The BMAP/G/1 queue with level dependent arrivals: An extended queueing model for stations with nonrenewal and state dependent input traffic

Dissertation

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Introduction

Over the last years high speed communication networks have been an area of intensive research for engineers and scientists, leading to continuous improvements in technology. Today there is a rapidly growing demand for mobile communication systems, e.g. cellular phones or wireless computer networks, including protocols like CDMA (Code Division Multiple Access) and (wireless) ATM (Asynchronous Transfer Mode).

Along with these developments grew, and still grows, the need for more sophisticated methods of performance evaluation. Queueing theory provides already classical models for analysing communication networks, e.g. [22, 23]. One subject of modern queueing theory is matrix analytic methods [30, 31, 35]. Today research concentrates on the Batch Markovian Arrival Process (BMAP) [24], which has been shown to be equivalent to the Neuts process (versatile Markovian point process [29, 31]), and queueing systems with this arrival process. The BMAP is a generalization of the Poisson process, the Markov Modulated Poisson Process (MMPP) and the Markovian Arrival Process (MAP) [26]. The BMAP/G/1 queue has been analysed by Ramaswami [33] (there still called N/G/1 queue) and Lucantoni [24], and many variants have been studied later. A comprehensive survey can be found in Lucantoni's tutorial [25].

There are many publications about applications of matrix analytic methods, we give three examples:

- Heffes and Lucantoni [16] used an MMPP to study the performance of a statistical multiplexer whose input consists of a superposition of packetized voice sources and data. They approximated the superpositioned input process by an MMPP with suitably chosen parameters, and modeled the multiplexer as an MMPP/G/1 queue, where the service time of a packet is its transmission time.

- Blondia [5] described the input processes into an M/1-stage in an ATM switching element by Markovian arrival processes (MAPs) to allow bursty input traffic. The input buffers and the bus of the M/1-stage are modeled as a single server finite capacity multi–queueing system with non–exhaustive cyclic service. Each input queue can then be approximated by a MAP/D/1/N queue with repeated vacations and limited service discipline. In his analysis Blondia also allowed for a general service time distribution.

- Choudhury, Lucantoni and Whitt [10] consider an ATM switch receiving fixed–size ATM cells from several sources and transmitting them over an output channel in a first–in–first–out fashion. They develop an approximation for the tail probabilities of the steady–state waiting time, which can be used for admission control. Their approximation is based on the analysis of the MAP/G/1 queue, where the arrival process is a superposition of independent heterogeneous MAPs.

The arrival processes considered so far are spatially homogeneous and hence independent of the current state of the queueing system. But routing in modern communication networks like ATM can be dynamic, so that the input into a node depends on the current state of this node, e.g. the number and type of connections already assigned to that node or the remaining available bandwidth. In ATM networks cells and frames can be discarded to avoid congestion [38], and possible congestion can be indicated in ATM cells to reduce the intensity of available bit rate (ABR) traffic [38]. Routing algorithms like e.g. "Intelligent Network based Dynamic Routing" (Siemens Austria and European Space Agency) [2] are supposed to avoid congestion in advance and to achieve an efficient use of network resources.

This was our motivation to define a level dependent batch Markovian arrival process and to analyse the BMAP/G/1 queue with level dependent arrivals. There are many publications about queueing systems with state dependent input. A comprehensive survey is given by Dshalalow [12]. Recently Bright and Taylor [6, 7] introduced and analysed a level dependent Quasi–Birth–Death–process (QBD) using the matrix geometric approach for Markov chains of G/M/1 type [30]. Ramaswami and Taylor [37] generalized this process by allowing a countable number of phases. Using these results they obtained a new class of product–form queueing networks [36].

The BMAP/G/1 queue with level dependent arrivals is a generalization of the classical BMAP/G/1 queue mentioned above. It also includes the finite capacity BMAP/G/1 queue.

Summary

In the first part (section 1) of this paper we define a level dependent batch Markovian arrival process and derive some properties of its generator matrix and its transition probabilities. We show that the transition probability matrix of the level dependent BMAP satisfies the forward and backward differential equations, and hence is given by the matrix exponential of the generator matrix.

The second part (sections 2-7) is devoted to the analysis of the BMAP/G/1 queue with level dependent arrivals. This queueing system and the corresponding stochastic process are introduced in section 2. To compute the limiting distribution of the queue length we apply the common method of the embedded Markov chain [22]. In section 3 we determine the entries of the transition matrix of the embedded Markov chain and the mean number of arrivals during a service time.

Stability conditions for the BMAP/G/1 queue with level dependent arrivals are presented in section 4. These conditions are obtained by applying a generalized Foster criterion [32].

Section 5 contains the analysis of the fundamental periods of the embedded Markov chain. In the level independent case the fundamental matrix G plays the key role in determining the steady state distributions. In our case the fundamental periods depend on the starting level k. So we have fundamental matrices $G^{(k)}$ for each level $k \geq 1$. We derive two algorithms for computing these fundamental matrices. Further, we show that the vectors of the mean numbers of service completions during a fundamental period are the unique solution of an infinite system of linear equations.

The stationary distribution of the queue length at service completion times is determined in section 6. We apply a result from the theory of semi–Markov processes [20] to obtain the steady state probabilities of level 0. To compute the steady state probabilities of the remaining levels we generalize Ramaswami's formulae [34]. The limiting distribution of the queue length at an arbitrary time is derived in section 7 by applying the key renewal theorem.

In the third part (sections 8-10) we consider some special cases. At first we assume the phase process to be level independent (section 8). In this special instance we can improve some of our results from the general case. In particular, we derive a stronger stability condition. We finish this part by showing that our results coincide with those for the classical level independent BMAP/G/1 queue [24, 33] (section 9) and the finite capacity BMAP/G/1 queue [4] (section 10).

Finally, we give some directions for future research.

Some of our results have already been published and are listed in the bibliography for completeness [17, 18, 19].

Notations and conventions

Scalars (i.e. real or complex numbers) are denoted by lower case latin letters, sometimes by lower case greek letters. In particular, i, j, k, l, m, n, r, and ν are usually integers, while t is usually a nonnegative real number ("time"). For vectors we use boldfaced lower case latin letters and sometimes also lower case greek letters. We do not distinguish in notation between row and column vectors. In particular, $\mathbf{e}$ ($\mathbf{e}_\infty$) is a column vector (sequence) of ones and $\mathbf{0}$ is a column vector of zeros. Matrices of finite size are denoted by upper case latin letters, block matrices of finite or infinite size by calligraphic upper case letters. In particular, I and $\mathcal{I}$ denote the identity matrix, O and $\mathcal{O}$ a matrix of zeros. When we represent block matrices we usually omit blocks of zeros.

The components of a vector are usually denoted by the same letter as the vector, but not boldfaced. So the ith component of $\mathbf{a}$ is a_i. If the vector $\mathbf{a}$ consists of blocks, its nth block is $[\mathbf{a}]_n$. The (i, j)th entry of a matrix M is marked by $(M)_{ij}$ and analogously the (n, ν)th block of a block matrix $\mathcal{M}$ is $(\mathcal{M})_{n\nu}$. If $\mathbf{a}$ and $\mathbf{b}$ are vectors, we say that $\mathbf{a} \leq \mathbf{b}$ if and only if $a_i \leq b_i$ for all i. Analogously, we say that $\mathbf{a} < \infty\mathbf{e}$ if and only if $a_i < \infty$ for all i.

Stochastic processes are also marked by calligraphic upper case letters.

The set of positive integers ("natural numbers") is denoted by $\mathbb{N}$ and the set of nonnegative integers by $\mathbb{N}_0$.

We define the empty sum to be zero and the empty product to be unity. A product of matrices shall always be arranged with the lowest index left, i.e. $\prod_{i=1}^{n} A_i = A_1 A_2 \cdots A_n$.

Probabilities $\mathrm{P}(\cdot)$ and moments of random variables $\mathrm{E}(X^n)$ are always defined on an underlying probability space.

The end of a proof is marked by a box ($\Box$).

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1 The arrival process

1.1 Definition and properties of the generator matrix

The classical Batch Markovian Arrival Process (BMAP) is defined by a sequence of $m \times m$ matrices $\{D_n : n \in \mathbb{N}_0\}$, where [24]

a) D_0 has negative diagonal elements and nonnegative off–diagonal elements,

b) D_n, $n \geq 1$, are nonnegative and

c) $D = \sum_{n=0}^{\infty} D_n$ is an irreducible generator matrix.

The BMAP is a 2-dimensional Markov process $\{N(t), J(t) : t \geq 0\}$ with state space $\mathbb{N}_0 \times \{1, \ldots, m\}$, where $N(t)$, the level, counts the number of arrivals up to time t and $J(t)$ is the socalled phase at time t. The number of phases, m, is usually assumed to be finite. The generator matrix $\mathcal{Q}$ of the BMAP $\{N(t), J(t) : t \geq 0\}$ is given by

$$\mathcal{Q} = \begin{pmatrix} D_0 & D_1 & D_2 & D_3 & \cdots \\ & D_0 & D_1 & D_2 & \cdots \\ & & D_0 & D_1 & \cdots \\ & & & D_0 & \cdots \\ & & & & \ddots \end{pmatrix}$$

(empty entries shall represent the zero matrix O).

We will now define an arrival process similar to the BMAP above, but with the additional property, that the phase process and the arrival rates depend on the current level, i.e. on the number of arrivals which have already taken place. Assume we are given a sequence $\{\mathbf{J}^{(k)} : k \in \mathbb{N}_0\}$ of finite nonempty sets with cardinalities $m^{(k)} = \sharp\mathbf{J}^{(k)}$, $k \in \mathbb{N}_0$, and

$$m := \sup\{m^{(k)} : k \in \mathbb{N}_0\} < \infty.$$

For simplicity we let $\mathbf{J}^{(k)} = \{1, \ldots, m^{(k)}\}$, this can be achieved by means of a suitable bijection $\chi^{(k)} : \mathbf{J}^{(k)} \to \{1, \ldots, m^{(k)}\}$. The set $\mathbf{J}^{(k)}$ shall be the state space of the phase process in level k.

Further, we are given sequences $\{D_n^{(k)} : n \in \mathbb{N}_0\}$ of $m^{(k)} \times m^{(k+n)}$ matrices

$$D_n^{(k)} = \left(d_{n,ij}^{(k)} \right)_{\substack{i=1,\ldots,m^{(k)},\\ j=1,\ldots,m^{(k+n)}}}$$

for $k \in \mathbb{N}_0$, that shall be the matrices of the arrival rates in level k. So

a) $D_0^{(k)}$ has negative diagonal elements and nonnegative off–diagonal elements,

b) $D_n^{(k)}$, $n \geq 1$, are nonnegative and

c) $\sum_{n=0}^{\infty} D_n^{(k)}\mathbf{e} = \mathbf{0}$, i.e. $\sum_{n=0}^{\infty} \sum_{j=1}^{m^{(k+n)}} d_{n,ij}^{(k)} = 0$, $i = 1, \ldots, m^{(k)}$,

for all k. We restrict ourselves to the nontrivial case that there exists some $n \geq 1$ such that $D_n^{(0)} \neq O$ or equivalently $D_0^{(0)}\mathbf{e} \neq \mathbf{0}$, i e. there are arrivals in level zero[1]. This need not hold for $k \geq 1$. Our assumptions imply that $D_0^{(0)}$ is nonsingular (corollary 1.17).

Further, we need to assume that all rates are bounded, i.e.

Assumption 1.1

$$\bar{d}_0 := \sup\left\{|d_{0,ii}^{(k)}| : k \in I\!N_0,\ i = 1, \ldots, m^{(k)}\right\} < \infty\,.$$

Assumption 1.1 should be fulfilled in all applications.

The sequences $\{D_n^{(k)} : n \in I\!N_0\}$ enable us to define for each $k \in I\!N_0$ a *level dependent BMAP* $(\mathcal{N}^{(k)}, \mathcal{J}^{(k)}) = \{N^{(k)}(t), J^{(k)}(t) : t \geq 0\}$ with generator matrix

$$\mathcal{Q}^{(k)} = \begin{pmatrix} D_0^{(k)} & D_1^{(k)} & D_2^{(k)} & D_3^{(k)} & \cdots \\ & D_0^{(k+1)} & D_1^{(k+1)} & D_2^{(k+1)} & \cdots \\ & & D_0^{(k+2)} & D_1^{(k+2)} & \cdots \\ & & & D_0^{(k+3)} & \cdots \\ & & & & \ddots \end{pmatrix}.$$

So $\mathcal{Q}^{(k)}$ describes a time–homogeneous 2-dimensional Markov process whose first component $N^{(k)}(t)$, the level, counts the number of arrivals during an interval of length t, and whose second component $J^{(k)}(t)$ is the phase at time t. The state space of $(\mathcal{N}^{(k)}, \mathcal{J}^{(k)})$ shall be the set of all pairs $(k+n, i)$ with $n \in I\!N_0$ and $i \in \mathbf{J}^{(k+n)}$.

Note that the processes $(\mathcal{N}^{(k)}, \mathcal{J}^{(k)})$ and $(\mathcal{N}^{(k+1)}, \mathcal{J}^{(k+1)})$ are "nested" in the following way:

$$\mathcal{Q}^{(k)} = \left(\begin{array}{c|ccc} D_0^{(k)} & D_1^{(k)} & D_2^{(k)} & \cdots \\ \hline O & & & \\ O & & \mathcal{Q}^{(k+1)} & \\ \vdots & & & \end{array}\right).$$

Thus $(\mathcal{N}^{(k)}, \mathcal{J}^{(k)})$ is from its level n on stochastically identical to $(\mathcal{N}^{(k+n)}, \mathcal{J}^{(k+n)})$.

[1] When we will consider queueing systems this condition assures that there are arrivals to the empty system.

1.1.1 The arrival rates

Let $\mathbf{d}^{(k)} = (d_i^{(k)})_{i=1,\ldots,m^{(k)}}$ denote the vector of the phase dependent mean arrival rates in level k, i.e.

$$\mathbf{d}^{(k)} := \sum_{n=1}^{\infty} nD_n^{(k)}\mathbf{e}$$

for $k \in I\!N_0$. We assume the arrival rates to be bounded, i.e.

Assumption 1.2

$$\sup\left\{d_i^{(k)} : k \in I\!N_0,\ i = 1, \ldots, m^{(k)}\right\} < \infty .$$

Note that assumption 1.2 implies assumption 1.1 if $d_{0,ij}^{(k)} < \infty$ for all $j \neq i$ and all $k \in I\!N_0$ (this should always be fulfilled).

The maximum arrival rate from level k on will be denoted by $\bar{\lambda}^{(k)}$, i.e.

$$\bar{\lambda}^{(k)} := \sup\left\{d_i^{(l)} : l \geq k,\ i = 1, \ldots, m^{(l)}\right\} .$$

Assumption 1.2 implies

$$\bar{\lambda}^{(0)} = \sup\left\{\bar{\lambda}^{(k)} : k \in I\!N_0\right\} < \infty . \tag{1.1}$$

1.1.2 Matrices of infinite size

To establish some properties of the matrices $\mathcal{Q}^{(k)}$ we need the following results concerning matrices of infinite size.

Definition 1.3 *A matrix $\mathcal{M} = (m_{ij})_{i,j \in I\!N}$ is called bounded, if*

$$\|\mathcal{M}\| := \sup_{i \in I\!N}\left[|\mathcal{M}|\,\mathbf{e}_\infty\right]_i := \sup_{i \in I\!N}\sum_{j=1}^{\infty}|m_{ij}| < \infty .$$

A sequence $\{\mathcal{M}_k : k \in I\!N_0\}$ of matrices is called uniformly bounded, if there exists $\theta < \infty$ such that $\|\mathcal{M}_k\| \leq \theta$ for all $k \in I\!N_0$.

Remark: If the matrix $\mathcal{M}$ is bounded, the product of $\mathcal{M}$ and any bounded sequence $\{\xi_0, \xi_1, \xi_2, \ldots\}$ of real or complex numbers ξ_i with $\sup_{i \in I\!N_0}|\xi_i| < \infty$ is again a bounded sequence (by theorem 2.6 in [27, p. 10]).

Lemma 1.4 *If $\mathcal{M}_1$ and $\mathcal{M}_2$ are bounded matrices, then $\mathcal{M}_1\mathcal{M}_2$ and $\mathcal{M}_1+\mathcal{M}_2$ are also bounded and*

$$\begin{aligned} \|\mathcal{M}_1\mathcal{M}_2\| &\leq \|\mathcal{M}_1\|\cdot\|\mathcal{M}_2\|, \\ \|\mathcal{M}_1+\mathcal{M}_2\| &\leq \|\mathcal{M}_1\|+\|\mathcal{M}_2\|. \end{aligned}$$

The exponential $e^{\mathcal{M}}$ of a bounded matrix $\mathcal{M}$ exists, i.e. its entries are all finite, and is also bounded with

$$\|e^{\mathcal{M}}\| \leq e^{\|\mathcal{M}\|}.$$

The first part of lemma 1.4 can be found in [11, p. 26]. The existence of the exponential of a bounded matrix can also be found in [11, p. 38], its boundedness is an immediate consequence of the first result:

$$\|e^{\mathcal{M}}\| = \left\|\sum_{\nu=0}^{\infty}\frac{1}{\nu!}\mathcal{M}^{\nu}\right\| \leq \sum_{\nu=0}^{\infty}\frac{1}{\nu!}\|\mathcal{M}\|^{\nu} = e^{\|\mathcal{M}\|}.$$

Remark: If M is a matrix of finite size, $\|M\|$ as defined in definition 1.3 is a norm. The assertions of lemma 1.4 also hold for matrices of finite size.

1.1.3 Properties of the generator matrix

We will need the following properties of the matrices $\mathcal{Q}^{(k)}$.

Lemma 1.5 *The matrices $\mathcal{Q}^{(k)}$, $k \in I\!N_0$, are uniformly bounded.*

Proof: Assumption 1.1 implies for all $k \in I\!N_0$ and $i = 1, \ldots, m^{(k)}$:

$$\sum_{n=0}^{\infty}\sum_{j=1}^{m^{(k+n)}} |d_{n,ij}^{(k)}| = |d_{0,ii}^{(k)}| + \sum_{\substack{j=1 \\ j\neq i}}^{m^{(k)}} |d_{0,ij}^{(k)}| + \sum_{n=1}^{\infty}\sum_{j=1}^{m^{(k+n)}} |d_{n,ij}^{(k)}| = 2\,|d_{0,ii}^{(k)}| \leq 2\bar{d}_0$$

since $\mathcal{Q}^{(k)}$ is a generator matrix. Thus the lth block of $|\mathcal{Q}^{(k)}|\mathbf{e}_\infty$ satisfies

$$\left[|\mathcal{Q}^{(k)}|\mathbf{e}_\infty\right]_l = \sum_{n=0}^{\infty}|D_n^{(k+l)}|\mathbf{e} \leq 2\bar{d}_0\,\mathbf{e}.$$

So the matrices $\mathcal{Q}^{(k)}$ are uniformly bounded. □

Corollary 1.6 *The exponentials of the matrices $\mathcal{Q}^{(k)}$ exist, i.e. their entries are all finite, and are uniformly bounded.*

Proof: The assertion is an immediate consequence of the lemmata 1.5 and 1.4 □

Lemma 1.7 *For all $j, l, n \in I\!N_0$ with $l \leq n$ it is*

$$\left((\mathcal{Q}^{(k)})^j\right)_{ln} = \left((\mathcal{Q}^{(k+l)})^j\right)_{0,n-l}.$$

Proof: At first we note that $(\mathcal{Q}^{(k)})^j$ is a block triangular matrix, since $\mathcal{Q}^{(k)}$ is block triangular. Let $l, n \in I\!N_0$, $l \leq n$, and perform induction over j: For $j = 0$ the result is obvious, and for $j = 1$ we have

$$\left(\mathcal{Q}^{(k)}\right)_{ln} = D_{n-l}^{(k+l)} = \left(\mathcal{Q}^{(k+l)}\right)_{0,n-l}.$$

Suppose the assertion is proven for j and consider $j + 1$:

$$\begin{aligned}
\left((\mathcal{Q}^{(k)})^{j+1}\right)_{ln} &= \sum_{\nu=0}^{\infty} \left(\mathcal{Q}^{(k)}\right)_{l\nu} \left((\mathcal{Q}^{(k)})^j\right)_{\nu n} \\
&= \sum_{\nu=l}^{n} \left(\mathcal{Q}^{(k)}\right)_{l\nu} \left((\mathcal{Q}^{(k)})^j\right)_{\nu n} \\
&= \sum_{\nu=l}^{n} \left(\mathcal{Q}^{(k+l)}\right)_{0,\nu-l} \left((\mathcal{Q}^{(k+\nu)})^j\right)_{0,n-\nu} \\
&\quad \text{by induction assumption} \\
&= \sum_{\nu=0}^{n-l} \left(\mathcal{Q}^{(k+l)}\right)_{0\nu} \left((\mathcal{Q}^{(k+\nu+l)})^j\right)_{0,n-\nu-l}
\end{aligned}$$

and

$$\begin{aligned}
\left((\mathcal{Q}^{(k+l)})^{j+1}\right)_{0,n-l} &= \sum_{\nu=0}^{\infty} \left(\mathcal{Q}^{(k+l)}\right)_{0\nu} \left((\mathcal{Q}^{(k+l)})^j\right)_{\nu,n-l} \\
&= \sum_{\nu=0}^{n-l} \left(\mathcal{Q}^{(k+l)}\right)_{0\nu} \left((\mathcal{Q}^{(k+l)})^j\right)_{\nu,n-l} \\
&= \sum_{\nu=0}^{n-l} \left(\mathcal{Q}^{(k+l)}\right)_{0\nu} \left((\mathcal{Q}^{(k+l+\nu)})^j\right)_{0,n-l-\nu} \\
&\quad \text{by induction assumption.}
\end{aligned}$$

So $\left((\mathcal{Q}^{(k)})^{j+1}\right)_{ln} = \left((\mathcal{Q}^{(k+l)})^{j+1}\right)_{0,n-l}$ and thus the assertion is proven for all $j \in I\!N_0$. □

Corollary 1.8 *For all $l, n \in I\!N_0$ with $l \leq n$ and all $t \geq 0$ it is*

$$\left(e^{\mathcal{Q}^{(k)}t}\right)_{ln} = \left(e^{\mathcal{Q}^{(k+l)}t}\right)_{0,n-l}.$$

Proof: Lemma 1.7 implies

$$\left(e^{\mathcal{Q}^{(k)}t}\right)_{ln} = \sum_{j=0}^{\infty} \frac{t^j}{j!}\left(\left(\mathcal{Q}^{(k)}\right)^j\right)_{ln} = \sum_{j=0}^{\infty} \frac{t^j}{j!}\left(\left(\mathcal{Q}^{(k+l)}\right)^j\right)_{0,n-l} = \left(e^{\mathcal{Q}^{(k+l)}t}\right)_{0,n-l}.$$

□

1.2 The transition probabilities

The transition probabilities of the Markov process $(\mathcal{N}^{(k)}, \mathcal{J}^{(k)})$ will play an important role in the subsequent analysis:

$$\begin{aligned}
p_{n,ij}^{(k)}(t) &:= \mathrm{P}\{N^{(k)}(t) = n,\ J^{(k)}(t) = j \mid N^{(k)}(0) = 0,\ J^{(k)}(0) = i\}, \\
P_n^{(k)}(t) &:= \left(p_{n,ij}^{(k)}(t)\right)_{\substack{i=1,\dots,m^{(k)}, \\ j=1,\dots,m^{(k+n)}}}, \\
\mathcal{P}^{(k)}(t) &:= \begin{pmatrix}
P_0^{(k)}(t) & P_1^{(k)}(t) & P_2^{(k)}(t) & P_3^{(k)}(t) & \cdots \\
 & P_0^{(k+1)}(t) & P_1^{(k+1)}(t) & P_2^{(k+1)}(t) & \cdots \\
 & & P_0^{(k+2)}(t) & P_1^{(k+2)}(t) & \cdots \\
 & & & P_0^{(k+3)}(t) & \cdots \\
 & & & & \ddots
\end{pmatrix}.
\end{aligned}$$

This definition implies that the matrices $\mathcal{P}^{(k)}(t)$ are uniformly bounded.

1.2.1 Properties of the transition matrices

As in the level independent case [24] the transition matrices fulfil the Chapman–Kolmogorov equation and the backward and forward differential equations.

Lemma 1.9 (Chapman–Kolmogorov equation)
The matrices $P_n^{(k)}(s+t)$ and $\mathcal{P}^{(k)}(s+t)$ satisfy

$$\begin{aligned}
P_n^{(k)}(s+t) &= \sum_{u=0}^{n} P_u^{(k)}(s)\, P_{n-u}^{(k+u)}(t), \\
\mathcal{P}^{(k)}(s+t) &= \mathcal{P}^{(k)}(s)\, \mathcal{P}^{(k)}(t).
\end{aligned}$$

Proof: By conditioning on the number of arrivals during $[0, s]$ and the phase at time s we obtain

$$p_{n,ij}^{(k)}(s+t) = \sum_{u=0}^{n} \sum_{l=1}^{m^{(k+u)}} p_{u,il}^{(k)}(s)\, p_{n-u,lj}^{(k+u)}(t)\,.$$

This yields the assertion. □

Lemma 1.10 (Backward differential equations)
The matrices $P_n^{(k)}(t)$ are the unique solution of the backward differential equations

$$\tfrac{d}{dt} P_n^{(k)}(t) = \sum_{u=0}^{n} D_u^{(k)} P_{n-u}^{(k+u)}(t) \tag{1.2}$$

with $P_0^{(k)}(0) = I$ and $P_n^{(k)}(0) = O$ for $n \geq 1$.

Proof: By lemma 1.9 we have for $\Delta t > 0$

$$\begin{aligned} p_{n,ij}^{(k)}(t+\Delta t) - p_{n,ij}^{(k)}(t) &= \sum_{u=0}^{n} \sum_{l=1}^{m^{(k+u)}} p_{u,il}^{(k)}(\Delta t)\, p_{n-u,lj}^{(k+u)}(t) - p_{n,ij}^{(k)}(t) \\ &= \sum_{u=0}^{n} \sum_{l=1}^{m^{(k+u)}} \left(p_{u,il}^{(k)}(\Delta t) - \delta_{(u,l),(0,i)}\right) p_{n-u,lj}^{(k+u)}(t)\,, \end{aligned}$$

where $\delta_{(u,l),(0,i)} = 1$ if $(u, l) = (0, i)$ and $\delta_{(u,l),(0,i)} = 0$ otherwise. Thus

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(p_{n,ij}^{(k)}(t+\Delta t) - p_{n,ij}^{(k)}(t)\right) = \sum_{u=0}^{n} \sum_{l=1}^{m^{(k+u)}} d_{u,il}^{(k)}\, p_{n-u,lj}^{(k+u)}(t) = \left(\sum_{u=0}^{n} D_u^{(k)} P_{n-u}^{(k+u)}(t)\right)_{ij}.$$

So we have shown that the matrices $P_n^{(k)}(t)$ fulfil the backward differential equations (1.2). From the definition of the matrices $P_n^{(k)}(t)$ we obtain immediately that $P_0^{(k)}(0) = I$ and $P_n^{(k)}(0) = O$ for $n \geq 1$.

To prove the claimed uniqueness we proceed similarly to Bellman [3, p. 167ff]. Assume there is another solution $S_n^{(k)}(t)$. Equation (1.2) yields

$$\begin{aligned} P_n^{(k)}(t) &= \delta_{n,0} I + \int_0^t \sum_{u=0}^{n} D_u^{(k)} P_{n-u}^{(k+u)}(s)\, ds\,, \\ S_n^{(k)}(t) &= \delta_{n,0} I + \int_0^t \sum_{u=0}^{n} D_u^{(k)} S_{n-u}^{(k+u)}(s)\, ds\,, \end{aligned}$$

where $\delta_{n,0} = 1$ if $n = 0$ and $\delta_{n,0} = 0$ otherwise. Thus

$$P_n^{(k)}(t) - S_n^{(k)}(t) \;=\; \sum_{u=0}^{n} D_u^{(k)} \int_0^t \left(P_{n-u}^{(k+u)}(s) - S_{n-u}^{(k+u)}(s)\right) ds\,.$$

Lemma 1.4 implies

$$\|P_n^{(k)}(t) - S_n^{(k)}(t)\| \;\le\; \sum_{u=0}^{n} \|D_u^{(k)}\| \int_0^t \|P_{n-u}^{(k+u)}(s) - S_{n-u}^{(k+u)}(s)\|\, ds\,.$$

We show that $\|P_n^{(k)}(t) - S_n^{(k)}(t)\| = 0$ for all $n, k \in I\!N_0$ and all $t \ge 0$ by induction over n. For $n = 0$ we obtain

$$\|P_0^{(k)}(t) - S_0^{(k)}(t)\| \;\le\; \|D_0^{(k)}\| \int_0^t \|P_0^{(k)}(s) - S_0^{(k)}(s)\|\, ds\,. \tag{1.3}$$

For any $t_0 \ge 0$ we have $c := \sup_{0 \le t \le t_0} \|P_0^{(k)}(t) - S_0^{(k)}(t)\| < \infty$, because $P_0^{(k)}(t)$ and $S_0^{(k)}(t)$ are differentiable. Thus from (1.3):

$$\|P_0^{(k)}(t) - S_0^{(k)}(t)\| \;\le\; \|D_0^{(k)}\|\, c\, t \quad \text{for all } 0 \le t \le t_0.$$

Using this in (1.3) yields

$$\|P_0^{(k)}(t) - S_0^{(k)}(t)\| \le \|D_0^{(k)}\|^2 c \int_0^t s\, ds \;=\; \frac{c\,\|D_0^{(k)}\|^2 t^2}{2} \quad \text{for all } 0 \le t \le t_0.$$

Repeating this substitution ν–times we get

$$\|P_0^{(k)}(t) - S_0^{(k)}(t)\| \;\le\; \frac{c\,\|D_0^{(k)}\|^{\nu+2} t^{\nu+2}}{(\nu+2)!} \quad \text{for all } 0 \le t \le t_0.$$

So $\lim_{\nu\to\infty} \|P_0^{(k)}(t) - S_0^{(k)}(t)\| \le 0$ for all $0 \le t \le t_0$ and thus, since t_0 was arbitrarily chosen, $\|P_0^{(k)}(t) - S_0^{(k)}(t)\| = 0$ for all $t \ge 0$.

Now assume $\|P_u^{(k)}(t) - S_u^{(k)}(t)\| = 0$ for all $u \le n, k \in I\!N_0$, $t \ge 0$, then

$$\begin{aligned}
\|P_{n+1}^{(k)}(t) - S_{n+1}^{(k)}(t)\| \;&\le\; \sum_{u=0}^{n+1} \|D_u^{(k)}\| \int_0^t \|P_{n+1-u}^{(k+u)}(s) - S_{n+1-u}^{(k+u)}(s)\|\, ds \\
&=\; \|D_0^{(k)}\| \int_0^t \|P_{n+1}^{(k)}(s) - S_{n+1}^{(k)}(s)\|\, ds\,.
\end{aligned}$$

By the same argument as above we obtain $\|P_{n+1}^{(k)}(t) - S_{n+1}^{(k)}(t)\| = 0$ for all $t \ge 0$. Altogether we have $P_n^{(k)}(t) = S_n^{(k)}(t)$ for all $n, k \in I\!N_0$ and all $t \ge 0$. $\square$

Lemma 1.11 (Forward differential equations)
The matrices $P_n^{(k)}(t)$ are the unique solution of the forward differential equations

$$\tfrac{d}{dt}P_n^{(k)}(t) = \sum_{u=0}^{n} P_u^{(k)}(t)\, D_{n-u}^{(k+u)} \tag{1.4}$$

with $P_0^{(k)}(0) = I$ and $P_n^{(k)}(0) = O$ for $n \geq 1$.

Proof: Lemma 1.9 yields for $\Delta t > 0$

$$\begin{aligned} p_{n,ij}^{(k)}(t+\Delta t) - p_{n,ij}^{(k)}(t) &= \sum_{u=0}^{n} \sum_{l=1}^{m^{(k+u)}} p_{u,il}^{(k)}(t)\, p_{n-u,lj}^{(k+u)}(\Delta t) - p_{n,ij}^{(k)}(t) \\ &= \sum_{u=0}^{n} \sum_{l=1}^{m^{(k+u)}} p_{u,il}^{(k)}(t) \left(p_{n-u,lj}^{(k+u)}(\Delta t) - \delta_{(u,l),(n,j)} \right), \end{aligned}$$

where $\delta_{(u,l),(n,j)} = 1$ if $(u,l) = (n,j)$ and $\delta_{(u,l),(n,j)} = 0$ otherwise. Thus

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \left(p_{n,ij}^{(k)}(t+\Delta t) - p_{n,ij}^{(k)}(t) \right) = \sum_{u=0}^{n} \sum_{l=1}^{m^{(k+u)}} p_{u,il}^{(k)}(t)\, d_{n-u,lj}^{(k+u)} = \left(\sum_{u=0}^{n} P_u^{(k)}(t)\, D_{n-u}^{(k+u)} \right)_{ij}.$$

So we have shown that the matrices $P_n^{(k)}(t)$ fulfil the forward differential equations (1.4). The definition of the matrices $P_n^{(k)}(t)$ implies that $P_0^{(k)}(0) = I$ and $P_n^{(k)}(0) = O$ for $n \geq 1$.

The proof of the claimed uniqueness is essentially the same as for the backward differential equations (lemma 1.10) and will just be given for completeness. Assume there is another solution $S_n^{(k)}(t)$. From (1.4) we obtain

$$\begin{aligned} P_n^{(k)}(t) &= \delta_{n,0} I + \int_0^t \sum_{u=0}^{n} P_u^{(k)}(s)\, D_{n-u}^{(k+u)}\, ds, \\ S_n^{(k)}(t) &= \delta_{n,0} I + \int_0^t \sum_{u=0}^{n} S_u^{(k)}(s)\, D_{n-u}^{(k+u)}\, ds, \end{aligned}$$

where $\delta_{n,0} = 1$ if $n = 0$ and $\delta_{n,0} = 0$ otherwise. Thus

$$P_n^{(k)}(t) - S_n^{(k)}(t) = \sum_{u=0}^{n} \int_0^t \left(P_u^{(k)}(s) - S_u^{(k)}(s) \right) ds\, D_{n-u}^{(k+u)}.$$

Lemma 1.4 implies

$$\| P_n^{(k)}(t) - S_n^{(k)}(t) \| \leq \sum_{u=0}^{n} \int_0^t \| P_u^{(k)}(s) - S_u^{(k)}(s) \|\, ds\, \| D_{n-u}^{(k+u)} \|.$$

We show that $\|P_n^{(k)}(t) - S_n^{(k)}(t)\| = 0$ for all $n, k \in I\!N_0$ and all $t \geq 0$ by induction over n. For $n = 0$ we obtain

$$\|P_0^{(k)}(t) - S_0^{(k)}(t)\| \;\leq\; \int_0^t \|P_0^{(k)}(s) - S_0^{(k)}(s)\|\, ds\, \|D_0^{(k)}\|\,. \tag{1.5}$$

For any $t_0 \geq 0$ we have $c' := \sup_{0\leq t\leq t_0} \|P_0^{(k)}(t) - S_0^{(k)}(t)\| < \infty$, and so

$$\|P_0^{(k)}(t) - S_0^{(k)}(t)\| \;\leq\; c'\, t\, \|D_0^{(k)}\| \quad \text{for all } 0 \leq t \leq t_0.$$

Using this in (1.5) yields

$$\|P_0^{(k)}(t) - S_0^{(k)}(t)\| \leq c' \int_0^t s\, ds\, \|D_0^{(k)}\|^2 \;=\; \frac{c'\, \|D_0^{(k)}\|^2 t^2}{2} \quad \text{for all } 0 \leq t \leq t_0.$$

Repeating this substitution ν–times we get

$$\|P_0^{(k)}(t) - S_0^{(k)}(t)\| \;\leq\; \frac{c'\, \|D_0^{(k)}\|^{\nu+2} t^{\nu+2}}{(\nu+2)!} \quad \text{for all } 0 \leq t \leq t_0.$$

So $\lim_{\nu\to\infty} \|P_0^{(k)}(t) - S_0^{(k)}(t)\| \leq 0$ for all $0 \leq t \leq t_0$ and thus, since t_0 was arbitrarily chosen, $\|P_0^{(k)}(t) - S_0^{(k)}(t)\| = 0$ for all $t \geq 0$.

Now assume $\|P_u^{(k)}(t) - S_u^{(k)}(t)\| = 0$ for all $u \leq n, k \in I\!N_0$, $t \geq 0$, then

$$\begin{aligned}
\|P_{n+1}^{(k)}(t) - S_{n+1}^{(k)}(t)\| \;&\leq\; \sum_{u=0}^{n+1} \int_0^t \|P_u^{(k)}(s) - S_u^{(k)}(s)\|\, ds\, \|D_{n+1-u}^{(k+u)}\| \\
&=\; \int_0^t \|P_{n+1}^{(k)}(s) - S_{n+1}^{(k)}(s)\|\, ds\, \|D_0^{(k+n+1)}\|\,.
\end{aligned}$$

By the same argument as above we obtain $\|P_{n+1}^{(k)}(t) - S_{n+1}^{(k)}(t)\| = 0$ for all $t \geq 0$. Altogether we have $P_n^{(k)}(t) = S_n^{(k)}(t)$ for all $n, k \in I\!N_0$ and all $t \geq 0$. $\square$

Corollary 1.12 *The matrices $\mathcal{P}^{(k)}(t)$ satisfy*

$$\tfrac{d}{dt}\mathcal{P}^{(k)}(t) \;=\; \mathcal{Q}^{(k)}\mathcal{P}^{(k)}(t) \;=\; \mathcal{P}^{(k)}(t)\, \mathcal{Q}^{(k)}$$

and $\mathcal{P}^{(k)}(0) = \mathcal{I}$.

Proof: The assertion follows immediately from lemmata 1.10 and 1.11. $\square$

Remark: Corollary 1.12 implies that the matrices $\frac{d}{dt}\mathcal{P}^{(k)}(t)$ are bounded, because $\mathcal{P}^{(k)}(t)$ and $\mathcal{Q}^{(k)}$ are bounded.

The forward differential equations assure that $P_n^{(k)}(t)$, $n \in I\!N_0$, is a proper distribution (cf. Feller [13, p. 451ff]):

Lemma 1.13 *The number of arrivals in time t is finite with probability 1, i.e.*

$$\sum_{n=0}^{\infty} P_n^{(k)}(t)\,\mathbf{e} = \mathbf{e}$$

for all $k \in I\!N_0$ and all $t \geq 0$.

Proof: Let

$$\mathbf{s}_N^{(k)}(t) := \sum_{n=0}^{N} P_n^{(k)}(t)\mathbf{e}\,,$$

with components $s_{N,i}^{(k)}(t)$, $i = 1, \ldots, m^{(k)}$, then

$$s_{N,i}^{(k)}(t) \;=\; \mathrm{P}\{N^{(k)}(t) \leq N \mid N^{(k)}(0) = 0, J^{(k)}(0) = i\}\,,$$

and so the sequence $\{\mathbf{s}_N^{(k)}(t) : N \in I\!N_0\}$ is componentwise nondecreasing and bounded by $\mathbf{e}$. Thus $\lim_{N\to\infty} \mathbf{s}_N^{(k)}(t) - \mathbf{e}$ exists. To prove the assertion we need to show that $\lim_{N\to\infty} \mathbf{s}_N^{(k)}(t) - \mathbf{e} = \mathbf{0}$.

Lemma 1.11 yields

$$\tfrac{d}{dt}\mathbf{s}_N^{(k)}(t) \;=\; \sum_{n=0}^{N}\sum_{u=0}^{n} P_u^{(k)}(t)\, D_{n-u}^{(k+u)}\mathbf{e}$$

and so (since $\mathbf{s}_N^{(k)}(0) = \mathbf{e}$)

$$\mathbf{s}_N^{(k)}(t) - \mathbf{e} \;=\; \int_0^t \tfrac{d}{ds}\mathbf{s}_N^{(k)}(s)\,ds \;=\; \sum_{n=0}^{N}\sum_{u=0}^{n}\left(\int_0^t P_u^{(k)}(s)\,ds\right) D_{n-u}^{(k+u)}\mathbf{e}\,.$$

The matrix $\mathcal{U} = \left(\int_0^t P_u^{(k)}(s)\,ds\right)_{k,u\in I\!N_0}$ is bounded, because

$$\sum_{u=0}^{\infty}\int_0^t P_u^{(k)}(s)\,\mathbf{e}\,ds \;=\; \int_0^t \sum_{u=0}^{\infty} P_u^{(k)}(s)\,\mathbf{e}\,ds \;\leq\; \int_0^t \mathbf{e}\,ds \;=\; t\mathbf{e}\,.$$

Therefore the matrix $\mathcal{U}\mathcal{Q}^{(k)}$ is bounded and this implies that the series

$$\sum_{n=0}^{\infty}\sum_{u=0}^{n}\left(\int_0^t P_u^{(k)}(s)\,ds\right) D_{n-u}^{(k+u)}\mathbf{e}$$

converges absolutely. So we can apply the theorem of Fubini:

$$\begin{aligned}
\lim_{N\to\infty} \mathbf{s}_N^{(k)}(t) - \mathbf{e} &= \sum_{n=0}^{\infty}\sum_{u=0}^{n}\left(\int_0^t P_u^{(k)}(s)\,ds\right) D_{n-u}^{(k+u)}\mathbf{e} \\
&= \sum_{u=0}^{\infty}\sum_{n=u}^{\infty}\left(\int_0^t P_u^{(k)}(s)\,ds\right) D_{n-u}^{(k+u)}\mathbf{e} \\
&= \sum_{u=0}^{\infty}\left(\int_0^t P_u^{(k)}(s)\,ds\right) \sum_{n=0}^{\infty} D_{n}^{(k+u)}\mathbf{e} \\
&= \mathbf{0}
\end{aligned}$$

since $\sum_{n=0}^{\infty} D_n^{(k+u)}\mathbf{e} = \mathbf{0}$ for all u. □

Remark: The forward differential equations need not hold for a general Markov process [13, p. 472]. In our case they do, because we are considering a counting process (cf. proof of lemma 1.11). Lemma 1.13 relies on the boundedness of the arrival rates which implies the boundedness of the matrices $\mathcal{Q}^{(k)}$.

1.2.2 Computation of the transition matrices

Now we are able to determine the matrices $P_n^{(k)}(t)$.

Theorem 1.14 *The matrices $P_n^{(k)}(t)$ are given by*

$$\begin{aligned}
P_n^{(k)}(t) &= \sum_{j=0}^{\infty}\frac{t^j}{j!}\left(\left(\mathcal{Q}^{(k)}\right)^j\right)_{0n} = \left(e^{\mathcal{Q}^{(k)}t}\right)_{0n} \quad \textit{and} \\
P_n^{(k)}(t) &= \sum_{j=0}^{\infty}\frac{t^j}{j!}\left(\left(\mathcal{Q}^{(0)}\right)^j\right)_{k,n+k} = \left(e^{\mathcal{Q}^{(0)}t}\right)_{k,n+k}.
\end{aligned}$$

Proof: Define $R_n^{(k)}(t) := \sum_{j=0}^{\infty}\frac{t^j}{j!}((\mathcal{Q}^{(k)})^j)_{0n}$. Differentiation yields

$$\begin{aligned}
\tfrac{d}{dt}R_n^{(k)}(t) &= \sum_{j=1}^{\infty}\frac{t^{j-1}}{(j-1)!}\left(\left(\mathcal{Q}^{(k)}\right)^j\right)_{0n} \\
&= \sum_{j=1}^{\infty}\frac{t^{j-1}}{(j-1)!}\sum_{l=0}^{n}\left(\mathcal{Q}^{(k)}\right)_{0l}\left(\left(\mathcal{Q}^{(k)}\right)^{j-1}\right)_{ln} \\
&= \sum_{j=1}^{\infty}\frac{t^{j-1}}{(j-1)!}\sum_{l=0}^{n}D_l^{(k)}\left(\left(\mathcal{Q}^{(k+l)}\right)^{j-1}\right)_{0,n-l} \quad \text{by lemma 1.7}
\end{aligned}$$

$$= \sum_{l=0}^{n} D_l^{(k)} \sum_{j=0}^{\infty} \frac{t^j}{j!} \left((\mathcal{Q}^{(k+l)})^j \right)_{0,n-l}$$

$$= \sum_{l=0}^{n} D_l^{(k)} R_{n-l}^{(k+l)}(t) \, .$$

So the matrices $R_n^{(k)}(t)$ satisfy the backward differential equations (1.2) and the condition

$$R_n^{(k)}(0) = \left((\mathcal{Q}^{(k)})^0 \right)_{0n} = \begin{cases} I & \text{if } n = 0 \\ O & \text{if } n \geq 1 \end{cases} .$$

Lemma 1.10 implies $R_n^{(k)}(t) = P_n^{(k)}(t)$. Corollary 1.8 yields the second assertion. □

Corollary 1.15 *The transition matrix $\mathcal{P}^{(k)}(t)$ is given by*

$$\mathcal{P}^{(k)}(t) = e^{\mathcal{Q}^{(k)} t} \, .$$

Proof: The assertion follows immediately from theorem 1.14 and corollary 1.8. □

Corollary 1.16 *The matrix $P_0^{(k)}(t)$ satisfies*

$$P_0^{(k)}(t) = e^{D_0^{(k)} t} \, .$$

Proof: The assertion follows immediately from theorem 1.14 and $((\mathcal{Q}^{(k)})^j)_{00} = (D_0^{(k)})^j$. □

This result implies an important property of the matrix $D_0^{(0)}$.

Corollary 1.17 *The matrix $D_0^{(0)}$ is nonsingular.*

Proof: For all $\mathbf{c} \in \mathbb{R}^{m^{(0)}}$ the unique solution of the differential equation $\frac{d}{dt}\mathbf{u}(t) = D_0^{(0)}\mathbf{u}(t)$ and $\mathbf{u}(0) = \mathbf{c}$ is given by $\mathbf{u}(t) = e^{D_0^{(0)} t}\mathbf{c}$ [3, p. 167ff]. Corollary 1.16 yields $\mathbf{u}(t) = P_0^{(0)}(t)\,\mathbf{c}$. From our assumption $D_0^{(0)}\mathbf{e} \neq \mathbf{0}$ we obtain $\lim_{t\to\infty} P_0^{(0)}(t) = O$ (because there are arrivals) and so $\lim_{t\to\infty} \mathbf{u}(t) = \lim_{t\to\infty} e^{D_0^{(0)} t}\mathbf{c} = \mathbf{0}$. Theorem 13.2.1 in Bellman [3, p. 250] states that all characteristic roots of $D_0^{(0)}$ have negative real parts, i.e. differ from zero. Thus $D_0^{(0)}$ is nonsingular. □

Remark: The proof of corollary 1.17 shows that all characteristic roots of $D_0^{(0)}$ have negative real parts. Thus $D_0^{(0)}$ is a stability matrix [3, p. 251].

1.3 The mean number of arrivals

The vector of the phase dependent mean number of arrivals in time t, $\mathbf{n}^{(k)}(t)$ with components $n_i^{(k)}(t)$, $i = 1, \ldots, m^{(k)}$, is defined by

$$\mathbf{n}^{(k)}(t) = \sum_{n=1}^{\infty} n P_n^{(k)}(t)\, \mathbf{e}\,.$$

Theorem 1.18 *The vector $\mathbf{n}^{(k)}(t)$ of the phase dependent mean number of arrivals in time t is given by*

$$\mathbf{n}^{(k)}(t) \;=\; \left[\sum_{j=1}^{\infty} \frac{t^j}{j!}\left(\mathcal{Q}^{(0)}\right)^{j-1} \begin{pmatrix} \mathbf{d}^{(0)} \\ \mathbf{d}^{(1)} \\ \mathbf{d}^{(2)} \\ \vdots \end{pmatrix}\right]_k \;=\; \left[\sum_{j=1}^{\infty} \frac{t^j}{j!}\left(\mathcal{Q}^{(k)}\right)^{j-1} \begin{pmatrix} \mathbf{d}^{(k)} \\ \mathbf{d}^{(k+1)} \\ \mathbf{d}^{(k+2)} \\ \vdots \end{pmatrix}\right]_0 .$$

Proof: Theorem 1.14 and assumption 1.2 yield

$$\begin{aligned}
\mathbf{n}^{(k)}(t) \;&=\; \sum_{n=1}^{\infty} n P_n^{(k)}(t)\mathbf{e} \;=\; \sum_{n=1}^{\infty} n \sum_{j=0}^{\infty} \frac{t^j}{j!}\left(\left(\mathcal{Q}^{(0)}\right)^j\right)_{k,n+k}\mathbf{e} \\
&=\; \sum_{j=1}^{\infty} \frac{t^j}{j!} \sum_{n=1}^{\infty} n \sum_{\nu=0}^{\infty} \left(\left(\mathcal{Q}^{(0)}\right)^{j-1}\right)_{k\nu} \left(\mathcal{Q}^{(0)}\right)_{\nu,n+k}\mathbf{e} \\
&=\; \sum_{j=1}^{\infty} \frac{t^j}{j!} \sum_{n=1}^{\infty} n \sum_{\nu=k}^{n+k} \left(\left(\mathcal{Q}^{(0)}\right)^{j-1}\right)_{k\nu} D_{n+k-\nu}^{(\nu)}\mathbf{e} \\
&=\; \sum_{j=1}^{\infty} \frac{t^j}{j!} \sum_{n=1}^{\infty} n \sum_{\nu=0}^{n} \left(\left(\mathcal{Q}^{(0)}\right)^{j-1}\right)_{k,k+\nu} D_{n-\nu}^{(k+\nu)}\mathbf{e} \\
&=\; \sum_{j=1}^{\infty} \frac{t^j}{j!} \sum_{\nu=0}^{\infty} \sum_{n=\nu}^{\infty} n \left(\left(\mathcal{Q}^{(0)}\right)^{j-1}\right)_{k,k+\nu} D_{n-\nu}^{(k+\nu)}\mathbf{e} \\
&=\; \sum_{j=1}^{\infty} \frac{t^j}{j!} \sum_{\nu=0}^{\infty} \left(\left(\mathcal{Q}^{(0)}\right)^{j-1}\right)_{k,k+\nu} \left(\sum_{n=0}^{\infty} n D_n^{(k+\nu)}\mathbf{e} + \nu \sum_{n=0}^{\infty} D_n^{(k+\nu)}\mathbf{e}\right) \\
&=\; \sum_{j=1}^{\infty} \frac{t^j}{j!} \sum_{\nu=0}^{\infty} \left(\left(\mathcal{Q}^{(0)}\right)^{j-1}\right)_{k,k+\nu} \mathbf{d}^{(k+\nu)} \\
&=\; \left[\sum_{j=1}^{\infty} \frac{t^j}{j!} \left(\mathcal{Q}^{(0)}\right)^{j-1} \begin{pmatrix} \mathbf{d}^{(0)} \\ \mathbf{d}^{(1)} \\ \mathbf{d}^{(2)} \\ \cdots \end{pmatrix}\right]_k .
\end{aligned}$$

Further, we obtain for $k \in \mathbb{N}_0$ by applying lemma 1.7

$$\begin{aligned}
\mathbf{n}^{(k)}(t) &= \left[\sum_{j=1}^{\infty} \frac{t^j}{j!} \left(\mathcal{Q}^{(0)}\right)^{j-1} \begin{pmatrix} \mathbf{d}^{(0)} \\ \mathbf{d}^{(1)} \\ \vdots \end{pmatrix}\right]_k = \sum_{n=k}^{\infty} \sum_{j=1}^{\infty} \frac{t^j}{j!} \left((\mathcal{Q}^{(0)})^{j-1}\right)_{kn} \mathbf{d}^{(n)} \\
&= \sum_{n=k}^{\infty} \sum_{j=1}^{\infty} \frac{t^j}{j!} \left((\mathcal{Q}^{(k)})^{j-1}\right)_{0,n-k} \mathbf{d}^{(n)} = \left[\sum_{j=1}^{\infty} \frac{t^j}{j!} \left(\mathcal{Q}^{(k)}\right)^{j-1} \begin{pmatrix} \mathbf{d}^{(k)} \\ \mathbf{d}^{(k+1)} \\ \vdots \end{pmatrix}\right]_0 .
\end{aligned}$$

□

In the level independent case the mean number of arrivals in time t equals the fundamental arrival rate (the inner product of the stationary distribution of the phase process and the vector of the phase dependent mean arrival rates) times t [31, p. 283]. This does not hold in our case, because the arrival rates and the phase process change according to the levels. We can only give an upper bound for the mean number of arrivals in time t.

Theorem 1.19 *The mean number of arrivals in time t satisfies*

$$\mathbf{n}^{(k)}(t) \leq \bar{\lambda}^{(k)} t \mathbf{e} .$$

Proof: Theorem 1.18 implies that $\mathbf{n}^{(k)}(s)$ is differentiable for all $s \in [0, t]$. Lemma 1.11 yields

$$\tfrac{d}{ds}\mathbf{n}^{(k)}(s) = \tfrac{d}{ds} \sum_{n=0}^{\infty} n P_n^{(k)}(s) \mathbf{e} = \sum_{n=0}^{\infty} n \sum_{u=0}^{n} P_u^{(k)}(s)\, D_{n-u}^{(k+u)} \mathbf{e} .$$

This series converges absolutely (since the power series in theorem 1.18 converges absolutely), so we obtain

$$\begin{aligned}
\tfrac{d}{ds}\mathbf{n}^{(k)}(s) &= \sum_{u=0}^{\infty} \sum_{n=u}^{\infty} n P_u^{(k)}(s)\, D_{n-u}^{(k+u)} \mathbf{e} \\
&= \sum_{u=0}^{\infty} P_u^{(k)}(s) \sum_{n=0}^{\infty} (n+u) D_n^{(k+u)} \mathbf{e} \\
&= \sum_{u=0}^{\infty} P_u^{(k)}(s)\, \mathbf{d}^{(k+u)} \\
&\leq \bar{\lambda}^{(k)} \sum_{u=0}^{\infty} P_u^{(k)}(s)\, \mathbf{e} = \bar{\lambda}^{(k)} \mathbf{e} .
\end{aligned}$$

Thus

$$\mathbf{n}^{(k)}(t) = \int_0^t \tfrac{d}{ds}\mathbf{n}^{(k)}(s)\, ds \leq \bar{\lambda}^{(k)} t \mathbf{e} .$$

□

2 The queueing system

If we use the level dependent BMAP defined in section 1 as the input process to a single server queueing system with a general service time distribution we obtain the *BMAP/G/1 queueing system with level dependent arrivals*. It is described by the stochastic process $(\mathcal{Y}, \mathcal{J}) = \{Y(t), J(t) : t \geq 0\}$, where $Y(t)$ is the number of customers in the system (waiting or in service) at time t, henceforth referred to as the level, and $J(t)$ is the phase of the arrival process at time t. So the state space $\mathbb{Y}$ of $(\mathcal{Y}, \mathcal{J})$ is

$$\mathbb{Y} = \bigcup_{k=0}^{\infty} \left(\{k\} \times \mathbf{J}^{(k)} \right).$$

If $Y(t) = k$, then arrivals occur according to $(\mathcal{N}^{(k)}, \mathcal{J}^{(k)})$. The arrival of a batch of size $\nu \geq 1$ implies an increase of the level by ν, while a service completion results in a decrease of the level by 1. The first effect corresponds to a state change in $(\mathcal{N}^{(k)}, \mathcal{J}^{(k)})$, the second effect causes that arrivals occur now according to $(\mathcal{N}^{(k-1)}, \mathcal{J}^{(k-1)})$. In order to keep the phase of the arrival process at a service completion time we need to assume that $\mathbf{J}^{(k)} \subseteq \mathbf{J}^{(k-1)}$. This assures that a service completion does not change the phase i of the arrival process, but the interarrival time "restarts" with a new rate, viz. $-d_{0,ii}^{(k-1)}$ instead of $-d_{0,ii}^{(k)}$.

Assumption 2.1

$$\mathbf{J}^{(k)} \supseteq \mathbf{J}^{(k+1)} \quad \textit{for all } k \in \mathbb{N}_0 .$$

Then $m^{(0)} \geq m^{(k)}$ for all $k \in \mathbb{N}_0$ and so $m = \sup\{m^{(k)} : k \in \mathbb{N}_0\} = m^{(0)} < \infty$.

In the ATM context the phases correspond to different input sources (e.g. CBR–, VBR–, ABR–traffic). Then assumption 2.1 means that we do not allow additional input sources if the level (i.e. the current load) increases. This should be a reasonable assumption.

Let $H(t)$ be the cumulative distribution function of the service time distribution and let μ^{-1} denote the mean service time, i.e.

$$\mu^{-1} = \int_0^{\infty} t \, dH(t) .$$

We are interested in the steady state distribution of the queue length at an arbitrary time, i.e. the limiting distribution of the stochastic process $(\mathcal{Y}, \mathcal{J})$. To compute this limiting distribution we apply the common method of the embedded Markov chain (cf. e.g. [22, pp. 167ff]).

3 The embedded Markov chain

If we observe the process $(\mathcal{Y}, \mathcal{J})$ immediately after service completion times T_ν, $\nu \in I\!N_0$ (with $T_0 := 0$), i.e. at $T_\nu+$, only, we obtain a discrete time–homogeneous Markov chain[2]

$$(\mathcal{X}, \mathcal{J}) \;:=\; \{X_\nu, J_\nu : \nu \in I\!N_0\} \;:=\; \{Y(T_\nu+), J(T_\nu+) : \nu \in I\!N_0\}$$

with state space $\boldsymbol{Y}$. But actually we have only $m^{(k+1)}$ phases in level k: If the process is in level k immediately after a service completion time T_ν, i.e. $(Y(T_\nu+), J(T_\nu+)) = (k, i)$, it was in level $k+1$ just before T_ν, i.e. $(Y(T_\nu-), J(T_\nu-)) = (k+1, i)$, and so the arrival process is still in the same phase $i \in \mathbf{J}^{(k+1)}$. Thus we can partition $\boldsymbol{Y}$ in two disjoint sets

$$\begin{aligned}
\boldsymbol{X} \;&:=\; \bigcup_{k=0}^{\infty} \left(\{k\} \times \mathbf{J}^{(k+1)}\right), \\
\boldsymbol{X}^c \;&:=\; \boldsymbol{Y} \setminus \boldsymbol{X} \;=\; \bigcup_{k=0}^{\infty} \left(\{k\} \times (\mathbf{J}^{(k)} \setminus \mathbf{J}^{(k+1)})\right).
\end{aligned}$$

Assumption 2.1 implies that $\boldsymbol{X} \subseteq \boldsymbol{Y}$. In applications $\boldsymbol{X}$ will be irreducible if the sets $\mathbf{J}^{(k)}$ are suitably chosen. Otherwise it suffices to consider the irreducible subsets of $\boldsymbol{X}$ and to add all other states to $\boldsymbol{X}^c$. For simplicity we restrict ourselves to the case that $\boldsymbol{X}$ is irreducible:

Assumption 3.1 *We assume $\boldsymbol{X}$ to be irreducible.*

So $\boldsymbol{X}$ is closed and irreducible, while $\boldsymbol{X}^c$ (if not empty) is open and transient.

3.1 The transition probabilities

The transition probability matrix of the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ is

$$\tilde{\mathcal{P}}(x) = \begin{pmatrix}
\tilde{B}_0(x) & \tilde{B}_1(x) & \tilde{B}_2(x) & \tilde{B}_3(x) & \cdots \\
\tilde{A}_0^{(1)}(x) & \tilde{A}_1^{(1)}(x) & \tilde{A}_2^{(1)}(x) & \tilde{A}_3^{(1)}(x) & \cdots \\
 & \tilde{A}_0^{(2)}(x) & \tilde{A}_1^{(2)}(x) & \tilde{A}_2^{(2)}(x) & \cdots \\
 & & \tilde{A}_0^{(3)}(x) & \tilde{A}_1^{(3)}(x) & \cdots \\
 & & & \ddots & \ddots
\end{pmatrix}.$$

The blocks $\tilde{B}_n(x)$ and $\tilde{A}_n^{(k)}(x)$ are of size $m^{(0)} \times m^{(n)}$ and $m^{(k)} \times m^{(k+n-1)}$, respectively, with their (i, j)th entries given by[3]

[2] To avoid an overwhelming notation we denote the phase process of the embedded Markov chain also by $\mathcal{J}$. So $(\mathcal{Y}, \mathcal{J})$ is a continuous time process, while $(\mathcal{X}, \mathcal{J})$ is a discrete time process.

[3] We use the notation $\tilde{B}_n(x)$ instead of $\tilde{A}_n^{(0)}(x)$ to stress the different character of these probabilities and to adopt the notation of Lucantoni [24] for the classical BMAP/G/1 queue as far as possible.

$$\begin{aligned}\left(\tilde{B}_n(x)\right)_{ij} &= \mathrm{P}\{X_{\nu+1} = n,\ J_{\nu+1} = j,\ T_{\nu+1} - T_\nu \le x \mid X_\nu = 0,\ J_\nu = i\} \\ &= \mathrm{P}\{N^{(0)}(T_{\nu+1} - T_\nu) = n+1,\ J^{(0)}(T_{\nu+1} - T_\nu) = j,\ T_{\nu+1} - T_\nu \le x \mid \\ &\qquad\qquad N^{(0)}(0) = 0,\ J^{(0)}(0) = i\}\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\left(\tilde{A}_n^{(k)}(x)\right)_{ij} &= \mathrm{P}\{X_{\nu+1} = k+n-1,\ J_{\nu+1} = j,\ T_{\nu+1} - T_\nu \le x \mid X_\nu = k,\ J_\nu = i\} \\ &= \mathrm{P}\{N^{(k)}(T_{\nu+1} - T_\nu) = n,\ J^{(k)}(T_{\nu+1} - T_\nu) = j,\ T_{\nu+1} - T_\nu \le x \mid \\ &\qquad\qquad N^{(k)}(0) = 0,\ J^{(k)}(0) = i\}\end{aligned} \tag{3.2}$$

for $n \in I\!N_0$, $k \in I\!N$, $x \ge 0$ and all $\nu \in I\!N_0$ (since $(\mathcal{X}, \mathcal{J})$ and $(\mathcal{N}^{(k)}, \mathcal{J}^{(k)})$ are time–homogeneous).

Note that $(\tilde{B}_n(x))_{ij} = 0$ for $(0, j) \in \boldsymbol{X}^c$, i.e. $j \in \mathbf{J}^{(0)} \setminus \mathbf{J}^{(1)}$, and $(\tilde{A}_n^{(k)}(x))_{ij} = 0$ for $(k+n-1, j) \in \boldsymbol{X}^c$, i.e. $j \in \mathbf{J}^{(k+n-1)} \setminus \mathbf{J}^{(k+n)}$.

The Laplace–Stieltjes transforms of these probabilities are defined by

$$A_n^{(k)}(s) := \int_0^\infty e^{-sx}\, d\tilde{A}_n^{(k)}(x) \quad \text{and} \quad B_n(s) := \int_0^\infty e^{-sx}\, d\tilde{B}_n(x)$$

for $\mathrm{Re}(s) \ge 0$. Further, we define the matrices

$$A_n^{(k)} := A_n^{(k)}(0) = \tilde{A}_n^{(k)}(\infty)\,, \qquad B_n := B_n(0) = \tilde{B}_n(\infty) \quad \text{and} \quad \mathcal{P} := \tilde{\mathcal{P}}(\infty)\,.$$

Conditioning on the duration of the service time we obtain

$$\tilde{A}_n^{(k)}(x) = \int_0^x \left[P_n^{(k)}(t) \mid O\right] dH(t)\,, \tag{3.3}$$

where $[P_n^{(k)}(t) \mid O]$ denotes the matrix of size $m^{(k)} \times m^{(k+n-1)}$ consisting of $P_n^{(k)}(t)$ and $m^{(k+n-1)} - m^{(k+n)}$ columns of zeros.

Lemma 1.13 yields that

$$\sum_{n=0}^{\infty} A_n^{(k)} \mathbf{e} = \mathbf{e} \tag{3.4}$$

for all $k \in I\!N$.

If all moments of the service time distribution exist, we can derive another expression for the matrices $A_n^{(k)}$.

Theorem 3.2 *If the moments $h_j := \mathrm{E}[H^j]$ exist for all $j \in \mathbb{N}_0$, the matrices $A_n^{(k)}$ are given by*

$$A_n^{(k)} = \sum_{j=0}^{\infty} \frac{h_j}{j!} \left[\left((\mathcal{Q}^{(k)})^j \right)_{0n} \mid O \right].$$

Proof: Using theorem 1.14 we obtain

$$\begin{aligned}\int_0^\infty P_n^{(k)}(t)\, dH(t) &= \int_0^\infty \sum_{j=0}^{\infty} \frac{t^j}{j!} \left((\mathcal{Q}^{(k)})^j \right)_{0n} dH(t) \\ &= \sum_{j=0}^{\infty} \int_0^\infty \frac{t^j}{j!}\, dH(t) \left((\mathcal{Q}^{(k)})^j \right)_{0n} = \sum_{j=0}^{\infty} \frac{h_j}{j!} \left((\mathcal{Q}^{(k)})^j \right)_{0n}.\end{aligned}$$

Equation (3.3) yields

$$A_n^{(k)} = \int_0^\infty \left[P_n^{(k)}(t) \mid O \right] dH(t) = \sum_{j=0}^{\infty} \frac{h_j}{j!} \left[\left((\mathcal{Q}^{(k)})^j \right)_{0n} \mid O \right].$$

□

We can now also determine the matrices B_n.

Lemma 3.3 *The matrix $sI - D_0^{(0)}$ is nonsingular for all s with* $\mathrm{Re}(s) \geq 0$.

Proof: Let C be the set of characteristic roots of $D_0^{(0)}$. Then $sI - D_0^{(0)}$ is nonsingular for all $s \notin C$. In the proof of corollary 1.17 we saw that all characteristic roots of $D_0^{(0)}$ have negative real parts. Thus $C \cap \{s : \mathrm{Re}(s) \geq 0\} = \emptyset$. □

Theorem 3.4 *The Laplace–Stieltjes transforms $B_n(s)$ are given by*

$$B_n(s) = \left(sI - D_0^{(0)} \right)^{-1} \sum_{l=1}^{n+1} D_l^{(0)} A_{n+1-l}^{(l)}(s)$$

for $\mathrm{Re}(s) \geq 0$.

Proof: By (3.1) the (i,j)th entry of $\tilde{B}_n(x)$ is

$$\left(\tilde{B}_n(x)\right)_{ij} = \mathrm{P}\{N^{(0)}(T_{\nu+1}-T_\nu) = n+1,\ J^{(0)}(T_{\nu+1}-T_\nu) = j,\ T_{\nu+1}-T_\nu \le x \mid \\ N^{(0)}(0) = 0,\ J^{(0)}(0) = i\}\,.$$

Given $X_\nu = 0$ the period $T_{\nu+1} - T_\nu$ consists of an idle–period and a service time. The idle–period is ended by the arrival of a batch of l customers. So during the service time exactly $n+1-l$ customers have to arrive to achieve $N^{(0)}(T_{\nu+1}-T_\nu) = n+1$. Suppose the idle period is of length u and the service time of length $t-u$ for some $t \le x$. Then

$$\tilde{B}_n(x) = \int_{t=0}^{x}\int_{u=0}^{t} P_0^{(0)}(u) \sum_{l=1}^{n+1} D_l^{(0)} \left[P_{n+1-l}^{(l)}(t-u) \mid O\right] H(t-du)\,dt\,,$$

and so

$$\tfrac{d}{dx}\tilde{B}_n(x) = \int_0^x P_0^{(0)}(u) \sum_{l=1}^{n+1} D_l^{(0)} \left[P_{n+1-l}^{(l)}(x-u) \mid O\right] H(x-du)\,.$$

So we obtain

$$\begin{aligned} B_n(s) &= \int_{x=0}^{\infty} e^{-sx} \int_{u=0}^{x} P_0^{(0)}(u) \sum_{l=1}^{n+1} D_l^{(0)} \left[P_{n+1-l}^{(l)}(x-u) \mid O\right] H(x-du)\,dx \\ &= \int_{u=0}^{\infty}\int_{x=u}^{\infty} e^{-sx}\, P_0^{(0)}(u) \sum_{l=1}^{n+1} D_l^{(0)} \left[P_{n+1-l}^{(l)}(x-u) \mid O\right] H(dx-u)\,du \\ &\quad \text{(by the theorem of Fubini [14, p. 122]).} \end{aligned}$$

The substitution $v = x-u$ yields

$$\begin{aligned} B_n(s) &= \int_{u=0}^{\infty} e^{-su}\, P_0^{(0)}(u) \sum_{l=1}^{n+1} D_l^{(0)} \int_{v=0}^{\infty} e^{-sv} \left[P_{n+1-l}^{(l)}(v) \mid O\right] H(dv)\,du \\ &= \int_0^{\infty} e^{-(sI-D_0^{(0)})u}\,du \sum_{l=1}^{n+1} D_l^{(0)} A_{n+1-l}^{(l)}(s) \\ &\quad \text{(by corollary 1.16 and (3.3)).} \end{aligned}$$

By lemma 3.3 $sI - D_0^{(0)}$ is nonsingular for all s with $\mathrm{Re}(s) \ge 0$. For these s we also know that all characteristic roots of $(sI - D_0^{(0)})$ have positive real parts (by the theorem of Gerŝgorin [28, p. 146]), and thus $\lim_{u\to\infty} e^{-(sI-D_0^{(0)})u} = O$ [3, p. 250,251]. So we have

$$B_n(s) = \left(sI - D_0^{(0)}\right)^{-1} \sum_{l=1}^{n+1} D_l^{(0)} A_{n+1-l}^{(l)}(s)$$

for all $\mathrm{Re}(s) \ge 0$. □

Corollary 3.5 *The matrices B_n are given by*

$$B_n = -D_0^{(0)^{-1}} \sum_{l=1}^{n+1} D_l^{(0)} A_{n+1-l}^{(l)} .$$

Proof: The assertion follows immediately from theorem 3.4. □

Remark: Corollary 3.5 implies that the (i,j)th entry of $-D_0^{(0)^{-1}} D_l^{(0)}$ gives the probability for an idle period starting in phase i to end in phase j with an arrival of a batch of size l.

Equation (3.4) yields

$$\begin{aligned}\sum_{n=0}^{\infty} B_n \mathbf{e} &= -D_0^{(0)^{-1}} \sum_{n=0}^{\infty}\sum_{l=0}^{n} D_{l+1}^{(0)} A_{n-l}^{(l+1)} \mathbf{e} = -D_0^{(0)^{-1}} \sum_{l=0}^{\infty} D_{l+1}^{(0)} \sum_{n=l}^{\infty} A_{n-l}^{(l+1)} \mathbf{e} \\ &= -D_0^{(0)^{-1}} (-D_0^{(0)}) \mathbf{e} = \mathbf{e} . \end{aligned} \tag{3.5}$$

3.2 The mean sojourn times

We are now able to determine the mean time (that passes in the embedding semi–Markov process $(\mathcal{Y},\mathcal{J})$) between two consecutive transitions ν and $\nu+1$ in the embedded Markov chain $(\mathcal{X},\mathcal{J})$, depending on $(X_\nu, J_\nu) = (k,i)$. According to Ramaswami [33] in the level independent case, we will call this the mean sojourn time of the embedded Markov chain $(\mathcal{X},\mathcal{J})$ in (k,i) (though this term is sometimes misleading) and denote it by $\delta_k = (\delta_{ki})_{i=1,\ldots,m^{(k)}}$.

Theorem 3.6 *The vectors δ_k are given by*

$$\begin{aligned}\delta_0 &= -D_0^{(0)^{-1}} \mathbf{e} + \mu^{-1} \mathbf{e} , \\ \delta_k &= \mu^{-1} \mathbf{e} , \quad k \geq 1. \end{aligned}$$

Proof: Using (3.3) we obtain for $k \geq 1$

$$\begin{aligned}\delta_k &= -\tfrac{d}{ds} \sum_{n=0}^{\infty} A_n^{(k)}(s) \mathbf{e} \Big|_{s=0} = -\sum_{n=0}^{\infty} \tfrac{d}{ds} \int_0^{\infty} e^{-sx} \Big[P_n^{(k)}(x) \mid O \Big] \, dH(x) \, \mathbf{e} \Big|_{s=0} \\ &= \sum_{n=0}^{\infty} \int_0^{\infty} x \Big[P_n^{(k)}(x) \mid O \Big] \mathbf{e} \, dH(x) = \int_0^{\infty} x \sum_{n=0}^{\infty} \Big[P_n^{(k)}(x) \mid O \Big] \mathbf{e} \, dH(x) \\ &= \mu^{-1} \mathbf{e} \quad \text{by lemma 1.13.} \end{aligned}$$

Theorem 3.4 yields

$$\begin{aligned}
\delta_0 &= -\tfrac{d}{ds}\sum_{n=0}^{\infty} B_n(s)\mathbf{e}\,\Big|_{s=0} \\
&= -\tfrac{d}{ds}\sum_{n=0}^{\infty}\Big(sI - D_0^{(0)}\Big)^{-1}\sum_{l=1}^{n+1} D_l^{(0)} A_{n+1-l}^{(l)}(s)\mathbf{e}\,\Big|_{s=0} \\
&= \Big(sI - D_0^{(0)}\Big)^{-2}\sum_{n=0}^{\infty}\sum_{l=0}^{n} D_{l+1}^{(0)} A_{n-l}^{(l+1)}(s)\mathbf{e}\,\Big|_{s=0} \\
&\quad + \Big(sI - D_0^{(0)}\Big)^{-1}\left(-\tfrac{d}{ds}\sum_{n=0}^{\infty}\sum_{l=0}^{n} D_{l+1}^{(0)} A_{n-l}^{(l+1)}(s)\mathbf{e}\right)\Big|_{s=0} \\
&= {D_0^{(0)}}^{-2}\sum_{l=0}^{\infty}\sum_{n=l}^{\infty} D_{l+1}^{(0)} A_{n-l}^{(l+1)}\mathbf{e} + \Big(sI - D_0^{(0)}\Big)^{-1}\left(-\tfrac{d}{ds}\sum_{l=0}^{\infty}\sum_{n=l}^{\infty} D_{l+1}^{(0)} A_{n-l}^{(l+1)}(s)\mathbf{e}\right)\Big|_{s=0} \\
&= {D_0^{(0)}}^{-2}\sum_{l=0}^{\infty} D_{l+1}^{(0)}\sum_{n=0}^{\infty} A_n^{(l+1)}\mathbf{e} + \Big(sI - D_0^{(0)}\Big)^{-1}\sum_{l=0}^{\infty} D_{l+1}^{(0)}\left(-\tfrac{d}{ds}\sum_{n=0}^{\infty} A_n^{(l+1)}(s)\mathbf{e}\right)\Big|_{s=0} \\
&= {D_0^{(0)}}^{-2}\sum_{l=0}^{\infty} D_{l+1}^{(0)}\mathbf{e} - {D_0^{(0)}}^{-1}\sum_{l=0}^{\infty} D_{l+1}^{(0)}\delta_{l+1} \\
&= {D_0^{(0)}}^{-2}\Big(-D_0^{(0)}\mathbf{e}\Big) - {D_0^{(0)}}^{-1}\mu^{-1}\Big(-D_0^{(0)}\mathbf{e}\Big) \\
&= -{D_0^{(0)}}^{-1}\mathbf{e} + \mu^{-1}\mathbf{e}\,.
\end{aligned}$$

□

Remark: The mean time between two consecutive transitions in the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ is a mean service time, plus a mean idle time if $(\mathcal{X}, \mathcal{J})$ was in level 0. Since

$$\Big[P_0^{(0)}(t)\mathbf{e}\Big]_i = \mathrm{P}\{\text{idle time} > t \mid N^{(0)}(0) = 0,\ J^{(0)}(0) = i\}$$

corollaries 1.16 and 1.17 yield that the vector of the phase dependent mean idle times is

$$\int_0^{\infty} P_0^{(0)}(t)\mathbf{e}\,dt = \int_0^{\infty} e^{D_0^{(0)}t}\mathbf{e}\,dt = -{D_0^{(0)}}^{-1}\mathbf{e}\,.$$

3.3 The mean number of arrivals during a service time

The vector of the mean number of arrivals during a sojourn time of the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ in (k, i) is given by

$$\tilde{\mathbf{b}} := \sum_{n=1}^{\infty} n B_{n-1} \mathbf{e} \qquad \text{for } k = 0, \tag{3.6}$$

$$\mathbf{a}^{(k)} := \sum_{n=1}^{\infty} n A_n^{(k)} \mathbf{e} \qquad \text{for } k \geq 1 \tag{3.7}$$

with components $\tilde{b}_i$, $i = 1, \ldots, m^{(0)}$, and $a_i^{(k)}$, $i = 1, \ldots, m^{(k)}$, respectively. So, for $k = 0$ $\tilde{\mathbf{b}}$ is the phase dependent mean number of arrivals during an idle period and the following service time, while for $k \geq 1$ $\mathbf{a}^{(k)}$ is the phase dependent mean number of arrivals during a service time which started in level k[4]. Further, let

$$\mathbf{b} = \sum_{n=1}^{\infty} n B_n \mathbf{e} \tag{3.8}$$

with components b_i, $i = 1, \ldots, m^{(0)}$, then we will see that $\tilde{\mathbf{b}} = \mathbf{b} + \mathbf{e}$.

We can obtain the phase dependent mean number of arrivals during a service time which started in level k by conditioning on the duration of the service time:

$$\mathbf{a}^{(k)} = \int_0^{\infty} \mathbf{n}^{(k)}(t)\, dH(t)\,. \tag{3.9}$$

Theorem 3.7 *If $h_j = \mathrm{E}[H^j]$ exists for all $j \in I\!N$, the vectors $\mathbf{a}^{(k)}$ are given by*

$$\begin{pmatrix} \mathbf{a}^{(1)} \\ \mathbf{a}^{(2)} \\ \vdots \end{pmatrix} = \sum_{j=1}^{\infty} \frac{h_j}{j!} \left(\mathcal{Q}^{(1)}\right)^{j-1} \begin{pmatrix} \mathbf{d}^{(1)} \\ \mathbf{d}^{(2)} \\ \vdots \end{pmatrix}.$$

Proof: Theorem 1.18 together with lemma 1.5 and assumption 1.2 yield

$$\begin{pmatrix} \mathbf{a}^{(1)} \\ \mathbf{a}^{(2)} \\ \vdots \end{pmatrix} = \int_0^{\infty} \begin{pmatrix} \mathbf{n}^{(1)}(t) \\ \mathbf{n}^{(2)}(t) \\ \vdots \end{pmatrix} dH(t) = \int_0^{\infty} \sum_{j=1}^{\infty} \frac{t^j}{j!} \left(\mathcal{Q}^{(1)}\right)^{j-1} \begin{pmatrix} \mathbf{d}^{(1)} \\ \mathbf{d}^{(2)} \\ \vdots \end{pmatrix} dH(t)$$

$$= \sum_{j=1}^{\infty} \frac{h_j}{j!} \left(\mathcal{Q}^{(1)}\right)^{j-1} \begin{pmatrix} \mathbf{d}^{(1)} \\ \mathbf{d}^{(2)} \\ \vdots \end{pmatrix}.$$

□

[4] Lucantoni [24] uses the notation β instead of $\mathbf{a}^{(k)}$.

Theorem 3.8 *The mean number of arrivals during a service time satisfies*

$$\mathbf{a}^{(k)} \leq \bar{\lambda}^{(k)} \mu^{-1} \mathbf{e}\,.$$

Proof: Equation (3.9) and theorem 1.19 imply

$$\mathbf{a}^{(k)} \;=\; \int_0^\infty \mathbf{n}^{(k)}(t)\, dH(t) \;\leq\; \int_0^\infty \bar{\lambda}^{(k)} t\mathbf{e}\, dH(t) \;=\; \bar{\lambda}^{(k)} \mu^{-1} \mathbf{e}\,.$$

□

Knowing the vectors $\mathbf{a}^{(k)}$ we can also determine $\tilde{\mathbf{b}}$.

Theorem 3.9 *The vector of the phase dependent mean number of arrivals during an idle period and the following service time is given by*

$$\tilde{\mathbf{b}} = \mathbf{b} + \mathbf{e}$$

with

$$\mathbf{b} = -D_0^{(0)^{-1}} \mathbf{d}^{(0)} - D_0^{(0)^{-1}} \sum_{k=1}^{\infty} D_k^{(0)} \mathbf{a}^{(k)} - \mathbf{e}\,.$$

Proof: At first, we have

$$\tilde{\mathbf{b}} \;=\; \sum_{n=1}^{\infty} n B_{n-1} \mathbf{e} \;=\; \sum_{n=1}^{\infty} (n-1) B_{n-1} \mathbf{e} + \sum_{n=1}^{\infty} B_{n-1} \mathbf{e} \;=\; \mathbf{b} + \mathbf{e}\,.$$

Further, corollary 3.5 implies

$$\begin{aligned}
\tilde{\mathbf{b}} \;&=\; \sum_{n=1}^{\infty} n B_{n-1} \mathbf{e} \;=\; \sum_{n=1}^{\infty} n \left(-D_0^{(0)^{-1}} \sum_{l=1}^{n} D_l^{(0)} A_{n-l}^{(l)} \mathbf{e} \right) \\
&=\; -D_0^{(0)^{-1}} \sum_{l=1}^{\infty} D_l^{(0)} \sum_{n=l}^{\infty} n A_{n-l}^{(l)} \mathbf{e} \\
&=\; -D_0^{(0)^{-1}} \sum_{l=1}^{\infty} D_l^{(0)} \left(\sum_{n=0}^{\infty} n A_n^{(l)} \mathbf{e} + \sum_{n=0}^{\infty} l A_n^{(l)} \mathbf{e} \right) \\
&=\; -D_0^{(0)^{-1}} \left(\sum_{l=1}^{\infty} D_l^{(0)} \mathbf{a}^{(l)} + \sum_{l=1}^{\infty} l D_l^{(0)} \mathbf{e} \right) \\
&=\; -D_0^{(0)^{-1}} \sum_{l=1}^{\infty} D_l^{(0)} \mathbf{a}^{(l)} - D_0^{(0)^{-1}} \mathbf{d}^{(0)}\,.
\end{aligned}$$

□

Theorem 3.10 *The mean number of arrivals during an idle period and the following service time satisfies*

$$\tilde{\mathbf{b}} \le \bar{\lambda}^{(0)} \delta_0 \,.$$

Proof: Let $U_i(t)$ be the cumulative distribution function of an idle period starting in phase i, then theorem 1.19 implies

$$\tilde{b}_i \;=\; \int_0^\infty n_i^{(0)}(t)\, d(U_i * H)(t) \;\le\; \int_0^\infty \bar{\lambda}^{(0)} t \, d(U_i * H)(t) \;=\; \bar{\lambda}^{(0)} \delta_{0i} \,.$$

Thus $\tilde{\mathbf{b}} \le \bar{\lambda}^{(0)} \delta_0$. □

4 Stability

Before computing the stationary distribution of the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ we need to derive some conditions for its existence, i.e. for the stability of the embedded Markov chain.

The classical level independent BMAP/G/1 queue is stable, i.e. the embedded Markov chain is ergodic and the limiting distribution of the queue length exists, if and only if the mean number of arrivals during a service time is less than 1 [33, Th. 2.2.11]. So we can suppose that the level dependent BMAP/G/1 queue is stable if the supremum of all mean numbers of arrivals during a service time ($a_i^{(k)}$) is less than 1. This can easily be seen by considering a superior level independent BMAP/G/1 queue with arrival rate $\bar{\lambda}^{(0)} = \sup \left\{ \bar{\lambda}^{(k)} : k \in \mathbb{N}_0 \right\}$ and applying theorem 3.8.

Further, the M/G/1 queue with state dependent arrivals is stable if the supremum of all but finitely many mean numbers of arrivals during a service time is less than 1 [40]. This suggests that it suffices to consider the limit superior of the sequence $\{\mathbf{a}^{(k)} : k \in \mathbb{N}_0\}$. In fact we will obtain a corresponding stability condition for the BMAP/G/1 queue with level dependent arrivals.

We have partitioned the state space of the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ in two sets $\boldsymbol{X}$ and $\boldsymbol{X}^c$, where $\boldsymbol{X}$ is closed and irreducible while $\boldsymbol{X}^c$ is (if not empty) open and transient (cf. section 3). We will call the embedded Markov chain *(positive) recurrent* if the class $\boldsymbol{X}$ is (positive) recurrent. Since $(\mathcal{X}, \mathcal{J})$ is aperiodic, positive recurrence ensures the existence of an unique stationary distribution which is strictly positive on $\boldsymbol{X}$ and zero on $\boldsymbol{X}^c$ [9, Th. 6.2.1, p. 152]. In this case we will also call the embedded Markov chain *ergodic*. For the rest of this section we will assume without loss of generality that $\boldsymbol{X}^c = \emptyset$.

Let us first state a generalized Foster criterion [32, Th. 2 + Th. 4]:

Theorem 4.1 *Let $\{Z_n : n \in \mathbb{N}_0\}$ be an irreducible and aperiodic time–homogeneous Markov chain with state space $\mathbb{N}_0$. Define $\gamma_i := \mathrm{E}[Z_{n+1} - Z_n \mid Z_n = i]$ for $i \in \mathbb{N}_0$.*

a) If $|\gamma_i| < \infty$ for all $i \in \mathbb{N}_0$ and $\limsup_{i\to\infty} \gamma_i < 0$ then $\{Z_n : n \in \mathbb{N}_0\}$ is ergodic.

b) If $|\gamma_i| < \infty$ for all $i \in \mathbb{N}_0$ and there exists $N \in \mathbb{N}$ such that $\gamma_i \le 0$ for all $i \ge N$ then $\{Z_n : n \in \mathbb{N}_0\}$ is recurrent.

This result enables us to derive a first stability condition for the BMAP/G/1 queue with level dependent arrivals.

Theorem 4.2 *If $\limsup_{k\to\infty} \max_{i=1,\dots,m^{(k)}} a_i^{(k)} < 1$ then the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ is positive recurrent.*

Proof: Define $\gamma_{ki} := \mathrm{E}[X_{\nu+1} - X_\nu \mid X_\nu = k,\ J_\nu = i]$, then $\gamma_{ki} \ge -1$ for all k, i. Further, by theorem 3.9

$$\begin{aligned}\gamma_{0i} &= \mathrm{E}[X_{\nu+1} \mid X_\nu = 0,\ J_\nu = i] = \sum_{n=1}^{\infty} n\Big[B_n \mathbf{e}\Big]_i = b_i \\ &= -\Big[D_0^{(0)^{-1}} \mathbf{d}^{(0)}\Big]_i - \Big[D_0^{(0)^{-1}} \sum_{k=1}^{\infty} D_k^{(0)} \mathbf{a}^{(k)}\Big]_i - 1\,.\end{aligned}$$

Theorem 3.8 and (1.1) yield

$$\sum_{k=1}^{\infty} D_k^{(0)} \mathbf{a}^{(k)} \le \sum_{k=1}^{\infty} D_k^{(0)} \bar{\lambda}^{(k)} \mu^{-1} \mathbf{e} \le \bar{\lambda}^{(0)} \mu^{-1} \sum_{k=1}^{\infty} D_k^{(0)} \mathbf{e} < \infty \cdot \mathbf{e}\,.$$

Thus $|\gamma_{0i}| < \infty$ for all $i = 1, \dots, m^{(0)}$. For $k \ge 1$

$$\begin{aligned}\gamma_{ki} &= \mathrm{E}[X_{\nu+1} \mid X_\nu = k,\ J_\nu = i] - k = \sum_{n=0}^{\infty} (n + k - 1)\Big[A_n^{(k)} \mathbf{e}\Big]_i - k \\ &= a_i^{(k)} - 1\,.\end{aligned}$$

So $|\gamma_{ki}| < \infty$ for all $k \in \mathbb{N}_0$, $i = 1, \dots, m^{(k)}$, and $\limsup_{k\to\infty} \max_{i=1,\dots,m^{(k)}} a_i^{(k)} < 1$ implies that $\limsup_{k\to\infty} \max_{i=1,\dots,m^{(k)}} \gamma_{ki} < 0$. By theorem 4.1a $(\mathcal{X}, \mathcal{J})$ is positive recurrent. □

Corollary 4.3 *If there exists some $N \in \mathbb{N}$ such that $\max_{i=1,\dots,m^{(k)}} a_i^{(k)} \le 1$ for all $k \ge N$ then the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ is recurrent.*

Proof: If we define γ_{ki} as in the proof of theorem 4.2 and apply theorem 4.1b we obtain that $(\mathcal{X}, \mathcal{J})$ is recurrent. $\square$

The stability conditions derived so far have the disadvantage to rely on the vectors $\mathbf{a}^{(k)}$, which are (in general) not easy to compute. So we would like to have a stability condition which only relies on the arrival rates and the service rate (or the mean service time).

Corollary 4.4 *If there exists some $N \in \mathbb{N}$ such that $\bar{\lambda}^{(N)} < \mu$ ($\bar{\lambda}^{(N)} \leq \mu$) then the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ is positive recurrent (recurrent).*

Proof: By theorem 3.8 $a_i^{(k)} \leq \bar{\lambda}^{(k)}\mu^{-1}$ for all $i = 1, \ldots, m^{(k)}$. The definition of the maximum arrival rates $\bar{\lambda}^{(k)}$ implies that $\bar{\lambda}^{(N)} \geq \bar{\lambda}^{(k)}$ for all $k \geq N$ and so

$$\limsup_{k\to\infty} \max_{i=1,\ldots,m^{(k)}} a_i^{(k)} \;\leq\; \limsup_{k\to\infty} \bar{\lambda}^{(k)}\mu^{-1} \;\leq\; \bar{\lambda}^{(N)}\mu^{-1} < 1$$

in the first case and

$$\max_{i=1,\ldots,m^{(k)}} a_i^{(k)} \;\leq\; \bar{\lambda}^{(k)}\mu^{-1} \;\leq\; \bar{\lambda}^{(N)}\mu^{-1} \leq 1$$

for all $k \geq N$ in the second case. Thus theorem 4.2 and corollary 4.3 yield the assertion. $\square$

5 The fundamental periods

In the level independent case the fundamental matrix G plays the key role in determining the steady state distributions (at service completion times and at arbitrary times) [24]. A fundamental period is the first passage time from level k to level $k-1$ in the embedded Markov chain.

In our case the first passage time from level k to level $k-1$ in the embedded Markov chain $(\mathcal{X}, \mathcal{J})$, i.e. the fundamental period, depends on the starting level k. Define $(\tilde{G}_l^{(k)}(x))_{ij}$ to be the probability that the first passage from state (k, i) to $(k-1, j)$ occurs in exactly l transitions, i.e. service completions, not later than time x, and that $(k-1, j)$ is the first state visited in level $k-1$[5]. Formally, $(\tilde{G}_l^{(k)}(x))_{ij}$ is defined by

$$\left(\tilde{G}_l^{(k)}(x)\right)_{ij} \;:=\; \mathrm{P}\{\; X_{\nu+l} = k-1,\; J_{\nu+l} = j,\; T_{\nu+l} - T_\nu \leq x, \\ \forall r = 1, \ldots, l-1 : X_{\nu+r} \neq k-1 \mid X_\nu = k,\; J_\nu = i \;\;\}\,.$$

[5] Do not confuse this notation with $G_{jj'}^{(r)}(k;x)$ in Neuts [31, p. 79] or $\tilde{G}_{jj'}^{(r)}(k;x)$ in Lucantoni [24] in the level independent case, which are the corresponding probabilities for going from level $i+r$ to i.

Let $\tilde{G}_l^{(k)}(x)$ be the $m^{(k)} \times m^{(k-1)}$ matrix with entries $(\tilde{G}_l^{(k)}(x))_{ij}$ and let $G_l^{(k)}(s)$ be its Laplace–Stieltjes transform. Further, define the joint transform $G^{(k)}(z,s)$ by

$$G^{(k)}(z,s) \;=\; \sum_{l=1}^{\infty} z^l\, G_l^{(k)}(s) \;=\; \sum_{l=1}^{\infty} z^l \int_0^{\infty} e^{-sx}\, d\tilde{G}_l^{(k)}(x)$$

for $|z| \le 1$ and $\mathrm{Re}(s) \ge 0$, and the matrices $G_l^{(k)}$ and $G^{(k)}$ by

$$G_l^{(k)} = G_l^{(k)}(0) = \tilde{G}_l^{(k)}(\infty)\,, \qquad G^{(k)} = G^{(k)}(1,0) = \sum_{l=1}^{\infty} G_l^{(k)}$$

for $k \ge 1$. So $(G^{(k)})_{ij}$ is the probability that starting in state (k,i) level $k-1$ will be reached and the first state visited there is $(k-1,j)$.

By conditioning on the first state visited after (k,i) we can obtain a functional equation for the matrices $G^{(k)}$ and the joint transforms $G^{(k)}(z,s)$ analogously to the level independent case.

Lemma 5.1 *The Laplace–Stieltjes transforms $G_l^{(k)}(s)$ satisfy*

$$\begin{aligned} G_1^{(k)}(s) &= A_0^{(k)}(s)\,, \\ G_l^{(k)}(s) &= \sum_{\nu=1}^{l-1} A_\nu^{(k)}(s) \sum_{\substack{l_1,\ldots,l_\nu \ge 1 \\ l_1+\ldots+l_\nu = l-1}} \left(\prod_{j=1}^{\nu} G_{l_j}^{(k+\nu-j)}(s) \right), \quad l \ge 2 \end{aligned}$$

for $\mathrm{Re}(s) \ge 0$.

Proof: The definition of $\tilde{G}_1^{(k)}(x)$ implies $\tilde{G}_1^{(k)}(x) = \tilde{A}_0^{(k)}(x)$ and hence $G_1^{(k)}(s) = A_0^{(k)}(s)$. For $l \ge 2$ the first transition will bring us to a level greater than or equal to k (otherwise we would visit the level $k-1$ in the first step, since the process is skip–free downward). So we have $l-1$ transitions left to reach level $k-1$ for the first time. If $A_\nu^{(k)}$ took us to level $k+\nu-1$, $G_{l_1}^{(k+\nu-1)}$ will bring us to level $k+\nu-2$ for the first time in exactly l_1 steps, $G_{l_2}^{(k+\nu-2)}$ to $k+\nu-3$ for the first time in exactly l_2 steps and so forth until we finally reach level $k-1$ for the first time in exactly $l-1$ steps. Note that we did not visit level $k-1$ before, because the process is skip–free downward. In matrix notation this gives

$$\tilde{G}_l^{(k)}(x) =$$

$$\sum_{\nu=1}^{l-1} \int_{t=0}^{x} \int_{u=0}^{t} d\tilde{A}_\nu^{(k)}(u) \sum_{\substack{l_1,\ldots,l_\nu \ge 1 \\ l_1+\ldots+l_\nu = l-1}} \int_{v_1=0}^{t-u} \int_{v_2=0}^{t-u-v_1} \cdots \int_{v_\nu=0}^{t-u-\sum_{\iota=1}^{\nu-1} v_\iota} \left(\prod_{j=1}^{\nu} d\tilde{G}_{l_j}^{(k+\nu-j)}(v_j) \right) dt\,.$$

Thus, for $l \geq 2$

$$G_l^{(k)}(s) \;=\; \sum_{\nu=1}^{l-1} A_\nu^{(k)}(s) \sum_{\substack{l_1,\ldots,l_\nu \geq 1 \\ l_1+\ldots+l_\nu = l-1}} \left(\prod_{j=1}^{\nu} G_{l_j}^{(k+\nu-j)}(s) \right) .$$

□

Theorem 5.2 *The joint transforms $G^{(k)}(z,s)$ satisfy*

$$G^{(k)}(z,s) \;=\; z \sum_{n=0}^{\infty} A_n^{(k)}(s) \left(\prod_{i=1}^{n} G^{(k+n-i)}(z,s) \right) \tag{5.1}$$

(here the empty product $\prod_{i=1}^{0} \cdot$ shall be the identity matrix I).

Proof: Lemma 5.1 yields

$$\begin{aligned}
G^{(k)}(z,s) \;&=\; \sum_{l=1}^{\infty} z^l \, G_l^{(k)}(s) \\
&=\; zA_0^{(k)}(s) + \sum_{l=1}^{\infty} z^{l+1} \sum_{\nu=1}^{l} A_\nu^{(k)}(s) \sum_{\substack{l_1,\ldots,l_\nu \geq 1 \\ l_1+\ldots+l_\nu = l}} \left(\prod_{j=1}^{\nu} G_{l_j}^{(k+\nu-j)}(s) \right) \\
&=\; zA_0^{(k)}(s) + z \sum_{\nu=1}^{\infty} A_\nu^{(k)}(s) \sum_{l=\nu}^{\infty} z^l \sum_{\substack{l_1,\ldots,l_\nu \geq 1 \\ l_1+\ldots+l_\nu = l}} \left(\prod_{j=1}^{\nu} G_{l_j}^{(k+\nu-j)}(s) \right) \\
&=\; zA_0^{(k)}(s) + z \sum_{\nu=1}^{\infty} A_\nu^{(k)}(s) \prod_{j=1}^{\nu} \left(\sum_{l_j=1}^{\infty} z^{l_j} G_{l_j}^{(k+\nu-j)}(s) \right) \\
&=\; zA_0^{(k)}(s) + z \sum_{\nu=1}^{\infty} A_\nu^{(k)}(s) \left(\prod_{j=1}^{\nu} G^{(k+\nu-j)}(z,s) \right) \\
&=\; z \sum_{\nu=0}^{\infty} A_\nu^{(k)}(s) \left(\prod_{j=1}^{\nu} G^{(k+\nu-j)}(z,s) \right) .
\end{aligned}$$

□

Corollary 5.3 *The matrices $G^{(k)}$, $k \geq 1$, satisfy*

$$G^{(k)} \;=\; \sum_{\nu=0}^{\infty} A_\nu^{(k)} \left(\prod_{j=1}^{\nu} G^{(k+\nu-j)} \right) .$$

Proof: Because of $G^{(k)} = G^{(k)}(1,0)$, the assertion follows immediately from theorem 5.2. $\square$

As an immediate consequence of their definition the fundamental matrices $G^{(k)}$ are substochastic, i.e. $G^{(k)}\mathbf{e} \le \mathbf{e}$. If the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ is recurrent[6] they are also stochastic, because the probability to reach level $k-1$ from any state in level k is 1.

Theorem 5.4 *The fundamental matrices $G^{(k)}$ are stochastic, i.e. $G^{(k)}\mathbf{e} = \mathbf{e}$ for all $k \in I\!N$, if and only if the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ is recurrent.*

Proof: Let us first prove that $(\mathcal{X}, \mathcal{J})$ is recurrent if $G^{(k)}\mathbf{e} = \mathbf{e}$ for all $k \in I\!N$. Since $(\mathcal{X}, \mathcal{J})$ is irreducible on $\boldsymbol{X}$ and the state space of $\{J_\nu : \nu \in I\!N_0\}$ is finite, it suffices to show that a return from level 0 to level 0 occurs with probability 1 [9, Th. 3.3.16, p. 131]. By conditioning on the first state visited after leaving level 0 we obtain

$$\begin{aligned} &\mathrm{P}\{\exists \nu \ge 1 \,:\, X_\nu = 0 \mid X_0 = 0,\ J_0 = i\} \\ &\quad = \Big[B_0\mathbf{e}\Big]_i + \left[\sum_{n=1}^{\infty} B_n \left(\prod_{l=0}^{n-1} G^{(n-l)}\right) \mathbf{e}\right]_i \;=\; \Big[B_0\mathbf{e}\Big]_i + \left[\sum_{n=1}^{\infty} B_n\mathbf{e}\right]_i \;=\; 1\,. \end{aligned}$$

Thus $(\mathcal{X}, \mathcal{J})$ is recurrent.

Conversely, if $(\mathcal{X}, \mathcal{J})$ is recurrent, suppose there exist $k \in I\!N$ and $i \in \mathbf{J}^{(k)}$ such that $[G^{(k)}\mathbf{e}]_i < 1$. Since $(\mathcal{X}, \mathcal{J})$ is irreducible and recurrent on $\boldsymbol{X}$, a transition from (k,i) to all $(k-1,j) \in \boldsymbol{X}$ must occur with probability 1 [9, L. 3.3.11, p. 129]. But

$$\begin{aligned} &\mathrm{P}\{\exists \nu \ge 1 \,:\, X_\nu = k-1,\ J_\nu = j \mid X_0 = k,\ J_0 = i\} \\ &\quad \le\; \mathrm{P}\{\exists \nu \ge 1 \,:\, X_\nu = k-1 \mid X_0 = k,\ J_0 = i\} \;=\; \Big[G^{(k)}\mathbf{e}\Big]_i \;<\; 1\,. \end{aligned}$$

Thus $G^{(k)}\mathbf{e} = \mathbf{e}$ for all $k \in I\!N$. $\square$

[6]Remember that we called the embedded Markov chain recurrent if it is recurrent on $\boldsymbol{X}$ (cf. sect. 4).

5.1 Computation of the fundamental matrices

5.1.1 Algorithm 1

To derive explicit expressions for the matrices $G^{(k)}$ we consider the taboo–probabilities $(V_l^{(n,k)})_{ij}$ that starting in state $(k+n,i)$ we reach (k,j) after exactly l transitions without visiting level $k-1$ in between. Note that $(V_l^{(n,k)})_{ij}$ does not describe a first passage time. Formally, $(V_l^{(n,k)})_{ij}$ is defined by

$$\left(V_l^{(n,k)}\right)_{ij} := \\ \mathrm{P}\{X_{\nu+l}=k,\ J_{\nu+l}=j,\ \forall r=1,\ldots,l-1: X_{\nu+r}\neq k-1 \mid X_\nu=k+n,\ J_\nu=i\}\,.$$

Let $V_l^{(n,k)}$ be the $m^{(k+n)}\times m^{(k)}$ matrix with entries $(V_l^{(n,k)})_{ij}$.

Lemma 5.5 *The matrices $G_l^{(k)}$ and $G^{(k)}$ are given by*

$$\begin{aligned} G_l^{(k)} &= V_{l-1}^{(0,k)}A_0^{(k)}\,, \quad l\geq 1,\\ G^{(k)} &= \sum_{l=0}^{\infty} V_l^{(0,k)}A_0^{(k)}\,. \end{aligned}$$

Proof: Since the process is skip–free downward we can reach level $k-1$ only from level k, if we started in k and did not visit $k-1$ in between. So we have to be back in level k after $l-1$ transitions without visiting $k-1$ in between and then go to $k-1$ for the first time. □

So we need to determine the matrices $V_l^{(0,k)}$.

Lemma 5.6 *The matrices $V_l^{(n,k)}$ satisfy*

$$\begin{aligned} V_0^{(0,k)} &= I\,,\\ V_0^{(n,k)} &= O \quad \textit{for } n\geq 1\,,\\ V_1^{(0,k)} &= A_1^{(k)}\,,\\ V_1^{(1,k)} &= A_0^{(k+1)}\,,\\ V_l^{(0,k)} &= \sum_{i=1}^{l} A_i^{(k)}V_{l-1}^{(i-1,k)} \quad \textit{for } l\geq 2\,,\\ V_l^{(n,k)} &= \sum_{i=0}^{l-n} A_i^{(k+n)}V_{l-1}^{(n+i-1,k)} \quad \textit{for } 1\leq n\leq l,\ l\geq 2\,,\\ V_l^{(n,k)} &= O \quad \textit{for } n>l\geq 1\,. \end{aligned}$$

Proof: The equations for $V_0^{(n,k)}$, $V_1^{(0,k)}$ and $V_1^{(1,k)}$ are obvious, the one for $V_l^{(n,k)}$, $n > l \geq 1$, is an immediate consequence of the process being skip–free downward. The other two equations are easily obtained by conditioning on the state reached after the first transition. □

Lemma 5.6 shows that the matrices $V_l^{(n,k)}$ are the analogues to the lower semi–convolutions of the matrices A_n, $n \in I\!N_0$, in the level independent case [1].

Remark: We can summarize lemma 5.6 as follows:

$$
\begin{aligned}
V_0^{(n,k)} &= \delta_{n,0} I \,, \\
V_l^{(0,k)} &= \sum_{i=1}^{l} A_i^{(k)} V_{l-1}^{(i-1,k)} \quad \text{for } l \geq 1 \,, \\
V_l^{(n,k)} &= \begin{cases} \displaystyle\sum_{i=0}^{l-n} A_i^{(k+n)} V_{l-1}^{(n+i-1,k)} \,, & 1 \leq n \leq l \\ O \,, & n > l \geq 1 \end{cases} \,,
\end{aligned}
$$

where $\delta_{n,0} = 1$ if $n = 0$ and $\delta_{n,0} = 0$ otherwise. Using matrix notation these equations can be written as

$$
\begin{aligned}
V_l^{(0,k)} &= \left(A_1^{(k)}, A_2^{(k)}, \ldots, A_l^{(k)} \right) \begin{pmatrix} V_{l-1}^{(0,k)} \\ V_{l-1}^{(1,k)} \\ \vdots \\ V_{l-1}^{(l-1,k)} \end{pmatrix} \quad \text{for } l \geq 1 \,, \\
V_l^{(n,k)} &= \left(A_0^{(k+n)}, A_1^{(k+n)}, \ldots, A_{l-n}^{(k+n)} \right) \begin{pmatrix} V_{l-1}^{(n-1,k)} \\ V_{l-1}^{(n,k)} \\ \vdots \\ V_{l-1}^{(l-1,k)} \end{pmatrix} \quad \text{for } 1 \leq n \leq l \,.
\end{aligned}
$$

So for $l \geq 1$

$$
\begin{pmatrix} V_l^{(0,k)} \\ V_l^{(1,k)} \\ V_l^{(2,k)} \\ \vdots \\ V_l^{(l,k)} \end{pmatrix} = \begin{pmatrix} A_1^{(k)} & A_2^{(k)} & \ldots & A_l^{(k)} \\ A_0^{(k+1)} & A_1^{(k+1)} & \ldots & A_{l-1}^{(k+1)} \\ & A_0^{(k+2)} & \ldots & A_{l-2}^{(k+2)} \\ & & \ddots & \vdots \\ & & & A_0^{(k+l)} \end{pmatrix} \begin{pmatrix} V_{l-1}^{(0,k)} \\ V_{l-1}^{(1,k)} \\ \vdots \\ V_{l-1}^{(l-1,k)} \end{pmatrix} . \tag{5.2}
$$

Notation: For $l \in I\!N$ the $(l+1) \times l$ block matrices $\mathcal{A}_l^{(k)}$ are defined by

$$\mathcal{A}_l^{(k)} := \begin{pmatrix} A_1^{(k)} & A_2^{(k)} & \dots & A_l^{(k)} \\ A_0^{(k+1)} & A_1^{(k+1)} & \dots & A_{l-1}^{(k+1)} \\ & A_0^{(k+2)} & \dots & A_{l-2}^{(k+2)} \\ & & \ddots & \vdots \\ & & & A_0^{(k+l)} \end{pmatrix} \tag{5.3}$$

(empty entries shall represent the zero matrix O).

Lemma 5.7 *The matrices $V_l^{(0,k)}$ are given by*

$$V_l^{(0,k)} = \left[\prod_{i=0}^{l-1} \mathcal{A}_{l-i}^{(k)} \right]_0 ,$$

where $[\cdot]_0$ denotes the 0th block of the vector.

Proof: Applying (5.2) recursively we obtain

$$\begin{pmatrix} V_l^{(0,k)} \\ V_l^{(1,k)} \\ \vdots \\ V_l^{(l,k)} \end{pmatrix} = \left(\prod_{i=0}^{l-1} \mathcal{A}_{l-i}^{(k)} \right) V_0^{(0,k)}$$

and so

$$V_l^{(0,k)} = \left[\prod_{i=0}^{l-1} \mathcal{A}_{l-i}^{(k)} \right]_0$$

since $V_0^{(0,k)} = I$ by lemma 5.6. □

Now we have explicit expressions for the matrices $G_l^{(k)}$ and $G^{(k)}$.

Theorem 5.8 *The matrices $G_l^{(k)}$ and $G^{(k)}$ are given by*

$$\begin{aligned} G_l^{(k)} &= \left[\prod_{i=0}^{l-2} \mathcal{A}_{l-1-i}^{(k)} \right]_0 A_0^{(k)}, \quad l \geq 1, \\ G^{(k)} &= \sum_{l=0}^{\infty} \left[\prod_{i=0}^{l-1} \mathcal{A}_{l-i}^{(k)} \right]_0 A_0^{(k)} . \end{aligned}$$

Proof: The assertion follows immediately from lemmata 5.5 and 5.7. □

5.1.2 Algorithm 2

Let us now define the taboo–probabilities $(W_l^{(n,k)})_{ij}$ to be the probability that starting in state (k,i) we reach $(k+n,j)$ after exactly l transitions without visiting level $k-1$ in between. Formally, $(W_l^{(n,k)})_{ij}$ is defined by

$$\left(W_l^{(n,k)}\right)_{ij} := \mathrm{P}\{X_{\nu+l}=k+n,\ J_{\nu+l}=j,\ \forall r=1,\dots,l-1: X_{\nu+r}\neq k-1 \mid X_\nu=k,\ J_\nu=i\}\,.$$

Further, $W_l^{(n,k)}$ shall be the $m^{(k)}\times m^{(k+n)}$ matrix with entries $(W_l^{(n,k)})_{ij}$. Then $W_0^{(0,k)}=I$ and $W_0^{(n,k)}=O$ for all $n\geq 1$.

Lemma 5.9 *The matrices $G_l^{(k)}$ and $G^{(k)}$ are given by*

$$\begin{aligned} G_l^{(k)} &= W_{l-1}^{(0,k)}A_0^{(k)}\,, \quad l\geq 1,\\ G^{(k)} &= \sum_{l=0}^{\infty} W_l^{(0,k)}A_0^{(k)}\,. \end{aligned}$$

Proof: Analogous to the proof of lemma 5.5. □

Lemma 5.10 *The matrices $W_l^{(n,k)}$ satisfy*

$$\begin{aligned} W_1^{(n,k)} &= A_{n+1}^{(k)} \quad \text{for } n\geq 0,\\ W_l^{(n,k)} &= \sum_{i=0}^{n+1} W_{l-1}^{(i,k)}A_{n+1-i}^{(k+i)} \quad \text{for } n\geq 0,\ l\geq 2. \end{aligned}$$

Proof: The equation for $W_1^{(n,k)}$ is obvious. By conditioning on the state reached after $l-1$ transitions we obtain the second equation. □

Lemma 5.10 shows that the matrices $W_l^{(n,k)}$ are the analogues to the upper semi–convolutions of the matrices A_n, $n\in \mathbb{N}_0$, in the level independent case [1].

Remark: In matrix notation lemma 5.10 can be summarized in the following way :

$$W_l^{(n,k)} = \left(W_{l-1}^{(0,k)}, W_{l-1}^{(1,k)},\dots,W_{l-1}^{(n+1,k)}\right)\begin{pmatrix} A_{n+1}^{(k)}\\ A_n^{(k+1)}\\ \vdots\\ A_0^{(k+n+1)}\end{pmatrix}, \quad l\geq 1,$$

and so for $l \geq 1$

$$\begin{aligned}
&\left(W_l^{(0,k)}, W_l^{(1,k)}, \ldots, W_l^{(n,k)}, \ldots\right) \\
&= \left(W_{l-1}^{(0,k)}, W_{l-1}^{(1,k)}, \ldots, W_{l-1}^{(n,k)}, \ldots\right) \begin{pmatrix} A_1^{(k)} & A_2^{(k)} & \ldots & A_{n+1}^{(k)} & \ldots \\ A_0^{(k+1)} & A_1^{(k+1)} & \ldots & A_n^{(k+1)} & \ldots \\ & A_0^{(k+2)} & \ldots & A_{n-1}^{(k+2)} & \ldots \\ & & \ddots & \vdots & \\ & & & A_0^{(k+n+1)} & \ldots \\ & & & & \ddots \end{pmatrix} \\
&= \left(W_{l-1}^{(0,k)}, W_{l-1}^{(1,k)}, \ldots, W_{l-1}^{(n,k)}, \ldots\right) \mathcal{A}_\infty^{(k)} \qquad (5.4)
\end{aligned}$$

with $\mathcal{A}_\infty^{(k)}$ being the matrix of infinite size given by (5.3).

Lemma 5.11 *The matrices $(\mathcal{A}_\infty^{(k)})^l$ are bounded and $\|(\mathcal{A}_\infty^{(k)})^l\| \leq 1$ for all $l \in \mathbb{N}_0$.*

Proof: At first, $\|(\mathcal{A}_\infty^{(k)})^0\| = \|\mathcal{I}\| = 1$. For $l = 1$ we have

$$\begin{aligned}
\sum_{n=1}^{\infty} A_n^{(k)} \mathbf{e} &= \mathbf{e} - A_0^{(k)} \mathbf{e} \leq \mathbf{e}\,, \\
\sum_{n=0}^{\infty} A_n^{(k+i)} \mathbf{e} &= \mathbf{e} \quad \text{for all } i \geq 1,
\end{aligned}$$

thus $\|(\mathcal{A}_\infty^{(k)})\| = 1$. Lemma 1.4 implies $\|(\mathcal{A}_\infty^{(k)})^l\| \leq \|(\mathcal{A}_\infty^{(k)})\|^l = 1$ for all $l \in \mathbb{N}$. □

These results enable us to derive a second set of explicit expressions for the matrices $G_l^{(k)}$ and $G^{(k)}$.

Theorem 5.12 *The matrices $G_l^{(k)}$ and $G^{(k)}$ are given by*

$$\begin{aligned}
G_l^{(k)} &= \left(\left(\mathcal{A}_\infty^{(k)}\right)^{l-1}\right)_{00} A_0^{(k)}\,, \quad l \geq 1\,, \\
G^{(k)} &= \sum_{l=0}^{\infty} \left(\left(\mathcal{A}_\infty^{(k)}\right)^{l}\right)_{00} A_0^{(k)} = \left(\left(\mathcal{I} - \mathcal{A}_\infty^{(k)}\right)'\right)_{00} A_0^{(k)}\,,
\end{aligned}$$

where $\left(\mathcal{I} - \mathcal{A}_\infty^{(k)}\right)'$ is the minimal nonnegative solution of $\left(\mathcal{I} - \mathcal{A}_\infty^{(k)}\right)\mathcal{Z} = \mathcal{I}$.

Proof: Applying (5.4) recursively we obtain

$$\left(W_l^{(0,k)}, W_l^{(1,k)}, \ldots, W_l^{(n,k)}, \ldots\right) = \left(I, O, O, \ldots\right)\left(\mathcal{A}_\infty^{(k)}\right)^l.$$

Thus $W_l^{(0,k)} = \left((\mathcal{A}_\infty^{(k)})^l\right)_{00}$ and lemma 5.9 implies

$$G_l^{(k)} = \left(\left(\mathcal{A}_\infty^{(k)}\right)^{l-1}\right)_{00} A_0^{(k)} \quad \text{and} \quad G^{(k)} = \sum_{l=0}^{\infty}\left(\left(\mathcal{A}_\infty^{(k)}\right)^{l}\right)_{00} A_0^{(k)}.$$

Let us now consider the Markov chain with state space $\{\Delta\} \cup \left(\bigcup_{l=k}^{\infty}(\{l\} \times \mathbf{J}^{(l)})\right)$, where Δ shall be an absorbing state, and transition matrix

$$\mathcal{T} = \left(\begin{array}{c|ccc} 1 & \mathbf{0}^T & \mathbf{0}^T & \ldots \\ \hline A_0^{(k)}\mathbf{e} & & & \\ \mathbf{0} & & \mathcal{A}_\infty^{(k)} & \\ \vdots & & & \end{array}\right).$$

Here all states (l,i), $l \geq k$, are transient. Then $\mathcal{S} := \sum_{l=0}^{\infty}(\mathcal{A}_\infty^{(k)})^l$ satisfies $(\mathcal{I} - \mathcal{A}_\infty^{(k)})\mathcal{S} = \mathcal{I}$ and $\mathcal{S}(\mathcal{I} - \mathcal{A}_\infty^{(k)}) = \mathcal{I}$. Further, $\mathcal{S}$ is the minimal nonnegative solution of $(\mathcal{I} - \mathcal{A}_\infty^{(k)})\mathcal{Z} = \mathcal{I}$ (by proposition 6.1.8 in [9, p. 146]). □

5.2 The mean number of service completions

Once we know the fundamental matrices $G^{(k)}$ we can also determine the mean number of service completions during a fundamental period starting in phase i of level k, which we will denote by $c_{F,i}^{(k)}$, and the mean duration[7] $t_{F,i}^{(k)}$ of a fundamental period starting in (k,i). If the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ is positive recurrent, all $c_{F,i}^{(k)}$ and $t_{F,i}^{(k)}$ are finite. Let $\mathbf{c}_F^{(k)} = (c_{F,i}^{(k)})_{i=1,\ldots,m^{(k)}}$ [8] and $\mathbf{t}_F^{(k)} = (t_{F,i}^{(k)})_{i=1,\ldots,m^{(k)}}$.

Lemma 5.13 *If the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ is positive recurrent, the vectors $\mathbf{c}_F^{(k)}$ of mean numbers of service completions during a fundamental period starting in level k satisfy the equation*

$$\mathbf{c}_F^{(k)} = \mathbf{e} + \sum_{l=1}^{\infty} A_l^{(k)}\left(\prod_{j=1}^{l-1} G^{(k+l-j)}\right)\mathbf{c}_F^{(k)} + \sum_{\nu=1}^{\infty}\sum_{l=1}^{\infty} A_{\nu+l}^{(k)}\left(\prod_{j=1}^{l-1} G^{(k+\nu+l-j)}\right)\mathbf{c}_F^{(k+\nu)}.$$

[7] The duration again refers to the time that passes in the embedding semi–Markov process $(\mathcal{Y}, \mathcal{J})$.

[8] Lucantoni [24] uses the notation μ instead of $\mathbf{c}_F$

Proof: The z-transform of the number of service completions during a fundamental period starting in level k is given by $G^{(k)}(z,0) = \sum_{l=1}^{\infty} G_l^{(k)} z^l$. Thus

$$\mathbf{c}_F^{(k)} = \frac{\partial}{\partial z} G^{(k)}(z,s)\Big|_{\substack{z=1\\s=0}} \mathbf{e}\,. \tag{5.5}$$

Differentiation in (5.1) yields

$$\begin{aligned}
&\frac{\partial}{\partial z} G^{(k)}(z,s)\\
&= \sum_{\nu=0}^{\infty} A_\nu^{(k)}(s) \left(\prod_{j=1}^{\nu} G^{(k+\nu-j)}(z,s) \right)\\
&\quad + z \sum_{\nu=1}^{\infty} A_\nu^{(k)}(s) \sum_{l=1}^{\nu} \left(\prod_{j=1}^{l-1} G^{(k+\nu-j)}(z,s) \right) \left(\frac{\partial}{\partial z} G^{(k+\nu-l)}(z,s) \right) \left(\prod_{j=l+1}^{\nu} G^{(k+\nu-j)}(z,s) \right).
\end{aligned}$$

Since we assumed $(\mathcal{X},\mathcal{J})$ to be positive recurrent, $G^{(k+\nu)}\mathbf{e} = \mathbf{e}$ for all ν (theorem 5.4). Thus

$$\begin{aligned}
\mathbf{c}_F^{(k)} &= \frac{\partial}{\partial z} G^{(k)}(z,s)\Big|_{\substack{z=1\\s=0}} \mathbf{e}\\
&= \sum_{\nu=0}^{\infty} A_\nu^{(k)} \left(\prod_{j=1}^{\nu} G^{(k+\nu-j)} \right) \mathbf{e}\\
&\quad + \sum_{\nu=1}^{\infty} A_\nu^{(k)} \sum_{l=1}^{\nu} \left(\prod_{j=1}^{l-1} G^{(k+\nu-j)} \right) \left(\frac{\partial}{\partial z} G^{(k+\nu-l)}(z,s)\Big|_{\substack{z=1\\s=0}} \right) \left(\prod_{j=l+1}^{\nu} G^{(k+\nu-j)} \right) \mathbf{e}\\
&= G^{(k)}\mathbf{e} + \sum_{\nu=1}^{\infty} A_\nu^{(k)} \sum_{l=1}^{\nu} \left(\prod_{j=1}^{l-1} G^{(k+\nu-j)} \right) \left(\frac{\partial}{\partial z} G^{(k+\nu-l)}(z,s)\Big|_{\substack{z=1\\s=0}} \right) \mathbf{e}\\
&\quad \text{by corollary 5.3}\\
&= \mathbf{e} + \sum_{\nu=1}^{\infty} A_\nu^{(k)} \sum_{l=1}^{\nu} \left(\prod_{j=1}^{l-1} G^{(k+\nu-j)} \right) \mathbf{c}_F^{(k+\nu-l)} \qquad (5.6)\\
&= \mathbf{e} + \sum_{l=1}^{\infty} \sum_{\nu=l}^{\infty} A_\nu^{(k)} \left(\prod_{j=1}^{l-1} G^{(k+\nu-j)} \right) \mathbf{c}_F^{(k+\nu-l)}\\
&= \mathbf{e} + \sum_{l=1}^{\infty} \sum_{\nu=0}^{\infty} A_{\nu+l}^{(k)} \left(\prod_{j=1}^{l-1} G^{(k+\nu+l-j)} \right) \mathbf{c}_F^{(k+\nu)}\\
&= \mathbf{e} + \sum_{l=1}^{\infty} A_l^{(k)} \left(\prod_{j=1}^{l-1} G^{(k+l-j)} \right) \mathbf{c}_F^{(k)} + \sum_{\nu=1}^{\infty} \sum_{l=1}^{\infty} A_{\nu+l}^{(k)} \left(\prod_{j=1}^{l-1} G^{(k+\nu+l-j)} \right) \mathbf{c}_F^{(k+\nu)}.
\end{aligned}$$

□

Remark: Equation 5.6 can also be obtained by conditioning on the state reached after the first service completion and adding the mean of all numbers of service completions during the fundamental periods required to reach level $k-1$.

We can solve the system of equations given in lemma 5.13 if the supremum of all $c_{F,i}^{(k)}$ is finite. This should be fulfilled in all applications.

For $k \geq 1$ and $\nu \geq 0$ let the $m^{(k)} \times m^{(k+\nu)}$ matrices $M^{(k,\nu)}$ be

$$\begin{aligned} M^{(k,0)} &:= I - \sum_{l=1}^{\infty} A_l^{(k)} \left(\prod_{j=1}^{l-1} G^{(k+l-j)} \right), \\ M^{(k,\nu)} &:= -\sum_{l=1}^{\infty} A_{\nu+l}^{(k)} \left(\prod_{j=1}^{l-1} G^{(k+\nu+l-j)} \right), \quad \nu \geq 1\,. \end{aligned}$$

If the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ is positive recurrent and $\sup_{k \in \mathbb{N}, i=1,\ldots,m^{(k)}} c_{F,i}^{(k)} < \infty$, lemma 5.13 yields

$$M^{(k,0)} \mathbf{c}_F^{(k)} + \sum_{\nu=1}^{\infty} M^{(k,\nu)} \mathbf{c}_F^{(k+\nu)} = \mathbf{e}\,. \tag{5.7}$$

Furthermore, we define a sequence $\{\mathcal{M}_n : n \in \mathbb{N}\}$ of $n \times n$ block–triangular matrices by

$$\begin{aligned} \mathcal{M}_1 &:= M^{(1,0)}, \\ \mathcal{M}_{n+1} &:= \left(\begin{array}{ccc|c} & & & M^{(1,n)} \\ & \mathcal{M}_n & & \vdots \\ & & & M^{(n,1)} \\ \hline O & \cdots & O & M^{(n+1,0)} \end{array} \right) = \begin{pmatrix} M^{(1,0)} & M^{(1,1)} & \cdots & M^{(1,n)} \\ & M^{(2,0)} & \cdots & M^{(2,n-1)} \\ & & \ddots & \vdots \\ & & & M^{(n+1,0)} \end{pmatrix} \end{aligned}$$

for $n \geq 1$.

Lemma 5.14 *The matrix of infinite size $\mathcal{M} := \mathcal{M}_\infty$ is bounded*[9].

Proof: The matrices $G^{(k)}$ are sub–stochastic[10]. Thus the kth block of $|\mathcal{M}|\,\mathbf{e}_\infty$ satisfies

$$\Big[|\mathcal{M}|\,\mathbf{e}_\infty \Big]_k = \sum_{\nu=0}^{\infty} |M^{(k,\nu)}|\,\mathbf{e}$$

[9] Cf. definition 1.3

[10] If the embedded Markov chain is (positive) recurrent they are even stochastic (theorem 5.4), but we do not require that here.

$$\begin{aligned}
&\le Ie + \sum_{l=1}^{\infty} A_l^{(k)} \left(\prod_{j=1}^{l-1} G^{(k+l-j)} \right) \mathbf{e} + \sum_{\nu=1}^{\infty} \sum_{l=1}^{\infty} A_{\nu+l}^{(k)} \left(\prod_{j=1}^{l-1} G^{(k+\nu+l-j)} \right) \mathbf{e} \\
&\le \mathbf{e} + \sum_{l=1}^{\infty} A_l^{(k)} \mathbf{e} + \sum_{\nu=1}^{\infty} \sum_{l=\nu}^{\infty} A_{l+1}^{(k)} \mathbf{e} \\
&= \mathbf{e} + \sum_{l=1}^{\infty} A_l^{(k)} \mathbf{e} + \sum_{l=1}^{\infty} \sum_{\nu=1}^{l} A_{l+1}^{(k)} \mathbf{e} \\
&= \mathbf{e} + \sum_{l=1}^{\infty} A_l^{(k)} \mathbf{e} + \sum_{l=1}^{\infty} l A_{l+1}^{(k)} \mathbf{e} \\
&= \mathbf{e} + \sum_{l=1}^{\infty} l A_l^{(k)} \mathbf{e} \\
&= \mathbf{e} + \mathbf{a}^{(k)} \\
&\le \mathbf{e} + \bar{\lambda}^{(0)} \mu^{-1} \mathbf{e}
\end{aligned}$$

by theorem 3.8 and (1.1). So the matrix $\mathcal{M}$ is bounded. □

Lemma 5.15 *The matrices $\mathcal{M}_n$ are nonsingular and their inverses are given by*

$$\mathcal{M}_1^{-1} = \left(M^{(1,0)} \right)^{-1},$$

$$\mathcal{M}_{n+1}^{-1} = \left(\begin{array}{c|c} \mathcal{M}_n^{-1} & -\mathcal{M}_n^{-1} \begin{pmatrix} M^{(1,n)} \\ \vdots \\ M^{(n,1)} \end{pmatrix} \left(M^{(n+1,0)} \right)^{-1} \\ \hline O \ \cdots \ O & \left(M^{(n+1,0)} \right)^{-1} \end{array} \right), \ n \ge 1 .$$

If the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ is positive recurrent and $\sup_{k \in \mathbb{N}, i=1,\ldots,m^{(k)}} c_{F,i}^{(k)} < \infty$, then $\mathcal{M}$ is invertible, and the matrix $\mathcal{M}' := \mathcal{M}_\infty^{-1}$ is bounded and the unique bounded inverse of $\mathcal{M}$.

Proof: To prove the first part of the lemma we start by showing that the matrices $M^{(k,0)}$ are nonsingular.

By corollary 1.16 $P_0^{(k)}(t) = e^{D_0^{(k)} t}$. So $P_0^{(k)}(t)$ is nonsingular [3, p. 170] and nonnegative, thus there must be a positive entry in each row. Therefore $A_0^{(k)} = \int_0^\infty [P_0^{(k)}(t) \mid O] \, dH(t)$ (by (3.3)) has a positive entry in each row and so

$$\left[\sum_{l=1}^{\infty} A_l^{(k)} \mathbf{e} \right]_i = \left[\mathbf{e} - A_0^{(k)} \mathbf{e} \right]_i < 1 \quad \text{for all } i = 1, \ldots, m^{(k)}. \tag{5.8}$$

Hence $\left[(I - M^{(k,0)})\mathbf{e}\right]_i = \left[\sum_{l=1}^{\infty} A_l^{(k)}\mathbf{e}\right]_i < 1$ for all $i = 1, \ldots, m^{(k)}$. This implies that $M^{(k,0)}$ is strictly diagonally dominant and therefore nonsingular [28, p. 146] with

$$\left(M^{(k,0)}\right)^{-1} = \sum_{\nu=0}^{\infty} \left(\sum_{l=1}^{\infty} A_l^{(k)} \left(\prod_{j=1}^{l-1} G^{(k+l-j)} \right) \right)^{\nu} . \tag{5.9}$$

Since the matrices $\mathcal{M}_n$ are block–triangular with nonsingular blocks on their diagonal, they are also nonsingular. Now we proceed by induction: $\mathcal{M}_1 = M^{(1,0)}$ and so $\mathcal{M}_1^{-1} = (M^{(1,0)})^{-1}$. Suppose $\mathcal{M}_n^{-1}$ is known, then

$$\begin{aligned}
&\mathcal{M}_{n+1} \cdot \left(\begin{array}{c|c} \mathcal{M}_n^{-1} & -\mathcal{M}_n^{-1} \begin{pmatrix} M^{(1,n)} \\ \vdots \\ M^{(n,1)} \end{pmatrix} \left(M^{(n+1,0)}\right)^{-1} \\ \hline O \ \cdots \ O & \left(M^{(n+1,0)}\right)^{-1} \end{array} \right) \\
&= \left(\begin{array}{c|c} \mathcal{M}_n & \begin{matrix} M^{(1,n)} \\ \vdots \\ M^{(n,1)} \end{matrix} \\ \hline O \ \cdots \ O & M^{(n+1,0)} \end{array} \right) \cdot \left(\begin{array}{c|c} \mathcal{M}_n^{-1} & -\mathcal{M}_n^{-1} \begin{pmatrix} M^{(1,n)} \\ \vdots \\ M^{(n,1)} \end{pmatrix} \left(M^{(n+1,0)}\right)^{-1} \\ \hline O \ \cdots \ O & \left(M^{(n+1,0)}\right)^{-1} \end{array} \right) \\
&= \left(\begin{array}{c|c} \mathcal{I}_n & -\mathcal{M}_n \mathcal{M}_n^{-1} \begin{pmatrix} M^{(1,n)} \\ \vdots \\ M^{(n,1)} \end{pmatrix} \left(M^{(n+1,0)}\right)^{-1} + \begin{pmatrix} M^{(1,n)} \\ \vdots \\ M^{(n,1)} \end{pmatrix} \left(M^{(n+1,0)}\right)^{-1} \\ \hline O \ \cdots \ O & I \end{array} \right) \\
&= \mathcal{I}_{n+1} \,,
\end{aligned}$$

where $\mathcal{I}_n$ denotes the identity matrix consisting of $n \times n$ blocks of appropriate size.

To prove the second part we first establish that the matrices $\mathcal{M}_n^{-1}$ are nonnegative. Equation (5.9) implies that $(M^{(k,0)})^{-1}$ is nonnegative for all $k \in \mathbb{N}$, therefore $\mathcal{M}_1^{-1} = (M^{(1,0)})^{-1}$ is nonnegative. Now suppose $\mathcal{M}_n^{-1}$ is nonnegative, then $\mathcal{M}_{n+1}^{-1}$ is also nonnegative, because $-M^{(\nu,n+1-\nu)})$, $\nu = 1, \ldots, n$, and $(M^{(n+1,0)})^{-1}$ are nonnegative and the product of nonnegative matrices is again nonnegative.

Let $C_F^{(n)}$ be the column vector with block entries $\mathbf{c}_F^{(1)}, \ldots, \mathbf{c}_F^{(n)}$ and $C_F := C_F^{(\infty)}$, then

$$\bar{\gamma} \ := \ \|C_F\|_\infty \ = \ \sup\{c_{F,i}^{(k)} : k \in \mathbb{N},\, i = 1, \ldots, m^{(k)}\} \ < \ \infty \,.$$

Equation (5.7) implies $\mathcal{M}_n C_F^{(n)} \geq \mathbf{e}$ for all $n \in I\!N$. Denote the (r,ν)th block of $\mathcal{M}_n^{-1}$ by $M'_{n;r\nu}$. Since $\mathcal{M}_n^{-1}$ is nonnegative, we obtain $C_F^{(n)} \geq \mathcal{M}_n^{-1}\mathbf{e}$, i.e.

$$\bar{\gamma}\mathbf{e} \;\geq\; \mathbf{c}_F^{(r)} \;\geq\; \sum_{\nu=1}^{n} M'_{n;r\nu}\mathbf{e}$$

for all $n \in I\!N$ and $r = 1, \ldots, n$. So $C_F \geq \mathcal{M}'\mathbf{e}_\infty$, i.e.

$$\bar{\gamma}\mathbf{e} \;\geq\; \mathbf{c}_F^{(r)} \;\geq\; \sum_{\nu=1}^{\infty} M'_{\infty;r\nu}\mathbf{e}$$

for all $r \in I\!N$ and

$$\bar{\gamma}\mathbf{e} \;\geq\; \sup_{n \in I\!N} \sum_{\nu=1}^{n} M'_{\infty;n\nu}\mathbf{e}\,.$$

Thus $\mathcal{M}'$ is bounded.

Altogether we have

$$\mathcal{M}' \cdot \mathcal{M} \;=\; \mathcal{M}_\infty^{-1} \cdot \mathcal{M}_\infty \;=\; \mathcal{I}_\infty$$

and

$$\mathcal{M} \cdot \mathcal{M}' \;=\; \mathcal{M}_\infty \cdot \mathcal{M}_\infty^{-1} \;=\; \mathcal{I}_\infty\,.$$

So $\mathcal{M}$ is invertible and by theorem 3.1.2 in [15, p. 54], $\mathcal{M}'$ is the unique inverse of $\mathcal{M}$ in the space of bounded linear operators on the space of bounded complex sequences ℓ_∞. $\square$

Theorem 5.16 *If the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ is positive recurrent and* $\sup\{c_{F,i}^{(k)} : k \in I\!N,\, i = 1, \ldots, m^{(k)}\} < \infty$, *the vectors $\mathbf{c}_F^{(k)}$ of mean numbers of service completions during a fundamental period starting in level k are the unique solution of the linear system of equations of lemma 5.13 and are given by*

$$\begin{pmatrix} \mathbf{c}_F^{(1)} \\ \mathbf{c}_F^{(2)} \\ \mathbf{c}_F^{(3)} \\ \vdots \end{pmatrix} \;=\; \mathcal{M}'\,\mathbf{e}_\infty\,.$$

Proof: Let again C_F be the column sequence consisting of the vectors $\mathbf{c}_F^{(1)}, \mathbf{c}_F^{(2)}, \ldots$. Since $\sup\{c_{F,i}^{(k)} : k \in I\!N,\, i = 1, \ldots, m^{(k)}\} < \infty$, lemmata 5.13 and 5.14 imply that $\mathcal{M}\, C_F = \mathbf{e}_\infty$. So the assertion follows from lemma 5.15. $\square$

We can now also determine the mean duration of a fundamental period. As one would expect, it is the mean service time μ^{-1} times the mean number of service completions during a fundamental period $\mathbf{c}_F^{(k)}$.

Theorem 5.17 *If the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ is positive recurrent, the vector $\mathbf{t}_F^{(k)}$ of the phase dependent mean duration of a fundamental period starting in level k is given by*

$$\mathbf{t}_F^{(k)} = \mu^{-1}\mathbf{c}_F^{(k)} .$$

Proof: Let $S_{\nu,i}^{(k)}$ be the random variable of the service time of the νth customer served and $C_{F,i}^{(k)}$ be the random variable of the number of service completions during a fundamental period starting in level k, phase i. Then the random variables $S_{1,i}^{(k)}, S_{2,i}^{(k)}, \ldots$ are independent and identically distributed, and $C_{F,i}^{(k)}$ is a stopping time for $S_{1,i}^{(k)}, S_{2,i}^{(k)}, \ldots$. Further, $\mathrm{E}[S_{1,i}^{(k)}] = \mu^{-1} < \infty$ and $\mathrm{E}[C_{F,i}^{(k)}] = c_{F,i}^{(k)} < \infty$, since $(\mathcal{X}, \mathcal{J})$ is positive recurrent. Wald's equation [39, p. 59] implies

$$t_{F,i}^{(k)} \;=\; \mathrm{E}\left[\sum_{\nu=1}^{C_{F,i}^{(k)}} S_{\nu,i}^{(k)}\right] \;=\; \mathrm{E}\left[C_{F,i}^{(k)}\right] \cdot \mathrm{E}\left[S_{1,i}^{(k)}\right] \;=\; c_{F,i}^{(k)}\,\mu^{-1} .$$

□

6 The queue length at service completion times

The distribution of the queue length[11] at service completion times is the stationary distribution of the embedded Markov chain $(\mathcal{X}, \mathcal{J})$, if the latter one is ergodic[12].

Henceforth we will assume that $(\mathcal{X}, \mathcal{J})$ is ergodic. Let $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$ with blocks $\mathbf{x}_k = (x_{ki})_{i=1,\ldots,m^{(k)}}$ be its stationary distribution, i.e.

$$x_{ki} = \lim_{\nu \to \infty} \mathrm{P}\{X_\nu = k,\ J_\nu = i \mid X_0 = l,\ J_0 = j\}$$

for all $k, l \in \mathbb{N}_0$ and $i = 1, \ldots, m^{(k)}$, $j = 1, \ldots, m^{(l)}$. Since the set $\boldsymbol{X}^c$ is transient we already know that $x_{ki} = 0$ for all $(k, i) \in \boldsymbol{X}^c$, i.e. $i \in \mathbf{J}^{(k)} \setminus \mathbf{J}^{(k+1)}$.

From $\mathbf{x}\mathcal{P} = \mathbf{x}$ we obtain

$$\mathbf{x}_k = \mathbf{x}_0 B_k + \sum_{\nu=1}^{k+1} \mathbf{x}_\nu A_{k+1-\nu}^{(\nu)}\,. \tag{6.1}$$

Unfortunately equation (6.1) is not feasible for determining $\mathbf{x}$, since the matrices $A_0^{(\nu)}$ need not be invertible. But the components $\mathbf{x}_k$, $k \geq 1$, of $\mathbf{x}$ can be computed by a recursion analogous to the one developed by Ramaswami for the level independent case [34]. This recursion has the additional advantage to be numerically stable, because all matrices involved are nonnegative.

Theorem 6.1 *The vectors* $\mathbf{x}_k$*,* $k \geq 1$*, satisfy*

$$\mathbf{x}_k = \left(\mathbf{x}_0 \bar{B}_k + \sum_{i=1}^{k-1} \mathbf{x}_i \bar{A}_{k+1-i}^{(i)} \right) \left(I - \bar{A}_1^{(k)} \right)^{-1},$$

where

$$\bar{A}_n^{(k)} = \sum_{\nu=n}^{\infty} A_\nu^{(k)} \left(\prod_{j=0}^{\nu-n-1} G^{(k+\nu-1-j)} \right) \qquad \textit{and} \qquad \bar{B}_k = \sum_{\nu=k}^{\infty} B_\nu \left(\prod_{j=0}^{\nu-k-1} G^{(\nu-j)} \right).$$

[11] By queue length we mean the number of customers in the system, including the one in service.
[12] Cf. section 4

Proof: For $k \geq 1$ we consider the Markov chain embedded at epochs of visits to the levels $0, \ldots, k$. Since we assumed $(\mathcal{X}, \mathcal{J})$ to be ergodic, the embedded chain is also positive recurrent and its transition probability matrix $\mathcal{P}_k$ is (cf. [8])

$$\mathcal{P}_k = \begin{pmatrix} B_0 & B_1 & B_2 & \cdots & B_{k-1} & \bar{B}_k \\ A_0^{(1)} & A_1^{(1)} & A_2^{(1)} & \cdots & A_{k-1}^{(1)} & \bar{A}_k^{(1)} \\ & A_0^{(2)} & A_1^{(2)} & \cdots & A_{k-2}^{(2)} & \bar{A}_{k-1}^{(2)} \\ & & A_0^{(3)} & \cdots & A_{k-3}^{(3)} & \bar{A}_{k-2}^{(3)} \\ & & & \ddots & \vdots & \vdots \\ & & & & A_0^{(k)} & \bar{A}_1^{(k)} \end{pmatrix}.$$

This structure of $\mathcal{P}_k$ is due to the fact that the process is skip–free downward. The entries of column k of $\mathcal{P}_k$ give the probabilities to reach level k or a higher level and then to return to k in the original Markov chain. Theorem 8.15 in [8, p. 166][13] implies that

$$(\mathbf{x}_0, \ldots, \mathbf{x}_k)\mathcal{P}_k = (\mathbf{x}_0, \ldots, \mathbf{x}_k)\,.$$

So we obtain

$$\mathbf{x}_k \;=\; \mathbf{x}_0 \bar{B}_k + \sum_{i=1}^{k} \mathbf{x}_i \bar{A}_{k+1-i}^{(i)}\,,$$

or equivalently

$$\mathbf{x}_k\left(I - \bar{A}_1^{(k)}\right) \;=\; \mathbf{x}_0 \bar{B}_k + \sum_{i=1}^{k-1} \mathbf{x}_i \bar{A}_{k+1-i}^{(i)}\,.$$

The matrix $A_0^{(k)} + \bar{A}_1^{(k)}$ is stochastic, and in the proof of lemma 5.15 we have seen that $A_0^{(k)}$ has a positive entry in each row. Thus all row sums of $\bar{A}_1^{(k)}$ are less than 1 and so $I - \bar{A}_1^{(k)}$ is nonsingular [28, p. 146]. $\square$

Remark: The matrix $(I - \bar{A}_1^{(k)})^{-1}$ in theorem 6.1 can be computed by

$$\left(I - \bar{A}_1^{(k)}\right)^{-1} = \sum_{\nu=0}^{\infty} \left(\bar{A}_1^{(k)}\right)^{\nu}.$$

So it only remains to determine the vector $\mathbf{x}_0$. To do so, we proceed similarly to the level independent case [24], [33]. First we note that x_{0i} is the reciprocal of the mean recurrence

[13] Theorem 8.15 in [8] implies: If $\mathcal{P} = \begin{pmatrix} C & D \\ E & F \end{pmatrix}$, where C is of size $(k+1) \times (k+1)$, then $\mathcal{P}_k = C + D(\sum_{n=0}^{\infty} F^n)E$. Thus, if $x = (a, b)$, where a is of size $k+1$, with $x\mathcal{P} = x$, then $aC + bE = a$ and $aD + bF = b$, so $a\mathcal{P}_k = aC + aD(\sum_{n=0}^{\infty} F^n)E = aC + (b - bF)(\sum_{n=0}^{\infty} F^n)E = aC + bE = a$.

time of the state $(0,i)$ in the Markov chain $(\mathcal{X},\mathcal{J})$ [9, p. 158]. Now we consider the embedded Markov chain at its visits to level 0 only. Let $(\tilde{K}_l(x))_{ij}$ be the probability that the first passage from state $(0,i)$ to state $(0,j)$ occurs in exactly l transitions not later than time x, and that $(0,j)$ is the first state visited in level 0. Formally, $(\tilde{K}_l(x))_{ij}$ is defined by

$$\left(\tilde{K}_l(x)\right)_{ij} := \\ \mathrm{P}\{X_{\nu+l}=0,\, J_{\nu+l}=j,\, T_{\nu+l}-T_\nu \le x,\, \forall r=1,\ldots,l-1 : X_{\nu+r}\neq 0 \mid X_\nu = 0,\, J_\nu = i\}\,.$$

Note that $(\tilde{K}_l(x))_{ij}=0$ for all $(0,j)\in \boldsymbol{\mathcal{X}}^c$, i.e. $j\in \mathbf{J}^{(0)}\setminus \mathbf{J}^{(1)}$. Let $\tilde{K}_l(x)$ be the matrix with entries $(\tilde{K}_l(x))_{ij}$ for $i,j=1,\ldots,m^{(0)}$, and define the transform matrices $K_l(s)$ and $K(z,s)$ by

$$\begin{aligned} K_l(s) &= \int_0^\infty e^{-sx}\, d\tilde{K}_l(x)\,, \\ K(z,s) &= \sum_{l=1}^{\infty} z^l\, K_l(s) = \sum_{l=1}^{\infty} z^l \int_0^\infty e^{-sx}\, d\tilde{K}_l(x) \end{aligned}$$

for $|z|\le 1$ and $\mathrm{Re}(s)\ge 0$. Further, let

$$K_l = K_l(0) = \tilde{K}_l(\infty) \qquad \text{and} \qquad K = K(1,0) = \sum_{l=1}^{\infty} K_l\,.$$

So $(K)_{ij}$ is the probability that starting in state $(0,i)$ the next state visited in level 0 is $(0,j)$. Since we assumed $(\mathcal{X},\mathcal{J})$ to be positive recurrent, K is stochastic. Furthermore, K is the transition probability matrix of the finite state Markov chain embedded at epochs of visits to level 0. Thus there exists a probability vector κ such that $\kappa K=\kappa$.

Let $c_{B,i}$ be the mean number of transitions between two consecutive visits to level 0 if the last state visited in level 0 was $(0,i)$. So $c_{B,i}$ is the phase dependent mean number of service completions during a busy period. Then $\mathbf{c}_B=(c_{B,i})_{i=1,\ldots,m^{(0)}}$ [14] is given by

$$\mathbf{c}_B = \frac{\partial}{\partial z} K(z,s)\Big|_{\substack{z=1\\ s=0}} \mathbf{e}\,.$$

Further, let $t_{B,i}$ be the mean time[15] between two consecutive visits to level 0 if the last state visited in level 0 was $(0,i)$, and let $\mathbf{t}_B=(t_{B,i})_{i=1,\ldots,m^{(0)}}$. Then we have

$$\mathbf{t}_B = -\frac{\partial}{\partial s} K(z,s)\Big|_{\substack{z=1\\ s=0}} \mathbf{e}\,.$$

[14] Lucantoni [24] uses the notation κ^* instead of $\mathbf{c}_B$.

[15] Again we mean the time passing in the embedding semi-Markov process $(\mathcal{Y},\mathcal{J})$.

Ramaswami [33, Th. 3.2.11] has shown that in the level independent case the vector $\mathbf{x}_0$ is given by

$$\mathbf{x}_0 = \frac{\kappa}{\langle \kappa, \mathbf{c}_B \rangle}\,, \tag{6.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product ("dot product") of two vectors. This is a general result for positive recurrent semi–Markov processes [20, Th. 2.9, Th. 2.11(a)] and hence also holds in our case. We only need to determine the matrix K and the vector $\mathbf{c}_B$.

Theorem 6.2 *The joint transform $K(z,s)$ is given by*

$$K(z,s) \;=\; \left(sI - D_0^{(0)}\right)^{-1} \sum_{j=1}^{\infty} D_j^{(0)} \left(\prod_{i=0}^{j-1} G^{(j-i)}(z,s) \right)$$

for $|z| \leq 1$ and $\mathrm{Re}(s) \geq 0$.

Proof: Analogously to the proof of lemma 5.1 we obtain $K_1(s) = B_0(s)$ and

$$K_l(s) \;=\; \sum_{n=1}^{l-1} B_n(s) \sum_{\substack{l_1,\ldots,l_n \geq 1 \\ l_1+\ldots+l_n = l-1}} \prod_{i=1}^{n} G_{l_i}^{(n+1-i)}(s)$$

for $l \geq 2$. Thus

$$\begin{aligned}
K(z,s) \;&=\; \sum_{l=1}^{\infty} K_l(s)\, z^l \\
&=\; zB_0(s) + \sum_{l=1}^{\infty} \sum_{n=1}^{l} B_n(s) \sum_{\substack{l_1,\ldots,l_n \geq 1 \\ l_1+\ldots+l_n = l}} \left(\prod_{i=1}^{n} G_{l_i}^{(n+1-i)}(s) \right) z^{l+1} \\
&=\; zB_0(s) + z \sum_{n=1}^{\infty} B_n(s) \left(\prod_{i=1}^{n} G^{(n+1-i)}(z,s) \right) \\
&\qquad \text{(cf. proof of theorem 5.2)} \\
&=\; z \left(sI - D_0^{(0)}\right)^{-1} D_1^{(0)} A_0^{(1)}(s) \\
&\qquad + z \sum_{n=1}^{\infty} \left(\left(sI - D_0^{(0)}\right)^{-1} \sum_{j=0}^{n} D_{j+1}^{(0)} A_{n-j}^{(j+1)}(s) \right) \left(\prod_{i=1}^{n} G^{(n+1-i)}(z,s) \right) \\
&\qquad \text{by theorem 3.4} \\
&=\; z \left(sI - D_0^{(0)}\right)^{-1} \left[\sum_{n=0}^{\infty} D_1^{(0)} A_n^{(1)}(s) \left(\prod_{i=1}^{n} G^{(n+1-i)}(z,s) \right) \right.
\end{aligned}$$

$$
\begin{aligned}
& \quad + \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} D_{j+1}^{(0)} A_{n-j}^{(j+1)}(s) \left(\prod_{i=1}^{n} G^{(n+1-i)}(z,s) \right) \Bigg] \\
= \; & z \left(sI - D_0^{(0)} \right)^{-1} \Bigg[D_1^{(0)} \sum_{n=0}^{\infty} A_n^{(1)}(s) \left(\prod_{i=1}^{n} G^{(n+1-i)}(z,s) \right) \\
& \quad + \sum_{j=2}^{\infty} D_j^{(0)} \sum_{n=0}^{\infty} A_n^{(j)}(s) \left(\prod_{i=1}^{n} G^{(n+j-i)}(z,s) \right) \left(\prod_{i=n+1}^{n+j-1} G^{(n+j-i)}(z,s) \right) \Bigg] \\
= \; & z \left(sI - D_0^{(0)} \right)^{-1} \Bigg[D_1^{(0)} \sum_{n=0}^{\infty} A_n^{(1)}(s) \left(\prod_{i=1}^{n} G^{(n+1-i)}(z,s) \right) \\
& \quad + \sum_{j=2}^{\infty} D_j^{(0)} \sum_{n=0}^{\infty} A_n^{(j)}(s) \left(\prod_{i=1}^{n} G^{(n+j-i)}(z,s) \right) \left(\prod_{i=1}^{j-1} G^{(j-i)}(z,s) \right) \Bigg] \\
= \; & \left(sI - D_0^{(0)} \right)^{-1} \sum_{j=1}^{\infty} D_j^{(0)} z \sum_{n=0}^{\infty} A_n^{(j)}(s) \left(\prod_{i=1}^{n} G^{(n+j-i)}(z,s) \right) \left(\prod_{i=1}^{j-1} G^{(j-i)}(z,s) \right) \\
= \; & \left(sI - D_0^{(0)} \right)^{-1} \sum_{j=1}^{\infty} D_j^{(0)} G^{(j)}(z,s) \left(\prod_{i=1}^{j-1} G^{(j-i)}(z,s) \right) \\
& \text{by theorem 5.2} \\
= \; & \left(sI - D_0^{(0)} \right)^{-1} \sum_{j=1}^{\infty} D_j^{(0)} \left(\prod_{i=0}^{j-1} G^{(j-i)}(z,s) \right) .
\end{aligned}
$$

□

Corollary 6.3 *The matrix K is given by*

$$
K \;=\; -{D_0^{(0)}}^{-1} \sum_{j=1}^{\infty} D_j^{(0)} \left(\prod_{i=0}^{j-1} G^{(j-i)} \right) .
$$

Proof: Since $K = K(1,0)$, the assertion follows immediately from theorem 6.2. □

Remark: The matrix $-D_0^{(0)} K = \sum_{l=1}^{\infty} D_l^{(0)} \prod_{\nu=0}^{l-1} G^{(l-\nu)}$ has an interpretation analogous to the one of $D[G] = \sum_{l=0}^{\infty} D_l G^l$ in the level independent case [24, p. 16]. Consider the arrival process $(\mathcal{N}^{(0)}, \mathcal{J}^{(0)})$ at a time epoch t_0 during an idle period, and let $J^{(0)}(t_0) = i$. During the infinitesimal time interval $(t_0, t_0 + dt)$, $(\mathcal{N}^{(0)}, \mathcal{J}^{(0)})$ could stay in phase i, or could change to phase $j' \neq i$ without an arrival with probability $(D_0^{(0)})_{ij'} dt$, or it could change to phase h with an arrival of a batch of size $l \geq 1$ with probability $(D_l^{(0)})_{ih} dt$.

That arriving batch initiates a busy period which ends in phase j with conditional probability $(\prod_{\nu=0}^{l-1} G^{(l-\nu)})_{hj}$. If we exclude the time interval corresponding to this busy period, we obtain a transition from i to j with probability $(\sum_{l=1}^{\infty} D_l^{(0)} \prod_{\nu=0}^{l-1} G^{(l-\nu)})_{ij} dt$. The matrix $\sum_{l=0}^{\infty} D_l^{(0)} \prod_{\nu=0}^{l-1} G^{(l-\nu)} = -D_0^{(0)} K + D_0^{(0)}$ can therefore be considered as the infinitesimal generator of a Markov process obtained by excluding the busy periods. Hence $K = I - {D_0^{(0)}}^{-1} \sum_{l=0}^{\infty} D_l^{(0)} \prod_{\nu=0}^{l-1} G^{(l-\nu)}$ is the corresponding transition probability matrix.

Now we can also determine the mean number of service completions during a busy period and the mean time between two consecutive visits to level 0.

Corollary 6.4 *The vector of the phase dependent mean number of service completions during a busy period is*

$$\mathbf{c}_B = -{D_0^{(0)}}^{-1} \sum_{j=1}^{\infty} D_j^{(0)} \sum_{l=0}^{j-1} \left(\prod_{i=0}^{l-1} G^{(j-i)} \right) \mathbf{c}_F^{(j-l)} .$$

Proof: Since we assumed $(\mathcal{X}, \mathcal{J})$ to be positive recurrent, $G^{(j)}$ is stochastic for all j (theorem 5.4). Theorem 6.2 yields

$$\begin{aligned}
&\mathbf{c}_B \\
&= \frac{\partial}{\partial z} K(z,s) \Big|_{\substack{z=1\\ s=0}} \mathbf{e} \\
&= \left(sI - D_0^{(0)}\right)^{-1} \sum_{j=1}^{\infty} D_j^{(0)} \sum_{l=0}^{j-1} \left(\prod_{i=0}^{l-1} G^{(j-i)}(z,s) \right) \left(\tfrac{\partial}{\partial z} G^{(j-l)}(z,s)\right) \left(\prod_{i=l+1}^{j-1} G^{(j-i)}(z,s) \right) \Big|_{\substack{z=1\\ s=0}} \mathbf{e} \\
&= -{D_0^{(0)}}^{-1} \sum_{j=1}^{\infty} D_j^{(0)} \sum_{l=0}^{j-1} \left(\prod_{i=0}^{l-1} G^{(j-i)} \right) \mathbf{c}_F^{(j-l)} \qquad \text{by (5.5).}
\end{aligned}$$

□

The mean time between two consecutive visits to level 0 is just the mean idle time plus the mean number of service completions during a busy period times the mean service time.

Corollary 6.5 *The vector of the phase dependent mean time between two consecutive visits to level 0 is*

$$\mathbf{t}_B = -{D_0^{(0)}}^{-1} \mathbf{e} + \mu^{-1} \mathbf{c}_B .$$

Proof: Analogously to the proof of corollary 6.4 we obtain

$$
\begin{aligned}
&\mathbf{t}_B \\
&= -\tfrac{\partial}{\partial s} K(z,s)\Big|_{\substack{z=1\\ s=0}} \mathbf{e} \\
&= \left(sI - D_0^{(0)}\right)^{-2} \sum_{j=1}^{\infty} D_j^{(0)} \left(\prod_{i=0}^{j-1} G^{(j-i)}(z,s) \right)\Bigg|_{\substack{z=1\\ s=0}} \mathbf{e} \\
&\quad - \left(sI - D_0^{(0)}\right)^{-1} \sum_{j=1}^{\infty} D_j^{(0)} \sum_{l=0}^{j-1} \left(\prod_{i=0}^{l-1} G^{(j-i)}(z,s) \right) \left(\tfrac{\partial}{\partial s} G^{(j-l)}(z,s)\right) \left(\prod_{i=l+1}^{j-1} G^{(j-i)}(z,s) \right)\Bigg|_{\substack{z=1\\ s=0}} \mathbf{e} \\
&= {D_0^{(0)}}^{-2} \sum_{j=1}^{\infty} D_j^{(0)} \mathbf{e} + {D_0^{(0)}}^{-1} \sum_{j=1}^{\infty} D_j^{(0)} \sum_{l=0}^{j-1} \left(\prod_{i=0}^{l-1} G^{(j-i)} \right) \left(-\mathbf{t}_F^{(j-l)} \right) \\
&= {D_0^{(0)}}^{-2} \left(-D_0^{(0)} \mathbf{e} \right) - \mu^{-1} {D_0^{(0)}}^{-1} \sum_{j=1}^{\infty} D_j^{(0)} \sum_{l=0}^{j-1} \left(\prod_{i=0}^{l-1} G^{(j-i)} \right) \mathbf{c}_F^{(j-l)} \\
&\qquad \text{(by theorem 5.17)} \\
&= -{D_0^{(0)}}^{-1} \mathbf{e} + \mu^{-1} \mathbf{c}_B \quad \text{(by corollary 6.4).}
\end{aligned}
$$

□

If we define the load ρ of the queueing system by the mean number of arrivals during a service time[16], we have

$$\rho = \mathbf{x}_0 \mathbf{b} + \sum_{k=1}^{\infty} \mathbf{x}_k \mathbf{a}^{(k)} ,$$

with $\mathbf{b}$ and $\mathbf{a}^{(k)}$ defined in (3.8) and (3.7). Now the probability that a departing customer leaves an empty system behind is equal to $1 - \rho$.

Theorem 6.6 *The steady state probability of level* 0 *satisfies*

$$\mathbf{x}_0 \mathbf{e} = 1 - \rho \,.$$

Proof: Using (6.1) we obtain

$$\sum_{k=0}^{\infty} k\, \mathbf{x}_k \mathbf{e} \quad = \quad \sum_{k=0}^{\infty} k \left(\mathbf{x}_0 B_k \mathbf{e} + \sum_{\nu=1}^{k+1} \mathbf{x}_\nu A_{k+1-\nu}^{(\nu)} \mathbf{e} \right)$$

[16]If the system was empty before, the load is given by the mean number of arrivals during an idle period and the following service time minus the one which is in service.

$$
\begin{aligned}
&= \mathbf{x}_0 \sum_{k=0}^{\infty} k B_k \mathbf{e} + \sum_{k=0}^{\infty} k \sum_{\nu=0}^{k} \mathbf{x}_{\nu+1} A_{k-\nu}^{(\nu+1)} \mathbf{e} \\
&= \mathbf{x}_0 \mathbf{b} + \sum_{\nu=0}^{\infty} \sum_{k=\nu}^{\infty} k\, \mathbf{x}_{\nu+1} A_{k-\nu}^{(\nu+1)} \mathbf{e} \\
&= \mathbf{x}_0 \mathbf{b} + \sum_{\nu=0}^{\infty} \mathbf{x}_{\nu+1} \left(\sum_{k=0}^{\infty} k A_k^{(\nu+1)} \mathbf{e} + \sum_{k=0}^{\infty} \nu A_k^{(\nu+1)} \mathbf{e} \right) \\
&= \mathbf{x}_0 \mathbf{b} + \sum_{\nu=0}^{\infty} \mathbf{x}_{\nu+1} \mathbf{a}^{(\nu+1)} + \sum_{\nu=0}^{\infty} \nu\, \mathbf{x}_{\nu+1} \mathbf{e} \\
&= \mathbf{x}_0 \mathbf{b} + \sum_{\nu=1}^{\infty} \mathbf{x}_{\nu} \mathbf{a}^{(\nu)} + \sum_{\nu=1}^{\infty} \nu\, \mathbf{x}_{\nu} \mathbf{e} - \sum_{\nu=1}^{\infty} \mathbf{x}_{\nu} \mathbf{e} \\
&= \rho + \sum_{\nu=1}^{\infty} \nu\, \mathbf{x}_{\nu} \mathbf{e} - (1 - \mathbf{x}_0 \mathbf{e})\,.
\end{aligned}
$$

Thus $1 - \mathbf{x}_0 \mathbf{e} = \rho$. □

Summary: The vector $\mathbf{x}$ of the queue length distribution at service completion times (in equilibrium) can be computed as follows:

$$
\mathbf{x}_0 = \frac{\kappa}{\langle \kappa, \mathbf{c}_B \rangle}\,,
$$

where κ is the stationary distribution of the Markov chain defined by the matrix K, and $\mathbf{c}_B$ is given by corollary 6.4. Further, for $k \geq 1$,

$$
\mathbf{x}_k = \left(\mathbf{x}_0 \bar{B}_k + \sum_{i=1}^{k-1} \mathbf{x}_i \bar{A}_{k+1-i}^{(i)} \right) \left(I - \bar{A}_1^{(k)} \right)^{-1},
$$

with the matrices $\bar{A}_n^{(k)}$ and $\bar{B}_k$ defined in theorem 6.1.

7 The queue length at an arbitrary time

The steady state distribution of the queue length[17] at an arbitrary time is the limiting distribution of the process $(\mathcal{Y}, \mathcal{J})$, if the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ is positive recurrent (cf. section 4).

Henceforth we will assume that $(\mathcal{X}, \mathcal{J})$ is positive recurrent. Let $\mathbf{y} = (\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \ldots)$, $\mathbf{y}_k = (y_{ki})_{i=1,\ldots,m^{(k)}}$ be the limiting distribution of $\{Y(t),\ J(t) : t \geq 0\}$, i.e.

$$y_{ki} = \lim_{t\to\infty} \mathrm{P}\{Y(t) = k,\ J(t) = i \mid Y(0) = l,\ J(0) = j\}$$

for all $k, l \in I\!N_0$ and $i = 1, \ldots, m^{(k)}$, $j = 1, \ldots, m^{(l)}$.

The components $\mathbf{y}_k$ of $\mathbf{y}$ can be obtained by applying the key renewal theorem [9, p. 295], similarly to the level independent case [33].

Theorem 7.1 *The vector* $\mathbf{y}_0$ *is given by*

$$\mathbf{y}_0 \;=\; -\frac{1}{\mu^{-1} - \mathbf{x}_0 {D_0^{(0)}}^{-1}\mathbf{e}}\, \mathbf{x}_0 {D_0^{(0)}}^{-1}.$$

Proof: At first we note that $(P_0^{(0)}(t))_{ij}$ is monotone nonincreasing and by corollary 1.16 continuous. Thus, by proposition 9.2.16(c) in [9, p. 296], $(P_0^{(0)}(t))_{ij}$ is directly Riemann integrable for all $i, j = 1, \ldots, m^{(0)}$.

We consider $\mathrm{P}\{Y(t) = 0,\ J(t) = i \mid Y(0) = 0,\ J(0) = j\}$ and condition on the state at the last service completion time before t, which must be a state in level 0. Say this last return to level 0 occurs at time $u \leq t$, and let $R_{(0,j),(0,\nu)}(u)$ be the mean number of visits to state $(0, \nu)$ up to time u, given $(0, j)$ at time 0. So $R_{(0,j),(0,\nu)}(u)$ is a Markov renewal function [9, p. 319]. Then

$$\mathrm{P}\{Y(t) = 0,\ J(t) = i \mid Y(0) = 0,\ J(0) = j\} = \sum_{\nu=1}^{m^{(0)}} \int_0^t \Big(P_0^{(0)}(t-u)\Big)_{\nu i}\, dR_{(0,j),(0,\nu)}(u)\,.$$

If we consider the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ at its visits to level 0 only, we obtain a Markov renewal process with state space $\{1, \ldots, m^{(0)}\}$. Now the key renewal theorem for Markov renewal processes with finite state space [9, p. 331] yields

$$y_{0i} \;=\; \left(\sum_{\nu=1}^{m^{(0)}} \kappa_\nu\, t_{B,\nu}\right)^{-1} \sum_{j=1}^{m^{(0)}} \kappa_j \left(\int_0^\infty P_0^{(0)}(t)\, dt\right)_{ji}.$$

[17] By queue length we mean the number of customers in the system, including the one in service.

Using corollaries 1.16 and 1.17 we obtain

$$\int_0^\infty P_0^{(0)}(t)\,dt \;=\; -D_0^{(0)^{-1}}\,. \tag{7.1}$$

Corollary 6.5 and (6.2) imply

$$\begin{aligned}\mathbf{y}_0 &= \frac{\kappa}{-\kappa D_0^{(0)^{-1}}\mathbf{e}+\mu^{-1}\langle\kappa,\mathbf{c}_B\rangle}\left(-D_0^{(0)^{-1}}\right)\\ &= -\frac{1}{-\frac{\kappa}{\langle\kappa,\mathbf{c}_B\rangle}D_0^{(0)^{-1}}\mathbf{e}+\mu^{-1}}\,\frac{\kappa}{\langle\kappa,\mathbf{c}_B\rangle}\,D_0^{(0)^{-1}}\\ &= -\frac{1}{-\mathbf{x}_0 D_0^{(0)^{-1}}\mathbf{e}+\mu^{-1}}\,\mathbf{x}_0 D_0^{(0)^{-1}}\,.\end{aligned}$$

□

In a similar way we can also determine the steady state probabilities of the levels $k \geq 1$.

Theorem 7.2 *For $k \geq 1$ the vector $\mathbf{y}_k$ is given by*

$$\mathbf{y}_k \;=\; \frac{1}{\mu^{-1}-\mathbf{x}_0 D_0^{(0)^{-1}}\mathbf{e}}\sum_{l=1}^{k}\left(\mathbf{x}_l-\mathbf{x}_0 D_0^{(0)^{-1}}D_l^{(0)}\right)\int_0^\infty P_{k-l}^{(l)}(t)\Big(1-H(t)\Big)\,dt\,.$$

Proof: We consider $\mathrm{P}\{Y(t)=k,\ J(t)=i \mid Y(0)=0,\ J(0)=j\}$ and condition on the state at the last service completion time before t. Again, $P_n^{(k)}(t)$ is directly Riemann integrable for all $n \in \mathbb{N}_0$ and so is $\sum_{n=0}^{\infty}\mathbf{x}_n P_n^{(k)}(t)$. The limit theorem for Markov renewal processes [9, p. 334] yields

$$\begin{aligned} y_{ki} \;=&\\ &\frac{1}{\langle\mathbf{x},\delta\rangle}\sum_{j=1}^{m^{(0)}}x_{0j}\int_{t=0}^{\infty}\int_{x=0}^{t}\Big(1-H(t-x)\Big)\sum_{l=1}^{k}\sum_{\nu=1}^{m^{(0)}}\left(P_0^{(0)}(x)\right)_{j\nu}\left(D_l^{(0)}P_{k-l}^{(l)}(t-x)\right)_{\nu i}dx\,dt\\ &+\frac{1}{\langle\mathbf{x},\delta\rangle}\sum_{l=1}^{k}\sum_{j=1}^{m^{(l)}}x_{lj}\int_0^\infty\left(P_{k-l}^{(l)}(t)\right)_{ji}\Big(1-H(t)\Big)dt\,,\end{aligned}$$

where $\delta=(\delta_0,\delta_1,\delta_2,\ldots)$ and δ_{ki} is the mean time between two consecutive transitions in the embedded Markov chain $(\mathcal{X},\mathcal{J})$, as defined in theorem 3.6. Again, $\langle\cdot,\cdot\rangle$ denotes the inner product of two vectors. In matrix–vector notation we have

$$\begin{aligned}\mathbf{y}_k \;=\;& \frac{\mathbf{x}_0}{\langle\mathbf{x},\delta\rangle}\int_{t=0}^{\infty}\int_{x=0}^{t}\Big(1-H(t-x)\Big)\sum_{l=1}^{k}P_0^{(0)}(x)\,D_l^{(0)}P_{k-l}^{(l)}(t-x)\,dx\,dt\\ &+\sum_{l=1}^{k}\frac{\mathbf{x}_l}{\langle\mathbf{x},\delta\rangle}\int_0^\infty P_{k-l}^{(l)}(t)\Big(1-H(t)\Big)dt\,.\end{aligned}$$

At first,

$$\begin{aligned}
&\int_{t=0}^{\infty}\int_{x=0}^{t}\Big(1-H(t-x)\Big)\sum_{l=1}^{k}P_0^{(0)}(x)\,D_l^{(0)}P_{k-l}^{(l)}(t-x)\,dx\,dt\\
&= \int_{x=0}^{\infty}\int_{t=x}^{\infty}\Big(1-H(t-x)\Big)\sum_{l=1}^{k}P_0^{(0)}(x)\,D_l^{(0)}P_{k-l}^{(l)}(t-x)\,dt\,dx\\
&= \int_{x=0}^{\infty}P_0^{(0)}(x)\,dx\,\int_{t=0}^{\infty}\Big(1-H(t)\Big)\sum_{l=1}^{k}D_l^{(0)}P_{k-l}^{(l)}(t)\,dt\\
&= -D_0^{(0)^{-1}}\sum_{l=1}^{k}D_l^{(0)}\int_{t=0}^{\infty}\Big(1-H(t)\Big)P_{k-l}^{(l)}(t)\,dt\,,
\end{aligned}$$

by (7.1). For $\langle \mathbf{x},\delta\rangle$ we obtain

$$\begin{aligned}
\langle \mathbf{x},\delta\rangle &= \mathbf{x}_0\Big(-D_0^{(0)^{-1}}\mathbf{e}+\mu^{-1}\mathbf{e}\Big)+\sum_{k=1}^{\infty}\mathbf{x}_k\mu^{-1}\mathbf{e} \;=\; -\mathbf{x}_0D_0^{(0)^{-1}}\mathbf{e}+\mu^{-1}\sum_{k=0}^{\infty}\mathbf{x}_k\mathbf{e}\\
&= -\mathbf{x}_0D_0^{(0)^{-1}}\mathbf{e}+\mu^{-1}\,.
\end{aligned}$$

Altogether we have

$$\begin{aligned}
\mathbf{y}_k &= \frac{1}{\langle \mathbf{x},\delta\rangle}\sum_{l=1}^{k}\Big(\mathbf{x}_l-\mathbf{x}_0D_0^{(0)^{-1}}D_l^{(0)}\Big)\int_0^{\infty}P_{k-l}^{(l)}(t)\Big(1-H(t)\Big)\,dt\\
&= \frac{1}{-\mathbf{x}_0D_0^{(0)^{-1}}\mathbf{e}+\mu^{-1}}\sum_{l=1}^{k}\Big(\mathbf{x}_l-\mathbf{x}_0D_0^{(0)^{-1}}D_l^{(0)}\Big)\int_0^{\infty}P_{k-l}^{(l)}(t)\Big(1-H(t)\Big)\,dt\,.
\end{aligned}$$

□

Summary: The vector $\mathbf{y}$ of the queue length distribution at an arbitrary time can be computed by

$$\begin{aligned}
\mathbf{y}_0 &= -\frac{1}{\mu^{-1}-\mathbf{x}_0D_0^{(0)^{-1}}\mathbf{e}}\,\mathbf{x}_0D_0^{(0)^{-1}} \quad \text{and}\\
\mathbf{y}_k &= \frac{1}{\mu^{-1}-\mathbf{x}_0D_0^{(0)^{-1}}\mathbf{e}}\sum_{l=1}^{k}\Big(\mathbf{x}_l-\mathbf{x}_0D_0^{(0)^{-1}}D_l^{(0)}\Big)\int_0^{\infty}P_{k-l}^{(l)}(t)\Big(1-H(t)\Big)\,dt
\end{aligned}$$

for $k\geq 1$.

8 A special case: Level independent phase process

Let us consider the special case when only the arrival rates dependent on the current level, but the phase process is level independent.

Assumption 8.1 *Suppose* $\mathbf{J}^{(k)} = \mathbf{J}^{(0)} =: \mathbf{J}$, *i.e.* $m^{(k)} = m < \infty$, *and*

$$D^{(k)} := \sum_{n=0}^{\infty} D_n^{(k)} = D^{(0)} =: D$$

for all $k \in \mathbb{N}_0$.

Let $D^{(k)}(z)$ denote the z–transform of the matrices $D_n^{(k)}$, $n \in \mathbb{N}_0$, i.e.

$$D^{(k)}(z) \;=\; \sum_{n=0}^{\infty} D_n^{(k)} z^n \,, \quad |z| \leq 1.$$

The vector $\mathbf{d}^{(k)}$ of the phase dependent mean arrival rates in level k (as defined in section 1) is then given by

$$\mathbf{d}^{(k)} \;=\; \frac{d}{dz} D^{(k)}(z)\Big|_{z=1} \mathbf{e}\,.$$

8.1 The arrival process

At first we will show that under assumption 8.1 all phase processes $\{J^{(k)}(t) : t \geq 0\}$ are stochastically identical, so that we can speak of just one phase process. Then the phase process possesses a stationary probability distribution, which we will denote by π according to the level independent case [24]. This enables us to obtain further results for the BMAP/G/1 queue with level dependent arrivals in this special case.

8.1.1 The generator matrices

Assumption 8.1 implies the following properties of the matrices $\mathcal{Q}^{(k)}$:

Lemma 8.2 *For all* $k, l, j \in \mathbb{N}_0$ *it is*

$$\sum_{n=l}^{\infty} \left((\mathcal{Q}^{(k)})^j \right)_{ln} \;=\; D^j\,.$$

Proof: At first we note that $(\mathcal{Q}^{(k)})^j$ is bounded (lemmata 1.5 + 1.4). For $j = 0$ the assertion is obvious, and for $j = 1$ we have

$$\sum_{n=l}^{\infty} \left(\mathcal{Q}^{(k)}\right)_{ln} = \sum_{n=0}^{\infty} D_n^{(k+l)} = D$$

for all $k, l \in I\!N_0$. Suppose the assertion holds for j and consider $j+1$:

$$\begin{aligned}
\sum_{n=l}^{\infty} \left((\mathcal{Q}^{(k)})^{j+1}\right)_{ln} &= \sum_{n=l}^{\infty}\sum_{\nu=l}^{n} \left(\mathcal{Q}^{(k)}\right)_{l\nu}\left((\mathcal{Q}^{(k)})^j\right)_{\nu n} \\
&= \sum_{n=0}^{\infty}\sum_{\nu=0}^{n} D_\nu^{(k+l)}\left((\mathcal{Q}^{(k+\nu+l)})^j\right)_{0,n-\nu} \quad \text{by lemma 1.7} \\
&= \sum_{\nu=0}^{\infty}\sum_{n=\nu}^{\infty} D_\nu^{(k+l)}\left((\mathcal{Q}^{(k+\nu+l)})^j\right)_{0,n-\nu} \\
&= \sum_{\nu=0}^{\infty} D_\nu^{(k+l)} D^j \quad \text{by assumption} \\
&= DD^j = D^{j+1}.
\end{aligned}$$

Thus the assertion holds for all $k, l, j \in I\!N_0$. □

Corollary 8.3 *For all $k, l \in I\!N_0$ it is*

$$\sum_{n=l}^{\infty} \left(e^{\mathcal{Q}^{(k)}t}\right)_{ln} = e^{Dt}.$$

Proof: By Corollary 1.6 the matrix–exponential of $\mathcal{Q}^{(k)}$ is bounded. Lemma 8.2 yields

$$\begin{aligned}
\sum_{n=l}^{\infty} \left(e^{\mathcal{Q}^{(k)}t}\right)_{ln} &= \sum_{n=l}^{\infty}\sum_{j=0}^{\infty} \frac{t^j}{j!}\left((\mathcal{Q}^{(k)})^j\right)_{ln} = \sum_{j=0}^{\infty} \frac{t^j}{j!}\sum_{n=l}^{\infty}\left((\mathcal{Q}^{(k)})^j\right)_{ln} \\
&= \sum_{j=0}^{\infty} \frac{t^j}{j!} D^j = e^{Dt}.
\end{aligned}$$

□

Let $\mathcal{Q}^{(k)}(z)$ denote the transform matrix

$$\mathcal{Q}^{(k)}(z) := \begin{pmatrix}
D_0^{(k)} & D_1^{(k)}z & D_2^{(k)}z^2 & D_3^{(k)}z^3 & \cdots \\
 & D_0^{(k+1)} & D_1^{(k+1)}z & D_2^{(k+1)}z^2 & \cdots \\
 & & D_0^{(k+2)} & D_1^{(k+2)}z & \cdots \\
 & & & D_0^{(k+3)} & \cdots \\
 & & & & \ddots
\end{pmatrix}.$$

Lemma 8.4 *The matrices $\mathcal{Q}^{(k)}(z)$, $k \in \mathbb{N}_0$, are uniformly bounded for all $|z| \leq 1$, i.e. there exists $\theta < \infty$ such that $\|\mathcal{Q}^{(k)}(z)\| \leq \theta$ for all $k \in \mathbb{N}_0$ and all $|z| \leq 1$.*

Proof: Assumption 1.1 implies for all $k \in \mathbb{N}_0$, $i = 1, \ldots, m$ and all $|z| \leq 1$:

$$\sum_{n=0}^{\infty} \sum_{j=1}^{m} |d_{n,ij}^{(k)} z^n| \;\leq\; |d_{0,ii}^{(k)}| + \sum_{\substack{j=1 \\ j \neq i}}^{m} |d_{0,ij}^{(k)}| + \sum_{n=1}^{\infty} \sum_{j=1}^{m} |d_{n,ij}^{(k)}| \;=\; 2\,|d_{0,ii}^{(k)}| \;\leq\; 2\bar{d}_0 \;<\; \infty$$

since $\mathcal{Q}^{(k)}$ is a generator matrix. Thus the lth block of $|\mathcal{Q}^{(k)}(z)|\mathbf{e}_\infty$ satisfies

$$\Big[|\mathcal{Q}^{(k)}(z)|\mathbf{e}_\infty\Big]_l \;=\; \sum_{n=0}^{\infty} |D_n^{(k+l)} z^n| \mathbf{e} \;\leq\; 2\bar{d}_0\, \mathbf{e}\,.$$

So the matrices $\mathcal{Q}^{(k)}(z)$ are uniformly bounded for all $|z| \leq 1$. □

Corollary 8.5 *The exponential of the matrix $\mathcal{Q}^{(k)}(z)$ exists for all $|z| \leq 1$, i.e. its entries are all finite.*

Proof: By lemma 8.4 $\mathcal{Q}^{(k)}(z)$ is bounded, and so by lemma 1.4 the matrix exponential $e^{\mathcal{Q}^{(k)}(z)}$ exists for all $|z| \leq 1$. □

Lemma 8.6 *For all $j, l, n \in \mathbb{N}_0$ with $l \leq n$ it is*

$$\left((\mathcal{Q}^{(k)}(z))^j\right)_{ln} = \left((\mathcal{Q}^{(k+l)}(z))^j\right)_{0,n-l}$$

for all $|z| \leq 1$.

Proof: The proof is completely analogous to the proof of lemma 1.7. For $j = 0$ the result is obvious, and for $j = 1$ we have

$$\left(\mathcal{Q}^{(k)}(z)\right)_{ln} \;=\; D_{n-l}^{(k+l)} z^{n-l} \;=\; \left(\mathcal{Q}^{(k+l)}(z)\right)_{0,n-l}.$$

Suppose the assertion is proven for j and consider $j+1$:

$$\begin{aligned}
\left((\mathcal{Q}^{(k)}(z))^{j+1}\right)_{ln} &= \sum_{\nu=l}^{n} \left(\mathcal{Q}^{(k)}(z)\right)_{l\nu} \left((\mathcal{Q}^{(k)}(z))^j\right)_{\nu n} \\
&= \sum_{\nu=l}^{n} \left(\mathcal{Q}^{(k+l)}(z)\right)_{0,\nu-l} \left((\mathcal{Q}^{(k+\nu)}(z))^j\right)_{0,n-\nu} \\
&\ \text{by assumption} \\
&= \sum_{\nu=0}^{n-l} \left(\mathcal{Q}^{(k+l)}(z)\right)_{0\nu} \left((\mathcal{Q}^{(k+\nu+l)}(z))^j\right)_{0,n-\nu-l}
\end{aligned}$$

and

$$\begin{aligned}\left(\left(\mathcal{Q}^{(k+l)}(z)\right)^{j+1}\right)_{0,n-l} &= \sum_{\nu=0}^{n-l} \left(\mathcal{Q}^{(k+l)}(z)\right)_{0\nu} \left(\left(\mathcal{Q}^{(k+l)}(z)\right)^{j}\right)_{\nu,n-l} \\ &= \sum_{\nu=0}^{n-l} \left(\mathcal{Q}^{(k+l)}(z)\right)_{0\nu} \left(\left(\mathcal{Q}^{(k+l+\nu)}(z)\right)^{j}\right)_{0,n-l-\nu} \\ & \text{by assumption.}\end{aligned}$$

So $\left(\left(\mathcal{Q}^{(k)}(z)\right)^{j+1}\right)_{ln} = \left(\left(\mathcal{Q}^{(k+l)}(z)\right)^{j+1}\right)_{0,n-l}$ and thus the assertion is proven for all $j \in I\!N_0$. □

Corollary 8.7 *For all $l, n \in I\!N_0$ with $l \leq n$ and all $t \geq 0$ it is*

$$\left(e^{\mathcal{Q}^{(k)}(z)\,t}\right)_{ln} = \left(e^{\mathcal{Q}^{(k+l)}(z)\,t}\right)_{0,n-l}.$$

Proof: Lemma 8.6 implies

$$\left(e^{\mathcal{Q}^{(k)}(z)\,t}\right)_{ln} = \sum_{j=0}^{\infty} \frac{t^j}{j!}\left(\left(\mathcal{Q}^{(k)}(z)\right)^{j}\right)_{ln} = \sum_{j=0}^{\infty} \frac{t^j}{j!}\left(\left(\mathcal{Q}^{(k+l)}(z)\right)^{j}\right)_{0,n-l} = \left(e^{\mathcal{Q}^{(k+l)}(z)\,t}\right)_{0,n-l}.$$

□

8.1.2 The phase process

We are now able to show that the transition probabilities of the phase process do not depend on the current level.

Theorem 8.8 *The transition probabilities of the phase processes $\{J^{(k)}(t) : t \geq 0\}$ are given by*

$$\mathrm{P}\{J^{(k)}(t) = j \mid J^{(k)}(0) = i\} = \left(e^{Dt}\right)_{ij}, \quad i, j = 1, \ldots, m$$

for all $k \in I\!N_0$.

Proof: From theorem 1.14 and corollary 8.3 we obtain

$$\sum_{n=0}^{\infty} P_n^{(k)}(t) = \sum_{n=0}^{\infty} \left(e^{\mathcal{Q}^{(0)}t}\right)_{k,n+k} = e^{Dt}$$

for all $k \in \mathbb{N}_0$. Thus for any $l \in \mathbb{N}_0$

$$\begin{aligned}
&\mathrm{P}\{J^{(k)}(t) = j \mid N^{(k)}(0) = l,\ J^{(k)}(0) = i\} \\
&= \sum_{n=0}^{\infty} \mathrm{P}\{N^{(k)}(t) = n + l,\ J^{(k)}(t) = j \mid N^{(k)}(0) = l,\ J^{(k)}(0) = i\} \\
&= \sum_{n=0}^{\infty} \left(P_n^{(k+l)}(t)\right)_{ij} \;=\; \left(e^{Dt}\right)_{ij}
\end{aligned}$$

and therefore

$$\begin{aligned}
&\mathrm{P}\{J^{(k)}(t) = j \mid J^{(k)}(0) = i\} \\
&= \sum_{l=0}^{\infty} \mathrm{P}\{J^{(k)}(t) = j \mid N^{(k)}(0) = l,\ J^{(k)}(0) = i\}\, \mathrm{P}\{N^{(k)}(0) = l \mid J^{(k)}(0) = i\} \\
&= \left(e^{Dt}\right)_{ij} \sum_{l=0}^{\infty} \mathrm{P}\{N^{(k)}(0) = l \mid J^{(k)}(0) = i\} \;=\; \left(e^{Dt}\right)_{ij}
\end{aligned}$$

for all $k \in \mathbb{N}_0$. □

So assumption 8.1 implies that the processes $\{J^{(k)}(t) : t \geq 0\}$ are stochastically identical for all $k \in \mathbb{N}_0$. Henceforth we will denote the phase process by $\{J(t) : t \geq 0\}$. Let $R(t)$ be its transition probability matrix, i.e.

$$\left(R(t)\right)_{ij} \;=\; \mathrm{P}\{J(t) = j \mid J(0) = i\}, \quad i, j = 1, \ldots, m\,.$$

Corollary 8.9 *The transition probability matrix $R(t)$ of the phase process $\{J(t) : t \geq 0\}$ is given by*

$$R(t) \;=\; e^{Dt}\,.$$

Proof: The assertion follows immediately from theorem 8.8. □

Corollary 8.10 *The matrix D is the generator matrix of the phase process $\{J(t) : t \geq 0\}$.*

Proof: Corollary 8.9 implies $\frac{d}{dt}R(t)|_{t=0} \;=\; D$ and thus D is the generator matrix of $\{J(t) : t \geq 0\}$. □

By assumption 3.1 the phase process is irreducible. Thus there exists a unique stationary probability vector π such that $\pi D = \mathbf{0}$ and $\pi R(t) = \pi$.

This enables us to define the mean arrival rate (or fundamental arrival rate [24]) $\lambda^{(k)}$ in level k by

$$\lambda^{(k)} \;:=\; \pi \mathbf{d}^{(k)} \;=\; \pi \sum_{n=1}^{\infty} n D_n^{(k)} \mathbf{e}\,.$$

8.1.3 The transition probabilities

In section 1 we considered the transition probabilities $P_n^{(k)}(t)$ of the level dependent BMAP $(\mathcal{N}^{(k)}, \mathcal{J}^{(k)}) = \{N^{(k)}(t), J^{(k)}(t) : t \geq 0\}$. Their z–transform $P^{(k)}(z,t)$ and the phase dependent mean number of arrivals in time t, $\mathbf{n}^{(k)}(t)$, can now be defined by

$$\begin{aligned} P^{(k)}(z,t) &= \sum_{n=0}^{\infty} P_n^{(k)}(t)\, z^n\,, \\ \mathbf{n}^{(k)}(t) &= \tfrac{\partial}{\partial z} P^{(k)}(z,t)\Big|_{z=1} \mathbf{e}\,. \end{aligned}$$

Theorem 8.11 *The z–transforms $P^{(k)}(z,t)$ are given by*

$$\begin{pmatrix} P^{(0)}(z,t) \\ P^{(1)}(z,t) \\ P^{(2)}(z,t) \\ \vdots \end{pmatrix} = e^{\mathcal{Q}^{(0)}(z)\, t} \begin{pmatrix} I \\ I \\ I \\ \vdots \end{pmatrix}$$

and

$$P^{(k)}(z,t) = \sum_{n=k}^{\infty} \left(e^{\mathcal{Q}^{(0)}(z)\, t} \right)_{kn} = \sum_{n=0}^{\infty} \left(e^{\mathcal{Q}^{(k)}(z)\, t} \right)_{0n}$$

for $|z| \leq 1$.

Proof: The backward differential equations (lemma 1.10) yield:

$$\tfrac{\partial}{\partial t} P^{(k)}(z,t) = \sum_{n=0}^{\infty} \left(\tfrac{\partial}{\partial t} P_n^{(k)}(t) \right) z^n = \sum_{n=0}^{\infty} \sum_{u=0}^{n} D_u^{(k)} P_{n-u}^{(k+u)}(t)\, z^n\,.$$

The matrices $\mathcal{Q}^{(k)}$ and $\mathcal{P}^{(k)}(t)$ are bounded, thus by lemma 1.4 $\mathcal{Q}^{(k)}\mathcal{P}^{(k)}(t)$ is bounded and so the series $\sum_{n=0}^{\infty} \sum_{u=0}^{n} D_u^{(k)} P_{n-u}^{(k+u)}(t)\, z^n$ converges absolutely for all $|z| \leq 1$. This implies:

$$\begin{aligned} \tfrac{\partial}{\partial t} P^{(k)}(z,t) &= \sum_{u=0}^{\infty} \sum_{n=u}^{\infty} D_u^{(k)} P_{n-u}^{(k+u)}(t)\, z^n = \sum_{u=0}^{\infty} D_u^{(k)}\, z^u\, P^{(k+u)}(z,t) \\ &= \left(D_0^{(k)}, D_1^{(k)} z, D_2^{(k)} z^2, \ldots \right) \begin{pmatrix} P^{(k)}(z,t) \\ P^{(k+1)}(z,t) \\ P^{(k+2)}(z,t) \\ \vdots \end{pmatrix}. \end{aligned}$$

Thus

$$\frac{\partial}{\partial t}\begin{pmatrix} P^{(0)}(z,t) \\ P^{(1)}(z,t) \\ P^{(2)}(z,t) \\ \vdots \end{pmatrix} = \begin{pmatrix} D_0^{(0)} & D_1^{(0)}z & D_2^{(0)}z^2 & \cdots \\ & D_0^{(1)} & D_1^{(1)}z & \cdots \\ & & D_0^{(2)} & \cdots \\ & & & \ddots \end{pmatrix}\begin{pmatrix} P^{(0)}(z,t) \\ P^{(1)}(z,t) \\ P^{(2)}(z,t) \\ \vdots \end{pmatrix}$$
$$= \mathcal{Q}^{(0)}(z)\begin{pmatrix} P^{(0)}(z,t) \\ P^{(1)}(z,t) \\ P^{(2)}(z,t) \\ \vdots \end{pmatrix}.$$

Further, we have $P^{(k)}(z,0) = I$ for all $k \in I\!N_0$. By lemma 8.4 the matrix $\mathcal{Q}^{(0)}(z)$ is bounded and by corollary 8.5 its exponential exists for all $|z| \leq 1$. Therefore the unique solution of the above differential equation is given by

$$\begin{pmatrix} P^{(0)}(z,t) \\ P^{(1)}(z,t) \\ P^{(2)}(z,t) \\ \vdots \end{pmatrix} = e^{\mathcal{Q}^{(0)}(z)\,t}\begin{pmatrix} I \\ I \\ I \\ \vdots \end{pmatrix}$$

for $|z| \leq 1$. The second equation follows by applying corollary 8.7. $\square$

By differentiating $P^{(k)}(z,t)\mathbf{e}$ with respect to z we obtain the result for $\mathbf{n}^{(k)}(t)$ given in theorem 1.18. The mean number of arrivals in time t is now $\pi\mathbf{n}^{(k)}(t)$.

Let $\bar{d}_i^{(k)}$ denote the maximum arrival rate in phase i from level k on, i.e.

$$\bar{d}_i^{(k)} := \sup\left\{d_i^{(l)} \,:\, l \geq k\right\},$$

and let $\bar{\mathbf{d}}^{(k)}$ be the vector with components $\bar{d}_i^{(k)}$, $i = 1, \ldots, m$.

Theorem 8.12 *The mean number of arrivals in time t satisfies*

$$\pi\mathbf{n}^{(k)}(t) \;\leq\; \pi\bar{\mathbf{d}}^{(k)}t\,.$$

Proof: In the proof of theorem 1.19 we have seen that

$$\tfrac{d}{ds}\mathbf{n}^{(k)}(s) \;=\; \sum_{u=0}^{\infty} P_u^{(k)}(s)\,\mathbf{d}^{(k+u)}\,.$$

Thus

$$\tfrac{d}{ds}\pi\mathbf{n}^{(k)}(s) \;\le\; \pi\sum_{u=0}^{\infty} P_u^{(k)}(s)\,\bar{\mathbf{d}}^{(k)} \;=\; \pi R(s)\,\bar{\mathbf{d}}^{(k)} \;=\; \pi\bar{\mathbf{d}}^{(k)}$$

for all $s \ge 0$. Hence

$$\pi\mathbf{n}^{(k)}(t) \;=\; \int_0^t \tfrac{d}{ds}\pi\mathbf{n}^{(k)}(s)\,ds \;\le\; \pi\bar{\mathbf{d}}^{(k)}t\,.$$

□

8.2 The embedded Markov chain

Assumption 8.1 implies that the state space of the embedded Markov chain $(\mathcal{X},\mathcal{J})$ is the set $\boldsymbol{X} = \boldsymbol{Y}$ defined in section 3. Thus the embedded Markov chain is irreducible.

8.2.1 The transition matrix

Let us define the joint transform matrices of the entries $\tilde{A}_n^{(k)}(x)$ and $\tilde{B}_n(x)$ of the transition probability matrix $\tilde{\mathcal{P}}(x)$ by

$$\begin{aligned} A^{(k)}(z,s) &:= \sum_{n=0}^{\infty} z^n\,A_n^{(k)}(s) \;=\; \sum_{n=0}^{\infty} z^n \int_0^{\infty} e^{-sx}\,d\tilde{A}_n^{(k)}(x)\,, \\ B(z,s) &:= \sum_{n=0}^{\infty} z^n\,B_n(s) \;=\; \sum_{n=0}^{\infty} z^n \int_0^{\infty} e^{-sx}\,d\tilde{B}_n(x) \end{aligned}$$

for $|z| \le 1$ and $\mathrm{Re}(s) \ge 0$. Further, we define the matrices

$$A^{(k)} \;:=\; A^{(k)}(1,0) \;=\; \sum_{n=0}^{\infty} A_n^{(k)} \quad\text{and}\quad B \;:=\; B(1,0) \;=\; \sum_{n=0}^{\infty} B_n\,.$$

Equations (3.4) and (3.5) imply that $A^{(k)}$ and B are stochastic.

Theorem 8.11 enables us to determine the joint transform matrices $A^{(k)}(z,s)$ and the matrices $A^{(k)}$.

Theorem 8.13 *The transform matrices $A^{(k)}(z,s)$ are given by*

$$\begin{pmatrix} A^{(1)}(z,s) \\ A^{(2)}(z,s) \\ A^{(3)}(z,s) \\ \vdots \end{pmatrix} \;=\; \int_0^{\infty} e^{-st} e^{\mathcal{Q}^{(1)}(z)\,t} \begin{pmatrix} I \\ I \\ I \\ \vdots \end{pmatrix} dH(t)$$

for $|z| \le 1$ and $\mathrm{Re}(s) \ge 0$.

Proof: Using (3.3) we obtain

$$\begin{aligned} A^{(k)}(z,s) &= \sum_{n=0}^{\infty} z^n A_n^{(k)}(s) = \sum_{n=0}^{\infty} z^n \int_0^{\infty} e^{-st} P_n^{(k)}(t)\, dH(t) \\ &= \int_0^{\infty} e^{-st} \sum_{n=0}^{\infty} z^n P_n^{(k)}(t)\, dH(t) = \int_0^{\infty} e^{-st} P^{(k)}(z,t)\, dH(t)\,. \end{aligned}$$

Theorem 8.11 yields

$$\begin{pmatrix} A^{(1)}(z,s) \\ A^{(2)}(z,s) \\ \vdots \end{pmatrix} = \int_0^{\infty} e^{-st} \begin{pmatrix} P^{(1)}(z,t) \\ P^{(2)}(z,t) \\ \vdots \end{pmatrix} dH(t) = \int_0^{\infty} e^{-st} e^{\mathcal{Q}^{(1)}(z)\,t} \begin{pmatrix} I \\ I \\ \vdots \end{pmatrix} dH(t)$$

for $|z| \leq 1$ and $\mathrm{Re}(s) \geq 0$. □

Corollary 8.14 *The matrices $A^{(k)}$ satisfy $A^{(k)} = A^{(1)} =: A$ for all $k \in I\!N$ and*

$$A = \int_0^{\infty} e^{Dt}\, dH(t)\,.$$

Proof: Theorem 8.13 and corollary 8.3 imply

$$A^{(k)} = A^{(k)}(1,0) = \int_0^{\infty} \sum_{n=k-1}^{\infty} \left(e^{\mathcal{Q}^{(1)}\,t}\right)_{k-1,n} dH(t) = \int_0^{\infty} e^{Dt}\, dH(t)$$

for all $k \in I\!N$ (note that we number the rows and columns of $e^{\mathcal{Q}^{(1)}\,t}$ beginning with 0). □

Corollary 8.15 *The stationary probability distribution π of the phase process is also the stationary vector of the matrix A, i.e. $\pi A = \pi$.*

Proof: Since $\pi D = \mathbf{0}$, corollary 8.14 yields

$$\pi A = \int_0^{\infty} \pi\, dH(t) = \pi\,.$$

□

The matrix A is the transition matrix of the phase process in the embedded Markov chain $(\mathcal{X}, \mathcal{J})$, if the queueing system was not empty after the last service completion, i.e.

$$\mathrm{P}\{J_{\nu+1} = j \mid X_\nu = k,\ J_\nu = i\} = (A)_{ij} \quad \text{for } k \geq 1.$$

So π is the stationary probability distribution of the phase process in the embedded Markov chain $(\mathcal{X}, \mathcal{J})$, if the queueing system was not empty after the last service completion.

We can now determine the joint transform $B(z,s)$ and the matrix B.

Theorem 8.16 *The joint transform matrix $B(z,s)$ is given by*

$$B(z,s) \;=\; \left(sI - D_0^{(0)}\right)^{-1} z^{-1} \sum_{l=1}^{\infty} D_l^{(0)} z^l A^{(l)}(z,s)$$

for $|z| \leq 1$ and $\mathrm{Re}(s) \geq 0$.

Proof: Using theorem 3.4 we obtain

$$\begin{aligned}
B(z,s) \;&=\; \sum_{n=0}^{\infty} B_n(s)\, z^n \;=\; \left(sI - D_0^{(0)}\right)^{-1} \sum_{n=0}^{\infty} \sum_{l=0}^{n} D_{l+1}^{(0)} A_{n-l}^{(l+1)}(s)\, z^n \\
&=\; \left(sI - D_0^{(0)}\right)^{-1} \sum_{l=0}^{\infty} \sum_{n=l}^{\infty} D_{l+1}^{(0)} A_{n-l}^{(l+1)}(s)\, z^n \\
&=\; \left(sI - D_0^{(0)}\right)^{-1} \sum_{l=0}^{\infty} D_{l+1}^{(0)}\, z^l\, A^{(l+1)}(z,s)
\end{aligned}$$

for all $|z| \leq 1$ and $\mathrm{Re}(s) \geq 0$. □

Corollary 8.17 *The matrix B is given by*

$$B \;=\; \left(I - {D_0^{(0)}}^{-1} D\right) A\,.$$

Proof: Theorem 8.16 implies

$$B \;=\; B(1,0) \;=\; -{D_0^{(0)}}^{-1} \sum_{l=1}^{\infty} D_l^{(0)} A \;=\; -{D_0^{(0)}}^{-1} \left(D - D_0^{(0)}\right) A \;=\; \left(I - {D_0^{(0)}}^{-1} D\right) A\,.$$

□

Note that $\pi B \neq \pi$, i.e. π is not the stationary distribution of the Markov chain defined by B. The matrix B is the transition matrix of the phase process during an idle time and the following service time. This period must contain an arrival time, and so it is not independent of the arrival process $(\mathcal{N}^{(0)}, \mathcal{J}^{(0)})$, i.e. it is not independent of the phase process.

8.2.2 The mean number of arrivals

The vectors $\mathbf{a}^{(k)}$ and $\mathbf{b}$ (cf. section 3) can be obtained by differentiating $A^{(k)}(z,s)\mathbf{e}$ and $B(z,s)\mathbf{e}$ with respect to z. Doing so, we obtain the results given in theorems 3.7 and 3.9.

Theorem 8.18 *The mean number of arrivals during a service time satisfies*

$$\pi\mathbf{a}^{(k)} \leq \pi\bar{\mathbf{d}}^{(k)}\mu^{-1}\,.$$

Proof: Theorem 8.12 implies

$$\pi\mathbf{a}^{(k)} \;=\; \int_0^\infty \pi\mathbf{n}^{(k)}(t)\,dH(t) \;\leq\; \int_0^\infty \pi\bar{\mathbf{d}}^{(k)}t\,dH(t) \;=\; \pi\bar{\mathbf{d}}^{(k)}\mu^{-1}\,.$$

□

8.3 Stability

In the special case of assumption 8.1 we can improve the stability condition for the BMAP/G/1 queue with level dependent arrivals given in theorem 4.2. To do so, we need to generalize a result of Pakes [32, Th. 1] which was the key to theorem 4.1.

Let P_{kl}, $k,l \in I\!N_0$, denote the (k,l)th block of the transition matrix $\mathcal{P}$. Then $P_{0l} = B_l$ for all $l \in I\!N_0$, $P_{kl} = A^{(k)}_{l-k+1}$ for $l \geq k-1 \geq 0$, and $P_{kl} = O$ otherwise.

Theorem 8.19 *If there exists some $N \in I\!N$ and nonnegative vectors $\zeta_k < \infty\mathbf{e}$, $k \in I\!N_0$, such that*

a) $\mathbf{e}\pi\zeta_k \;\leq\; \zeta_k$ *for all $k \geq N$,*

b) $\pi\sum_{l=0}^{\infty} P_{kl}\zeta_l \;\leq\; \pi\zeta_k - 1$ *for all $k \geq N$,*

c) $\pi\sum_{l=0}^{\infty} P_{kl}\zeta_l \;<\; \infty$ *for all $k = 0,\ldots,N-1$,*

then the embedded Markov chain $(\mathcal{X},\mathcal{J})$ is positive recurrent.

Proof: Let $P_{kl}^{[n]}$ denote the (k,l)th block of $\mathcal{P}^n$. Define $R_{kl} := \mathbf{e}\pi P_{kl}$ for $k,l \in I\!N_0$, and $\mathcal{R} := (R_{kl})_{k,l\in I\!N_0}$, and let $R_{kl}^{[n]}$ denote the (k,l)th block of $\mathcal{R}^n$. Then $\mathcal{R}$ is the transition matrix of the discrete Markov chain $(\mathcal{X}', \mathcal{J}')$ obtained by starting with the stationary distribution of $\{J(t) : t \geq 0\}$ and applying the transition matrix $\mathcal{P}$.

We will first show that $(\mathcal{X}', \mathcal{J}')$ is positive recurrent, proceeding similarly to Pakes [32, Th. 1]. Let $\mathbf{z}_l = (z_{li})_{i=1,\dots,m}$ be the lth block of the limiting distribution of $(\mathcal{X}', \mathcal{J}')$. Then $\lim_{n\to\infty} \mathcal{R}^n = (\mathbf{e}\mathbf{z}_l)_{k,l\in I\!N_0}$, i.e. $\lim_{n\to\infty} R_{kl}^{[n]} = \mathbf{e}\mathbf{z}_l$ for all $k,l \in I\!N_0$, and $(\mathcal{X}', \mathcal{J}')$ is positive recurrent if and only if there exists some (l,i) such that $z_{li} > 0$ (and hence $z_{li} > 0$ for all $(l,i) \in \boldsymbol{X}$) [13, pp. 389 + 393].

Let

$$\theta_k \ := \ \pi \sum_{l=0}^{\infty} P_{kl}\zeta_l \quad \text{for } k = 0,\dots,N-1,$$

then $\theta_k < \infty$ by assumption. Further, define the sequences $\zeta_k^{[n]}$ for $n \in I\!N$ and $k \in I\!N_0$ by

$$\zeta_k^{[1]} := \zeta_k \quad \text{and} \quad \zeta_k^{[n+1]} := \sum_{l=0}^{\infty} R_{kl}^{[n]}\zeta_l\,.$$

Then

$$\begin{aligned}
\zeta_k^{[n+2]} &= \sum_{l=0}^{\infty}\sum_{\nu=0}^{\infty} R_{k\nu}^{[n]} R_{\nu l}\zeta_l \\
&= \sum_{\nu=0}^{N-1} R_{k\nu}^{[n]}\mathbf{e}\pi \sum_{l=0}^{\infty} P_{\nu l}\zeta_l + \sum_{\nu=N}^{\infty} R_{k\nu}^{[n]}\mathbf{e}\pi \sum_{l=0}^{\infty} P_{\nu l}\zeta_l \\
&\leq \sum_{\nu=0}^{N-1} R_{k\nu}^{[n]}\mathbf{e}\theta_\nu + \sum_{\nu=N}^{\infty} R_{k\nu}^{[n]}\mathbf{e}(\pi\zeta_\nu - 1) \quad \text{(by assumption)} \\
&\leq \sum_{\nu=0}^{N-1} R_{k\nu}^{[n]}\mathbf{e}\theta_\nu + \sum_{\nu=N}^{\infty} R_{k\nu}^{[n]}\zeta_\nu - \sum_{\nu=0}^{\infty} R_{k\nu}^{[n]}\mathbf{e} + \sum_{\nu=0}^{N-1} R_{k\nu}^{[n]}\mathbf{e} \quad \text{(by assumption)} \\
&\leq \sum_{\nu=0}^{N-1} R_{k\nu}^{[n]}\mathbf{e}(1+\theta_\nu) + \zeta_k^{[n+1]} - \mathbf{e}\,.
\end{aligned} \tag{8.1}$$

So $\zeta_k^{[n+1]} < \infty\mathbf{e}$ implies $\zeta_k^{[n+2]} < \infty\mathbf{e}$, and from $\zeta_k^{[1]} = \zeta_k < \infty\mathbf{e}$ we get $\zeta_k^{[n]} < \infty\mathbf{e}$ for all $n \in I\!N$. Applying (8.1) recursively we obtain

$$\zeta_k^{[n+2]} \ \leq \ \sum_{r=1}^{n}\sum_{\nu=0}^{N-1} R_{k\nu}^{[r]}\mathbf{e}(1+\theta_\nu) + \zeta_k^{[2]} - n\mathbf{e}\,,$$

so

$$\frac{1}{n}\,\zeta_k^{[n+2]} \;\le\; \frac{1}{n}\sum_{r=1}^{n}\sum_{\nu=0}^{N-1} R_{k\nu}^{[r]}\mathbf{e}(1+\theta_\nu) + \frac{1}{n}\,\zeta_k^{[2]} - \mathbf{e}\,,$$

and letting $n \to \infty$

$$0 \;\le\; \sum_{\nu=0}^{N-1} \mathbf{e}\mathbf{z}_\nu\mathbf{e}(1+\theta_\nu) - \mathbf{e}\,.$$

Thus

$$\mathbf{e} \;\le\; \sum_{\nu=0}^{N-1} \mathbf{e}\mathbf{z}_\nu\mathbf{e}(1+\theta_\nu)$$

and hence there must be some (l,i), $l \in \{0,\dots,N-1\}$, such that $z_{li} > 0$. So $(\mathcal{X}',\mathcal{J}')$ is positive recurrent.

Now, for $k \ge 1$ let $(\mathbf{a}_{\mathcal{R}}^{(k)})_i = \mathrm{E}[X'_{n+1} - X'_n \mid X'_n = k,\ J'_n = i]$. Then $\mathbf{a}_{\mathcal{R}}^{(k)}$ can be considered to be the vector of the phase dependent mean numbers of arrivals in $(\mathcal{X}',\mathcal{J}')$ during a service time starting in level $k \ge 1$. Thus, for $k \ge 1$

$$\mathbf{a}_{\mathcal{R}}^{(k)} \;=\; \sum_{n=1}^{\infty} nR_{kn}\mathbf{e} \;=\; \sum_{n=1}^{\infty} n\mathbf{e}\pi A_n^{(k)}\mathbf{e}\,.$$

Further, let $\alpha_{\mathcal{R}}^{(k)}$ be the mean number of arrivals in $(\mathcal{X}',\mathcal{J}')$ during a service time starting in level $k \ge 1$. Then $\alpha_{\mathcal{R}}^{(k)} = \pi_{\mathcal{R}}^{(k)}\mathbf{a}_{\mathcal{R}}^{(k)}$, where $\pi_{\mathcal{R}}^{(k)}$ denotes the stationary distribution of the phase process of $(\mathcal{X}',\mathcal{J}')$ in level k. So, for $k \ge 1$

$$\alpha_{\mathcal{R}}^{(k)} \;=\; \pi_{\mathcal{R}}^{(k)}\mathbf{a}_{\mathcal{R}}^{(k)} \;=\; \sum_{n=1}^{\infty} n\pi A_n^{(k)}\mathbf{e}$$

and therefore $\alpha_{\mathcal{R}}^{(k)} = \pi\mathbf{a}^{(k)}$.

In our original process the mean number $\bar{\alpha}$ of arrivals during a time period which corresponds to the length of a service time but starts at an arbitrary time is given by

$$\bar{\alpha} \;=\; \int_0^{\infty} \pi\mathbf{n}^{(k)}(t)\,dH(t) \;=\; \pi\mathbf{a}^{(k)} \;=\; \alpha_{\mathcal{R}}^{(k)}\,.$$

Hence $(\mathcal{X},\mathcal{J})$ evolves in the same way as $(\mathcal{X}',\mathcal{J}')$ and so the positive recurrence of $(\mathcal{X}',\mathcal{J}')$ implies the positive recurrence of $(\mathcal{X},\mathcal{J})$. □

Theorem 8.20 *If* $\limsup_{k\to\infty} \pi\mathbf{a}^{(k)} < 1$ *then the embedded Markov chain* $(\mathcal{X},\mathcal{J})$ *is positive recurrent.*

Proof: Again we proceed similarly to Pakes [32, Th. 2]. Since $\limsup_{k\to\infty} \pi\mathbf{a}^{(k)} < 1$ there exist $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $\pi\mathbf{a}^{(k)} \leq 1-\varepsilon$ for all $k \geq N$. Define $\zeta_k := \frac{k}{\varepsilon}\,\mathbf{e}$ for $k \in \mathbb{N}_0$, then

$$\mathbf{e}\pi\zeta_k \;=\; \frac{k}{\varepsilon}\,\mathbf{e}\pi\mathbf{e} \;=\; \frac{k}{\varepsilon}\,\mathbf{e} \;=\; \zeta_k$$

for all $k \in \mathbb{N}_0$. Further, for $k \geq N$:

$$\begin{aligned}
\pi\sum_{l=0}^{\infty} P_{kl}\zeta_l &= \pi\sum_{l=k-1}^{\infty} A_{l-k+1}^{(k)}\frac{l}{\varepsilon}\,\mathbf{e} \\
&= \frac{1}{\varepsilon}\,\pi\sum_{l=0}^{\infty} lA_l^{(k)}\mathbf{e} + \frac{k-1}{\varepsilon}\,\pi\sum_{l=0}^{\infty} A_l^{(k)}\mathbf{e} \\
&= \frac{1}{\varepsilon}\,\pi\mathbf{a}^{(k)} + \frac{k-1}{\varepsilon} \;\leq\; \frac{1}{\varepsilon}(1-\varepsilon) + \frac{k-1}{\varepsilon} \\
&= \frac{k}{\varepsilon} - 1 \;=\; \frac{k}{\varepsilon}\,\pi\mathbf{e} - 1 \;=\; \pi\zeta_k - 1\,,
\end{aligned}$$

and for $k = 1, \ldots, N-1$:

$$\begin{aligned}
\pi\sum_{l=0}^{\infty} P_{kl}\zeta_l &= \pi\sum_{l=k-1}^{\infty} A_{l-k+1}^{(k)}\frac{l}{\varepsilon}\,\mathbf{e} \;=\; \frac{1}{\varepsilon}\,\pi\Big(\mathbf{a}^{(k)} + (k-1)\mathbf{e}\Big) \\
&\leq \frac{1}{\varepsilon}\,\pi\Big(\bar{\lambda}^{(0)}\mu^{-1}\mathbf{e} + (N-1)\mathbf{e}\Big) \quad \text{(by theorem 3.8 and (1.1))} \\
&= \frac{1}{\varepsilon}\Big(\bar{\lambda}^{(0)}\mu^{-1} + (N-1)\Big) \;<\; \infty\,.
\end{aligned}$$

Finally, theorems 3.9 and 3.10 imply

$$\pi\sum_{l=0}^{\infty} P_{0l}\zeta_l \;=\; \pi\sum_{l=0}^{\infty} B_l\frac{l}{\varepsilon}\,\mathbf{e} \;=\; \frac{1}{\varepsilon}\,\pi\mathbf{b} \;\leq\; \frac{1}{\varepsilon}\,\pi\Big(\bar{\lambda}^{(0)}\delta_0 - 1\Big) \;<\; \infty\,.$$

Theorem 8.19 yields that $(\mathcal{X}, \mathcal{J})$ is positive recurrent. $\square$

Again we can derive a weaker stability condition which only relies on the arrival rates and the service rate.

Corollary 8.21 *If $\limsup_{k\to\infty} \lambda^{(k)} < \mu$ then the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ is positive recurrent.*

Proof: Remember that $\lambda^{(k)} = \pi\mathbf{d}^{(k)}$ and $\pi\mathbf{a}^{(k)} \leq \pi\bar{\mathbf{d}}^{(k)}\mu^{-1}$ (theorem 8.18). So we start by showing that $\limsup_{k\to\infty} \bar{\mathbf{d}}^{(k)} = \limsup_{k\to\infty} \mathbf{d}^{(k)}$. The definition of $\bar{\mathbf{d}}^{(k)}$ implies $\bar{\mathbf{d}}^{(k)} \geq \mathbf{d}^{(k)}$ for all $k \in \mathbb{N}_0$ and so $\limsup_{k\to\infty} \bar{\mathbf{d}}^{(k)} \geq \limsup_{k\to\infty} \mathbf{d}^{(k)}$.

Let $\bar{\mathbf{c}} := \limsup_{k\to\infty} \bar{\mathbf{d}}^{(k)}$ with components $\bar{c}_i$, $i = 1,\ldots,m$, then there are sequences $\{k_{i,n}\}_{n\in\mathbb{N}} \subset \mathbb{N}_0$ such that $\lim_{n\to\infty} \bar{d}_i^{(k_{i,n})} = \bar{c}_i$ for $i = 1,\ldots,m$. Thus for any $\varepsilon_i > 0$ there exists some $N_i \in \mathbb{N}$ such that $|\bar{d}_i^{(k_{i,n})} - \bar{c}_i| < \frac{\varepsilon_i}{2}$ for all $n \geq N_i$. Further, since $\bar{d}_i^{(k_{i,n})} = \sup\{d_i^{(r)} : r \geq k_{i,n}\}$, there is $r_{i,n} \geq k_{i,n}$ with $|d_i^{(r_{i,n})} - \bar{d}_i^{(k_{i,n})}| < \frac{\varepsilon_i}{2}$ for all $n \in \mathbb{N}$. This implies for all $n \geq N_i$:

$$\left|d_i^{(r_{i,n})} - \bar{c}_i\right| \;\leq\; \left|d_i^{(r_{i,n})} - \bar{d}_i^{(k_{i,n})}\right| + \left|\bar{d}_i^{(k_{i,n})} - \bar{c}_i\right| \;<\; \varepsilon_i\,.$$

So we have $\lim_{n\to\infty} d_i^{(r_{i,n})} = \bar{c}_i$ for $i = 1,\ldots,m$, and thus $\limsup_{k\to\infty} \mathbf{d}^{(k)} \geq \bar{\mathbf{c}}$. Altogether, $\limsup_{k\to\infty} \bar{\mathbf{d}}^{(k)} = \limsup_{k\to\infty} \mathbf{d}^{(k)}$.

Now theorem 8.18 yields

$$\limsup_{k\to\infty} \pi\mathbf{a}^{(k)} \;\leq\; \limsup_{k\to\infty} \pi\bar{\mathbf{d}}^{(k)}\mu^{-1} \;=\; \limsup_{k\to\infty} \pi\mathbf{d}^{(k)}\mu^{-1} \;=\; \limsup_{k\to\infty} \lambda^{(k)}\mu^{-1} \;<\; 1$$

and therefore $(\mathcal{X},\mathcal{J})$ is positive recurrent by theorem 8.20. $\square$

8.4 The queue length at service completion times

The stationary distribution of the embedded Markov chain $(\mathcal{X},\mathcal{J})$, i.e. the distribution of the queue length at service completion times, can be computed according to theorem 6.1 and equation (6.2). The sum of the components $\mathbf{x}_k$, $k \in \mathbb{N}_0$, is just the stationary distribution of the phase process at service completion times, which we will denote by $\tilde{\pi}$.

Theorem 8.22 *The stationary distribution of the phase process at service completion times is given by*

$$\tilde{\pi} \;=\; \Big(-\mathbf{x}_0 {D_0^{(0)}}^{-1} DA + \pi\Big)\Big(I - A + \mathbf{e}\pi\Big)^{-1}.$$

Proof: By definition of the stationary distribution, $\tilde{\pi} = \sum_{k=0}^{\infty} \mathbf{x}_k$. Equation (6.1) implies

$$\begin{aligned}
\tilde{\pi} \;&=\; \sum_{k=0}^{\infty} \mathbf{x}_k \;=\; \sum_{k=0}^{\infty}\left(\mathbf{x}_0 B_k + \sum_{\nu=1}^{k+1} \mathbf{x}_\nu A_{k+1-\nu}^{(\nu)}\right)\\
&=\; \mathbf{x}_0 B + \sum_{\nu=0}^{\infty}\sum_{k=\nu}^{\infty} \mathbf{x}_{\nu+1} A_{k-\nu}^{(\nu+1)}\\
&=\; \mathbf{x}_0\Big(I - {D_0^{(0)}}^{-1} D\Big)A + \sum_{\nu=0}^{\infty} \mathbf{x}_{\nu+1} A \quad \text{by corollaries 8.17 and 8.14}\\
&=\; -\mathbf{x}_0 {D_0^{(0)}}^{-1} DA + \sum_{\nu=0}^{\infty} \mathbf{x}_\nu A\\
&=\; -\mathbf{x}_0 {D_0^{(0)}}^{-1} DA + \tilde{\pi} A
\end{aligned}$$

and so

$$\tilde{\pi}(I - A) \;=\; -\mathbf{x}_0 D_0^{(0)^{-1}} DA\,.$$

Because $\mathbf{x}_k$, $k \in \mathbb{N}_0$, is a probability distribution, $\tilde{\pi}\mathbf{e} = \sum_{k=0}^{\infty} \mathbf{x}_k \mathbf{e} = 1$. Thus

$$\tilde{\pi}(I - A + \mathbf{e}\pi) \;=\; \pi - \mathbf{x}_0 D_0^{(0)^{-1}} DA\,.$$

Since A is the transition matrix of a finite state Markov chain with stationary distribution π (corollary 8.15), $I - A + \mathbf{e}\pi$ is nonsingular [20, Th. 2.13] and therefore

$$\tilde{\pi} \;=\; \Big(\pi - \mathbf{x}_0 D_0^{(0)^{-1}} DA\Big)\Big(I - A + \mathbf{e}\pi\Big)^{-1}.$$

□

8.5 The queue length at an arbitrary time

The distribution of the queue length at an arbitrary time, i.e. the limiting distribution $\mathbf{y} = (\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, ...)$ of the stochastic process $(\mathcal{Y}, \mathcal{J}) = \{Y(t),\, J(t) : t \geq 0\}$, is given by theorems 7.1 and 7.2. We will now show that in the case of assumption 8.1 the vector π is the marginal distribution of the phase process $\{J(t) : t \geq 0\}$.

Theorem 8.23 *The stationary distribution π of the phase process $\{J(t) : t \geq 0\}$ is also its marginal limiting distribution.*

Proof: The marginal limiting distribution of the phase process $\{J(t) : t \geq 0\}$ is $\sum_{k=0}^{\infty} \mathbf{y}_k$. Theorem 7.2 yields

$$\begin{aligned}
&\Big(\mu^{-1} - \mathbf{x}_0 D_0^{(0)^{-1}} \mathbf{e}\Big) \sum_{k=1}^{\infty} \mathbf{y}_k \\
&= \sum_{k=1}^{\infty} \sum_{l=1}^{k} \Big(\mathbf{x}_l - \mathbf{x}_0 D_0^{(0)^{-1}} D_l^{(0)}\Big) \int_0^{\infty} P_{k-l}^{(l)}(t)\Big(1 - H(t)\Big)\, dt \\
&= \sum_{l=1}^{\infty} \sum_{k=l}^{\infty} \Big(\mathbf{x}_l - \mathbf{x}_0 D_0^{(0)^{-1}} D_l^{(0)}\Big) \int_0^{\infty} P_{k-l}^{(l)}(t)\Big(1 - H(t)\Big)\, dt \\
&= \sum_{l=1}^{\infty} \Big(\mathbf{x}_l - \mathbf{x}_0 D_0^{(0)^{-1}} D_l^{(0)}\Big) \int_0^{\infty} e^{Dt}\Big(1 - H(t)\Big)\, dt \\
&\qquad \text{by corollary 8.9} \\
&= \Big(\Big(\pi - \mathbf{x}_0 D_0^{(0)^{-1}} DA\Big)\Big(I - A + \mathbf{e}\pi\Big)^{-1} - \mathbf{x}_0\Big) \int_0^{\infty} e^{Dt}\Big(1 - H(t)\Big)\, dt \\
&\qquad - \mathbf{x}_0 D_0^{(0)^{-1}} \Big(D - D_0^{(0)}\Big) \int_0^{\infty} e^{Dt}\Big(1 - H(t)\Big)\, dt
\end{aligned}$$

by theorem 8.22. First we note that $\pi(I - A + \mathbf{e}\pi) = \pi$, so $\pi(I - A + \mathbf{e}\pi)^{-1} = \pi$, and $\pi e^{Dt} = \pi$. Further, $D(I - A + \mathbf{e}\pi) = D(I - A)$, and since $AD = DA$ by corollary 8.14, $D(I - A + \mathbf{e}\pi) = (I - A + \mathbf{e}\pi)D$. This implies

$$
\begin{aligned}
&\Big(\mu^{-1} - \mathbf{x}_0 D_0^{(0)^{-1}} \mathbf{e}\Big) \sum_{k=1}^{\infty} \mathbf{y}_k \\
&= \pi \int_0^\infty e^{Dt}\Big(1 - H(t)\Big)\, dt \\
&\quad - \mathbf{x}_0 D_0^{(0)^{-1}} DA\Big(I - A + \mathbf{e}\pi\Big)^{-1} \int_0^\infty e^{Dt}\Big(1 - H(t)\Big)\, dt \\
&\quad - \mathbf{x}_0 D_0^{(0)^{-1}} D \int_0^\infty e^{Dt}\Big(1 - H(t)\Big)\, dt \\
&= \mu^{-1}\pi - \mathbf{x}_0 D_0^{(0)^{-1}} DA\Big(I - A + \mathbf{e}\pi\Big)^{-1} \int_0^\infty e^{Dt}\Big(1 - H(t)\Big)\, dt \\
&\quad - \mathbf{x}_0 D_0^{(0)^{-1}} D\Big(I - A\Big)\Big(I - A + \mathbf{e}\pi\Big)^{-1} \int_0^\infty e^{Dt}\Big(1 - H(t)\Big)\, dt \\
&= \mu^{-1}\pi - \mathbf{x}_0 D_0^{(0)^{-1}} D \int_0^\infty e^{Dt}\Big(1 - H(t)\Big)\, dt\, \Big(I - A + \mathbf{e}\pi\Big)^{-1}.
\end{aligned}
$$

Partial integration [21, Th. 8.10, pp. 209f] yields

$$
\begin{aligned}
\int_0^\infty De^{Dt}\Big(1 - H(t)\Big)\, dt &= e^{Dt}\Big(1 - H(t)\Big)\Big|_0^\infty - \int_0^\infty e^{Dt}\, d\Big(1 - H(t)\Big) \\
&= -I + \int_0^\infty e^{Dt}\, dH(t) \;=\; -I + A
\end{aligned}
$$

by corollary 8.14. Further, $\mathbf{e}\pi(I - A + \mathbf{e}\pi) = \mathbf{e}\pi$. Thus

$$
\begin{aligned}
&\Big(\mu^{-1} - \mathbf{x}_0 D_0^{(0)^{-1}} \mathbf{e}\Big) \sum_{k=1}^{\infty} \mathbf{y}_k \\
&= \mu^{-1}\pi + \mathbf{x}_0 D_0^{(0)^{-1}} \Big(I - A\Big)\Big(I - A + \mathbf{e}\pi\Big)^{-1} \\
&= \mu^{-1}\pi + \mathbf{x}_0 D_0^{(0)^{-1}} \Big(I - A + \mathbf{e}\pi\Big)\Big(I - A + \mathbf{e}\pi\Big)^{-1} - \mathbf{x}_0 D_0^{(0)^{-1}} \mathbf{e}\pi\Big(I - A + \mathbf{e}\pi\Big)^{-1} \\
&= \mu^{-1}\pi + \mathbf{x}_0 D_0^{(0)^{-1}} - \mathbf{x}_0 D_0^{(0)^{-1}} \mathbf{e}\pi\,.
\end{aligned}
$$

So we obtain by applying theorem 7.1

$$\begin{aligned}
\sum_{k=0}^{\infty} \mathbf{y}_k &= -\frac{1}{\mu^{-1} - \mathbf{x}_0 D_0^{(0)^{-1}} \mathbf{e}} \mathbf{x}_0 D_0^{(0)^{-1}} \\
&\quad + \frac{1}{\mu^{-1} - \mathbf{x}_0 D_0^{(0)^{-1}} \mathbf{e}} \left(\mu^{-1}\pi + \mathbf{x}_0 D_0^{(0)^{-1}} - \mathbf{x}_0 D_0^{(0)^{-1}} \mathbf{e}\pi \right) \\
&= \frac{\mu^{-1} - \mathbf{x}_0 D_0^{(0)^{-1}} \mathbf{e}}{\mu^{-1} - \mathbf{x}_0 D_0^{(0)^{-1}} \mathbf{e}} \pi \;=\; \pi \, .
\end{aligned}$$

□

Ramaswami [33, L. 3.3.1] and Lucantoni [24, p. 29] have shown, that in the level independent case $\mu^{-1} - \mathbf{x}_0 D_0{}^{-1}\mathbf{e} = \lambda^{-1}$. This does not hold in our case (with λ being $\lambda^{(0)}$ or any suitable combination of the $\lambda^{(k)}$), as we can see from the following simple example. Suppose arrivals in level 0 occur according to a Poisson process with rate $\lambda > 0$, and there are no arrivals in all other levels $k \geq 1$. So $m = 1$, $D_0^{(0)} = -\lambda$, $D_1^{(0)} = \lambda$ and all other $D_n^{(k)} = 0$. Then a departing customer always leaves an empty system behind and so $\mathbf{x}_0 = 1$. Thus $\mu^{-1} - \mathbf{x}_0 D_0^{(0)^{-1}} \mathbf{e} = \mu^{-1} + \lambda^{-1}$.

9 The level independent BMAP/G/1 queue

The classical level independent BMAP/G/1 queue has been analysed by Ramaswami [33] and Lucantoni [24]. We want to compare our results with theirs.

Assumption 9.1 *Suppose* $\mathbf{J}^{(k)} = \mathbf{J}^{(0)} =: \mathbf{J}$, *i.e.* $m^{(k)} = m < \infty$, *and* $D_n^{(k)} = D_n^{(0)} = D_n$ *for all* $n, k \in I\!N_0$.

In this case it is also $\mathcal{Q}^{(k)}(z) = \mathcal{Q}^{(0)}(z) =: \mathcal{Q}(z)$ for all $k \in I\!N_0$ and $|z| \leq 1$. Further, we have $D^{(k)}(z) = D^{(0)}(z) =: D(z)$ and $\mathbf{d}^{(k)} = \mathbf{d}^{(0)} =: \mathbf{d}$ as well as $\lambda^{(k)} = \lambda^{(0)} =: \lambda$ for all $k \in I\!N_0$.

Now we can simplify some of our results for this special instance.

Lemma 9.2 *For all* $l \in I\!N_0$ *it is*

$$\sum_{n=l}^{\infty} \left(e^{\mathcal{Q}(z)\,t}\right)_{ln} \;=\; e^{D(z)\,t}\,.$$

Proof: Since

$$\sum_{n=l}^{\infty} \left(e^{\mathcal{Q}(z)\,t}\right)_{ln} \;=\; \sum_{n=l}^{\infty}\sum_{j=0}^{\infty} \frac{t^j}{j!}\left((\mathcal{Q}(z))^j\right)_{ln} \;=\; \sum_{j=0}^{\infty} \frac{t^j}{j!}\sum_{n=l}^{\infty}\left((\mathcal{Q}(z))^j\right)_{ln}$$

it suffices to show that

$$\sum_{n=l}^{\infty} \left((\mathcal{Q}(z))^j\right)_{ln} \;=\; (D(z))^j \tag{9.1}$$

for all $j \in I\!N_0$. This is obviously true for $j = 0$, so assume (9.1) holds for j and consider $j+1$:

$$\begin{aligned} \sum_{n=l}^{\infty} \left((\mathcal{Q}(z))^{j+1}\right)_{ln} \;&=\; \sum_{n=l}^{\infty}\sum_{\nu=l}^{n} \left((\mathcal{Q}(z))^j\right)_{l\nu}\left(\mathcal{Q}(z)\right)_{\nu n} \\ &=\; \sum_{\nu=l}^{\infty}\sum_{n=\nu}^{\infty} \left((\mathcal{Q}(z))^j\right)_{l\nu} D_{n-\nu} z^{n-\nu} \\ &=\; (D(z))^j D(z) \;=\; (D(z))^{j+1}\,. \end{aligned}$$

Thus (9.1) holds for all $j \in I\!N_0$, and so the assertion is proven. $\square$

9.1 The arrival process and the embedded Markov chain

Theorems 8.11 and 8.13 together with lemma 9.2 yield

$$P(z,t) \ := \ P^{(k)}(z,t) \ = \ e^{D(z)\,t} \quad \text{for all } k \in I\!N_0 \tag{9.2}$$

and

$$A(z,s) \ := \ A^{(k)}(z,s) \ = \ \int_0^\infty e^{-sx} e^{D(z)\,t}\, dH(t) \quad \text{for all } k \in I\!N \tag{9.3}$$

in accordance with the results of Lucantoni [24, pp. 7, 11]. So the transition probabilities do not dependent on the current level. Let $P_n(t) := P_n^{(k)}(t)$ and $A_n := A_n^{(k)}$ for all $k,n \in I\!N_0$ ($k \in I\!N$ respectively).

Now we can also determine the mean number of arrivals.

Theorem 9.3 *The vectors of the phase dependent mean numbers of arrivals in time t and during a service time are given by*

$$\mathbf{n}(t) \ := \ \mathbf{n}^{(k)}(t) \ = \ \lambda t\mathbf{e} - (e^{Dt} - I)(\mathbf{e}\pi - D)^{-1}\mathbf{d}$$

and

$$\mathbf{a} \ := \ \mathbf{a}^{(k)} \ = \ \lambda\mu^{-1}\mathbf{e} - (A - I)(\mathbf{e}\pi - D)^{-1}\mathbf{d}$$

for all $k \in I\!N_0$ ($k \in I\!N$ respectively).

Proof: The vector of the phase dependent mean numbers of arrivals in time t is given by

$$\mathbf{n}(t) \ = \ \tfrac{\partial}{\partial z} P(z,t)\Big|_{z=0} \mathbf{e} \ = \ \sum_{j=0}^{\infty} \frac{t^j}{j!} \tfrac{d}{dz}(D(z))^j\Big|_{z=0} \mathbf{e}\,.$$

Because of $D\mathbf{e} = \mathbf{0}$ we obtain

$$\mathbf{n}(t) \ = \ \sum_{j=1}^{\infty} \frac{t^j}{j!} D^{j-1} D'(z)\Big|_{z=0} \mathbf{e} \ = \ \sum_{j=1}^{\infty} \frac{t^j}{j!} D^{j-1}\mathbf{d}\,.$$

So $D\mathbf{n}(t) = (e^{Dt} - I)\mathbf{d}$ and $\pi\mathbf{n}(t) = t\pi\mathbf{d} = t\lambda$, thus

$$(\mathbf{e}\pi - D)\mathbf{n}(t) \ = \ \lambda t\mathbf{e} - (e^{Dt} - I)\mathbf{d}\,.$$

Since D is the generator matrix of a finite state Markov process with stationary distribution π, $D - \mathbf{e}\pi$ is nonsingular [20, Th. 2.13]. Further, $(\mathbf{e}\pi - D)\mathbf{e} = \mathbf{e}$, so $(\mathbf{e}\pi - D)^{-1}\mathbf{e} = \mathbf{e}$, and $(\mathbf{e}\pi - D)(e^{Dt} - I) = -De^{Dt} + D = (e^{Dt} - I)(\mathbf{e}\pi - D)$. This implies

$$\mathbf{n}(t) \ = \ \lambda t\mathbf{e} - (e^{Dt} - I)(\mathbf{e}\pi - D)^{-1}\mathbf{d}\,.$$

Now

$$\begin{aligned}\mathbf{a} &= \int_0^\infty \mathbf{n}(t)\, dH(t) = \int_0^\infty \lambda t \mathbf{e}\, dH(t) - \int_0^\infty (e^{Dt} - I)\, dH(t)\, (\mathbf{e}\pi - D)^{-1}\mathbf{d} \\ &= \lambda\mu^{-1}\mathbf{e} - (A - I)(\mathbf{e}\pi - D)^{-1}\mathbf{d}\end{aligned}$$

by corollary 8.14. □

Theorem 9.3 implies $\pi\mathbf{n}(t) = \lambda t$ and $\pi\mathbf{a} = \lambda\mu^{-1}$. By theorem 8.20 and corollary 8.21 the BMAP/G/1 queue is stable, i.e. the embedded Markov chain $(\mathcal{X}, \mathcal{J})$ is positive recurrent, if $\lambda < \mu$. Ramaswami has shown that this condition is not only sufficient, but also necessary [33, Th. 2.2.16]. Henceforth we will assume that $\lambda < \mu$.

9.2 The fundamental period

We will now show that our algorithms for computing the fundamental matrix $G := G^{(k)}$ for all $k \in I\!N$ are equivalent to the corresponding algorithms for the classical BMAP/G/1 queue derived by Baum [1]. At first we state the definition of the upper and lower semi–convolution of a sequence of matrices [1].

Definition 9.4 *Let* $\mathbf{A}$ *denote the sequence* $\{A_0, A_1, A_2, \ldots\}$.

a) For $k \in I\!N_0$ *the sequence* ${}^{[*k]}\mathbf{A} = \{{}^{[*k]}A_0, {}^{[*k]}A_1, {}^{[*k]}A_2, \ldots\}$ *with*

$$\begin{aligned}{}^{[*0]}A_0 &= I, \\ {}^{[*0]}A_j &= O \quad \text{for } j \geq 1, \\ {}^{[*1]}A_j &= A_j \quad \text{for } j \geq 1, \\ {}^{[*k]}A_j &= O \quad \text{for } 0 \leq j < k, \\ {}^{[*k]}A_j &= \sum_{l=k-1}^{j} {}^{[*k-1]}A_l\, A_{j-l} \quad \text{for } j \geq k \geq 1\end{aligned}$$

is called the kth upper semi–convolution of $\mathbf{A}$ with itself.

b) For $k \in I\!N_0$ *the sequence* ${}_{[*k]}\mathbf{A} = \{{}_{[*k]}A_0, {}_{[*k]}A_1, {}_{[*k]}A_2, \ldots\}$ *with*

$$\begin{aligned}{}_{[*0]}A_0 &= I, \\ {}_{[*0]}A_j &= O \quad \text{for } j \geq 1, \\ {}_{[*1]}A_j &= A_j \quad \text{for } j \leq 1, \\ {}_{[*k]}A_j &= O \quad \text{for } j > k \geq 1, \\ {}_{[*k]}A_j &= \sum_{l=0}^{j} A_l\, {}_{[*k-1]}A_{j-l} \quad \text{for } 0 \leq j \leq k,\ k \geq 1\end{aligned}$$

is called the kth lower semi–convolution of $\mathbf{A}$ with itself.

The matrices $G_l := G_l^{(k)}$ for all $k \in I\!N$ are then given by [1]:

$$G_l \;=\; {}_{[*l-1]}A_{l-1}\,A_0 \;=\; {}^{[*l-1]}A_{l-1}\,A_0\,, \quad l \geq 1.$$

So we need to show that the matrices $\mathcal{A}_l := \mathcal{A}_l^{(k)}$ and $\mathcal{A}_\infty := \mathcal{A}_\infty^{(k)}$ for all $k \in I\!N$ (cf. section 5) satisfy $\left[\prod_{i=1}^{l-1} \mathcal{A}_{l-i}\right]_0 = {}_{[*l-1]}A_{l-1}$ and $\left((\mathcal{A}_\infty)^{l-1}\right)_{00} = {}^{[*l-1]}A_{l-1}$ for all $l \in I\!N$.

Lemma 9.5 *The matrices $\mathcal{A}_l$ satisfy*

$$\left[\prod_{i=1}^{l-1} \mathcal{A}_{l-i}\right]_\nu \;=\; {}_{[*l-1]}A_{l-1-\nu}$$

for all $l \in I\!N$ and $\nu = 0, \ldots, l-1$.

Proof: For $l = 1$ the assertion is true (the empty product $\prod_{i=1}^{0} \cdot$ shall be the identity matrix I), so suppose it holds for l and consider $l+1$:

$$\left[\prod_{i=1}^{l} \mathcal{A}_{l+1-i}\right]_\nu \;=\; \left[\mathcal{A}_l \prod_{i=2}^{l} \mathcal{A}_{l+1-i}\right]_\nu \;=\; \left[\mathcal{A}_l \prod_{i=1}^{l-1} \mathcal{A}_{l-i}\right]_\nu .$$

Since ${}_{[*l-1]}A_l = O$ we obtain for $\nu = 0$

$$\left[\prod_{i=1}^{l} \mathcal{A}_{l+1-i}\right]_0 \;=\; \sum_{n=0}^{l-1} A_{n+1}\;{}_{[*l-1]}A_{l-1-n} \;=\; \sum_{n=0}^{l} A_n\;{}_{[*l-1]}A_{l-n} \;=\; {}_{[*l]}A_l\,,$$

and for $\nu > 0$

$$\left[\prod_{i=1}^{l} \mathcal{A}_{l+1-i}\right]_\nu \;=\; \sum_{n=0}^{l-\nu} A_n\;{}_{[*l-1]}A_{l-1-(\nu-1+n)} \;=\; \sum_{n=0}^{l-\nu} A_n\;{}_{[*l-1]}A_{l-\nu-n} \;=\; {}_{[*l]}A_{l-\nu}\,.$$

So the assertion holds for all $l \in I\!N$ and $\nu = 0, \ldots, l-1$. □

Lemma 9.6 *The matrix $\mathcal{A}_\infty$ satisfies*

$$\left((\mathcal{A}_\infty)^{l-1}\right)_{0\nu} \;=\; {}^{[*l-1]}A_{l-1+\nu}$$

for all $l \in I\!N$ and $\nu \geq 0$.

Proof: For $l = 1$ the assertion is obviously true, so suppose it holds for l and consider $l+1$:

$$\begin{aligned}\left((\mathcal{A}_\infty)^l\right)_{0\nu} &= \sum_{n=0}^{\nu+1}\left((\mathcal{A}_\infty)^{l-1}\right)_{0n} A_{\nu+1-n} = \sum_{n=0}^{\nu+1} {}^{[*l-1]}A_{l-1+n}\, A_{\nu+1-n} \\ &= \sum_{n=l-1}^{l+\nu} {}^{[*l-1]}A_n\, A_{l+\nu-n} = {}^{[*l]}A_{l+\nu}\,.\end{aligned}$$

So the assertion holds for all $l \in I\!N$ and $\nu \geq 0$. □

Theorem 9.7 *The matrices G_l are given by*

$$\begin{aligned} G_l &= \left[\prod_{i=1}^{l-1} \mathcal{A}_{l-i}\right]_0 A_0 = {}_{[*l-1]}A_{l-1}\, A_0 \quad \textit{and} \\ G_l &= \left((\mathcal{A}_\infty)^{l-1}\right)_{00} A_0 = {}^{[*l-1]}A_{l-1}\, A_0 \end{aligned}$$

for all $l \in I\!N$.

Theorem 9.7 is a restatement of our theorems 5.8 and 5.12 and lemma 4.2 in [1].

Let $\mathbf{g}$ denote the stationary probability vector of G, i.e. $\mathbf{g}G = \mathbf{g}$ and $\mathbf{ge} = 1$. We can now also determine the phase dependent mean number of service completions during a fundamental period, $\mathbf{c}_F := \mathbf{c}_F^{(k)}$ for all $k \in I\!N$. This result is due to Neuts [31, Th. 3.1.1, p. 126] and Ramaswami [33, Th. 2.3.1].

Theorem 9.8 *The phase dependent mean number of service completions during a fundamental period is given by*

$$\mathbf{c}_F = \left(I - G + \mathbf{eg}\right)\left(I - A + (\mathbf{e} - \mathbf{a})\mathbf{g}\right)^{-1}\mathbf{e}\,.$$

Proof: Neuts [31, p. 127] proved that the matrix $(I - A + (\mathbf{e} - \mathbf{a})\mathbf{g})$ is nonsingular. So we only need to show that $\mathbf{c}_F = (I - G + \mathbf{eg})(I - A + (\mathbf{e} - \mathbf{a})\mathbf{g})^{-1}\mathbf{e}$ is the solution of the system of linear equations given in lemma 5.13:

$$\begin{aligned} &\mathbf{e} + \sum_{l=1}^{\infty} A_l G^{l-1}\mathbf{c}_F + \sum_{\nu=1}^{\infty}\sum_{l=1}^{\infty} A_{\nu+l} G^{l-1}\mathbf{c}_F \\ &= \mathbf{e} + \sum_{\nu=0}^{\infty}\sum_{l=\nu}^{\infty} A_{l+1} G^{l-\nu}\mathbf{c}_F \end{aligned}$$

$$\begin{aligned}
&= \mathbf{e} + \sum_{l=0}^{\infty}\sum_{\nu=0}^{l} A_{l+1} G^{l-\nu}\mathbf{c}_F \\
&= \mathbf{e} + \sum_{l=0}^{\infty} A_{l+1}\sum_{\nu=0}^{l} G^{l-\nu}\Big(I - G + \mathbf{eg}\Big)\Big(I - A + (\mathbf{e}-\mathbf{a})\mathbf{g}\Big)^{-1}\mathbf{e}\,.
\end{aligned}$$

Now, $\sum_{\nu=0}^{l} G^{l-\nu}(I-G) = I - G^{l+1}$ and $\sum_{\nu=0}^{l} G^{l-\nu}\mathbf{eg} = (l+1)\mathbf{eg}$. Thus

$$\begin{aligned}
&\mathbf{e} + \sum_{l=1}^{\infty} A_l G^{l-1}\mathbf{c}_F + \sum_{\nu=1}^{\infty}\sum_{l=1}^{\infty} A_{\nu+l} G^{l-1}\mathbf{c}_F \\
&= \mathbf{e} + \sum_{l=0}^{\infty} A_{l+1}\Big(I - G^{l+1} + (l+1)\mathbf{eg}\Big)\Big(I - A + (\mathbf{e}-\mathbf{a})\mathbf{g}\Big)^{-1}\mathbf{e} \\
&= \Big(I - A + (\mathbf{e}-\mathbf{a})\mathbf{g} + (A - A_0) - (G - A_0) + \mathbf{ag}\Big)\Big(I - A + (\mathbf{e}-\mathbf{a})\mathbf{g}\Big)^{-1}\mathbf{e} \\
&\quad \text{by corollary 5.3} \\
&= \Big(I - G + \mathbf{eg}\Big)\Big(I - A + (\mathbf{e}-\mathbf{a})\mathbf{g}\Big)^{-1}\mathbf{e} \;=\; \mathbf{c}_F\,.
\end{aligned}$$

So lemma 5.13 and theorem 5.16 yield the assertion. □

9.3 The queue length distribution

The stationary distribution $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots)$ of the queue length at service completion times can be computed recursively by Ramaswami's formulae (theorem 6.1). In the case of assumption 9.1 this yields the algorithm given by Lucantoni [24, p. 25]. Further, our result for the joint transform $K(z,s)$ (theorem 6.2) coincides with the one of Lucantoni [24, p. 15]. Corollary 6.4 and theorem 9.8 imply that the vector of the phase dependent mean number of service completions during a busy period, $\mathbf{c}_B$, is given by

$$\begin{aligned}
\mathbf{c}_B &= -{D_0}^{-1}\sum_{j=1}^{\infty} D_j \sum_{l=0}^{j-1} G^l \mathbf{c}_F \\
&= -{D_0}^{-1}\sum_{j=1}^{\infty} D_j \sum_{l=0}^{j-1} G^l\Big(I - G + \mathbf{eg}\Big)\Big(I - A + (\mathbf{e}-\mathbf{a})\mathbf{g}\Big)^{-1}\mathbf{e} \\
&= -{D_0}^{-1}\sum_{j=1}^{\infty} D_j\Big(I - G^j + j\mathbf{eg}\Big)\Big(I - A + (\mathbf{e}-\mathbf{a})\mathbf{g}\Big)^{-1}\mathbf{e} \\
&= -{D_0}^{-1}\Big(D - D[G] + \mathbf{dg}\Big)\Big(I - A + (\mathbf{e}-\mathbf{a})\mathbf{g}\Big)^{-1}\mathbf{e}\,,
\end{aligned}$$

where $D[G] = \sum_{j=0}^{\infty} D_j G^j$. This is also the result of Lucantoni [24, p. 16].

Theorem 9.9 *The z-transform* $X(z) = \sum_{k=0}^{\infty} \mathbf{x}_k z^k$ *of* $\mathbf{x}$ *satisfies*

$$X(z)\Big(zI - A(z,0)\Big) \;=\; -\mathbf{x}_0 {D_0}^{-1} D(z) A(z,0)$$

for $|z| \leq 1$.

Proof: Equation (6.1) and theorem 8.16 imply

$$\begin{aligned}
X(z) &= \sum_{k=0}^{\infty} \mathbf{x}_k z^k \;=\; \sum_{k=0}^{\infty} \mathbf{x}_0 B_k z^k + \sum_{k=0}^{\infty}\sum_{\nu=0}^{k} \mathbf{x}_{\nu+1} A_{k-\nu} z^k \\
&= \mathbf{x}_0 B(z,0) + \sum_{\nu=0}^{\infty}\sum_{k=\nu}^{\infty} \mathbf{x}_{\nu+1} A_{k-\nu} z^k \\
&= -\mathbf{x}_0 {D_0}^{-1} z^{-1} \sum_{l=1}^{\infty} D_l z^l A(z,0) + \sum_{\nu=0}^{\infty} \mathbf{x}_{\nu+1} z^{\nu} A(z,0) \\
&= -\mathbf{x}_0 {D_0}^{-1} z^{-1} \Big(D(z) - D_0\Big) A(z,0) + z^{-1}\Big(X(z) - \mathbf{x}_0\Big) A(z,0) \\
&= -\mathbf{x}_0 {D_0}^{-1} z^{-1} D(z) A(z,0) + z^{-1} X(z) A(z,0)\,.
\end{aligned}$$

So $X(z)(zI - A(z,0)) = -\mathbf{x}_0 {D_0}^{-1} D(z) A(z,0)$. □

It remains to show that our results for the limiting distribution $\mathbf{y} = (\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \ldots)$ of the queue length at an arbitrary time coincide with those of Lucantoni [24, p. 18]. At first we note that

$$\mu^{-1} - \mathbf{x}_0 {D_0}^{-1} \mathbf{e} \;=\; \lambda^{-1}$$

as shown by Ramaswami [33, L. 3.3.1] and Lucantoni [24, p. 29] (cf. section 8.5).

Theorem 9.10 *The z-transform* $Y(z) = \sum_{k=0}^{\infty} \mathbf{y}_k z^k$ *of* $\mathbf{y}$ *satisfies*

$$\lambda^{-1} Y(z) D(z) \;=\; X(z)(z-1)$$

for $|z| \leq 1$.

Proof: Theorems 7.1 and 7.2 yield:

$$\begin{aligned}
&\lambda^{-1} Y(z) D(z) \\
&= \lambda^{-1} \sum_{k=0}^{\infty}\sum_{l=0}^{k} \mathbf{y}_l D_{k-l} z^k
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty}\Big(-\mathbf{x}_0 D_0{}^{-1}\Big) D_k z^k \\
&\quad + \sum_{k=1}^{\infty}\sum_{l=1}^{k}\sum_{\nu=1}^{l}\Big(\mathbf{x}_\nu - \mathbf{x}_0 D_0{}^{-1} D_\nu\Big)\int_0^\infty P_{l-\nu}(t)\Big(1-H(t)\Big)\,dt\,D_{k-l}z^k \\
&= -\mathbf{x}_0 D_0{}^{-1} D(z) \\
&\quad + \sum_{l=1}^{\infty}\sum_{k=l}^{\infty}\sum_{\nu=1}^{l}\Big(\mathbf{x}_\nu - \mathbf{x}_0 D_0{}^{-1} D_\nu\Big)\int_0^\infty P_{l-\nu}(t)\Big(1-H(t)\Big)\,dt\,D_{k-l}z^k \\
&= -\mathbf{x}_0 D_0{}^{-1} D(z) + \sum_{l=1}^{\infty}\sum_{\nu=1}^{l}\Big(\mathbf{x}_\nu - \mathbf{x}_0 D_0{}^{-1} D_\nu\Big)\int_0^\infty P_{l-\nu}(t)\Big(1-H(t)\Big)\,dt\,z^l D(z) \\
&= -\mathbf{x}_0 D_0{}^{-1} D(z) + \sum_{\nu=1}^{\infty}\sum_{l=\nu}^{\infty}\Big(\mathbf{x}_\nu - \mathbf{x}_0 D_0{}^{-1} D_\nu\Big)\int_0^\infty P_{l-\nu}(t)\Big(1-H(t)\Big)\,dt\,z^l D(z) \\
&= -\mathbf{x}_0 D_0{}^{-1} D(z) + \sum_{\nu=1}^{\infty}\Big(\mathbf{x}_\nu - \mathbf{x}_0 D_0{}^{-1} D_\nu\Big)\int_0^\infty P(z,t) z^\nu\Big(1-H(t)\Big)\,dt\,D(z) \\
&= -\mathbf{x}_0 D_0{}^{-1} D(z) + \sum_{\nu=1}^{\infty}\Big(\mathbf{x}_\nu - \mathbf{x}_0 D_0{}^{-1} D_\nu\Big) z^\nu \int_0^\infty e^{D(z)\,t} D(z)\Big(1-H(t)\Big)\,dt
\end{aligned}$$

by (9.2). Partial integration [21, Th. 8.10, pp. 209f] yields

$$\begin{aligned}
\int_0^\infty e^{D(z)\,t} D(z)\Big(1-H(t)\Big)\,dt &= e^{D(z)\,t}\Big(1-H(t)\Big)\Big|_0^\infty - \int_0^\infty e^{D(z)\,t}\,d\Big(1-H(t)\Big) \\
&= -I + \int_0^\infty e^{D(z)\,t}\,dH(t) \;=\; -I + A(z,0)
\end{aligned}$$

by (9.3). Thus

$$\begin{aligned}
&\lambda^{-1} Y(z) D(z) \\
&\quad = -\mathbf{x}_0 D_0{}^{-1} D(z) + \sum_{\nu=1}^{\infty}\Big(\mathbf{x}_\nu - \mathbf{x}_0 D_0{}^{-1} D_\nu\Big) z^\nu \Big(A(z,0) - I\Big) \\
&\quad = -\mathbf{x}_0 D_0{}^{-1} D(z) + \Big(X(z) - \mathbf{x}_0 - \mathbf{x}_0 D_0{}^{-1}(D(z) - D_0)\Big)\Big(A(z,0) - I\Big) \\
&\quad = X(z)\Big(A(z,0) - I\Big) - \mathbf{x}_0 D_0{}^{-1} D(z) A(z,0) \\
&\quad = X(z)\Big(A(z,0) - I\Big) + X(z)\Big(zI - A(z,0)\Big) \quad \text{by theorem 9.9} \\
&\quad = X(z)(z-1)\,.
\end{aligned}$$

□

This result coincides with the one of Lucantoni [24, p. 18]. Note that $Y(z)$ is uniquely determined by theorem 9.10 for $|z| < 1$, since $D(z)$ is nonsingular for $|z| < 1$ (by the theorem of Gerŝgorin [28, p. 146]). Further, theorem 8.23 yields $Y(1) = \pi$.

So we have seen that our results for the BMAP/G/1 queue with level dependent arrivals include the classical results for the level independent BMAP/G/1 queue.

10 The finite capacity queue

A special case of the BMAP/G/1 queue with level dependent arrivals is the finite capacity queue. This is of great importance for applications, especially when considering systems with limited buffering memory where the loss probability is an important performance measure.

Suppose the buffer has $N-1$ waiting places available, so there can be up to N customers in the system. Further, arrivals shall occur according to a level dependent BMAP defined by a family of sequences $\{D_n^{(k)} : n \in \mathbb{N}_0\}$ (cf. section 1). Then $D_n^{(k)} = O$ for all $n > N-k$ (in particular for $k > N$). This queueing system is always stable, because the state space $\mathbb{Y}$ of the stochastic process $(\mathcal{Y}, \mathcal{J})$ is finite:

$$\mathbb{Y} = \bigcup_{k=0}^{N} \left(\{k\} \times \mathbf{J}^{(k)} \right).$$

The level independent BMAP/G/1/N queue has been analysed by Blondia [4] using the method of the embedded Markov chain. He determines the stationary distribution $\mathbf{x}$ of the queue length at service completion times by solving the finite system of equations $\mathbf{x}\mathcal{P} = \mathbf{x}$ and the normalising condition $\mathbf{xe} = 1$. He also derives the limiting distribution $\mathbf{y}$ of the queue length at an arbitrary time by computing the conditional joint distribution of the queue length and the remaining service time for the customer that is in service, given that the server is busy. We will show that his and our approach yield the same results for the distributions $\mathbf{x}$ and $\mathbf{y}$.

10.1 The arrival process

Suppose arrivals occur according to a BMAP with generator matrix

$$\mathcal{Q} = \begin{pmatrix} D_0 & D_1 & D_2 & D_3 & \cdots \\ & D_0 & D_1 & D_2 & \cdots \\ & & D_0 & D_1 & \cdots \\ & & & D_0 & \cdots \\ & & & & \ddots \end{pmatrix}$$

and transition probabilities $P_n(t)$, $n \in \mathbb{N}_0$, with $P(z,t) = e^{D(z)\,t}$ (cf. section 9).

Using our approach this BMAP/G/1/N queue can be described by a BMAP/G/1 queue with level dependent arrivals, whose arrival process is defined by

$$\begin{aligned} D_n^{(k)} &= D_n \quad \text{for } k = 0, \ldots, N-1 \text{ and } n = 0, \ldots, N-k-1, \\ D_{N-k}^{(k)} &= \sum_{n=N-k}^{\infty} D_n \quad \text{for } k = 0, \ldots, N, \\ D_n^{(k)} &= O \quad \text{otherwise.} \end{aligned}$$

The nonzero part of the matrix $\mathcal{Q}^{(0)}$ is then given by (note that $D_0^{(N)} = D$)

$$\mathcal{Q}^{(0)} = \left(\begin{array}{cccc|c} D_0 & D_1 & \cdots & D_{N-1} & D_N^{(0)} \\ & D_0 & \cdots & D_{N-2} & D_{N-1}^{(1)} \\ & & \ddots & \vdots & \vdots \\ & & & D_0 & D_1^{(N-1)} \\ \hline & & & & D_0^{(N)} \end{array}\right) = \left(\begin{array}{c|c} & D_N^{(0)} \\ & D_{N-1}^{(1)} \\ \mathcal{Q}_{N-1} & \vdots \\ & D_1^{(N-1)} \\ \hline & D \end{array}\right).$$

Lemma 10.1 *For $n \in \mathbb{N}_0$ the nth power of $\mathcal{Q}^{(0)}$ satisfies*

$$\left(\mathcal{Q}^{(0)}\right)^n = \left(\begin{array}{c|c} & Q_N^{[n]} \\ \left(\mathcal{Q}_{N-1}\right)^n & Q_{N-1}^{[n]} \\ & \vdots \\ & Q_1^{[n]} \\ \hline & D^n \end{array}\right),$$

where

$$Q_{N-k}^{[n]} = \sum_{\nu=0}^{n-1} \sum_{l=k}^{N-1} \left(\left(\mathcal{Q}_{N-1}\right)^{n-1-\nu}\right)_{kl} D_{N-l}^{(l)} D^{\nu}.$$

Proof: For $n=0$ and $n=1$ the assertion is obviously true. So suppose it holds for n and consider $n+1$:

$$\begin{aligned}
Q_{N-k}^{[n+1]} &= \sum_{l=k}^{N-1}\left((\mathcal{Q}_{N-1})^n\right)_{kl} D_{N-l}^{(l)} + Q_{N-k}^{[n]} D \\
&= \sum_{l=k}^{N-1}\left((\mathcal{Q}_{N-1})^n\right)_{kl} D_{N-l}^{(l)} + \sum_{\nu=0}^{n-1}\sum_{l=k}^{N-1}\left((\mathcal{Q}_{N-1})^{n-1-\nu}\right)_{kl} D_{N-l}^{(l)} D^{\nu+1} \\
&= \sum_{l=k}^{N-1}\left((\mathcal{Q}_{N-1})^n\right)_{kl} D_{N-l}^{(l)} + \sum_{\nu=1}^{n}\sum_{l=k}^{N-1}\left((\mathcal{Q}_{N-1})^{n-\nu}\right)_{kl} D_{N-l}^{(l)} D^{\nu} \\
&= \sum_{\nu=0}^{n}\sum_{l=k}^{N-1}\left((\mathcal{Q}_{N-1})^{n-\nu}\right)_{kl} D_{N-l}^{(l)} D^{\nu}\,.
\end{aligned}$$

So the assertion holds for all $n \in \mathbb{N}_0$. □

Corollary 10.2 *The transition probability matrices $P_n^{(k)}(t)$ are given by*

$$P_n^{(k)}(t) \;=\; P_n(t) \;=\; \left(e^{\mathcal{Q}_{N-1}t}\right)_{k,k+n}$$

for $k=0,\ldots,N-1$ and $n=0,\ldots,N-k-1$.

Proof: Lemma 10.1 and theorem 1.14 imply

$$P_n^{(k)}(t) \;=\; \left(e^{\mathcal{Q}_{N-1}t}\right)_{k,k+n} \;=\; \left(e^{\mathcal{Q}t}\right)_{k,k+n} \;=\; P_n(t)$$

for $k=0,\ldots,N-1$ and $n=0,\ldots,N-k-1$. □

10.2 The embedded Markov chain

We will now show that our approach yields the same embedded Markov chain as obtained by Blondia [4], whose transition probability matrix is given by (we mark its entries with an upper case B for distinction):

$$\tilde{\mathcal{P}}^{\mathrm{B}}(x) = \begin{pmatrix} \tilde{B}_0^{\mathrm{B}}(x) & \tilde{B}_1^{\mathrm{B}}(x) & \tilde{B}_2^{\mathrm{B}}(x) & \cdots & \sum_{n=N-1}^{\infty} \tilde{B}_n^{\mathrm{B}}(x) \\ \tilde{A}_0^{\mathrm{B}}(x) & \tilde{A}_1^{\mathrm{B}}(x) & \tilde{A}_2^{\mathrm{B}}(x) & \cdots & \sum_{n=N-1}^{\infty} \tilde{A}_n^{\mathrm{B}}(x) \\ & \tilde{A}_0^{\mathrm{B}}(x) & \tilde{A}_1^{\mathrm{B}}(x) & \cdots & \sum_{n=N-2}^{\infty} \tilde{A}_n^{\mathrm{B}}(x) \\ & & \ddots & \cdots & \vdots \\ & & & \tilde{A}_0^{\mathrm{B}}(x) & \sum_{n=1}^{\infty} \tilde{A}_n^{\mathrm{B}}(x) \end{pmatrix}.$$

Note that the state space of the embedded Markov chain is

$$\boldsymbol{\mathcal{X}} \;=\; \bigcup_{k=0}^{N-1}\left(\{k\}\times \mathbf{J}^{(k)}\right) \;=\; \Big\{(k,i)\;:\;k=0,\ldots,N-1,\;i=1,\ldots,m\Big\}\,,$$

because there cannot be more than $N-1$ customers in the system immediately after a service completion. The matrices $\tilde{A}^{\mathrm{B}}_n(x)$ and $\tilde{B}^{\mathrm{B}}_n(x)$ are given by

$$\begin{aligned}
\tilde{A}^{\mathrm{B}}_n(x) &= \int_0^x P_n(t)\,dH(t)\,,\\
\tilde{B}^{\mathrm{B}}_n(x) &= \sum_{l=1}^{n+1}(\tilde{U}_l * \tilde{A}^{\mathrm{B}}_{n+1-l})(x)\,,
\end{aligned}$$

where the (i,j)th entry of $\tilde{U}_l(x)$ is the probability that an idle period which started in phase i is ended by the arrival of a batch of size l and phase j not later than time x. The Laplace–Stieltjes transform $U_l(s)$ of $\tilde{U}_l(x)$ is given by (cf. theorem 3.4):

$$U_l(s) \;=\; (sI-D_0)^{-1}D_l\,.$$

Theorem 10.3 *The transition probability matrices $\tilde{\mathcal{P}}(x)$ and $\tilde{\mathcal{P}}^{\mathrm{B}}(x)$ are equal.*

Proof: We need to show that $(\tilde{\mathcal{P}}(x))_{kn} = (\tilde{\mathcal{P}}^{\mathrm{B}}(x))_{kn}$ for all $k,n=0,\ldots,N-1$, i.e.

$$\begin{aligned}
\tilde{B}_n(x) &= \tilde{B}^{\mathrm{B}}_n(x) \quad \text{for } n=0,\ldots,N-2,\\
\tilde{B}_{N-1}(x) &= \sum_{n=N-1}^{\infty}\tilde{B}^{\mathrm{B}}_n(x)\,,\\
\tilde{A}^{(k)}_n(x) &= \tilde{A}^{\mathrm{B}}_n(x) \quad \text{for } k=1,\ldots,N-1 \text{ and } n=0,\ldots,N-k-1,\\
\tilde{A}^{(k)}_{N-k}(x) &= \sum_{n=N-k}^{\infty}\tilde{A}^{\mathrm{B}}_n(x) \quad \text{for } k=1,\ldots,N-1.
\end{aligned}$$

Corollary 10.2 implies $\tilde{A}^{(k)}_n(x) = \tilde{A}^{\mathrm{B}}_n(x)$ for $n=0,\ldots,N-k-1$. Lemma 10.1 and theorem 1.14 yield

$$\begin{aligned}
P^{(k)}_{N-k}(t) &= \left(e^{\mathcal{Q}^{(0)}t}\right)_{kN} = \sum_{j=0}^{\infty}\frac{t^j}{j!}\left((\mathcal{Q}^{(0)})^j\right)_{kN}\\
&= \sum_{j=1}^{\infty}\frac{t^j}{j!}\sum_{\nu=0}^{j-1}\sum_{n=k}^{N-1}\left((\mathcal{Q}_{N-1})^{j-1-\nu}\right)_{kn}D^{(n)}_{N-n}D^{\nu}\,,
\end{aligned}$$

while

$$\sum_{n=N-k}^{\infty} P_n(t) \;=\; \sum_{n=N-k}^{\infty} \Big(e^{\mathcal{Q}t}\Big)_{k,k+n} \;=\; \sum_{n=N}^{\infty}\sum_{j=0}^{\infty} \frac{t^j}{j!}\Big(\mathcal{Q}^j\Big)_{kn} \;=\; \sum_{j=1}^{\infty} \frac{t^j}{j!}\sum_{n=N}^{\infty}\Big(\mathcal{Q}^j\Big)_{kn}.$$

So it suffices to show that

$$\sum_{\nu=0}^{j-1}\sum_{n=k}^{N-1}\Big((\mathcal{Q}_{N-1})^{j-1-\nu}\Big)_{kn} D^{(n)}_{N-n} D^{\nu} \;=\; \sum_{n=N}^{\infty}\Big(\mathcal{Q}^j\Big)_{kn} \tag{10.1}$$

for all $j \in \mathbb{N}$. For $j = 1$ we have

$$\sum_{n=k}^{N-1}\Big((\mathcal{Q}_{N-1})^{0}\Big)_{kn} D^{(n)}_{N-n} D^{0} \;=\; D^{(k)}_{N-k} \;=\; \sum_{n=N-k}^{\infty} D_n \;=\; \sum_{n=N}^{\infty}\Big(\mathcal{Q}\Big)_{kn}.$$

Suppose (10.1) holds for j and consider $j+1$:

$$\begin{aligned}
&\sum_{\nu=0}^{j}\sum_{n=k}^{N-1}\Big((\mathcal{Q}_{N-1})^{j-\nu}\Big)_{kn} D^{(n)}_{N-n} D^{\nu} \\
&= \sum_{\nu=0}^{j-1}\sum_{n=k}^{N-1}\sum_{l=k}^{N-1}\Big(\mathcal{Q}_{N-1}\Big)_{kl}\Big((\mathcal{Q}_{N-1})^{j-\nu-1}\Big)_{ln} D^{(n)}_{N-n} D^{\nu} + \sum_{n=k}^{N-1}\Big((\mathcal{Q}_{N-1})^{0}\Big)_{kn} D^{(n)}_{N-n} D^{j} \\
&= \sum_{l=k}^{N-1}\Big(\mathcal{Q}_{N-1}\Big)_{kl}\sum_{\nu=0}^{j-1}\sum_{n=l}^{N-1}\Big((\mathcal{Q}_{N-1})^{j-\nu-1}\Big)_{ln} D^{(n)}_{N-n} D^{\nu} + D^{(k)}_{N-k} D^{j} \\
&= \sum_{l=k}^{N-1}\Big(\mathcal{Q}_{N-1}\Big)_{kl}\sum_{n=N}^{\infty}\Big(\mathcal{Q}^j\Big)_{ln} + \sum_{l=N-k}^{\infty} D_l D^j \\
&= \sum_{l=k}^{N-1}\Big(\mathcal{Q}\Big)_{kl}\sum_{n=N}^{\infty}\Big(\mathcal{Q}^j\Big)_{ln} + \sum_{l=N}^{\infty}\Big(\mathcal{Q}\Big)_{kl}\sum_{n=N}^{\infty}\Big(\mathcal{Q}^j\Big)_{ln} \qquad \text{by lemma 8.2} \\
&= \sum_{l=k}^{\infty}\Big(\mathcal{Q}\Big)_{kl}\sum_{n=N}^{\infty}\Big(\mathcal{Q}^j\Big)_{ln} \;=\; \sum_{n=N}^{\infty}\Big(\mathcal{Q}^{j+1}\Big)_{kn}.
\end{aligned}$$

Thus (10.1) holds for all $j \in \mathbb{N}$ and so $\tilde{A}^{(k)}_{N-k}(x) = \sum_{n=N-k}^{\infty} \tilde{A}^{\mathrm{B}}_n(x)$.

Now theorem 3.4 implies $B_n(s) = B^{\mathrm{B}}_n(s)$ for $n = 0, \ldots, N-2$ and

$$B_{N-1}(s) \;=\; \Big(sI - D^{(0)}_0\Big)^{-1} \sum_{l=1}^{N} D^{(0)}_l A^{(l)}_{N-l}(s),$$

where $A_0^{(N)}(s) = \sum_{n=0}^{\infty} A_n^{\mathrm{B}}(s)$ describes the evolution of the phase process during a service time (note that $P_0^{(N)}(t) = e^{Dt} = \sum_{n=0}^{\infty} P_n(t)$). So

$$\begin{aligned}
B_{N-1}(s) &= \Big(sI - D_0\Big)^{-1} \sum_{l=1}^{N-1} D_l \sum_{n=N-l}^{\infty} A_n^{\mathrm{B}}(s) + \Big(sI - D_0\Big)^{-1} D_N^{(0)} \sum_{n=0}^{\infty} A_n^{\mathrm{B}}(s) \\
&= \Big(sI - D_0\Big)^{-1} \sum_{l=1}^{N-1} D_l \sum_{n=N-l}^{\infty} A_n^{\mathrm{B}}(s) + \Big(sI - D_0\Big)^{-1} \sum_{l=N}^{\infty} D_l \sum_{n=0}^{\infty} A_n^{\mathrm{B}}(s) \\
&= \Big(sI - D_0\Big)^{-1} \sum_{l=1}^{\infty} D_l \sum_{n=N-l}^{\infty} A_n^{\mathrm{B}}(s) \quad \text{with } A_\nu(s) \equiv O \text{ for } \nu < 0 \\
&= \Big(sI - D_0\Big)^{-1} \sum_{l=1}^{\infty} D_l \sum_{n=N}^{\infty} A_{n-l}^{\mathrm{B}}(s) \\
&= \Big(sI - D_0\Big)^{-1} \sum_{n=N}^{\infty} \sum_{l=1}^{n} D_l A_{n-l}^{\mathrm{B}}(s) \\
&= \Big(sI - D_0\Big)^{-1} \sum_{n=N-1}^{\infty} \sum_{l=1}^{n+1} D_l A_{n+1-l}^{\mathrm{B}}(s) \;=\; \sum_{n=N-1}^{\infty} B_n^{\mathrm{B}}(s)\,.
\end{aligned}$$

Altogether we have $\tilde{\mathcal{P}}(x) = \tilde{\mathcal{P}}^{\mathrm{B}}(x)$. □

10.3 The queue length distribution

Theorem 10.3 implies that our approach yields the same embedded Markov chain as obtained by Blondia [4]. Consequently, both approaches give the same results for the queue length distribution at service completion times. This implies that the results for the queue length distribution at an arbitrary time also coincide:

Theorem 10.4 *The limiting distribution of the queue length at an arbitrary time is given by*

$$\begin{aligned}
\mathbf{y}_0 &= -\frac{1}{\mu^{-1} - \mathbf{x}_0 {D_0}^{-1}\mathbf{e}}\, \mathbf{x}_0 {D_0}^{-1}\,, \\
\mathbf{y}_k &= \frac{1}{\mu^{-1} - \mathbf{x}_0 {D_0}^{-1}\mathbf{e}} \sum_{l=1}^{k} \Big(\mathbf{x}_l - \mathbf{x}_0 {D_0}^{-1} D_l\Big) \left(\sum_{\nu=0}^{k-l} A_\nu \Big(\mathcal{Q}_{N-1}^{-1}\Big)_{\nu+l,k} - \Big(\mathcal{Q}_{N-1}^{-1}\Big)_{lk} \right) \\
&\quad \textit{for } k = 1, \ldots, N-1, \\
\mathbf{y}_N &= \pi - \sum_{k=0}^{N-1} \mathbf{y}_k\,.
\end{aligned}$$

Proof: Theorem 7.1 yields

$$\mathbf{y}_0 = -\frac{1}{\mu^{-1} - \mathbf{x}_0 {D_0^{(0)}}^{-1}\mathbf{e}} \mathbf{x}_0 {D_0^{(0)}}^{-1} = -\frac{1}{\mu^{-1} - \mathbf{x}_0 {D_0}^{-1}\mathbf{e}} \mathbf{x}_0 {D_0}^{-1}$$

by definition of the matrices $D_n^{(k)}$ for this case.

By corollary 1.17 D_0 is nonsingular and hence $\mathcal{Q}_{N-1}$ is also nonsingular. This enables us to simplify our results for $\mathbf{y}_k$, $k \geq 1$ (theorem 7.2). Partial integration [21, Th. 8.10, pp. 209f] yields for $k = 1, \ldots, N-1$ and $l = 1, \ldots, k$:

$$\begin{aligned}
&\int_0^\infty P_{k-l}^{(l)}(t)\Big(1 - H(t)\Big)\,dt \\
&= \int_0^\infty \Big(e^{\mathcal{Q}_{N-1}t}\Big)_{lk}\Big(1 - H(t)\Big)\,dt \quad \text{by corollary 10.2} \\
&= \left(e^{\mathcal{Q}_{N-1}t}\mathcal{Q}_{N-1}^{-1}\Big(1 - H(t)\Big)\Big|_0^\infty - \int_0^\infty e^{\mathcal{Q}_{N-1}t}\mathcal{Q}_{N-1}^{-1}\,d\Big(1 - H(t)\Big)\right)_{lk} \\
&= -\Big(\mathcal{Q}_{N-1}^{-1}\Big)_{lk} + \sum_{\nu=0}^{N-1}\int_0^\infty \Big(e^{\mathcal{Q}_{N-1}t}\Big)_{l\nu}\,dH(t)\Big(\mathcal{Q}_{N-1}^{-1}\Big)_{\nu k} \\
&= -\Big(\mathcal{Q}_{N-1}^{-1}\Big)_{lk} + \sum_{\nu=l}^{k}\int_0^\infty P_{\nu-l}\,dH(t)\Big(\mathcal{Q}_{N-1}^{-1}\Big)_{\nu k} \\
&= -\Big(\mathcal{Q}_{N-1}^{-1}\Big)_{lk} + \sum_{\nu=0}^{k-l} A_\nu\Big(\mathcal{Q}_{N-1}^{-1}\Big)_{\nu+l,k},
\end{aligned}$$

where $A_\nu = A_\nu^{\mathrm{B}}$ by theorem 10.3. Thus

$$\mathbf{y}_k = \frac{1}{\mu^{-1} - \mathbf{x}_0 {D_0}^{-1}\mathbf{e}} \sum_{l=1}^{k}\Big(\mathbf{x}_l - \mathbf{x}_0 {D_0}^{-1} D_l\Big)\left(\sum_{\nu=0}^{k-l} A_\nu\Big(\mathcal{Q}_{N-1}^{-1}\Big)_{\nu+l,k} - \Big(\mathcal{Q}_{N-1}^{-1}\Big)_{lk}\right)$$

for $k = 1, \ldots, N-1$ by theorem 7.2. Finally, theorem 8.23 implies $\sum_{k=0}^{N}\mathbf{y}_k = \pi$ and therefore $\mathbf{y}_N = \pi - \sum_{k=0}^{N-1}\mathbf{y}_k$. □

Remark: Blondia gives the same results for $\mathbf{y}_0$ and $\mathbf{y}_N$ [4, eq. (18),(20)]. For $\mathbf{y}_k$, $k = 1, \ldots, N-1$, he obtains [4, before eq. (19)] $\mathbf{y}_k = \mathrm{P}_{\text{busy}}\,\bar{\omega}_k^*(0)$, where (in our notation)

$$\begin{aligned}
\mathrm{P}_{\text{busy}} &= \frac{\mu^{-1}}{\mu^{-1} - \mathbf{x}_0 {D_0}^{-1}\mathbf{e}}, \\
\bar{\omega}_k^*(s) &= -\mathbf{x}_0 {D_0}^{-1}\sum_{l=1}^{k} D_l H_{k-l}(s) + \sum_{l=1}^{k}\mathbf{x}_l H_{k-l}(s),
\end{aligned}$$

$$\begin{aligned} H_n(s) &= \mu \left(\sum_{\nu=0}^{n} A_\nu R_{n-\nu}(s) - H^*(s)\, R_n(s) \right), \\ \sum_{n=0}^{\infty} R_n(s)\, z^n &= \Big(D(z) + sI \Big)^{-1} \quad \text{for } \mathrm{Re}(s) \geq 0 \text{ and } |z| < 1, \\ H^*(s) &= \int_0^\infty e^{-st}\, dH(t)\,. \end{aligned}$$

Thus

$$\bar{\omega}_k^*(s) \;=\; \mu \sum_{l=1}^{k} \Big(\mathbf{x}_l - \mathbf{x}_0 {D_0}^{-1} D_l \Big) \left(\sum_{\nu=0}^{k-l} A_\nu R_{k-l-\nu}(s) - H^*(s)\, R_{k-l}(s) \right),$$

and so

$$\mathbf{y}_k \;=\; \frac{1}{\mu^{-1} - \mathbf{x}_0 {D_0}^{-1} \mathbf{e}} \sum_{l=1}^{k} \Big(\mathbf{x}_l - \mathbf{x}_0 {D_0}^{-1} D_l \Big) \left(\sum_{\nu=0}^{k-l} A_\nu R_{k-l-\nu}(0) - R_{k-l}(0) \right).$$

From

$$\mathcal{Q} \;=\; \left(\begin{array}{ccc|cc} D_0 & \cdots & D_{N-1} & D_N & \cdots \\ & \ddots & \vdots & \vdots & \\ & & D_0 & D_1 & \cdots \\ \hline & & & D_0 & \cdots \\ & & & & \ddots \end{array} \right) \;=\; \left(\begin{array}{c|cc} & D_N & \cdots \\ \mathcal{Q}_{N-1} & \vdots & \\ & D_1 & \cdots \\ \hline & D_0 & \cdots \\ & & \ddots \end{array} \right)$$

we obtain $(\mathcal{Q}_{N-1}^{-1})_{\nu n} = R_{n-\nu}(0)$ for $\nu = 0, \ldots, N-1$ and $n = \nu, \ldots, N-1$. Thus Blondia's result coincides with ours of theorem 10.4.

Directions for future work

In this paper we defined a level dependent batch Markovian arrival process and analysed the BMAP/G/1 queue with level dependent arrivals. We derived analytical results for the queue length distributions at service completion times and at an arbitrary time.

Future research could start with the following ideas:

a) The results of this paper give rise to numerical algorithms for computing the queue length distributions. To implement these algorithms one needs to think about the appropriate truncation of sequences and matrices of infinite size. Numerical algorithms are needed to use the BMAP/G/1 queue with level dependent arrivals for the performance evaluation of communication systems. The results could then be compared with simulations.

b) The BMAP/G/1 queue with level dependent arrivals can be extended by allowing the service time distribution to depend on the state of the queueing system at the beginning of a service. This would not change the structure of the transition probability matrix $\tilde{\mathcal{P}}(x)$ of the embedded Markov chain. But our derivations of the mean sojourn times (theorem 3.6) and the mean duration of a fundamental period (theorem 5.17) would not longer hold. These vectors were needed to determine the queue length distribution at an arbitrary time (theorems 7.1 and 7.2).

c) The level dependent batch Markovian arrival process can be generalized by allowing an infinite number of phases. It might then be possible to extend the results of Ramaswami and Taylor concerning product–form queueing networks [36].

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