Stable elements in topological algebras

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Preface

The present work is inspired by the following question, posed by the British mathematician G. R. Allan in 1998 (cf. [5, p. 94]):

"If x, y are stable elements of a commutative ring R, is xy necessarily stable [...]?"

Stability is a term which is often used in different branches of science, and even in mathematics there are many definitions of stability. The similarity between these definitions is mostly the examination of certain solutions under small perturbations of the starting conditions.
This thesis is dealing with a form of stability that was introduced by Allan in [5]. In the sense of Allan, an element x of a commutative ring R is stable, if for every choice of a sequence $(b_n)_{n\in\mathbb{N}}$ in R there exists a solution $(a_n)_{n\in\mathbb{N}}$ in R for the following infinite system of equations:

$$\begin{aligned} a_1 &= xa_2 + b_1 \\ a_2 &= xa_3 + b_2 \\ a_3 &= xa_4 + b_3 \\ &\vdots \end{aligned}$$

Here the perturbation mentioned above is represented by the sequence $(b_n)_{n\in\mathbb{N}}$.

Allan's motivation for working with this kind of stability goes back to his paper [2] from 1972, where he worked with the embedding of the algebra of formal power series into a Banach algebra. There he proved that for a commutative unital Banach algebra A the following are equivalent:

1. There exists a homomorphism $\theta_x : \mathbb{C}[[X]] \to A$ with $\theta_x(X) = x$.
2. $x \in rad(A)$ and x is stable,

where $\mathbb{C}[[X]]$ denotes the algebra of formal power series with complex coefficients and $rad(A)$ denotes the radical of A, i.e. the intersection of all maximal ideals in A. Surely Allan did not speak of stability, but he used a property of the element x, which turned out to be equivalent to stability in the case of Banach algebras (cf. [5, Theorem 4.7]). He applied this theorem to show the existence of a discontinuous homomorphism between Banach algebras and studied further consequences in connection with the theory of automatic continuity in [2] and [6]. Since he was able to show an analogous theorem for Fréchet algebras [7, Theorem 19], his studies also include these spaces. So this theory might also be of interest for the famous unsolved Michael problem, whether every homomorphism from a commutative Fréchet algebra to the complex numbers is necessarily continuous.

As mentioned in the quotation at the beginning of this introduction, one aspect of Allan's work with stable elements was the question whether the product of two stable elements is stable again. Allan did not mention any motivation for this problem, however this question is of interest in itself. He found out that the answer is affirmative in the case of commutative Banach and Fréchet algebras, where he proved suitable characterizations for stability.

The main topic of this thesis is the extension of Allan's work with stable elements to more general topological algebras. For this purpose we use some methods from the theory of homological algebra that were introduced by V. P. Palamodov in the 1960s and 1970s (cf. [18] and [19]). On the one hand these methods give an easier approach to Allan's results about stability in Fréchet algebras and on the other hand they are still applicable in more general settings.

Although these techniques do not lead to a final answer of Allan's question, that would be either a proof that in every commutative ring the product of two stable elements is stable or a counterexample of two stable elements whose product is not stable, we will show in this thesis that the answer is affirmative in all important situations that are usually considered in functional analysis.

In the first chapter we will bring together Allan's definition of stability and Palamodov's work with the projective limit functor $\mathrm{Proj}\mathcal{A}$. After defining stability and considering several conclusions and examples, we will observe that the stability of an element x is equivalent to a condition concerning the derived functor of the projective limit functor, usually written as $\mathrm{Proj}^1\mathcal{A}_x = 0$, of a particular projective spectrum $\mathcal{A}_x$. This projective spectrum depends on the multiplication M_x that maps an element y to the product xy. Because of this equivalence it is convenient to use some results about this derived functor, especially we will present a necessary condition for $\mathrm{Proj}^1\mathcal{A} = 0$ (cf. [23]). The considerations in this chapter will be purely algebraic without any topological requirements.

After this we will characterize stability under further topological assumptions. To this end we will deduce a sufficient condition for $\mathrm{Proj}^1\mathcal{A} = 0$ in the context of complete metrizable groups from a Mittag-Leffler lemma due to R. Arens [9, Theorem 2.4]. This will lead us to Allan's results concerning Banach algebras and Fréchet algebras without using the theory of embedding the algebra of formal power series. Furthermore we obtain as a direct consequence that in Fréchet algebras the product of two stable elements is again stable. Additionally we will analyze stability on some typical examples of Banach and Fréchet algebras.

Chapter 3 is devoted to the study of LB-algebras. There we will make use of a characterization of the condition $\mathrm{Proj}^1\mathcal{A} = 0$ due to V. S. Retakh [20] and V. P. Palamodov [19, Theorem 5.4] to find a suitable characterization of stability from which we can deduce that again the product of stable elements is stable. Besides we will determine all stable elements in some examples of LB-algebras and at the end of this chapter we will transfer a result of Allan for Fréchet algebras to the LB-algebra case, namely that for a stable element x and the ideal $I(x) = \bigcap_{n\in\mathbb{N}} x^n A$ in an LB-algebra A we have $M_x(I(x)) = I(x)$.

In the last chapter of this thesis we will work with webbed locally convex topological algebras that generalize both Fréchet algebras and LB-algebras and were introduced by M. de Wilde in [12] in the context of closed graph theorems. For these spaces L. Frerick, D. Kunkle and J. Wengenroth [13] found a characterization for $\mathrm{Proj}^1\mathcal{A} = 0$. Under a further assumption on the webs we can state the main result of this thesis:

In every commutative locally convex Hausdorff topological algebra that has a multiplicative web the product of two stable elements is stable.

In the end we will prove several hereditary properties of these topological algebras to show that Allan's result concerning the product of stable elements remains true for a large class of topological algebras.

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1 Stable elements in commutative rings

In the first part of this thesis we present two different approaches to the kind of stability defined by G. R. Allan in [5]. So we start with the introduction of Allan's notion.

Definition 1.1.
Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of groups and for all positive integers m and n with $m \geq n$ let $\varrho_m^n : A_m \to A_n$ be a homomorphism such that $\varrho_n^n = id$ and $\varrho_m^n \circ \varrho_k^m = \varrho_k^n$ for all natural numbers $n \leq m \leq k$. Then $\mathcal{A} = (\mathcal{A}_n, \varrho_m^n)$ is called a *projective spectrum* (of groups and homomorphisms) and the set

$$\text{Proj}\mathcal{A} = \left\{(a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n : \varrho_{n+1}^n(a_{n+1}) = a_n \text{ for all } n\right\}$$

is called the *projective limit* of $\mathcal{A}$.

It is possible to define this limit in every category where countable products can be formed. Besides the category of groups or the category of rings these are for example the categories of linear spaces, metric spaces or topological spaces. Then every A_n in the above definition is an object of the category and every ϱ_m^n is a morphism between objects A_m and A_n.

Definition 1.2.
Let $\mathcal{A} = (\mathcal{A}_n, \varrho_m^n)$ be a projective spectrum of groups and homomorphisms. Then $\mathcal{A}$ is called *stable* if for every sequence $(b_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n$ the set

$$\left\{(a_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n : a_n = \varrho_{n+1}^n(a_{n+1}) + b_n\right\}$$

is nonempty. If in addition $A_n = A$ for all $n \in \mathbb{N}$ and T is an endomorphism on A, we say that T *acts stably* on A if the projective spectrum $\mathcal{A}_T = (A, T)$ is stable, where $\varrho_m^n = T^{m-n} = T\circ\ldots\circ T$ is the $(m-n)$times composition of T with itself for all $m \geq n$.

If we set $\sigma_{n+1}^n(x) = \varrho_{n+1}^n(x) + b_n$, $\sigma_m^n = \sigma_{n+1}^n \circ \ldots \circ \sigma_m^{m-1}$ and $\mathcal{A}_b = (A_n, \sigma_m^n)$ for $b = (b_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n$, the spectrum $\mathcal{A}$ is stable if and only if for all $b \in \prod_{n\in\mathbb{N}} A_n$ the corresponding projective limit Proj $\mathcal{A}_b$ is nonempty.

Now we come to the core notion of this thesis, the stability of an element of a ring.

Definition 1.3.
Let A be a ring. An element $x \in A$ is *stable* if the projective spectrum $\mathcal{A}_x = (A, M_x)$ is stable, where every A_n is equal to A and every ϱ_{n+1}^n is the multiplication $M_x : A \to A$ defined by $y \mapsto xy$. In other words $x \in A$ is stable if M_x acts stably on A.

Thus the stability of x just describes the fact that for every sequence $(b_n)_{n\in\mathbb{N}}$ in A we find a sequence $(a_n)_{n\in\mathbb{N}}$ in A such that for all $n \in \mathbb{N}$ we have $a_n = xa_{n+1} + b_n$. Naturally we need to distinguish between the left multiplication M_x and the right multiplication defined by $y \mapsto yx$. But since we will mainly be concerned with commutative rings and commutative topological algebras, it suffices to work with the mapping M_x.
It is clear that the zero element of a ring is always stable since in this case we can choose $a_n = b_n$ for every $n \in \mathbb{N}$. The same holds for an invertible element $x \in A$, because in this situation we can solve the equations $a_n = xa_{n+1} + b_n$ recursively. So in a field every element is stable.
In the following examples we will determine the stable elements in the set of polynomials and in the set of formal power series (cf. [5, Proposition 2.5 and Theorem 2.6]).

Example 1.4.
Let $A = \mathbb{C}[X]$ be the ring of all polynomials with complex coefficients. We will show that the zero polynomial and the constant polynomials are the only stable elements in A: Let $p \in A$ have degree greater than or equal to 1. If we assume that p is stable and choose $b_n = 1$ for every natural number n, the stability of p yields a sequence of polynomials $(a_n)_{n\in\mathbb{N}}$ that satisfies

$$a_n = a_{n+1}p + 1$$

for every choice of n. For an arbitrary $k \geq 1$ there exists $m \geq k$ such that $a_m \neq 0$ and so the degree of a_m is larger than the degree of a_{m+1}. But then the degree of a_1 must be at least k and since k was chosen arbitrarily this contradicts the fact that a_1 has finite degree.

Example 1.5.
Let $A = \mathbb{C}[[X]]$ be the ring of all formal power series with coefficients in $\mathbb{C}$. Then every element of A is stable: For some $f \in A$ with $f(X) = \sum_{\nu=0}^{\infty} f_\nu X^\nu$ and a sequence $(b^n)_{n\in\mathbb{N}}$ in A we have to find a sequence $(a^n)_{n\in\mathbb{N}}$ in A such that for all $n \in \mathbb{N}$

$$\sum_{\nu=0}^{\infty} a_\nu^n X^\nu = \sum_{\nu=0}^{\infty} f_\nu X^\nu \cdot \sum_{\nu=0}^{\infty} a_\nu^{n+1} X^\nu + \sum_{\nu=0}^{\infty} b_\nu^n X^\nu.$$

To show this we will consider the νth coefficients of the series in this equation for all $\nu \in \mathbb{N}_0$. For $\nu = 0$ the stability of $f_0 \in \mathbb{C}$ yields a sequence $(a_0^n)_{n\in\mathbb{N}}$ such that

$$a_0^n = f_0 a_0^{n+1} + b_0.$$

If we now assume that we have found sequences $(a_j^n)_{n\in\mathbb{N}}$ for all $1 \leq j \leq \nu$ such that

$$a_j^n = \sum_{k=0}^{j} a_k^{n+1} f_{j-k} + b_j^n,$$

we have to find a sequence $(a_{\nu+1}^n)_{n\in\mathbb{N}}$ such that

$$a_{\nu+1}^n = \sum_{k=0}^{\nu+1} a_k^{n+1} f_{\nu+1-k} + b_{\nu+1}^n = a_{\nu+1}^{n+1} f_0 + \sum_{k=0}^{\nu} a_k^{n+1} f_{\nu+1-k} + b_{\nu+1}^n = a_{\nu+1}^{n+1} f_0 + c_n$$

where the $c_n = \sum_{k=0}^{\nu} a_k^{n+1} f_{\nu+1-k} + b_{\nu+1}^n$ are given by the induction hypothesis. Then again the stability of f_0 implies the existence of a suitable sequence $(a_{\nu+1}^n)_{n\in\mathbb{N}}$. So by induction the assertion is proved.

Remark 1.6.
The examples above represent two opposite cases that can occur: In $\mathbb{C}[X]$ only the zero element and the invertible elements are stable, while in $\mathbb{C}[[X]]$ every element is stable. Actually every natural example of a topological algebra considered in this thesis belongs to one of these two cases. Nevertheless we can easily construct a ring that belongs to neither of these extremes: Let $R = \mathbb{C}[X]$, $S = \mathbb{C}[[X]]$ and $A = R \times S$. Then, with coordinate-wise addition and multiplication, A is a ring and an element $(f,g) \in A$ is stable if and only if f is stable in R and g is stable in S. The same is true if we replace "stable" by "invertible". So we can choose $f_1, f_2 \in R$ such that f_1 is invertible and f_2 is not stable and thus not invertible and also $g_1, g_2 \in S$ such that g_1 is not invertible but stable and g_2 is invertible. Then neither (f_1, g_1) nor (f_2, g_2) are invertible, but (f_1, g_1) is stable while (f_2, g_2) is not stable.

We continue with the following result of Allan [5, Lemma 2.2.] concerning stable elements and ring homomorphisms.

Proposition 1.7.
Let R, S be commutative rings and $T : R \to S$ a surjective ring homomorphism.

1. If x is stable in R, then $T(x)$ is stable in S.
2. If $T(x)$ is stable in S and M_x acts stably on the kernel of T, then x is stable in R.

Proof. 1. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in S. Since T is surjective, for all $n \in \mathbb{N}$ there exists $x_n \in R$ such that $T(x_n) = y_n$. The stability of x yields a sequence $(a_n)_{n\in\mathbb{N}}$ in R such that for all $n \in \mathbb{N}$

$$a_n = xa_{n+1} + x_n.$$

Then we have

$$T(a_n) = T(xa_{n+1} + x_n) = T(x)T(a_{n+1}) + y_n$$

and thus $T(x)$ is stable in S.
2. We consider an arbitrary sequence $(b_n)_{n\in\mathbb{N}}$ in R. Since $T(x)$ is stable in S and T is surjective, there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in R such that for all natural numbers n we have

$$T(a_n) = T(x)T(a_{n+1}) + T(b_n) = T(xa_{n+1} + b_n).$$

Hence $z_n = b_n + xa_{n+1} - a_n$ is an element of $\operatorname{Ker} T$ for all $n \in \mathbb{N}$, where $\operatorname{Ker} T$ denotes the kernel of T. Then the fact that M_x acts stably on $\operatorname{Ker} T$ implies the existence of a sequence $(d_n)_{n\in\mathbb{N}}$ in $\operatorname{Ker} T \subseteq R$ such that

$$d_n = xd_{n+1} + z_n$$

for all $n \in \mathbb{N}$. Thus we have

$$d_n + a_n = xd_{n+1} + b_n + xa_{n+1} - a_n + a_n = x(d_{n+1} + a_{n+1}) + b_n$$

and so the sequence $(e_n)_{n\in\mathbb{N}}$ defined by $e_n = d_n + a_n$ yields the stability of x in R. □

Now we turn our attention from stability of elements of a commutative ring to the more general notion of stability of projective spectra of abelian groups. To this purpose we will use some results from homological algebra without going into the details of this theory, adopting the notation from [23]. In this context the following remark will show the connection between Allan's definition of stability and a property of the derived functor of the functor Proj.

Remark 1.8.
The stability of projective spectra can also be characterized in a category-theoretical sense: If we have two projective spectra $\mathcal{A} = (A_n, \varrho^n_m)$ and $\mathcal{B} = (B_n, \tau^n_m)$ in a suitable category $\mathcal{K}$ and a morphism $f = (f_n)_{n\in\mathbb{N}} : \mathcal{A} \to \mathcal{B}$, i.e. morphisms $f_n : A_n \to B_n$ such that $f_n \circ \varrho^n_m = \tau^n_m \circ f_m$ for all $m \geq n$, let $\mathrm{Proj} f : \mathrm{Proj}\mathcal{A} \to \mathrm{Proj}\mathcal{B}$ be defined by

$$(x_n)_{n\in\mathbb{N}} \mapsto (f_n(x_n))_{n\in\mathbb{N}}.$$

Then Proj can be considered as a functor acting on the category of projective spectra with values in the category $\mathcal{K}$. To avoid more category-theoretical details we now make the following ad hoc definition: If $\mathcal{A} = (A_n, \varrho^n_m)$ is a projective spectrum of abelian groups and homomorphisms, let $\psi = \psi_{\mathcal{A}} : \prod_{n\in\mathbb{N}} A_n \to \prod_{n\in\mathbb{N}} A_n$ be defined by

$$(x_n)_{n\in\mathbb{N}} \mapsto (x_n - \varrho^n_{n+1}(x_{n+1}))_{n\in\mathbb{N}}.$$

Then we set

$$\mathrm{Proj}^1\mathcal{A} = \Big(\prod_{n\in\mathbb{N}} A_n\Big)\Big/ \mathrm{Im}\,\psi\ ,$$

where $\mathrm{Im}\,\psi$ denotes the image of ψ. In category-theoretical sense Proj^1 is the derived functor of the functor Proj.

One immediately notices that the stability of a projective spectrum $\mathcal{A}$ is just the same as the surjectivity of the mapping ψ and hence also the same as the condition $\mathrm{Proj}^1\mathcal{A} = 0$.

Having this characterization of stability, we can make use of several results about the derived functor. The first one will be the following theorem due to Palamodov [19, Theorem 5.1], that will be applied later to prove a necessary condition for stability. The proof presented here is much more elementary than that of Palamodov.

Theorem 1.9.
Let $\mathcal{A} = (A_n, \varrho_m^n)$ be a projective spectrum of groups and homomorphisms. If we equip the product $\prod_{n\in\mathbb{N}} A_n$ with the group topology $\mathcal{F}$ whose basis of zero neighborhoods is given by the sets
$$U_n = \Big\{(\varrho_n^1 x_n, \dots, \varrho_n^{n-1} x_n, x_n, x_{n+1}, \dots) : x_k \in A_k \text{ for all } k \geq n\Big\},$$
then $\mathrm{Proj}^1\mathcal{A} = 0$ if and only if $(\prod_{n\in\mathbb{N}} A_n, \mathcal{F})$ is complete.

Proof. First of all we consider the topology $\mathcal{G} = \prod_{n\in\mathbb{N}} \mathcal{D}_n$ on $\prod_{n\in\mathbb{N}} A_n$, where $\mathcal{D}_n$ is the discrete topology on A_n for all $n \in \mathbb{N}$, and
$$\psi : (\prod_{n\in\mathbb{N}} A_n, \mathcal{F}) \to (\prod_{n\in\mathbb{N}} A_n, \mathcal{G})$$
defined by
$$(x_n)_{n\in\mathbb{N}} \mapsto (x_n - \varrho_{n+1}^n(x_{n+1}))_{n\in\mathbb{N}}.$$
Obviously, ψ is a group homomorphism. As a first step we will show that ψ is continuous, open onto its image, and has dense image. The sets
$$V_n = 0 \times \dots \times 0 \times \prod_{k\geq n} A_k$$
form a basis of zero neighborhoods in $\mathcal{G}$ and
$$\begin{aligned}
&V_n \cap \mathrm{Im}(\psi) = \Big\{(0, \dots, 0, x_n - \varrho_{n+1}^n(x_{n+1}), x_{n+1} - \varrho_{n+2}^{n+1}(x_{n+2}), \dots) : x_k \in A_k (k \geq n)\Big\}\\
&= \Big\{(\varrho_n^1(x_n) - \varrho_2^1(\varrho_n^2(x_n)), \dots, \varrho_n^{n-1}(x_n) - \varrho_n^{n-1}(x_n), x_n - \varrho_{n+1}^n(x_{n+1}), \dots), x_k \in A_k (k \geq n)\Big\}\\
&= \psi(U_n).
\end{aligned}$$
Since for every zero neighborhood V in $\mathcal{G}$ there exists $n \in \mathbb{N}$ such that $V_n \subseteq V$ we have
$$\psi(U_n) = V_n \cap \mathrm{Im}(\psi) \subseteq V$$
which implies the continuity of ψ. Furthermore, for each zero neighborhood U in $\mathcal{F}$ we find some $n \in \mathbb{N}$ such that $U_n \subseteq U$ and so the fact that
$$\psi(U) \supseteq \psi(U_n) = V_n \cap \mathrm{Im}(\psi)$$
implies that ψ is open onto its image. Moreover $\mathrm{Im}(\psi)$ is dense:
If we choose some $x = (x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} A_n$, then for
$$y_n = \Big(\sum_{k=1}^n \varrho_k^1(x_k), \sum_{k=2}^n \varrho_k^2(x_k), \dots, x_n, 0, \dots\Big)$$
we have
$$\psi(y_n) = (x_1, \dots, x_n, 0, \dots)$$
so that $\psi(y_n) - x \in V_n$ for all $n \in \mathbb{N}$. Additionally we can state that
$$\mathrm{Proj}\mathcal{A} = \mathrm{Ker}(\psi) = \bigcap_{n\in\mathbb{N}} U_n = \overline{\{0\}}^{\mathcal{F}}.$$

If we assume that $\operatorname{Proj}^1\mathcal{A} = 0$, then ψ is surjective and thus a quotient mapping. This implies that

$$(\prod_{n\in\mathbb{N}} A_n, \mathcal{F})/\overline{\{0\}}^{\mathcal{F}} \cong (\prod_{n\in\mathbb{N}} A_n, \mathcal{G})$$

is complete. We want to conclude that $(\prod_{n\in\mathbb{N}} A_n, \mathcal{F})$ is complete. To this end let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $(\prod_{n\in\mathbb{N}} A_n, \mathcal{F})$ and $q : (\prod_{n\in\mathbb{N}} A_n, \mathcal{F}) \to (\prod_{n\in\mathbb{N}} A_n, \mathcal{F})/\overline{\{0\}}^{\mathcal{F}}$ the quotient mapping. Then $(q(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in $(\prod_{n\in\mathbb{N}} A_n, \mathcal{F})/\overline{\{0\}}^{\mathcal{F}}$ that converges to some $q(x)$. Thus for an arbitrary $U \in \mathcal{F}$ and n sufficiently large we have

$$q(x_n) - q(x) = q(x_n - x) \in q(U)$$

and therefore $(x_n - x) - u \in \overline{\{0\}}^{\mathcal{F}}$ for some $u \in U$. So, using the continuity of addition, for all $V \in \mathcal{F}$ there exists $U \in \mathcal{F}$ such that for sufficiently large n

$$(x_n - x) \in U + \overline{\{0\}}^{\mathcal{F}} \subseteq U + U \subseteq V.$$

Hence $(x_n)_{n\in\mathbb{N}}$ converges to x and the assertion is proved.
Now let $(\prod_{n\in\mathbb{N}} A_n, \mathcal{F})$ be complete. Since ψ is continuous and open onto its image, $(\operatorname{Im}(\psi), \mathcal{G}\cap\operatorname{Im}(\psi))$ is complete. But then the fact that $\operatorname{Im}(\psi)$ is dense in $(\prod_{n\in\mathbb{N}} A_n, \mathcal{G})$ implies that $\operatorname{Im}(\psi) = \prod_{n\in\mathbb{N}} A_n$. Hence ψ is surjective and $\operatorname{Proj}^1\mathcal{A} = 0$. □

As a direct consequence we obtain the following interesting result:

Corollary 1.10.
Let $\mathcal{A}$ and $(\prod_{n\in\mathbb{N}} A_n, \mathcal{F})$ be as in 1.9. Then the condition $\operatorname{Proj}^1\mathcal{A} = 0$ implies that every A_n equipped with the topology $\mathcal{F}_n$ whose basis of zero neighborhoods is given by the sets $U_{n,m} = \varrho^n_m(A_m)$ for $m \geq n$ is complete for all $n \in \mathbb{N}$.

Proof. Since every $\mathcal{F}_n$ is the final group topology on A_n with respect to the projection

$$\pi_n : (\prod_{n\in\mathbb{N}} A_n, \mathcal{F}) \to A_n,$$

which can be checked by the definition of the $\mathcal{F}_n$, the assertion follows directly from 1.9. □

We want to finish this chapter with a necessary condition for $\operatorname{Proj}^1\mathcal{A} = 0$ (cf. [23, Proposition 3.2.6]). Using 1.9 we obtain this condition in a more general case, namely for projective spectra of groups. Nevertheless the proof will be similar to the one in [23] and we will present the original result as a corollary.

Proposition 1.11.
Let $\mathcal{A} = (A_n, \varrho_m^n)$ be a projective spectrum of groups and homomorphisms with $A_n = \bigcup_{l\in\mathbb{N}} A_{n,l}$ for all $n \in \mathbb{N}$. If $\operatorname{Proj}^1\mathcal{A} = 0$, then there exists a sequence $(N(n))_{n\in\mathbb{N}}$ of natural numbers such that

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m : \varrho_m^n(A_m) \subseteq \varrho_k^n(A_k) + \bigcap_{j=1}^{n} (\varrho_n^j)^{-1}(A_{j,N(j)} - A_{j,N(j)}).$$

Proof. With the notation of 1.9 $\operatorname{Proj}^1\mathcal{A} = 0$ implies that $(\prod_{n\in\mathbb{N}} A_n, \mathcal{F})$ is complete and hence a Baire space . For all $n \in \mathbb{N}$ we have $A_n = \bigcup_{l\in\mathbb{N}} A_{n,l}$ and thus

$$\prod_{n\in\mathbb{N}} A_n = \bigcup_{l\in\mathbb{N}} A_{1,l} \times \prod_{k\geq 2} A_k.$$

So there exists a natural number $N(1)$ such that $A_{1,N(1)} \times \prod_{k\geq 2} A_k$ is of second category. Inductively we find a sequence $(N(n))_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$B_n = A_{1,N(1)} \times A_{2,N(2)} \times \cdots \times A_{n,N(n)} \times \prod_{k\geq n+1} A_k$$

is of second category, hence there exists some x in the interior of $\overline{B_n}^{\mathcal{F}}$. Thus we find a zero neighborhood U such that $x + U \subseteq \overline{B_n}^{\mathcal{F}}$ and we can conclude that

$$0 \in U \subseteq \overline{B_n}^{\mathcal{F}} - x = \overline{B_n - x}^{\mathcal{F}} \subseteq \overline{B_n - B_n}^{\mathcal{F}}.$$

Hence for all $n \in \mathbb{N}$ there exists some $m \in \mathbb{N}$ such that for $k \geq n$

$$U_m \subseteq \overline{B_n - B_n}^{\mathcal{F}} = \bigcap_{l\geq n} (B_n - B_n) + U_l \subseteq (B_n - B_n) + U_k.$$

Now let $x_m \in A_m$. Then $x = (\varrho_m^1(x_m), \ldots, \varrho_m^{m-1}(x_m), x_m, 0, \ldots)$ is an element of U_m. Thus the inclusion above implies that we can write x as

$$x = (a_1 - \tilde{a}_1, \ldots, a_n - \tilde{a}_n, *, \ldots) + (\varrho_k^1(y_k), \ldots, \varrho_k^{k-1}(y_k), y_k, *, \ldots)$$

with $a_j, \tilde{a}_j \in A_{j,N(j)}$ for $j \leq n$, $y_k \in A_k$ and the elements $*$ chosen suitably. Thus, for all $j \leq n$, we have

$$\varrho_n^j(a_n - \tilde{a}_n) = \varrho_n^j(\varrho_m^n(x_m) - \varrho_k^n(y_k)) = \varrho_m^j(x_m) - \varrho_k^j(y_k) = a_j - \tilde{a}_j \in A_{j,N(j)} - A_{j,N(j)}.$$

So we have shown that for all $j \leq n$

$$\varrho_m^n(x_m) - \varrho_k^n(y_k) \in \bigcap_{j=1}^{n} (\varrho_n^j)^{-1}(A_{j,N(j)} - A_{j,N(j)})$$

which completes the proof. □

Corollary 1.12.
If the projective spectrum $\mathcal{A} = (A_n, \varrho_m^n)$ in 1.11 consists of linear spaces and linear mappings and every A_n is the countable union of absolutely convex sets $A_{n,l}$, then $\operatorname{Proj}^1 \mathcal{A} = 0$ implies that

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m : \varrho_m^n(A_m) \subseteq \varrho_k^n(A_k) + \bigcap_{j=1}^{n} (\varrho_n^j)^{-1}(A_{j,N(j)}).$$

Proof. Each $A_{n,l}$ is absolutely convex and thus $A_{j,N(j)} - A_{j,N(j)} = 2A_{j,N(j)}$. Then multiplying the inclusion in 1.11 by $\frac{1}{2}$ yields the conclusion. □

2 Banach and Fréchet algebras

In the case of commutative Fréchet algebras Allan used a characterization of stability to show that the product of two stable elements is again stable. The proof of this characterization was very involved and it turns out that dealing with $\text{Proj}^1\mathcal{A}$ leads to a much easier proof.
As we have seen in the last chapter, stability corresponds to the condition $\text{Proj}^1\mathcal{A} = 0$. After showing a necessary condition for this property in 1.11 we now want to deduce a sufficient one. To this end we will use a Mittag-Leffler lemma due to R. Arens [9, Theorem 2.4] and thus we need some topological requirements for the projective spectrum. The proof of Arens' theorem will follow the one in [22, Proposition 1].

Theorem 2.1.
Let $\mathcal{A} = (A_n, \varrho_m^n)$ be a projective spectrum of complete metric spaces (A_n, d_n) with $A_n \neq \emptyset$ and continuous mappings $\varrho_m^n : A_m \to A_n$ for all $n, m \in \mathbb{N}$ such that

$$\forall\ n \in \mathbb{N}, \varepsilon > 0\ \exists\ m(n) \geq n\ \forall\ k \geq m(n) : \varrho_{m(n)}^n(A_{m(n)}) \subseteq [\varrho_k^n(A_k)]_\varepsilon, \tag{1}$$

where $[M]_\varepsilon = \{x \in A_n : \exists\ y \in M \quad d_n(x, y) < \varepsilon\}$ for any $M \subseteq A_n$. Then $\text{Proj}\mathcal{A}$ is nonempty.

Proof. Let $A = \prod_{n\in\mathbb{N}}(A_n, d_n)$ and for $a, b \in A$ let

$$d_\infty(a, b) = \sup\{\min\{d_n(a_n, b_n), 2^{-n}\} : n \in \mathbb{N}\}.$$

Then (A, d_∞) is a complete metric space.
If $S : A \to A$ is defined by $(a_n)_{n\in\mathbb{N}} \mapsto (\varrho_{n+1}^n(a_{n+1}))_{n\in\mathbb{N}}$, then

$$\text{Proj}\mathcal{A} = \text{Fix}(S) = \Big\{(a_n)_{n\in\mathbb{N}} : S((a_n)_{n\in\mathbb{N}}) = (a_n)_{n\in\mathbb{N}}\Big\}$$

is the set of all fixed points of S. We define for $\varepsilon > 0$

$$\text{Fix}_\varepsilon(S) = \Big\{(a_n)_{n\in\mathbb{N}} : d_\infty(S((a_n)_{n\in\mathbb{N}}), (a_n)_{n\in\mathbb{N}}) \leq \varepsilon\Big\}.$$

Our aim is to build a Cauchy sequence in A such that its limit belongs to $\text{Fix}(S)$. To this end we start with an arbitrary $x_{m(1)}$ in $A_{m(1)}$ and choose $b^1 \in A$ to be

$$b^1 = (\varrho_{m(1)}^1(x_{m(1)}), *, *, \dots),$$

where $*$ can be any element of the corresponding space A_n. Since the condition (1) holds and the mapping $\varrho_{m(2)}^1$ is continuous, we can choose some $x_{m(2)} \in A_{m(2)}$ such that

$$d_1(\varrho_{m(2)}^1(x_{m(2)}), \varrho_{m(1)}^1(x_{m(1)})) < \frac{1}{2^2}.$$

Then we set

$$b^2 = (\varrho_{m(2)}^1(x_{m(2)}), \varrho_{m(2)}^2(x_{m(2)}), *, *, \dots).$$

Using condition (1) and the continuity of the mappings $\varrho_{m(n)}^k$, we can inductively choose for every $n \in \mathbb{N}$ an element $x_{m(n)} \in A_{m(n)}$ such that

$$d_k(\varrho_{m(n)}^k(x_n), \varrho_{m(n-1)}^k(x_{n-1})) < \frac{1}{2^n}$$

for all $k \leq n-1$. Now we define

$$b^n := (\varrho^1_{m(n)}(x_{m(n)}), \varrho^2_{m(n)}(x_{m(n)}), \dots, \varrho^n_{m(n)}(x_{m(n)}), *, *, \dots).$$

Obviously every b^n is an element of $\mathrm{Fix}_{2^{-n}}(S)$. In addition, the definition of the $x_{m(n)}$ ensures that

$$d_\infty(b^n, b^{n+1}) < \frac{1}{2^{n+1}}$$

holds for all $n \in \mathbb{N}$ and therefore $(b^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in A and converges to an element $b \in A$. Evidently this limit b belongs to $\mathrm{Fix}(S) = \mathrm{Proj}\mathcal{A}$ and thus the assertion is proved. □

Corollary 2.2.
Let $\mathcal{A} = (A_n, \varrho^n_m)$ be a projective spectrum of complete metric spaces (A_n, d_n) with $A_n \neq \emptyset$ and continuous mappings such that $\varrho^n_{n+1}(A_{n+1})$ is dense in A_n for all $n \in \mathbb{N}$. Then $\mathrm{Proj}\mathcal{A} \neq \emptyset$.

In the following proposition we will show that under the assumptions of 2.1 the preceding corollary already implies that the projections

$$\varrho^n : \mathrm{Proj}\mathcal{A} \to A_n, (x_k)_{k\in\mathbb{N}} \mapsto x_n$$

have dense image. Actually, as mentioned in [22], this was the original statement in Arens' theorem.

Proposition 2.3.
Let $\mathcal{A} = (A_n, \varrho^n_m)$ be a projective spectrum of complete metric spaces (A_n, d_n) and continuous mappings that satisfies condition (1). Then $\varrho^n : \mathrm{Proj}\mathcal{A} \to A_n$ has dense image for every $n \in \mathbb{N}$.

Proof. For some $n \in \mathbb{N}$ let $\varepsilon > 0$ and $a \in A_n$. Then with $B = U_\varepsilon(a) = \{x \in A_n : d_n(x, a) < \varepsilon\}$ we set $X_m = (\varrho^n_m)^{-1}(B)$ for all $m \geq n$ and consider the projective spectrum

$$\mathcal{X} = (X_m, \varrho^m_{m+1}|_{X_{m+1}})$$

for the sequence $(X_m)_{m\geq n}$. Since every X_m is open, the theorem of Alexandroff (cf. [17, Theorem 12.1]) yields that X_m is completely metrizable and that we can assume without loss of generality that the restriction of d_m to X_m is complete. Thus to apply 2.2 we will verify that $\varrho^m_{m+1} : X_{m+1} \to X_m$ has dense image for all $m \geq n$. So let $x_m \in X_m \subseteq A_m$, hence

$$\alpha = d_n(\varrho^n_m(x_m), a) < \varepsilon.$$

Then the continuity of ϱ^n_m yields a $\delta > 0$ such that $d_m(x, y) < \delta$ implies that

$$d_n(\varrho^n_m(x), \varrho^n_m(y)) < \varepsilon - \alpha$$

for all $x, y \in X_m$. Since $\varrho^m_{m+1} : A_{m+1} \to A_m$ has dense image, for every $0 < \beta < \delta$ there exists $a_{m+1} \in A_{m+1}$ such that

$$d_m(\varrho^m_{m+1}(a_{m+1}), x_m) < \beta.$$

Because $d_n(\varrho^n_{m+1}(a_{m+1}), a) = d_n(\varrho^n_m(\varrho^m_{m+1}(a_{m+1})), a)$ we can conclude that

$$\begin{aligned} d_n(\varrho^n_{m+1}(a_{m+1}), a) &\leq d_n(\varrho^n_m(\varrho^m_{m+1}(a_{m+1})), \varrho^n_m(x_m)) + d_n(\varrho^n_m(x_m), a) \\ &< \varepsilon - \alpha + \alpha = \varepsilon. \end{aligned}$$

Thus $a_{m+1} \in X_{m+1}$ and we have shown that $\varrho^m_{m+1|X_{m+1}}$ has dense image. Hence 2.2 applies and we have $\mathrm{Proj}\mathcal{X} \neq \emptyset$. This means that there exists a sequence $(x_m)_{m\geq n} \in \prod_{m\geq n} X_m$ such that $x_m = \varrho^m_{m+1}(x_{m+1})$ for all $m \geq n$. If we set

$$x = (\varrho^1_n(x_n), \ldots, \varrho^{n-1}_n(x_n), x_n, x_{n+1}, \ldots)$$

it is clear that $x \in \mathrm{Proj}\mathcal{A}$ and $\varrho^n(x) = x_n \in U_\varepsilon(a)$. Since we have chosen a and ε arbitrarily, $\varrho^n(\mathrm{Proj}\mathcal{A})$ is dense in A_n. □

Corollary 2.4.
Let $\mathcal{A} = (A_n, \varrho^n_m)$ be a projective spectrum of complete metric spaces (A_n, d_n) and continuous mappings. Then the condition (1) is equivalent to

$$\forall\, n \in \mathbb{N}, \varepsilon > 0\ \exists\ m(n) \geq n : \varrho^n_{m(n)}(A_{m(n)}) \subseteq [\varrho^n(\mathrm{Proj}\mathcal{A})]_\varepsilon.$$

Proof. Since $\varrho^n(\mathrm{Proj}\mathcal{A})$ is contained in $\varrho^n_k(A_k)$ for every $k \geq n$, the necessity of (1) is trivial. If we assume that the condition (1) holds, 2.3 yields that for every $\varepsilon > 0$ and $a \in A_{m(n)}$ we can find an element $x \in \mathrm{Proj}\mathcal{A}$ such that

$$d_n(\varrho^n_{m(n)}(a), \varrho^n(x)) < \varepsilon$$

and thus the assertion is proved. □

Now we present the following theorem due to Palamodov [18], from which we can deduce a sufficient condition for stability. In the proof we use the fact that in a metrizable topological group we can assume without loss of generality that the corresponding metric is invariant under translation (cf. [21, Proposition 7.4]).

Theorem 2.5.
Let $\mathcal{A} = (A_n, \varrho^n_m)$ be a projective spectrum of complete metrizable topological groups and continuous homomorphisms that satisfies the condition (1). Then $\mathrm{Proj}^1\mathcal{A} = 0$.

Proof. Again we set $A := \prod_{n\in\mathbb{N}}(A_n, d_n)$. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in A. Then we define $\sigma^n_{n+1} = \varrho^n_{n+1} + y_n$ for every natural number n. If (1) holds for the spectrum $\mathcal{A} = (A_n, \varrho^n_m)$, the same is true for the perturbed spectrum $\mathcal{A}_y = (A_n, \sigma^n_m)$ since the metric on every A_n is invariant. Hence 2.2 implies that $\mathrm{Proj}\mathcal{A}$ is nonempty and thus there exists a sequence $(a_n)_{n\in\mathbb{N}}$ in A such that for every $n \in \mathbb{N}$

$$a_n = \sigma^n_{n+1}(a_{n+1}) = \varrho^n_{n+1}(a_{n+1}) + y_n.$$

Therefore we have shown the surjectivity of the mapping $\psi_\mathcal{A}$ defined in 1.8, consequently

$$\mathrm{Proj}^1\mathcal{A} = 0.$$

□

The sufficient condition in the above theorem can again be replaced by a condition about the projections ϱ^n:

Proposition 2.6.
Let $\mathcal{A} = (A_n, \varrho^n_m)$ be a projective spectrum of complete metrizable abelian topological groups and continuous homomorphisms. Then the following are equivalent:

1. $\forall\, n \in \mathbb{N},\ U \in U_0(A_n)\ \exists\, m \geq n\ \forall\, k \geq m : \varrho^n_m(A_m) \subseteq \varrho^n_k(A_k) + U$.
2. $\forall\, n \in \mathbb{N},\ U \in U_0(A_n)\ \exists\, m \geq n : \varrho^n_m(A_m) \subseteq \varrho^n(\mathrm{Proj}\mathcal{A}) + U$.

Proof. Obviously the second condition implies the first one. The proof of 2.5 shows that the first condition implies (1) and so 2.3 yields that every ϱ^n has dense range. This obviously implies the second condition. □

From now on we will consider stability in topological algebras. A *topological algebra* is an algebra A endowed with a topology τ such that (A, τ) is a topological vector space and the multiplication $(x, y) \mapsto xy$ is jointly continuous. Recalling that an element x of a topological algebra A is stable if the corresponding projective spectrum $\mathcal{A} = (A, M_x)$ satisfies $\mathrm{Proj}^1\mathcal{A} = 0$ and observing that the inclusion

$$M_{x^{m-n}}(A) \subseteq M_{x^{k-n}}(A) + U$$

is equivalent to $x^m A \subseteq x^k A + U$ for all $k \geq m \geq n$, we immediately obtain the following corollary:

Corollary 2.7.
Assume that for an element x of a complete metrizable commutative topological algebra A we have

$$\forall\, U \in U_0(A)\ \exists\, m \in \mathbb{N}\ \forall\, k \geq m : x^m A \subseteq x^k A + U$$

or, equivalently,

$$\forall\, U \in U_0(A),\ n \in \mathbb{N}\ \exists\, m \in \mathbb{N} : x^m A \subseteq \varrho^n(\mathrm{Proj}\mathcal{A}) + U.$$

Then x is stable.

Allan studied stability in Banach and Fréchet algebras and so we will apply the previous results to these special cases of complete metrizable topological groups. We start with some notation:

A *Banach algebra* is a topological algebra whose topology is induced by a complete submultiplicative norm, while a *Fréchet algebra* is a complete metrizable topological algebra whose topology can be defined by an increasing sequence $(p_n)_{n\in\mathbb{N}}$ of submultiplicative seminorms. We should also mention that every Fréchet algebra can be represented as the projective limit of a projective spectrum of Banach algebras and continuous mappings: Let A_n be the completion of $A/\ker p_n$ and $d_m : A_{m+1} \to A_m$ the extensions of the mappings $\tilde{d}_m$ defined by $\tilde{d}_m(x + \ker p_{m+1}) = x + \ker p_m$. If we set $\varrho^n_m = d_n \circ \ldots \circ d_{m-1}$, then A is the projective limit of the projective spectrum (A_m, ϱ^n_m) and this spectrum is called the Arens-Michael representation of the Fréchet algebra A. The details are shown for example in [4].
Furthermore a topological algebra is called *locally m-convex* if there exist a basis of zero neighborhoods that consists of absolutely convex and multiplicative sets.
For Allan's characterization of stable elements in Banach and Fréchet algebras we need the following definitions introduced in his papers [3] and [4].

Definition 2.8.
Let A be a locally m-convex topological algebra. Then $x \in A$ has

1. *finite closed descent* (FCD) if there is a positive integer m such that $x^{m+1}A$ is dense in $x^m A$ (and the the least of this integers is called the closed descent $\delta(x)$ of x),

2. *locally finite closed descent* (LFCD) if for each continuous submultiplicative seminorm p on A the element x has FCD relative to the p-topology.

Remark 2.9.
If A is a Fréchet algebra and its topology is defined by seminorms $(p_n)_{n\in\mathbb{N}}$, we can easily show that $x \in A$ has LFCD if and only if x has FCD relative to every p_n. The sufficiency is trivial since LFCD just means FCD relative to every continuous seminorm on A. If otherwise x has FCD relative to every p_n and p is an arbitrary continuous seminorm on A, there exist $l \in \mathbb{N}$ and $C > 0$ such that $p \leq Cp_l$. Hence there exists $m \in \mathbb{N}$ such that

$$x^m A \subseteq \overline{x^{m+1}A}^{p_l} \subseteq \overline{x^{m+1}A}^{p}.$$

So x has FCD relative to p and therefore x has LFCD.
Analogously it is clear that in normed algebras FCD and LFCD are equivalent properties.

For the proof that these properties characterize stability in the corresponding spaces Allan used some theory of embedding the algebra of formal power series into Banach and Fréchet algebras. In contrast we will make use of the following result due to Palamodov [19, Theorem 5.2], which is a direct consequence of 1.12, 2.5 and 2.6.

Theorem 2.10.
Let $\mathcal{A} = (A_n, \varrho_m^n)$ be a projective spectrum of Fréchet spaces and continuous linear mappings. Then the following conditions are equivalent:

1. $\operatorname{Proj}^1 \mathcal{A} = 0$.
2. $\forall\, n \in \mathbb{N}, U \in U_0(A_n)\ \exists\, m \geq n\ \forall\, k \geq m : \varrho_m^n(A_m) \subseteq \varrho_k^n(A_k) + U$.
3. $\forall\, n \in \mathbb{N}, U \in U_0(A_n)\ \exists\, m \geq n : \varrho_m^n(A_m) \subseteq \varrho^n(\operatorname{Proj}\mathcal{A}) + U$.

Proof. The equivalence of 2. and 3. is given by 2.6. If 2. holds, 2.5 yields $\operatorname{Proj}^1 \mathcal{A} = 0$. To prove the other implication, let $\operatorname{Proj}^1 \mathcal{A} = 0$ and $n \in \mathbb{N}$. For an arbitrary $U \in U_0(A_n)$ we have $A_n = \bigcup_{m \in \mathbb{N}} A_{n,m}$ with $A_{n,m} = mU$, hence 1.12 applies and therefore

$$\exists\, m \geq n\ \forall\, k \geq m : \varrho_m^n(A_m) \subseteq \varrho_k^n(A_k) + \bigcap_{j=1}^{n} (\varrho_n^j)^{-1}(A_{j,N(j)}) \subseteq \varrho_k^n(A_k) + N(n)U.$$

Since $\varrho_m^n(A_m)$ and $\varrho_k^n(A_k)$ are linear spaces, we obtain 2. by multiplying the inclusion above with $\frac{1}{N(n)}$. □

Theorem 2.11.
Let A be a commutative topological algebra and $x \in A$.

1. If A is a Banach algebra, then x is stable if and only if x has FCD.
2. If A is a Fréchet algebra, then x is stable if and only if x has LFCD.

Proof. In the case of Banach algebras FCD and LFCD are equivalent, hence it suffices to verify the second statement. So let $x \in A$ be stable. Then the projective spectrum $\mathcal{A}_x = (A, M_x)$ satisfies $\operatorname{Proj}^1 \mathcal{A}_x = 0$ and from 2.10 we obtain that

$$\forall\, U \in U_0(A)\ \exists\, m \in \mathbb{N}\ \forall\, k \geq m : x^m A \subseteq x^k A + U. \tag{2}$$

Just like in the proof of the previous theorem we multiply both sides of this relation with some $\varepsilon > 0$ and obtain

$$\forall\, U \in U_0(A)\ \exists\, m \in \mathbb{N}\ \forall\, k \geq m : x^m A \subseteq x^k A + \varepsilon U.$$

For $k = m + 1$ this implies

$$\forall\, U \in U_0(A)\ \exists\, m \in \mathbb{N}\ \forall\, \varepsilon > 0 : x^m A \subseteq x^{m+1} A + \varepsilon U. \tag{3}$$

Since A is a Fréchet algebra, its topology is induced by an increasing sequence of seminorms $(p_n)_{n \in \mathbb{N}}$ and thus a basis of zero neighborhoods is given by the sets $U_n = \{y \in A : p_n(y) \leq \frac{1}{n}\}$. Thus (3) implies that x has FCD in every p_n-topology, hence 2.9 yields that x has LFCD. If on the other hand x has LFCD, x has FCD relative to every p_n-topology. This implies (3) and since the multiplication $M_x : A \to A, y \mapsto xy$ is continuous in the p_n-topology, we can conclude that

$$\overline{Ax^{m+2}}^{p_n} \supseteq \overline{Ax^{m+1}}^{p_n} x \supseteq Ax^{m+1}$$

and therefore we have

$$\overline{Ax^{m+2}}^{p_n} \supseteq \overline{Ax^{m+1}}^{p_n} \supseteq Ax^{m}.$$

A simple induction shows that Ax^{m+k} is p_n-dense in Ax^m for all natural numbers k, and since the sets U_n defined above form a basis of zero neighborhoods, (2) is fulfilled and 2.10 implies that x is stable. □

Now we can deduce Allan's main result [5, Proposition 6.5] about the product of stable elements:

Corollary 2.12.
Let A be a commutative Fréchet algebra and $x, y \in A$. If x and y are stable, then xy is stable.

Proof. Since x and y are stable, for an arbitrary submultiplicative seminorm p there is a positive integer m such that $x^{m+1}A$ is p-dense in x^mA and $y^{m+1}A$ is p-dense in y^mA. Thus, using the commutativity of A, we have

$$\overline{(xy)^{m+1}A}^{p} = \overline{x^{m+1}y^{m+1}A}^{p} \supseteq x^{m+1}\overline{y^{m+1}A}^{p} \supseteq x^{m+1}y^{m}A$$

and thus

$$\overline{(xy)^{m+1}A}^{p} \supseteq \overline{x^{m+1}y^{m}A}^{p} = \overline{y^{m}x^{m+1}A}^{p} \supseteq y^{m}\overline{x^{m+1}A}^{p} \supseteq y^{m}x^{m}A = (xy)^{m}A,$$

i.e. xy has LFCD and is therefore stable. □

Remark 2.13.
Since we claimed that the seminorms that define a Fréchet algebra must be submultiplicative, it is clear that every Fréchet algebra is locally m-convex. Actually the assertion in 2.12 remains true if we omit this requirement:
If we consider some $U \in U_0(A)$, the continuity of the multiplication yields some $V \in U_0(A)$ such that $V^2 \subseteq \frac{1}{2}U$. Since x and y are stable there exists a natural number m such that

$$x^mA \subseteq x^kA + V \quad \text{and} \quad y^mA \subseteq y^kA + V$$

for all $k \geq m$. For every k we find some $\varepsilon > 0$ such that both εy^k and εx^m are elements of V. Thus we have

$$\begin{aligned}(xy)^mA &= \varepsilon((xy)^mA) \subseteq \varepsilon(x^m(y^kA + V)) \subseteq \varepsilon((xy)^kA + y^kV + x^mV) \\ &= (xy)^kA + \varepsilon y^kV + \varepsilon x^mV \subseteq (xy)^kA + V^2 + V^2 \subseteq (xy)^kA + U\end{aligned}$$

which yields the stability of the product xy.

In [5] Allan posed another question in the context of the stability of an element x in a topological algebra A. He asked if for a stable element x the two ideals $I(x)$ and $I_0(x)$ coincide, where $I(x) = \bigcap_{n\in\mathbb{N}} x^n A$ and

$$I_0(x) = \Big\{a \in A : \exists\ (a_n)_{n\in N} : a_1 = a \text{ and } a_n = a_{n+1}x \text{ for all } n \geq 0\Big\}.$$

Equivalently one may ask if $M_x(I(x)) = I(x)$. We also note that obviously $I_0(x) = \varrho^n(\mathrm{Proj}\mathcal{A})$ for all $n \in \mathbb{N}$.
Allan answered this question for commutative Banach algebras [2, Lemma 1] and commutative Fréchet algebras [4, Corollary 4]. We will omit the proof here and present an analogous proof of this result in the case of LB-algebras in the next chapter.

Proposition 2.14.
Let A be a commutative Fréchet algebra and let $x \in A$ be stable. Then $I(x) = I_0(x)$.

We close this chapter with some examples of Banach and Fréchet algebras.

Example 2.15.
For a compact Hausdorff space X we consider $C(X)$, the space of all continuous functions on X with values in $\mathbb{C}$. Equipped with the uniform norm and pointwise multiplication, $C(X)$ is a Banach algebra. To find out which elements are stable in this algebra, we will make use of the following characterization of the closed ideals in $C(X)$ (cf. [11, Theorem 4.2.1]). On the one hand, for a compact subset $K \subseteq X$

$$I(K) = \{g \in C(X) : g\,|_K = 0\}$$

is a closed ideal in $C(X)$. On the other hand, for a given closed ideal I and K defined by

$$K = \bigcap_{g\in I} g^{-1}(\{0\})$$

we have $I = I(K)$. Hence the closed ideals in $C(X)$ correspond to the compact subsets of X. It is obvious that for every $f \in C(X)$ both f and f^2 have the same zero set. Thus we have

$$\bigcap_{g\in\overline{fC(X)}} g^{-1}(\{0\}) = f^{-1}(\{0\}) = (f^2)^{-1}(\{0\}) = \bigcap_{g\in\overline{f^2C(X)}} g^{-1}(\{0\}).$$

So the above characterization of the closed ideals yields that $\overline{fC(X)} = \overline{f^2C(X)}$ and we have shown that every $f \in C(X)$ has FCD and is therefore stable.

The following two interesting examples will show that the stability of an element depends strongly on the corresponding multiplication.

Example 2.16.
For $d \in \mathbb{N}$ and a connected open set $\Omega \subseteq \mathbb{C}^d$ let $A = H(\Omega)$ be the space of functions holomorphic in Ω. If an exhaustion by compact sets of Ω is given by the sequence $(K_n)_{n\in\mathbb{N}}$, the seminorms p_n defined by

$$p_n(f) = \sup_{z\in K_n} |f(z)|$$

are submultiplicative seminorms that define a complete metrizable topology known as the topology of uniform convergence on compact sets or as the compact open topology. Then equipped with pointwise multiplication A is a Fréchet algebra. We now claim that the zero function and the invertible holomorphic functions on Ω are the only stable elements of A.
To show this we assume that $f \in A \setminus \{0\}$ is stable but not invertible. That means that f has at least one zero. If we assume that $I(f) = \bigcap_{n\in\mathbb{N}} f^n A$ is nontrivial, there exists a $g \in A \setminus \{0\}$ such that for all $n \in \mathbb{N}$ we can find some $h_n \in A$ with $g = f^n h_n$. But then g has a zero with multiplicity at least n, which contradicts the fact that $g \in A \setminus \{0\}$. Thus we have shown that $I(f) = \{0\}$ and using 2.14 and the continuity of f we can conclude that

$$\{0\} = I(f) = I_0(f) = \varrho^n(\mathrm{Proj}\mathcal{A}_f)$$

where $\mathcal{A}_f = (A, M_f)$. Hence 2.10 implies that

$$\forall\ n \in \mathbb{N}, U \in U_0(A)\ \exists\ m \in \mathbb{N} : x^m A \subseteq \varrho^n(\mathrm{Proj}\mathcal{A}_f) + U.$$

This contradicts $\varrho^n(\mathrm{Proj}\mathcal{A}_f) = \{0\}$, hence the assertion is proved.
Note that we obtain the same result for any closed subalgebra $A \subseteq H(\Omega)$, the proof is just analogous to the one above.

On the contrary, in the space of functions holomorphic in the complex unit disc equipped with the Hadamard product every function is stable. To show this we will adopt some definitions and results shown in [10].

Example 2.17.
Let $\mathbb{D} \subseteq \mathbb{C}$ be the unit disc and $A = H(\mathbb{D})$ the set of functions holomorphic in $\mathbb{D}$. Furthermore we consider the Hadamard product of two functions $f = \sum_{\nu=0}^{\infty} f_\nu z^\nu$ and $g = \sum_{\nu=0}^{\infty} g_\nu z^\nu$ in A defined by

$$(f * g)(z) = \sum_{\nu=0}^{\infty} f_\nu g_\nu z^\nu.$$

This power series converges locally uniformly and so with $*$ as multiplication and the topology of locally uniform convergence A is a Fréchet algebra. Thus 2.11 shows that every element of A that has LFCD is already stable. We now prove that every element of A indeed has FCD. For this purpose we need the following characterization of the closed ideals in A due to Brück and Müller [10, Theorem 1]:
An ideal $I \subseteq A$ is closed if and only if $I = I_B$ with $B = \{n \in \mathbb{N}_0 : f_n = 0 \text{ for all } f \in I\}$ and $I_B = \{f \in A : f_n = 0 \text{ for all } n \in B\}$.
So let $f \in A$. Since $f_n = 0$ is equivalent to $f^{(n)}(0) = 0$ we have

$$\overline{fA} = I_B$$

for $B = \{n \in \mathbb{N}_0 : g^{(n)}(0) = 0 \text{ for all } g \in \overline{fA}\}$. We claim that for $C = \{n \in \mathbb{N}_0 : f^{(n)}(0) = 0\}$ we have $B = C$. It is obvious that B is contained in C. So let $n \in C$ and $g \in fA$ be arbitrary. Then there exists some $h \in A$ such that $g = f * h$ and hence

$$g^{(n)}(0) = (f * h)^{(n)}(0) = n! f_n h_n = 0.$$

If now $g \in \overline{fA}$, then $g = \lim_{k\to\infty} f * h_k$ for a sequence $(h_k)_{k\in\mathbb{N}}$ in A and since this limit implies uniform convergence in a neighborhood of 0 we can conclude that

$$g^{(n)}(0) = \lim_{k\to\infty} (f * h_k)^{(n)}(0) = 0.$$

So we have shown that $B = C$ and since $f^{(n)}(0) = 0$ if and only if $(f^m)^{(n)}(0) = 0$ for every $m \in \mathbb{N}$ we can conclude that

$$\overline{fA} = I_B = I_C = \overline{f^m A}.$$

Thus f has FCD and is therefore stable.

In the last example of this chapter we consider the Fréchet algebra of smooth functions on an open subset of $\mathbb{R}^d$.

Example 2.18.
For an open set $\Omega \subseteq \mathbb{R}^d$ let $C^\infty(\Omega)$ be the space of all smooth functions on Ω. Furthermore let $(K_n)_{n\in\mathbb{N}}$ be an exhaustion by compact sets of Ω such that for every $n \in \mathbb{N}$ we have $K_n \subseteq \overset{\circ}{K}_{n+1}$. Then the seminorms

$$p_n(f) = 2^n \sup_{x\in K_n} \Big\{ |D^\alpha f(x)| : |\alpha| \le n \Big\}$$

define a topology on $C^\infty(\Omega)$ which, together with the pointwise multiplication, turns $C^\infty(\Omega)$ into a commutative Fréchet algebra. The factor 2^n ensures that every seminorm p_n is submultiplicative. We now show that in this algebra all elements are stable. Of course $0 \in C^\infty(\Omega)$ is stable and the same holds for all invertible elements. So let $f \in C^\infty(\Omega) \setminus \{0\}$ be non-invertible, that means f has at least one zero in Ω. If we denote by $N_n(g)$ the zero set of a function $g \in C^\infty(\Omega)$ on K_n, we obtain the following result: For all $n, m \in \mathbb{N}_0$

$$\overline{f^m C^\infty(\Omega)}^{p_n} \supseteq \Big\{ h \in C^\infty(\Omega) : N_n(f) \subseteq N_n(D^\alpha h) \text{ for } |\alpha| \le n \Big\}. \tag{4}$$

To prove this, choose arbitrary $n, m \in \mathbb{N}_0$. For an element h of the right-hand side of (4) Taylor's theorem yields that for some $x_0 \in \Omega$, for all $|\alpha| \le n$ and all $x \in \Omega$ such that the line segment $[x, x_0]$ is contained in Ω there exists $z \in [x, x_0]$ such that

$$D^\alpha h(x) = \sum_{|\beta|\le n-|\alpha|} \frac{D^\beta D^\alpha h(x_0)}{\beta!}(x - x_0)^\beta + \sum_{|\beta|=n-|\alpha|+1} \frac{D^\beta D^\alpha h(z)}{\beta!}(x - x_0)^\beta.$$

So for $x_0 \in N_n(f)$ the first part of the sum vanishes. If we set

$$K_\delta = \Big\{x \in \Omega : |x-y| \le \delta \text{ for some } y \in N_n(f)\Big\},$$

then for a sufficiently small $\delta > 0$ and every $x \in K_\delta$ there exist $x_0 \in N_n(f)$ and $z \in [x, x_0]$ such that

$$|D^\alpha h(x)| = \Big|\sum_{|\beta|=n-|\alpha|+1} \frac{D^\beta D^\alpha h(z)}{\beta!}(x-x_0)^\beta\Big| \le \delta^{|\beta|} \sum_{|\beta|=n-|\alpha|+1} \Big|\frac{D^\beta D^\alpha h(z)}{\beta!}\Big|.$$

Since $\sum_{|\beta|=n-|\alpha|+1} |\frac{D^\beta D^\alpha h(z)}{\beta!}|$ is bounded on K_δ we can conclude that there exists $M_\alpha > 0$ such that for every $x \in K_\delta$

$$|D^\alpha h(x)| \le \delta^{n-|\alpha|+1} M_\alpha.$$

This inequality will be applied later. We now want to approximate h with respect to p_n by elements of $f^m C^\infty(\Omega)$. For that purpose we need an appropriate function in $C^\infty(\Omega)$, which we can obtain from Theorem 1.4.1 and the subsequent remark in [14]: For every $\delta > 0$ there exists $\varphi_\delta \in C^\infty(\Omega)$ such that

1. $\varphi_\delta \equiv 0$ on $K_{\frac{\delta}{3}}$
2. $\varphi_\delta \equiv 1$ on K_δ^C
3. for all $|\alpha| \le n$ there exists $C_\alpha > 0$ such that $|D^\alpha \varphi_\delta(x)| \le C_\alpha \frac{1}{(\frac{1}{3}\delta)^{|\alpha|}} \le C \frac{1}{(\frac{1}{3}\delta)^{|\alpha|}}$

with $C = \max_{|\alpha|\le n} C_\alpha$. If we iterate this procedure for $n+1$ instead of n, we obtain a smooth function $\tilde{\varphi}_\delta$ and we can assume without loss of generality that $\tilde{\varphi}_\delta|_{K_n} = \varphi_\delta$ for sufficiently small δ. We now want to define an appropriate function $g_\delta \in C^\infty(\Omega)$. On K_{n+1} we set

$$g_\delta(x) = \begin{cases} \frac{h(x)\tilde{\varphi}_\delta(x)}{f^m(x)} & , x \notin N_{n+1}(f) \\ 0 & , x \in N_{n+1}(f) \end{cases}.$$

Since $K_n \subseteq \mathring{K}_{n+1}$ there exists a smooth function ψ on $\mathring{K}_{n+1}$ with compact support such that $\psi \equiv 1$ on K_n. Thus $\tilde{g}_\delta = \psi g_\delta$ can be extended smoothly on Ω and for all $x \in K_n$ we have

$$\tilde{g}_\delta(x) = \begin{cases} \frac{h(x)\varphi_\delta(x)}{f^m(x)} & , x \notin N_n(f) \\ 0 & , x \in N_n(f) \end{cases}.$$

To show that we can approximate h by means of $\tilde{g}_\delta$, let $\varepsilon > 0$. Then we find $\overline{\delta} > 0$ such that for all $|\alpha| \le n$, for all $\beta \le \alpha$ and for all $x \in K_{\overline{\delta}}$

1. $|D^\alpha h(x)| < \frac{\varepsilon}{2^{n+1}}$.
2. $\sum_{\beta \le \alpha} 3^{|\beta|} \binom{\alpha}{\beta} \bar{\delta}^{n-|\alpha|+1} M_{\alpha-\beta} C < \frac{\varepsilon}{2^{n+1}}$.

Now we will verify that $p_n(h - f^m g_{\bar{\delta}}) < \varepsilon$. For $x \in K_{\bar{\delta}}^C$ we have

$$(h - f^m \tilde{g}_{\bar{\delta}})(x) = h(x) - h(x)\varphi_{\bar{\delta}}(x) = 0$$

and since $K_{\bar{\delta}}^C$ is open we can conclude that $D^\alpha(h - f^m g_{\bar{\delta}})(x) = 0$ for all α. If on the other hand $x \in K_{\bar{\delta}} \cap K_n$, then

$$\begin{aligned}
|D^\alpha(h - f^m \tilde{g}_{\bar{\delta}})(x)| &= |D^\alpha h(x) - \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^{\alpha-\beta} h(x) D^\beta \varphi_{\bar{\delta}}(x)| \\
&\le |D^\alpha h(x)| + \sum_{\beta \le \alpha} \binom{\alpha}{\beta} |D^{\alpha-\beta} h(x)||D^\beta \varphi_{\bar{\delta}}(x)| \\
&< \frac{\varepsilon}{2^{n+1}} + \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \bar{\delta}^{n-|\alpha|+|\beta|+1} M_{\alpha-\beta} C \frac{1}{(\frac{1}{3}\bar{\delta})^{|\beta|}} \\
&= \frac{\varepsilon}{2^{n+1}} + \sum_{\beta \le \alpha} \binom{\alpha}{\beta} 3^{|\beta|} \bar{\delta}^{n-|\alpha|+1} M_{\alpha-\beta} C \\
&< \frac{\varepsilon}{2^{n+1}} + \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2^n}.
\end{aligned}$$

Thus $p_n(h - f^m g_{\bar{\delta}}) < \varepsilon$ and $h \in \overline{f^m C^\infty(\Omega)}^{p_n}$.
To show the stability of f, we observe that for $m > n$ we also have

$$\overline{f^m C^\infty(\Omega)}^{p_n} \subseteq \{h \in C^\infty(\Omega) : N_n(f) \subseteq N_n(D^\alpha h) \text{ for all } |\alpha| \le n\}, \tag{5}$$

since in this case every element of $f^m C^\infty(\Omega)$ is an element of the set on the right-hand side and the latter is closed with respect to every seminorm p_n. In particular we obtain for every $n \in \mathbb{N}$ some $m > n$ such that

$$\overline{f^{m+1} C^\infty(\Omega)}^{p_n} = \overline{f^m C^\infty(\Omega)}^{p_n}.$$

Thus f has FCD relative to every p_n, so according to 2.9 f has LFCD, i.e. f is stable.

3 LB-algebras

As we have mentioned in the last chapter, every Fréchet algebra is a projective limit of Banach algebras. Besides projective limits, the most common concept of building limits of linear spaces is that of inductive limits. So it would be consequential to study stability on inductive limits of Banach algebras. Thus we consider a sequence $(A_n)_{n\in\mathbb{N}}$ of Banach spaces such that A_n is a subspace of A_{n+1} for all $n \in \mathbb{N}$ and the inclusion mapping $i_n : A_n \hookrightarrow A_{n+1}$ is continuous. Then $A = \bigcup_{n\in N} A_n$ is the inductive limit of the spaces A_n. If we equip this inductive limit with the finest locally convex topology on A such that each embedding $j_n : A_n \hookrightarrow A$ is continuous, A is called an *LB-space*. If every A_n is a Banach algebra and also a subalgebra of A_{n+1}, A is called an *LB-algebra*. Note that Akkar and Nacir have shown in [1] that in this case A is locally m-convex. This property obviously implies the joint continuity of the multiplication and thus A is indeed a topological algebra.

We want to present a characterization of stable elements of LB-algebras. To this end we will show a general result for projective spectra of Hausdorff LB-spaces due to Retakh [20] and Palamodov [19, Theorem 5.2] and afterwards we consider the special case $\mathcal{A} = \mathcal{A}_x = (A, M_x)$, where A is a commutative Hausdorff LB-algebra. In this context we need to work with Banach discs and so we start with the following remark:

Remark 3.1.
In a locally convex space a *Banach disc* B is a bounded and absolutely convex set whose linear span $[B]$, equipped with the Minkowski functional p_B of B, is a Banach space. If $A = \bigcup_{n\in\mathbb{N}} A_n$ is an LB-algebra and D_n denotes the unit ball of A_n, we can assume without loss of generality that the sequence $(D_n)_{n\in\mathbb{N}}$ is increasing. Then Grothendieck's factorization theorem (cf. [16, Theorem 24.33]) implies that the sequence $(B_n)_{n\in\mathbb{N}}$ defined by

$$B_n = nD_n$$

for every $n \in \mathbb{N}$ is an increasing fundamental sequence of Banach discs in A, which means that every Banach disc in A is contained in some B_n.

Now we can state the theorem of Retakh and Palamodov:

Theorem 3.2.
Let $\mathcal{A} = (A_n, \varrho_m^n)$ be a projective spectrum of Hausdorff LB-spaces and continuous linear mappings. Then $\mathrm{Proj}^1\mathcal{A} = 0$ if and only if there is a sequence $(B_n)_{n\in\mathbb{N}}$ of Banach discs $B_n \subseteq A_n$ such that

1. $\forall\, m \geq n : \varrho_m^n(B_m) \subseteq B_n$.
2. $\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m : \varrho_m^n(A_m) \subseteq \varrho_k^n(A_k) + B_n$.

Analogously to the case of Fréchet spaces considered in the last chapter we can replace property 2. by

3. $\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ : \varrho_m^n(A_m) \subseteq \varrho^n(\mathrm{Proj}\mathcal{A}) + B_n$.

Proof. Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of Banach discs that satisfies 1. and 2. For all $n \in \mathbb{N}$ and $B_n \subseteq A_n$ let $\mathcal{T}_n$ be the group topology on A_n whose basis of zero neighborhoods is given by the sets $\{\varepsilon B_n : \varepsilon > 0\}$. Then each $(A_n, \mathcal{T}_n)$ is a complete metrizable group. Thus 2.6 implies the equivalence of 2. and 3. Further we obtain from 2.5 that $\mathrm{Proj}^1\mathcal{A} = 0$.
To show the other implication suppose that $\mathrm{Proj}^1\mathcal{A} = 0$. For all $n \in \mathbb{N}$ let $(B_k^n)_{k\in\mathbb{N}}$ be the fundamental sequence of Banach discs in A_n defined in 3.1. Since

$$A_n = \bigcup_{k\in\mathbb{N}} B_k^n,$$

1.12 yields a sequence $(N(k))_{k\in\mathbb{N}}$ of natural numbers such that

$$\forall\, n \in \mathbb{N}\ \exists\, m \geq n\ \forall\, k \geq m : \varrho_m^n(A_m) \subseteq \varrho_k^n(A_k) + \bigcap_{j=1}^{n} (\varrho_n^j)^{-1}(B_{N(j)}^j).$$

We set

$$B_n = \bigcap_{j=1}^{n} (\varrho_n^j)^{-1}(B_{N(j)}^j).$$

Then every B_n is a Banach disc, which can be shown with the following result that can be found for example in [23, Lemma 3.2.10]:

Let X, Y be Hausdorff locally convex spaces, $f : X \to Y$ linear and continuous and $A \subseteq X$, $B \subseteq Y$ Banach discs. Then $A \cap f^{-1}(B)$ is a Banach disc.

To show this we consider the linear map $g : [A \cap f^{-1}(B)] \to [A] \times [B]$ defined by

$$g(x) = (x, -f(x)).$$

An easy computation shows that

$$p_A \leq p_{A\cap f^{-1}(B)}$$

and

$$p_B \circ (-f) \leq p_{A\cap f^{-1}(B)}.$$

Hence for the projections pr_1, pr_2 we can conclude that $pr_1 \circ g$ and $pr_2 \circ g$ are continuous, which implies the continuity of g. By definition g is injective, hence it is bijective onto its image and one can easily show that for all $x \in A \cap f^{-1}(B)$ we have

$$p_{A\cap f^{-1}(B)}(x) = p_{A\times B}(g(x)).$$

Thus we can conclude that g^{-1} is continuous and g is an isomorphism onto its image. Now it remains to show that the image of g is closed in $[A] \times [B]$ because in this case it is a Banach

space as a closed subspace of the Banach space $[A]\times[B]$ and then $[A\cap f^{-1}(B)]$ is also a Banach space. So let $(x_n, -f(x_n))_{n\in\mathbb{N}}$ be a convergent sequence in $[A]\times[B]$. Then for $n\to\infty$ we have $x_n\to x$ in $[A]$ and also in X, since the embedding of $[A]$ into X is continuous. Hence the continuity of f implies $f(x_n)\to f(x)$ in Y. In addition $f(x_n)\to -y$ in $[B]$ and analogously in Y for $n\to\infty$. But then the fact that Y is Hausdorff implies that $y=-f(x)$ and therefore the image of g is closed.
So inductively we obtain that B_n is a Banach disc and obviously the sequence $(B_n)_{n\in\mathbb{N}}$ satisfies 2. Furthermore, if $x\in(\varrho_m^j)^{-1}(B^j_{N(j)})$ for some $j\leq n$, then

$$\varrho_m^n(x)\in(\varrho_n^j)^{-1}(B^j_{N(j)})$$

and thus $(B_n)_{n\in\mathbb{N}}$ also satisfies the condition in 1. □

We consider again the projective spectrum $\mathcal{A}=\mathcal{A}_x=(A,M_x)$. 3.2 yields the following characterization of stable elements:

Theorem 3.3.
Let $A=\bigcup_{n\in\mathbb{N}}A_n$ be a commutative, Hausdorff LB-algebra and let $x\in A$. Then the following are equivalent:

1. x is stable .
2. There exists a Banach disc B and a natural number m such that for all $k\geq m$

$$x^mA\subseteq x^kA+B.$$

3. For all $n\in\mathbb{N}$ there exists a Banach disc B and a natural number m such that

$$x^mA\subseteq\varrho^n(\mathrm{Proj}\mathcal{A})+B.$$

Proof. In condition 2. and 3. we can assume without loss of generality that $B=lD_l$, where D_l is the unit ball in A_l, and $x\in B$ for some $l\in\mathbb{N}$. Then obviously $xB\subseteq lB$, which yields the continuity of the multiplication M_x in the topology whose basis of neighborhoods of zero is given by the sets $\{\varepsilon B:\varepsilon>0\}$. Thus the equivalence of 2. and 3. is again a direct consequence of 2.6.
It remains to show that 1. and 2. are equivalent. Without loss of generality we can assume that x is contained in every Banach algebra A_n of the inductive limit A. If x is stable, the projective spectrum $\mathcal{A}$ satisfies $\mathrm{Proj}^1\mathcal{A}=0$ and condition 2. of 3.2 holds for a sequence $(C_n)_{n\in\mathbb{N}}$ of Banach discs. This implies that for every $n\in\mathbb{N}$ there exists $m\in\mathbb{N}$ such that for all $k\geq m$

$$x^mA\subseteq x^kA+C_n,$$

hence we have shown that 2. holds for every Banach disc C_n.
If on the other hand there exists a Banach disc B that satisfies 2. for some $m\in\mathbb{N}$ and all $k\geq m$, 3.1 implies that there exists a positive integer l such that B is contained in a multiple

of the unit ball D_l of the Banach algebra A_l. Then obviously 2. implies that there exists a natural number m such that for all $k \geq m$ and all $\varepsilon > 0$

$$x^m A \subseteq x^k A + \varepsilon D_l.$$

We denote the norm on A_l by $\|\cdot\|_l$ and define

$$C_n = \frac{1}{\|x\|_l^{n-1}} D_l$$

for all $n \in \mathbb{N}$. By the above, the sequence of Banach discs $(C_n)_{n\in\mathbb{N}}$ satisfies condition 2. of 3.2, but this sequence also satisfies 1. because if $z \in C_m$, there exists some $y \in D_l$ with $z = \frac{1}{\|x\|_l^{m-1}} y$ and we can conclude that

$$\|x^{m-n}z\|_l = \frac{1}{\|x\|_l^{m-1}} \|x^{m-n}y\|_l \leq \frac{1}{\|x\|_l^{m-1}} \|x^{m-n}\|_l \|y\|_l \leq \frac{1}{\|x\|_l^{n-1}} \|y\|_l$$

and thus $x^{m-n}z \in C_n$. Hence for all $m \geq n$ we have

$$(M_x)^{m-n}(C_m) = x^{m-n} C_m \subseteq C_n.$$

Thus 3.2 yields the stability of x. □

Now we come back to Allan's question whether the product of two stable elements is stable:

Corollary 3.4.
Let x and y be two stable elements of a commutative Hausdorff LB-algebra A. Then xy is stable.

Proof. Since x and y are stable, 3.3 implies the existence of two Banach discs B_x and B_y and natural numbers m_x and m_y such that for $k_x \geq m_x$ and $k_y \geq m_y$ we have the inclusions

$$x^{m_x} A \subseteq x^{k_x} A + B_x$$

and

$$y^{m_y} A \subseteq y^{k_y} A + B_y.$$

B_x and B_y are Banach discs, hence there exist $l \in \mathbb{N}$ and $\lambda > 0$ such that B_x and B_y are both contained in $D = \lambda D_l$, where again D_l is the unit ball of the Banach algebra A_l. Since we can assume without loss of generality that $x, y \in A_l$, the mappings M_x and M_y are continuous in the topology whose basis of neighborhoods of zero is given by the sets $\{\varepsilon D : \varepsilon > 0\}$. Hence the same computation as in the proof of 2.12 yields that there exists $m \in \mathbb{N}$ such that for all $k \geq m$

$$(xy)^m A \subseteq (xy)^k A + D.$$

Since D is a Banach disc the assertion follows from 3.3. □

We will continue with some examples of LB-algebras. In the first one we determine the stable elements of an inductive limit of Banach algebras of weighted continuous functions.

Example 3.5.
Let $(v_n)_{n\in\mathbb{N}}$ be a decreasing sequence of continuous functions

$$v_n : \mathbb{R} \to [1, \infty).$$

Further let A_n be the space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{\mathbb{R}} |f| v_n < \infty.$$

Then the space A_n endowed with the norm $\|\cdot\|_n$ defined by

$$\|f\|_n = \sup_{\mathbb{R}} |f| v_n$$

is a Banach algebra since $v_n \geq 1$ and thus the norm is submultiplicative. Obviously every A_n is a subalgebra of A_{n+1} and every inclusion mapping is continuous. We set

$$A = \bigcup_{n\in\mathbb{N}} A_n,$$

equipped with the inductive limit topology described in the introduction of this chapter. Note that this topology is Hausdorff since the embedding of A into $C(\mathbb{R})$, the space of all continuous functions on $\mathbb{R}$, equipped with the topology of pointwise convergence, is continuous. We claim that every $f \in A$ is stable. To prove this, let $f \in A$ be arbitrary. As we have shown in 3.3 it suffices to verify that there exist a Banach disc B and a positive integer m such that for all $k \geq m$

$$f^m A \subseteq f^k A + B.$$

We can assume without loss of generality that $f \in A_1$. Let $m = 2$ and

$$B = \overline{D_1} = \{\eta \in A_1 : \|\eta\|_1 \leq 1\}.$$

For an arbitrary $g \in A$ we have to show that there exists some $h \in A$ such that for all $k \geq 2$

$$f^2 g - f^k h \in B.$$

So let $g \in A_n$ for some $n \in \mathbb{N}$. To construct a suitable function h, we set

$$U = \{x \in \mathbb{R} : |f^2(x) g(x)| v_1(x) < 1\}.$$

If $U = \mathbb{R}$, then $f^2 g \in B$ and the assertion is true. Otherwise U is the disjoint union of open intervals (α_i, β_i) with $i \in I$ for some index set I. For each such interval we define

$$U_i = (\alpha_i, \alpha_i + \frac{\beta_i - \alpha_i}{3}) \cup (\beta_i - \frac{\beta_i - \alpha_i}{3}, \beta_i)$$

and

$$\tilde{U} = \bigcup_{i\in I} U_i.$$

Now we can define the desired function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} \frac{g(x)}{f^{k-2}(x)} & x \notin U \\ \psi(x) & x \in \tilde{U} \\ 0 & x \in U \setminus \tilde{U} \end{cases},$$

where ψ shall be a continuous continuation of h on $\tilde{U}$ such that for $x \in (\alpha_i, \alpha_i + \frac{\beta_i - \alpha_i}{3})$

$$|\psi(x)| \leq \min\left\{ \frac{|h(\alpha_i)|}{v_n(x)} v_n(\alpha_i), \left| \frac{g(x)}{f^{k-2}(x)} \right| \right\}$$

(note that $\alpha_i \notin U$ so that $h(\alpha_i)$ is defined) and analogously for $x \in (\beta_i - \frac{\beta_i - \alpha_i}{3}, \beta_i)$ the same inequality with α_i replaced by β_i, and such that $\operatorname{sign}(\psi) = \operatorname{sign}(\frac{g}{f^{k-2}})$ on $\tilde{U}$. First we will verify that $h \in A_n \subseteq A$: If $x \in U^C$ we have

$$|f^2(x)g(x)|v_1(x) \geq 1$$

and thus

$$\frac{1}{|f(x)|} \leq |f(x)g(x)|v_1(x).$$

Taking the $(k-2)$th power of this inequality we get

$$\begin{aligned} |h(x)|v_n(x) &= \left| \frac{g(x)}{f^{k-2}(x)} \right| v_n(x) \leq |f^{k-2}(x)||g^{k-1}(x)|v_1^{k-2}(x)v_n(x) \\ &= |f^{k-2}(x)|v_1^{k-2}(x)|g^{k-1}(x)|v_n(x) \leq \|f\|_1^{k-2}\|g\|_n^{k-1} \end{aligned}$$

since $f \in A_1$, $g \in A_n$ and $v_n \leq v_n^{k-1}$. For $x \in \tilde{U}$ we can assume without loss of generality that $x \in (\alpha_i, \alpha_i + \frac{\beta_i - \alpha_i}{3})$ and then we have

$$|h(x)|v_n(x) \leq \frac{|h(\alpha_i)|}{v_n(x)} v_n(\alpha_i) v_n(x) = |h(\alpha_i)|v_n(\alpha_i)$$

and thus

$$\sup_{\mathbb{R}} |h|v_n = \sup_{U^C} |h|v_n < \infty.$$

It remains to prove that $\varphi = f^2 g - f^k h \in B$. Since φ vanishes on U^C, $|h(x)| \leq |\frac{g(x)}{f^{k-2}(x)}|$ and $\operatorname{sign}(h) = \operatorname{sign}(\frac{g}{f^{k-2}})$ on $\tilde{U}$, we can conclude that

$$\sup_{\mathbb{R}} |\varphi|v_1 = \sup_{U} |\varphi|v_1 \leq \sup_{U} |f^2 g|v_1 \leq 1.$$

Hence 3.3 yields the stability of f.

In the next example we will consider what happens if the weight functions v_n can have any positive value.

Example 3.6.
Let again $(v_n)_{n\in\mathbb{N}}$ be a decreasing sequence of continuous functions, but now

$$v_n : \mathbb{R} \to (0, \infty).$$

In this case the spaces A_n defined in 3.5 may fail to be Banach algebras, since it may happen that $fg \notin A_n$ for two elements $f, g \in A_n$. Then $(A_n)_{n\in\mathbb{N}}$ is just an increasing sequence of Banach spaces and the inductive limit $A = \bigcup_{n\in\mathbb{N}} A_n$ with the corresponding topology is an LB-space. Nevertheless it is possible that A is an algebra, we just need an additional requirement for the sequence $(v_n)_{n\in\mathbb{N}}$. In this context we show the following assertion:

For natural numbers n, m and k we have $A_n A_m \subseteq A_k$ if and only if there exists some $C > 0$ such that $v_k \leq C v_n v_m$.

So let $A_n A_m$ be contained in A_k. Since

$$\frac{1}{v_n} \in A_n, \ \frac{1}{v_m} \in A_m,$$

we can conclude that

$$\frac{1}{v_n v_m} \in A_k.$$

Then

$$\sup_{\mathbb{R}} |\frac{1}{v_n v_m}| v_k < \infty$$

and hence there exists $C > 0$ such that

$$\frac{1}{v_n v_m} v_k \leq C.$$

If we assume that $v_k \leq C v_n v_m$ for some $C > 0$ and $f \in A_n$ and $g \in A_m$ are arbitrary, we can conclude that

$$\sup_{\mathbb{R}} |fg| v_k \leq C \sup_{\mathbb{R}} |fg| v_n v_m \leq C \sup_{\mathbb{R}} |f| v_n \sup_{\mathbb{R}} |g| v_m < \infty.$$

Hence, if for all $n, m \in \mathbb{N}$ there exists some $k \in \mathbb{N}$ and $C > 0$ such that $v_k \leq C v_n v_m$, $A = \bigcup_{n\in\mathbb{N}} A_n$ is an algebra and we will show that it remains true that every $f \in A$ is stable. If $f \in A$, we can assume without loss of generality that $f \in A_1$. As the hypothesis of 3.3 is not fulfilled, we will make use of 3.2 to verify the stability of f. For this purpose we define for all $n \in \mathbb{N}$

$$B_n = \bigcap_{j=1}^{n} (M_f^{n-j})^{-1}(\overline{D_1}) = \{g \in A : f^{n-j} g \in \overline{D_1} \text{ for all } 1 \leq j \leq n\},$$

where D_1 is again the unit ball of the Banach space A_1. Then every B_n is a Banach disc and obviously for all $m \geq n$

$$f^{m-n} B_m \subseteq B_n.$$

We will prove that for an arbitrary $n \in \mathbb{N}$ and all $k \geq 2$ we have

$$f^2 A \subseteq f^k A + B_n.$$

Let $g \in A$ and for $1 \leq j \leq n$ we set

$$U_j = \{x \in \mathbb{R} : |f^{n-j+2}(x)||g(x)|v_1(x) < 1\}$$

and

$$U = \bigcap_{1 \leq j \leq n} U_j.$$

Then again U is the disjoint union of open intervals and analogously to 3.5 we define the set $\tilde{U}$ and the functions h and ψ, the only difference is that ψ has to fulfill

$$|\psi(x)| \leq \min\Big\{\frac{|h(\alpha_i)|}{v_m(x)} v_m(\alpha_i), \Big|\frac{g(x)}{f^{k-2}(x)}\Big|\Big\},$$

where m is any sufficiently large natural number such that for all $j \leq n$ we have

$$f^{(k-2)(n-j)} g^{k-1} \in A_m.$$

Then for $x \in U^C$ there exists some $j \leq n$ such that $x \in U_j^C$ and thus

$$|f^{n-j+2}(x)||g(x)|v_1(x) \geq 1.$$

Hence

$$\begin{aligned} |h(x)|v_m(x) &= \Big|\frac{g(x)}{f^{k-2}(x)}\Big| v_m(x) \leq |f^{(n-j+1)(k-2)}(x)||g^{k-1}(x)|v_1^{k-2}(x)v_m(x) \\ &= |f^{k-2}(x)|v_1^{k-2}(x)|f^{(k-2)(n-j)}(x)||g^{k-1}(x)|v_m(x) \leq \|f\|_1^{k-2}\|f^{(k-2)(n-j)}g^{k-1}\|_m. \end{aligned}$$

This yields once again that $h \in A$ and as in 3.5 we can conclude that $f^2 g - f^k h \in B_n$. Therefore 3.2 implies the stability of f.

Analogous to the last example the characterization of Palamodov and Retakh in 3.2 yields an abstract characterization of stability for another type of algebras. We call a topological algebra A a graded LB-algebra if it is an inductive limit of Banach spaces A_n such that $A_n A_m \subseteq A_{n+m}$ for every $n, m \in \mathbb{N}$. Although we cannot apply 3.3 and 3.4 to analyze stability on this algebras, the first condition of 3.2 implies that for a stable element $x \in A$ there has to be a sequence of Banach disc $(B_n)_{n \in \mathbb{N}}$ with $B_n \subseteq A_n$ such that particularly

$$x^n B_{n+1} \subseteq B_1.$$

This condition can be very restrictive, as we will see in the following example.

Example 3.7.
Let $A = C_c(\mathbb{R})$ be the convolution algebra of all complex-valued continuous function with compact support. Then we have $A = \bigcup_{n\in\mathbb{N}} C([-n,n])$, where $A_n = C([-n,n])$ is the Banach space of complex-valued continuous functions with support in the interval $[-n,n]$ equipped with the maximum norm. We assume that $f \in A \setminus \{0\}$ is stable. The theorem of support (cf. [14, Theorem 4.3.3]) yields that

$$\mathrm{Conv}\,(\mathrm{supp}\, f * g) = \mathrm{Conv}\,(\mathrm{supp}\, f) + \mathrm{Conv}\,(\mathrm{supp}\, g),$$

where $\mathrm{Conv}\, M$ denotes the convex hull of a set $M \subseteq \mathbb{R}$ and $\mathrm{supp}\, f = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$ denotes the support of $f \in A$. Thus there exists some $N \in \mathbb{N}$ such that for all $n \geq N$ the length of the interval $\mathrm{Conv}\,(\mathrm{supp}\, f^n)$ is larger than 2. But then for all $g \in A \setminus \{0\}$ the convolution $f^n * g$ cannot be an element of A_1, so that the condition

$$f^n B_{n+1} \subseteq f^{n-1} f B_{n+1} \subseteq f^{n-1} B_n \subseteq \cdots \subseteq B_1$$

implies that for all $n \geq N$ we have $B_n = \{0\}$. Hence the second condition of 3.2 implies that there exists some $m \in \mathbb{N}$ such that for all $k > m$

$$f^m A \subseteq f^k A. \tag{6}$$

If we set $h = f^{m+1} \in f^m A$, then $h \in A_l$ for some $l \in \mathbb{N}$ and (6) implies that there exists a sequence $(g_n)_{n\in\mathbb{N}}$ such that $h = f^{m+n} g_n$ for all $n \in \mathbb{N}$. But this contradicts the theorem of support, since $\mathrm{supp}\, f^{m+n} g_n$ is not a subset of the interval $[-l,l]$ for sufficiently large natural numbers n. Thus $f = 0$ is the only stable element in A.

We now turn our attention back to the question considered in 2.14, whether for a stable element x we have $I(x) = I_0(x)$. In the following theorem we will show that the assertion remains true in the case of LB-algebras and after a short remark we will present an application of this result. The proof complies with Allan's proof of the Banach algebra case in [2, Lemma 1].

Theorem 3.8.
Let $A = \bigcup_{n\in\mathbb{N}} A_n$ be a commutative LB-algebra and let $x \in A$ be stable. Then

$$M_x(I(x)) = I(x).$$

Proof. Since $I(x)$ is an ideal, it is clear that $M_x(I(x))$ is contained in $I(x)$. So if z is an element of $I(x)$ we have to show that there exists an element $\tilde{z} \in I(x)$ such that $z = x\tilde{z}$. Let $z \in I(x)$ and $S_j = \{y \in A : yx^j = z\}$ for $j \in \mathbb{N}$. By definition of $I(x)$ the set S_j is nonempty and an easy computation shows that $M_x(S_{j+1})$ is contained in S_j. As we have shown in the proof of 3.3 the stability of x yields a Banach disc $D = \lambda D_l$ and an integer m such that for all $k > m$ and for all $\varepsilon > 0$ we have

$$x^m A \subseteq x^k A + \varepsilon D. \tag{7}$$

For $y, \tilde{y} \in S_j$ we have $(y-\tilde{y})x^j A = \{0\}$. Thus for $k \geq m$, $\varepsilon > 0$ and $y, \tilde{y} \in S_{k+1}$

$$(y-\tilde{y})x^m A \subseteq (y-\tilde{y})x^{k+1}A + (y-\tilde{y})\varepsilon D = (y-\tilde{y})\varepsilon D.$$

Hence

$$(y-\tilde{y})x^m A \subseteq (y-\tilde{y}) \bigcap_{\varepsilon>0} \varepsilon D = (y-\tilde{y})\overline{\{0\}}^{\mathfrak{T}_D},$$

where $\mathfrak{T}_D$ denotes the topology whose basis of zero neighborhoods is given by the sets $\{\varepsilon D : \varepsilon > 0\}$. Since $D = \lambda D_l$ for some $l \in \mathbb{N}$, $\mathfrak{T}$ is the topology of the Banach space A_l and obviously

$$\overline{\{0\}}^{\mathfrak{T}_D} = \bigcap_{\varepsilon>0} \varepsilon D = \{0\}.$$

Thus we have shown that $(y-\tilde{y})x^m A = \{0\}$ for $y, \tilde{y} \in S_{k+1}$. Especially $yx^{m+1} = \tilde{y}x^{m+1}$ and therefore $M_{x^{m+1}}(S_{k+1})$ is a singleton for all $k > m$. This defines a sequence $(z_n)_{n\in\mathbb{N}}$ such that

$$\{z_n\} = M_{x^{m+1}}(S_{n+m+1})$$

for all $n \in \mathbb{N}$. Thus we have $z_n = yx^{m+1}$ for some y with $yx^{n+m+1} = z$. This implies

$$z_n x^n = yx^{m+1}x^n = yx^{m+n+1} = z,$$

which results in $z_n \in S_n$. Moreover, since $z_{n+1} = yx^{m+1}$ with $y \in S_{n+m+2}$, we have

$$xz_{n+1} = xyx^{m+1} = x^{m+1}xy$$

and

$$xyx^{n+m+1} = yx^{n+m+2} = z,$$

which means that $xy \in S_{n+m+1}$ and thus $M_x(z_{n+1}) \in M_{x^{m+1}}(S_{n+m+1})$. Particularly for $z_1 = x^{m+1}y$ with $y \in S_{m+2}$ we have

$$M_x(z_1) = xz_1 = xx^{m+1}y = x^{m+2}y = z.$$

Furthermore, if we combine the two facts $M_x(S_{j+1})$ being contained in S_j for all $j \in \mathbb{N}$ and $\{z_n\} = M_{x^{m+1}}(S_{n+m+1})$ for every $n \in \mathbb{N}$, we can conclude that

$$z_1 = z_2 x = z_3 x^2 = \cdots$$

and thus z_1 belongs to $I(x)$. □

Remark 3.9.
Actually the proof of 3.8 shows that for every topological vector space A and every $x \in A$ that is stable we have $I(x) = I_0(x)$ whenever there exists a submultiplicative norm p on A. In this case (7) remains true for

$$D = \{x \in A : p(x) \leq 1\}$$

due to 1.12 and the rest of the proof is identical.

In the last example of this chapter we will show that in the algebra of germs of holomorphic functions in 0 there are no stable elements besides the zero function and the invertible functions.

Example 3.10.
For all $n \in \mathbb{N}$ let $\Omega_n = \{z \in \mathbb{C} : |z| < \frac{1}{n}\}$. Then $H^\infty(\Omega_n)$, the space of all holomorphic and bounded functions on Ω_n, equipped with the supremum norm and pointwise multiplication, is a Banach algebra. The inductive limit $A = \bigcup_{n\in\mathbb{N}} H^\infty(\Omega_n)$, where the inclusion mappings $i_n : A_n \to A_{n+1}$ are defined by $f \mapsto f|_{\Omega_{n+1}}$, can be identified with the algebra of germs of holomorphic functions on 0 denoted by $H(\{0\})$ (cf. [8, Example 2.6]).
Obviously an element $f \in A = H(\{0\})$ is invertible if and only if $f(0) \neq 0$. So let $f \in A$ be not invertible. We claim that

$$I(x) = \bigcap_{n\in\mathbb{N}} f^n A = \{0\}.$$

If we consider some $g \in f^n A$, the vanishing of f in 0 implies that g has a zero of multiplicity larger or equal to n in 0 and thus we have

$$g^{(n-1)}(0) = 0.$$

So for some $h \in \bigcap_{n\in\mathbb{N}} f^n A$ we can conclude that for all $n \in \mathbb{N}$

$$h^{(n)}(0) = 0.$$

Since h is holomorphic in a neighborhood of 0, h must be identically zero there, thus $h = 0 \in A$. Hence $\bigcap_{n\in\mathbb{N}} f^n A = \{0\}$ and if we now assume that f is stable, 3.8 implies that for all $n \in \mathbb{N}$ we have

$$\{0\} = I(x) = I_0(x) = \varrho^n(\mathrm{Proj}\mathcal{A}).$$

Together with 3.3 we can conclude that there exists a Banach disc B and a positive integer m such that for all $\varepsilon > 0$

$$f^m A \subseteq \{0\} + \varepsilon B.$$

Since B is bounded this is a contradiction and thus f is not stable.

4 Topological algebras with multiplicative webs

The theory of webbed locally convex spaces was introduced by M. De Wilde in [12] for the purpose of a generalization of the classical closed graph theorem. Further studies in this context can be found for example in [15] or [16]. Since Fréchet spaces and LB-spaces are both examples for webbed locally convex spaces, this theory is also of interest for our work with stability. We start with the definition of a web in the way it was introduced by Meise and Vogt in [16].

Definition 4.1.
Let A be a locally convex space. A family $\mathcal{C} = \{C_{k_1,\ldots,k_\nu} : k_1, \ldots, k_\nu, \nu \in \mathbb{N}\}$ of absolutely convex subsets of A is called a *web*, if the following properties are satisfied:

(1) $\bigcup_{n\in\mathbb{N}} C_n = A$.

(2) $\bigcup_{n\in\mathbb{N}} C_{k_1,\ldots,k_\nu,n} = C_{k_1,\ldots,k_\nu}$.

(3) For every sequence $(k_\nu)_{\nu\in\mathbb{N}}$ in $\mathbb{N}$ there exists a sequence $(\lambda_\nu)_{\nu\in\mathbb{N}}$ in $]0,1[$ such that for every $(x_\nu)_{\nu\in\mathbb{N}} \in \prod_{\nu\in\mathbb{N}} C_{k_1,\ldots,k_\nu}$ the series $\sum_{\nu=1}^{\infty} \lambda_\nu x_\nu$ converges in A.

If such a web exists, A is called a *webbed space*. Furthermore, a web is called *ordered*, if for two sequences of natural numbers $(k_n)_{n\in\mathbb{N}}$ and $(l_n)_{n\in\mathbb{N}}$ with $k_n \leq l_n$ for all $n \in \mathbb{N}$ we have

$$C_{k_1,\ldots,k_\nu} \subseteq C_{l_1,\ldots,l_\nu}$$

for all $\nu \in \mathbb{N}$.

Remark 4.2.
With this definition it is possible to show the following versions of the closed graph theorem and the open mapping theorem (cf. [12, Chapter 4]). We recall that a locally convex space A is called ultra-bornological if every linear mapping T from A into any locally convex space $\tilde{A}$ such that $T(B)$ is bounded in $\tilde{A}$ for all Banach discs $B \subseteq A$ is already continuous.

Let E be a webbed locally convex space and F an ultra-bornological locally convex space. Then the following are valid:

1. Every $T : F \to E$ that is linear and has a closed graph is continuous.
2. Every $T : E \to F$ that is linear, continuous and surjective is open.

The following result of Frerick, Kunkle and Wengenroth [13, Lemma 2.4] shows that we can always choose a special sequence $(\lambda_\nu)_{\nu\in\mathbb{N}}$ in the third condition of 4.1:

Proposition 4.3.
Let A be a locally convex space and $\mathcal{C}$ be a web on A. Then for all $(k_\nu)_{\nu\in\mathbb{N}}$ in $\mathbb{N}$ and all $(x_\nu)_{\nu\in\mathbb{N}} \in \prod_{\nu\in\mathbb{N}} C_{k_1,\dots,k_\nu}$ the series $\sum_{\nu=1}^{\infty} \frac{1}{2^{\nu+1}} x_\nu$ converges in A.

Proof. Let $(\lambda_\nu)_{\nu\in\mathbb{N}}$ be a sequence in $]0,1[$ such that $\sum_{\nu=1}^{\infty} \lambda_\nu x_\nu$ converges for all $(x_\nu)_{\nu\in\mathbb{N}} \in \prod_{\nu\in\mathbb{N}} C_{k_1,\dots,k_\nu}$. Then we can find a strictly increasing sequence $(\alpha_\nu)_{\nu\in\mathbb{N}}$ of natural numbers such that for all $\nu \in \mathbb{N}$

$$\sum_{j=\alpha_\nu}^{\infty} \frac{1}{2^j} \leq \lambda_{\nu+1}.$$

For $(x_\nu)_{\nu\in\mathbb{N}} \in \prod_{\nu\in\mathbb{N}} C_{k_1,\dots,k_\nu}$ we set $y_1 = 0$ and for all $v \geq 2$

$$y_\nu = \frac{1}{\lambda_\nu} \sum_{j=\alpha_{\nu-1}+1}^{\alpha_\nu} \frac{1}{2^j} x_j.$$

Every $C_{k_1,\dots,k_\nu}$ is absolutely convex and for $j > \alpha_{\nu-1} \geq \nu - 1$ we have $x_j \in C_{k_1,\dots,k_j} \subseteq C_{k_1,\dots,k_\nu}$. Since $\sum_{j=\alpha_{\nu-1}}^{\infty} \frac{1}{2^j} \leq \lambda_\nu$ we can conclude that $y_\nu \in C_{k_1,\dots,k_\nu}$ for all $\nu \in \mathbb{N}$. Hence $\sum_{\nu=1}^{\infty} \lambda_\nu y_\nu$ converges and thus also the sequence

$$\sum_{j=1}^{\alpha_\nu} \frac{1}{2^{j+1}} x_j = \frac{1}{2}\left(\sum_{j=1}^{\alpha_1} \frac{1}{2^j} x_j + \sum_{j=1}^{\nu} \lambda_j y_j\right)$$

converges. For a given $m \in \mathbb{N}$ let $n(m)$ be the largest natural number such that $\alpha_{n(m)} \leq m$. Then we define for all $m \in \mathbb{N}$

$$r_m = \sum_{j=\alpha_{n(m)+1}}^{m} \frac{1}{2^{j+1}} x_j \in \lambda_{n(m)+1} C_{k_1,\dots,k_{n(m)+1}}.$$

If $(m(k))_{k\in\mathbb{N}}$ is a subsequence of the natural numbers, there exists a further subsequence $(m(k(l))_{l\in\mathbb{N}}$ such that $n(m(k(l)))$ is strictly increasing. So for $l \to \infty$ we have $r_{m(k(l))} \to 0$, which implies that $r_m \to 0$ for $m \to \infty$. Thus $\sum_{\nu=1}^{\infty} \frac{1}{2^{\nu+1}} x_\nu$ converges in A. □

As mentioned above we will now show that Fréchet spaces and LB-spaces are webbed spaces.

Remark 4.4.
If A is a Fréchet space and $(U_n)_{n\in\mathbb{N}}$ a basis of zero neighborhoods, we set

$$C_{k_1,\dots,k_\nu} = \bigcap_{l=1}^{\nu} k_l U_l.$$

Then we have

$$\bigcup_{n\in\mathbb{N}} C_n = \bigcup_{n\in\mathbb{N}} n U_1 = A$$

and

$$\bigcup_{n\in\mathbb{N}} C_{k_1,\ldots,k_\nu,n} = \bigcup_{n\in\mathbb{N}} (\bigcap_{l=1}^{\nu} k_l U_l \cap nU_{\nu+1}) = (\bigcap_{l=1}^{\nu} k_l U_l) \cap (\bigcup_{n\in\mathbb{N}} nU_{\nu+1}) = C_{k_1,\ldots,k_\nu}.$$

To show that the sets $\bigcap_{l=1}^{\nu} k_l U_l$ form a web in A it remains to verify the third condition of 4.1. To this end let $(k_\nu)_{\nu\in\mathbb{N}}$ be an arbitrary sequence of natural numbers and for all $\nu \in \mathbb{N}$ we set $\lambda_\nu = \frac{1}{2^{\nu+1}}$. The topology of A is induced by an increasing family of seminorms $(p_n)_{n\in\mathbb{N}}$, thus we can again assume that $U_n = \{y \in A : p_n(y) \leq \frac{1}{n}\}$. If we choose an arbitrary sequence

$$(x_\nu)_{\nu\in\mathbb{N}} \in \prod_{\nu\in\mathbb{N}} \bigcap_{l=1}^{\nu} k_l U_l$$

and fix some $n \in \mathbb{N}$, then for all $m \geq n$ we have

$$x_m \in \bigcap_{l=1}^{m} k_l U_l \subseteq k_n U_n.$$

Thus $p_n(x_m) \leq \frac{k_n}{n}$ for all $m \geq n$ and therefore $\sum_{\nu=1}^{\infty} p_n(\lambda_\nu x_\nu)$ converges. Since n was chosen arbitrarily, we can conclude that $\sum_{\nu=1}^{\infty} \lambda_\nu x_\nu$ is Cauchy, hence the completeness of A implies the convergence of the series. So the sets $\bigcap_{l=1}^{\nu} k_l U_l$ form a web in A which is also ordered, since for two sequences $(k_\nu)_{\nu\in\mathbb{N}}$ and $(\tilde{k}_\nu)_{\nu\in\mathbb{N}}$ of natural numbers with $k_\nu \leq \tilde{k}_\nu$ for all $\nu \in \mathbb{N}$ it is clear that

$$\bigcap_{l=1}^{\nu} k_l U_l \subseteq \bigcap_{l=1}^{\nu} \tilde{k}_l U_l.$$

If on the other hand A is an LB-space, let $(B_n)_{n\in\mathbb{N}}$ be an increasing fundamental sequence of Banach discs. Then we define

$$C_{k_1,\ldots,k_\nu} = \min\{k_1,\ldots,k_\nu\} B_{k_1}.$$

It is clear that

$$\bigcup_{n\in\mathbb{N}} C_n = \bigcup_{n\in\mathbb{N}} nB_n = A$$

and that

$$\bigcup_{n\in\mathbb{N}} C_{k_1,\ldots,k_\nu,n} = \bigcup_{n\in\mathbb{N}} \min\{k_1,\ldots,k_\nu,n\} B_{k_1} = \min\{k_1,\ldots,k_\nu\} B_{k_1} = C_{k_1,\ldots,k_\nu}.$$

Again let $(k_\nu)_{\nu\in\mathbb{N}}$ be a sequence of natural numbers and $\lambda_\nu = \frac{1}{2^{\nu+1}}$ for all $\nu \in \mathbb{N}$. If we choose an arbitrary sequence

$$(x_\nu)_{\nu\in\mathbb{N}} \in \prod_{\nu\in\mathbb{N}} \min\{k_1,\ldots,k_\nu\} B_{k_1},$$

every x_ν lies in a multiple of the Banach disc B_{k_1}, hence in the Banach space A_{k_1}. So it is obvious that $\sum_{\nu=1}^{\infty} \lambda_\nu x_\nu$ converges in A_{k_1}. Since the topology on A is the finest locally convex topology on A such that the embeddings $A_n \hookrightarrow A$ are continuous, it is clear that the series $\sum_{\nu=1}^{\infty} \lambda_\nu x_\nu$ converges also in the inductive limit A. The web

$$\mathcal{C} = \{\min\{k_1,\ldots,k_\nu\} B_{k_1} : k_1,\ldots,k_\nu, \nu \in \mathbb{N}\}$$

is also ordered, due to the fact that for two sequences $(k_\nu)_{\nu\in\mathbb{N}}$ and $(\tilde{k}_\nu)_{\nu\in\mathbb{N}}$ in $\mathbb{N}$ with $k_\nu \leq \tilde{k}_\nu$ we have

$$\min\{k_1,\dots,k_\nu\}B_{k_1} \subseteq \min\{\tilde{k}_1,\dots,\tilde{k}_\nu\}B_{\tilde{k}_1}$$

since the sequence $(B_n)_{n\in\mathbb{N}}$ can without loss of generality be chosen increasingly as shown in 3.1.

We now come back to the theory of stability. In this context we need, as in the chapters before, a characterization of the condition $\operatorname{Proj}^1\mathcal{A}_x = 0$ for $\mathcal{A}_x = (A, M_x)$. For webbed spaces the following result was established by Frerick, Kunkle and Wengenroth in [13].

Theorem 4.5.
Let $\mathcal{A} = (A_n, \varrho^n_m)$ be a projective spectrum of Hausdorff locally convex spaces that have ordered webs $\mathcal{C}^n = \{C^n_{k_1,\dots,k_\nu} : k_1,\dots,k_\nu,\nu \in \mathbb{N}\}$ and continuous linear mappings. Then $\operatorname{Proj}^1\mathcal{A} = 0$ if and only if there exists a sequence $(k_\nu)_{\nu\in\mathbb{N}}$ in $\mathbb{N}$ such that

$$\forall\, n\in\mathbb{N}\ \exists\, m\geq n\ \forall\, k\geq m : \varrho^n_m(A_m) \subseteq \varrho^n_k(A_k) + \bigcap_{j=1}^{n}(\varrho^j_n)^{-1}(C^j_{k_j,\dots,k_n}). \tag{8}$$

Proof. The fact that $\operatorname{Proj}^1\mathcal{A} = 0$ implies (8) can be proved as 1.11. Since the webs are ordered, we have for example

$$\bigcup_{n\in\mathbb{N}}(C^1_{k_1,n}\times C^2_n) = C^1_{k_1}\times A_2.$$

Thus we can conclude analogously that the closure of

$$\prod_{j=1}^{n} C^j_{k_j,\dots,k_n}\times\prod_{j>n}A_j$$

contains interior points and in the same way as in the proof of 1.11 we obtain the second condition.
If (8) is true, we can assume without loss of generality that the condition holds with $m = n+1$, otherwise we may go over to a subsequence of the projective spectrum. To show that $\operatorname{Proj}^1\mathcal{A} = 0$ we have to verify that $\psi = \psi_{\mathcal{A}} : \prod_{n\in\mathbb{N}}A_n \to \prod_{n\in\mathbb{N}}A_n$ defined by

$$(x_n)_{n\in\mathbb{N}} \mapsto (x_n - \varrho^n_{n+1}(x_{n+1}))_{n\in\mathbb{N}}$$

is surjective, i.e. for a given $(y_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}}A_n$ we need to find a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\psi((x_n)_{n\in\mathbb{N}}) = (y_n)_{n\in\mathbb{N}}$. To this end we set $w_1 = w_2 = 0$ and choose inductively $w_n \in A_n$ and

$$v_n \in \frac{1}{2^{n+1}}\bigcap_{j=1}^{n}(\varrho^j_n)^{-1}(C^j_{k_j,\dots,k_n})$$

such that for all $n\in\mathbb{N}$

$$\varrho^n_{n+1}(w_{n+1}-y_{n+1}) = \varrho^n_{n+2}(w_{n+2}) + v_n.$$

Since $\varrho_m^n(v_m) \in \frac{1}{2^{m+1}}(C^n_{k_n,\ldots,k_m})$ for all $m \geq n$, we can conclude with 4.3 that the series

$$z_n = \sum_{m=n}^{\infty} \varrho_m^n(v_m)$$

converges. Furthermore we have

$$z_n - \varrho_{n+1}^n(z_{n+1}) = v_n$$

and we define the desired sequence $(x_n)_{n\in\mathbb{N}}$ as

$$x_n = \varrho_{n+1}^n(w_{n+1}) - z_n + y_n.$$

Then for all $n \in \mathbb{N}$ we obtain

$$\begin{aligned}
\varrho_{n+1}^n(x_{n+1}) &= \varrho_{n+2}^n(w_{n+2}) - \varrho_{n+1}^n(z_{n+1}) + \varrho_{n+1}^n(y_{n+1}) \\
&= \varrho_{n+1}^n(w_{n+1}) - v_n - \varrho_{n+1}^n(z_{n+1}) \\
&= \varrho_{n+1}^n(w_{n+1}) - z_n \\
&= x_n - y_n.
\end{aligned}$$

Thus ψ is surjective and therefore $\operatorname{Proj}^1\mathcal{A} = 0$. □

Remark 4.6.
4.5 implies that for a commutative locally convex Hausdorff topological algebra A that has an ordered web $\mathcal{C} = \{C_{k_1,\ldots,k_\nu} : k_1,\ldots,k_\nu, \nu \in \mathbb{N}\}$ an element $x \in A$ is stable if and only if there is a sequence $(k_\nu)_{\nu\in\mathbb{N}}$ of natural numbers such that

$$\forall\ n \in \mathbb{N}\ \exists\ m \geq n\ \forall\ k \geq m : x^m A \subseteq x^k A + \bigcap_{j=1}^{n} (M_x^{n-j})^{-1}(C_{k_j,\ldots,k_n}).$$

To apply this characterization of stable elements we need a further requirement for the web.

Definition 4.7.
A web on a topological algebra A is called *multiplicative* if for every sequence $k = (k_\nu)_{\nu\in\mathbb{N}}$ of natural numbers and every $\nu \in \mathbb{N}$ there exists a $\lambda = \lambda(k,\nu) > 0$ such that

$$C_{k_1,\ldots,k_\nu} \cdot C_{k_1,\ldots,k_\nu} \subseteq \lambda C_{k_1,\ldots,k_\nu}.$$

This definition is reasonable since Fréchet algebras and LB-algebras have multiplicative webs:

Remark 4.8.
If A is a commutative Fréchet algebra, the sets $C_{k_1,\ldots,k_\nu} = \bigcap_{l=1}^{\nu} k_l U_l$ defined in 4.4 form a web in A. This web is also multiplicative: The topology of A is induced by an increasing sequence of submultiplicative seminorms $(p_n)_{n\in\mathbb{N}}$ and as in the proof of 2.11 we assume that the basis of zero neighborhoods consists of the sets $U_n = \{x \in A : p_n(x) \le \frac{1}{n}\}$. Since every p_n is submultiplicative, for two elements $x, y \in U_n$ we have

$$p_n(x \cdot y) \le p_n(x) \cdot p_n(y) \le \frac{1}{n},$$

hence xy is again an element of U_n. Thus it is easy to see that

$$\bigcap_{l=1}^{\nu} k_l U_l \cdot \bigcap_{l=1}^{\nu} k_l U_l \subseteq \max_{1\le l\le \nu} k_l \bigcap_{l=1}^{\nu} k_l U_l,$$

hence the web is multiplicative.
If A is a commutative LB-algebra and $(B_n)_{n\in\mathbb{N}}$ defined as in 3.1, the web defined by the sets $C_{k_1,\ldots,k_\nu} = \min\{k_1,\ldots,k_\nu\}B_{k_1}$ is multiplicative: Every B_{k_1} is a subset of the Banach algebra A_{k_1} and so for two elements $x, y \in B_{k_1}$ we have

$$\|x \cdot y\|_{k_1} \le \|x\|_{k_1} \cdot \|y\|_{k_1} \le k_1^2.$$

Thus

$$\min\{k_1,\ldots,k_\nu\}B_{k_1} \cdot \min\{k_1,\ldots,k_\nu\}B_{k_1} \subseteq k_1^2 \min\{k_1,\ldots,k_\nu\}B_{k_1}.$$

Now we present the main result of this chapter that generalizes Allans work from the 1990s.

Theorem 4.9.
Let A be a commutative locally convex Hausdorff topological algebra with an ordered multiplicative web $\mathcal{C} = \{C_{k_1,\ldots,k_\nu} : k_1,\ldots,k_\nu,\nu \in \mathbb{N}\}$ and let $x, y \in A$ be stable. Then xy is stable.

Proof. Since x and y are stable, 4.6 yields that there are sequences $(k_\nu)_{\nu\in\mathbb{N}}$ and $(l_\nu)_{\nu\in\mathbb{N}}$ of natural numbers such that for all $n \in \mathbb{N}$ there exist $m_x, m_y \ge n$ such that for all natural numbers $k_x \ge m_x$ and $k_y \ge m_y$

$$x^{m_x}A \subseteq x^{k_x}A + \bigcap_{j=1}^{n} (M_x^{n-j})^{-1}(C_{k_j,\ldots,k_n})$$

and

$$y^{m_y}A \subseteq y^{k_y}A + \bigcap_{j=1}^{n} (M_y^{n-j})^{-1}(C_{l_j,\ldots,l_n}).$$

We now need to find a sequence of natural numbers that satisfies the condition in 4.6 for the product xy. To this end we choose a sequence $(r_\nu)_{\nu\in\mathbb{N}}$ such that $r_j \ge \max\{k_j, l_j\}$ for all $j \in \mathbb{N}$

and $x, y \in C_{r_1,\dots,r_n}$ for all $n \in \mathbb{N}$. As we will see later in this proof, the required sequence shall be increasing, so we consider the sequence $(\tilde{r}_\nu)_{\nu\in\mathbb{N}}$ defined by

$$\tilde{r}_j = \max\{r_1, \dots r_j\} \geq r_j$$

for all $j \in \mathbb{N}$. Since the web is ordered and $x, y \in C_{r_1,\dots,r_n} \subseteq C_{\tilde{r}_j,\dots,\tilde{r}_n}$, the inclusions above remain true if we replace the sets $C_{k_j,\dots,k_n}$ and $C_{l_j,\dots,l_n}$ by $C_{\tilde{r}_j,\dots,\tilde{r}_n}$. If we fix $n \in \mathbb{N}$ and set $m = \max\{m_x, m_y\} \geq n$, we obtain for an arbitrary $k \geq m$

$$x^m A \subseteq x^k A + \bigcap_{j=1}^{n} (M_x^{n-j})^{-1}(C_{\tilde{r}_j,\dots,\tilde{r}_n})$$

and

$$y^m A \subseteq y^k A + \bigcap_{j=1}^{n} (M_y^{n-j})^{-1}(C_{\tilde{r}_j,\dots,\tilde{r}_n}).$$

Thus we have

$$\begin{aligned}(xy)^m A = x^m y^m A &\subseteq x^m (y^k A + \bigcap_{j=1}^{n} (M_y^{n-j})^{-1}(C_{\tilde{r}_j,\dots,\tilde{r}_n}))\\ &= y^k x^m A + x^m \bigcap_{j=1}^{n} (M_y^{n-j})^{-1}(C_{\tilde{r}_j,\dots,\tilde{r}_n})\\ &\subseteq y^k x^k A + y^k \bigcap_{j=1}^{n} (M_x^{n-j})^{-1}(C_{\tilde{r}_j,\dots,\tilde{r}_n}) + x^m \bigcap_{j=1}^{n} (M_y^{n-j})^{-1}(C_{\tilde{r}_j,\dots,\tilde{r}_n})).\end{aligned}$$

We set $I_x = \bigcap_{j=1}^{n} (M_x^{n-j})^{-1}(C_{\tilde{r}_j,\dots,\tilde{r}_n})$ and $I_y = \bigcap_{j=1}^{n} (M_y^{n-j})^{-1}(C_{\tilde{r}_j,\dots,\tilde{r}_n}))$. If z is an element of I_x, we obtain for $1 \leq j \leq n$

$$x^{n-j} z \in C_{\tilde{r}_j,\dots,\tilde{r}_n}.$$

Hence

$$(xy)^{n-j} y^{k+j-n} z = y^k x^{n-j} z \in y^k C_{\tilde{r}_j,\dots,\tilde{r}_n}$$

and

$$(xy)^{n-j} y^k z \in y^{k+n-j} C_{\tilde{r}_j,\dots,\tilde{r}_n}.$$

Since $(\tilde{r}_\nu)_{\nu\in\mathbb{N}}$ is increasing we have

$$C_{\tilde{r}_1,\dots,\tilde{r}_n} \subseteq C_{\tilde{r}_j,\dots,\tilde{r}_n}$$

for $1 \leq j \leq n$. So we can conclude that

$$\begin{aligned}y C_{\tilde{r}_j,\dots,\tilde{r}_n} &\subseteq C_{r_1,\dots,r_n} \cdot C_{\tilde{r}_j,\dots,\tilde{r}_n} \subseteq C_{\tilde{r}_1,\dots,\tilde{r}_n} \cdot C_{\tilde{r}_j,\dots,\tilde{r}_n}\\ &\subseteq C_{\tilde{r}_j,\dots,\tilde{r}_n} \cdot C_{\tilde{r}_j,\dots,\tilde{r}_n} \subseteq \lambda_j C_{\tilde{r}_j,\dots,\tilde{r}_n},\end{aligned}$$

where the last inclusion holds for some $\lambda_j > 0$ since the web is multiplicative. Inductively we see that for $1 \leq j \leq n$

$$y^{k+n-j} C_{\tilde{r}_j,\dots,\tilde{r}_n} \subseteq \lambda_j^{k+n-j} C_{\tilde{r}_j,\dots,\tilde{r}_n}.$$

So we have shown that for $\lambda = \max_{1 \le j \le n} \lambda_j^{k+n-j}$

$$(xy)^{n-j} y^k z \in \lambda C_{\tilde{r}_j,\dots,\tilde{r}_n}$$

for all $1 \le j \le n$, which means that

$$y^k z \in \bigcap_{j=1}^{n} (M_{xy}^{n-j})^{-1}(\lambda C_{\tilde{r}_j,\dots,\tilde{r}_n}) = \lambda \bigcap_{j=1}^{n} (M_{xy}^{n-j})^{-1}(C_{\tilde{r}_j,\dots,\tilde{r}_n}).$$

Analogously we can show that for $z \in I_y$ and all $1 \le j \le n$ there exists $\mu > 0$ such that

$$x^m z \in \bigcap_{j=1}^{n} (M_{xy}^{n-j})^{-1}(\mu C_{\tilde{r}_j,\dots,\tilde{r}_n}) = \mu \bigcap_{j=1}^{n} (M_{xy}^{n-j})^{-1}(C_{\tilde{r}_j,\dots,\tilde{r}_n}).$$

With these computations we can now state that

$$(xy)^m A \subseteq (xy)^k A + \lambda \bigcap_{j=1}^{n} (M_{xy}^{n-j})^{-1}(C_{\tilde{r}_j,\dots,\tilde{r}_n}) + \mu \bigcap_{j=1}^{n} (M_{xy}^{n-j})^{-1}(C_{\tilde{r}_j,\dots,\tilde{r}_n}).$$

The set $C_{\tilde{r}_j,\dots,\tilde{r}_n}$ is convex, hence

$$\lambda C_{\tilde{r}_j,\dots,\tilde{r}_n} + \mu C_{\tilde{r}_j,\dots,\tilde{r}_n} = (\lambda + \mu) C_{\tilde{r}_j,\dots,\tilde{r}_n}.$$

Thus for some $z = \lambda a + \mu b \in \lambda \bigcap_{j=1}^{n} (M_{xy}^{n-j})^{-1}(C_{\tilde{r}_j,\dots,\tilde{r}_n}) + \mu \bigcap_{j=1}^{n} (M_{xy}^{n-j})^{-1}(C_{\tilde{r}_j,\dots,\tilde{r}_n})$ it is easy to check that

$$(xy)^{n-j} z \in (\lambda + \mu) C_{\tilde{r}_j,\dots,\tilde{r}_n}.$$

So we have

$$(xy)^m A \subseteq (xy)^k A + (\lambda + \mu) \bigcap_{j=1}^{n} (M_{xy}^{n-j})^{-1}(C_{\tilde{r}_j,\dots,\tilde{r}_n}).$$

Multiplying this inclusion with $\frac{1}{\lambda+\mu}$ we obtain

$$(xy)^m A \subseteq (xy)^k A + \bigcap_{j=1}^{n} (M_{xy}^{n-j})^{-1}(C_{\tilde{r}_j,\dots,\tilde{r}_n}),$$

hence 4.6 implies the stability of xy. □

Remark 4.10.
With 4.8 we can conclude that 4.9 generalizes 2.12 and 3.4.

In the following propositions we want to show that topological algebras with multiplicative webs have some notable hereditary properties. The following considerations are inspired by the work of Meise and Vogt in [16] and in most of the cases we will adopt the construction of the webs on the corresponding spaces.

Proposition 4.11.
Let A be a commutative locally convex topological algebra with an ordered multiplicative web $\mathcal{C} = \{C_{k_1,\dots,k_\nu} : k_1,\dots,k_\nu,\nu \in \mathbb{N}\}$ and let $F \subseteq A$ be a closed subalgebra of A. Then

$$\tilde{\mathcal{C}} = \left\{\tilde{C}_{k_1,\dots,k_\nu} = F \cap C_{k_1,\dots,k_\nu} : k_1,\dots,k_\nu,\nu \in \mathbb{N}\right\}$$

is an ordered and multiplicative web on F.

Proof. It is clear that the sets $F \cap C_{k_1,\dots,k_\nu}$ are absolutely convex and that

$$\bigcup_{n\in\mathbb{N}} \tilde{C}_n = \bigcup_{n\in\mathbb{N}} F \cap C_n = F \cap \bigcup_{n\in\mathbb{N}} C_n = F \cap A = F.$$

Furthermore we have for all $\nu \in \mathbb{N}$

$$\bigcup_{n\in\mathbb{N}} \tilde{C}_{k_1,\dots,k_\nu,n} = \bigcup_{n\in\mathbb{N}} F \cap C_{k_1,\dots,k_\nu,n} = F \cap \bigcup_{n\in\mathbb{N}} C_{k_1,\dots,k_\nu,n} = F \cap C_{k_1,\dots,k_\nu} = \tilde{C}_{k_1,\dots,k_\nu}.$$

If we consider a sequence $(k_\nu)_{\nu\in\mathbb{N}}$ of natural numbers and a sequence $(x_\nu)_{\nu\in\mathbb{N}} \in \prod_{\nu\in\mathbb{N}} \tilde{C}_{k_1,\dots,k_\nu}$ and if we observe that

$$\prod_{\nu\in\mathbb{N}} \tilde{C}_{k_1,\dots,k_\nu} = \prod_{\nu\in\mathbb{N}} F \cap C_{k_1,\dots,k_\nu} \subseteq \prod_{\nu\in\mathbb{N}} C_{k_1,\dots,k_\nu},$$

the fact that $\mathcal{C}$ is a web implies that $\sum_{\nu=1}^{\infty} \lambda_\nu x_\nu$ converges in A for $\lambda_\nu = \frac{1}{2^{v+1}}$. Since F is a closed subalgebra, $\sum_{\nu=1}^{\infty} \lambda_\nu x_\nu$ also converges in F. Thus $\tilde{\mathcal{C}}$ is a web that is obviously ordered because $\mathcal{C}$ is ordered. Hence it remains to prove that $\tilde{\mathcal{C}}$ is multiplicative. To this end let $\tilde{C}_{k_1,\dots,k_\nu} \in \tilde{\mathcal{C}}$. For

$$x, y \in \tilde{C}_{k_1,\dots,k_\nu} = F \cap C_{k_1,\dots,k_\nu}$$

we have $xy \in F$ since $F \subseteq A$ is a subalgebra and the fact that $\mathcal{C}$ is multiplicative implies that $xy \in \lambda C_{k_1,\dots,k_\nu}$ for some $\lambda > 0$. So xy belongs to $F \cap \lambda C_{k_1,\dots,k_\nu}$ and therefore we can conclude

$$\tilde{C}_{k_1,\dots,k_\nu} \cdot \tilde{C}_{k_1,\dots,k_\nu} \subseteq F \cap \lambda C_{k_1,\dots,k_\nu} = \lambda(F \cap C_{k_1,\dots,k_\nu}) = \lambda \tilde{C}_{k_1,\dots,k_\nu}.$$

□

Proposition 4.12.
Let A be a commutative locally convex topological algebra with an ordered multiplicative web $\mathcal{C} = \{C_{k_1,\dots,k_\nu} : k_1,\dots,k_\nu,\nu \in \mathbb{N}\}$ and let $I \subseteq A$ be a closed ideal. Then for the quotient algebra A/I equipped with the quotient topology an ordered and multiplicative web is defined by

$$\tilde{\mathcal{C}} = \left\{\tilde{C}_{k_1,\dots,k_\nu} = q(C_{k_1,\dots,k_\nu}) : k_1,\dots,k_\nu,\nu \in \mathbb{N}\right\},$$

where $q : A \to A/I$ is the quotient mapping.

Proof. Since q is linear, every $\tilde{C}_{k_1,\dots,k_\nu}$ is absolutely convex. Moreover it is clear that

$$\bigcup_{n\in\mathbb{N}} \tilde{C}_n = \bigcup_{n\in\mathbb{N}} q(C_n) = \bigcup_{n\in\mathbb{N}} \{x + I : x \in C_n\} = \{x + I : x \in A\} = A/I$$

and that for all $k_1, \dots, k_\nu, \nu \in \mathbb{N}$

$$\bigcup_{n\in\mathbb{N}} \tilde{C}_{k_1,\dots,k_\nu,n} = \bigcup_{n\in\mathbb{N}} q(C_{k_1,\dots,k_\nu,n}) = \bigcup_{n\in\mathbb{N}} \{x + I : x \in C_{k_1,\dots,k_\nu,n}\} = \{x + I : x \in C_{k_1,\dots,k_\nu}\}$$
$$= q(C_{k_1,\dots,k_\nu}) = \tilde{C}_{k_1,\dots,k_\nu}.$$

To show the last condition of 4.1 let $(k_\nu)_{\nu\in\mathbb{N}}$ be a sequence of natural numbers and $(x_\nu)_{\nu\in\mathbb{N}}$ an element of $\prod_{\nu\in\mathbb{N}} \tilde{C}_{k_1,\dots,k_\nu}$. We have

$$\prod_{\nu\in\mathbb{N}} \tilde{C}_{k_1,\dots,k_\nu} = \prod_{\nu\in\mathbb{N}} q(C_{k_1,\dots,k_\nu}) = \prod_{\nu\in\mathbb{N}} \{x + I : x \in C_{k_1,\dots,k_\nu}\}$$

and therefore we obtain for all $\nu \in \mathbb{N}$

$$x_\nu = y_\nu + I$$

for some $y_\nu \in C_{k_1,\dots,k_\nu}$, which means that $q(y_\nu) = x_\nu$. Since $\mathcal{C}$ is a web on A, the series $\sum_{\nu=1}^{\infty} \lambda_\nu y_\nu$ converges to some $y \in A$. Thus, using the linearity and the continuity of q, we have

$$\sum_{\nu=1}^{\infty} \lambda_\nu x_\nu = \sum_{\nu=1}^{\infty} q(\lambda_\nu y_\nu) = q(\sum_{\nu=1}^{\infty} \lambda_\nu y_\nu) = q(y).$$

The fact that $\mathcal{C}$ is ordered implies that the same holds for $\tilde{\mathcal{C}}$ and since the quotient mapping q and the web $\mathcal{C}$ are both multiplicative we have

$$\begin{aligned} \tilde{C}_{k_1,\dots,k_\nu} \cdot \tilde{C}_{k_1,\dots,k_\nu} &= q(C_{k_1,\dots,k_\nu}) \cdot q(C_{k_1,\dots,k_\nu}) \\ &= q(C_{k_1,\dots,k_\nu} \cdot C_{k_1,\dots,k_\nu}) \\ &\subseteq q(\lambda C_{k_1,\dots,k_\nu}) \\ &= \lambda \tilde{C}_{k_1,\dots,k_\nu} \end{aligned}$$

for some $\lambda > 0$ that only depends on the set $C_{k_1,\dots,k_\nu}$. □

In the following case of a countable product of topological algebras the definition of the web on the product differs from the one of Meise and Vogt. Using 4.3 we can give a slightly more accessible definition that will preserve the multiplicativity.

Proposition 4.13.
For all $j \in \mathbb{N}$ let A_j be a commutative locally convex topological algebra with an ordered multiplicative web $\mathcal{C}^j = \{C^j_{k_1,\dots,k_\nu} : k_1, \dots, k_\nu, \nu \in \mathbb{N}\}$. Then

$$\tilde{\mathcal{C}} = \left\{ \tilde{C}_{k_1,\dots,k_\nu} = C^1_{k_1,\dots,k_\nu} \times C^2_{k_2,\dots,k_\nu} \times \dots \times C^\nu_{k_\nu} \times \prod_{j>\nu} A_j : k_1, \dots, k_\nu, \nu \in \mathbb{N} \right\}$$

defines an ordered and multiplicative web on the product algebra $\prod_{j\in\mathbb{N}} A_j$ equipped with pointwise multiplication and the product topology.

Proof. Obviously every $\tilde{C}_{k_1,\dots,k_\nu}$ is absolutely convex and we have

$$\bigcup_{n\in\mathbb{N}} \tilde{C}_n = \bigcup_{n\in\mathbb{N}} (C_n^1 \times \prod_{j>1} A_j) = (\bigcup_{n\in\mathbb{N}} C_n^1) \times \prod_{j>1} A_j = \prod_{j\in\mathbb{N}} A_j.$$

Furthermore we have for all $\nu \in \mathbb{N}$

$$\begin{aligned}\bigcup_{n\in\mathbb{N}} \tilde{C}_{k_1,\dots,k_\nu,n} &= \bigcup_{n\in\mathbb{N}} (C^1_{k_1,\dots,k_\nu,n} \times C^2_{k_2,\dots,k_\nu,n} \times \cdots \times C_n^{\nu+1} \times \prod_{j>\nu+1} A_j) \\ &\subseteq C^1_{k_1,\dots,k_\nu} \times C^2_{k_2,\dots,k_\nu} \times \cdots \times C^\nu_{k_\nu} \times \prod_{j>\nu} A_j \\ &= \tilde{C}_{k_1,\dots,k_\nu}.\end{aligned}$$

To show the other inclusion let $x = (x_j)_{j\in\mathbb{N}} \in \tilde{C}_{k_1,\dots,k_\nu}$. Then $x_j \in C^j_{k_j,\dots,k_\nu}$ for all $j \le \nu$ and thus there exists a natural number n_j such that $x_j \in C^j_{k_j,\dots,k_\nu,n_j}$. So for $n = \max\limits_{1\le j\le\nu} n_j$ we have $x_j \in C^j_{k_j,\dots,k_\nu,n}$ for all $j \le \nu$ and hence

$$\tilde{C}_{k_1,\dots,k_\nu} \subseteq \bigcup_{n\in\mathbb{N}} \tilde{C}_{k_1,\dots,k_\nu,n}.$$

If $(k_\nu)_{\nu\in\mathbb{N}}$ is a sequence of natural numbers and $(x_\nu)_{\nu\in\mathbb{N}} \in \prod_{\nu\in\mathbb{N}} \tilde{C}_{k_1,\dots,k_\nu}$, then

$$\begin{aligned}&x_1 \in \tilde{C}_{k_1} = C^1_{k_1} \times \prod_{j>1} A_j \\ &x_2 \in \tilde{C}_{k_1,k_2} = C^1_{k_1,k_2} \times C^2_{k_2} \times \prod_{j>2} A_j \\ &\vdots \\ &x_\nu \in \tilde{C}_{k_1,\dots,k_\nu} = C^1_{k_1,\dots,k_\nu} \times \cdots \times C^\nu_{k_\nu} \times \prod_{j>\nu} A_j \\ &\vdots\end{aligned}$$

We now want to show that for $(\lambda_\nu)_{\nu\in\mathbb{N}}$ defined by $\lambda_\nu = \frac{1}{2^{\nu+1}}$ the series $\sum_{\nu=1}^\infty \lambda_\nu x_\nu$ converges in A or, equivalently, that its projection onto A_j converges in A_j for every $j \in \mathbb{N}$. Thus we consider an arbitrary $j \in \mathbb{N}$. Then we have

$$\begin{aligned}\sum_{\nu=1}^\infty \lambda_\nu x_\nu^j &= \sum_{\nu=1}^{j-1} \lambda_\nu x_\nu^j + \sum_{\nu=j}^\infty \lambda_\nu x_\nu^j \\ &= \sum_{\nu=1}^{j-1} \lambda_\nu x_\nu^j + \sum_{\nu=1}^\infty \lambda_\nu \lambda_{j-2} x^j_{\nu+j-1}.\end{aligned}$$

The first part of the sum is an element of A_j and $x^j_{\nu+j-1}$ belongs to the absolutely convex set $C^j_{k_j,\dots,k_{\nu+j-1}}$ for all $\nu \in \mathbb{N}$. Hence it is clear that $\lambda_{j-2} x_{\nu+j-1} \in C^j_{k_j,\dots,k_{\nu+j-1}}$ and we have

$$(\lambda_{j-2} x^j_{\nu+j-1})_{\nu\in\mathbb{N}} \in \prod_{\nu\in\mathbb{N}} C^j_{k_j,\dots,k_{\nu+j-1}}.$$

Thus the series $\sum_{\nu=1}^{\infty} \lambda_\nu \lambda_{j-2} x_{\nu+j-1}^j$ converges in A_j because $\mathcal{C}^j$ is a web, hence the same holds for the series $\sum_{\nu=1}^{\infty} \lambda_\nu x_\nu^j$. Since $j \in \mathbb{N}$ was chosen arbitrarily this yields the convergence of the series $\sum_{\nu=1}^{\infty} \lambda_\nu x_\nu$ in A. So we have shown that $\tilde{\mathcal{C}}$ is a web on $\prod_{j\in\mathbb{N}} A_j$ that is obviously ordered since every $\mathcal{C}^j$ is ordered and it remains to show that $\tilde{\mathcal{C}}$ is also multiplicative. To this end let $\tilde{C}_{k_1,\ldots,k_\nu}$ be an element of $\tilde{\mathcal{C}}$. Every web C^j is multiplicative, hence for every positive integer j there exists some $\lambda_j > 0$ such that

$$C_{k_1,\ldots,k_\nu}^j \cdot C_{k_1,\ldots,k_\nu}^j \subseteq \lambda_j C_{k_1,\ldots,k_\nu}^j.$$

So with $\lambda = \max\limits_{1\le j\le \nu} \lambda_j$ we can conclude that

$$\begin{aligned}
\tilde{C}_{k_1,\ldots,k_\nu} \cdot \tilde{C}_{k_1,\ldots,k_\nu} &= (C_{k_1,\ldots,k_\nu}^1 \times \cdots \times C_{k_\nu}^\nu \times \prod_{j>\nu} A_j) \cdot (C_{k_1,\ldots,k_\nu}^1 \times \cdots \times C_{k_\nu}^\nu \times \prod_{j>\nu} A_j) \\
&\subseteq \lambda_1 C_{k_1,\ldots,k_\nu}^1 \times \cdots \times \lambda_\nu C_{k_\nu}^\nu \times \prod_{j>\nu} A_j. \\
&\subseteq \lambda C_{k_1,\ldots,k_\nu}^1 \times \cdots \times \lambda C_{k_\nu}^\nu \times \prod_{j>\nu} A_j. \\
&= \lambda \tilde{C}_{k_1,\ldots,k_\nu}.
\end{aligned}$$

□

As a direct consequence of 4.11 and 4.13 we obtain:

Corollary 4.14.
For all $n \in \mathbb{N}$ let A_n be a commutative locally convex topological algebra with an ordered multiplicative web and let $\varrho_m^n : A_m \to A_n$ be linear and continuous for all $m, n \in \mathbb{N}$. Then the projective limit Proj$\mathcal{A}$ of the projective spectrum (A_n, ϱ_m^n) has an ordered multiplicative web.

Remark 4.15.
After presenting a result concerning projective limits of topological algebras with multiplicative webs it would be consequential again to analyze inductive limits of these topological algebras. The matter with this situation is that the inductive limit of topological algebras may fail to be a topological algebra again. For this reason it is not possible to present an analogous result to 4.14 for inductive limits. Nevertheless Jarchow showed in [15] that the Hausdorff inductive limit of webbed topological vector spaces is webbed again and we can transfer the previous hereditary property to the case of inductive limits of Fréchet algebras:
If $(A_n)_{n\in\mathbb{N}}$ is a sequence of Fréchet algebras with ordered multiplicative webs $\mathcal{C}^n = \{C_{k_1,\ldots,k_\nu}^n : k_1, \ldots, k_\nu, \nu \in \mathbb{N}\}$ such that every A_n is a subalgebra of A_{n+1} and every inclusion mapping $i_n : A_n \hookrightarrow A_{n+1}$ is continuous, let $A = \bigcup_{n\in\mathbb{N}} A_n$ be equipped with the finest locally convex topology such that each embedding $j_n : A_n \hookrightarrow A$ is continuous. Then A is called an LF-algebra and it is easy to verify that $\mathcal{C} = \{C_{k_1,\ldots,k_\nu} : k_1, \ldots, k_\nu, \nu \in \mathbb{N}\}$ defined by

$$C_{k_1,\ldots,k_\nu} = C_{k_2,\ldots,k_\nu}^{k_1}$$

is a multiplicative web on A. But in general this web is not ordered since it may occur that

$$C_n^m \nsubseteq C_n^{m+1}$$

for some $n, m \in \mathbb{N}$. Thus we have to modify the bases of zero neighborhoods of the Fréchet algebras A_n:
If $\mathcal{U}_1 = \{U_{1,k} : k \in \mathbb{N}\}$ is a decreasing basis of zero neighborhoods of in A_1 and $\mathcal{U}_2 = \{U_{2,k} : k \in \mathbb{N}\}$ a corresponding basis in A_2, we set

$$\tilde{\mathcal{U}}_2 = \{\tilde{U}_{2,k} : k \in \mathbb{N}\} = \{X_2, \dots, X_2, U_{2,1} \dots U_{2,1}, U_{2,2} \dots\},$$

where the length of the blocks have to be chosen in a way that for all $k \in \mathbb{N}$ we have $U_{1,k} \subseteq \tilde{U}_{2,k}$. Inductively we can form a basis of zero neighborhoods on every A_n such that $\tilde{U}_{n,k} \subseteq \tilde{U}_{n+1,k}$ for all $n, k \in \mathbb{N}$ and as a direct consequence we obtain that the web $\mathcal{C}$ is ordered. Hence every LF-algebra has an ordered multiplicative web.

Proposition 4.16.
For all $j \in \mathbb{N}$ let A_j be a commutative locally convex topological algebra with an ordered multiplicative web $\mathcal{C}^j = \{C^j_{k_1,\dots,k_\nu} : k_1, \dots, k_\nu, \nu \in \mathbb{N}\}$. Then the direct sum $\bigoplus_{j\in\mathbb{N}} A_j$ equipped with the pointwise multiplication and the topology defined by the basis of zero neighborhoods

$$\mathcal{U}_0^{\oplus} = \Big\{ \bigoplus_{j\in\mathbb{N}} U_j = \prod_{j\in\mathbb{N}} U_j \cap \bigoplus_{j\in\mathbb{N}} A_j : U_j \in U_0(A_j) \Big\}$$

is a locally convex topological algebra and an ordered and multiplicative web $\mathcal{C} = \{C^{\oplus}_{k_1,\dots,k_\nu} : k_1, \dots, k_\nu, \nu \in \mathbb{N}\}$ on $\bigoplus_{j\in\mathbb{N}} A_j$ is defined by the sets

$$C^{\oplus}_{k_1} = \bigoplus_{j=1}^{k_1} A_j = \prod_{j=1}^{k_1} A_j \times \prod_{j>k_1} \{0\}$$

for $\nu = 1$,

$$C^{\oplus}_{k_1,\dots,k_\nu} = C^1_{k_2,\dots,k_\nu} \times C^2_{k_3,\dots,k_\nu} \times \cdots \times C^{\nu-1}_{k_\nu} \times \prod_{j=\nu}^{k_1} A_j \times \prod_{j>k_1} \{0\}$$

for $1 < \nu \leq k_1$ and

$$C^{\oplus}_{k_1,\dots,k_\nu} = C^1_{k_2,\dots,k_\nu} \times C^2_{k_3,\dots,k_\nu} \times \cdots \times C^{k_1}_{k_{k_1+1},\dots,k_\nu} \times \prod_{j>k_1} \{0\}$$

for $\nu > k_1$.

Proof. Since every A_j is a topological algebra it is easy to check that for every $V = \bigoplus_{j\in\mathbb{N}} V_j \in \mathcal{U}_0^{\oplus}$ there exists some $U = \bigoplus_{j\in\mathbb{N}} U_j \in \mathcal{U}_0^{\oplus}$ such that $U \cdot U \subseteq V$. Thus the multiplication is jointly continuous in this topology and $\bigoplus_{j\in\mathbb{N}} A_j$ is a topological algebra. Again it is clear that the sets $C^{\oplus}_{k_1,\dots,k_\nu}$ are absolutely convex and that

$$\bigcup_{n\in\mathbb{N}} C^{\oplus}_n = \bigcup_{n\in\mathbb{N}} \bigoplus_{j=1}^{n} A_j = \bigoplus_{j\in\mathbb{N}} A_j.$$

Moreover, analogously to the considerations in the proof of 4.13, we can conclude that

$$\bigcup_{n\in\mathbb{N}} C^{\oplus}_{k_1,\ldots,k_\nu,n} = C^{\oplus}_{k_1,\ldots,k_\nu},$$

we only have to note that for $j > k_1$ the j-th component of the sets $C^{\oplus}_{k_1,\ldots,k_\nu,n}$ and $C^{\oplus}_{k_1,\ldots,k_\nu}$ is equal to $\{0\}$. If $(x_\nu)_{\nu\in\mathbb{N}}$ is an element of $\prod_{\nu\in\mathbb{N}} C^{\oplus}_{k_1,\ldots,k_\nu}$ for a sequence $(k_\nu)_{\nu\in\mathbb{N}}$ in $\mathbb{N}$, we have

$$
\begin{aligned}
&x_1 \in C^{\oplus}_{k_1} = \prod_{j=1}^{k_1} A_j \times \prod_{j>k_1} \{0\}\\
&x_2 \in C^{\oplus}_{k_1,k_2} = C^1_{k_2} \times \prod_{j=2}^{k_1} A_j \times \prod_{j>k_1} \{0\}\\
&\vdots\\
&x_{k_1} \in C^{\oplus}_{k_1,\ldots,k_{k_1}} = C^1_{k_2,\ldots,k_{k_1}} \times \cdots \times C^{k_{k_1}-1}_{k_{k_1}} \times A_{k_1} \times \prod_{j>k_1} \{0\}\\
&\vdots\\
&x_\nu \in C^{\oplus}_{k_1,\ldots,k_\nu} = C^1_{k_2,\ldots,k_\nu} \times \cdots \times C^{k_{k_1}}_{k_{k_1+1},\ldots,k_\nu} \times \prod_{j>k_1} \{0\}\\
&\vdots
\end{aligned}
$$

To show the convergence of $\sum_{\nu=1}^{\infty} \lambda_\nu x_\nu$ we consider again the projections $(x^j_m)_{m\in\mathbb{N}}$ of $(x_m)_{m\in\mathbb{N}}$ for $j \in \mathbb{N}$. If $j > k_1$, every x^j_m is equal to zero, hence the convergence is trivial. For $j \le k_1$ we can use the same method as in the proof of 4.13 to conclude that the series $\sum_{\nu=1}^{\infty} \lambda_\nu x^j_\nu$ converges in A_j and thus the series $\sum_{\nu=1}^{\infty} \lambda_\nu x_\nu$ converges in $\bigoplus_{j\in\mathbb{N}} A_j$. It is clear that $\mathcal{C}$ is ordered and for all $n \in \mathbb{N}$ we have

$$C^{\oplus}_k \cdot C^{\oplus}_k = \prod_{j=1}^{k} A_j \cdot \prod_{j=1}^{k} A_j \subseteq \prod_{j=1}^{k} A_j = C^{\oplus}_k.$$

If we choose some $C^{\oplus}_{k_1,\ldots,k_\nu} \in \mathcal{C}$ the multiplicativity of every web $\mathcal{C}^j$ yields again analogously to the proof of 4.13 some $\lambda > 0$ such that

$$C^{\oplus}_{k_1,\ldots,k_\nu} \cdot C^{\oplus}_{k_1,\ldots,k_\nu} \subseteq \lambda C^{\oplus}_{k_1,\ldots,k_\nu},$$

the only difference to 4.13 is that for $j > k_1$ in the j-th component of this implication we use the trivial fact that $\{0\} \cdot \{0\} = \{0\}$. □

Remark 4.17.
With 4.9 and the subsequent results we have shown that in all common locally convex topological algebras that occur in standard analysis the product of two stable elements is again stable. This includes for example projective limits of LB-algebras, the so-called PLB-algebras. One important PLB-algebra is the algebra of real analytic functions $A(\mathbb{R})$, which can be written as

$$A(\mathbb{R}) = \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} H^\infty((-n,n) \times i(-\frac{1}{m}, \frac{1}{m})),$$

where $H^\infty(U)$ is the Banach space of bounded holomorphic functions on an open set U. Although the webs in $A(\mathbb{R})$ look quite complicated, the resulting phenomena concerning stability are the same as for the algebra of holomorphic functions in 2.16, namely, only the invertible elements are stable. This can be shown as in 2.16 by means of the order of a zero of every non-invertible function.

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Zusammenfassung

Diese Arbeit ist von folgender Frage inspiriert worden, die der britische Mathematiker G. R. Allan in einem Artikel aus dem Jahre 1998 (vgl. [5, S. 94]) gestellt hat:

"Ist für zwei stabile Elemente x, y eines kommutativen Ringes R stets das Produkt xy wieder stabil [...]?"

Die Stabilität ist ein Begriff, der in vielen Bereichen in der Naturwissenschaft benutzt wird, und sogar in der Mathematik gibt es viele unterschiedliche Definitionen der Stabilität. Diese Definitionen haben üblicherweise gemeinsam, dass man die Lösungen eines Prozesses unter kleinen Veränderungen der Anfangsbedingungen untersucht.
In dieser Dissertation wird eine Form der Stabilität behandelt, die Allan in [5] eingeführt hat. Im Sinne von Allan ist ein Element x eines kommutativen Ringes R genau dann stabil, wenn man für jede Folge $(b_n)_{n\in\mathbb{N}}$ aus R eine Lösung $(a_n)_{n\in\mathbb{N}}$ für das folgende unendliche Gleichungssystem findet:

$$\begin{aligned} a_1 &= xa_2 + b_1 \\ a_2 &= xa_3 + b_2 \\ a_3 &= xa_4 + b_3 \\ &\vdots \end{aligned}$$

Hierbei beschreibt die Folge $(b_n)_{n\in\mathbb{N}}$ die oben erwähnte Störung.

Allans Motivation, sich mit dieser Art der Stabilität zu beschäftigen, hat seinen Ursprung in einem seiner Artikel [2] aus dem Jahre 1972. Dort untersuchte er die Möglichkeit, die Algebra der formalen Potenzreihen in eine Banachalgebra einzubetten. Dabei konnte er beweisen, dass für eine kommutative Banachalgebra mit Einselement die folgenden Aussagen äquivalent sind:

1. Es gibt einen Homomorphismus $\theta_x : \mathbb{C}[[X]] \to A$, so dass $\theta_x(X) = x$.
2. $x \in rad(A)$ und x ist stabil,

wobei $\mathbb{C}[[X]]$ die Algebra der formalen Potenzreihen mit komplexen Koeffizienten ist und $rad(A)$ das Radikal von A, d.h. der Schnitt aller maximalen Ideale in A. Natürlich hat Allan damals noch nicht den Begriff der Stabilität verwendet, allerdings beschrieb er eine Eigenschaft des Elementes x, die im Falle einer Banachalgebra äquivalent zur Stabilität ist (vgl. [5, Theorem 4.7]). Er konnte das obige Theorem nutzen, um die Existenz eines unstetigen Homomorphismus zwischen zwei Banachalgebren nachzuweisen. Außerdem untersuchte er weitere Konsequenzen daraus im Zusammenhang mit der Theorie der automatischen Stetigkeit in

[2] und [6]. Da er zudem ein analoges Resultat für Fréchet-Algebren beweisen konnte (vgl. [7, Theorem 19]), beinhalteten seine Untersuchungen ebenfalls jene Algebren. Somit könnte dieses Thema auch im Zusammenhang mit dem bekannten ungelösten Problem von Michael von Interesse sein. Dieses Problem behandelt die Frage, ob jeder Homomorphismus von einer Fréchet-Algebra in die Menge der komplexen Zahlen stetig ist.

Wie bereits im Zitat zu Beginn erwähnt wurde, war ein Aspekt der Arbeit Allans mit stabilen Elementen die Frage, ob das Produkt zweier stabiler Elemente wieder stabil ist. Der Urprung dieser Fragestellung wurde von Allan nicht beschrieben, wenngleich diese Frage an sich bereits interessant ist. Er konnte zeigen, dass sich im Falle kommutativer Banachalgebren und kommutativer Fréchet-Algebren die Stabilität tatsächlich auf das Produkt überträgt. Dies gelang ihm durch geeignete Charakterisierungen dieser Eigenschaft in den entsprechenden Algebren.

Das Hauptthema dieser Dissertation ist die Erweiterung der Ergebnisse Allans über stabile Elemente auf allgemeinere topologische Algebren. Dazu werden Methoden aus dem Gebiet der homologischen Algebra verwendet, welche von V. P. Palamodov in den 1960er und 1970er Jahren eingeführt wurden (vgl. [18] and [19]). Auf der einen Seite ermöglichen diese Methoden einen leichteren Zugang zu Allans Ergebnissen über stabile Elemente in Fréchet-Algebren, andererseits sind sie auch noch unter allgemeineren Voraussetzungen anwendbar.

Zwar führt auch dieser Ansatz nicht zu einer endgültigen Antwort auf die Ausgangsfrage der Stabilität von Produkten, d.h. entweder zu einem Beweis, dass das Produkt von stabilen Elementen stests stabil ist oder zu einem Gegenbeispiel eines kommutativen Ringes, indem es zwei stabile Elemente gibt, deren Produkt nicht stabil ist. Allerdings wird in der vorliegenden Arbeit gezeigt, dass in allen wichtigen Situationen, die man üblicherweise in der Funktionalanalysis betrachtet, die Antwort positiv ist.

Im ersten Kapitel der Dissertation werden Allans Definition der Stabilität und Palamodovs Arbeiten mit dem projektiven Limes Funktor $\mathrm{Proj}\mathcal{A}$ zusammengeführt. Dazu werden nach der Definition der Stabilität einige Folgerungen und Beispiele betrachtet und schließlich wird gezeigt, dass die Stabilität eines Elementes x äquivalent zu einer Bedingung an den abgeleiteten Funktor des projektiven Limes Funktors eines speziellen projektiven Spektrums $\mathcal{A}_x$ ist. Diese Bedingung bezeichnet man üblicherweise mit $\mathrm{Proj}^1\mathcal{A} = 0$. Dabei hängt das projektive Spektrum $\mathcal{A}_x$ von der Multiplikation M_x ab, die ein Element y auf das Produkt xy abbildet. Aufgrund der Äquivalenz ist es zweckdienlich, Resultate über den abgeleiteten Funktor anzuwenden, insbesondere wird eine notwendige Bedingung für $\mathrm{Proj}^1\mathcal{A} = 0$ gezeigt. Dabei werden die Betrachtungen in diesem Abschnitt der Arbeit rein algebraisch bleiben, d.h. ohne jede topologische Voraussetzung.

Danach wird die Stabilität unter weiteren topologischen Annahmen charakterisiert. Zu diesem Zweck wird aus einem sogenannten Mittag-Leffler-Lemma von R. Arens [9, Theorem 2.4] eine hinreichende Bedingung für $\mathrm{Proj}^1\mathcal{A} = 0$ im Kontext vollständiger metrisierbarer Gruppen abgeleitet. Dies führt dann zu den Ergbnissen Allans ohne die Anwendung der Theorie der

Einbettung der formalen Potenzreihen. Ferner erhält man als unmittelbare Konsequenz, dass in Fréchet-Algebren das Produkt stabiler Elemente stabil ist. Abschließend wird noch die Stabilität in einigen typischen Beispielen von Banach- und Fréchet-Algebren untersucht.

Das dritte Kapitel beinhaltet dann die Untersuchung von LB-Algebren. Dabei wird eine Charakterisierung der Bedingung $\mathrm{Proj}^1\mathcal{A} = 0$ von V. S. Retakh [20] und V. P. Palamodov [19, Theorem 5.4] verwendet, um eine geeignete Charakterisierung der Stabilität zu finden, von der man ableiten kann, dass wieder das Produkt zweier stabiler Elemente stabil ist. Zusätzlich werden die stabilen Elemente in einigen Beispielen von LB-Algebren bestimmt und zum Abschluss des Abschnittes wird ein weiteres Ergebnis von Allan für Fréchet-Algebren auf LB-Algebren übertragen. Dieses Ergebnis besagt, dass in einer LB-Algebra A für ein stabiles x aus A das Ideal $I(x) = \bigcap_{n\in\mathbb{N}} x^n A$ die Bedingung $M_x(I(x)) = I(x)$ erfüllt.

Im letzten Kapitel der Arbeit werden dann lokalkonvexe topologische Algebren mit Geweben betrachtet, die Verallgemeinerungen von Fréchet-Algebren und LB-Algebren darstellen. Diese topologischen Algebren wurden von M. de Wilde in [12] im Zusammenhang mit dem Satz vom abgeschlossenen Graphen eingeführt. Für solche Räume konnten L. Frerick, D. Kunkle und J. Wengenroth in [13] eine Charakterisierung der Bedingung $\mathrm{Proj}^1\mathcal{A} = 0$ beweisen. Unter einer weiteren Voraussetzung an die Gewebe wird dann das Hauptresultat der Dissertation formuliert:

In jeder kommutativen lokalkonvexen Hausdorffschen topologischen Algebra, die ein multiplikatives Gewebe besitzt, ist das Produkt stabiler Elemente stabil.

Zum Abschluß werden dann einige Vererbungseigenschaften dieser topologischen Algebren bewiesen, was zeigt, dass sich das Ergebnis von Allan auf eine große Klasse topologischer Algebren überträgt.