Surjectivity of augmented linear partial differential operators with constant coefficients and a conjecture of Trèves

Habilitationsschrift

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Contents

Introduction

The question of solvability of a linear partial differential equation with constant coefficients in some open set $X \subseteq \mathbb{R}^d$

$$P(D)u = f$$

is a classical problem. Depending on the properties of the right hand side f this problem leads in a natural way to the question of surjectivity of $P(D)$ on various spaces of functions and distributions. Malgrange [25] proved in 1955 that if the above equation has a distributional solution for every $f \in C^\infty(X)$ then there is always a solution $u \in C^\infty(X)$. Moreover, Malgrange was able to give a complete characterization of surjectivity of $P(D)$ on $C^\infty(X)$ by a certain kind of geometric condition involving the adjoint $P(-D)$ of $P(D)$. $P(D)$ is surjective on $C^\infty(X)$ if and only if X is P-convex for supports. Roughly speaking this means that for compactly supported $u \in C^\infty(X)$ the location of their support is determined by the location of the support of $P(-D)u$. As is well-known, in general surjectivity of the differential operator $P(D)$ on $C^\infty(X)$ does not imply surjectivity on $\mathscr{D}'(X)$ (see e.g. [13, Section 6]). In order to have surjectivity on $\mathscr{D}'(X)$ a second condition apart from P-convexity for supports has to be satisfied. As proved by Hörmander [13] in 1962, $P(D)$ is surjective on $\mathscr{D}'(X)$ if and only if X is P-convex for supports as well as P-convex for singular supports. The later means, roughly, that for compactly supported distributions u in X the location of their singular support is determined by the location of the singular support of $P(-D)u$. Despite the ingenious elegance of these characterizations, in concrete examples it can be rather involved to verify their validity.

Although P-convexity for supports does not imply P-convexity for singular supports in general, Trèves conjectured in [35, Problem 2, page 389] that in the special case of a differential operator in two independent variables, i.e. when $X \subseteq \mathbb{R}^2$, P-convexity for supports of $X \subseteq \mathbb{R}^2$ already implies P-convexity for singular supports of X, or equivalently surjectivity of $P(D)$ on $C^\infty(X)$ implies surjectivity on $\mathscr{D}'(X)$.

Another problem concerning the notion of P-convexity for singular supports considered in this treatise arises from the work of Bonet and Domański. In [5], Bonet and Domański investigated the parameter dependence problem for solutions of partial differential equations with constant coefficients, that is the problem whether for a given family (f_λ) of distributions on X depending "nicely" on a (real) parameter λ it is possible to solve

$$P(D)u_\lambda = f_\lambda$$

in such a way that the solution family (u_λ) depends on the parameter in the same way. Their results lead to the following problem [5, Problem 9.1]. Does surjectivity of $P(D)$ on $\mathscr{D}'(X)$ imply surjectivity of the augmented operator $P^+(D)$ on $\mathscr{D}'(X \times \mathbb{R})$, where $P^+(x_1, \ldots, x_{d+1}) = P(x_1, \ldots, x_d)$? The motivation of this problem will also be discussed in chapter 1. Clearly, one has to investigate whether P-convexity for supports and singular supports of X implies P^+-convexity for supports and singular supports of $X \times \mathbb{R}$. It will be shown in chapter 1 that convexity for supports does not cause any problem. P-convexity for supports of X always implies P^+-convexity for supports of $X \times \mathbb{R}$. So, as for Trèves' conjecture, the problem of Bonet and Domański leads again to the question if some open set, $X \times \mathbb{R}$ in this case, is convex for singular supports with respect to the polynomial P^+.

Motivated by these two problems, we will prove sufficient conditions for P-convexity for singular supports in chapter 2. Our main tool will be a deep result about the continuation of differentiability for distributional solutions of the differential equation $P(D)u = 0$ due to Hörmander [18, Section 11.3]. This result involves the zeros of a certain function σ_P defined on the non-trivial subspaces of $\mathbb{R}^d$. After a careful analysis of this function in section 2.1 we give sufficient conditions for P-convexity for singular supports in section 2.2. The most important one of these conditions for our purposes is an exterior cone condition for every boundary point of the set X under consideration. Although this sufficient condition is far from being necessary in general, we show in section 2.3 that for certain sets X the exterior cone condition in fact characterizes P-convexity for singular supports. Moreover, it will be shown in section 4.1 that in the two dimensional case the exterior cone condition characterizes P-convexity for singular supports no matter the geometry of the open (connected) set $X \subseteq \mathbb{R}^2$. Analogous conditions for P-convexity for supports will also be proved in chapter 2.

In chapter 3 we turn our attention to the problem of Bonet and Domański. In section 3.1 we first present some results showing that for special classes of differential operators (namely semi-elliptic differential operators and operators of principal type) the problem has a positive solution if the underlying set X satisfies certain geometrical properties. Furthermore, we give an alternative proof of a result due to Vogt [39] stating that the problem of Bonet and Domański has a positive solution for every elliptic operator. However, in section 3.2 we present an example of a surjective differential operator $P(D)$ on $\mathscr{D}'(X)$ for some open $X \subseteq \mathbb{R}^d$, for arbitrary $d \geq 3$, such that the augmented operator $P^+(D)$ is not surjective on $\mathscr{D}'(X \times \mathbb{R})$. Thus, we solve the problem of Bonet and Domański in the negative. Moreover, the differential operator in this example is even hypoelliptic so that it also answers a problem posed by Varol in [38, Section 3].

Additionally, it will be shown in section 4.2 that in the two dimensional case the problem of Bonet and Domański has a positive solution. But before we do so, we prove in the affirmative Trèves' conjecture in section 4.1. Moreover, using results due to Langenbruch [23] about the continuation of ultradifferentiability of ultradistributional solutions u of Beurling type of the differential equation $P(D)u = 0$ we prove an analogue result of Trèves' conjecture in the setting of ultradistributions of Beurling type $\mathscr{D}'_{(\omega)}(X)$ for non-quasianalytic weights ω in section 4.3. These spaces of distributions generalize classical distributions by allowing more flexible growth conditions for the Fourier transforms of the corresponding test functions than the classical Paley-Wiener weights. In particular, we prove that contrary to $d \geq 3$ in the case of $d = 2$ surjectivity of $P(D)$ on $\mathscr{D}'_{(\omega)}(X)$ does not depend on the specific weight function ω. These results of chapter 4 complement results of Zampieri [44,45] who proved that, again in contrast to the case $d \geq 3$, in two dimensions surjectivity of a differential operator $P(D)$ on $C^\infty(X)$ is equivalent to surjectivity on the space of real analytic functions $\mathscr{A}(X)$ on X. Contrary to $d \geq 3$, there is a geometrical characterization of surjectivity on $C^\infty(X)$ due to Hörmander for $X \subseteq \mathbb{R}^2$ [18, Theorem 10.8.3] which can easily be applied to concrete examples. Due to the equivalences of the surjectivity of a differential operator on the various spaces of functions and distributions mentioned above this characterization is now also applicable in all the previously mentioned settings.

Throughout this treatise we will use standard notation from the theory of linear partial differential operators and functional analysis. In particular, we denote by $\mathscr{E}(X)$ the space $C^\infty(X)$ equipped with its standard Fréchet space topology. For any notion not explained explicitly, see [17, 18] and/or [30].

It is a great pleasure for me to express my deep gratitude to Prof. Dr. Leonhard Frerick and to Prof. Dr. Jochen Wengenroth not only for turning my attention to the subject of this treatise but also for constant encouragement and helpful discussions for quite a long time. Moreover, I want to thank Prof. Dr. Michael Langenbruch for valueable hints concerning the literature and Prof. Dr. Dietmar Vogt for interesting and stimulating discussions.

1 The problem of surjectivity of augmented linear partial differential operators with constant coefficients

In this short chapter we will introduce one of the problems considered in this treatise. Let $P_1 \in \mathbb{C}[X_1 \ldots, X_{d_1}], P_2 \in \mathbb{C}[X_1, \ldots, X_{d_2}]$ be polynomials and $X \subseteq \mathbb{R}^{d_1}$ as well as $Y \subseteq \mathbb{R}^{d_2}$ be open sets such that the differential operators

$$P_1(D) : \mathscr{D}'(X) \to \mathscr{D}'(X),\ P_2(D) : \mathscr{D}'(Y) \to \mathscr{D}'(Y)$$

are both surjective. It is a natural question whether

$$P_1 \otimes P_2 : \mathscr{D}'(X \times Y) \to \mathscr{D}'(X \times Y)$$

is surjective again, where $P_1 \otimes P_2$ is the polynomial in $d_1 + d_2$ variables defined as $P_1 \otimes P_2(x, y) = P_1(x)P_2(y)$.

$\mathscr{D}'(X \times Y)$ and $\mathscr{D}'(X)\hat{\otimes}_\varepsilon\mathscr{D}'(Y)$, the complete ε-tensor product, are canonically isomorphic as locally convex spaces (see e.g. [36, Theorem 51.7]) and for this isomorphism we have

$$P_1 \otimes P_2(D) = (P_1(D) \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes P_2(D)),$$

where id_X and id_Y denote the identity operator on $\mathscr{D}'(X)$ and $\mathscr{D}'(Y)$, respectively. Obviously, $P_1(D) \otimes \mathrm{id}_Y$ and $\mathrm{id}_X \otimes P_2(D)$ commute so that the following proposition is trivial.

Proposition 1.1. *$P_1 \otimes P_2(D)$ is surjective on $\mathscr{D}'(X \times Y)$ if and only if both $P_1(D) \otimes id_Y$ and $id_X \otimes P_2(D)$ are surjective on $\mathscr{D}'(X \times Y)$.*

It turns out that as a model space for the above, it suffices to investigate the special case when $Y = \mathbb{R}$ which is a consequence of the next theorem proved independently by Valdivia [37] and Vogt [40].

Theorem 1.2. *For every open $X \subseteq \mathbb{R}^d$ we have $\mathscr{D}'(X) \cong (s')^{\mathbb{N}}$ as locally convex spaces, where s' denotes the strong dual of the nuclear Fréchet space of rapidly decreasing sequences. In particular, $\mathscr{D}'(X) \cong \mathscr{D}'(\mathbb{R})$.*

As the tensor product is commutative modulo canonical isomorphism, by the above $P_1 \otimes P_2(D)$ is surjective on $\mathscr{D}'(X \times Y)$ if and only if both operators $P_1(D) \otimes \mathrm{id}_\mathbb{R}$ and $P_2(D) \otimes \mathrm{id}_\mathbb{R}$ are surjective on $\mathscr{D}'(X \times \mathbb{R})$ and $\mathscr{D}'(Y \times \mathbb{R})$, respectively. Moreover, if we denote for a polynomial $P \in \mathbb{C}[X_1, \ldots, X_d]$ by P^+ the polynomial defined as

$$P^+(x_1, \ldots, x_{d+1}) := P(x_1, \ldots, x_d)$$

we have $P(D) \otimes \mathrm{id}_\mathbb{R} = P^+(D)$. We call the differential operator $P^+(D)$ the *augmented operator of* $P(D)$. With this notion, the original question leads in a natural way to the following problem posed by Bonet and Domański [5, Problem 9.1]:

Does surjectivity of a differential operator on $\mathscr{D}'(X)$ imply surjectivity of the augmented operator on $\mathscr{D}'(X \times \mathbb{R})$?

This question is also connected with the parameter dependence of solutions of the differential equation

$$P(D)u_\lambda = f_\lambda,$$

see [5]. By the above considerations a positive answer to the problem of Bonet and Domański also implies the surjectivity of the vector valued operators

$$P(D) : \mathscr{D}'(X, \mathscr{D}'(Y)) \to \mathscr{D}'(X, \mathscr{D}'(Y)),$$

where $Y \subseteq \mathbb{R}^{d'}$ is any non-empty open set.

There are various results about surjectivity of vector valued differential operators. Just to mention a few, surjectivity of $P(D) : \mathscr{D}'(X, F) \to \mathscr{D}'(X, F)$, where X is convex and F is a nuclear Fréchet space with the linear topological invariant (Ω) is considered by Bonet and Domański in [6, Section 5]. Examples for such F are nuclear power series spaces $\Lambda_r(\alpha)$, see [30, Section 29]. The case of F being a Banach space is treated in [4, Theorem 36]. Moreover, Domański investigates in [9] surjectivity of $P(D) : \mathscr{D}'(X, \mathscr{A}(U)) \to \mathscr{D}'(X, \mathscr{A}(U))$, where $\mathscr{A}(U)$ denotes the space of real analytic functions on a real analytic manifold U.

Clearly, the problem of Bonet and Domański has a positive solution when the operator $P(D)$ has a continuous linear right inverse R on $\mathscr{D}'(X)$ for then $R \otimes \mathrm{id}_{\mathbb{R}}$ is a continuous linear right inverse of $P^+(D)$ on $\mathscr{D}'(X \times \mathbb{R})$. The existence of such a right inverse has been characterized by Meise, Taylor, and Vogt in [28] and [29] via the existence of shifted fundamental solutions with additional properties. Moreover, as shown in [29], $P(D)$ has a continuous linear right inverse on $\mathscr{D}'(X)$ if and only if $P(D)$ has a continuous linear right inverse on $\mathscr{E}(X)$. It was already shown by Grothendieck (see e.g. [36, Theorem 52.4]) that elliptic operators $P(D)$ for $d \geq 2$ never possess such a right inverse. More generally, the same holds for hypoelliptic operators, as shown by Vogt [39], [41].

On the other hand, Bonet and Domański proved in [5, Proposition 8.3] that a positive solution of their problem is equivalent to the kernel of $P(D)$ in $\mathscr{D}'(X)$ having $(P\Omega)$, a linear topological invariant introduced by them in [5]. As is well-known, for hypoelliptic polynomials the kernel of $P(D)$ in $\mathscr{D}(X)$ and in $\mathscr{E}(X)$ coincide as locally convex spaces (this follows for example from [17, Theorem 4.4.2]), so that it is a nuclear Fréchet space and hence has property $(P\Omega)$ if and only if it has the linear topological invariant (Ω). It has already been shown by Vogt [39] that the kernel of any elliptic operator on $\mathscr{E}(X)$ has (Ω), where X is an arbitrary open subset of $\mathbb{R}^d$. Thus, in case of an elliptic operator the problem of Bonet and Domański has always a positive solution. Moreover, it is well-known that for convex open sets X every differential operator which is not identically zero is surjective on $\mathscr{D}'(X)$. Because $X \times \mathbb{R}$ is convex if this holds for X the problem of Bonet and Domański also has a positive solution for convex sets.

By a classical result due to Hörmander [13] $P(D)$ is surjective on $\mathscr{D}'(X)$ if and only if X is $P(D)$-convex for supports as well as for singular supports. Since these notions will be important throughout the whole text we recall their definition.

Definition 1.3. Let $X \subseteq \mathbb{R}^d$ be open and let $P \in \mathbb{C}[X_1, \ldots, X_d]$ be a polynomial.

i) X is called $P(D)$*-convex for supports* if to every compact set $K \subseteq X$ there is another compact set $L \subseteq X$ such that $u \in \mathscr{E}'(X)$ and $\operatorname{supp} P(-D)u \subseteq K$ implies $\operatorname{supp} u \subseteq L$.

ii) X is called $P(D)$*-convex for singular supports* if to every compact set $K \subseteq X$ there is another compact set $L \subseteq X$ such that $u \in \mathscr{E}'(X)$ and $\operatorname{sing\,supp} P(-D)u \subseteq K$ implies $\operatorname{sing\,supp} u \subseteq L$.

It is well-known that for $P(D)$-convexity for supports it is enough to consider $u \in \mathscr{D}(X)$ and that X is $P(D)$-convex for (singular) supports if and only if for every $u \in \mathscr{E}'(X)$ the distance of X^c to (sing)supp u coincides with the distance to (sing)supp $P(-D)u$ (see e.g. [18, Theorem 10.6.3 and Theorem 10.7.3]).

A different characterization without the notions of convexity for (singular) supports for the surjectivity of $P(D)$ on $\mathscr{D}'(X)$ in the spirit of [28], [29] via the existence of shifted fundamental solutions with additional properties was given only recently by Wengenroth [43]. Because his characterization of surjectivity seems difficult to apply in concrete situations, we will treat the problem of Bonet and Domański by using Hörmanders classical approach. Thus we are interested in whether $X \times \mathbb{R}$ is P^+-convex for supports as well as P^+-convex for singular supports in case of X being P-convex for supports and P-convex for singular supports. We will see immediately that P-convexity for supports of X is passed on to P^+-convexity for supports of $X \times \mathbb{R}$. Recall, that $P(D)$ is surjective on $\mathscr{E}(X)$ if and only if X is P-convex for supports, as was proved by Malgrange [25]. For two locally convex spaces E and F we denote their complete π-tensor product by $E\hat{\otimes}_\pi F$.

Proposition 1.4. *Let E and F be locally convex spaces, E complete, $F \neq \{0\}$. Moreover let $T : E \to F$ be continuous and linear.*

i) If $T\hat{\otimes}_\pi id_F : E\hat{\otimes}_\pi F \to E\hat{\otimes}_\pi F$ is surjective the same holds for T.

ii) If E and F are Fréchet spaces and T is surjective the same holds for $T\hat{\otimes}_\pi id_F$.

Proof. i) Let $y_0 \in F\backslash\{0\}$. We denote by $[y_0]$ the subspace of F generated by y_0 and by $P : F \to [y_0]$ the corresponding continuous projection. Clearly, via $\Psi(x) := x \otimes y_0$ a topological isomorphism from E onto $E \otimes_\pi [y_0]$ is defined, so that $E \otimes_\pi [y_0]$ is a closed subspace of $E\hat{\otimes}_\pi F$ by the completeness of E. Moreover, $E \otimes_\pi [y_0]$ is a complemented subspace of $E\hat{\otimes}_\pi F$ with continuous projection $\mathrm{id}_E\hat{\otimes}_\pi P$. Because the later commutes with the surjection $T\hat{\otimes}_\pi \mathrm{id}_F$ we obtain
$$T\hat{\otimes}_\pi \mathrm{id}_F(E \otimes_\pi [y_0]) = E \otimes_\pi [y_0].$$
As $\Psi^{-1} \circ (T\hat{\otimes}_\pi \mathrm{id}_F) \circ \Psi = T$ the surjectivity of T follows.

ii) is a well-known result about the π-tensor product (see e.g. [36, Proposition 43.9]). □

Using the nuclearity of $\mathscr{E}(X)$ (see e.g. [36, Corollary of Theorem 51.5], [30, Example 28.9], or [12, Corollaire of Théorème 3]) and $\mathscr{E}(X)\hat{\otimes}_\varepsilon\mathscr{E}(\mathbb{R}) \cong \mathscr{E}(X\times\mathbb{R})$ (see e.g. [36, Theorem 51.6] or [12, Section 5]) as well as the nuclearity of $\mathscr{D}'(X)$ (see e.g. [36, Corollary of Theorem 51.5] or [12, Corollaire of Théorème 3]) and $\mathscr{D}'(X)\hat{\otimes}_\varepsilon\mathscr{D}'(\mathbb{R}) \cong \mathscr{D}'(X \times \mathbb{R})$ proposition 1.4 immediately yields the following result.

Theorem 1.5. *Let $X \subseteq \mathbb{R}^d$ be open and $P \in \mathbb{C}[X_1, \ldots, X_d]$.*

i) $P(D)$ is surjective on $\mathscr{E}(X)$ if and only if $P^+(D)$ is surjective on $\mathscr{E}(X \times \mathbb{R})$.

ii) $P(D)$ is surjective on $\mathscr{D}'(X)$ if $P^+(D)$ is surjective on $\mathscr{D}'(X \times \mathbb{R})$.

Remark 1.6. It should be mentioned that proposition 1.4 immediately implies that $P_1 \otimes P_2(D)$ is surjective on $\mathscr{E}(X \times Y)$ if and only if both operators $P_1(D)$ and $P_2(D)$ are surjective on $\mathscr{E}(X)$ and $\mathscr{E}(Y)$, respectively.

In order to solve the problem of Bonet and Domański, some preparations have to be made, which will be presented in the following chapters. It will be shown in section 3.2 that contrary to theorem 1.5 i) in general surjectivity of $P(D)$ on $\mathscr{D}'(X)$ does not imply surjectivity of the augmented operator $P^+(D)$ on $\mathscr{D}'(X \times \mathbb{R})$ (see also example 3.5). So in general the problem has a negative solution. However, we will provide some special cases when the answer is positive (see theorem 3.16 and theorem A at the beginning of chapter 4).

2 Conditions for P-convexity

The main purpose of this chapter is to prove some sufficient conditions for P-convexity for singular supports of an open subset X of $\mathbb{R}^d$ in section 2.2. Moreover, we give some sufficient conditions for P-convexity for supports of X as well. In the third section of this chapter we characterize these properties for arbitrary polynomials and certain open subsets of $\mathbb{R}^d$ having a rather special geometric form. As a starting point for all this, in the first section we consider a result of Hörmander about the the continuation of differentiability from $P(D)u$ to u for distributions $u \in \mathscr{D}'(X)$ with $P(D)u = 0$. This result involves a certain function σ_P defined on the subspaces of $\mathbb{R}^d$ which we carefully analyse.

Apart from standard notation we use the following. For an affine subspace V of $\mathbb{R}^d$ we denote by $V^\perp$ the orthogonal space of the subspace parallel to V. In particular, for a hyperplane $H = \{x \in \mathbb{R}^d;\, \langle x, N\rangle = \alpha\}$ in $\mathbb{R}^d$, where $N \in \mathbb{R}^d\backslash\{0\}$ and $\alpha \in \mathbb{R}$ we have that $H^\perp$ is the one-dimensional subspace spanned by N. Moreover, for $x = (x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1}$ we set $x' = (x_1, \dots, x_d) \in \mathbb{R}^d$ and more generally, we write $M' = \{x';\, x \in M\}$ for a subset M of $\mathbb{R}^{d+1}$. This notation will be used throughout the whole text. Furthermore, a cone is always assumed to be non-empty.

2.1 Continuation of differentiability

As is well-known (see e.g. [18, Theorem 10.7.3]) X is P-convex for singular supports if and only if for every $u \in \mathscr{E}'(X)$ the distance of $\operatorname{sing\,supp} P(-D)u$ and $\operatorname{sing\,supp} u$ to X^c coincide. Obviously $\operatorname{sing\,supp} P(-D)u$ is always contained in $\operatorname{sing\,supp} u$ so the problem we consider is related to deriving bounds for $\operatorname{sing\,supp} u$ by knowledge of $\operatorname{sing\,supp} P(-D)u$. If E is a fundamental solution of $\check{P}$ (where as usual $\check{P}(x) = P(-x)$) we have $u = P(-D)u * E$ so that for the wave front set $WF(u)$ of u one has

$$WF(u) \subseteq \{(x + y, \xi);\, (x, \xi) \in WF(P(-D)u) \text{ and } (y, \xi) \in WF(E)\} \qquad (1)$$

(cf. [17, p. 270, Formula (8.2.16)]), where the wave front set of a distribution v is a subset of $\mathbb{R}^d \times S^{d-1}$ whose projection onto $\mathbb{R}^d$ is precisely $\operatorname{sing\,supp} v$. Therefore, knowledge about $WF(P(-D)u)$ as well as $WF(E)$ will allow to obtain bounds for $\operatorname{sing\,supp} u$.

For every polynomial P there is a specific fundamental solution $E(P)$ for which the location of its wave front set is well understood. This specific fundamental solution is given by

$$\forall\, \varphi \in \mathscr{D}(X) : \langle E(P), \varphi\rangle = (2\pi)^{-d} \int_{\mathbb{R}^d} d\xi \int_{\mathbb{C}^d} d\zeta\, \hat{\varphi}(-\xi - \zeta) \frac{\Phi(P_\xi, \zeta)}{P_\xi(\zeta)},$$

where $\hat{\varphi}$ denotes the Fourier transform of φ, $P_\xi(x) = P(\xi + x)$ and Φ is a certain function of polynomials Q and $\zeta \in \mathbb{C}^d$ such that $\Phi(Q, \zeta)$ vanishes "in a controlled manner" if $Q(\zeta) = 0$ (see [17, Section 7.3]).

The location of the wave front set of $E(P)$ is described by means of the so called localizations at infinity of P whose definition we want to recall. For a polynomial P and $\xi \in \mathbb{R}^d$ we set $P_\xi(\eta) = P(\eta + \xi)$. We denote by $L(P)$ the set of limits (in the unique Hausdorff linear topology on the space of polynomials of degree not exceeding $\deg P$, the degree of P) of the normalized polynomials

$$\eta \mapsto \frac{P_\xi(\eta)}{\tilde{P}_\xi(0)}$$

as ξ tends to infinity, where $\tilde{P}_\xi(0) = \sqrt{\sum_\alpha |P_\xi^{(\alpha)}(0)|^2}$. More precisely, if $N \in \mathbb{R}^d\backslash\{0\}$ then the set of limits where ξ tends to infinity and $\xi/|\xi| \to N/|N|$ is denoted by $L_N(P)$. Hence, $L_N(P) = L_{\alpha N}(P)$ for every $N \in \mathbb{R}^d\backslash\{0\}, \alpha > 0$, so that we can assume without loss of generality that $|N| = 1$. Obviously, $L(P)$ as well as $L_N(P)$ are closed subsets of the unit sphere of all polynomials in d variables of degree not exceeding the degree of P, equipped with the norm $Q \mapsto \tilde{Q}(0)$. The non-zero multiples of elements of $L(P)$ (resp. of $L_N(P)$) are called *localizations of P at infinity* (resp. *localizations of P at infinity in direction N*). Clearly, $Q \in L_N(\check{P})$ if and only if $\check{Q} \in L_{-N}(P)$.

We define for a polynomial Q

$$\Lambda(Q) = \{\eta \in \mathbb{R}^d; \forall\, \xi \in \mathbb{R}^d, \forall\, t \in \mathbb{R} : Q(\xi + t\eta) = Q(\xi)\},$$

which is obviously a subspace of $\mathbb{R}^d$. Moreover, denote by $\Lambda'(Q)$ the orthogonal space of $\Lambda(Q)$. Clearly, by an appropriate linear change of variables Q does only depend on k variables where $k = \dim \Lambda'(Q)$. In particular, Q is constant if and only if $\Lambda'(Q) = \{0\}$ and by an application of the Tarski-Seidenberg Theorem Hörmander proved $N \in \Lambda(Q)$ if $Q \in L_N(P)$ (see [18, Theorem 10.2.8]). Therefore, $\Lambda(Q) \neq \{0\}$ and hence $\Lambda'(Q) \neq \mathbb{R}^d$ for every $Q \in L(P)$.

By a result due to Hörmander (cf. [18, Theorem 10.2.11]) the wave front set $WF(E(\check{P}))$ of the above mentioned fundamental solution $E(\check{P})$ is contained in the closure of the set

$$\{(x, N) \in \mathbb{R}^d \times S^{d-1};\ x \in \Lambda'(Q) \text{ for some } Q \in L_N(\check{P})\}.$$

From this and equation (1) above it clearly follows that for $u \in \mathscr{E}'(X)$ the non-constant elements of $L(\check{P})$, or better the non-trivial subspaces $\Lambda'(Q)$, are the ones which may cause $\operatorname{sing\,supp} u$ to be much larger than $\operatorname{sing\,supp} P(-D)u$.

Define for a polynomial Q, a subspace V of $\mathbb{R}^d$, and $t \geq 1$

$$\tilde{Q}_V(\xi, t) = \sup\{|Q(\xi + \eta)|;\ \eta \in V, |\eta| \leq t\}$$

and

$$\tilde{Q}(\xi, t) = \tilde{Q}_{\mathbb{R}^d}(\xi, t).$$

Clearly, for every $\xi \in \mathbb{R}^d$ and $t \geq 1$ $\tilde{Q}_V(\xi, t)$ is a continuous seminorm, while $\tilde{Q}(\xi, t)$ is a continuous norm depending continuously on ξ and t. If $Q \in L(\check{P})$ is non-constant then

$$0 = \inf_{t \geq 1} \frac{\tilde{Q}_{\Lambda(Q)}(0, t)}{\tilde{Q}(0, t)}.$$

Moreover, since $Q \in L(\check{P})$ it follows that there is a sequence $(\xi_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^d$ tending to infinity such that $Q = \lim_{n \to \infty} \check{P}_{\xi_n}/\tilde{\check{P}}_{\xi_n}(0)$, hence

$$0 = \inf_{t \geq 1} \frac{\tilde{Q}_{\Lambda(Q)}(0, t)}{\tilde{Q}(0, t)} = \inf_{t \geq 1} \lim_{n \to \infty} \frac{\tilde{\check{P}}_{\Lambda(Q)}(\xi_n, t)}{\tilde{\check{P}}(\xi_n, t)}.$$

Defining for an arbitrary subspace V of $\mathbb{R}^d$

$$\sigma_{\check{P}}(V) = \inf_{t\geq 1} \liminf_{\xi\to\infty} \frac{\tilde{\check{P}}_V(\xi,t)}{\tilde{\check{P}}(\xi,t)},$$

it follows immediately that $\sigma_{\check{P}}(V) = \sigma_P(V)$. Moreover, for $y \in \mathbb{R}^d$ we shall simply write $\sigma_P(y)$ instead of $\sigma_P(span\{y\})$.

Remark 2.1. a) Clearly, if $V_1 \subseteq V_2$ are subspaces of $\mathbb{R}^d$ it follows from the definition that we have $\sigma_P(V_1) \leq \sigma_P(V_2)$.

b) Recall that a polynomial P is hypoelliptic if and only if all of its localizations at infinity are constant (cf. proof of [18, Theorem 11.1.11]). Therefore it follows that $\sigma_P(V) = 1$ for every subspace V of $\mathbb{R}^d$ if P is hypoelliptic. Moreover, observe that a polynomial P is hypoelliptic if and only if the polynomial $\check{P}(\xi) = P(-\xi)$ is hypoelliptic (this follows e.g. from [18, Theorem 11.1.11]) which together with [18, Corollary 11.3.3] and the above observations gives the equivalence of the following properties of a polynomial P.

i) Every open set $X \subseteq \mathbb{R}^d$ is P-convex for singular supports.

ii) P is hypoelliptic.

iii) $\sigma_P(V) \neq 0$ for every subspace V of $\mathbb{R}^d$.

iv) $\sigma_P(y) \neq 0$ for every $y \in \mathbb{R}^d\backslash\{0\}$.

c) Let $V, W \subseteq \mathbb{R}^d$ be two subspaces and

$$d(V,W) = \sup_{x\in V,|x|=1} (\inf_{y\in W,|y|=1} |x-y|).$$

Then

$$|\sigma_P(V) - \sigma_P(W)| \leq C \max\{d(V,W), d(W,V)\},$$

where $C > 0$ is a constant depending only on P (cf. [18, Section 11.3]). Since for unit vectors $N_1, N_2 \in S^{d-1}$ we have

$$d(\text{span}\{N_1\}, \text{span}\{N_2\}) \leq |N_1 - N_2|$$

it follows in particular that σ_P is a continuous function on the d-dimensional unit sphere S^{d-1}.

The function σ_P is much more powerful than simply indicating the existence of non-constant elements of $L(\check{P})$. The values of σ_P govern the possibility to continue differentiability of zero solutions of $P(D)$ across a hyperplane

$$H = \{x \in \mathbb{R}^d; \langle x, N\rangle = \alpha\}, N \in S^{d-1}, \alpha \in \mathbb{R}.$$

Let $X \subseteq \mathbb{R}^d$ be open, $x_0 \in X$ and $N \in S^{d-1}$ be such that $\sigma_P(N) \neq 0$. Then there is a neighborhood $U \subseteq X$ of x_0 such that $u \in \mathscr{E}(U)$ whenever $u \in \mathscr{D}'(X)$ with $P(D)u = 0$ as well as $u_{|X_-} \in \mathscr{E}(X_-)$, where $X_- = \{x \in X; \langle x, N\rangle < \langle x_0, N\rangle\}$. This is only a very special case of the following deep theorem due to Hörmander (cf. [18, Theorem 11.3.6]).

Theorem 2.2. *Let X be an open subset of $\mathbb{R}^d$, $x_0 \in X$ and let $\phi_1, \ldots, \phi_k \in C^1(X)$ be real valued functions such that $\nabla\phi_1(x_0), \ldots, \nabla\phi_k(x_0)$ are linearly independent. Assume that $\sigma_P(W) \neq 0$ for the linear space W spanned by the gradients $\nabla\phi_1(x_0), \ldots, \nabla\phi_k(x_0)$ and set*

$$X_- = \{x \in X;\ \phi_j(x) < \phi_j(x_0) \text{ for some } j = 1, \ldots, k\}.$$

If $u \in \mathscr{D}'(X), P(D)u \in \mathscr{E}(X)$ and $u \in \mathscr{E}(X_-)$ then $u \in \mathscr{E}(U)$ in a neighborhood U of x_0 which is independent of u.

Some kind of converse of the above theorem is also true. Again it is due to Hörmander (cf. [18, Theorem 11.3.1]).

Theorem 2.3. *Let V be a linear subspace of $\mathbb{R}^d$ such that $\sigma_P(V^\perp) = 0$. For every non-negative integer k one can find $u \in C^k(\mathbb{R}^d)$ with $P(D)u = 0$ and $\operatorname{sing\,supp} u = V$. More precisely, we can find u so that $u \neq C^{k+1}(U)$ for any open set U intersecting V.*

As a consequence of the previous two deep results one obtains the following corollary (see [18, Corollary 11.3.7]). This will be the starting point of deriving sufficient conditions for P-convexity for singular supports.

Corollary 2.4. *Let $X_1 \subseteq X_2$ be open convex sets in $\mathbb{R}^d$ and $P(D)$ a differential operator with constant coefficients. Then the following conditions are equivalent:*

i) Every solution $u \in \mathscr{D}'(X_2)$ of the equation $P(D)u = 0$ with $u|_{X_1} \in \mathscr{E}(X_1)$ already belongs to $\mathscr{E}(X_2)$.

ii) Every hyperplane H with $\sigma_P(H^\perp) = 0$ which intersects X_2 already intersects X_1.

One way we use $\sigma_P(V)$ is given by the following result which is a reformulation of Corollary 2.4 from [10] more suitable for our aims.

Proposition 2.5. *Let $X_1 \subseteq X_2$ be open and convex, and let P be a non-constant polynomial. Then the following are equivalent:*

i) Every $u \in \mathscr{D}'(X_2)$ satisfying $P(D)u \in \mathscr{E}(X_2)$ as well as $u|_{X_1} \in \mathscr{E}(X_1)$ already belongs to $\mathscr{E}(X_2)$.

ii) Every hyperplane H with $\sigma_P(H^\perp) = 0$ which intersects X_2 already intersects X_1.

Proof. That i) implies ii) is just a special case of Corollary 2.4. Let $u \in \mathscr{D}'(X_2)$ satisfy $P(D)u \in \mathscr{E}(X_2)$ as well as $u|_{X_1} \in \mathscr{E}(X_1)$. By the convexity of X_2 we find $v \in \mathscr{E}(X_2)$ such that $P(D)v = P(D)u$. Therefore $w := u - v \in \mathscr{D}'(X_2)$ satisfies $P(D)w = 0$ and $w|_{X_1} \in \mathscr{E}(X_1)$. Now if ii) holds it follows from Corollary 2.4 that $w \in \mathscr{E}(X_2)$, thus $u \in \mathscr{E}(X_2)$. □

In order to apply the above proposition it is crucial to know the zeros of σ_P on S^{d-1}, or more generally, to identify the subspaces $V \subseteq \mathbb{R}^d$ with $\sigma_P(V) = 0$. Since the very definition of σ_P seems not very appropriate to investigate this question we give a different representation of σ_P. At the beginning of this section we have already indicated the connection between the localizations of P at infinity and the function σ_P. The next lemma strengthens this connection.

Its usefulness will be shown in chapter 3 when dealing with a problem posed by Bonet and Domański as well as in chapter 4 where we will prove a conjecture by Trèves.

Lemma 2.6. *Let P be of degree m, $P=\sum_{j=0}^m P_j$ with P_j being a homogeneous polynomial of degree j, P_m the principal part of P.*

i) For every subspace V of $\mathbb{R}^d$ and $t\geq 1$ we have

$$\liminf_{\xi\to\infty}\frac{\tilde{P}_V(\xi,t)}{\tilde{P}(\xi,t)}=\inf_{Q\in L(P)}\frac{\tilde{Q}_V(0,t)}{\tilde{Q}(0,t)}.$$

ii) Let $N\in S^{d-1}$ and $Q\in L_N(P)$. If $P_m(N)\neq 0$ then Q is constant.

iii) If P is non-elliptic then for every subspace V of $\mathbb{R}^d$ and $t\geq 1$ we have

$$\liminf_{\xi\to\infty}\frac{\tilde{P}_V(\xi,t)}{\tilde{P}(\xi,t)}=\inf_{N\in S^{d-1},P_m(N)=0}\ \inf_{Q\in L_N(P)}\frac{\tilde{Q}_V(0,t)}{\tilde{Q}(0,t)}.$$

iv) With the convention that the infimum taken over an empty subset of $[0,1]$ equals 1 we have

$$\sigma_P(V)=\inf_{t\geq 1}\ \inf_{N\in S^{d-1},P_m(N)=0}\ \inf_{Q\in L_N(P)}\frac{\tilde{Q}_V(0,t)}{\tilde{Q}(0,t)}.$$

Proof. i) Since for every subspace V and each $t\geq 1$ the maps $R\mapsto\tilde{R}_V(0,t)$ are continuous seminorms on the space of all polynomials R in d variables and because $\tilde{P}_V(\xi,t)=(\tilde{P_\xi})_V(0,t)$ it follows immediately from the definition that

$$\frac{\tilde{Q}_V(0,t)}{\tilde{Q}(0,t)}\geq\liminf_{\xi\to\infty}\frac{\tilde{P}_V(\xi,t)}{\tilde{P}(\xi,t)}$$

for every $Q\in L(P)$.

Moreover, if $(\xi_n)_{n\in\mathbb{N}}$ tends to infinity such that

$$\liminf_{\xi\to\infty}\frac{\tilde{P}_V(\xi,t)}{\tilde{P}(\xi,t)}=\lim_{n\to\infty}\frac{\tilde{P}_V(\xi_n,t)}{\tilde{P}(\xi_n,t)}=\lim_{n\to\infty}\frac{(\tilde{P}_{\xi_n})_V(0,t)}{\tilde{P}_{\xi_n}(0,t)}$$

we can extract a subsequence of $(\xi_n)_{n\in\mathbb{N}}$ which we again denote by $(\xi_n)_{n\in\mathbb{N}}$ such that the sequence of normalized polynomials $P_{\xi_n}/\tilde{P}_{\xi_n}(0)$ converges in the compact unit sphere of all polynomials in d variables of degree at most m. This limit belongs to $L(P)$ and we get

$$\liminf_{\xi\to\infty}\frac{\tilde{P}_V(\xi,t)}{\tilde{P}(\xi,t)}\geq\inf_{Q\in L(P)}\frac{\tilde{Q}_V(0,t)}{\tilde{Q}(0,t)}$$

completing the proof of i).

The proof of ii) is a consequence of Taylor's formula. For $\xi_n\in\mathbb{R}^d$ we have

$$\begin{aligned}P_{\xi_n}(x)&=\sum_{0\leq|\alpha|\leq j\leq m}\frac{P_j^{(\alpha)}(\xi_n)}{\alpha!}x^\alpha\\&=|\xi_n|^m\left(\sum_{0\leq j\leq m}\frac{|\xi_n|^j}{|\xi_n|^m}P_j(\frac{\xi_n}{|\xi_n|})+\sum_{0<|\alpha|\leq j\leq m}\frac{|\xi_n|^{j-|\alpha|}}{|\xi_n|^m\alpha!}P_j^{(\alpha)}(\frac{\xi_n}{|\xi_n|})x^\alpha\right).\end{aligned}$$

Moreover

$$\begin{aligned}\tilde{P}_{\xi_n}(0) &= \sqrt{\sum_{0\le|\alpha|\le m} |\sum_{j=|\alpha|}^{m} P_j^{(\alpha)}(\xi_n)|^2} \\ &= |\xi_n|^m \sqrt{|\sum_{j=0}^{m} P_j(\frac{\xi_n}{|\xi_n|})\frac{|\xi_n|^j}{|\xi_n|^m}|^2 + \sum_{0<|\alpha|\le m} |\sum_{j=|\alpha|}^{m} P_j^{(\alpha)}(\frac{\xi_n}{|\xi_n|})\frac{|\xi_n|^{j-|\alpha|}}{|\xi_n|^m}|^2},\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \frac{P_{\xi_n}(x)}{\tilde{P}_{\xi_n}(0)} = \frac{P_m(N)}{|P_m(N)|}$$

for all $x \in \mathbb{R}^d$ if $\lim_{n\to\infty} \frac{\xi_n}{|\xi_n|} = N$. This proves ii).

iii) is an immediate consequence of i), ii), and $\liminf_{\xi\to\infty} \tilde{P}_V(\xi,t)/\tilde{P}(\xi,t) \le 1$ while iv) follows directly from iii). □

When treating the problem of Bonet and Domański from chapter 1 we will be interested in the P^+-convexity for singular supports of $X\times\mathbb{R}$. Of course, one could simply use the function σ_{P^+} ignoring the fact that P^+ does not depend on the last variable as well as the special geometric form of $X\times\mathbb{R}$. Instead of considering σ_{P^+} it will turn out to be more convenient to consider for a subspace V of $\mathbb{R}^d$

$$\sigma_P^0(V) := \inf_{t>1,\xi\in\mathbb{R}^d} \tilde{P}_V(\xi,t)/\tilde{P}(\xi,t).$$

This function has already been used by Hörmander in [15, Section 5] to discuss "Hölder estimates" for solutions of partial differential equations. The reason for considering this quantity here is given by the following lemma from [10] which reveals a first connection between σ_{P^+} and σ_P^0. Again we write $\sigma_P^0(y)$ instead of $\sigma_P^0(span\{y\})$ for simplicity.

Lemma 2.7. *Let $P \in \mathbb{C}[X_1,\dots,X_d]$ and let $\pi_d : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}, \pi_d(x) = (x_1,\dots,x_d,0)$. For a subspace W of $\mathbb{R}^{d+1}$ we identify the subspace W' of $\mathbb{R}^d$ and $\pi_d(W)$. Then the following hold.*

i) $\sigma_{P^+}(W'\times\{0\}) = \sigma_{P^+}(W'\times\mathbb{R}) = \sigma_P^0(W')$.

ii) $\sigma_{P^+}(W) = 0$ *if and only if* $\sigma_P^0(W') = 0$.

Proof. By definition we have for $(\xi,\eta)\in\mathbb{R}^d\times\mathbb{R}$

$$\begin{aligned}\tilde{P}^+_{W'\times\mathbb{R}}((\xi,\eta),t) &= \sup\{|P(\xi+x')|;\ (x',x_{d+1})\in W'\times\mathbb{R}, |(x',x_{d+1})|\le t\} \\ &= \sup\{|P(\xi+x')|;\ x'\in W', |x'|\le t\} \\ &= \tilde{P}_{W'}(\xi,t) = \tilde{P}^+_{W'\times\{0\}}((\xi,\eta),t).\end{aligned}$$

In particular, this implies

$$\tilde{P}^+((\xi,\eta),t) = \tilde{P}(\xi,t).$$

Hence

$$\begin{aligned}\liminf_{(\xi,\eta)\to\infty}\frac{\tilde{P}^+_{W'\times\mathbb{R}}((\xi,\eta),t)}{\tilde{P}^+((\xi,\eta),t)} &= \sup_{r>0}\inf_{|(\xi,\eta)|>r}\frac{\tilde{P}^+_{W'\times\mathbb{R}}((\xi,\eta),t)}{\tilde{P}^+((\xi,\eta),t)}\\ &= \sup_{r>0}\inf_{|(\xi,\eta)|>r}\frac{\tilde{P}_{W'}(\xi,t)}{\tilde{P}(\xi,t)}\\ &= \inf_{\xi\in\mathbb{R}^d}\frac{\tilde{P}_{W'}(\xi,t)}{\tilde{P}(\xi,t)}\end{aligned}$$

as well as

$$\liminf_{(\xi,\eta)\to\infty}\frac{\tilde{P}^+_{W'\times\{0\}}((\xi,\eta),t)}{\tilde{P}^+((\xi,\eta),t)} = \inf_{\xi\in\mathbb{R}^d}\frac{\tilde{P}_{W'}(\xi,t)}{\tilde{P}(\xi,t)}$$

which gives

$$\sigma_{P^+}(W'\times\mathbb{R}) = \inf_{t>1}\liminf_{(\xi,\eta)\to\infty}\frac{\tilde{P}^+_{W'\times\mathbb{R}}((\xi,\eta),t)}{\tilde{P}^+((\xi,\eta),t)} = \sigma^0_P(W'),$$

as well as

$$\sigma_{P^+}(W'\times\{0\}) = \sigma^0_P(W').$$

Thus i) is proved.

In order to prove ii) assume first that W is contained in the kernel of π_d, i.e. $W\subseteq\{0\}\times\mathbb{R}$. Then we have for $(\xi,\eta)\in\mathbb{R}^d\times\mathbb{R}$

$$\tilde{P}^+_W((\xi,\eta),t) = \sup\{|P(\xi)|;(0,x_{d+1})\in W,|x_{d+1}|\leq t\} = |P(\xi)| = \tilde{P}_{W'}(\xi,t).$$

As in the proof of i) it then follows that

$$\sigma_{P^+}(W) = \inf_{t>1,\xi\in\mathbb{R}^d}\frac{\tilde{P}_{W'}(\xi,t)}{\tilde{P}(\xi,t)} = \sigma^0_P(W').$$

Hence, without loss of generality, let $W\not\subseteq\{0\}\times\mathbb{R}$. Then, by setting $p_1:=\|\Pi_{|W}\|$ we get $p_1>0$ as well as

$$\begin{aligned}\tilde{P}^+_W((\xi,\eta),t) &= \sup\{|P(\xi+x')|;\,(x',x_{d+1})\in W,|(x',x_{d+1})|\leq t\}\\ &\leq \sup\{|P(\xi+x')|;\,x'\in W',|x'|\leq tp_1\}\\ &= \tilde{P}_{W'}(\xi,tp_1).\end{aligned}$$

Now we distinguish two cases. If $\pi_{d|W}:W\to W'$ is not injective we clearly have $\{(0,y);\,y\in\mathbb{R}\}\subseteq W$. Therefore, recalling that π_d as an orthogonal projection satisfies $p_1=\|\pi_{d|W}\|\leq\|\pi_d\|\leq 1$

$$\sup\{|P(\xi+x')|;x'\in W',|x'|\leq tp_1\} = \sup\{|P(\xi+x')|;(x',x_{d+1})\in W,|(x',x_{d+1})|\leq t\}$$

because if $x'_0\in W'$ with $|x'_0|\leq tp_1$ is a point where the supremum on the left hand side is attained then $(x'_0,0)\in W$ with $|(x'_0,0)|\leq t$. Therefore

$$\tilde{P}_{W'}(\xi,tp_1) = \tilde{P}^+_W((\xi,\eta),t).$$

In case of $\pi_{d|W}: W \to W'$ being injective $(\pi_{d|W})^{-1}: W' \to W$ is well-defined and continuous and we get

$$\begin{aligned}\tilde{P}_{W'}(\xi, t\,\|(\Pi_{|W})^{-1}\|^{-1}) &= \sup\{|P(\xi + x')|;\ x' \in W', |x'| \leq t\,\|(\pi_{d|W})^{-1}\|^{-1}\}\\ &\leq \sup\{|P(\xi + x')|;\ (x', x_{d+1}) \in W, |(x', x_{d+1})| \leq t\}\\ &= \tilde{P}^{+}_{W}((\xi, \eta), t).\end{aligned}$$

Hence, in both cases there are $p_1, p_2 > 0$ such that

$$\tilde{P}_{W'}(\xi, tp_2) \leq \tilde{P}^{+}_{W}((\xi, \eta), t) \leq \tilde{P}_{W'}(\xi, tp_1)$$

for all $\xi \in \mathbb{R}^d, \eta \in \mathbb{R}, t \geq 1$. Altogether this yields

$$\inf_{\xi \in \mathbb{R}^d} \frac{\tilde{P}_{W'}(\xi, tp_2)}{\tilde{P}(\xi, t)} \leq \liminf_{(\xi,\eta)\to\infty} \frac{\tilde{P}^{+}_{W}((\xi, \eta), t)}{\tilde{P}^{+}((\xi, \eta), t)} \leq \inf_{\xi \in \mathbb{R}^d} \frac{\tilde{P}_{W'}(\xi, tp_1)}{\tilde{P}(\xi, t)},$$

so that

$$\inf_{t \geq 1, \xi \in \mathbb{R}^d} \frac{\tilde{P}_{W'}(\xi, tp_2)}{\tilde{P}(\xi, t)} \leq \sigma_{P^+}(W) \leq \inf_{t \geq 1, \xi \in \mathbb{R}^d} \frac{\tilde{P}_{W'}(\xi, tp_1)}{\tilde{P}(\xi, t)}. \tag{2}$$

Now, as on the finite dimensional vector space

$$\{Q_{|W'}; Q \in \mathbb{C}[X_1, \ldots, X_d], \deg Q \leq \deg P\}$$

all norms are equivalent, there are $C_j > 0, j = 1, 2$, such that for every $Q \in \mathbb{C}[X_1, \ldots, X_d]$ with $\deg Q \leq \deg P$ we have for $j = 1, 2$

$$1/C_j \sup_{x' \in W', |x'| \leq p_j} |Q(x')| \leq \sup_{x' \in W', |x'| \leq 1} |Q(x')| \leq C_j \sup_{x' \in W', |x'| \leq p_j} |Q(x')|.$$

Since for arbitrary $\xi \in \mathbb{R}^d$, and $t > 1$ the degree of the polynomial $y \mapsto P(\xi + ty)$ equals that of P it follows that for $j = 1, 2$

$$1/C_j \frac{\tilde{P}_{W'}(\xi, tp_j)}{\tilde{P}(\xi, t)} \leq \frac{\tilde{P}_{W'}(\xi, t)}{\tilde{P}(\xi, t)} \leq C_j \frac{\tilde{P}_{W'}(\xi, tp_j)}{\tilde{P}(\xi, t)} \tag{3}$$

for all $\xi \in \mathbb{R}^d$ and $t > 1$. Now ii) follows from the inequalities (2) and (3) completing the proof of the lemma. □

Lemma 2.7 will be used in the next two sections to give sufficient conditions, respectively to give a characterization for certain sets X, of the P^+-convexity for singular supports of $X \times \mathbb{R}$ in terms of P and X. As an application of these results it will be shown in example 3.5 that for the hypoelliptic polynomial in two variables inducing the heat operator $P(\xi_1, \xi_2) = i\xi_1 + \xi_2^2$ there are open subsets $X \subseteq \mathbb{R}^2$ such that $X \times \mathbb{R}$ is not P^+-convex for singular supports. As P is hypoelliptic this set X is P-convex for singular supports. However, the set X in this example is not P-convex for supports.

2.2 Sufficient conditions for P-convexity

In this section we will mainly give sufficient conditions for P-convexity for singular supports. Moreover, we also present sufficient conditions for P-convexity for supports. Part i) of theorem 2.9, a sufficient condition for P-convexity for supports, is originally due to Tervo [34, Theorem 4.3]. We give a simplified and more transparent proof here. Moreover, our proof has the advantage that it can easily be modified in such a way as to give a similar sufficient conditions for P-convexity for singular supports. We denote by $B(0,r)$ the open ball about the origin with radius $r > 0$. Recall that a hyperplane

$$H = \{x \in \mathbb{R}^d; \langle x, N\rangle = \alpha\}$$

with $N \in S^{d-1}, \alpha \in \mathbb{R}$ is called *characteristic* for a polynomial P if $P_m(N) = 0$, where P_m is the principal part of P. The next well-known theorem will be used several times in this section so we state it here for the reader's convenience (see e.g. [17, Theorem 8.6.8]).

Theorem 2.8. *Let X_1 and X_2 be open convex sets in $\mathbb{R}^d$ such that $X_1 \subseteq X_2$ and let $P \in \mathbb{C}[X_1, \ldots, X_d]$ be non-constant. Then the following are equivalent.*

i) Every $u \in \mathscr{D}'(X_2)$ with $P(D)u = 0$ and $u_{|X_1} = 0$ vanishes in X_2.

ii) Every characteristic hyperplane for P which intersects X_2 already intersects X_1.

It should be noted that in part iii) of the following theorem the P^+-convexity for singular supports of $X \times \mathbb{R}$ is derived from properties of P and X rather than from properties of P^+ and $X \times \mathbb{R}$.

Theorem 2.9. *Let X be an open, connected subset of $\mathbb{R}^d$ and let $P \in \mathbb{C}[X_1, \ldots, X_d]$ be a non-constant polynomial with principal part P_m.*

i) X is P-convex for supports if for every $x \in \partial X$ and every $r > 0$ there are convex sets $C_1 \subseteq C_2 \subseteq \mathbb{R}^d \backslash X$ such that $x \in C_2, C_1 \subseteq \mathbb{R}^d \backslash B(0,r)$, and every characteristic hyperplane for P which intersects C_2 also intersects C_1.

ii) X is P-convex for singular supports if for every $x \in \partial X$ and every $r > 0$ there are convex sets $C_1 \subseteq C_2 \subseteq \mathbb{R}^d \backslash X$ such that $x \in C_2, C_1 \subseteq \mathbb{R}^d \backslash B(0,r)$, and every hyperplane H with $\sigma_P(H^\perp) = 0$ intersecting C_2 already intersects C_1.

iii) $X \times \mathbb{R}$ is P^+-convex for singular supports if for every $x \in \partial X$ and every $r > 0$ there are convex sets $C_1 \subseteq C_2 \subseteq \mathbb{R}^d \backslash X$ such that $x \in C_2, C_1 \subseteq \mathbb{R}^d \backslash B(0,r)$, and every hyperplane H with $\sigma_P^0(H^\perp) = 0$ intersecting C_2 already intersects C_1.

Proof. In order to prove i) we fix $u \in \mathscr{E}'(X)$ and set $K := \operatorname{supp} P(-D)u$ and $\delta := \operatorname{dist}(K, X^c)$. Moreover, set $L := \operatorname{supp} u$ and let $r > \delta$ be such that $L \subseteq B(0, r - \delta)$. Let $x \in \partial X$ be arbitrary and choose C_1 and C_2 according to i) for x and r. Then $X_j := B(0,\delta) + C_j, j = 1, 2,$ are open convex sets.

Recall that by extending any compactly supported distribution by zero to all of $\mathbb{R}^d$ we have $\mathscr{E}'(X) \subseteq \mathscr{D}'(\mathbb{R}^d)$ and thus $\mathscr{E}'(X) \subseteq \mathscr{D}'(\mathbb{R}^d) \subseteq \mathscr{D}'(Y)$ for every open subset $Y \subseteq \mathbb{R}^d$.

By choice of r we have $u|_{X_1} = 0$ and $X_2 \cap K = \emptyset$ because of $C_2 \subseteq \mathbb{R}^d \backslash X$ so that also $P(-D)u|_{X_2} = 0$. Hence, by the hypothesis and theorem 2.8 we conclude $u|_{X_2} = 0$, in particular u vanishes in $B(x, \delta)$ and as $x \in \partial X$ was chosen arbitrarily

$$\mathrm{dist}(\mathrm{supp}\, u, X^c) \geq \delta = \mathrm{dist}(\mathrm{supp}\, P(-D)u, X^c)$$

follows. Since $\mathrm{supp}\, P(-D)u \subseteq \mathrm{supp}\, u$ we have equality of the distances, thus X is P-convex for supports by [18, Theorem 10.6.3].

Referring to proposition 2.5 instead of theorem 2.8 and replacing supports by singular supports in the proof of part i) immediately gives ii).

In order to proof iii) we observe that with $C_1, C_2 \subseteq \mathbb{R}^d$ trivially $C_1 \times \mathbb{R}$ and $C_2 \times \mathbb{R}$ are convex, open, and non-empty. Assume that for some unit vector from $\mathbb{R}^{d+1}$ $N = (N', N_{d+1}) \in S^d$ and $\alpha \in \mathbb{R}$ the hyperplane

$$H = \{x \in \mathbb{R}^{d+1}; \langle x, N \rangle = \alpha\}$$

satisfies $\sigma_{P^+}(N) = 0$, $H \cap (C_2 \times \mathbb{R}) \neq \emptyset$ and $H \cap (C_1 \times \mathbb{R}) = \emptyset$. Without loss of generality let $C_1 \times \mathbb{R} \subseteq \{x \in \mathbb{R}^{d+1}; \langle x, N \rangle > \alpha\}$. But this implies $N_{d+1} = 0$ because otherwise we had

$$(x, \frac{1}{N_{d+1}}(\alpha - \langle x, N' \rangle)) \in (C_1 \times \mathbb{R}) \cap H$$

for any $x \in C_1$. Thus, $N_{d+1} = 0$ so that $H = \{x \in \mathbb{R}^d; \langle x, N' \rangle = \alpha\} \times \mathbb{R}$ implying $H' \cap C_2 \neq \emptyset$ as well as $H' \cap C_1 = \emptyset$. By $\sigma_{P^+}(N) = 0$ and lemma 2.7 ii) we conclude $\sigma_P^0(N') = 0$.

Now, if $(x, t) \in \partial X \times \mathbb{R} = \partial(X \times \mathbb{R})$ and $r > 0$ are arbitrary let C_1 and C_2 be as in iii) for x and r. From the hypothesis and what we have observed above $C_1 \times \mathbb{R} \subseteq C_2 \times \mathbb{R}$ are convex, $(x, t) \in C_2 \times \mathbb{R} \subseteq \mathbb{R}^{d+1} \backslash (X \times \mathbb{R})$, $C_1 \times \mathbb{R} \subseteq \mathbb{R}^{d+1} \backslash B(0, R)$ and every hyperplane $H \in \mathbb{R}^{d+1}$ with $\sigma_{P^+}(H^\perp) = 0$ intersecting $C_2 \times \mathbb{R}$ also intersects $C_1 \times \mathbb{R}$. By ii) $X \times \mathbb{R}$ is therefore P^+-convex for singular supports. □

In the above theorem, the condition that $C_1 \subseteq \mathbb{R}^d \backslash B(0, r)$ together with the condition that certain hyperplanes intersecting C_2 should also intersect C_1 may sometimes be hard to verify in concrete examples. Therefore we consider in the sequel convex cones as special cases for C_2 and we will see that in this case one can give a sufficient condition for the convexity properties solely in terms of a single convex set.

Recall that a cone C is called *proper* if it does not contain any affine subspace of dimension one. Moreover, recall that for an open convex cone $\Gamma \subseteq \mathbb{R}^d$ its *dual cone* is defined as

$$\Gamma^\circ := \{\xi \in \mathbb{R}^d; \forall\, y \in \Gamma : \langle y, \xi \rangle \geq 0\}.$$

It is a closed proper convex cone in $\mathbb{R}^d$. On the other hand, every closed proper convex cone C in $\mathbb{R}^d$ is the dual cone of a unique open convex cone which is given by

$$\Gamma := \{y \in \mathbb{R}^d; \forall \xi \in C \backslash \{0\} : \langle y, \xi \rangle > 0\} = \{y \in \mathbb{R}^d; \forall \xi \in C \cap S^{d-1} : \langle y, \xi \rangle > 0\}.$$

Γ is obviously a convex cone. To see that $\mathbb{R}^d\backslash\Gamma$ is closed one uses the compactness of $C\cap S^{d-1}$ while the proof of $C=\Gamma^\circ$ can be done with the Hahn-Banach Theorem (cf. [17, p. 257]). Therefore, we use the notation Γ° also for arbitrary closed convex proper cones.

Before we continue to prove sufficient conditions for P-convexity we need some more preparations. We begin with the following proposition containing some elementary geometric results which will be useful in the sequel.

Proposition 2.10. *a) If $C\subseteq\mathbb{R}^d$ is closed, convex, and unbounded, then for every $x\in C$ there is $\omega\in S^{d-1}$ such that $x+t\omega\in C$ for every $t\geq 0$.*

b) Let $\Gamma^\circ\neq\{0\}$ be a closed proper convex cone in $\mathbb{R}^d$ and $N\in S^{d-1}$. For $c\in\mathbb{R}$ let $H_c:=\{x\in\mathbb{R}^d;\langle x,N\rangle=c\}$. Then the following are equivalent.

- *i) $H_0\cap\Gamma^\circ=\{0\}$.*
- *ii) $N\in\Gamma$ or $-N\in\Gamma$.*
- *iii) If $x\in\mathbb{R}^d$ and $H_c\cap(x+\Gamma^\circ)\neq\emptyset$ then $H_c\cap(x+\Gamma^\circ)$ is bounded.*
- *iv) If $x\in H_c$ then $H_c\cap(x+\Gamma^\circ)=\{x\}$.*

Proof. a) Let $x\in C$. Replacing C by $C-x$ we may assume without loss of generality that $x=0$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in C with $|x_n|\geq n$ for all $n\in\mathbb{N}$. Because $0\in C$ we have $x_n/|x_n|\in C$ for every $n\in\mathbb{N}$. Passing to a subsequence if necessary, we can assume that $(x_n/|x_n|)_{n\in\mathbb{N}}$ converges to $\omega\in S^{d-1}$. For every $t\geq 0$ we have $t/|x_n|<1$ for n sufficiently large, hence $tx_n/|x_n|\in C$. Since C is closed it follows that $t\omega\in C$.

b) By translating and changing c appropriately, we can assume throughout the proof that $x=0$. Obviously, i) is then equivalent to iv).

To show that i) implies ii) let

$$H^+:=\{x\in\mathbb{R}^d;\ \langle x,N\rangle>0\}\text{ and }H^-:=\{x\in\mathbb{R}^d;\ \langle x,N\rangle<0\}.$$

If $H^+\cap\Gamma^\circ\neq\emptyset$ then $H^-\cap\Gamma^\circ=\emptyset$. Indeed, assume there are $x\neq y$ in Γ° such that $\langle x,N\rangle>0$ and $\langle y,N\rangle<0$. Convexity of Γ° together with $H_0\cap\Gamma^\circ=\{0\}$ imply the existence of $\lambda\in(0,1)$ such that $\lambda x+(1-\lambda)y=0$, hence $-x=(1-\lambda)/\lambda\,y$. Since Γ° is a cone and $(1-\lambda)/\lambda>0$ it follows that $-x\in\Gamma^\circ$. Hence $\{0\}\neq\operatorname{span}\{x\}\subseteq\Gamma^\circ$ contradicting that Γ° is proper.

Analogously one shows that $H^-\cap\Gamma^\circ\neq\emptyset$ implies $H^+\cap\Gamma^\circ=\emptyset$. Moreover, assuming $H^+\cap\Gamma^\circ=\emptyset$ as well as $H^-\cap\Gamma^\circ=\emptyset$ implies $\Gamma^\circ\subseteq H_0$. This yields $\Gamma^\circ=\{0\}$ because of $\Gamma^\circ\cap H_0=\{0\}$, contradicting $\Gamma^\circ\neq\{0\}$.

Without loss of generality we therefore may assume that $H^+\cap\Gamma^\circ\neq\emptyset$. From the above we obtain $\Gamma^\circ\subseteq\{x\in\mathbb{R}^d;\ \langle x,N\rangle\geq 0\}$. Since $H_0\cap\Gamma^\circ=\{0\}$ it follows that for all $x\in\Gamma^\circ\backslash\{0\}$ we have $\langle x,N\rangle>0$ which shows ii).

That ii) implies i) is trivial.

In order to show that iii) implies i) assume that $H_0\cap\Gamma^\circ\neq\{0\}$. Then, there is $\omega\in S^{d-1}$ such that $t\omega\in H_0\cap\Gamma^\circ$ for every $t\geq 0$. If $x\in H_c\cap\Gamma^\circ$ it follows that $x+t\omega\in H_c$. Moreover, because of $x\in\Gamma^\circ$ we have

$$\forall\,y\in\Gamma,t\geq 0:\ \langle y,x+t\omega\rangle=\langle y,x\rangle+t\langle y,\omega\rangle\geq 0,$$

hence $x+t\omega\in H_c\cap\Gamma^\circ$ for all $t\geq 0$ contradicting the boundedness of $H_c\cap\Gamma^\circ$.

To show that i) implies iii) assume that $H_c \cap \Gamma^\circ \neq \emptyset$ is unbounded. It follows from a) that for $x \in H_c \cap \Gamma^\circ \backslash \{0\}$ there is $\omega \in S^{d-1}$ such that $x + t\omega \in H_c \cap \Gamma^\circ$ for all $t \geq 0$. Thus

$$c = \langle x, N \rangle = \langle x, N \rangle + t\langle \omega, N \rangle,$$

i.e. $\omega \in H_0$, and

$$\forall y \in \Gamma, t \geq 0 : \ 0 \leq \langle y, x + t\omega \rangle.$$

Since Γ is a cone, this implies

$$\forall y \in \Gamma, t \geq 0, \varepsilon > 0 : \ 0 \leq \langle \varepsilon y, x + t/\varepsilon\, \omega \rangle = \varepsilon \langle y, x \rangle + t\langle y, \omega \rangle.$$

The special case $t := \langle y, x \rangle$ gives

$$\forall y \in \Gamma, \varepsilon > 0 : \ 0 \leq (\varepsilon + \langle y, \omega \rangle)\langle y, x \rangle.$$

Because $x \in \Gamma^\circ \backslash \{0\}$ we have $\langle y, x \rangle > 0$ for every $y \in \Gamma$, so that the above inequality yields $\langle y, \omega \rangle \geq 0$ for all $y \in \Gamma$, thus $\omega \in \Gamma^\circ$. We conclude that $\omega \in H_0 \cap \Gamma^\circ \cap S^{d-1}$ contradicting i). □

With the aid of the above proposition and theorem 2.9 we can now prove the next theorem.

Theorem 2.11. *Let X be an open, connected subset of $\mathbb{R}^d$ and let P be a non-constant polynomial with principal part P_m.*

i) X is P-convex for supports if for every $x \in \partial X$ there is an open convex cone $\Gamma \neq \mathbb{R}^d$ such that $(x + \Gamma^\circ) \cap X = \emptyset$ and $P_m(y) \neq 0$ for all $y \in \Gamma$.

ii) X is P-convex for singular supports if for every $x \in \partial X$ there is an open convex cone $\Gamma \neq \mathbb{R}^d$ such that $(x + \Gamma^\circ) \cap X = \emptyset$ and $\sigma_P(y) \neq 0$ for all $y \in \Gamma$.

iii) $X \times \mathbb{R}$ is P^+-convex for singular supports if for every $x \in \partial X$ there is an open convex cone $\Gamma \neq \mathbb{R}^d$ such that $(x + \Gamma^\circ) \cap X = \emptyset$ and $\sigma_P^0(y) \neq 0$ for all $y \in \Gamma$.

Proof. We begin with a general observation. Let $x \in \mathbb{R}^d$ be arbitrary and let $\Gamma^\circ \neq \{0\}$ be a closed proper convex cone in $\mathbb{R}^d$. Moreover, let π be a supporting hyperplane of $C_2 := x + \Gamma^\circ$ with $\pi \cap (x + \Gamma^\circ) = \{x_0\}$. If $r > 0$ let $\tilde{\pi}$ be a halfspace with boundary parallel to π such that $C_1 := (x + \Gamma^\circ) \cap \tilde{\pi} \subseteq \mathbb{R}^d \backslash B(0, r)$ is unbounded. From the choice of π it follows that $C_2 \backslash C_1$ is bounded.

Now, if $H = \{\xi \in \mathbb{R}^d; \langle \xi, N \rangle = \alpha\}$ is a hyperplane intersecting C_2 then by proposition 2.10 b) $H \cap C_2$ is unbounded if and only if $\{N, -N\} \cap \Gamma = \emptyset$. As $C_2 \backslash C_1$ is bounded we obtain that the hyperplane H intersecting C_2 also intersects C_1 if and only if $\{N, -N\} \cap \Gamma = \emptyset$.

In order to prove i) let $x \in \partial X$ be arbitrary and let Γ be as in the hypothesis of i). Setting C_2 as above we have $x \in C_2 \subseteq \mathbb{R}^d \backslash X$. For $r > 0$ arbitary let C_1 be as above, too. Then $C_1 \subseteq \mathbb{R}^d \backslash B(0, r)$. Moreover, $C_1 \subseteq C_2$ are convex sets and by hypothesis and the fact that $P_m(N) \neq 0$ if and only if $P_m(-N) \neq 0$ it follows from the above observation that every characteristic hyperplane for P which intersects C_2 also intersects C_1. From theorem 2.9 i) it follows that X is P-convex for supports.

Taking into account that $\sigma_P(y) = \sigma_P(-y)$ and $\sigma_P^0(y) = \sigma_P^0(-y)$ for all $y \in \mathbb{R}^d$ the proofs of ii) and iii) are obvious modifications of the above. □

We provide an alternative way to proof theorem 2.11 as was originally done in [22]. This proof will involve two results which are interesting in their own right, see propositions 2.12 and 2.15 below.

Proposition 2.12. *Let Γ be an open proper convex cone in $\mathbb{R}^d$, $x_0 \in \mathbb{R}^d$, and let P be a non-constant polynomial. If for $X := x_0 + \Gamma$ no hyperplane*

$$H = \{x \in \mathbb{R}^d; \langle x, N\rangle = \alpha\}$$

with $\sigma_P(H^\perp) = 0$ intersects $\overline{X}$ only in x_0, the following holds.

Each $u \in \mathscr{D}'(X)$ with $P(D)u \in \mathscr{E}(X)$ which is C^∞ outside a bounded subset of X already belongs to $\mathscr{E}(X)$.

Proof. Let $u \in \mathscr{D}'(X)$ satisfy $P(D)u \in \mathscr{E}(X)$ and assume that u is C^∞ outside a bounded subset of X. Since Γ is a proper cone, there is a hyperplane π intersecting $\overline{X}$ only in x_0. Let H_π be a halfspace with boundary parallel to π such that $X_1 := X \cap H_\pi \neq \emptyset$ is unbounded and $u|_{X_1} \in \mathscr{E}(X_1)$. Denoting $X_2 := X$ we have convex sets $X_1 \subseteq X_2$ and by the hypothesis and proposition 2.10, each hyperplane H with $\sigma_P(H^\perp) = 0$ and $H \cap X_2 \neq \emptyset$ already intersects X_1. Proposition 2.5 now gives $u \in \mathscr{E}(X)$. □

In order to obtain an analogous result of the above for P^+ and $X \times \mathbb{R}$ only involving properties of P and X we continue with some geometrical considerations. Recall that for $M \in \mathbb{R}^{d+1}$ and $A \subseteq \mathbb{R}^{d+1}$ we write

$$M' = (M_1, \dots, M_d) \in \mathbb{R}^d \text{ and } A' = \{\xi'; \xi \in A\}.$$

Proposition 2.13. *Let Γ be an open proper convex cone in $\mathbb{R}^d$, $x_0 \in \mathbb{R}^d$, and $N \in S^{d-1}$ such that $\pi := \{x \in \mathbb{R}^d; \langle x, N\rangle = \alpha\}$ is a supporting hyperplane of $x_0 + \overline{\Gamma}$ intersecting $x_0 + \overline{\Gamma}$ only in x_0 and $x_0 + \Gamma \subseteq \{x \in \mathbb{R}^d; \langle x, N\rangle > \alpha\}$. For $\beta > \alpha$ set $\tilde{X}_1 := \{x \in x_0 + \Gamma; \langle x, N\rangle > \beta\}$, $X_1 := \tilde{X}_1 \times \mathbb{R}$, and $X_2 := (x_0 + \Gamma) \times \mathbb{R}$.*

If $H = \{x \in \mathbb{R}^{d+1}; \langle x, M\rangle = c\}$ is a hyperplane with $X_2 \cap H \neq \emptyset$ as well as $X_1 \cap H = \emptyset$ then the hyperplane $H_{x_0} := \{x \in \mathbb{R}^{d+1}; \langle x, M\rangle = \langle x_0, M'\rangle\}$ is a supporting hyperplane of $\overline{X_2}$ with $H_{x_0} \cap \overline{X_2} = \{x_0\} \times \mathbb{R}$ and $M_{d+1} = 0$. Moreover, $H'_{x_0} = \{x \in \mathbb{R}^d; \langle x, M'\rangle = \langle x_0, M'\rangle\}$ is a supporting hyperplane of $x_0 + \overline{\Gamma}$ such that $H'_{x_0} \cap (x_0 + \overline{\Gamma}) = \{x_0\}$.

Proof. Without loss of generality, let $x_0 = 0$. In this case, $\alpha = 0$ and H_0 contains 0. Suppose H_0 is not a supporting hyperplane of $\overline{X_2}$. Because of $0 \in H_0 \cap \overline{X_2}$ this means that there are $v, w \in \overline{X_2} = \overline{\Gamma} \times \mathbb{R}$ such that $\langle v, M\rangle < 0 < \langle w, M\rangle$, hence $\langle x, M\rangle < 0 < \langle y, M\rangle$ for some $x, y \in \Gamma \times \mathbb{R}$.

Set $R := (N, 0) \in \mathbb{R}^{d+1}$. Then $|R| = 1$ and because of

$$\Gamma \subseteq \{v \in \mathbb{R}^d; \langle v, N\rangle > 0\}$$

we have

$$X_2 \subseteq \{v \in \mathbb{R}^{d+1}; \langle v, R\rangle > 0\}.$$

Therefore, $\lambda_1 := \langle x, R\rangle > 0$ as well as $\lambda_2 := \langle y, R\rangle > 0$. Since X_2 is a cone we have $x_1 := \frac{\beta+1}{\lambda_1}x, y_1 := \frac{\beta+1}{\lambda_2}y \in X_2$ and from $X_1 = \{v \in X_2; \langle v, R\rangle > \beta\}$ we get $x_1, y_1 \in X_1$.

Because $\langle x_1, M\rangle < 0 < \langle y_1, M\rangle$ it follows for some $t > 1$

$$\langle tx_1, M\rangle < c < \langle ty_1, M\rangle.$$

Hence there is $\lambda \in (0,1)$ with

$$\langle \lambda t x_1 + (1-\lambda) t y_1, M\rangle = c,$$

i.e. $\lambda t x_1 + (1-\lambda)ty_1 \in H$. Obviously, X_1 is convex and for every $x \in X_1$ and $t > 1$ we have $tx \in X_1$. Therefore we have $\lambda t x_1 + (1-\lambda)ty_1 \in H \cap X_1$ which contradicts our hypothesis.

So, H_0 is a supporting hyperplane of $\overline{X_2} = \overline{\Gamma} \times \mathbb{R}$. This immediately implies that $M_{d+1} = 0$ and that H_0' is a supporting hyperplane of $\overline{\Gamma}$. Moreover, $M_{d+1} = 0$ implies that $H' = \{x \in \mathbb{R}^d; \langle x, M'\rangle = c\}$ intersects Γ but not $X_1' = \tilde{X}_1$. Because Γ is a proper cone and $\Gamma\backslash X_1' = \{x \in \Gamma; \langle x, N\rangle \leq \beta\}$ this implies that $H' \cap \overline{\Gamma}$ is bounded. Since H_0' is a supporting hyperplane of $\overline{\Gamma}$ this yields $H_0' \cap \overline{\Gamma} = \{0\}$ by proposition 2.10 *b*), hence $H_0 \cap \overline{X_2} = (H_0' \times \mathbb{R}) \cap (\overline{\Gamma} \times \mathbb{R}) = \{0\} \times \mathbb{R}$. □

Proposition 2.14. *Let Γ be an open proper convex cone in $\mathbb{R}^d$, $x_0 \in \mathbb{R}^d$, and let X_1 and X_2 be as in proposition 2.13. Moreover, let P be a non-constant polynomial. Assume that no hyperplane H in $\mathbb{R}^d$ with $\sigma_P^0(H^\perp) = 0$ intersects $x_0 + \overline{\Gamma}$ only in x_0.*

Then for every hyperplane H in $\mathbb{R}^{d+1}$ with $H \cap X_2 \neq \emptyset$ and $\sigma_{P^+}(H^\perp) = 0$ it follows that $H \cap X_1 \neq \emptyset$.

Proof. Let $H = \{x \in \mathbb{R}^{d+1}; \langle x, M\rangle = \beta\}$ be a hyperplane with $H \cap X_2 \neq \emptyset$ but $H \cap X_1 = \emptyset$. We have to show that $\sigma_{P^+}(M) \neq 0$.

From proposition 2.13 it follows that $M = (M', 0)$ and that

$$H'_{x_0} = \{x \in \mathbb{R}^d; \langle x, M'\rangle = \langle x_0, M'\rangle\}$$

is a supporting hyperplane of $x_0 + \overline{\Gamma}$ with

$$H'_{x_0} \cap (x_0 + \overline{\Gamma}) = \{x_0\}.$$

In particular, the hypothesis gives $\sigma_P^0(M') \neq 0$. With lemma 2.7 we get

$$0 \neq \sigma_P^0(M') = \sigma_{P^+}(span\{M'\} \times \{0\}) = \sigma_{P^+}(M),$$

proving the proposition. □

Now, we can prove an analogous result to proposition 2.12 for P^+ and $X \times \mathbb{R}$ which only relies on properties of P and X.

Proposition 2.15. *Let Γ be an open proper convex cone in $\mathbb{R}^d$, $x_0 \in \mathbb{R}^d$, and let $P \in \mathbb{C}[X_1, \ldots, X_d]$ be a non-constant polynomial. Assume that no hyperplane H in $\mathbb{R}^d$ with $\sigma_P^0(H^\perp) = 0$ intersects $x_0 + \overline{\Gamma}$ only in x_0.*

Then, every $u \in \mathscr{D}'((x_0+\Gamma)\times\mathbb{R})$ with $P^+(D)u \in \mathscr{E}((x_0+\Gamma)\times\mathbb{R})$ for which there is a bounded subset B of $x_0 + \Gamma$ such that u is C^∞ outside $B \times \mathbb{R}$ already satisfies $u \in \mathscr{E}((x_0 + \Gamma) \times \mathbb{R})$.

Proof. Without restriction, assume $x_0 = 0$. Let $u \in \mathscr{D}'(\Gamma \times \mathbb{R})$ with $P^+(D)u \in \mathscr{E}(\Gamma \times \mathbb{R})$ and let $B \subseteq \Gamma$ be bounded such that $u_{|\Gamma\backslash B \times \mathbb{R}} \in \mathscr{E}(\Gamma \backslash B \times \mathbb{R})$. Because Γ is a proper cone in $\mathbb{R}^d$ there is a hyperplane H_1 intersecting $\overline{\Gamma}$ only in 0. Let $\tilde{X}_1$ be the intersection of Γ with a halfspace whose boundary is parallel to H_1 such that $\tilde{X}_1$ is unbounded and $B \subseteq \Gamma\backslash\tilde{X}_1$.

Let $X_1 := \tilde{X}_1 \times \mathbb{R}$, and $X_2 := \Gamma \times \mathbb{R}$. Then $X_1 \subseteq X_2$ are open convex subsets of $\mathbb{R}^{d+1}$ and it follows from proposition 2.14 that for every hyperplane H in $\mathbb{R}^{d+1}$ with $\sigma_{P^+}(H^\perp) = 0$ and $H \cap X_2 \neq \emptyset$ already $H \cap X_1 \neq \emptyset$. Since $u \in \mathscr{D}'(X_2), P^+(D)u \in \mathscr{E}(X_2)$ and $u_{|X_1} \in \mathscr{E}(X_1)$ it follows from proposition 2.5 that $u \in \mathscr{E}(X_2)$. □

We now give an alternative proof of theorem 2.11.

Alternative proof of theorem 2.11. Again, the alternative proofs of all three parts are very similar, so we give the proof of part iii) and only sketch the proofs of i) and ii).

In order to prove iii), let $u \in \mathscr{E}'(X \times \mathbb{R})$. We set $K := \operatorname{sing\,supp} P^+(-D)u$ and $\delta := \operatorname{dist}(K, X^c \times \mathbb{R})$. Let $x_0 \in \partial(X \times \mathbb{R}) = \partial X \times \mathbb{R}$ and let Γ be as in the hypothesis for $x_0' \in \partial X$. Then $(x_0 + (\Gamma^\circ \times \mathbb{R})) \cap (X \times \mathbb{R}) = \emptyset$, thus $(x_0 + y + (\Gamma^\circ \times \mathbb{R})) \cap K = \emptyset$ for all $y \in \mathbb{R}^{d+1}$ with $|y| < \delta$. Therefore, for fixed y with $|y| < \delta$, there is an open proper convex cone $\tilde{\Gamma}$ in $\mathbb{R}^d$ with $\tilde{\Gamma} \supset \Gamma^\circ \backslash \{0\}$ such that $(x_0 + y + (\tilde{\Gamma} \times \mathbb{R})) \cap K = \emptyset$. Hence, $u \in \mathscr{E}'(X \times \mathbb{R}) \subseteq \mathscr{D}'(x_0 + y + (\tilde{\Gamma} \times \mathbb{R}))$ satisfies $P^+(-D)u \in \mathscr{E}(x_0 + y + (\tilde{\Gamma} \times \mathbb{R}))$. We show that $u \in \mathscr{E}(x_0 + y + (\tilde{\Gamma} \times \mathbb{R}))$ by applying proposition 2.15.

Let $H = \{v \in \mathbb{R}^d; \langle v, N\rangle = \alpha\}$ be a hyperplane with $\sigma_P^0(N) = 0$. As $\overline{\tilde{\Gamma}}$ is a closed proper convex cone with non-empty interior, it is the dual cone of some open proper convex cone Γ_1. It follows from $\Gamma_1^\circ = \overline{\tilde{\Gamma}} \supset \Gamma^\circ$ that $\Gamma_1 \subseteq \Gamma$. Because $\sigma_P^0(N) = 0$ it follows from the hypothesis on Γ that $\{N, -N\} \cap \Gamma = \emptyset$, hence $\{N, -N\} \cap \Gamma_1 = \emptyset$, so that by proposition 2.10 b) H does not intersect $x_0' + y' + \overline{\tilde{\Gamma}}$ only in $x_0' + y'$. Now $\operatorname{sing\,supp} u$ is compact since $u \in \mathscr{E}'(X \times \mathbb{R})$. Because

$$P^+(-D)u \in \mathscr{E}(x_0 + y + (\tilde{\Gamma} \times \mathbb{R})),$$

we have

$$u \in \mathscr{E}(x_0 + y + (\tilde{\Gamma} \times \mathbb{R}))$$

by proposition 2.15. Since $x_0 \in \partial X \times \mathbb{R}$ and y with $|y| < \delta$ were chosen arbitrarily, it follows that

$$\operatorname{dist}(\operatorname{sing\,supp} u, X^c \times \mathbb{R}) \geq \delta = \operatorname{dist}(\operatorname{sing\,supp} P(-D)u, X^c \times \mathbb{R}),$$

which proves iii).

To prove ii) let $u \in \mathscr{E}'(X)$. We set again $K := \operatorname{sing\,supp} P(-D)u$ and $\delta := \operatorname{dist}(K, X^c)$. Again, we have to show that $\operatorname{dist}(\operatorname{sing\,supp} u, X^c) \geq \delta$ in order to prove P-convexity for singular supports of X.

Let $x_0 \in \partial X$ and let Γ be as in the hypothesis for $x_0 \in \partial X$. As above, for $y \in \mathbb{R}^d$ with $|y| < \delta$ we find an open proper convex cone $\tilde{\Gamma}$ such that $u \in \mathscr{E}'(X) \subseteq \mathscr{D}'(x_0 + y + \tilde{\Gamma})$ satisfies $P(-D)u \in \mathscr{E}(x_0 + y + \tilde{\Gamma})$. Using proposition 2.12 instead of proposition 2.15 the proof of ii) is now completed exactly as the one of iii).

In order to prove i), let $u \in \mathscr{E}'(X)$, $K := \operatorname{supp} P(-D)u$ and $\delta := \operatorname{dist}(K, X^c)$. By [18, Theorem 10.6.3] one has to show $\operatorname{dist}(\operatorname{supp} u, X^c) \geq \delta$ which is done as in the proof of iii) and ii), respectively, by using [17, Corollary 8.6.11] instead of proposition 2.12. □

We close this section with yet another pair of sufficient conditions for P-convexity for supports and singular supports. These will be used to give an alternative proof of a result due to Vogt [39] stating that for elliptic P the operator $P^+(D)$ is always surjective on $\mathscr{D}'(X \times \mathbb{R})$. For $x, y \in \mathbb{R}^d$ we define

$$[x, y] = \{\gamma x + (1 - \gamma)y;\ \gamma \in [0, 1]\}.$$

Moreover, for $X \subseteq \mathbb{R}^d$ open, $x \in X$, $r \in \mathbb{R}^d\backslash\{0\}$, we define

$$\lambda_X(x, r) := \sup\{\lambda > 0;\ \forall\, 0 \leq \mu < \lambda :\ [x, x + \mu r] \subseteq X\}.$$

In case of $\lambda_X(x, r) = \infty$ we write $[x, x + \lambda_X(x, r)r]$ instead of

$$\cup_{0<\lambda<\lambda_X(x,r)}[x, x + \lambda r].$$

The next lemma gives a sufficient condition for P-convexity for supports.

Lemma 2.16. *Let X be an open subset of $\mathbb{R}^d$ and let P be a non-zero polynomial of degree m. Assume that for each compact subset K of X there is another compact subset L of X such that for every $x \in X\backslash L$ one can find $r \in \{x \in \mathbb{R}^d;\ P_m(x) = 0\}^\perp\backslash\{0\}$ satisfying*

$$[x_0, x_0 + \lambda_X(x_0, r)r] \cap K = \emptyset.$$

Then X is P-convex for supports.

Proof. Let $\phi \in \mathscr{D}(X)$ and $K := \operatorname{supp} P(-D)\phi$. Choose L for K as stated in the hypothesis. For $x_0 \in X\backslash L$ there is $r \in \{\xi \in \mathbb{R}^d;\ P_m(\xi) = 0\}^\perp\backslash\{0\}$ such that

$$[x_0, x_0 + \lambda_X(x_0, r)r] \cap K = \emptyset.$$

From the compactness of $\operatorname{supp}\phi$ it follows that there is $\lambda \in (0, \lambda_X(x_0, r))$ with $x_1 := x_0 + \lambda r \notin \operatorname{supp}\phi$. From the definition of $\lambda_X(x_0, r)$ we have $[x_0, x_1] \subseteq X$ and we can find $\rho > 0$ such that

$$X_1 := B(x_1, \rho) \subseteq X\backslash\operatorname{supp}\phi \text{ and } X_2 := [x_0, x_1] + B(0, \rho) \subseteq X\backslash K.$$

$X_1 \subseteq X_2$ are open and convex, and $\phi_{|X_1} = 0$ as well as $P(-D)\phi_{|X_2} = 0$. Let $H = \{x \in \mathbb{R}^d;\ \langle x, N\rangle = \alpha\}$ be a characteristic hyperplane for P. If H intersects X_2 there are $\gamma \in [0, 1], b \in B(0, \rho)$ satisfying

$$\begin{aligned}\alpha &= \langle \gamma x_0 + (1 - \gamma)x_1 + b, N\rangle = \langle x_0 + (1 - \gamma)\lambda r + b, N\rangle \\ &= \langle x_0 + b, N\rangle = \langle x_1 - \lambda r + b, N\rangle = \langle x_1 + b, N\rangle\end{aligned}$$

where we used $r \in \{\xi \in \mathbb{R}^d;\ P_m(\xi) = 0\}^\perp$. So H already intersects X_1. Theorem 2.8 now gives $\phi_{|X_2} = 0$ so that $x_0 \notin \operatorname{supp}\phi$. Since $x_0 \in X\backslash L$ was arbitrary it follows $\operatorname{supp}\phi \subseteq L$ proving the lemma. □

In order to formulate a similar condition for P-convexity for singular supports we introduce for a non-zero polynomial P the subspace

$$S_P := \{y \in \mathbb{R}^d;\ \sigma_P(y) = 0\}^\perp.$$

The non-zero elements r of S_P are the directions which lie in every hyperplane $H = \{x \in \mathbb{R}^d;\ \langle x, N\rangle = \alpha\}$ with $\sigma_P(N) = 0$. Hence, using these directions and proposition 2.5 instead of theorem 2.8 the next lemma can be proved in a very similar way to the previous one. Indeed, the proof is mutatis mutandis the same. Nevertheless, we include it for the reader's convenience.

Lemma 2.17. *Let X be an open subset of $\mathbb{R}^d$ and let P be a non-zero polynomial. Assume that for each compact subset K of X there is another compact subset L of X such that for every $x \in X \backslash L$ one can find $r \in S_P \backslash \{0\}$ with*

$$[x, x + \lambda_X(x, r)r] \cap K = \emptyset.$$

Then X is P-convex for singular supports.

Proof. Let $u \in \mathscr{E}'(X)$ and $K := \operatorname{sing\,supp} P(-D)u$. Choose L for K as stated in the hypothesis. For a fixed $x_0 \in X \backslash L$ we can find $r \in S_P \backslash \{0\}$ with

$$[x_0, x_0 + \lambda_X(x_0, r)r] \cap K = \emptyset.$$

The compactness of $\operatorname{sing\,supp} u$ implies that there is $\lambda \in (0, \lambda_X(x_0, r))$ such that $x_1 := x_0 + \lambda r \notin \operatorname{sing\,supp} u$. Therefore, $[x_0, x_1] \subseteq X$ and we can find $\rho > 0$ such that $X_1 := B(x_1, \rho) \subseteq X \backslash \operatorname{sing\,supp} u$ and

$$X_2 := [x_0, x_1] + B(0, \rho) \subseteq X \backslash K.$$

We will show that $u_{|X_2} \in \mathscr{E}(X_2)$ implying $x_0 \notin \operatorname{sing\,supp} u$. Since $x_0 \in X \backslash L$ was chosen arbitrarily this will show $\operatorname{sing\,supp} u \subseteq L$ proving P-convexity for singular supports of X.

By definition of K we have $P(-D)u_{|X_2} \in \mathscr{E}(X_2)$. Moreover, X_1 is convex and $\operatorname{sing\,supp}\, u_{|X_2} \subseteq X_2 \backslash X_1$. To show that $u_{|X_2} \in \mathscr{E}(X_2)$, let

$$H = \{x \in \mathbb{R}^d;\ \langle x, N \rangle = \alpha\}$$

be a hyperplane with $\sigma_P(N) = 0$. Since $r \in S_P$ we have $\langle r, N \rangle = 0$. If H intersects X_2 it follows exactly as in the proof of lemma 2.16 that H already intersects X_1. Now Corollary 2.5 gives $u_{|X_2} \in \mathscr{E}(X_2)$ thus proving the lemma. □

As $\{\xi \in \mathbb{R}^d;\ P_m(\xi) = 0\}^\perp$ as well as S_P are subspaces the next proposition might be helpful in applying lemma 2.16 and lemma 2.17.

Proposition 2.18. *Let $X \subseteq \mathbb{R}^d$ be open and let $M \subseteq S^{d-1}$ be such that with $r \in M$ also $-r \in M$. Then the following condition i) implies ii).*

i) For each $x \in X$ there is $r \in M$ such that $\operatorname{dist}(x, X^c) \geq \operatorname{dist}(y, X^c)$ for all $y \in [x, x + \lambda_X(x, r)r]$

ii) For each compact subset K of X there is a compact subset L of X such that for any $x \in X \backslash L$ there is $r \in M$ satisfying $[x, x + \lambda_X(x, r)r] \cap K = \emptyset$.

Proof. For $m \in \mathbb{N}$ let $X_m := \{x \in X;\ |x| < m, \operatorname{dist}(x, X^c) > 1/m\}$. For $K \subseteq X$ compact choose m such that $K \subseteq X_m$ and set $L := \overline{X_m}$.

Fix $x \in X \backslash L$ and let r be as in i). If $|x| > m$ either

$$\{x + \lambda r; \lambda > 0\} \subseteq \mathbb{R}^d \backslash \overline{B(0, m)}$$

or

$$\{x - \lambda r; \lambda > 0\} \subseteq \mathbb{R}^d \backslash \overline{B(0, m)}$$

so that ii) follows with r or $-r$. If $|x| \leq m$ we have $1/m \geq \operatorname{dist}(x, X^c) \geq \operatorname{dist}(y, X^c)$ for every $y \in [x, x + \lambda_X(x, r)r]$ because of $x \in X \backslash L$, hence ii) follows in this case, too. □

We close this section with an example showing that, in general, the sufficient condition for P-convexity for singular supports from theorem 2.11 ii) is not necessary. However, it will be shown in section 4.1 that in case of X being a (connected) subset of $\mathbb{R}^2$ this sufficient condition is indeed necessary as well.

Example 2.19. Let $0 < r < R$ and define

$$f_{r,R} : \mathbb{R}^3 \to \mathbb{R}, f(x_1, x_2, x_3) = (\sqrt{x_1^2 + x_2^2} - R)^2 + x_3^2 - r^2,$$

and

$$\begin{aligned} T_{r,R} &:= \{x : f_{r,R}(x) = 0\} \\ &= \{((R + r\cos\varphi)\cos\psi, (R + r\cos\varphi)\sin\psi, r\sin\varphi) : \varphi, \psi \in \mathbb{R}\} \end{aligned}$$

as well as

$$\begin{aligned} V_{r,R} &:= \{x : f_{r,R}(x) \leq 0\} \\ &= \{((R + \rho\cos\varphi)\cos\psi, (R + \rho\cos\varphi)\sin\psi, \rho\sin\varphi) : \varphi, \psi \in \mathbb{R}, \rho \in [0, r]\}. \end{aligned}$$

Then, $T_{r,R} = \partial V_{r,R}$ is the torus with inner radius $R - r$ and outer radius $R + r$.

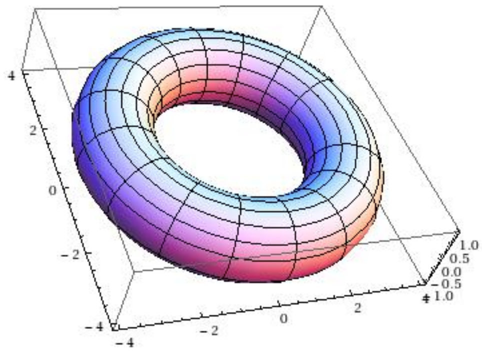

Figure 1: Torus for $r = 1$ and $R = 3$

We first show that for the wave operator $P(D)$ where $P(\xi) = \xi_1^2 + \xi_2^2 - \xi_3^2$ the interior $V_{r,R}^\circ$ of $V_{r,R}$ is P-convex for singular supports whenever $2r < R$. The polynomial P is of real principal type, i.e. the principal part of P has real coefficients and $\nabla P(\xi) \neq 0$ whenever $P(\xi) = 0$. Therefore, by [18, Theorem 10.8.9] it suffices to show that the boundary distance satisfies the minimum principle in any bicharacteristic line in order to prove P-convexity for singular supports. So we have to show that

$$d : V_{r,R}^\circ \to [0, \infty), a \mapsto \operatorname{dist}(a, \mathbb{R}^3 \backslash V_{r,R}^\circ) = \operatorname{dist}(a, T_{r,R})$$

satisfies the minimum principle in any bicharacteristic line, i.e. for every $x \neq 0$ with $P(x) = 0$ and any $a \in V_{r,R}^\circ$ the function

$$t \mapsto d(a + t\nabla P(x))$$

does not have a strict local minimum in the open set of those $t \in \mathbb{R}$ for which we have $a + t\nabla P(x) \in V_{r,R}^\circ$. Since $\nabla P(x) = 2(x_1, x_2, -x_3)$ we have $P(\nabla P(x)) =$

$4P(x)$ so that for $a \in V^\circ_{r,R}$ the bicharacteristic lines through a are precisely the sets of the form

$$\{a + tx : t \in \mathbb{R}, x \in \mathbb{R}^3 \backslash \{0\}, P(x) = 0\}.$$

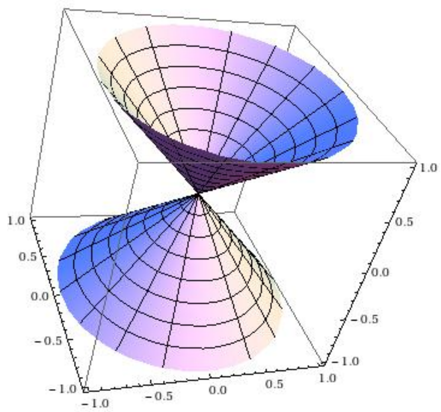

Figure 2: The set $\{\xi \in \mathbb{R}^3; P(\xi) = 0\}$ in a neighborhood of the origin

We have for each $\varphi, \psi \in \mathbb{R}$ and $\rho \in [0, r)$

$$\text{dist}\left(((R + \rho\cos\varphi)\cos\psi, (R + \rho\cos\varphi)\sin\psi, \rho\sin\varphi), T_{r,R}\right) = r - \rho.$$

If $a \in V^\circ_{r,R}$ there are $\varphi, \psi \in \mathbb{R}$ and a unique $\rho \in [0, r)$ such that

$$a = ((R + \rho\cos\varphi)\cos\psi, (R + \rho\cos\varphi), \sin\psi, \rho\sin\varphi).$$

We want to express ρ in terms of $|a|$:

$$\begin{aligned} |a|^2 &= R^2 + 2R\rho\cos\varphi + \rho^2 \\ &= R^2 + 2R(|a'| - R) + \rho^2 \\ &= \rho^2 + 2R|a'| - R^2, \end{aligned}$$

where we used the relation $\rho\cos\varphi = |a'| - R$ and where as usual $a' = (a_1, a_2) \in \mathbb{R}^2$. So we obtain

$$\rho^2 = a_3^2 + (|a'| - R)^2 = a_3^2 + (|a'| - R)^2.$$

Hence we have for $a \in V^\circ_{r,R}$

$$\text{dist}(a, T_{r,R}) = r - \sqrt{a_3^2 + (|a'| - R)^2}.$$

Now, given $x \neq 0$ with $P(x) = 0$ we have $a + tx \in V^\circ_{r,R}$ for $|t|$ sufficiently small such that

$$d_{a,x}(t) := \text{dist}(a + tx, T_{r,R}) = r - \sqrt{(a_3 + tx_3)^2 + (|a' + tx'| - R)^2}$$

is well-defined. Moreover, $d_{a,x}(t) = d_{a+t_0x,x}(t-t_0)$ for all $t_0 \in \mathbb{R}$ with $a+t_0x \in V^\circ_{r,R}$ so that it suffices to consider $d_{a,x}$ in a neighborhood of $t=0$.

Let $(\alpha,\beta) \subseteq \mathbb{R}$ be an open intervall containing zero. Since

$$[0,r^2) \to \infty, t \mapsto r - \sqrt{t}$$

is strictly decreasing, it follows that $d_{a,x} : (\alpha,\beta) \to \mathbb{R}$ does not have a local minimum if and only if

$$g_{a,x} : (\alpha,\beta) \to [0,\infty), t \mapsto (a_3+tx_3)^2 + (|a'+tx'|-R)^2$$

does not have a local maximum.

Because $P(x)=0$ we have $|x'|^2 = x_3^2$ so that a straight forward calculation gives

$$g_{a,x}(t) = |a|^2 + 2t\langle a,x\rangle + 2|x'|^2t^2 + R^2 - 2R|a'+tx'|,$$

thus for $a' \neq 0$ we get

$$g'_{a,x}(t) = 2\langle a,x\rangle + 4|x'|^2t - 2R\frac{\langle a',x'\rangle + t|x'|^2}{|a'+tx'|}$$

and

$$\begin{aligned} g''_{a,x}(t) &= 4|x'|^2 - 2R\frac{|x'|^2|a'+tx'|^2 - (\langle a',x'\rangle + t|x'|^2)^2}{|a'+tx'|^3} \\ &= \frac{2|x'|^2|a'|^2}{|a'+tx'|^3}\left(2\frac{|a'+tx'|^3}{|a'|^2} - R(1-\langle\frac{a'}{|a'|},\frac{x'}{|x'|}\rangle^2)\right). \end{aligned}$$

Using

$$R(1-\langle\frac{a'}{|a'|},\frac{x'}{|x'|}\rangle^2) \in [0,R]$$

as well as $2|a'| \in (2(R-r),2(R+r))$ it follows for $2r<R$

$$\begin{aligned} g''_{a,x}(0) &= \frac{2|x'|^2|a'|^2}{|a'|^3}\left(2|a'| - R(1-\langle\frac{a'}{|a'|},\frac{x'}{|x'|}\rangle^2)\right) \\ &> \frac{2|x'|^2|a'|^2}{|a'|^3}(2R-2r-R) > 0. \end{aligned}$$

so that $g_{a,x}$ is strictly convex in a neighborhood of $t=0$ in case of $a' \neq 0$ and $2r<R$.

Moreover, in case of $a'=0$ we have $g''_{a,x}(t) = 4|x'|^2 > 0$ so that in any case $g_{a,x}$ is strictly convex in a neighborhood of $t=0$ if $2r<R$. Therefore, replacing a by $a+t_0x$ and hence $g_{a,x}$ by $g_{a+t_0x,x}$ if necessary, we obtain that $g_{a,x}$ is strictly convex. Thus, $g_{a,x}$ has no local maximum so that $d_{a,x}$ does not have a local minimum.

We conclude that in case of $2r<R$ the boundary distance for $V^\circ_{r,R}$ satisfies the minimum principle in any bicharacteristic line which implies the P-convexity for singular supports of $V^\circ_{r,R}$. It should be noted that by [18, Corollary 10.8.10] $P(D)$ is in fact surjective on $\mathscr{D}'(V^\circ_{r,R})$ whenever $2r<R$.

Next we show that for the boundary point $(R-r,0,0)$ of $V^\circ_{r,R}$ there is no open convex cone $\Gamma \neq \mathbb{R}^d$ such that

$$((R-r,0,0)+\Gamma^\circ) \cap V^\circ_{r,R} = \emptyset \text{ and } \sigma_P(y) \neq 0 \text{ for all } y \in \Gamma.$$

In order to do so we observe that $(-1,0,0)$ is the outer normal vector in $(R-r,0,0)$ with respect to $V^\circ_{r,R}$. This implies that for any closed convex cone C with $((R-r,0,0)+C)\cap V^\circ_{r,R}=\emptyset$ we must have

$$C\subseteq\{x\in\mathbb{R}^3;\,\xi_1\le 0\}.$$

Let $\Gamma\ne\mathbb{R}^d$ be an open convex cone with

$$((R-r,0,0)+\Gamma^\circ)\cap V^\circ_{r,R}=\emptyset.$$

By the above we have $\Gamma^\circ\subseteq\{x_1\le 0\}$. Moreover for every $x\in\Gamma^\circ\backslash\{0\}$ we have

$$\forall\, t>0:(\sqrt{(R-r+tx_1)^2+(tx_2)^2}-R)^2+(tx_3)^2-r^2\ge 0.$$

This implies $x_3\ne 0$ because otherwise we had

$$\forall\, t>0:(\sqrt{(R-r+tx_1)^2+(tx_2)^2}-R)^2-r^2\ge 0$$

which is equivalent to

$$\forall\, t>0:\sqrt{(R-r+tx_1)^2+(tx_2)^2}\ge R+r \text{ or } \sqrt{(R-r+tx_1)^2+(tx_2)^2}\le R-r.$$

But this holds if and only if

$$\forall\, t>0: 2(R-r)tx_1+t^2|x'|^2\ge 4rR \text{ or } 2(R-r)tx_1+t^2|x'|^2\le 0,$$

where as usual $x'=(x_1,x_2)$. $x'\ne 0$ as $x\in\Gamma^\circ\backslash\{0\}$ and $x_3=0$ so the above is equivalent to

$$\begin{aligned}\forall\, t>0:\quad &(\frac{x_1(R-r)}{|x'|}+t|x'|)^2\ge(\frac{x_1(R-r)}{|x'|})^2+4rR\\ \text{or } &(\frac{x_1(R-r)}{|x'|}+t|x'|)^2\le(\frac{x_1(R-r)}{|x'|})^2.\end{aligned}$$

Because $x_1\le 0$ there are always $t>0$ for which none of the two above conditions is satisfied. Hence we must have $x_3\ne 0$.

Because P and $V^\circ_{r,R}$ are invariant under the transformation $\xi\mapsto(\xi_1,\xi_2,-\xi_3)$ we can assume without loss of generality that

$$\Gamma^\circ\backslash\subseteq\{\xi_1\le 0\}\cap\{\xi_3>0\}.$$

Herefrom and from

$$\Gamma=\{\xi\in\mathbb{R}^3;\,\langle\xi,y\rangle>0 \text{ for all } y\in\Gamma^\circ\backslash\{0\}\}$$

we conclude $(-1,0,1)\in\Gamma$. Because $P(-1,0,1)=0$ we have $\sigma_P(-1,0,1)=0$ by theorem 3.14 ii) so that the condition in theorem 2.11 ii) is not satisfied for the boundary point $(R-r,0,0)$ of $V^\circ_{r,R}$.

2.3 Characterizing P-convexity in the complement of closed proper convex cones

In the previous section we gave sufficient criteria for the various P-convexity conditions of an open set X. In this section we show that these sufficient criteria are also necessary for arbitrary P in case of X being of a certain geometrical form.

Recall that a real valued function f defined on a subset M of $\mathbb{R}^d$ is said to *satisfy the minimum principle in the closed subset F of* $\mathbb{R}^d$ if for every compact subset $K \subseteq F \cap M$ it holds that

$$\inf_{x\in K} f(x) = \inf_{x\in \partial_F K} f(x),$$

where $\partial_F K$ denotes the boundary of K relative F. For a subset M of $\mathbb{R}^d$ let

$$d_M : M \to \mathbb{R}, x \mapsto \operatorname{dist}(x, \mathbb{R}^d \backslash M)$$

be the Euclidean distance to its complement.

It is well-known that for an open subset $X \subseteq \mathbb{R}^d$ to be P-convex for supports it is necessary that d_X satisfies the minimum principle in every characteristic hyperplane for P, see [18, Theorem 10.8.1]. Moreover, for the P-convexity for singular supports of X it is necessary that d_X satisfies the minimum principle in every affine subspace $V \subseteq \mathbb{R}^d$ with $\sigma_P(V^\perp) = 0$, see [18, Corollary 11.3.2].

Having in mind theorem 2.11 it is no surprise that the next geometric result will be helpful.

Proposition 2.20. *Let $\Gamma^\circ \neq \{0\}$ be a closed proper convex cone in $\mathbb{R}^d$ and $N \in S^{d-1}$. Assume that $d_{\mathbb{R}^d\backslash\Gamma^\circ}$ satisfies the minimum principle in every hyperplane $H_c = \{x \in \mathbb{R}^d; \langle x, N\rangle = c\}, c \in \mathbb{R}$. Then $\{N, -N\} \cap \Gamma = \emptyset$.*

Proof. If $\{N, -N\} \cap \Gamma \neq \emptyset$ it follows from proposition 2.10 that $H_0 \cap \Gamma^\circ = \{0\}$.

Let $c \neq 0$ be arbitrary. We first show that $H_c \cap \Gamma^\circ = \emptyset$ if and only if $H_{-c} \cap \Gamma^\circ \neq \emptyset$. Indeed, if $H_c \cap \Gamma^\circ = \emptyset$ the convexity of Γ° implies that either $\Gamma^\circ \subseteq \{x \in \mathbb{R}^d; \langle x, N\rangle < c\}$ or $\Gamma^\circ \subseteq \{x \in \mathbb{R}^d; \langle x, N\rangle > c\}$. Without restriction we only consider the first case. Since $0 \in \Gamma^\circ$ we have $0 < c$. Moreover, because Γ° is a cone, it follows for every $x \in \Gamma^\circ\backslash\{0\}$ and $t > 0$ that $t\langle x, N\rangle < c$. Obviously, this implies $\langle x, N\rangle < 0$ for every $x \in \Gamma^\circ\backslash\{0\}$. Therefore, $-c/\langle x, N\rangle > 0$ so that $-c/\langle x, N\rangle\, x \in \Gamma^\circ$ for every $x \in \Gamma^\circ\backslash\{0\}$. In particular, there is $x \in \Gamma^\circ \cap H_{-c}$.

On the other hand, let $H_{-c}\cap\Gamma^\circ \neq \emptyset$. If $H_c\cap\Gamma^\circ \neq \emptyset$ it follows from $c \neq 0$ that there are $x, y \in \Gamma^\circ\backslash\{0\}$ such that for some $\lambda \in (0, 1)$ we have $\lambda x+(1-\lambda)y \in H_0$. The convexity of Γ° together with $H_0 \cap \Gamma^\circ = \{0\}$ implies $\lambda x + (1 - \lambda)y = 0$. Therefore, $-x \in \Gamma^\circ\backslash\{0\}$ which contradicts the fact that Γ° is proper.

So, for arbitrary $c \neq 0$ we can therefore assume that $H_c \cap \Gamma^\circ = \emptyset$ as well as $H_{-c} \cap \Gamma^\circ \neq \emptyset$. Because of $H_0 \cap \Gamma^\circ = \{0\}$ it follows from proposition 2.10 that the non-empty set $H_{-c} \cap \Gamma^\circ$ is bounded. So there is $R > |c|$ such that $H_{-c}\cap\Gamma^\circ$ is contained in the closed R-ball $\overline{B(0, R)}$ about the origin. In particular, $K := H_c \cap \overline{B_R(0)}$ is a non-empty, compact subset of $H_c \cap \mathbb{R}^d\backslash\Gamma^\circ$ with

$$\inf_{x\in K} d_{\mathbb{R}^d\backslash\Gamma^\circ}(x) = \inf_{x\in K} \operatorname{dist}(x, \Gamma^\circ) \le \inf_{x\in K} |x| = |c|.$$

Obviously, $x - cN \in H_0$ for all $x \in H_c$, so that

$$M := \{x - cN;\, x \in H_c \cap \partial B_R(0)\} \subseteq H_0$$

is compact, and because $R > |c|$, M does not contain 0. Since $H_0\backslash\{0\}\cap\Gamma^\circ = \emptyset$ we obtain

$$\delta := \inf_{v\in M} \operatorname{dist}(v,\Gamma^\circ) > 0.$$

We have

$$\begin{aligned} \forall\, x\in H_c, y\in\Gamma^\circ:\ |x-y|^2 &= |(x-cN)-(y-cN)|^2 \\ &= c^2 + |(x-cN)-y|^2 - 2c\langle N,y\rangle. \end{aligned}$$

Again, by the convexity of Γ° and $H_c\cap\Gamma^\circ=\emptyset$ we have either

$$\Gamma^\circ \subseteq \{x\in\mathbb{R}^d; \langle x,N\rangle < c\} \text{ or } \Gamma^\circ \subseteq \{x\in\mathbb{R}^d; \langle x,N\rangle > c\}.$$

As we have seen above in the first case $\langle x,N\rangle < 0$ for every $x\in\Gamma^\circ\backslash\{0\}$ as well as $0<c$. Hence $c\langle N,y\rangle\leq 0$ for all $y\in\Gamma^\circ$ if $\Gamma^\circ\subseteq\{x\in\mathbb{R}^d;\langle x,N\rangle<c\}$. In the same way we conclude $c\langle N,y\rangle\leq 0$ in case of $\Gamma^\circ\subseteq\{x\in\mathbb{R}^d;\langle x,N\rangle>c\}$. Therefore, $c\langle N,y\rangle\leq 0$ for all $y\in\Gamma^\circ$ so that we get

$$\forall\, x\in H_c, y\in\Gamma^\circ:\ |x-y|^2 \geq c^2 + |(x-cN)-y|^2.$$

Hence,

$$\begin{aligned} \inf_{x\in\partial_{H_c}K} d_{\mathbb{R}^d\backslash\Gamma^\circ}(x) &= \inf_{x\in\partial_{H_c}K}\operatorname{dist}(x,\Gamma^\circ) = \inf_{x\in H_c\cap\partial B_R(0)}\operatorname{dist}(x,\Gamma^\circ) \\ &\geq \big(c^2 + \inf_{x\in H_c\cap\partial B_R(0)}\operatorname{dist}(x-cN,\Gamma^\circ)^2\big)^{1/2} \\ &= \big(c^2+\inf_{v\in M}\operatorname{dist}(v,\Gamma^\circ)^2\big)^{1/2} \\ &= (c^2+\delta^2)^{1/2} > |c| \geq \inf_{x\in K}\operatorname{dist}_{\mathbb{R}^d\backslash\Gamma^\circ}(x), \end{aligned}$$

so that $d_{\mathbb{R}^d\backslash\Gamma^\circ}$ does not satisfy the minimum principle in H_c contradicting the hypothesis. □

Combining the previous proposition with theorem 2.11 gives the next result which characterizes P-convexity in the complement of convex cones.

Theorem 2.21. *Let $\Gamma\neq\mathbb{R}^d$ be an open convex cone in $\mathbb{R}^d$ and $X:=\mathbb{R}^d\backslash\Gamma^\circ$. Let P be a non-constant polynomial with principal part P_m.*

i) X is P-convex for supports if and only if $P_m(y)\neq 0$ for all $y\in\Gamma$.

ii) X is P-convex for singular supports if and only if $\sigma_P(y)\neq 0$ for all $y\in\Gamma$.

iii) $X\times\mathbb{R}$ is P^+-convex for singular supports if and only if $\sigma_P^0(y)\neq 0$ for all $y\in\Gamma$.

Proof. If X is P-convex for (singular) supports it follows that d_X satisfies the minimum principle in every hyperplane $H=\{x\in\mathbb{R}^d;\langle x,N\rangle=c\}$ with $P_m(N)=0$ or $\sigma_P(N)=0$, respectively. Hence, necessity of the conditions in i) and ii) follow from 2.20. On the other hand, sufficiency of these conditions follows immediately from theorem 2.11. Thus, i) and ii) are proved.

Finally, to prove iii) observe that by [18, Corollary 11.3.2] P^+-convexity for singular supports of $X\times\mathbb{R}$ in particular implies that $d_{X\times\mathbb{R}}$ satisfies the minimum principle in every affine subspace $H=\{x\in\mathbb{R}^d;\langle x,N\rangle=c\}\times\{0\}$

with $0=\sigma_{P^+}(span\{N\}\times\mathbb{R})=\sigma_P^0(N)$, where we used lemma 2.7. Hence, if $X\times\mathbb{R}$ is P^+-convex for singular supports d_X satisfies the minimum principle in every hyperplane $H=\{x\in\mathbb{R}^d;\langle x,N\rangle=c\}$ with $\sigma_P^0(N)=0$, so that $\sigma_P^0(y)\neq 0$ for every $y\in\Gamma$ due to proposition 2.20. This proves necessity in iii) while sufficiency is again an immediate consequence of theorem 2.11. □

As an immediate consequence we obtain the next result.

Corollary 2.22. *Let $X_0\subseteq\mathbb{R}^d$ be open and convex and let $\Gamma_1,\Gamma_2,\dots$ be a sequence of open convex cones, all different from $\mathbb{R}^d$. Moreover, let $x_1,x_2\dots$ be a sequence in X_0. Denote by X the interior of $X_0\cap\bigcap_{n=1}^\infty(x_n+\Gamma_n^\circ)^c$ and assume that for every $n\in\mathbb{N}$ we have $\varepsilon_n>0$ such that*

$$B_{\varepsilon_n}(x_n)\cap(x_n+\Gamma_n^\circ)^c\subseteq X. \tag{4}$$

Then the following holds for a non-constant polynomial P.

i) X is P-convex for supports if and only if $P_m(y)\neq 0$ for every $y\in\cup_{n=1}^\infty\Gamma_n$, where P_m is the principal part of P.

ii) X is P-convex for singular supports if and only if $\sigma_P(y)\neq 0$ for every $y\in\cup_{n=1}^\infty\Gamma_n$.

iii) $X\times\mathbb{R}$ is P^+-convex for singular supports if and only if $\sigma_P^0(y)\neq 0$ for every $y\in\cup_{n=1}^\infty\Gamma_n$.

Proof. Since for non-constant polynomials Q convex sets are Q-convex for (singular) supports and the interior of arbitrary intersections of Q-convex sets for (singular) supports are again Q-convex for (singular) supports (cf. [18, Theorems 10.6.4 and 10.7.4]) the sufficiency of the conditions follows from theorem 2.21.

We only prove necessity in iii) since the corresponding proofs for parts i) and ii) are the same modulo obvious changes.

Let $X\times\mathbb{R}$ be P^+-convex for singular supports. Assume that there is $j\in\mathbb{N}$ and $y\in\Gamma_j$ such that $\sigma_P^0(y)=0$. Without restriction let $|y|=1$. Then $H:=\{x\in\mathbb{R}^{d+1};\langle x,y\rangle=\langle x_j,y\rangle\}$ is a hyperplane through x_j with $\sigma_P^0(H^\perp)=0$ and $H\cap(x_j+\Gamma_j^\circ)=\{x_j\}$ by proposition 2.10. Without loss of generality we can assume that $x_j+\Gamma_j^\circ\subseteq\{x\in\mathbb{R}^{d+1};\langle x,y\rangle\geq\langle x_j,y\rangle\}$.

For $c>0$ set $H_c:=\{x\in\mathbb{R}^{d+1};\langle x,y\rangle=\langle x_j,y\rangle-c\}$ and $K_c:=H_c\cap B_{2c}(x_j)$. Then $K_c\neq\emptyset$ is compact and due to condition (4) we have

$$\forall\, 0<c<\varepsilon_j/4:\ K_c\subseteq X$$

as well as

$$\inf_{x\in K_c}d_X(x)=\inf_{x\in K_c}d_{\mathbb{R}^d\backslash(x_j+\Gamma_j^\circ)}(x).$$

As in the proof of proposition 2.20 it follows that

$$\inf_{x\in K_c}d_{\mathbb{R}^d\backslash(x_j+\Gamma_j^\circ)}(x)=c<\inf_{x\in\partial_{H_c}K_c}d_{\mathbb{R}^d\backslash(x_j+\Gamma_j^\circ)}(x).$$

Hence by lemma 2.7 for $0<c<\varepsilon/4$ the affine subspace $H_c\times\{0\}$ of $\mathbb{R}^{d+1}$ satisfies $\sigma_{P^+}((H_c\times\{0\})^\perp)=\sigma_P^0(H_c^\perp)=\sigma_P^0(y)=0$ but for the compact subset

$K_c \times \{0\}$ of $(H_c \times \{0\}) \cap (X \times \mathbb{R})$ we have

$$\begin{aligned}\inf_{x \in K_c \times \{0\}} d_{X \times \mathbb{R}}(x) &= \inf_{x \in K_c} d_X(x) = \inf_{x \in K_c} d_{\mathbb{R}^d \setminus (x_j + \Gamma_j^\circ)}(x) = c \\ &< \inf_{x \in \partial_{H_c} K_c} d_{\mathbb{R}^d \setminus (x_j + \Gamma_j^\circ)}(x) \\ &= \inf_{x \in \partial_{H_c \times \{0\}} K_c \times \{0\}} d_{X \times \mathbb{R}}(x).\end{aligned}$$

So the minimum principle for $d_{X \times \mathbb{R}}$ is not valid in $H_c \times \{0\}$ which contradicts the P^+-convexity for singular supports of $X \times \mathbb{R}$ by [18, Corollary 11.3.2]. □

Remark 2.23. It should be noted that for sufficiency of the above conditions instead of X_0 being convex, in part i) one only needs X_0 to be P-convex for supports while in parts ii) and iii) it suffices to let X_0 be P-convex for singular supports, respectively $X_0 \times \mathbb{R}$ be P^+-convex for singular supports. For necessity of the above conditions, X_0 can be arbitrary.

Example 2.24. Let $d > 2$ and $P(x_1, \ldots, x_d) = x_1^2 - x_2^2 - \ldots - x_d^2$ be the polynomial inducing the wave operator. Moreover, let

$$\Gamma := \{x \in \mathbb{R}^d;\ x_d > (x_1^2 + \ldots + x_{d-1}^2)^{1/2}\}.$$

Then Γ is an open convex cone with $\Gamma^\circ = \overline{\Gamma}$. Set $X := \mathbb{R}^d \setminus \overline{\Gamma}$. Since no zero of the principal part of P belongs to Γ it follows from theorem 2.21 i) that X is P-convex for supports.

On the other hand, one easily checks that $Q(\xi_1, \ldots, \xi_d) = (\xi_1 - \xi_2)/2$ is a localization of P at infinity in direction $1/\sqrt{2}\,(1, 1, 0, \ldots, 0)$. Hence it follows for $e_d = (0, \ldots, 0, 1)$ that $\tilde{Q}_{span\{e_d\}}(0, t) = 0$ for every $t \geq 1$ so that $\sigma_P(e_d) = 0$ by lemma 2.6 iv). From $e_d \in \Gamma$ and theorem 2.21 ii) we conclude that X is not P-convex for singular supports. Hence, the wave operator is surjective on $\mathscr{E}(X)$ but not surjective $\mathscr{D}'(X)$. It will be shown in chapter 4 that for this example $d > 2$ is essential.

3 Surjectivity of augmented linear partial differential operators with constant coefficients

In chapter 1 we already encountered the problem posed by Bonet and Domański [5, Problem 9.1] whether for a surjective differential operator $P(D)$ on $\mathscr{D}'(X)$ its augmented operator $P^+(D)$ is surjective on $\mathscr{D}'(X\times\mathbb{R})$, where $P^+(x_1,\ldots,x_{d+1}) := P(x_1,\ldots,x_d)$. In the first section of this chapter we give some non-trivial examples of open sets X in $\mathbb{R}^d$ such that surjectivity of special classes of operators $P(D)$ on $\mathscr{D}'(X)$ implies surjectivity of the augmented operator $P^+(D)$ on $\mathscr{D}'(X\times\mathbb{R})$. That this implication is not true in general will be shown in the second section of the present chapter thus answering the above problem in the negative.

3.1 Some positive results

As proved by Bonet and Domański in [5] for a surjective partial differential operator

$$P(D):\mathscr{D}'(X)\to\mathscr{D}'(X)$$

the augmented operator $P^+(D)$ is surjective on $\mathscr{D}'(X\times\mathbb{R})$ if and only if the kernel of $P(D)$ possesses the linear topological invariant $(P\Omega)$. Since for elliptic polynomials P, or more general hypoelliptic P, the kernels of

$$P(D):\mathscr{E}(X)\to\mathscr{E}(X) \text{ and } P(D):\mathscr{D}'(X)\to\mathscr{D}'(X)$$

coincide as locally convex spaces, it is a Fréchet-Schwartz space and therefore it has $(P\Omega)$ if and only if it has the linear topological invariant (Ω). It was proved by Vogt in [39] that the kernel of an elliptic operator always has (Ω). As elliptic operators are always surjective it follows therefrom that $P^+(D)$ is always surjective in case of P being elliptic.

As we have seen in section 2.2 it is possible to give sufficient conditions for P^+-convexity for singular supports of $X\times\mathbb{R}$ in terms of P and X involving the function σ_P^0. So in order to investigate the above problem of Bonet and Domański it will be helpful to investigate the connection between σ_P and σ_P^0. Having at our disposal the alternative representation of σ_P given in lemma 2.6 this will be accomplished in the next lemma. Part iii) is [15, Lemma 6.1]. Part ii) in particular implies that for every non-elliptic polynomial P there is a non-trivial subspace $V\subseteq\mathbb{R}^d$ such that $\sigma_P^0(V)=0$.

Lemma 3.1. *Let $P\in\mathbb{C}[X_1,\ldots,X_d]$ be a non-constant polynomial with principal part P_m and let $V\subseteq\mathbb{R}^d$ be a subspace.*

i) $\sigma_P^0(V)\leq\sigma_P(V)$.

ii) If $V\subseteq\{\xi\in\mathbb{R}^d;\, P_m(\xi)=0\}$ *then* $\sigma_P^0(V)=0$.

iii) $\sigma_P^0(V)\leq\sigma_{P_m}^0(V)$.

Proof. i) is obvious from the definitions.

Obviously $\sigma_P^0(V)\leq\frac{\tilde{P}_V(0,t)}{\tilde{P}(0,t)}$ for every $t>1$. If $P(\xi)=\sum_{0\leq|\alpha|\leq m}c_\alpha\xi^\alpha$ with $c_\alpha\neq 0$ for some α with $|\alpha|=m$, we define $P_j(\xi):=\sum_{|\alpha|=j}c_\alpha\xi^\alpha, 0\leq j\leq m$.

Thus, $P(\xi) = \sum_{j=0}^m P_j(\xi)$, where each P_j is a homogeneous polynomial of degree j and P_m is the principal part of P.

If $V \subseteq \{\xi \in \mathbb{R}^d;\, P_m(\xi) = 0\}$ it follows for $t > 1$

$$\frac{\tilde{P}_V(0,t)}{t^m} = \sup_{x\in V, |x|\le t} |\sum_{j=0}^{m} \frac{1}{t^m} P_j(x)| = \sup_{x\in V, |x|\le 1} |\sum_{j=0}^{m-1} \frac{1}{t^{m-j}} P_j(x)|.$$

Moreover, for $t > 1$ we have

$$\tilde{P}(0,t) = t^m \sup_{|x|\le 1} |\sum_{j=0}^{m} \frac{1}{t^{m-j}} P_j(x)|,$$

so that

$$\lim_{t\to\infty} \frac{\tilde{P}_V(0,t)}{\tilde{P}(0,t)} = 0$$

proving ii).

In order to show iii) we note that

$$\tilde{P}_V(\xi,t) \ge \sigma_P^0(V)\tilde{P}(\xi,t) \tag{5}$$

for any $\xi \in \mathbb{R}^d, t \ge 1$. For any $s \ge 1$ and any subspace $W \subseteq \mathbb{R}^d$ we have

$$\tilde{P}_W(s\xi, st) = \sup_{x\in W, |x|\le st} |P(s\xi + x)| = s^m \sup_{x\in W, |x|\le t} |P_m(\xi + x) + O(s^{-1})|.$$

Using this for $W = V$ and $W = \mathbb{R}^d$ and inserting the results into inequality (5) we obtain after division by s^m

$$\sup_{x\in V, |x|\le t} |P_m(\xi + x) + O(s^{-1})| \ge \sigma_P^0(V) \sup_{|x|\le t} |P_m(\xi + x) + O(s^{-1})|.$$

Letting s tend to infinity yields

$$\tilde{P}_{mV}(\xi,t) \ge \sigma_P^0(V)\tilde{P}_m(\xi,t)$$

for every $\xi \in \mathbb{R}^d, t \ge 1$ which implies iii)

□

Next we consider special classes of polynomials to which we want to apply the results from chapter 2 in order to examine surjectivity of augmented differential operators. The notion of equal strength of operators will be used in several of our considerations so that we recall the definition here.

For two polynomials $P, Q \in \mathbb{C}[X_1, \ldots, X_d]$ P is called *stronger than* Q (and Q *weaker than* P) if there is $C > 0$ such that

$$\tilde{Q}(\xi,1) \le C\tilde{P}(\xi,1)$$

for every $\xi \in \mathbb{R}^d$. We write $Q \prec P$ if P is stronger than Q. P and Q are called *equally strong* if P is stronger than Q and vice versa. Moreover, we say that P *dominates* Q and write $Q \prec\prec P$ if

$$\lim_{t\to\infty} \sup_{\xi\in\mathbb{R}^d} \frac{\tilde{Q}(\xi,t)}{\tilde{P}(\xi,t)} = 0.$$

Obviously, P is stronger than Q whenever P dominates Q. Furthermore, P and $P + \alpha Q$ are equally strong for every $\alpha \in \mathbb{C}$ if and only if P dominates Q by [18, Corollary 10.4.8].

The next theorem is the reason why these notions are important for our concerns. Part i) is [18, Theorem 11.3.14].

Theorem 3.2. *Let P and Q be equally strong polynomials and $V \subseteq \mathbb{R}^d$ a subspace.*

i) $\sigma_P(V) = 0$ *if and only if* $\sigma_Q(V) = 0$.

ii) $\sigma^0_P(V) = 0$ *if and only if* $\sigma^0_Q(V) = 0$.

Proof. For every $R \in \mathbb{C}[X_1, \dots, X_d]$ we have

$$\forall\, \xi \in \mathbb{R}^{d+1} : \tilde{R^+}(\xi, 1) = \tilde{R}(\xi', 1),$$

where $\xi' = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$. Thus, P^+ is stronger than Q^+ if P is stronger than Q. By lemma 2.7 we have $\sigma^0_P(V) = 0$ if and only if $\sigma_{P^+}(V \times \{0\}) = 0$ and $\sigma^0_Q(V) = 0$ if and only if $\sigma_{Q^+}(V \times \{0\}) = 0$. Therefore, ii) follows from i). □

Next we consider semi-elliptic polynomials. Recall that a polynomial P is called *semi-elliptic* if it is possible to represent P as

$$P(\xi) = \sum_{|\alpha:\mathbf{m}|\leq 1} a_\alpha \xi^\alpha$$

with $P^0(\xi) := \sum_{|\alpha:\mathbf{m}|=1} a_\alpha \xi^\alpha \neq 0$ for any $\xi \in \mathbb{R}^d \backslash \{0\}$. Here $\mathbf{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$ and $|\alpha : \mathbf{m}| = \sum_{j=1}^d \alpha_j / m_j$. If P is an elliptic polynomial of degree m, it is easily seen that P is semi-elliptic by taking $m_j = m$ for every $1 \leq j \leq d$. On the other hand, the polynomial $P(\xi) = i\xi_1 + \xi_2^2 + \ldots + \xi_d^2$ inducing the heat operator is not elliptic but semi-elliptic ($m_1 = 1, m_2 = \ldots = m_d = 2$).

In the next proposition we recall some obvious properties of semi-elliptic polynomials. In fact, parts iii) and v) are taken from the proof of [18, Theorem 11.1.11]. We include its proof for the sake of completeness.

Proposition 3.3. *Let $P(\xi) = \sum_{|\alpha:\mathbf{m}|\leq 1} a_\alpha \xi^\alpha$ be a semi-elliptic polynomial of degree m, $P^0(\xi) = \sum_{|\alpha:\mathbf{m}|=1} a_\alpha \xi^\alpha$. Then the following properties hold.*

i) The degree m of P equals $\max_{1\leq j\leq d} m_j$.

ii) The principal part P_m is a part of P^0, i.e. there is a polynomial R of degree $\leq m-1$ such that $P^0 = P_m + R$ and $P(\xi) - P_m(\xi) - R(\xi) = \sum_{|\alpha:\mathbf{m}|<1} a_\alpha \xi^\alpha$.

iii) There is $C > 0$ such that $\sum_{j=1}^d |\xi_j|^{m_j} \leq C|P^0(\xi)|$ for every $\xi \in \mathbb{R}^d$.

iv) $P_m(x) = 0$ for $x \in \mathbb{R}^d$ if and only if $x_j = 0$ for every j with $m_j = m$. In particular, $\{\xi \in \mathbb{R}^d ; P_m(\xi) = 0\}$ is a subspace of $\mathbb{R}^d$.

v) For α with $|\alpha : \mathbf{m}| \leq 1$ we have $|\xi^\alpha| \leq 1 + \sum_{j=1}^d |\xi_j|^{m_j}$.

vi) P^0 dominates $P - P^0$. In particular, P^0 and P are equally strong.

Proof. In case of $d = 1$ part i) is trivial so let $d > 1$. Not every monomial appearing in P^0 depends on ξ_1, for if this was true then $P^0(0, \xi_2, \ldots, \xi_d) = 0$ for every choice of $\xi_2, \ldots, \xi_d \in \mathbb{R}$ contradicting the semi-ellipticity of P. If $d > 2$ from these monomials independent of ξ_1, not every monomial depends of ξ_2 for this would yield $P^0(0, 0, \xi_3, \ldots, \xi_d) = 0$ for all $\xi_3, \ldots, \xi_d \in \mathbb{R}$ again contradicting the semi-ellipticity of P. Continuing in that way we finally find a monomial in P^0 which only depends on ξ_d. For the exponent α of this monomial we have, since it is part of P^0, that $1 = |\alpha : \mathbf{m}| = \alpha_d/m_d$. Because $|\alpha| \leq m$ this gives $m_d \leq m$. In the same way we get $m_j \leq m$ for every $j = 1, \ldots, d$.

Now, for every α with $|\alpha| = m$ and $a_\alpha \neq 0$ we have $1 \geq |\alpha : \mathbf{m}|$. If $m > m_j$ for some j with $\alpha_j \neq 0$ we get $1 \geq \sum \frac{\alpha_l}{m_l} > \sum \frac{\alpha_l}{m}$ contradicting $|\alpha| = m$. This shows $m = \max m_j$ and $m_j = m$ for every j such that there is α with $|\alpha| = m, a_\alpha \neq 0, \alpha_j \neq 0$ which implies i) and ii). Moreover, if α is the exponent of a monomial in P_m we have $m_j = m$ for every j with $\alpha_j \neq 0$. Therefore, $P_m(x) = 0$ if $x_j = 0$ for every j with $m_j = m$, i.e. this proves sufficiency in iv).

In order to prove iii) we observe that due to the semi-ellipticity of P and the compactness of $K := \{\xi \in \mathbb{R}^d; \sum |\xi_j|^{m_j} = 1\}$ there is some $C > 0$ such that $C|P^0(\xi)| \geq 1$ for every $\xi \in K$. For arbitrary $\xi \in \mathbb{R}^d \backslash \{0\}$ we have $1 = \sum |t^{1/m_j} \xi_j|^{m_j}$ with $t := (\sum |\xi_j|^{m_j})^{-1}$ so that

$$1 \leq C P^0(t^{1/m_1}\xi_1, \ldots, t^{1/m_d}\xi_d) = C\, t P^0(\xi),$$

proving iii).

To prove necessity in iv), note that by iii) there is some $C > 0$ such that $\sum |\xi_j|^{m_j} \leq C|P^0(\xi)|$ for all $\xi \in \mathbb{R}^d$. If $P_m(x) = 0$ it follows from the homogeneity of P_m and ii) that for l with $m_l = m$ and $t > 0$ sufficiently large

$$t^m |x_l|^m \leq \sum_{j=1}^{d} |t x_j|^{m_j} \leq C|P^0(tx)| \leq C' t^{m-1}$$

which shows $x_l = 0$.

In order to prove v), the trivial inequality $|\xi_k|^{m_k} \leq \sum_{j=1}^{d} |\xi_j|^{m_j}$ implies for $\alpha \neq 0$ with $|\alpha : \mathbf{m}| \leq 1$ that

$$|\xi^\alpha| \leq \Big(\sum_{j=1}^{d} |\xi_j|^{m_j} \Big)^{|\alpha:\mathbf{m}|} \leq \sum_{j=1}^{d} |\xi_j|^{m_j}.$$

This proves v).

Finally, to prove vi) we set $S := P - P^0$. For $\xi \in \mathbb{R}^d$ we have for some constant $C_1 > 0$

$$|S(\xi)|^2 \leq C_1 \sum_{|\alpha:\mathbf{m}|<1} |a_\alpha|^2 |\xi^\alpha|^2.$$

Without loss of generality, let $m_1 = m$ so that for $t > 0$ we have with iii) for some $C > 0$

$$\begin{aligned}\tilde{P^0}(\xi,t)^2 &= \sup_{|\eta|<1}|P^0(\xi+t\eta)|^2 \geq C\sup_{|\eta|<1}(\sum_{j=1}^d|\xi_j+t\eta_j|^{m_j})^2\\ &\geq C_2\sup_{|\eta|<1}(\sum_{j=1}^d|\xi_j+t\eta_j|^{2m_j}) \geq C_3\sup_{\sigma\in\{-1,1\}}(\sum_{j=2}^d\xi_j^{2m_j}+(\xi_1+\sigma t)^{2m})\\ &\geq C_4(\sum_{j=1}^d\xi^{2m_j}+t^{2m}),\end{aligned}$$

for suitable constants $C_2, C_3, C_4 > 0$ independent of ξ and t.

From this and the fact that for α with $|\alpha : \mathbf{m}| < 1$ we have $\alpha_l < m_l \leq m$ for some l we get for $t \geq 1$

$$\begin{aligned}\frac{|S(\xi)|^2}{\tilde{P^0}(\xi,t)^2} &\leq C'\sum_{|\alpha:\mathbf{m}|<1}|a_\alpha|^2\prod_{j=1}^d\frac{\xi_j^{2\alpha_j}}{\sum_{k=1}^d\xi_k^{2m_k}+t^{2m}}\\ &\leq C'\sum_{|\alpha:\mathbf{m}|<1}|a_\alpha|^2\frac{\xi_l^{2(m_l-1)}}{\xi_l^{2m_l}+t^{2m}}\\ &\leq C''\sum_{|\alpha:\mathbf{m}|<1}|a_\alpha|^2(t^{2m})^{-1/m_l}\leq C'''t^{-2}\end{aligned}$$

where in the third inequality we used that

$$f:[0,\infty)\to\mathbb{R}, f(x) := x^{2m_l-2}/(x^{2m_l}+c)$$

for $c > 0$ is bounded by Mc^{-1/m_l} for some constant M.

It follows that

$$\inf_{t>1}(\sup_{\xi\in\mathbb{R}^d}\frac{|S(\xi)|}{\tilde{P^0}(\xi,t)}) = 0$$

so that by [18, Theorem 10.4.6] P^0 dominates S. This proves v). □

Theorem 3.4. *Let $P(\xi) = \sum_{|\alpha:\mathbf{m}|\leq 1} a_\alpha\xi^\alpha$ be a semi-elliptic polynomial of degree m on $\mathbb{R}^d$ and V a subspace of $\mathbb{R}^d$. Then we have $\sigma_P^0(V) = 0$ if and only if V is a subspace of $\{\xi \in \mathbb{R}^d;\ P_m(\xi) = 0\}$.*

Proof. By proposition 3.3 the polynomials $P^0(\xi) = \sum_{|\alpha:\mathbf{m}|=1} a_\alpha\xi^\alpha$ and P are equally strong, thus $\sigma_P^0(V) = 0$ if and only if $\sigma_{P^0}^0(V) = 0$ by theorem 3.2 ii). If $V \subseteq \{\xi \in \mathbb{R}^d;\ P_m(\xi) = 0\}$ it follows from lemma 3.1 ii) that $\sigma_P^0(V) = 0$ so that we only have to show $\sigma_{P^0}^0(V) > 0$ if V is not contained in $\{\xi \in \mathbb{R}^d;\ P_m(\xi) = 0\}$.

By proposition 3.3 iii) V is a subspace of $\{\xi \in \mathbb{R}^d;\ P_m(\xi) = 0\}$ if and only if for each $x \in V$ we have $x_j = 0$ for every j with $m_j = m$.

Assume there is $x \in V$ such that $x_l \neq 0$ for some l with $m_l = m$. Without loss of generality let $|x| = 1$. Then by proposition 3.3 iii) we have for suitable constants $C, C' > 0$

$$\begin{aligned}
\tilde{P^0}_V(\xi,t)^2 &\geq \sup_{|\lambda|\leq t} |P^0(\xi+\lambda x)|^2 \\
&\geq C \sup_{|\lambda|\leq t} (\sum_{j=1}^d |\xi_j+\lambda x_j|^{m_j})^2 \\
&\geq C' \sum_{j=1}^d ((\xi_j+tx_j)^{2m_j} + (\xi_j-tx_j)^{2m_j}) \\
&\geq 2C' \left(\sum_{j=1}^d \xi_j^{2m_j} + \sum_{j=1}^d t^{2m_j} x_j^{2m_j}\right) \\
&\geq 2C' \left(\sum_{j=1}^d \xi_j^{2m_j} + t^{2m} x_l^{2m}\right).
\end{aligned}$$

Since for α with $|\alpha : \mathbf{m}| \leq 1$ we have $|\xi^\alpha| \leq 1 + \sum_{j=1}^d |\xi_j|^{m_j}$ by proposition 3.3 v) we get for $t \geq 1$ with suitable constants $C'', C''' > 0$

$$\begin{aligned}
\tilde{P^0}(\xi,t)^2 &= \sup_{|y|\leq t} |P^0(\xi+y)|^2 \leq C'' \left(1 + \sup_{|y|\leq t} (\sum_{j=1}^d |\xi_j+y_j|^{m_j})\right)^2 \\
&\leq C'''(1+\sum_{j=1}^d \xi_j^{2m_j} + dt^{2m}) \\
&\leq C'''(\sum_{j=1}^d \xi_j^{2m_j} + (d+1)t^{2m}).
\end{aligned}$$

Observing that $x_l \leq 1$, these estimates give for some constant $D > 0$

$$\frac{\tilde{P^0}_V(\xi,t)^2}{\tilde{P^0}(\xi,t)^2} \geq D \frac{\sum_{j=1}^d \xi_j^{2m_j} + t^{2m} x_l^{2m}}{\sum_{j=1}^d \xi_j^{2m_j} + (d+1)t^{2m}} \geq D \frac{x_l^{2m}}{d+1} > 0,$$

so $\sigma_P^0(V) > 0$. This finishes the proof. □

A different proof of the above result can be found in [15, Theorem 6.8]. The proof we presented here is taken from [10, Theorem 1]. As a first application of theorem 3.4 we show that contrary to the case of P-convexity for supports, P-convexity for singular supports of some open set X does not imply P^+-convexity for singular supports of $X \times \mathbb{R}$ in general. This example is from [10].

Example 3.5. Consider $P(\xi_1, \xi_2) = i\xi_1 + \xi_2^2$, i.e. the heat polynomial in one spatial dimension. Taking $\mathbf{m} = (1,2), a_{(1,0)} = i, a_{(0,2)} = 1$, and $a_\alpha = 0$ otherwise it follows from

$$P(\xi) = P^0(\xi) = \sum_{|\alpha:\mathbf{m}|=\mathbf{1}} a_\alpha \xi^\alpha$$

that P is semi-elliptic hence hypoelliptic by [18, Theorem 11.1.11]. Therefore

$$X := \mathbb{R}^2 \backslash \Gamma^\circ,$$

where $\Gamma = \{x \in \mathbb{R}^2; x_1 < 0, |x_1| > |x_2|\}$, is P-convex for singular supports. It is easily seen that $\Gamma^\circ = \overline{\Gamma}$. Since $P_2(1,0) = 0$ we conclude $\sigma_P^0((-1,0)) = 0$ from lemma 3.1 iv). As $(-1,0) \in \Gamma$ theorem 2.21 iii) implies that $X \times \mathbb{R}$ is not P^+-convex for singular supports. However, by theorem 2.21 i) X is not P-convex for supports.

Combining theorem 3.4, lemma 2.16, lemma 2.17, proposition 2.18, and theorem 1.5 we obtain the following result from [10].

Theorem 3.6. *Let $X \subseteq \mathbb{R}^d$ be open and let P be a non-zero polynomial with principal part P_m. If for every $x \in X$ there is $r \in \{\xi \in \mathbb{R}^d; P_m(\xi) = 0\}^\perp \backslash \{0\}$ such $dist(x, \partial X) \geq dist(y, \partial X)$ for every $y \in \{x + \lambda r; \lambda \in (0, \lambda_X(x,r))\}$ then X is P-convex for supports.*

Moreover, if additionally P is semi-elliptic then $X \times \mathbb{R}$ is P^+-convex for singular supports, hence $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ as well as $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ are surjective.

As explained at the beginning of this section, as a consequence of results due to Vogt [39] on the one hand and Bonet and Domański [5] on the other hand, $P^+(D)$ is surjective on $\mathscr{D}'(X \times \mathbb{R})$ whenever P is elliptic. Vogt's proof relied on Grothendieck's duality theory and a generalization of Hadamard's Three Circles Theorem to certain sheaves of real analytic functions. As an application of the above theorem we will now give an alternative proof from [10] of the consequence of Vogt's result.

Corollary 3.7. *Let $X \subseteq \mathbb{R}^d$ be open and let P be an elliptic polynomial. Then $P^+(D)$ is surjective on $\mathscr{D}'(X \times \mathbb{R})$.*

Proof. This follows immediately from theorem 3.6 since elliptic polynomials are semi-elliptic and $\{\xi \in \mathbb{R}^d; P_m(\xi) = 0\}^\perp = \mathbb{R}^d$. □

After having dealt with semi-elliptic polynomials we now turn our attention to homogeneous polynomials. Again we start with a simple observation.

Proposition 3.8. *Let P be a homogenous polynomial and let $V \subseteq \mathbb{R}^d$ be a subspace such that P vanishes on V. Then $V \subseteq \Lambda(P)$. In particular $V \subseteq \Lambda(Q)$ for every $Q \in L(P)$.*

Proof. By an appropriate linear change of coordinates we may assume without loss of generality that $V = \mathbb{R}^k \times \{0\}^{d-k}$ with $k = \dim V$. Using the homogeneity of P an easy induction on the degree m of P yields that $\partial^\alpha P$ vanishes on V if $|\alpha| < m$ or if $|\alpha| = m$ and $\alpha \notin V^\perp = \{0\}^k \times \mathbb{R}^{d-k}$. Since $x_{k+1} = \ldots x_d = 0$ for every $x \in V$ this implies

$$P(\xi + x) = \sum_\alpha \frac{\partial^\alpha P(0)}{\alpha!}(\xi + x)^\alpha = \sum_{|\alpha| = m, \alpha \in V^\perp} \frac{\partial^\alpha P(0)}{\alpha!}\xi^\alpha = P(\xi)$$

for every $\xi \in \mathbb{R}^d$ and $x \in V$. Hence $V \subseteq \Lambda(P)$ and therefore $V \subseteq \Lambda(Q)$ for every $Q \in L(P)$, too. □

Lemma 3.9. *Let P be a homogeneous polynomial and let $V \subseteq \mathbb{R}^d$ be a subspace. Then the following are equivalent.*

i) $\sigma_P(V) = 0$.

ii) $\sigma_P^0(V) = 0$.

Proof. Let m be the degree of P. Without loss of generality we may assume that $\{0\} \neq V$. Assume that $\sigma_P(V) > 0$. If P is not elliptic there is $N \in S^{d-1}$ such that $P(nN) = 0$ for every $n \in \mathbb{N}$. It follows with [18, Theorem 11.1.3 Ia)] that P cannot be hypoelliptic, so there is a non-constant $Q \in L(P)$. With lemma 2.6 we obtain

$$0 < \sigma_P(V) \leq \inf_{t \geq 1} \frac{\tilde{Q}_V(0,t)}{\tilde{Q}(0,t)}.$$

As Q is not constant we conclude that Q cannot be constant on V. Hence, P does not vanish identically on V by proposition 3.8.

In case of P being elliptic it follows from the homogeneity of P that $P(x) \neq 0$ for every $x \in V \backslash \{0\}$, in particular, P does not vanish identically on V.

Since $P|_V \neq 0$ the same is true for P_ξ for every $\xi \in \mathbb{R}^d$. For otherwise we had for any $x \in V \backslash \{0\}$ and every $\lambda \in \mathbb{R}$

$$0 = P(\xi + \lambda x) = \lambda^m P(x) + O(\lambda^{m-1}),$$

for $\lambda \to \infty$, so that $P(x) = 0$, i.e. $P|_V = 0$. So for every $\xi \in \mathbb{R}^d$ we have

$$0 < \tilde{P}_V(\xi, 1).$$

In particular, for every $r > 0$ there is some constant $C(r) > 0$ such that for every $|\xi| \leq r$

$$\tilde{P}_V(\xi, 1) \geq C(r)$$

since $\xi \mapsto \tilde{P}_V(\xi, 1)$ is continuous. Hence

$$\forall r > 0 \, \exists C(r) > 0 \, \forall \, |\xi| \leq r : \; \frac{\tilde{P}_V(\xi,1)}{\tilde{P}(\xi,1)} \geq \frac{C(r)}{\tilde{P}(0, r+1)} > 0.$$

Now, as $\sigma_P(V) > 0$ this immediately implies

$$0 < \inf_{\xi \in \mathbb{R}^d} \frac{\tilde{P}_V(\xi,1)}{\tilde{P}(\xi,1)}.$$

Using the homogeneity of P again we finally conclude

$$\sigma_P^0(V) = \inf_{t \geq 1} \inf_{\xi \in \mathbb{R}^d} \frac{\tilde{P}_V(\xi,t)}{\tilde{P}(\xi,t)} = \inf_{t \geq 1} \inf_{\xi \in \mathbb{R}^d} \frac{t^m \tilde{P}_V(\frac{\xi}{t},1)}{t^m \tilde{P}(\frac{\xi}{t},1)} = \inf_{\xi \in \mathbb{R}^d} \frac{\tilde{P}_V(\xi,1)}{\tilde{P}(\xi,1)} > 0$$

so that ii) implies i). As i) implies ii) by lemma 3.1 i) the lemma is proved. □

Recall that a polynomial $P \in \mathbb{C}[X_1, \ldots, X_d]$ of degree m with principal part P_m is of ***principal type*** if $\nabla P_m(\xi) \neq 0$ for all $\xi \in \{x \in \mathbb{R}^d; P_m(x) = 0\}$. As $\langle \xi, \nabla P_m(\xi) \rangle = m P_m(\xi)$ by Euler's identity for homogeneous functions, we then have $\nabla P_m(\xi) \neq 0$ for all $\xi \in \mathbb{R}^d \backslash \{0\}$. If moreover P_m has real coefficients then P is said to be of ***real principal type***. As is well-known, for polynomials P of principal type the principal part P_m and P are equally strong, see e.g. [18, Theorem 10.4.10].

The next theorem generalizes one half of [15, Theorem 6.9].

Theorem 3.10. *Let P be a polynomial of degree m with principal part P_m and let $V \subseteq \mathbb{R}^d$ be a subspace. If P and P_m are equally strong then $\sigma_P(V) = 0$ if and only if $\sigma_P^0(V) = 0$.*

Proof. It follows from the hypothesis and theorem 3.2 i) that $\sigma_P(V) = 0$ if and only if $\sigma_{P_m}(V) = 0$. By lemma 3.9 the latter is equivalent to $\sigma^0_{P_m}(V) = 0$ which in turn is equivalent to $\sigma^0_P(V) = 0$ by theorem 3.2 ii) and the hypothesis. □

The above theorem applies in particular to polynomials of principal type. For this reason as well as for notational convenience we introduce the following notion.

Definition 3.11. Let $P \in \mathbb{C}[X_1, \ldots, X_d]$ be of degree m with principal part P_m. P is said to be of *generalized principal type* if P and P_m are equally strong.

Obviously, every polynomial of principal type is of generalized principal type. More generally, if P is a polynomial acting along a subspace V and being of principal type there, then P is of generalized principal type. Additionally, every polynomial of degree 1 is of generalized principal type, or more general, every homogeneous polynomial plus some constant term is.

We now examine the class of polynomials of principal type more closely. Since by definition the characteristics of a polynomial of principal type are all simple and since by lemma 2.6 ii) we know that for a non-constant localization Q of P at infinity its direction N has to be a characteristic vector of P we now describe the localizations at infinity in direction of a simple characteristic in the next lemma. This is a specification of [14, Example 1.4.4].

Lemma 3.12. *For $N \in S^{d-1}$ we denote by $\omega(N)$ the set of all sequences $(\xi_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ tending to infinity such that $\lim_{n\to\infty} \xi_n/|\xi_n| = N$. Moreover, let P be a non-constant polynomial. We set*

$$\lambda_1(N) := \{\beta \in \mathbb{C}; \exists\, (\xi_n)_{n\in\mathbb{N}} \in \omega(N) : \ \beta = \lim_{n\to\infty} |\xi_n| P_m(\frac{\xi_n}{|\xi_n|})\}$$

If $N \in S^{d-1}$ is a simple characteristic vector for P, then we have

$$\{Q \in L_N(P); \quad Q \text{ is not constant }\} = \{x \mapsto \frac{\beta + P_{m-1}(N) + \langle \nabla P_m(N), x\rangle}{\sqrt{|\beta + P_{m-1}(N)|^2 + |\nabla P_m(N)|^2}}; \beta \in \lambda_1(N)\}.$$

In particular, $L_N(P)$ contains the set of non-constant polynomials

$$\{Q \in \mathbb{C}[X_1, \ldots, X_d]; \, Q(x) = \frac{P_{m-1}(N) + \langle \nabla P_m(N), x\rangle}{\sqrt{|P_{m-1}(N)|^2 + |\nabla P_m(N)|^2}}\}.$$

Proof. We have for $\xi, x \in \mathbb{R}^d$ by Taylor's Theorem

$$\begin{aligned} P_\xi(x) &= P_m(\xi) + P_{m-1}(\xi) + P_{m-2}(\xi) + \langle \nabla P_m(\xi) + \nabla P_{m-1}(\xi), x\rangle \\ &\quad + \frac{1}{2}\langle x, \nabla^2 P_m(\xi) x\rangle + O(|\xi|^{m-3}) \qquad (6) \\ &= P_m(\xi) + P_{m-1}(\xi) + \langle \nabla P_m(\xi), x\rangle + O(|\xi|^{m-2}) \end{aligned}$$

as $\xi \to \infty$, uniformly for $|x|$ bounded.

Moreover, with $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^d$

$$\begin{aligned}
\tilde{P}(\xi) &= \Big(|P(\xi)|^2 + |\nabla P(\xi)|^2 + \sum_{2\le|\alpha|\le m} |P^{(\alpha)}(\xi)|^2\Big)^{1/2} \\
&= \Big(|P_m(\xi) + P_{m-1}(\xi) + O(|\xi|^{m-2})|^2 + |\nabla P_m(\xi) + O(|\xi|^{m-2})\mathbf{1}|^2 \\
&\quad + O(|\xi|^{2m-4})\Big)^{1/2} \\
&= |\xi|^{m-1}\Big(|\,|\xi| P_m(\frac{\xi}{|\xi|}) + P_{m-1}(\frac{\xi}{|\xi|}) + O(|\xi|^{-1})|^2 \qquad (7)\\
&\quad + |\nabla P_m(\frac{\xi}{|\xi|}) + O(|\xi|^{-1})\mathbf{1}|^2 + O(|\xi|^{-2})\Big)^{1/2} \\
&=: |\xi|^{m-1}\, D_1(\xi),
\end{aligned}$$

as $\xi \to \infty$.

Now, let $N \in S^{d-1}$ be a simple characteristic of P. Equations (6) and (7) imply

$$\begin{aligned}
\frac{P(x+\xi)}{\tilde{P}_\xi(0)} &= \frac{\frac{1}{|\xi|^{m-1}} P(x+\xi)}{D_1(\xi)} \qquad (8)\\
&= \frac{|\xi| P_m(\frac{\xi}{|\xi|}) + P_{m-1}(\frac{\xi}{|\xi|}) + O(|\xi|^{-1}) + \langle \nabla P_m(\frac{\xi}{|\xi|}), x\rangle}{D_1(\xi)}
\end{aligned}$$

for $\xi \to \infty$, uniformly for bounded $|x|$. Hence, for any $\beta \in \lambda_1(N)$, by choosing a corresponding $(\xi_n)_{n\in\mathbb{N}} \in \omega(N)$, the polynomial

$$Q(x) = \frac{\beta + P_{m-1}(N) + \langle \nabla P_m(N), x\rangle}{\sqrt{|\beta + P_{m-1}(N)|^2 + |\nabla P_m(N)|^2}}$$

is contained in $L_N(P)$.

On the other hand, let $Q \in L_N(P)$ be a non-constant localization with corresponding $(\xi_n)_{n\in\mathbb{N}} \in \omega(N)$. We first show that for every $x \in \mathbb{R}^d$ with $Q(x) \neq Q(0)$ we have $\langle \nabla P_m(N), x\rangle \neq 0$. For if there is x with $Q(x) \neq Q(0)$ such that $\langle \nabla P_m(N), x\rangle = 0$ it follows with Taylor's Theorem

$$\begin{aligned}
0 \neq Q(x) - Q(0) &= \lim_{n\to\infty} \frac{P(\xi_n + x) - P(\xi_n)}{\tilde{P}(\xi_n)} \\
&= \lim_{n\to\infty} \frac{\langle \nabla P_m(\xi_n), x\rangle + O(|\xi_n|^{m-2})}{|\xi_n|^{m-1} D_1(\xi_n)} \qquad (9)\\
&= \lim_{n\to\infty} \frac{\langle \nabla P_m(\frac{\xi_n}{|\xi_n|}), x\rangle + O(|\xi_n|^{-1})}{D_1(\xi_n)}.
\end{aligned}$$

By assumption on x the numerator in the last line of the above expression converges to zero so that $(D_1(\xi_n))_{n\in\mathbb{N}}$ cannot have a subsequence which is bounded from below by some $\varepsilon > 0$. Therefore $(D_1(\xi_n))_{n\in\mathbb{N}}$ tends to zero. From the definition of $D_1(\xi_n)$ we immediately obtain

$$0 = \lim_{n\to\infty} |\nabla P_m(\frac{\xi_n}{|\xi_n|})| = |\nabla P_m(N)|,$$

contradicting $\nabla P_m(N) \neq 0$. Hence, if $Q \in L_N(P)$ is not constant, we have $\langle \nabla P_m(N), x\rangle \neq 0$ for every x with $Q(x) \neq Q(0)$.

For x with $Q(x) \neq Q(0)$ we therefore obtain that the numerator in (9) converges to $\langle \nabla P_m(N), x\rangle \neq 0$, so that again by (9) and the definition of $D_1(\xi_n)$ we conclude that $(D_1(\xi_n))_{n\in\mathbb{N}}$ does not have any unbounded subsequence. Therefore, by passing to a subsequence if necessary, we may assume that $(D_1(\xi_n))_{n\in\mathbb{N}}$ converges in $[0,\infty)$ and that $(|\xi_n|P_m(\xi_n/|\xi_n|))_{n\in\mathbb{N}}$ converges to some $\beta \in \mathbb{C}$, hence

$$\lim_{n\to\infty} D_1(\xi_n) = \sqrt{|\beta + P_{m-1}(N)|^2 + |\nabla P_m(N)|^2}.$$

From this and equation (8)

$$Q(x) = \lim_{n\to\infty} \frac{P_{\xi_n}(x)}{\tilde{P}_{\xi_n}(0)} = \frac{\beta + P_{m-1}(N) + \langle \nabla P_m(N), x\rangle}{\sqrt{|\beta + P_{m-1}(N)|^2 + |\nabla P_m(N)|^2}}$$

finally follows. Thus, every non-constant polynomial $Q \in L_N(P)$ is of the desired form. To finish the proof we observe that choosing $\xi_n = nN$ yields $0 \in \lambda_1(N)$.

□

Remark 3.13. In general, nothing specific can be said about the set $\lambda_1(N)$. For example, let $P(x_1, x_2) = x_1x_2$ and consider $\eta_n := (\sqrt{1-\frac{1}{n^2}}, \frac{1}{n})$. Then $|\eta_n| = 1$ and (η_n) converges to the simple zero $(1,0)$ of P. For $\beta > 0$ arbitrary and $\xi_n := n\beta\,\eta_n$ it follows

$$(\xi_n)_{n\in\mathbb{N}} \in \omega\big((1,0)\big) \text{ and } |\xi_n|P(\frac{\xi_n}{|\xi_n|}) = \beta\sqrt{1-\frac{1}{n^2}} \longrightarrow_{n\to\infty} \beta.$$

Moreover, considering $\xi_n := e^n\,\eta_n$ we obtain $\lim_{n\to\infty} |\xi_n|P(\frac{\xi_n}{|\xi_n|}) = \infty$. One easily checks that then the corresponding localization at infinity is constant.

Lemma 3.12 will now be used to derive the following result. Part ii) corrects a notational inaccuracy in [15, Theorem 6.9].

Theorem 3.14. *Let P be a polynomial with principal part P_m and let $V \subseteq \mathbb{R}^d$ be a subspace.*

i) If $N \in S^{d-1}$ is a simple characteristic vector for P with

$$\{\mathrm{Re}\nabla P_m(N), \mathrm{Im}\nabla P_m(N)\} \subseteq V^\perp$$

then $\sigma_P(V) = 0$.

ii) If P is of principal type then $\sigma_P(V) = 0$ if and only if $\sigma_P^0(V) = 0$ if and only if

$$\{\mathrm{Re}\nabla P_m(N), \mathrm{Im}\nabla P_m(N)\} \subseteq V^\perp \text{ for some } N \in S^{d-1}\cap\{\xi \in \mathbb{R}^d;\, P_m(\xi) = 0\}.$$

In particular, if P is of real principal type then $\sigma_P(V) = 0$ if and only if there is $N \in \{\xi \in \mathbb{R}^d; P_m(\xi) = 0\}\backslash\{0\}$ with $\nabla P_m(N) \in V^\perp$ which is the case e.g. for $V = span\{N\}$ for $N \neq 0, P_m(N) = 0$.

Proof. By lemma 3.12 for every simply characteristic $N \in S^{d-1}$ there is a non-constant $Q \in L_N(P)$ of the form

$$Q(x) = \frac{\beta + P_{m-1}(N) + \langle \nabla P_m(N), x\rangle}{\sqrt{|\beta + P_{m-1}(N)|^2 + |\nabla P_m(N)|^2}}$$

for some $\beta \in \lambda_1(N)$. Hence, if $\{\mathrm{Re}\nabla P_m(N), \mathrm{Im}\nabla P_m(N)\} \subseteq V^\perp$ it follows that Q is constant in V but not constant in $\mathbb{R}^d$. Therefore

$$\inf_{t\geq 1} \frac{\tilde{Q}_V(0,t)}{\tilde{Q}(0,t)} = 0,$$

thus $\sigma_P(V) = 0$ by lemma 2.6. This proves i).

Now if P is of principal type it follows from theorem 3.10 and the remark preceding it that $\sigma_P(V) = 0$ if and only if $\sigma_P^0(V) = 0$. Moreover, if $\sigma_P(V) = 0$ it follows from lemma 2.6 and lemma 3.12 i) that there are sequences $(t_n)_{n\in\mathbb{N}}$ in $[1,\infty)$, $(N_n)_{n\in\mathbb{N}}$ in $S^{d-1} \cap \{\xi \in \mathbb{R}^d;\, P_m(\xi) = 0\}$ and $(\beta_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ such that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{\sup_{x\in V, |x|\leq t_n} |\beta_n + P_{m-1}(N_n) + \langle \nabla P_m(N_n), x\rangle|}{\sup_{|x|\leq t_n} |\beta_n + P_{m-1}(N_n) + \langle \nabla P_m(N_n), x\rangle|} \\ &= \lim_{n\to\infty} \frac{\sup_{x\in V, |x|\leq 1} |\frac{\beta_n + P_{m-1}(N_n)}{t_n} + \langle \nabla P_m(N_n), x\rangle|}{\sup_{|x|\leq 1} |\frac{\beta_n + P_{m-1}(N_n)}{t_n} + \langle \nabla P_m(N_n), x\rangle|}, \end{aligned} \tag{10}$$

in particular, for suitable $c > 0$

$$0 = \lim_{n\to\infty} \frac{|\frac{\beta_n + P_{m-1}(N_n)}{t_n}|}{\sup_{|x|\leq 1} |\frac{\beta_n + P_{m-1}(N_n)}{t_n}| + c},$$

as the $(N_n)_{n\in\mathbb{N}}$ are unit vectors. We conclude

$$0 = \lim_{n\to\infty} \frac{\beta_n + P_{m-1}(N_n)}{t_n}.$$

Choosing a converging subsequence of $(N_n)_{n\in\mathbb{N}}$ with limit N we have $N \in S^{d-1} \cap \{\xi \in \mathbb{R}^d;\, P_m(\xi) = 0\}$ and equation (10) yields

$$0 = \frac{\sup_{x\in V, |x|\leq 1} |\langle \nabla P_m(N), x\rangle|}{\sup_{|x|\leq 1} |\langle \nabla P_m(N), x\rangle|} = \frac{\sup_{x\in V, |x|\leq 1} |\langle \nabla P_m(N), x\rangle|}{|\nabla P_m(N)|}$$

so that

$$\forall\, x \in V:\ 0 = \langle \mathrm{Re}\nabla P_m(N), x\rangle + i\langle \mathrm{Im}\nabla P_m(N), x\rangle$$

showing $\{\mathrm{Re}\nabla P_m(N), \mathrm{Im}\nabla P_m(N)\} \subseteq V^\perp$.

On the other hand, applying i) it follows that $\sigma_P(V) = 0$ if

$$\{\mathrm{Re}\nabla P_m(N), \mathrm{Im}\nabla P_m(N)\} \subseteq V^\perp$$

for some $N \in S^{d-1} \cap \{\xi \in \mathbb{R}^d;\, P_m(\xi) = 0\}$. This proves ii).

□

As shown in the proof of [18, Theorem 10.4.5] there is a constant C depending only on d and the maximal degree of the polynomials P_1, and P_2 such that for all $\xi \in \mathbb{R}^d, t \geq 1$

$$\frac{1}{C}\tilde{P}_1(\xi,t)\tilde{P}_2(\xi,t) \leq \widetilde{P_1P_2}(\xi,t) \leq C\tilde{P}_1(\xi,t)\tilde{P}_2(\xi,t).$$

Herefrom follows immediately the next theorem which is [15, Theorem 6.7].

Theorem 3.15. *For $P_1, P_2 \in \mathbb{C}[X_1,\dots,X_d]$ and $P(x) := P_1(x)P_2(x)$ the following hold for any subspace $V \subseteq \mathbb{R}^d$.*

i) $\sigma_P(V) = 0$ if and only if $\sigma_{P_1}(V) = 0$ or $\sigma_{P_2}(V) = 0$.

ii) $\sigma_P^0(V) = 0$ if and only if $\sigma_{P_1}^0(V) = 0$ or $\sigma_{P_2}^0(V) = 0$.

As a consequence of the above theorem together with theorems 3.4 and 3.10 as well as the results of chapter 2 we obtain the following theorem giving further sufficient conditions for the augmented operator of a surjective differential operator to be surjective again. It will be seen in the next section that this implication is not always true in general.

Theorem 3.16. *Let X be an open subset of $\mathbb{R}^d$ and let the polynomials $Q_1,\dots,Q_n$ be semi-elliptic or of generalized principal type. Set $P := Q_1\cdots Q_n$ and denote its principal part by P_m.*

a) If for each $x \in \partial X$ there is an open convex cone $\Gamma \neq \mathbb{R}^d$ such that $(x + \Gamma^\circ) \cap X = \emptyset$ and $P_m(y)\sigma_P(y) \neq 0$ for all $y \in \Gamma$ then

$$P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X) \text{ as well as } P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$$

are surjective.

b) Let $X_0 \subseteq \mathbb{R}^d$ be open and convex and let $\Gamma_1, \Gamma_2, \dots$ be a sequence of open convex cones, all different from $\mathbb{R}^d$. Moreover, let $x_1, x_2 \dots$ be a sequence in X_0. Denote by X the interior of $X_0 \cap \bigcap_{n=1}^\infty (x_n + \Gamma_n^\circ)^c$ and assume that for every $n \in \mathbb{N}$ we have $\varepsilon_n > 0$ such that

$$B_{\varepsilon_n}(x_n) \cap (x_n + \Gamma_n^\circ)^c \subseteq X.$$

Then the following are equivalent.

i) $P(D)$ is surjective on $\mathscr{D}'(X)$.

ii) $P^+(D)$ is surjective on $\mathscr{D}'(X \times \mathbb{R})$.

iii) $P_m(y)\sigma_P(y) \neq 0$ for all $y \in \cup_{n=1}^\infty \Gamma_n$.

iv) $\sigma_P^0(y) \neq 0$ for all $y \in \cup_{n=1}^\infty \Gamma_n$.

Proof. The surjectivity of $P(D)$ on $\mathscr{D}'(X)$ in a) follows from the hypothesis and theorem 2.11. Moreover, we obtain from theorem 3.15 together with theorem 3.4 and theorem 3.10 that the hypothesis in a) together with theorem 2.11 imply the P^+-convexity for singular supports of $X \times \mathbb{R}$. Thus a) follows since $X \times \mathbb{R}$ is also P-convex for supports by theorem 1.5.

In order to prove b) we observe that i) is equivalent to iii) by corollary 2.22. It follows from theorem 3.15, theorem 3.4, and theorem 3.10 that iii) and iv) are equivalent. Moreover, iii) and iv) together with corollary 2.22 imply ii). Finally, ii) implies i) by theorem 1.5. □

3.2 A surjective differential operator on $\mathscr{D}'(X)$ with non-surjective augmented operator

In the previous section we gave some sufficient conditions on P as well as X such that surjectivity of $P(D)$ on $\mathscr{D}'(X)$ implies surjectivity of the augmented operator $P^+(D)$ on $\mathscr{D}'(X\times\mathbb{R})$. Moreover, it will be shown in section 4.2 that this implication always holds in case of $X\subseteq\mathbb{R}^2$. However, in general $P^+(D)$ need not inherit surjectivity from $P(D)$ as will be shown in this section. This answers in the negative a problem posed by Bonet and Domański in [5, Problem 9.1]. The example presented in this section will appear in [20].

As a special case of lemma 3.12 i) we have the following proposition.

Proposition 3.17. *Let P be a homogeneous polynomial of degree m and $\xi\in\mathbb{R}^d$ with $P(\xi)=0$ and $\nabla P(\xi)\neq 0$. Then $x\mapsto\sum_{j=1}^d\partial_jP(\xi)x_j=\langle\nabla P(\xi),x\rangle$ is a localization of P at infinity.*

In the example we now provide the polynomial P is hypoelliptic. Since for hypoelliptic P the kernels of

$$P(D):\mathscr{E}(X)\to\mathscr{E}(X)\text{ and }P(D):\mathscr{D}'(X)\to\mathscr{D}'(X)$$

coincide as locally convex spaces it is a Fréchet-Schwartz space. Hence it has property (Ω) if and only if it has property $(P\Omega)$. By the explanations given in chapter 1 the following theorem therefore also gives an example of a surjective hypoelliptic differential operator

$$P(D):\mathscr{E}(X)\to\mathscr{E}(X)$$

such that its kernel does not have property (Ω). Therefore it also solves an open problem from Varol [38, Section 3]. This should be compared with Vogt's classical result [39] that the kernel of an elliptic differential operators always has (Ω).

Theorem 3.18. *For any $d\geq 3$ there are an open subset $X\subseteq\mathbb{R}^d$ and a hypoelliptic polynomial $P\in\mathbb{C}[X_1,\dots,X_d]$ such that $P(D)$ is surjective on $\mathscr{D}'(X)$ but $P^+(D)$ is not surjective on $\mathscr{D}'(X\times\mathbb{R})$. Therefore its kernel $\mathscr{N}_P(X)$ does not have (Ω).*

Proof. Let Q be a homogeneous polynomial of real principal type of degree m which is not elliptic and let $x\in\mathbb{R}^d, |x|=1$ such that $Q(x)\neq 0$ but $\sigma_Q(x)=0$. It will be shown in section 4.1 that we have to choose $d\geq 3$ for this. For example, take $Q(\xi)=\xi_1^2-\xi_2^2-\ldots-\xi_d^2$ and $x=e_d=(0,\dots,0,1)$. Indeed, by proposition 3.17 applied to Q and $\xi=(1,1,0,\dots,0)$ it follows that

$$x\mapsto\langle\nabla Q(\xi),x\rangle=2x_1-2x_2$$

is a localization of Q at infinity. Because $\langle\nabla Q(\xi),e_d\rangle=0$ since $d\geq 3$ we have $\sigma_Q(e_d)=0$ by lemma 2.6 on the one hand and obviously $Q(e_d)\neq 0$ on the other hand.

By [18, Theorem 11.1.12] there is a polynomial R of degree $4m-2$ such that

$$P(\xi):=Q(\xi)^4+R(\xi) \tag{11}$$

is a hypoelliptic polynomial of degree $4m$. Clearly, for its principal part P_{4m} we have $P_{4m} = Q^4$ so $P_{4m}(x) = Q^4(x) \neq 0$. Moreover, we have

$$\sigma_P^0(x) \leq \sigma_{P_{4m}}^0(x) = \sigma_{Q^4}^0(x) = (\sigma_Q^0(x))^4 \leq (\sigma_Q(x))^4 = 0, \tag{12}$$

where we have used lemma 3.1 iii), the obvious fact that $\sigma_{P^k}^0(V) = (\sigma_P^0(V))^k$ for any $k \in \mathbb{N}$, and lemma 3.1 i).

Since P_{4m} is homogeneous and $P_{4m}(x) \neq 0$ there is an open proper convex cone $\Gamma \neq \mathbb{R}^d$ with $x \in \Gamma$ such that $P_{4m}(y) \neq 0$ for every $y \in \Gamma$. If we set $X := \mathbb{R}^d \backslash \Gamma^\circ$ it follows from theorem 2.21 i) that X is P-convex for supports. Since P is hypoelliptic X is P-convex for singular supports as well. But because $x \in \Gamma$ and $\sigma_P^0(x) = 0$ by inequality (12) $X \times \mathbb{R}$ is not P^+-convex for singular supports by theorem 2.21 iii). Thus, for the hypoelliptic polynomial P in (11)

$$P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X) \text{ is surjective}$$

but

$$P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R}) \text{ is not surjective.}$$

□

4 Linear partial differential operators with constant coefficients in two independent variables

A constant coefficient linear partial differential operator $P(D)$ can be considered as an endomorphism on various spaces of functions or distributions defined on an open subset $X \subseteq \mathbb{R}^d$. It is well-known and no surprise that the question of surjectivity of this endomorphism depends on the space of functions or distributions under consideration. From results of Malgrange [25, Théorème 4] and Hörmander [13, Theorem 3.10] it follows that surjectivity of $P(D)$ on $\mathscr{D}'(X)$ implies surjectivity on $\mathscr{E}(X)$. On the other hand, it is well-known, that in general surjectivity of $P(D)$ on $\mathscr{E}(X)$ is not sufficient to guarantee surjectivity on $\mathscr{D}'(X)$, as remarked in [13, Section 6]. (A concrete example for this is given in example 2.24.) Moreover, convexity of X is sufficient for $P(D)$ to be surjective on $\mathscr{E}(X)$, [25, Théorème 3], as well as on $\mathscr{D}'(X)$, [26].

Proving a conjecture of De Giorgi and Cattabriga [8], it was shown by Piccinini [33] that $\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ is not surjective on the space $\mathscr{A}(\mathbb{R}^d)$ of real analytic functions on $\mathbb{R}^d$ for every $d \geq 3$. In particular, convexity of X is not sufficient for $P(D)$ to be surjective on $\mathscr{A}(X)$. Moreover, one can consider $P(D)$ as an endomorphism on the space of ultradistributions of Beurling type $\mathscr{D}'_{(\omega)}(X)$ for a non-quasianalytic weight function ω. In this setting, it was shown by Langenbruch [23, Example 3.13] that in general surjectivity of $P(D)$ on $\mathscr{D}'_{(\omega)}(X)$ for fixed X depends explicitly on the weight function ω under consideration.

For all the above mentioned spaces of functions and distributions, characterizations of surjectivity of the endomorphism $P(D)$ are available, although evaluating these conditions in concrete examples is not an easy task, in general. While surjectivity of $P(D)$ on $\mathscr{E}(X)$ and $\mathscr{D}'(X)$ was characterized by Malgrange in [25] and Hörmander in [13], respectively, a characterization of the surjectivity of $P(D)$ in the setting of ultradistributions of Beurling type has been given by Björck [1]. Moreover, a characterization of surjectivity of $P(D)$ on $\mathscr{A}(X)$ by means of an application of the Proj^k-functors of Palamodov [31,32] (see also Vogt [42]) is due to Langenbruch [24]. In case of a convex open set X a different characterization of surjectivity of $P(D)$ on $\mathscr{A}(X)$ by means of a Phragmén-Lindelöff condition valid on the complex variety of P was given by Hörmander [16].

However, despite of the differences in the surjectivity on $\mathscr{E}(X)$ and $\mathscr{A}(X)$ it was shown by Zampieri in [44, 45] that in case of $d = 2$ surjectivity of $P(D)$ on $\mathscr{E}(X)$ is equivalent to surjectivity on $\mathscr{A}(X)$. Moreover, Trèves conjectured in [35, Problem 2, page 389] that for $X \subseteq \mathbb{R}^2$ surjectivity of $P(D)$ on $\mathscr{E}(X)$ implies surjectivity on $\mathscr{D}'(X)$.

The content of the present chapter is to prove the following result stating that all the above mentioned differences in the surjectivity of $P(D)$ on the various spaces of functions and distributions vanish in case of open subsets X of $\mathbb{R}^2$.

Theorem A. *Let $X \subseteq \mathbb{R}^2$ be open and $P \in \mathbb{C}[X_1, X_2]$. Then the following are equivalent.*

i) $P(D) : \mathscr{E}(X) \to \mathscr{E}(X)$ *is surjective.*

ii) $P(D) : \mathscr{A}(X) \to \mathscr{A}(X)$ *is surjective.*

iii) $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ *is surjective.*

iv) $P^+(D) : \mathscr{D}'(X \times \mathbb{R}) \to \mathscr{D}'(X \times \mathbb{R})$ *is surjective.*

v) $P(D) : \mathscr{D}'_{(\omega)}(X) \to \mathscr{D}'_{(\omega)}(X)$ *is surjective for every non-quasianalytic weight function* ω.

vi) $P(D) : \mathscr{D}'_{(\omega)}(X) \to \mathscr{D}'_{(\omega)}(X)$ *is surjective for some non-quasianalytic weight function* ω.

vii) The intersection of every connected component of X *with any characteristic line for* P *is convex.*

Herein, the equivalence of i) and vii) is due to Hörmander (see e.g. [18, Theorem 10.8.3]). As mentioned above, iii) always implies i) no matter the dimension d. Moreover, as already stated above, the equivalence of i) and ii) is due to Zampieri [44, Theorem 2]. The equivalence of i) and iii) proves in the affirmative Trèves' conjecture, while the sufficiency of iii) for iv) shows that, again contrary to arbitrary dimension, the problem of Bonet and Domański introduced in chapter 1 (see [5, Problem 9.1]) has a positive solution in the two dimensional case (compare with theorem 3.18). Note that iv) always implies iii) by theorem 1.5, again in arbitrary dimension d. That vi) always implies i) was shown by Björck in [1, Theorem 3.4.12] (see also the remark preceding theorem 4.11).

The aim of this chapter is to give proofs of the implications which are not yet proved. To be more precise, in section 4.1 we will prove Trèves' conjecture, thus showing that i) implies iii). Section 4.2 will be devoted to show that iii) implies iv), while in section 4.3 we will deal with the remaining non-trivial implication that i) suffices for v) to hold. The content of this chapter has been published in [21], [22], and [20].

4.1 On a conjecture of Trèves

Let $X \subseteq \mathbb{R}^2$ be open and $P \in \mathbb{C}[X_1, X_2]\backslash\{0\}$ be of degree m. This section is devoted to prove that surjectivity of $P(D)$ on $\mathscr{E}(X)$ implies surjectivity of $P(D)$ on $\mathscr{D}'(X)$, thus proving a conjecture of Trèves [35, Problem 2, page 389] and showing that i) implies iii) in theorem A. As usual, we denote the principal part of P by P_m. Applying the Fundamental Theorem of Algebra to the one-variable polynomial $z \mapsto P_m(z, 1)$ of degree $k \leq m$ yields for $\xi_2 \neq 0$

$$P_m(\xi_1, \xi_2) = \xi_2^m P_m(\frac{\xi_1}{\xi_2}, 1) = \xi_2^m c \prod_{j=1}^{k}(\frac{\xi_1}{\xi_2} - \alpha_j) = c\xi_2^{m-k} \prod_{j=1}^{k}(\xi_1 - \alpha_j \xi_2)$$

for some $c \in \mathbb{C}\backslash\{0\}, \alpha_j \in \mathbb{C}$. In particular it follows that

$$\{\xi \in S^1;\, P_m(\xi) = 0\}$$

is a finite set. Therefore, the only characteristic surfaces for P are hyperplanes and there are only a finite number of them, up to translations. We call them characteristic lines for obvious reasons.

In $\mathbb{R}^2$ P-convexity for supports of X is completely characterized by the following theorem due to Hörmander (cf. [18, Theorem 10.8.3]). Recall that for an elliptic polynomial P every open set is P-convex for supports (cf. [18, Corollary 10.8.2]).

Theorem 4.1. *If P is non-elliptic then the following conditions on an open connected set $X \subseteq \mathbb{R}^2$ are equivalent:*

i) X is P-convex for supports.

ii) The intersection of X with any characteristic line is convex.

iii) Every $x \in \partial X$ is the vertex of a closed proper convex cone $C \subseteq \mathbb{R}^2 \backslash X$ such that no characteristic line intersects C only at x.

By proposition 2.10 it follows that the above condition iii) is equivalent to

iii') *For every $x \in \partial X$ there is an open convex cone Γ such that $(x+\Gamma^\circ)\cap X = \emptyset$ and $P_m(y) \neq 0$ for all $y \in \Gamma$.*

Next, we want to prove a similar characterization for P-convexity for singular supports of an open, connected set $X \subseteq \mathbb{R}^2$. The following lemma will be useful not only in this task but also in proving that P-convexity for singular supports of $X \subseteq \mathbb{R}^2$ follows from P-convexity for supports.

Lemma 4.2. *For a non-constant polynomial $P \in \mathbb{C}[X_1, X_2]$ of degree m with principal part P_m we have*

$$\{y \in \mathbb{R}^2\backslash\{0\};\, \sigma_P(y) = 0\} \subseteq \{y \in \mathbb{R}^2\backslash\{0\};\, P_m(y) = 0\}.$$

In particular, $\{y \in S^1;\, \sigma_P(y) = 0\}$ is finite.

Proof. As for a hypoelliptic polynomial P the function σ_P is constantly equal to 1 we can assume without loss of generality that P is not hypoelliptic, hence not elliptic.

As observed at the beginning of this section $\{N \in S^1;\, P_m(N) = 0\}$ is finite. Let us denote its elements by $N_1, \ldots, N_l$. For each $1 \leq j \leq l$ choose $x_j \in S^1$ orthogonal to N_j. Take an arbitrary, non-constant $Q \in L(P)$ which exists because P is not hypoelliptic. By lemma 2.6 ii) there is $1 \leq j \leq l$ such that $Q \in L_{N_j}(P)$. By [18, Theorem 10.2.8] we have $Q(\xi + sN_j) = Q(\xi)$ for any $\xi \in \mathbb{R}^2, s \in \mathbb{R}$. Hence $Q(\xi) = Q(\langle \xi, x_j\rangle x_j)$ for all $\xi \in \mathbb{R}^2$. Defining

$$q : \mathbb{R} \to \mathbb{C}, s \mapsto Q(sx_j)$$

it follows that for fixed $y \in S^1$

$$\begin{aligned} \tilde{Q}_{span\{y\}}(0,t) &= \sup\{|Q(\lambda y)|;\, |\lambda| \leq t\} = \sup\{|Q(\lambda\langle y, x_j\rangle x_j)|;\, |\lambda| \leq t\} \\ &= \sup\{|q(\lambda t\langle y, x_j\rangle)|;\, |\lambda| \leq 1\}, \end{aligned}$$

and because $|x_j| = 1$ we also have

$$\begin{aligned} \tilde{Q}(0,t) &= \sup\{|Q(\xi)|;\, \xi \in \mathbb{R}^2, |\xi| \leq t\} = \sup\{|Q(\langle \xi, x_j\rangle x_j)|;\, \xi \in \mathbb{R}^2, |\xi| \leq t\} \\ &= \sup\{|Q(\lambda x_j)|;\, |\lambda| \leq t\} = \sup\{|q(\lambda t)|;\, |\lambda| \leq 1\}. \end{aligned}$$

Since $Q \in L(P)$ it follows that q is a polynomial of degree at most m. Because of the fact that on the finite dimensional space of all polynomials in one variable of degree at most m the norms $\sup_{|s|\leq 1} |p(s)|$ and $\sum_{k=0}^m |p^{(k)}(0)|$ are equivalent, there is $C > 0$ such that

$$C \sup_{|s|\leq 1} |p(s)| \geq \sum_{k=0}^{m} |p^{(k)}(0)| \geq 1/C \sup_{|s|\leq 1} |p(s)|$$

for all $p \in \mathbb{C}[X]$ with degree at most m. Applying this to the polynomials $s \mapsto q(st)$ and $s \mapsto q(st\langle y, x_j\rangle)$ gives

$$\begin{aligned}\frac{\tilde{Q}_{span\{y\}}(0,t)}{\tilde{Q}(0,t)} &\geq \frac{\sum_{k=0}^{m}|q^{(k)}(0)|t^k|\langle y,x_j\rangle|^k}{C^2\sum_{k=0}^{m}|q^{(k)}(0)|t^k}\\ &\geq |\langle y,x_j\rangle|^m/C^2,\end{aligned}$$

where we used $|\langle y, x_j\rangle| \leq 1$ in the last inequality. We conclude for every $1 \leq j \leq l$

$$\inf_{Q\in L_{N_j}(P)}\frac{\tilde{Q}_{span\{y\}}(0,t)}{\tilde{Q}(0,t)} \geq \frac{|\langle y,x_j\rangle|^m}{C^2},$$

where C only depends on the degree m of P. It follows from lemma 2.6 iii) and $\{N \in S^1;\, P_m(N) = 0\} = \{N_1, \ldots, N_l\}$ that for all $t \geq 1$

$$\liminf_{\xi\to\infty}\frac{\tilde{P}_{span\{y\}}(\xi,t)}{\tilde{P}(\xi,t)} = \min_{1\leq j\leq l}\inf_{Q\in L_{N_j}(P)}\frac{\tilde{Q}_{span\{y\}}(0,t)}{\tilde{Q}(0,t)} \geq \min_{1\leq j\leq l}\frac{|\langle y,x_j\rangle|^m}{C^2}.$$

Therefore, if for $y \in \mathbb{R}^d\backslash\{0\}$

$$0 = \sigma_P(y) = \inf_{t\geq 1}\liminf_{\xi\to\infty}\frac{\tilde{P}_{span\{y\}}(\xi,t)}{\tilde{P}(\xi,t)}$$

it follows that y is orthogonal to some x_j, hence y is a non-zero multiple of N_j which shows $P_m(y) = 0$. □

As a first application of the above lemma we characterize P-convexity for singular supports of open, connected $X \subseteq \mathbb{R}^2$ similar to the characterization of P-convexity for supports in theorem 4.1. Its proof is mutatis mutandis identical to that of theorem 4.1 but we include it for completeness' sake.

Theorem 4.3. *For $P \in \mathbb{C}[X_1, X_2]$ and an open connected set $X \subseteq \mathbb{R}^2$ the following are equivalent.*

i) X is P-convex for singular supports.

ii) The intersection of X with every hyperplane H satisfying $\sigma_P(H^\perp) = 0$ is convex.

iii) For every $x \in \partial X$ there is an open convex cone $\Gamma \neq \mathbb{R}^2$ with $\sigma_P(y) \neq 0$ for all $y \in \Gamma$ and $(x + \Gamma^\circ) \cap X = \emptyset$.

Proof. We first show that i) implies ii). It is enough to show that if $(\pm 1, 0) \in X$ and $\sigma_P((0,1)) = 0$ (i.e. parallels to the x-axis are hyperplanes H with $\sigma_P(H^\perp) = 0$), then $I = [-1,1] \times \{0\} \subseteq X$. We join $(-1,0)$ and $(1,0)$ by a polygon γ in X without self-intersection, where we can assume that γ intersects the x_1-axis only at its end points. For if this is not the case we can decompose γ into several polygons meeting the x_1-axis only at the end points and treat them separately. Then I and γ are the boundary of a connected and compact set C. We define

$$Y = \{y \in \mathbb{R};\, (x,y) \in C \text{ for some } x \in \mathbb{R}\}$$

$$Y_0 = \{y \in Y;\, (x,y) \in C \Rightarrow (x,y) \in X\}.$$

Y is a closed interval with non-empty interior and Y_0 is not empty since the end point of Y which is different from 0 belongs to Y_0. Since X is P-convex for singular supports it follows from [18, Corollary 11.3.2] that d_X satisfies the minimum principle in the hyperplane $\mathbb{R} \times \{y\}$ for arbitrary $y \in \mathbb{R}$. Therefore, if $y \in Y_0$ then from the definition of Y_0 $(x, y) \in C$ implies $(x, y) \in X$ so that $\emptyset \neq C \cap (\mathbb{R} \times \{y\}) \subseteq X \cap (\mathbb{R} \times \{y\})$ is compact. Hence for $y \in Y_0$ and x with $(x, y) \in C$ we have due to the minimum principle

$$\begin{aligned} d_X(x,y) &\geq d_X(C \cap (\mathbb{R} \times \{y\})) = d_X(\partial C \cap (\mathbb{R} \times \{y\})) \geq d_X(\gamma \cap (\mathbb{R} \times \{y\})) \\ &\geq \operatorname{dist}(\gamma, X^c). \end{aligned}$$

Because $\gamma \subseteq X$ we have $\operatorname{dist}(\gamma, X^c) > 0$, i.e. if $y \in Y_0$ then $(x, y) \in C$ implies that the distance form (x, y) to X^c is bounded below by the positive constant $\operatorname{dist}(\gamma, X^c)$. From this it follows that Y_0 is closed in Y. Since X is open, Y_0 is also open in the interval Y. Because Y_0 is not empty this implies $Y = Y_0$, hence $0 \in Y = Y_0$, so that $I = [-1, 1] \times \{0\} \subseteq X$.

Next, we prove that ii) implies iii). If $x \in \partial X$ and H is a hyperplane through x with $\sigma_P(H^\perp) = 0$ then one half ray H_1 of H bounded by x is contained in X^c by ii). If there is another hyperplane I through x with $\sigma_P(I^\perp) = 0$ such that $H_1 \cap I = \{x\}$ then one of its half rays I_1 bounded by x is contained in X^c by ii) and since X is connected it can be chosen so that the convex hull Γ° of H_1 and I_1 is contained in X^c (and obviously is a proper convex cone by $H_1 \cap I = \{x\}$). If there is a hyperplane K through x with $\sigma_P(K^\perp) = 0$ and with $K \cap \Gamma^\circ = \{x\}$ we continue extending Γ° until there is no hyperplane L with $\sigma_P(L^\perp) = 0$ intersecting Γ° only in x. Observe that by lemma 4.2 this procedure stops after a finite number of extensions so that the resulting closed convex cone is indeed proper! From proposition 2.10 it follows that for no $y \in \Gamma$ we have $\sigma_P(y) = 0$.

To finish the proof, we show that iii) implies i). But this follows from theorem 2.11 ii) which itself was inspired by the proof of the corresponding implication of [18, Theorem 10.8.3]. □

With the aid of theorem 4.1, lemma 4.2, and theorem 4.3 we give now a proof of Trèves' conjecture.

Theorem 4.4. *Let $X \subseteq \mathbb{R}^2$ be open and $P \in \mathbb{C}[X_1, X_2]$ be a non-constant polynomial with principal part P_m. Then the following are equivalent.*

i) $P(D) : \mathscr{E}(X) \to \mathscr{E}(X)$ is surjective.

ii) $P(D) : \mathscr{D}'(X) \to \mathscr{D}'(X)$ is surjective.

iii) The intersection of every characteristic line for P and any connected component of X is convex.

iv) For every connected component X_0 of X and every $x \in \partial X_0$ there is an open convex cone Γ such that $(x + \Gamma^\circ) \cap X_0 = \emptyset$ and $P_m(y) \neq 0$ for all $y \in \Gamma \backslash \{0\}$.

Proof. Without loss of generality, let X be connected. If P is elliptic i) and ii) are always satisfied. Moreover, in this case there is no characteristic line for P so that iii) is also satisfied. Choosing $\Gamma = \mathbb{R}^2$ in iv) we see that iv) is then always satisfied, too.

We therefore assume that P is not elliptic. The equivalence of i), iii) and iv) is theorem 4.1 and we only have to show that i) implies ii).

If P is hypoelliptic, X is P-convex for singular support so that i) and ii) are equivalent. If P is not hypoelliptic it follows from the equivalence of i) and iii) together with lemma 4.2 and theorem 4.3 that X is P-convex for singular supports. So i) implies ii) and the proof is finished. $\square$

4.2 The augmented differential operator of a surjective differential operator in two variables is again surjective

The purpose of this section is to prove that iii) implies iv) in theorem A, thus showing that, contrary to dimension $d \geq 3$, for open sets $X \subseteq \mathbb{R}^2$ the problem of Bonet and Domański on surjectivity of augmented differential operators posed in [5, Problem 9.1] has a positive solution (compare with section 3.2).

So let $X \subseteq \mathbb{R}^2$ open and $P \in \mathbb{C}[X_1, X_2]$. By theorem 1.5 we have to show that P-convexity for supports and singular supports of X implies P^+-convexity for singular supports of $X \times \mathbb{R}$. Recall that even for $d = 2$ it has been shown in example 3.5 that P-convexity for singular supports of X is not enough to ensure P^+-convexity for singular supports of $X \times \mathbb{R}$ in general. Without loss of generality we can assume that X is connected so that the same is true for $X \times \mathbb{R}$. [1, Theorem 3.4.12]

As in the proof of Trèves' conjecture, it will follow from theorem 4.1 that the sufficient condition given in theorem 2.11 iii) for $X \times \mathbb{R}$ to be P^+-convex for singular supports is always satisfied if X is P-convex for supports once we have proved that $\sigma_P^0(y) = 0$ implies $P_m(y) = 0$, where as usual P_m is the principal part of P.

Lemma 4.5. *Let $P \in \mathbb{C}[X_1, X_2]$ be a non-constant polynomial of degree m with principal part P_m. Then we have*

$$\{y \in \mathbb{R}^2 \backslash \{0\};\, \sigma_P^0(y) = 0\} \subseteq \{y \in \mathbb{R}^2 \backslash \{0\};\, P_m(y) = 0\}.$$

In particular, $\{y \in S^1;\, \sigma_P^0(y) = 0\}$ is finite.

Proof. Let y be a unit vector in $\mathbb{R}^2$ such that $P_m(y) \neq 0$. Then $0 \neq \sigma_P(y)$ by lemma 4.2. We assume that $\sigma_P^0(y) = 0$. Thus, denoting the span of y by $[y]$, there are sequences $(\xi_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^2$ and $(t_n)_{n\in\mathbb{N}}$ in $[1, \infty)$ such that

$$\lim_{n\to\infty} \frac{\tilde{P}_{[y]}(\xi_n, t_n)}{\tilde{P}(\xi_n, t_n)}.$$

If $(\xi_n)_{n\in\mathbb{N}}$ is bounded, we can assume without restriction, that $\lim_{n\to\infty} \xi_n = \xi$. Moreover, we can assume that $(t_n)_{n\in\mathbb{N}}$ is unbounded - and therefore tends to infinity without loss of generality. For if $(t_n)_{n\in\mathbb{N}}$ is bounded, without restriction $\lim_{n\to\infty} t_n = t$ for some $t \geq 1$, so that

$$0 = \lim_{n\to\infty} \frac{\tilde{P}_{[y]}(\xi_n, t_n)}{\tilde{P}(\xi_n, t_n)} = \frac{\tilde{P}_{[y]}(\xi, t)}{\tilde{P}(\xi, t)}.$$

But this means that

$$0 = \sup_{|\theta| \leq t} |P(\xi + \theta y)|,$$

i.e. the polynomial $P(\xi + sy)$ in $s \in \mathbb{R}$ is identically zero. Replacing ξ_n by ξ and t_n by n for each $n \in \mathbb{N}$ we have a bounded sequence $(\xi_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^2$ and a sequence $(t_n)_{n\in\mathbb{N}}$ in $[1, \infty)$ tending to infinity such that

$$0 = \lim_{n\to\infty} \frac{\tilde{P}_{[y]}(\xi_n, t_n)}{\tilde{P}(\xi_n, t_n)}.$$

Since $(\xi_n)_{n\in\mathbb{N}}$ is bounded and $(t_n)_{n\in\mathbb{N}}$ tends to infinity, there is a constant $C > 0$ such that $\sup_{|x|\leq 1}\sum_{j=0}^m |P_j(\frac{\xi_n}{t_n}+x)| \leq C$, where $P = \sum_{j=0}^m P_j$ with P_j being homogeneous of degree j. Using that $t_n \geq 1$ we obtain

$$\sup_{|x|\leq t_n} |P(\xi_n + x)| \leq \sup_{|x|\leq 1} \sum_{j=0}^{m} t_n^j |P_j(\frac{\xi_n}{t_n} + x)| \leq t_n^m C.$$

This gives

$$\begin{aligned}\frac{\tilde{P}_{[y]}(\xi_n, t_n)}{\tilde{P}(\xi_n, t_n)} &\geq \frac{|P(\xi_n + t_n y)|}{\sup_{|x|\leq t_n}|P(\xi_n + x)|}\\ &\geq \frac{t_n^{-m}|P(\xi_n + t_n y)|}{C}\\ &= \frac{|\sum_{j=0}^m t_n^{j-m} P_j(\frac{\xi_n}{t_n} + y)|}{C}.\end{aligned}$$

Since $(\xi_n/t_n)_{n\in\mathbb{N}}$ converges to zero due to the boundedness of $(\xi_n)_{n\in\mathbb{N}}$ and the fact that $(t_n)_{n\in\mathbb{N}}$ tends to infinity, it follows that for every $0 \leq j \leq m$ we have

$$\lim_{n\to\infty} P_j(y + \xi_n/t_n) = P_j(y).$$

Therefore, the right hand side of the above inequality converges to $|P_m(y)|/C$ while the left hand side converges to zero contradicting $P_m(y) \neq 0$.

So $(\xi_n)_{n\in\mathbb{N}}$ has to be unbounded. It follows from [18, Proposition 10.2.10] that for sufficiently large $\eta \in \mathbb{R}^2$ and $t \geq 1$ we have

$$\inf\{\sum_\alpha |\frac{P_\eta^{(\alpha)}(0)}{\tilde{P}_\eta(0)} - Q^{(\alpha)}(0)|t^{|\alpha|}; Q \in L(P)\} \leq C|\eta|^{-2b},$$

where C and b are positive constants. Thus, from the continuity of $(\xi, t) \mapsto \tilde{P}_{[y]}(\xi, t)$ and the equivalences of $\tilde{P}_{[y]}(\xi, t)$ and $\sum_j |\langle D, y\rangle^j P(\xi)|t^j$ as well as $\tilde{P}(\xi, t)$ and $\sum_\alpha |P^{(\alpha)}(\xi)|t^{|\alpha|}$, for n sufficiently large there are $Q_n \in L(P)$ with

$$|\frac{\tilde{P}_{[y]}(\xi_n, t_n)}{\tilde{P}(\xi_n, t_n)} - \frac{\tilde{Q}_{n\,[y]}(0, t_n)}{\tilde{Q}_n(0, t_n)}| < \sigma_P(y)/2$$

which implies

$$\inf_{t\geq 1}\inf_{Q\in L(P)} \frac{\tilde{Q}_{[y]}(0, t)}{\tilde{Q}(0, t)} \leq \limsup_{n\to\infty} |\frac{\tilde{P}_{[y]}(\xi_n, t_n)}{\tilde{P}(\xi_n, t_n)} - \frac{\tilde{Q}_{n\,[y]}(0, t_n)}{\tilde{Q}_n(0, t_n)}| \leq \sigma_P(y)/2.$$

But by lemma 2.6 we have

$$\inf_{t\geq 1}\inf_{Q\in L(P)} \frac{\tilde{Q}_{[y]}(0, t)}{\tilde{Q}(0, t)} = \sigma_P(y),$$

contradicting $\sigma_P(y) > 0$.

□

As an immediate consequence we obtain analogously to theorem 4.3 in section 4.1 a characterization of when $X \times \mathbb{R}$ is P^+-convex for singular supports. The proof is mutatis mutandis the same so that we omit it.

Theorem 4.6. *For $P \in \mathbb{C}[X_1, X_2]$ and an open connected set $X \subseteq \mathbb{R}^2$ the following are equivalent.*

i) *$X \times \mathbb{R}$ is P^+-convex for singular supports.*

ii) *The intersection of X with every hyperplane H satisfying $\sigma_P^0(H^\perp) = 0$ is convex.*

iii) *For every $x \in \partial X$ there is an open convex cone $\Gamma \neq \mathbb{R}^2$ with $\sigma_P^0(y) \neq 0$ for all $y \in \Gamma$ and $(x + \Gamma^\circ) \cap X = \emptyset$.*

Moreover, combining the above theorem with theorem 4.1 and theorem 4.4 we obtain as in section 4.1 the following, proving that iii) implies iv) in theorem A.

Theorem 4.7. *Let $X \subseteq \mathbb{R}^2$ be open and $P \in \mathbb{C}[X_1, X_2]$. If $P(D)$ is surjective on $\mathscr{D}'(X)$ then $P^+(D)$ is surjective on $\mathscr{D}'(X \times \mathbb{R})$.*

4.3 Partial differential operators on non-quasianalytic ultradistributions of Beurling type in two variables

The main purpose of the present section is to prove an adaption of Trèves' conjecture to the setting of ultradistributions of Beurling type associated with a non-quasianalytic weight function ω. Thus, we will show that in theorem A condition i) implies v). Furthermore, we will give some results dealing with P-convexity for ω-singular supports which are valid in arbitrary dimension.

Spaces of ultradistributions of Beurling type generalize classical distributions by allowing more flexible growth conditions for the Fourier transforms of the corresponding test functions than the Paley-Wiener weights. We choose the ultradistributional framework in the sense of Braun, Meise, and Taylor, as introduced in [7]. We begin by recalling some well-known facts about ultradistributions.

Definition 4.8. A continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ is called a *(non-quasianalytic) weight function* if it satisfies the following properties

(α) there exists $K \geq 1$ with $\omega(2t) \leq K(1 + \omega(t))$ for all $t \geq 0$,

(β) $\int_0^\infty \frac{\omega(t)}{1+t^2}\, dt < \infty$,

(γ) $\lim_{t\to\infty} \frac{\log t}{\omega(t)} = 0$,

(δ) $\varphi = \omega \circ \exp$ is convex.

ω is extended to $\mathbb{C}^d$ by setting $\omega(z) := \omega(|z|)$. Since we are not dealing with quasianalytic weight functions we simply speak of weight functions for brevity.

For $K \subseteq \mathbb{R}^d$ compact let

$$\mathscr{D}_{(\omega)}(K) = \{f \in \mathscr{E}(\mathbb{R}^d);\ \operatorname{supp} f \subseteq K \text{ and } \int_{\mathbb{R}^d} |\hat{f}(x)| \exp(\lambda\omega(x))\, dx < \infty \text{ for all } \lambda \geq 1\}$$

be equipped with its natural Fréchet space topology, and $\mathscr{D}_{(\omega)}(X) = \bigcup \mathscr{D}_{(\omega)}(K)$, the union being taken over all compact subsets of the open subset X of $\mathbb{R}^d$, equipped with its natural (LF)-space topology. The elements of its dual space $\mathscr{D}'_{(\omega)}(X)$ are the *ultradistributions of Beurling type*.

The associated local space in the sense of Hörmander [18, Theorem 10.1.19]

$$\mathscr{E}_{(\omega)}(X) = \mathscr{D}_{(\omega)}(X)^{loc} = \{u \in \mathscr{D}'_{(\omega)}(X);\ \varphi u \in \mathscr{D}_{(\omega)}(X) \text{ for all } \varphi \in \mathscr{D}_{(\omega)}(X)\}$$

is the space of *ultradifferentiable functions of Beurling type*.

Remark 4.9. i) For each weight function ω we have $\lim_{t\to\infty} \omega(t)/t = 0$ by the remark following 1.3 of Meise, Taylor, and Vogt in [27].

ii) It is shown in [7] that condition (β) guarantees that $\mathscr{D}_{(\omega)}(X) \neq \{0\}$ and that there are partitions of unity consisting of elements of $\mathscr{D}_{(\omega)}(X)$.

iii) By [7] we have

$$\mathscr{E}_{(\omega)}(X) = \{f \in \mathscr{E}(X);\ \text{for all } k \in \mathbb{N} \text{ and } K \subseteq X, K \text{ compact } |f|_{k,K} := \sup_{\alpha\in\mathbb{N}_0^d, x\in K} |f^{(\alpha)}(x)| \exp\left(-k\varphi^*\left(\frac{|\alpha|}{k}\right)\right) < \infty\},$$

where $\varphi^*(s) = \sup\{st - \varphi(t);\, t \geq 0\}$ is the Young conjugate of φ.

iv) For $\delta > 1$ the function $\omega(t) = t^{1/\delta}$ is a weight function for which the corresponding class of ultradifferentiable functions coincides with the small Gevrey class

$$\gamma^\delta(X) = \{f \in \mathscr{E}(X);\, \forall\, K \subseteq X, K \text{ compact } \forall\, C \geq 1 : \sup_{x \in K, \alpha \in \mathbb{N}_0^d} \frac{|f^{(\alpha)}(x)|}{\alpha!^\delta C^{|\alpha|}} < \infty\}.$$

Definition 4.10. $\mathscr{E}_{(\omega)}(X)$ equipped with the seminorms

$$\{|\cdot|_{k,K} :\, k \in \mathbb{N}, K \subseteq X,\, K \text{ compact}\}$$

is a nuclear Fréchet space. Its dual $\mathscr{E}'_{(\omega)}(X)$ is equal to the space of $u \in \mathscr{D}'_{(\omega)}(X)$ for which

$$\operatorname{supp} u = \mathbb{R}^d \backslash \bigcup \{B \subseteq \mathbb{R}^d \text{ open; } u(\varphi) = 0 \text{ for all } \varphi \in \mathscr{D}_{(\omega)}(B)\}$$

is a compact subset of X.

The next theorem characterizes surjectivity of a differential operator on spaces of ultradistributions of Beurling type which is due to Björck [1, Theorem 3.4.12]. It should be noted that, although the weight functions considered here are slightly more general than the ones used in [1] the theorem is valid. More generally, complementing a result of Bonet, Galbis, and Meise [3], surjectivity of convolution operators between spaces of ultradistributions of Beurling type has been characterized by Frerick and Wengenroth in [11] and contains the following theorem as a special case. The formulation in [11] uses the notion of P-convexity for (ω)-supports instead of P-convexity for supports but these coincide, as is easily seen (see e.g. [19, Remark 2.5 i)]).

Theorem 4.11. *For $X \subseteq \mathbb{R}^d$ open, $P \in \mathbb{C}[X_1, \ldots, X_d]$ and a weight function ω the following are equivalent.*

i) $P(D) : \mathscr{D}'_{(\omega)}(X) \to \mathscr{D}'_{(\omega)}(X)$ is surjective.

ii) X is P-convex for supports as well as P-convex for (ω)-singular supports.

Recall, that an open subset X of $\mathbb{R}^d$ is called P-convex for (ω)-singular supports if for every compact subset K of X there is a compact subset L of X such that for every $u \in \mathscr{E}'_{(\omega)}(X)$ we have $\operatorname{sing\,supp}_{(\omega)} u \subseteq L$ whenever $\operatorname{sing\,supp}_{(\omega)} P(-D)u \subseteq K$.

Remark 4.12. If P is elliptic the same is obviously true for $\check{P}$. Hence $P(-D)$ has a fundamental solution E which is analytic in $\mathbb{R}^d \backslash \{0\}$. Since analytic functions are contained in $\mathscr{E}_{(\omega)}(X)$ for each weight function ω (cf. [7, Proposition 4.10]) we have in particular

$$ch(\operatorname{sing\,supp}_{(\omega)} E) = ch(\operatorname{sing\,supp}_{(\omega)} P(-D)\delta_0),$$

where $ch(A)$ denotes the convex hull of a set $A \subseteq \mathbb{R}^d$. By [2, Theorem 2.1] it therefore follows for each open set $X \subseteq \mathbb{R}^d$ and every $u \in \mathscr{D}'_{(\omega)}(X)$ that

$$\operatorname{sing\,supp}_{(\omega)} P(-D)u = \operatorname{sing\,supp}_{(\omega)} u.$$

In particular, X is P-convex for (ω)-singular supports. This and the well-known fact that every open subset X of $\mathbb{R}^d$ is P-convex for supports for elliptic P imply by theorem 4.11 the surjectivity of

$$P(D) : \mathscr{D}'_{(\omega)}(X) \to \mathscr{D}'_{(\omega)}(X)$$

whenever P is elliptic.

As in the classical case, P-convexity for (ω)-singular supports is closely related to the continuation of (ω)-ultradifferentiability of $P(-D)u$ to u. Analogously to the tools introduced by Hörmander in order to deal with the classical case (see e.g. [18, Section 11.3] and section 2.1) Langenbruch introduced the following notions in [23]. For a polynomial P, a subspace V of $\mathbb{R}^d$, and $t > 0, \xi \in \mathbb{R}^d$ let

$$\sigma_{P,(\omega)}(V) := \inf_{t \geq 1} \liminf_{\xi \to \infty} \frac{\tilde{P}_V(\xi, t\omega(\xi))}{\tilde{P}(\xi, t\omega(\xi))}.$$

If we formally set $\omega \equiv 1$, we obtain Hörmander's classical definition of $\sigma_P(V)$ as studied in section 2.1. In order to simplify notation we write $\sigma_{P,(\omega)}(y)$ instead of $\sigma_{P,(\omega)}(\text{span}\{y\})$ for $y \in \mathbb{R}^d$.

The next proposition is an immediate consequence of [23, Theorem 2.5] and the ultradistributional analogue of the part of proposition 2.5 used in section 2.2.

Proposition 4.13. *Let $X_1 \subseteq X_2$ be open convex subsets of $\mathbb{R}^d$. Assume that every hyperplane $H = \{x \in \mathbb{R}^d; \langle x, N\rangle = \alpha\}$, $N \in S^{d-1}, \alpha \in \mathbb{R}$ with $\sigma_{P,(\omega)}(N) = 0$ which intersects X_2 already intersects X_1.*

Then for every $u \in \mathscr{D}'_{(\omega)}(X_2)$ with $P(D)u \in \mathscr{E}_{(\omega)}(X_2)$ as well as $u_{|X_1} \in \mathscr{E}_{(\omega)}(X_1)$ we already have $u \in \mathscr{E}_{(\omega)}(X_2)$.

PROOF. Let $u \in \mathscr{D}'_{(\omega)}(X_2)$ satisfy $P(D)u \in \mathscr{E}_{(\omega)}(X_2)$ and $u|_{X_1} \in \mathscr{E}_{(\omega)}(X_1)$. Since X_2 is convex it follows from the Theorem of supports (see e.g. [17, Theorem 4.3.3]) and [3, Theorem A] that there is $v \in \mathscr{E}_{(\omega)}(X_2)$ such that $P(D)v = P(D)u$ so that $w := u - v \in \mathscr{D}'_{(\omega)}(X_2)$ satisfies $P(D)w = 0$ as well as $w|_{X_1} \in \mathscr{E}_{(\omega)}(X_1)$. Hence, by [23, Theorem 2.5] it follows that $w \in \mathscr{E}_{(\omega)}(X_2)$ which proves the theorem. □

As in the classical case, when investigating P-convexity for (ω)-singular supports by means of the above proposition it is necessary to study the zeros of $\sigma_{P,(\omega)}$ in S^{d-1}. In order to do so, recall the definition of ω-localizations of P at infinity, as introduced by Langenbruch in [23]. For a polynomial P and $\xi \in \mathbb{R}^d$ we set

$$P_{\xi,\omega}(x) := P(\xi + \omega(\xi)x)$$

which is again a polynomial of the same degree as P. The set of all limits in $\mathbb{C}[X_1, \ldots, X_d]$ of the normalized polynomials

$$x \mapsto \frac{P_{\xi,\omega}(x)}{\tilde{P}_{\xi,\omega}(0)}$$

as ξ tends to infinity is denoted by $L_\omega(P)$. More precisely, if $N \in S^{d-1}$ then the set of limits where $\xi/|\xi| \to N$ (with ξ tending to infinity) is denoted by $L_{\omega,N}(P)$.

Obviously, $L_\omega(P)$ as well as $L_{\omega,N}(P)$ are closed subsets of the unit sphere of all polynomials in d variables of degree not exceeding the degree of P, equipped with the norm $Q \mapsto \tilde{Q}(0)$. The non-zero multiples of elements of $L_\omega(P)$ (resp. of $L_{\omega,N}(P)$) are called *ω-localizations of P at infinity* (resp. *ω-localizations of P at infinity in direction N*). Since $\omega(\xi) = \omega(|\xi|)$, $Q \in L_{\omega,N}(\check{P})$ if and only if $\check{Q} \in L_{\omega,-N}(P)$. Again, if we formally set $\omega \equiv 1$ we obtain the well-known set $L(P)$ of localizations of P at infinity (see Hörmander [18, Definition 10.2.6] or section 2.1).

For the classical case, i.e. if formally $\omega \equiv 1$, the next result is lemma 2.6. The proof is exactly the same as the one of 2.6 so that we omit it.

Lemma 4.14. *Let P be of degree m with principal part P_m.*

i) For every subspace V of $\mathbb{R}^d$ and $t \geq 1$ we have

$$\liminf_{\xi\to\infty} \frac{\tilde{P}_V(\xi, t\omega(\xi))}{\tilde{P}(\xi, t\omega(\xi))} = \inf_{Q\in L_\omega(P)} \frac{\tilde{Q}_V(0,t)}{\tilde{Q}(0,t)}.$$

ii) Let $N \in S^{d-1}$ and $Q \in L_{\omega,N}(P)$. If $P_m(N) \neq 0$ then Q is constant.

iii) If P is non-elliptic then for every subspace V of $\mathbb{R}^d$ and $t \geq 1$ we have

$$\liminf_{\xi\to\infty} \frac{\tilde{P}_V(\xi, t\omega(\xi))}{\tilde{P}(\xi, t\omega(\xi))} = \inf_{N\in S^{d-1}, P_m(N)=0} \ \inf_{Q\in L_{\omega,N}(P)} \frac{\tilde{Q}_V(0,t)}{\tilde{Q}(0,t)}.$$

iv) With the convention that the infimum taken over an empty subset of $[0,1]$ equals 1 we have

$$\sigma_{P,(\omega)}(V) = \inf_{t\geq 1} \ \inf_{N\in S^{d-1}, P_m(N)=0} \ \inf_{Q\in L_{\omega,N}(P)} \frac{\tilde{Q}_V(0,t)}{\tilde{Q}(0,t)}.$$

In case of $\omega \equiv 1$ the corresponding result of the next proposition is due to Hörmander [18, Theorem 10.2.8] and its proof uses the Tarski-Seidenberg theorem. In our case, the proof is rather elementary.

Lemma 4.15. *If $Q \in L_{\omega,N}(P)$ then $N \in \Lambda(Q)$.*

Proof. Since $\omega(\xi) = \omega(|\xi|)$, by a linear change of coordinates we can assume without loss of generality that $N = e_1 = (1, 0, \ldots, 0)$. We denote the degree of P by m. In case of $P^{(e_1)} \equiv 0$ we clearly have by Taylor's theorem that $e_1 \in \Lambda(P)$ which clearly implies $e_1 \in \Lambda(Q)$ by the definition of $L_\omega(P)$.

Now, if $P^{(e_1)}$ does not vanish identically it follows that $P^{(e_1)}_{\xi,\omega}$ does not vanish identically either, for every $\xi \in \mathbb{R}^d$. Since $P \mapsto \sum_\alpha |P^{(\alpha)}(0)|$ is a norm on the space of all polynomials in d variables, it follows that for every $\xi \in \mathbb{R}^d$

$$0 \neq \sum_\alpha |P^{(e_1)}_{\xi,\omega}(0)| = \sum_\alpha |P^{(\alpha+e_1)}(\xi)| \omega(\xi)^{|\alpha|} = \sum_{0\leq|\alpha|\leq m-1} |P^{(\alpha+e_1)}(\xi)| \omega(\xi)^{|\alpha|},$$

because P has degree m. Hence, for every $\xi \in \mathbb{R}^d, t \in \mathbb{R}$ we have by Taylor's theorem

$$\begin{aligned}
0 &\leq \frac{|P^{(e_1)}(\xi + \omega(\xi)(x + se_1))|}{\sum_\alpha |P^{(\alpha)}(\xi)|\omega(\xi)^{|\alpha|}} \\
&= \frac{|\sum_{0\leq|\alpha|\leq m-1} P^{(\alpha+e_1)}(\xi)\omega(\xi)^{|\alpha|}\frac{1}{\alpha!}(x+se_1)^\alpha|}{\sum_\alpha |P^{(\alpha)}(\xi)|\omega(\xi)^{|\alpha|}} \\
&\leq \frac{\sum_{0\leq|\alpha|\leq m-1} |P^{(\alpha+e_1)}(\xi)|\omega(\xi)^{|\alpha|}\frac{1}{\alpha!}|(x+se_1)^\alpha|}{\sum_{0\leq|\alpha|\leq m-1} |P^{(\alpha+e_1)}(\xi)|\omega(\xi)^{1+|\alpha|}} \\
&\leq \frac{\max_{0\leq|\alpha|\leq m-1}\frac{1}{\alpha!}|(x+se_1)^\alpha|}{\omega(\xi)}.
\end{aligned}$$

Since $Q \in L_\omega(P)$ there is $(\xi_n)_{n\in\mathbb{N}}$ tending to infinity such that

$$Q(x) = \lim_{n\to\infty} \frac{P(\xi_n + \omega(\xi_n)x)}{\tilde{P}_{\xi_n,\omega}(0)}.$$

In particular, we also have

$$Q^{(e_1)}(x) = \lim_{n\to\infty} \frac{P^{(e_1)}(\xi_n + \omega(\xi_n)x)}{\tilde{P}_{\xi_n,\omega}(0)}.$$

The space of all polynomials in d variables of degree not exceeding m being finite dimensional, all norms on it are equivalent. Therefore, by passing to a subsequence of $(\xi_n)_{n\in\mathbb{N}}$ if necessary, there is $c > 0$ such that for every $x \in \mathbb{R}^d$ and $s \in \mathbb{R}$

$$\begin{aligned}
|Q^{(e_1)}(x + se_1)| &= \lim_{n\to\infty} \frac{|P^{(e_1)}(\xi_n + \omega(\xi_n)(x + se_1))|}{\tilde{P}_{\xi_n,\omega}(0)} \\
&\leq c \lim_{n\to\infty} \frac{|P^{(e_1)}(\xi_n + \omega(\xi_n)(x + se_1))|}{\sum_\alpha |P^{(\alpha+e_1)}(\xi_n)|\omega(\xi_n)^{|\alpha|}} \\
&\leq c \lim_{n\to\infty} \frac{\max_{0\leq|\alpha|\leq m-1}\frac{1}{\alpha!}|(x+se_1)^\alpha|}{\omega(\xi_n)} \\
&= 0.
\end{aligned}$$

Hence, for each $x \in \mathbb{R}^d$ the polynomial $q_x : \mathbb{R} \to \mathbb{C}, s \mapsto Q(x + se_1)$ satisfies $q_x'(s) = Q^{(e_1)}(x+se_1) = 0$. Thus q_x is constant which shows that $e_1 \in \Lambda(Q)$. □

With the aid of the previous lemma we can prove the next result exactly as lemma 4.2. Again we omit the proof.

Lemma 4.16. *Let $P \in \mathbb{C}[X_1, X_2]$ be a non-constant polynomial with principal part P_m. Then*

$$\{y \in \mathbb{R}^2\backslash\{0\};\, \sigma_{P,(\omega)}(y) = 0\} \subseteq \{y \in \mathbb{R}^2\backslash\{0\};\, P_m(y) = 0\}.$$

In particular, the set $\{y \in S^1;\, \sigma_{P,(\omega)}(y) = 0\}$ is finite.

Before we continue to discuss the two-dimensional case we provide a general sufficient condition for P-convexity for (ω)-singular supports in arbitrary dimension similar to theorems 2.9 and 2.11. In order to do this, we first recall an analogue to the classical characterization of P-convexity for singular supports in the context of ultradistributions of Beurling type which is due to Björck [1, Theorem 3.4.2 and Theorem 3.4.4]. Again, although the weight functions considered here are slightly more general than the ones in [1], the theorem is still valid in our context (see also [19, Theorem 4.2]).

Theorem 4.17. *i) If $u \in \mathscr{E}'_{(\omega)}(\mathbb{R}^d)$ then*

$$ch(sing\,supp_{(\omega)}u) = ch(sing\,supp_{(\omega)}P(-D)u).$$

ii) For an open subset X of $\mathbb{R}^d$ the following are equivalent.

a) X is P-convex for (ω)-singular supports.

b) For each $u \in \mathscr{E}'_{(\omega)}(X)$ one has

$$dist(sing\,supp_{(\omega)}u, X^c) = dist(sing\,supp_{(\omega)}P(-D)u, X^c).$$

From the above theorem it follows immediately that convex open sets are P-convex for ω-singular supports (see [1, Corollary 3.4.3]). Moreover, as in the classical case (cf. [18, Theorem 10.6.4 and/or Theorem 10.7.4]) it follows that the interior of the intersection of any family of P-convex sets for ω-singular supports (see [1, Theorem 3.4.5]) as well as the set of points having a neighborhood contained in all but finitely many members of a family of sets being P-convex for ω-singular supports is again P-convex for ω-singular supports.

Using the characterization of P-convexity for (ω)-singular supports given in 4.17 ii) and theorem 4.13 we can prove part i) of the next theorem by exactly the same kind of arguments as theorem 2.9 ii). Part ii) follows from i) as theorem 2.11 ii) follows from theorem 2.9 ii). Because the proofs are verbatim the same we omit them.

Theorem 4.18. *Let $X \subseteq \mathbb{R}^d$ be open and connected, $P \in \mathbb{C}[X_1, \ldots, X_d]$.*

i) X is P-convex for (ω)-singular supports if for every $x \in \partial X$ and any $r > 0$ there are convex sets $C_1 \subseteq C_2 \subseteq \mathbb{R}^d \backslash X$ such that $x \in C_2, C_1 \subseteq \mathbb{R}^d \backslash B(0,r)$ and every hyperplane H with $\sigma_{P,(\omega)}(H^\perp) = 0$ intersecting C_2 already intersects C_1.

ii) X is P-convex for (ω)-singular supports if for every $x \in \partial X$ there is an open convex cone $\Gamma \neq \mathbb{R}^d$ such that $(x + \Gamma^\circ) \cap X = \emptyset$ and $\sigma_{P,(\omega)}(y) \neq 0$ for all $y \in \Gamma$.

Now, changing the obvious in the proof of theorem 4.3 we obtain a characterization of P-convexity for (ω)-singular supports of open subsets $X \subseteq \mathbb{R}^2$.

Theorem 4.19. *For a non-constant polynomial $P \in \mathbb{C}[X_1, X_2]$ and an open connected set $X \subseteq \mathbb{R}^2$ the following are equivalent.*

i) X is P-convex for (ω)-singular supports.

ii) The intersection of X with every hyperplane H satisfying $\sigma_{P,(\omega)}(H^\perp) = 0$ is convex.

iii) For every $x \in \partial X$ there is an open convex cone $\Gamma \neq \mathbb{R}^2$ with $\sigma_{P,(\omega)}(y) \neq 0$ for all $y \in \Gamma$ and $(x + \Gamma^\circ) \cap X = \emptyset$.

As in section 4.1 we derive our next result from the above theorem, theorem 4.1 and the fact that $P(D)$ is surjective on $\mathscr{D}'_{(\omega)}(X)$ if and only if X is P-convex for supports as well as P-convex for (ω)-singular supports.

Theorem 4.20. *Let $X \subseteq \mathbb{R}^2$ be open and let $P \in \mathbb{C}[X_1, X_2]$ be a non-constant polynomial. The following are equivalent.*

i) $P(D) : \mathscr{E}(X) \to \mathscr{E}(X)$ is surjective.

ii) $P(D) : \mathscr{D}'_{(\omega)}(X) \to \mathscr{D}'_{(\omega)}(X)$ is surjective for some non-quasianalytic weight function ω.

iii) $P(D) : \mathscr{D}'_{(\omega)}(X) \to \mathscr{D}'_{(\omega)}(X)$ is surjective for each non-quasianalytic weight function ω.

iv) The intersection of every characteristic line with any connected component of X is convex.

The next example shows that for $d \geq 3$ an analogous result to the above theorem is not true in general. See also Langenbruch [23, Example 3.13], where it is even shown that surjectivity of $P(D)$ on $\mathscr{D}'_{(\omega)}(X)$ for $d \geq 3$ depends explicitly on the weight function ω in general.

Example 4.21. Let $d > 2$ and $P(x_1, \ldots, x_d) = x_1^2 - x_2^2 - \ldots - x_d^2$. Moreover, let $\Gamma := \{x \in \mathbb{R}^d;\, x_d > (x_1^2 + \ldots + x_{d-1}^2)^{1/2}\}$. Then Γ is an open convex cone with $\Gamma^\circ = \overline{\Gamma}$. Set $X := \mathbb{R}^d \backslash \overline{\Gamma}$. As seen in 2.24, X is P-convex for supports but not P-convex for singular supports. Hence, $P(D)$ is surjective on $\mathscr{E}(X)$ but $P(D)$ is not surjective on $\mathscr{D}'(X)$.

Moreover, as in 2.24 one checks that $Q(\xi) = (\xi_1 - \xi_2)/\sqrt{2}$ is a ω-localization at infinity in direction $1/\sqrt{2}(1, 1, 0, \ldots, 0)$. It therefore follows from lemma 4.14 that

$$\liminf_{\xi \to \infty} \frac{\tilde{P}_{\mathrm{span}\{e_d\}}(\xi, \omega(\xi))}{\tilde{P}(\xi, \omega(\xi))} = 0$$

where $e_d = (0, \ldots, 0, 1)$.

Setting $H = \{x \in \mathbb{R}^d;\, \langle x, e_d \rangle = -1\}$ and

$$K := H \cap \{x \in \mathbb{R}^d; |x| \leq 2\}$$

it is easily seen that the distance of $\partial X = \partial \Gamma$ to K is 1 while the distance of $\partial \Gamma$ to $\partial_H K$, i.e. to the boundary of K relative H, strictly increases 1. Hence, it follows from [23, Corollary 2.7] that $P(D)$ cannot be surjective on $\mathscr{D}'_{(\omega)}(X)$.

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