omega)$ according to Proposition 3.6.4. These properties justify the following definition.

7.2.1 Definition:
Let $\Omega \subset \mathbb{C}$ be a non-empty open set, $\psi \in H(\mathbb{C}_\infty \setminus \{1\})$, and $g \in \mathrm{EXP}(0)$. We define $g(\vartheta_\Omega) : H(\Omega) \to H(\Omega)$ and $H_{\psi,\Omega} : H(\Omega) \to H(\Omega)$ by

$$g(\vartheta_\Omega)(f) \;:=\; \sum_{k=0}^{\infty} g_k \cdot \vartheta_\Omega^k(f)$$

and

$$H_{\psi,\Omega}(f) \;:=\; \psi *_{\mathbb{C}_\infty \setminus \{1\},\Omega} f,$$

The operator $g(\vartheta_\Omega)$ is called *Euler differential operator* or *hyper-differential operator.* ◇

The operators defined in Definition 7.2.1 have the following properties.

7.2.2 Lemma:
Let $\Omega \subset \mathbb{C}$ be a non-empty open set, $\psi \in H(\mathbb{C}_\infty \setminus \{1\})$, and $g \in \mathrm{EXP}(0)$. Then $g(\vartheta_\Omega)$ and $H_{\psi,\Omega}$ are linear and continuous.

Proof: The linearity and continuity of $H_{\psi,\Omega}$ were proved in Lemma 3.7.1. The linearity of $g(\vartheta_\Omega)$ follows from the linearity of all the operators ϑ_Ω^k ($k \in \mathbb{N}_0$). The continuity of $g(\vartheta_\Omega)$ is proved in [Hille, Theorem 11.2.3]. □

7.3 The connection with the Hadamard product

Let g be an entire function of exponential type zero and ψ according to (7.1). In [Fr97, Proposition 8] it was shown that $H_{\psi,D} = g(\vartheta_D)$ holds for every simply connected domain $D \subset \mathbb{C}$ with $0 \in D$ (see also [Hille, Theorem 11.2.3]). The Hadamard product enables us to generalize this result (see Corollary 7.3.3).

At first, we shall show the connection between the iterates of the lacuna preserving operator and the Hadamard product. To this end, we have to define the operator ϑ for the function Θ. According to Lemma 7.1.5, we have for each $z \in \mathbb{C} \setminus \{1\}$ and each $k \in \mathbb{N}_0$:

$$\vartheta^k_{\mathbb{C}\setminus\{1\}}\left(\Theta\big|_{\mathbb{C}\setminus\{1\}}\right)(z) \;=\; \sum_{\mu=1}^{k} \frac{\alpha_{\mu,k}\cdot k!\cdot z^\mu}{(1-z)^{\mu+1}} \xrightarrow{z\to\infty} 0.$$

Therefore, we set

$$\vartheta^k_{\mathbb{C}_\infty\setminus\{1\}}(\Theta)(\infty) \;:=\; \begin{cases} \vartheta^k_{\mathbb{C}\setminus\{1\}}\left(\Theta\big|_{\mathbb{C}\setminus\{1\}}\right)(z) & ,\ z\neq\infty \\ 0 & ,\ z=\infty \end{cases}$$

for each $k \in \mathbb{N}_0$. We have $\vartheta^k_{\mathbb{C}_\infty\setminus\{1\}}(\Theta) \in H\big(\mathbb{C}_\infty \setminus \{1\}\big)$ for each $k \in \mathbb{N}_0$.

7.3.1 Lemma:
Let $\Omega \subset \mathbb{C}$ be a non-empty open set and $k \in \mathbb{N}_0$. Then for each $f \in H(\Omega)$ we have

$$\vartheta^k_\Omega(f) \;=\; \big(\vartheta^k_{\mathbb{C}_\infty\setminus\{1\}}(\Theta)\big) *_{\mathbb{C}_\infty\setminus\{1\},\Omega} f,$$

or in other words

$$\vartheta^k_\Omega \;=\; H_{\vartheta^k_{\mathbb{C}_\infty\setminus\{1\}}(\Theta),\Omega}.$$

Proof: 1. For $k = 0$, we have

$$\vartheta^0_\Omega = \mathrm{id}_{H(\Omega)} = H_{\Theta,\Omega} = H_{\vartheta^0_{\mathbb{C}_\infty\setminus\{1\}}(\Theta),\Omega}.$$

2. Now let $k \geq 1$, $f \in H(\Omega)$, $z \in \Omega * (\mathbb{C}_\infty \setminus \{1\}) = \Omega$, and γ a Hadamard cycle

for $z \cdot (\mathbb{C}_\infty \setminus \{1\})^* = \{z\}$ in Ω. By applying Lemma 7.1.5, we obtain

$$\begin{aligned}
\vartheta_\Omega^k(f)(z) &= \sum_{\mu=1}^{k} \alpha_{\mu,k}\, f^{(\mu)}(z)\, z^\mu \;=\; \sum_{\mu=1}^{k} \alpha_{\mu,k} \cdot \left(\frac{\mu!}{2\pi i} \cdot \int\limits_\gamma \frac{f(\zeta)}{(\zeta - z)^{\mu+1}}\, d\zeta \right) \cdot z^\mu \\
&= \sum_{\mu=1}^{k} \alpha_{\mu,k} \cdot \left(\frac{1}{2\pi i} \cdot \int\limits_\gamma \frac{f(\zeta)}{\zeta} \cdot \left(\frac{z}{\zeta}\right)^\mu \cdot \frac{\mu!}{(1 - z/\zeta)^{\mu+1}}\, d\zeta \right) \\
&= \left(f *_{\Omega, \mathbb{C}_\infty \setminus \{1\}} \left(\sum_{\mu=1}^{k} \alpha_{\mu,k} \cdot \mathrm{id}^\mu_{\mathbb{C}_\infty \setminus \{1\}} \cdot \Theta^{(\mu)} \right) \right)(z) \\
&= \Big(f *_{\Omega, \mathbb{C}_\infty \setminus \{1\}} \big(\vartheta^k_{\mathbb{C}_\infty \setminus \{1\}}(\Theta) \big) \Big)(z) \;=\; \Big(\big(\vartheta^k_{\mathbb{C}_\infty \setminus \{1\}}(\Theta) \big) *_{\mathbb{C}_\infty \setminus \{1\}, \Omega} f \Big)(z).
\end{aligned}$$

This proves the lemma. $\square$

After these preparations, we are able to state and prove the connection between the Hadamard product and the Euler differential operators.

7.3.2 Proposition:
Let $\Omega \subset \mathbb{C}$ be a non-empty open set. Then for each $f \in H(\Omega)$ we have

$$g(\vartheta_\Omega)(f) \;=\; \big(g(\vartheta_{\mathbb{C}_\infty \setminus \{1\}}(\Theta) \big) *_{\mathbb{C}_\infty \setminus \{1\}, \Omega} f,$$

or in other words

$$g(\vartheta_\Omega) \;=\; H_{g(\vartheta_{\mathbb{C}_\infty \setminus \{1\}}(\Theta)), \Omega}.$$

Proof: Let $f \in H(\Omega)$. By applying Lemma 7.3.1 and the continuity theorem, we obtain

$$\begin{aligned}
g(\vartheta_\Omega)(f) &= \sum_{k=0}^{\infty} g_k \cdot \Big(\big(\vartheta^k_{\mathbb{C}_\infty \setminus \{1\}}(\Theta) \big) *_{\mathbb{C}_\infty \setminus \{1\}, \Omega} f \Big) \\
&= \left(\sum_{k=0}^{\infty} g_k \cdot \vartheta^k_{\mathbb{C}_\infty \setminus \{1\}}(\Theta) \right) *_{\mathbb{C}_\infty \setminus \{1\}, \Omega} f \;=\; \big(g \left(\vartheta_{\mathbb{C}_\infty \setminus \{1\}} \right) (\Theta) \big) *_{\mathbb{C}_\infty \setminus \{1\}, \Omega} f.
\end{aligned}$$

This proves the proposition. $\square$

A consequence of this proposition is the following result.

7.3.3 Corollary:
Let $\Omega \subset \mathbb{C}$ be a non-empty open set, $\psi \in H(\mathbb{C}_\infty \setminus \{1\})$, and $g \in \mathrm{EXP}(0)$ according to (7.1). Then we have $H_{\psi,\Omega} = g(\vartheta_\Omega)$.

Proof: According to Example 7.1.3, we have

$$\vartheta^k_{\mathbb{C}_\infty \setminus \{1\}}(\Theta)(z) \;=\; \sum_{\nu=0}^{\infty} \nu^k \, z^\nu \qquad (z \in \mathbb{D}).$$

Thus, for $z \in \mathbb{D}$ we obtain

$$g\big(\vartheta_{\mathbb{C}_\infty \setminus \{1\}}(\Theta)\big)(z) \;=\; \sum_{k=0}^{\infty} g_k \cdot \vartheta^k_{\mathbb{C}_\infty \setminus \{1\}}(\Theta)(z)$$

$$=\; \sum_{\nu=0}^{\infty} \left(\sum_{k=0}^{\infty} g_k \, \nu^k \right) z^\nu \;=\; \sum_{\nu=0}^{\infty} g(\nu) \, z^\nu \;=\; \psi(z).$$

By the identity theorem we get $g\big(\vartheta_{\mathbb{C}_\infty \setminus \{1\}}(\Theta)\big) = \psi$. For $f \in H(\Omega)$, we finally get by Proposition 7.3.2:

$$g(\vartheta_\Omega)(f) \;=\; g\left(\vartheta_{\mathbb{C}_\infty \setminus \{1\}}(\Theta)\right) *_{\mathbb{C}_\infty \setminus \{1\},\,\Omega} f \;=\; \psi *_{\mathbb{C}_\infty \setminus \{1\},\,\Omega} f \;=\; H_{\psi,\Omega}(f).$$

This proves the assertion. □

Chapter 8

Open problems

In this chapter, we will state some problems that have arisen in the thesis.

8.1 Derived universality without further restrictions

In section 6.5, we posed Problem 6.5.1. An answer to this was the strong universality preservation theorem. But this required the condition that for every $G \in \mathcal{G}(\Omega * D)$ there exists a $\tilde{G} \in \mathcal{G}(D)$ with $\Omega * \tilde{G} = G$. It is not known whether this condition is also necessary to solve Problem 6.5.1. This gives rise to the following question.

Open problem no. 1:
Let $D \subsetneq \mathbb{C}$ be an open simply connected set with $0 \in D$. Moreover, let $\Omega \subset \mathbb{C}_\infty$ be an open set with $0 \in \Omega$ and $\infty \in \Omega$, $\psi \in H(\Omega)$, and $\varphi \in \mathscr{U}(D)$. Under what conditions is $\psi *_{\Omega,D} \varphi \in \mathscr{U}(\Omega * D)$? $\diamond$

8.2 Boundary behavior of universal functions

In chapter 6, we considered for an open simply connected set $D \subsetneq \mathbb{C}$ containing the origin the set $\mathscr{U}(D, G)$ of universal functions with respect to $G \in \mathcal{G}(D)$. In particular, this set G is open; and open sets are convenient for the Hadamard product. If we are given a compact exhaustion (with connected complements) of G, each of the compact sets is disjoint to the closure of D. In Proposition 5.4.5, we showed that $\tilde{\mathscr{U}}(D) = \mathscr{U}(D)$ for all open simply connected sets so that $\mathbb{C}_\infty \setminus \overline{D}^\infty$ is a simply connected domain. Therefore, for considerations merely concerning universality it does not matter whether we have the universal property on open simply connected sets outside D or whether we have it on compact sets with connected complements outside $\overline{D}$—at least for those D having the above property. But what happens if we allow the compact sets to meet the boundary? Let us introduce the following sets of universal functions (cf. [Nes96] and [MN01]):

- For $K \in \mathcal{M}(D)$, we denote by $\mathscr{U}_\partial(D, K)$ the set of all functions $\varphi \in H(\mathbb{D})$ so that for each $f \in A(K)$ there exists a subsequence of $(s_n(\varphi))_{n\in\mathbb{N}_0}$ that converges to f uniformly on K.
- By $\mathscr{U}_\partial(D)$ we denote the set of all functions $\varphi \in H(\mathbb{D})$ so that for each $K \in \mathcal{M}(D)$ and each $f \in A(K)$ there exists a subsequence of $(s_n(\varphi))_{n\in\mathbb{N}_0}$ that converges to f uniformly on K.

The only difference to the sets $\tilde{\mathscr{U}}(D, K)$ and $\tilde{\mathscr{U}}(D)$ is that the compact sets are now allowed to intersect the boundary of D.

We are interested in the following question.

Open problem no. 2:
Let $D \subsetneq \mathbb{C}$ be an open simply connected set with $0 \in D$. Moreover, let $K \in \mathcal{M}(D)$, $\Omega \subset \mathbb{C}_\infty$ an open set with $0 \in \Omega$ and $\infty \in \Omega$, $\psi \in H(\Omega)$, and $\varphi \in \mathscr{U}_\partial(D, K)$. Under what conditions is $\psi *_{\Omega,D} \varphi \in \mathscr{U}_\partial(\Omega * D, \Omega * K)$? $\diamond$

We remark that $\Omega * K \in \mathcal{M}(\Omega * D)$ if $K \in \mathcal{M}(D)$.

Intimately connected with this question is the following one.

Open problem no. 3:
Let $D \subsetneq \mathbb{C}$ be an open simply connected set with $0 \in D$. Moreover, let $\Omega \subset \mathbb{C}_\infty$ be an open set with $0 \in \Omega$ and $\infty \in \Omega$, $\psi \in H(\Omega)$, and $\varphi \in \mathscr{U}_\partial(D)$. Under what conditions is $\psi *_{\Omega,D} \varphi \in \mathscr{U}_\partial(\Omega * D)$? $\diamond$

Appendix A

On Hadamard cycles

The following table gives an overview of the possible Hadamard cycles (see also Table 2.1). Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be open and star-eligible sets, and let $z \in (\Omega_1 * \Omega_2) \setminus \{0, \infty\}$. The first row shows whether the origin or the point at infinity are contained in Ω_1. If the corresponding cell is empty, none of these points is contained in Ω_1. The first column has to be understood in the same way. The symbol "cc" stands for Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1; "cc$^+$" stands for Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 with $\operatorname{ind}(\gamma, 0) = 1$; "acc" stands for anti-Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1; and "acc$^-$" stands for anti-Cauchy cycle for $z \cdot \Omega_2^*$ in Ω_1 with $\operatorname{ind}(\gamma, 0) = -1$. A "/" means that this case cannot occur.

Ω_2 \ Ω_1	$0, \infty$	∞	0	
$0, \infty$	cc$^+$ *or* acc$^-$	acc$^-$	cc$^+$	cc
∞	acc$^-$	acc$^-$	/	/
0	cc$^+$	/	cc$^+$	/
	acc	/	/	/

Appendix B

On the compatibility theorem

We want to give an overview of the cases that can occur in the compatibility theorem. Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ as well as $D_1 \subset \Omega_1$ and $D_2 \subset \Omega_2$ be open and star-eligible sets. Furthermore, let $z \in (D_1 * D_2) \setminus \{0, \infty\}$ and γ a Hadamard cycle for $z \cdot D_2^*$ in D_1. We are interested in the question whether γ is also a Hadamard cycle for $z \cdot \Omega_2^*$ in Ω_1. In the following sections, we will list the possible cases.

B.1 The case $\mathbf{0} \in \mathbf{D_1} \cap \mathbf{D_2}$ and $\infty \notin \mathbf{D_1} \cap \mathbf{D_2}$

In this case, we get $0 \in \Omega_1 \cap \Omega_2$. Moreover, γ is a Cauchy cycle for $z \cdot D_2^*$ in D_1 with $\operatorname{ind}(\gamma, 0) = 1$. The first row of the following table shows in which of the sets the point at infinity is contained. The columns show what kind of cycle γ has to be to be a Hadamard cycle for $z \cdot \Omega_2^*$ in Ω_1.

∞	$\Omega_1 \cap \Omega_2$	$(\Omega_1 \cup \Omega_2)^C$	$\Omega_1 \setminus \Omega_2$	$\Omega_2 \setminus \Omega_1$
	cc$^+$ or acc$^-$	cc$^+$	cc$^+$	cc$^+$

B.2 The case $\mathbf{0} \notin \mathbf{D_1} \cap \mathbf{D_2}$ and $\infty \in \mathbf{D_1} \cap \mathbf{D_2}$

In this case, we get $\infty \in \Omega_1 \cap \Omega_2$. Moreover, γ is an anti-Cauchy cycle for $z \cdot D_2^*$ in D_1 with $\operatorname{ind}(\gamma, 0) = -1$. The following table has to be understood in the same way as the one in section B.1.

0	$\Omega_1 \cap \Omega_2$	$(\Omega_1 \cup \Omega_2)^C$	$\Omega_1 \setminus \Omega_2$	$\Omega_2 \setminus \Omega_1$
	cc^+ or acc^-	acc^-	acc^-	acc^-

B.3 The case $\mathbf{0 \in D_1 \cap D_2}$ and $\mathbf{\infty \in D_1 \cap D_2}$

In this case, we have $0 \in \Omega_1 \cap \Omega_2$ and $\infty \in \Omega_1 \cap \Omega_2$, and thus γ is a Hadamard cycle for $z \cdot \Omega_2^*$ in Ω_1.

B.4 The case $\mathbf{0 \in D_2 \setminus D_1}$ and $\mathbf{\infty \in D_2 \setminus D_1}$

In this case, we have $0 \in \Omega_2$ and $\infty \in \Omega_2$. Moreover, γ is a Cauchy cycle for $z \cdot D_2^*$ in D_1. The first row and column show which of the sets contain the origin and the point at infinity, respectively. The columns show what kind of cycle γ has to be to be a Hadamard cycle for $z \cdot \Omega_2^*$ in Ω_1.

∞ \ 0	Ω_1	Ω_1^C
Ω_1	cc^+ or acc^-	acc^-
Ω_1^C	cc^+	cc

B.5 The case $\mathbf{0 \in D_1 \setminus D_2}$ and $\mathbf{\infty \in D_1 \setminus D_2}$

In this case, we have $0 \in \Omega_1$ and $\infty \in \Omega_1$. Moreover, γ is an anti-Cauchy cycle for $z \cdot D_2^*$ in D_1. The following table has to be understood in the same way as the one in section B.4.

∞ \ 0	Ω_2	Ω_2^C
Ω_2	cc^+ or acc^-	acc^-
Ω_2^C	cc^+	acc

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